Math 33A - Midterm 2

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Discussion session: 1B

Problems	Points	Score
1	35	35
2	30	27
3	25	25
4	10	10
Total	100	97

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Problem 1. *(35 points)* Let V be the subspace of $\mathbb{R}^4$ with basis:

$$\mathcal{B} = \left\{ v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\4\\6\\8\end{bmatrix} \right\}.$$

15 (a) *(15 points)* Construct an orthonormal basis of V from the basis $\mathcal{B}$ and write the matrix (relative to the standard basis) that gives the orthogonal projection operator on V.

20 (b) *(20 points)* Find a basis for $V^\perp$ (by any method). → on back.

a). $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$

$v_2^{\parallel} = \langle v_2, u_1\rangle u_1 = \left(\begin{bmatrix}2\\4\\6\\8\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right) \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$= \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$

$v_2^{\perp} = v_2 - v_2^{\parallel} = \begin{bmatrix}2\\4\\6\\8\end{bmatrix} - \begin{bmatrix}5\\5\\5\\5\end{bmatrix} = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix}.$

$\|v_2^{\perp}\| = \sqrt{9+1+1+9} = \sqrt{20} = 2\sqrt{5}.$

$u_2 = \dfrac{v_2^{\perp}}{\|v_2^{\perp}\|} = \dfrac{1}{2\sqrt{5}}\begin{bmatrix}-3\\-1\\1\\3\end{bmatrix}.$ ✓

$\mathcal{B}_{\text{ortho}} = \left\{ \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2\sqrt{5}}\begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \right\}.$ let $w_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $w_2 = \frac{1}{2\sqrt{5}}\begin{bmatrix}-3\\-1\\1\\3\end{bmatrix}.$

since $\mathcal{B}_{\text{ortho}}$ is an orthonormal basis,

$\text{proj}_V(x) = \text{proj}_{w_1}(x) + \text{proj}_{w_2}(x).$ ✓

$= w_1\langle w_1, x\rangle + w_2\langle w_2, x\rangle = w_1 w_1^T x + w_2 w_2^T x$

$= (w_1 w_1^T + w_2 w_2^T)\, x.$

$\Rightarrow w_1 w_1^T = \dfrac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$

$= \dfrac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}.$

$w_2 w_2^T = \dfrac{1}{20}\begin{bmatrix}-3\\-1\\1\\3\end{bmatrix}\begin{bmatrix}-3 & -1 & 1 & 3\end{bmatrix}.$

$= \dfrac{1}{20}\begin{bmatrix}9&3&-3&-9\\3&1&-1&-3\\-3&-1&1&3\\-9&-3&3&9\end{bmatrix}.$

$\text{proj}_V(x) = Px$

$P = \dfrac{1}{20}\begin{bmatrix}5&5&5&5\\5&5&5&5\\5&5&5&5\\5&5&5&5\end{bmatrix} + \dfrac{1}{20}\begin{bmatrix}9&3&-3&-9\\3&1&-1&-3\\-3&-1&1&3\\-9&-3&3&9\end{bmatrix}$

✓ $= \dfrac{1}{20}\begin{bmatrix}14&8&2&-4\\8&6&4&2\\2&4&6&8\\-4&2&8&14\end{bmatrix}.$

$\text{Im}(V^{\perp}) = \ker(V^T)$

let $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix}$.

$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$,

$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

free vars.

$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

let $x_3 = 1$, $x_4 = 0$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

let $x_3 = 0$, $x_4 = 1$.

$\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

basis for $V^{\perp}$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 2. (*30 points*)

(a) (*20 points*) Are the following vectors of $\mathbb{R}^3$

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$$w_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad w_2 = \begin{bmatrix}2\\5\\7\end{bmatrix}, \quad w_3 = \begin{bmatrix}3\\9\\13\end{bmatrix},$$

linearly independent or not? Justify your answer.

Remark: You cannot use determinants.

7 (b) (*10 points*) Let V be a subspace of $\mathbb{R}^n$ such that $dim(V) < n$ and let A be an $n \times n$ matrix such that $ker(A) = V^{\perp}$. What is $dim(Im(A))$? Is A invertible? Justify your answer.

a) $c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + c_2 \begin{bmatrix}2\\5\\7\end{bmatrix} = \begin{bmatrix}3\\9\\13\end{bmatrix}$.

$$\begin{cases} c_1 + 2c_2 = 3 \\ c_1 + 5c_2 = 9 \\ c_1 + 7c_2 = 13 \end{cases} \Rightarrow \quad 3c_2 = 6 \Rightarrow \underline{c_2 = 2}.$$

$c_1 + 2(2) = 3 \Rightarrow \underline{c_1 = -1}$

plug into 3rd eq. $-1 + 7(2) = 14$

w_1, w_2, w_3 not L.I ✓

a bit more explanation here would be nice

b). we know that $dim(V^{\perp}) + dim(V) = n$ ✓

$dim(ker(A)) = dim(V^{\perp})$ ✓

since $dim(V) < n \Rightarrow dim(V^{\perp}) = n - dim(V) > 0$. ✓

$\Rightarrow dim(ker(A)) > 0$ ✓

~~by rank-nullity, $dim(Im(A)) + dim(ker(A)) = n$~~

~~$\Rightarrow dim(Im(A)) = n - dim(ker(A))$~~

~~$= n - dim(V^{\perp}) < n$~~ ✓

since A is invertible only if $dim(ker(A)) = 0$,

$\therefore$ A is not invertible ✓

$dim(Im(A)) = \ldots$?

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Problem 3. *(25 points)* Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that is given by

$$L(x, y) = (3x + 4y, 4x - 3y).$$

Find a basis C of $\mathbb{R}^2$ such the matrix of L with respect to C is given by the matrix

$$C = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

Remark: You do not need to verify that the basis that you will give in the end is correct by direct computation as long as you explain clearly your method.

$L(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$.

$\Rightarrow A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+4y \\ 4x-3y \end{bmatrix} \Rightarrow A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

want $[L(x,y)]_C = C\begin{bmatrix} x \\ y \end{bmatrix}_C$.

let $S = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ st basis C of $\mathbb{R}^2 = \left\{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right\}$.

$\Rightarrow S\,L(x,y) = AS\begin{bmatrix} x \\ y \end{bmatrix}$. since S is a basis & is square, it is invertible.

$\Rightarrow L(x,y) = S^{-1}AS\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow C = S^{-1}AS$.

$\Rightarrow SC = AS$

$\Rightarrow \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$,

$\Rightarrow \begin{bmatrix} 5v_1 & -5w_1 \\ 5v_2 & -5w_2 \end{bmatrix} = \begin{bmatrix} 3v_1+4v_2 & 3w_1+4w_2 \\ 4v_1-3v_2 & 4w_1-3w_2 \end{bmatrix}$.

$$\begin{cases} 5v_1 = 3v_1 + 4v_2 \\ 5v_2 = 4v_1 - 3v_2 \\ -5w_1 = 3w_1 + 4w_2 \\ -5w_2 = 4w_1 - 3w_2 \end{cases} \Rightarrow \begin{cases} 2v_1 = 4v_2 \\ 8v_2 = 4v_1 \\ -8w_1 = 4w_2 \\ 4w_1 = -2w_2 \end{cases} \Rightarrow \begin{cases} v_1 = 2v_2 \\ w_2 = -2w_1 \end{cases}$$

let $v_2 = 1$, $w_1 = 1$

basis $\Rightarrow \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$

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Problem 4. *(10 points)* Find a 2×2 matrix A such that

$$Im(A) = ker(A).$$

$\text{rank}(A) = 1$

$$\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \Rightarrow Im(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \Rightarrow ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$