MATH 33A - MIDTERM 1

Name: Yijing Zhou

UID: 404-186-693.

Discussion session: 2F

Question 1: 17 / 25

Question 2: 23 / 25

Question 3: 25 / 25

Question 4: 25 / 25

Total: 90 / 100

Problem 1. *(25 points)*

(a) *(10 points)* Let A be a square and invertible matrix. Show that

$$(A^T)^{-1} = (A^{-1})^T.$$

(b) *(15 points)* Find the inverse of the matrix

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

2

15

How to deal with general $n \times n$ matrices.

show that

a) $(A^T)^{-1} = (A^{-1})^T$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the assumption is that A is invertible.

$(A^T)^{-1} = \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)^{-1}$

then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$(A^{-1})^T = \left[\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\right]^T$ $= \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$= \left[\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\right]$ $\therefore (A^{-1})^T = (A^T)^{-1}$ # shown.

b) $B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$\Rightarrow B^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ Augmented matrix form

$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$

performing Gauss - Jordan elimination.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$ Switch R_1 with R_2

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$ $R_3 - 4R_1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$ $R_3 + 3R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$ $R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \quad \text{(cont'd)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \quad \xrightarrow{R_3 \times \frac{1}{2}}$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]}_{= I \qquad\qquad = B^{-1}} \quad \xrightarrow{R_1 - 3R_3}$$

$$\therefore B^{-1} = \begin{bmatrix} \cancel{-\frac{2}{9}}\ {-9/2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

Problem 2. *(25 points)*

(a) *(10 points)* Find the matrix of the linear transformation that first projects a vector $v \in \mathbb{R}^2$ on the line $y = x$, then rotates it counterclockwise by a $\pi/4$ angle, and then scales it by a factor $k > 0$.

(b) *(15 points)* Let w be a vector on a line L in $\mathbb{R}^2$ that passes through the origin. Consider the $proj_L(v)$ which is the projection of another vector $v \in \mathbb{R}^2$ onto L. Finally let A be the 2×2 matrix whose columns are the vectors w and $proj_L(v)$. Is A invertible? Justify your answer.

a) line $L \Rightarrow y = x$

$\therefore$ a unit vector on the line

$$\vec{l} = \frac{\langle 1, 1 \rangle}{\sqrt{1+1}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle. \qquad l_1 = \frac{1}{\sqrt{2}} \quad l_2 = \frac{1}{\sqrt{2}}$$

$$\therefore proj_L = \begin{bmatrix} l_1^2 & l_1 l_2 \\ l_1 l_2 & l_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$R^{\theta} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

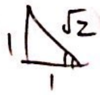
$$S_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

let L denote the matrix that undergoes all linear transformations.

$$L = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

$\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



$$L = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right).$$

b) let $\mu, \lambda \in \mathbb{R}$ $\quad y_1, y_2 \in \mathbb{R}^3$

$$F(\lambda y_1 + \mu y_2) = F(\lambda y_1) + F(\mu y_2)$$

$$= \lambda F(y_1) + \mu F(y_2) \text{ if } F \text{ is } \underline{L.T}$$

$$F(\lambda y_1 + \mu y_2) = \underbrace{v_1 \times (\lambda y_1 + \mu y_2)}_{①} + \underbrace{A(\lambda y_1 + \mu y_2)}_{②}$$

$$\lambda F(y_1) + \mu F(y_2) = \lambda(v_1 \times y_1 + Ay_1) + \mu(v_1 y_2 + \mu A y_2)$$

$$= \lambda v_1 \times y_1 + \mu v_2 \times y_2 + \lambda A y_1 + \mu A y_2$$

$$= \underbrace{\lambda v_1 \times y_1 + \mu v_1 \times y_2}_{③} + \underbrace{A(\lambda y_1 + \mu y_2)}_{④}$$

$\Rightarrow$ ② = ④

Showing ① $\Leftrightarrow$ ③.

$\frac{y_3}{v_3}$ ① $\Rightarrow$ y_3 of y_1

① $\Rightarrow$ $v_1 \times (\lambda y_1 + \mu y_2)$

$$= \begin{bmatrix} v_2(\lambda y_{31} + \mu y_{32}) - v_3(\lambda y_{21} + \mu y_{22}) \\ v_3(\lambda y_{11} + \mu y_{12}) - v_1(\lambda y_{31} + \mu y_{32}) \\ v_1(\lambda y_{21} + \mu y_{22}) - v_2(\lambda y_{11} + \mu y_{12}) \end{bmatrix}$$

ok

③ $\Rightarrow$ $\lambda v_1 y_1 + \mu v_1 y_2$

$$= \begin{bmatrix} v_2 \lambda y_{31} - v_3 \lambda y_1 \\ v_3 \lambda y_{11} - v_1 \lambda y_{31} \\ v_1 \lambda y_{21} - v_2 \lambda y_{11} \end{bmatrix} + \begin{bmatrix} v_2 \mu y_{32} - v_3 \mu y_{22} \\ v_3 \mu y_{12} - v_1 \mu y_{32} \\ v_1 \mu y_{22} - v_2 \mu y_{12} \end{bmatrix}$$

$$= \begin{bmatrix} v_2(\lambda y_{31} + \mu y_{32}) - v_3(\lambda y_{21} + \mu y_{22}) \\ v_3(\lambda y_{11} + \mu y_{12}) - v_1(\lambda y_{31} + \mu y_{32}) \\ v_1(\lambda y_{21} + \mu y_{22}) - v_2(\lambda y_{11} + \mu y_{12}) \end{bmatrix}$$

$\therefore$ ③ = ①

$\therefore F(\lambda y_1 + \mu y_2) = \lambda F(y_1) + \mu F(y_2)$

$\therefore$ it is a linear transformation.

Problem 4. *(25 points)*

(a) *(10 points)* Is the set

$$V = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$$

nothing to do with subspace U, h ii)

a subspace of $\mathbb{R}^2$?

(b) *(15 points)* Let V, W be two subspaces of $\mathbb{R}^n$. Show that

$$V \cap W = \{x \in \mathbb{R}^n \mid x \in V \text{ and } x \in W\}$$

is also a subspace of $\mathbb{R}^n$.

A) No, the set is not a subspace of $\mathbb{R}^2$.

$\because$ take $(2,-2)$ and $(2,2) \in V$.

$(2,-2) + (2,2) = (4,0) \notin V$

$\Rightarrow$ not a subspace of $\mathbb{R}^2$.

10

b). take $x_1 \in V$ and $\in W$ $\quad \mu, \lambda \in \mathbb{R}$.

$x_2 \in U$ and $\in W$.

15

$\mu x_1 + \lambda x_2 \in V$ and $\in W$. is true

$\because V, W$ is a subspace $\Rightarrow$ all $x \in V$

and $\mu x \in V$ $\forall$ $\mu \in \mathbb{R}$.

$\Rightarrow$ all $x \in W$

and $\lambda x \in W$ $\forall \in \lambda \in \mathbb{R}$.

$\therefore$ $\mu x_1 + \lambda x_2 \in V$ and $\in W$ $\forall$ all μ and $\lambda \in \mathbb{R}$.

$\therefore$ by definition, V and W are subspace of $\mathbb{R}^n$.