Math 33A - Midterm 1

Name: Stacy Li

UID: 905137772

Discussion session: 1C

Problems	Points	Score
1	30	30
2	25	20
3	30	29
4	15	14
Total	100	93

30

Problem 1. *(30 points)*

(a) *(20 points)* For which $a \in \mathbb{R}$ is the matrix A given below invertible?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 8 & a \end{bmatrix}.$$

Remark: You can't use determinants and you don't have to compute the inverse.

(b) *(10 points)* Set $a = 1$ in the matrix A given in part (a) above and solve the system

$$Ax = b$$

for

$$b = \begin{bmatrix} 6 \\ 8 \\ 9 \end{bmatrix}.$$

a. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & \cancel{1} & \cancel{0} & \cancel{0} \\ 2 & 5 & 9 & \cancel{0} & \cancel{1} & \cancel{0} \\ 3 & 8 & a & \cancel{0} & \cancel{0} & \cancel{1} \end{array}\right] \xrightarrow[iii-3i]{ii-2i} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & a-9 \end{bmatrix} \xrightarrow{iii-2ii} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & a-15 \end{bmatrix}$

A is invertible if $a \neq 15$ ✓

b. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 5 & 9 & 8 \\ 3 & 8 & 1 & 9 \end{array}\right] \xrightarrow[iii-3i]{ii-2i} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & -8 & -9 \end{array}\right] \xrightarrow[iii-2ii]{i-2ii} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 14 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & -14 & -1 \end{array}\right]$

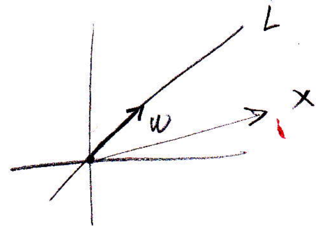
$\xrightarrow{-1/14\, iii} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 14 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 1/14 \end{array}\right] \xrightarrow[ii-3iii]{i+3iii} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 14 + 3/14 \\ 0 & 1 & 0 & -4 - 3/14 \\ 0 & 0 & 1 & 1/14 \end{array}\right]$

$\frac{196}{14} + 3/14$ $199/14$ $\frac{-56}{14} - \frac{3}{14}$ $\frac{-59}{14}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 + 3/14 \\ -4 - 3/14 \\ 1/14 \end{bmatrix}$$ ✓

$14 \times 14 = 56 + 140 = 196$

20



$w = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$W = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$

Problem 2. *(25 points)*

(a) *(15 points)* Consider a line L in $\mathbb{R}^2$ that passes through the origin and w a unit vector starting from $(0,0)$ that is parallel to L. Write down the 2×2 matrix R_L that gives the reflection operation:

$$ref_L(x) = R_L \cdot x \text{ for all } x \in \mathbb{R}^2,$$

that reflects a vector $x \in \mathbb{R}^2$ across L (you only have to write down R_L, you don't have to derive it). Show that

$$R_L^2 = R_L \cdot R_L = I_2.$$

Does the above computation mean that R_L invertible or not? If yes what is it its inverse?

Hint: For the computation $R_L^2 = I_2$ you can just argue geometrical.

(b) *(10 points)* Consider the following subset of $\mathbb{R}^2$:

$$S = \{(x, y) | y^4 = x^{16}\}.$$

Is S a subspace of $\mathbb{R}^2$ or not? Justify your answer.

a. $ref_L(x) = 2\, proj_L(x) - x$

? $= 2 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

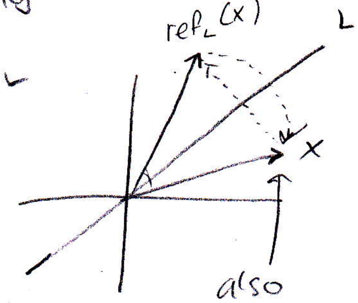
$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

R_L

-5

R_L is invertible as shown by that calculation because reflecting a vector across a line and then doing it again gets you the original vector. inverse of R_L is R_L

$(R_L)^{-1} = R_L$



also $ref_L(ref_L(x))$

$y = x^4$

b. $(1,1), (2,16) \rightarrow$ Yes, in subspace

$(1,1) + (2,16) = (3, 17) \rightarrow$ no! not in subspace.

No because it doesn't respect the addition rule

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Problem 3. *(30 points)*

14 (a) *(15 points)* Find the matrix of the linear transformation that first rotates a vector $v \in \mathbb{R}^2$ by an angle of $\pi/4$ and then scales it by 10. Does the order of operations matter in this particular case?

counterclockwise

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

15 (b) *(15 points)* Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by

$$f(x, y, z) = (y + z, x + z, x + y).$$

Is f a linear transformation or not? Justify your answer. Moreover, if your answer is yes, find a 3×3 matrix A such that

$$f(X) = AX \text{ for any } X \in \mathbb{R}^3.$$

there exist X, Y invertible with $XY \neq YX$.

a. Order ~~doesn't~~ matter b/c both operations are ~~invertible~~

A = transformation matrix

$$A = 10 \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \text{ or } \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2}/2 \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

or

$$\boxed{\begin{bmatrix} 5\sqrt{2} & -5\sqrt{2} \\ 5\sqrt{2} & 5\sqrt{2} \end{bmatrix}}$$

b. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$ let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$f(\lambda_1 x + \lambda_2 y) \stackrel{?}{=} \lambda_1 f(x) + \lambda_2 f(y)$$

$$f\left(\begin{bmatrix} \lambda_1 x_1 + \lambda_2 y_1 \\ \lambda_1 x_2 + \lambda_2 y_2 \\ \lambda_1 x_3 + \lambda_2 y_3 \end{bmatrix}\right)$$

$$\begin{bmatrix} \lambda_1 x_2 + \lambda_2 y_2 + \lambda_1 x_3 + \lambda_2 y_3 \\ \lambda_1 x_1 + \lambda_2 y_1 + \lambda_1 x_3 + \lambda_2 y_3 \\ \lambda_1 x_1 + \lambda_2 y_1 + \lambda_1 x_2 + \lambda_2 y_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 (x_2 + x_3) + \lambda_2 (y_2 + y_3) \\ \lambda_1 (x_1 + x_3) + \lambda_2 (y_1 + y_3) \\ \lambda_1 (x_1 + x_2) + \lambda_2 (y_1 + y_2) \end{bmatrix}$$

$$\lambda_1 f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \lambda_2 f\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

$$\lambda_1 \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} y_2 + y_3 \\ y_1 + y_3 \\ y_1 + y_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 (x_1 + x_3) + \lambda_2 (y_2 + y_3) \\ \lambda_1 (x_1 + x_3) + \lambda_2 (y_1 + y_3) \\ \lambda_1 (x_1 + x_2) + \lambda_2 (y_1 + y_2) \end{bmatrix}$$

These are equal yes it's linear

thanks!

MORE

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Problem 4. *(15 points)*
Let A be an $n \times n$ matrix such that for some $b \in \mathbb{R}^n$, $b \neq 0_n$, the equation

$$Ax = b$$

has a *unique* solution. Can the equation

$$\left[\begin{array}{cc|c} n & n & b_1 \\ n & n & b_2 \end{array}\right]$$

$$Ax = 0_n$$

have more than one solutions? Justify your answer.

No because if $Ax=b$ has a unique solution (meaning there's no case where a row is $0=0 \leftarrow$ infinite solutions because of free variable nor a case where a row gets $0=1 \leftarrow$ no solution) Then for $Ax=0_n \leftarrow$ which is $\left[\begin{array}{c|c} Ax & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}\right]$, there won't be a case of $0=0$ nor $0=1$.

$Ax=0_n$ has a unique solution (just one)

very informal.
why won't "$0=0$" or "$0=1$" ~~[illegible]~~ happen?