Math 33A - Midterm 1

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UID: 

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Discussion session: 1B

Problems	Points	Score
1	30	30
2	25	21
3	30	30
4	15	15
Total	100	96

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Problem 1. *(30 points)*

(a) *(20 points)* For which $a \in \mathbb{R}$ is the matrix A given below invertible?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 8 & a \end{bmatrix}.$$

Remark: You can't use determinants and you don't have to compute the inverse.

(b) *(10 points)* Set $a = 1$ in the matrix A given in part (a) above and solve the system

$$Ax = b$$

for

$$b = \begin{bmatrix} 6 \\ 8 \\ 9 \end{bmatrix}.$$

a). A is invertible if its rows are independent.

$\Rightarrow [3\ 8\ a] = x[1\ 2\ 3] + y[2\ 5\ 9]$ must not have a solution for x, y

$\Rightarrow \begin{cases} x + 2y = 3 \quad \rightarrow 2x + 4y = 6. \\ 2x + 5y = 8 \\ 3x + 9y = a \end{cases}$

(ii) − 2(i) $\quad y = 2 \quad \Rightarrow a = 3(-1) + 9(2) = 15.$

$x = 3 - 2y = 3 - 4 = -1$

$\therefore$ A invertible when $a \neq 15$. ✓

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 5 & 9 & 8 \\ 3 & 8 & 1 & 9 \end{array}\right] \begin{array}{l} \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & -8 & -9 \end{array}\right] \begin{array}{l} \\ \\ -2(\mathrm{II}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & -14 & -1 \end{array}\right] \begin{array}{l} \\ \\ / -\tfrac{1}{14} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & \tfrac{1}{14} \end{array}\right] \begin{array}{l} -3(\mathrm{III}) \\ -3(\mathrm{III}) \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & \tfrac{81}{14} \\ 0 & 1 & 0 & -\tfrac{59}{14} \\ 0 & 0 & 1 & \tfrac{1}{14} \end{array}\right] \begin{array}{l} -2(\mathrm{II}) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \tfrac{199}{14} \\ 0 & 1 & 0 & -\tfrac{59}{14} \\ 0 & 0 & 1 & \tfrac{1}{14} \end{array}\right] \Rightarrow x = \frac{1}{14}\begin{bmatrix} 199 \\ -59 \\ 1 \end{bmatrix}$$ ✓

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Problem 2. *(25 points)*

(a) *(15 points)* Consider a line L in $\mathbb{R}^2$ that passes through the origin and w a unit vector starting from $(0,0)$ that is parallel to L. Write down the 2×2 matrix R_L that gives the reflection operation:

$$ref_L(x) = R_L \cdot x \text{ for all } x \in \mathbb{R}^2,$$

that reflects a vector $x \in \mathbb{R}^2$ across L (you only have to write down R_L, you don't have to derive it). Show that

$$R_L^2 = R_L \cdot R_L = I_2.$$

Does the above computation mean that R_L invertible or not? If yes what is it its inverse?

Hint: For the computation $R_L^2 = I_2$ you can just argue geometrical.

(b) *(10 points)* Consider the following subset of $\mathbb{R}^2$:

$$S = \{(x,y) | y^4 = x^{16}\}.$$

Is S a subspace of $\mathbb{R}^2$ or not? Justify your answer.

a) $R_L = 2P_L - I$

$= 2\frac{ww^T}{w^Tw} - I$

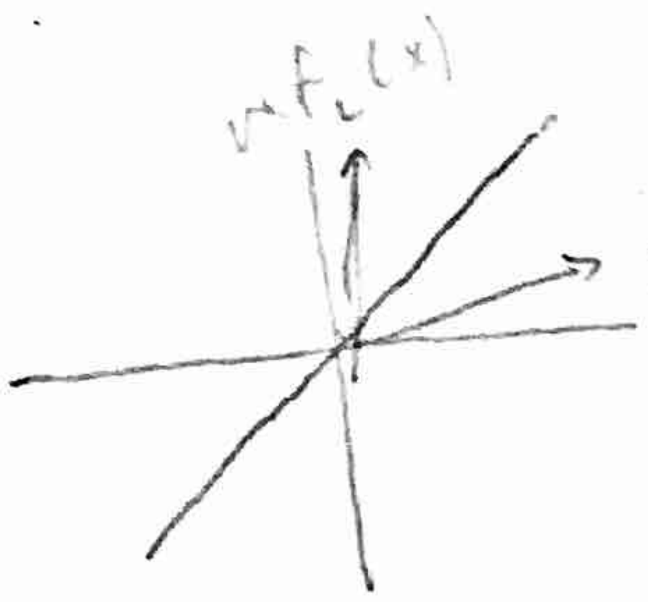
$\text{proj}_w v = w\frac{w^Tv}{w^Tw}$

$\Rightarrow P_L = \frac{ww^T}{w^Tw}$

let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\Rightarrow R_L = \frac{2}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 - \frac{w_1^2+w_2^2}{2} & w_1w_2 \\ w_1w_2 & w_2^2 - \frac{w_1^2+w_2^2}{2} \end{bmatrix}$

$= \frac{2}{w_1^2+w_2^2}\begin{bmatrix} \frac{w_1^2-w_2^2}{2} & w_1w_2 \\ w_1w_2 & \frac{w_2^2-w_1^2}{2} \end{bmatrix}$ ok



reflecting over a line twice gives back the same vector, so $R_L^2 = I_2$. R_L is invertible

$\Rightarrow R_L^{-1} = R_L$

b) need to check if $c\begin{bmatrix} x \\ y \end{bmatrix} \in S \Rightarrow$ if $y^4 = x^{16}$ then $(cy)^4 = c^4y^4 \neq (cx)^{16} = c^{16}x^{16}$

$c \in \mathbb{R}, \begin{bmatrix} x \\ y \end{bmatrix} \in S$

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So S is not a subspace of $\mathbb{R}^2$. Why?

30.

Problem 3. *(30 points)*

(a) *(15 points)* Find the matrix of the linear transformation that first rotates a vector $v \in \mathbb{R}^2$ by an angle of $\pi/4$ and then scales it by 10. Does the order of operations matter in this particular case?

(b) *(15 points)* Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by

$$f(x, y, z) = (y + z, x + z, x + y).$$

Is f a linear transformation or not? Justify your answer. Moreover, if your answer is yes, find a 3×3 matrix A such that

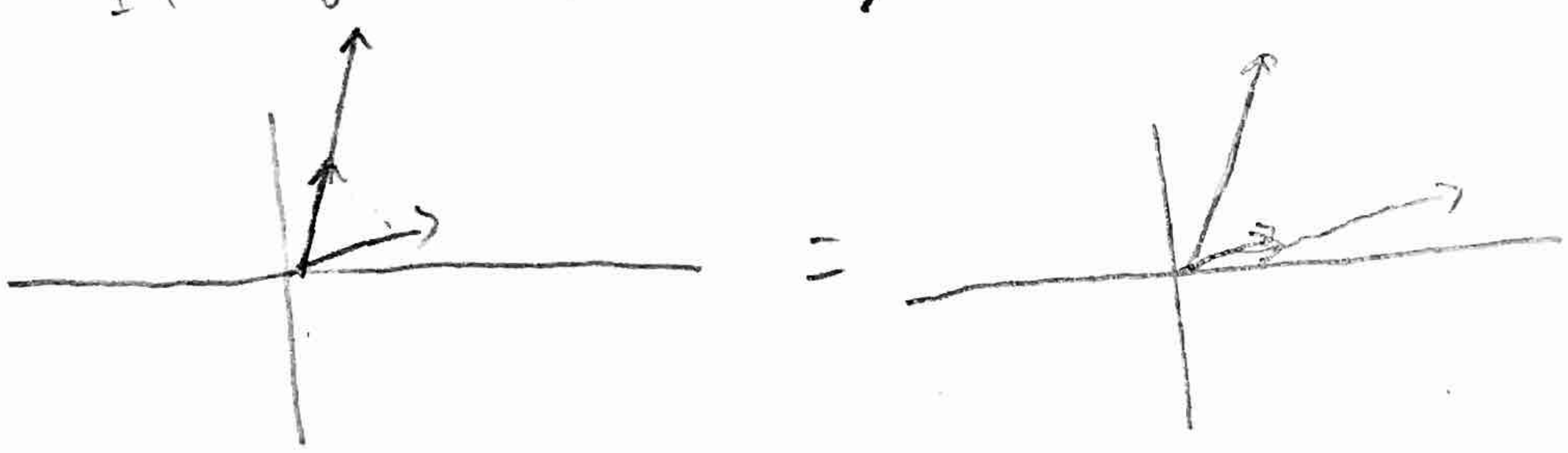
$$f(X) = AX \text{ for any } X \in \mathbb{R}^3.$$

a). rotation : $\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

scale : $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

rotate, then scale : $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} & -5\sqrt{2} \\ 5\sqrt{2} & 5\sqrt{2} \end{bmatrix}$

It doesn't matter the order.



or, we know the scale matrix $= 10 I_2$ and for any matrix A, $AI = IA$.

$\Rightarrow 10I \cdot R = R \cdot 10I$, so order doesn't matter

b) need to check

$f(x+y) = f(x) + f(y) \Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} x_2+y_2+x_3+y_3 \\ x_1+y_1+x_3+y_3 \\ x_1+y_1+x_2+y_2 \end{bmatrix}$

$= \begin{bmatrix} x_2+x_3 \\ x_1+x_3 \\ x_1+x_2 \end{bmatrix} + \begin{bmatrix} y_2+y_3 \\ y_1+y_3 \\ y_1+y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + f\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$

$f(cx) = cf(x) \Rightarrow f\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} cx_2+cx_3 \\ cx_1+cx_3 \\ cx_1+cx_2 \end{bmatrix} = c\begin{bmatrix} x_2+x_3 \\ x_1+x_3 \\ x_1+x_2 \end{bmatrix} = cf\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$

$\therefore$ f is a linear transformation

$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

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Problem 4. *(15 points)*
Let A be an $n \times n$ matrix such that for some $b \in \mathbb{R}^n$, $b \neq 0_n$, the equation

$$Ax = b$$

has a *unique* solution. Can the equation

$$Ax = 0_n$$

have more than one solutions? Justify your answer.

no. If there were some solution x_1 such that $Ax_1 = 0$, then for some $c \in \mathbb{R}$

nice

$$A(x + cx_1) = Ax + cAx_1$$
$$= b + 0 = b.$$

which means $Ax = b$ would not have a unique solution, which is a contradiction ✓