∞ rank=n rank< m
1 rank = m = n

1. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

   Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

   (a) Every vector in $\mathbb{R}^4$ is a linear combination of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{e}_4$.

   True False

   (b) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if rref(A) contains a row of zeros.

   True False

   counter example:

   $$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   not inconsistent, has infinitely many solutions.

   (c) If A is a 3×4 matrix of rank 3 then the system $A\vec{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ has infinitely many solutions.

   True False

   (d) There exists a nonzero lower triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   $$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots \\ \cdot & \end{bmatrix}$$

   $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \cdot \\ \cdot & \cdot \end{bmatrix}$$

   True False

   Nonzero low triangular 2x2 matrix must have a 1 in the top left (a_{11}) position, so A^2 will always have a 1.

   (e) If A and B are invertible $n \times n$ matrices then $A^{-1}B^{-1}AB = I_n$.

   True False

   $A^{-1}B^{-1}AB = I$
   $AA^{-1}B^{-1}AB = AI$
   $B^{-1}AB = A$
   $BB^{-1}AB = BA$
   $AB = BA$ → not always true because A, B may not commute

2. (10 points) Show your work.

(a) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_2 \end{bmatrix}.$$

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$.

$$\underset{(4\times 3)}{A}\ \underset{(3\times 1)}{\vec{x}} = \underset{(4\times 1)}{\vec{b}}$$

$$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & \frac{1}{2} & 0 \end{bmatrix}}$$

check: $\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_2 \end{bmatrix}$ ✓

(b) If A is a 4×4 matrix such that

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

then what is

$$A\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}?$$

$$A = \begin{bmatrix} | & | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \\ | & | & | & | \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & | & 1 & | \\ 0 & T(\vec{e}_2) & -1 & T(\vec{e}_4) \\ -1 & | & 0 & | \\ 1 & | & 0 & | \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ a linear comb. of $\vec{e}_1$ and $\vec{e}_3$

By linearity, $A\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = -1 \cdot T(\vec{e}_1) + 2 \cdot T(\vec{e}_3) = -\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 0-2 \\ 1+0 \\ -1+0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}}$

$T(a\vec{v}_1 + b\vec{v}_2)$
$= aT(\vec{v}_1) + bT(\vec{v}_2)$

3. (10 points) For each of the matrices below, either find the inverse or explain why no inverse exists **without using determinants**. Show your work.

(a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 0&1&0&0&1&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 1&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right] \text{ swap } R_1, R_2, R_3 \longrightarrow \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&1&0 \\ 0&1&0&0&1&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right]$$

$$\text{rref}\begin{bmatrix} 0&0&1&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$

The inverse of the identity matrix is itself

(b) $\begin{bmatrix} 1 & -1 & 5 \\ 1 & 8 & -12 \\ 4 & 5 & 3 \end{bmatrix}$

$3-4(5)$
$3-20=-17$

$$\left[\begin{array}{ccc|ccc} 1&-1&5&1&0&0 \\ 1&8&-12&0&1&0 \\ 4&5&3&0&0&1 \end{array}\right] R_2-R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1&-1&5&1&0&0 \\ 0&9&-17&-1&1&0 \\ 4&5&3&0&0&1 \end{array}\right] \xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|ccc} 1&-1&5&1&0&0 \\ 0&9&-17&-1&1&0 \\ 0&9&-17&-4&0&1 \end{array}\right] \xrightarrow{R_3-R_2}$$

$$\left[\begin{array}{ccc|ccc} 1&-1&5&1&1&0 \\ 0&9&-17&-1&1&0 \\ 0&0&0&-3&-1&1 \end{array}\right]$$

There exists no inverse for the matrix, because $\text{rref}\begin{bmatrix} 1&-1&5 \\ 1&8&-12 \\ 4&5&3 \end{bmatrix}$ is not I_3, it has a row of 0's.

4. (10 points) Find all lower triangular 2×2 matrices of the form

$$X = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$$

such that X commutes with **every** 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with a, b, c, d all nonzero. Show your work!

A, X commute $\Rightarrow AX = XA$

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} ax+by & bz \\ cx+dy & dz \end{bmatrix}$$

$$XA = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} xa & xb \\ ay+cz & by+dz \end{bmatrix}$$

$xa = ax + by$

$0 = by$

$b = 0$ or $y = 0$

↑

arbitrary, so $y = 0$

$xb = bz$

$x = z$ because b is arbitrary.

$cx + dy = ay + cz$

$dy = ay$

$y = 0$

$dz = by + dz$

$0 = by$

$y = 0$

Thus $\boxed{X = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}}$

5. (10 points) Multiple Choice. Write your answers (A, B, C, etc.) in the boxes provided.

Each of the linear transformations listed below corresponds to one (and only one) of the matrices A through J. Match them up.

I	scaling
J	rotation
A	reflection
G	orthogonal projection
H	shear

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad J = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

scaling $= \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

shear $= \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

rotation $= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $a^2+b^2=1$

reflection $= \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ $a^2+b^2=1$

projection $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$