1. (10 points) (a) Which of the following is the QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 9 \\ -1 & 2 & -4 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}?$$

Circle your answer.

$$\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 12 \\ 0 & 12 & 18 \\ 0 & 0 & 6\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 44 \\ 1 & -4 & 22 \\ 1 & -4 & 29 \\ 1 & -5 & 37 \end{bmatrix} \begin{bmatrix} 1 & 6 & 10 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

(b) Write down the result of the Gram-Schmidt process on the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 9 \\ -4 \\ 3 \\ 2 \end{bmatrix}.$$

You do not have to show your work (if you need more room, use Page 0).

Box your answer

$12 \quad 6 - \frac{5}{2}$

$-\frac{2}{2} - \frac{5}{2} - 1 - \frac{5}{2}$

$9+4-3+2$

$13-3+2$

$12/4 = 3$

$\frac{2}{4}(1-2-2+1)$

$-\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

$\|\vec{v}_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix} \qquad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix}1\\2\\2\\1\end{bmatrix} - \left(\begin{bmatrix}1\\2\\2\\1\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$\frac{1}{4}$

$9-4+3+2$

$5+3+2$

10

$$= \begin{bmatrix}1\\2\\2\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\3/2\\3/2\end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{3}\begin{bmatrix}3/2\\3/2\\3/2\\3/2\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$\sqrt{\frac{9}{4}+\frac{9}{4}+\frac{9}{4}+\frac{9}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{7}\begin{bmatrix}7/2\\-7/2\\7/2\\-7/2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 = \begin{bmatrix}9\\-4\\3\\2\end{bmatrix} - \left(\begin{bmatrix}9\\-4\\3\\2\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} - \left(\begin{bmatrix}9\\-4\\3\\2\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}9\\-4\\3\\2\end{bmatrix} - \begin{bmatrix}3\\-3\\-3\\3\end{bmatrix} - \begin{bmatrix}5/2\\5/2\\5/2\\5/2\end{bmatrix} = \begin{bmatrix}7/2\\-7/2\\7/2\\-7/2\end{bmatrix}$$

$$\boxed{\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}, \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}}$$ form an o.n.b.

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

(a) Suppose that $\vec{v}$ is *any* nonzero vector in $\mathbb{R}^2$. Explain why the vectors $\vec{v}$, $A\vec{v}$, and $A^2\vec{v}$ must be linearly dependent. (*Note:* do not use any numerical examples in your answer; your reasoning must be valid no matter what $\vec{v}$ is.)

2 linearly independent vectors span $\mathbb{R}^2$, so ≥ 1 of the vectors in $\vec{v}$, $A\vec{v}$, and $A^2\vec{v}$ must be a linear combination of the other two, thus they are linearly dependent. The max. number of linearly independent vectors in the basis of $\mathbb{R}^2$ is 2.

(b) Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Part (a) shows that there are scalars c_0, c_1, and c_2 such that

$$c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} = \vec{0}. \tag{1}$$

Explain why this shows that the matrix $c_0 I + c_1 A + c_2 A^2$ is not invertible.

$A\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$A^2\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$C_0 I + C_1 A + C_2 A^2$ is not invertible because there exist nonzero scalars that cause the sum to be the zero matrix, meaning that there are infinitely many vectors in $\operatorname{Ker}(C_0 I + C_1 A + C_2 A^2)$, so $\operatorname{Ker} \neq \{\vec{0}\}$, implying it is not invertible.

(c) Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find scalars c_0, c_1, and c_2 such that (1) holds. $\boxed{\text{Box your answer}}$

$A\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$A^2\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$$C_0\vec{v} + C_1 A\vec{v} + C_2 A^2\vec{v} = \vec{0}$$

$$C_0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \vec{0}$$

$$6\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \vec{0}$$

$$\boxed{\begin{array}{l} C_0 = 6 \\ C_1 = 1 \\ C_2 = -1 \end{array}}$$

3. (10 points) $\boxed{\text{You do not need to show work on this page.}}$ Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 \\ 1 & 1 & -1 & 4 & -2 \end{bmatrix}.$$

redundant redundant

(a) Find a basis for the image of A. $\boxed{\text{Box your answer}}$

basis of im $A = \boxed{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right)}$

(b) What does the rank-nullity theorem say in this case?
(Your answer should be a simple equation of the form $1+1=2$).

$\boxed{3 + 2 = 5}$

rank(A), nullity(A) m where A is $n \times m$ matrix

(c) Find a basis for the kernel of A. $\boxed{\text{Box your answer}}$

$$\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

basis of ker $A = \boxed{\left(\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/2 \\ 1 \end{bmatrix} \right)}$

4. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) If A is an invertible matrix then $\ker(A) = \ker(A^{-1})$.

$\ker(A) = A\vec{x}=\vec{0}$

$\ker(A^{-1}) = A^{-1}\vec{x}=\vec{0}$

True **False**

$A\vec{x}=A^{-1}\vec{v}$

$A\vec{x}=0$

$A^{-1}\vec{v}=0$

$A^2\vec{x}=AA^{-1}\vec{v}$

$A^2\vec{x}=\vec{v}$

$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ $A^{-1}=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$

$ad-bc=1$

$A^2=[1\ 4$

(b) If A is an orthogonal matrix then A^TA is also an orthogonal matrix.

$A^TA=I$

True False

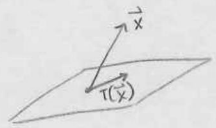
(c) There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

True **False**

(d) If V is a two-dimensional subspace of $\mathbb{R}^3$ (a plane) then there is a basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of the linear transformation $T = \text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ has the form

$V=\text{span}(\vec{v}_1,\vec{v}_2)$

$B=\begin{bmatrix}| & | \\ A\vec{v}_1 & A\vec{v}_2 \\ | & |\end{bmatrix}$



$$B = \begin{array}{c} \begin{array}{ccc} A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \end{array} \\ \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \end{array}.$$

$[\vec{x}]=S[\vec{x}]_{\mathcal{B}}$ $B=S^{-1}AS$

$T(\vec{x})=B[T(\vec{x})]_{\mathcal{B}}$

True **False**

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"'] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"'] \end{tikzcd}$$

$\text{proj}_V(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + 0\vec{u}_3$

(e) If A is a symmetric $n \times n$ matrix and S is an orthogonal $n \times n$ matrix then the matrix $S^{-1}AS$ is symmetric.

True False

$A^T=A$

$S^{-1}=S^T$

$$(S^{-1}AS)^T = S^TA^T(S^{-1})^T = S^{-1}A(S^T)^T = S^{-1}AS$$

5. (10 points) The following statements are the possible answers to questions (a) and (b) below.

(A) The linear system $A\vec{x} = \vec{0}$ has a unique solution

(B) $\dim(\operatorname{im} A) + \dim(\ker A) = n$

(C) $\operatorname{rank}(A) = n$

(D) A is upper triangular

(E) The column vectors of A span $\mathbb{R}^n$

(F) $A^T A = I_n$ A is orth. $\Rightarrow$ A invert.

(G) $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

(H) $A^n = 0$

(I) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$ A is orth. $\Rightarrow$ A invert.

(a) Which of the statements above implies that the $n \times n$ matrix A is invertible? There are five correct answers.

A	C	E	I	F

(b) Suppose that A is an invertible $n \times n$ matrix. Which of the properties above does A satisfy? There are five correct answers.

B	C	E	A	G