Math 33A Final Exam

Sei Hoon Chang

TOTAL POINTS

99 / 100

QUESTION 1

1 Vector in kernel 5 / 5

✓ - **0 pts** Correct

- **5 pts** Blank or completely incorrect

- **1 pts** Computation mistakes or incorrect form

QUESTION 2

2 Im and Ker of orthogonal projection 4 / 5

- **0 pts** Correct

- **5 pts** Blank or completely incorrect

- **1 pts** Computation mistakes or wrong dimension

✓ - **1 pts** Incorrect form (eg. basis not span)

- **3 pts** Show some efforts but not enough (eg. unfamiliar with the definitions or did not find a basis for the plane)

QUESTION 3

Diagonal or rotation-scaling 8 pts

3.1 (a) 4 / 4

✓ + **4 pts** Correct

+ **2 pts** Correct rotation-scaling with real entries based on given complex eigenvalues (or diagonal with complex eigenvalues)

+ **1 pts** Rotation-scaling with real entries (or diagonal with complex eigenvalues)

+ **2 pts** Eigenvalues 4-i, 4+i

+ **1 pts** Used characteristic equation but incorrect or missing eigenvalues

+ **0 pts** Incorrect

3.2 (b) 4 / 4

✓ + **4 pts** Correct

+ **2 pts** Correct matrix based on computed eigenvalues

+ **2 pts** Eigenvalues 1 and 3

+ **1 pts** Used characteristic equation but incorrect or missing eigenvalues

+ **0 pts** Incorrect

QUESTION 4

Change of basis 10 pts

4.1 (a) 6 / 6

✓ + **6 pts** Correct

+ **5 pts** Minor error (matrix off by one or two entries)

+ **3 pts** Stated B = S^-1 A S

+ **2 pts** Computed S^-1

+ **3 pts** Stated columns = [A v_i]_B

+ **2 pts** B-coordinate vector computation

+ **1 pts** Computed AS or S^-1 A

+ **0 pts** Incorrect

4.2 (b) 4 / 4

✓ + **4 pts** Correct (diagonal entries of upper triangular B from part (a))

+ **2 pts** Characteristic equation of A or B

+ **0 pts** Incorrect

QUESTION 5

Eigenspaces of symmetric matrices are orthogonal 10 pts

5.1 (a) 4 / 4

✓ - **0 pts** Correct

- **4 pts** Blank or completely incorrect

- **3 pts** Show efforts but not correct (eg. mistakenly took the transpose or unfamiliar with the definition of symmetry matrix)

- **2 pts** Incomplete solution (eg. not using A is symmetric)

5.2 (b) 6 / 6

✓ - **0 pts** Correct

- **6 pts** Blank or completely incorrect

- **1 pts** Not rigorous for some details (eg. what if one the eigenvalues is 0 or one is the additive inverse of the other one) or missing conclusion

- **4 pts** Show efforts but not enough/correct (eg. use spectral theorem directly instead of proving from part a)

- **5 pts** Try something but not complete or incorrect (eg. A is not necessarily invertible or orthogonal)

QUESTION 6

Eigenstuff 12 pts

6.1 (a) 3 / 3

✓ - **0 pts** Correct

- **2 pts** Missing eigenvalue

- **1 pts** One incorrect multiplicity

- **3 pts** Incorrect

6.2 (b) 5 / 5

✓ - **0 pts** Correct

- **2 pts** Incorrect basis for eigenvalue 1

- **2 pts** Incorrect basis for eigenvalue 0

- **5 pts** Incorrect

- **1 pts** Incorrect notation

6.3 (c) 2 / 2

✓ - **0 pts** Correct

- **2 pts** Incorrect, based on part b

6.4 (d) 2 / 2

✓ - **0 pts** Correct

- **1 pts** Incorrect justification

- **2 pts** Incorrect, based on previous parts

QUESTION 7

7 SVD 10 / 10

✓ - **0 pts** Correct

- **8 pts** Serious mistakes; serious conceptual

misunderstanding; perhaps U or V isn't orthogonal or the dimensions are wrong.

- **3 pts** A computational mistake

- **10 pts** Incorrect

- **5 pts** Multiple computational mistakes

- **7 pts** Wrong dimensions but otherwise close

QUESTION 8

TF 20 pts

8.1 (a) 2 / 2

✓ - **0 pts** True

- **2 pts** False

8.2 (b) 2 / 2

- **2 pts** True

✓ - **0 pts** False

8.3 (c) 2 / 2

✓ - **0 pts** True

- **2 pts** False

- **2 pts** No answer

8.4 (d) 2 / 2

- **2 pts** True

✓ - **0 pts** False

8.5 (e) 2 / 2

- **2 pts** True

✓ - **0 pts** False

8.6 (f) 2 / 2

✓ - **0 pts** True

- **2 pts** False

8.7 (g) 2 / 2

- **2 pts** True

✓ - **0 pts** False

8.8 (h) 2 / 2

✓ - **0 pts** True

- **2 pts** False

8.9 (i) 2 / 2

- **2 pts** True

✓ **- 0 pts False**

8.10 (j) 2 / 2

✓ **- 0 pts True**

- **2 pts** False

9.9 (i) 2 / 2

✓ **- 0 pts J**

- **2 pts** not J

9.10 (j) 2 / 2

✓ **- 0 pts D**

- **2 pts** not D

QUESTION 9

Fill in the blanks 20 pts

9.1 (a) 2 / 2

✓ **- 0 pts J**

- **2 pts** Not J

9.2 (b) 2 / 2

✓ **- 0 pts A**

- **2 pts** not A

9.3 (c) 2 / 2

✓ **- 0 pts F**

- **2 pts** not F

9.4 (d) 2 / 2

✓ **- 0 pts B**

- **2 pts** not B

9.5 (e) 2 / 2

✓ **- 0 pts None**

- **2 pts** not None

9.6 (f) 2 / 2

✓ **- 0 pts H**

- **2 pts** not H

9.7 (g) 2 / 2

✓ **- 0 pts E**

- **2 pts** not E

9.8 (h) 2 / 2

✓ **- 0 pts H and/or K**

- **2 pts** neither H nor K

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2D		R	ROYCE 162
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Section	2	B

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Page:	1	2	3	4	5	6	7	8	Total
Points:	10	8	10	10	12	10	20	20	100
Score:									

1. (5 points) Suppose that A is a 5×4 matrix of the form

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

Given that the vector $\begin{bmatrix} 7 \\ 3 \\ -2 \\ 8 \end{bmatrix}$ is in the kernel of A, write $\vec{v}_4$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Box your answer.

Since in kernel, we know $7v_1 + 3v_2 - 2v_3 + 8v_4 = 0$

So $\boxed{v_4 = \dfrac{2v_3 - 3v_2 - 7v_1}{8}}$

2. (5 points) Suppose that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto the plane $3x_1 + x_2 - 2x_3 = 0$. Find a basis for $\text{im}(T)$ and a basis for $\text{ker}(T)$. Box your answers.

kernel consists of normal vector to plane so

$\boxed{\ker(T) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\right)}$

image consists of 2 ~~time~~ L.I vectors in plane

So $\boxed{\text{im}(T) = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right)}$

3. (8 points) For each of the 2×2 matrices A below, there is an invertible matrix S such that $B = S^{-1}AS$ is either a diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or a rotation-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Find B in each case (you do not have to find S).

(a) $A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$

$(15-(-2))$

17

$$\lambda^2 - 8\lambda + (15+2) = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-4(17)}}{2} \overset{68}{=} \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = \underline{4 \pm i}$$

Since 2 complex, we know $a=4$, $b=1$ so

$$B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

3

trace $= 5-1 = 4$

$$\lambda^2 - 4\lambda + (-5+8) = 0$$

$$(\lambda - 3)(\lambda - 1) = 0 \qquad \lambda = 3, 1$$

Since 2 eigenvals, $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$\downarrow$

can flip eigenvals

4. (10 points) Let $T(\vec{x}) = A\vec{x}$ be the linear transformation with matrix

$$A = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}.$$

(a) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Find the matrix of T in the basis $\mathcal{B}$.
(Another way to say this is: find a matrix B such that $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$.)

$$A\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \qquad A\vec{v}_3 = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}$$

$$= \vec{v}_1 \qquad\qquad = 2\vec{v}_1 + 4\vec{v}_2 \qquad\qquad = 3\vec{v}_1 + 6\vec{v}_3 + 5\vec{v}_2$$

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = B$$

(b) Find the eigenvalues of A, repeating any eigenvalues according to their algebraic multiplicities. (So if $\lambda_2 = 17$ has algebraic multiplicity 2, list it twice.)
Hint: use your answer from part (a), which should be nice enough that you don't need to compute a determinant for this part.

The eigenvalues of A are $1, 4, 6$
(eigenvals of similar matrices are same)
algebraic
all multipl.
of 1

5. (10 points) Make sure to fully justify your answers on this page.

Suppose that A is a symmetric $n \times n$ matrix (this is an assumption for both (a) and (b) below).

(a) Show that $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$ for any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$.

Hint: remember that another way to write the dot product is $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$.

$$A\vec{v} \cdot \vec{w} = (A\vec{v})^T \vec{w} = \vec{v}^T A^T \vec{w} = \vec{v}^T A \vec{w} = \vec{v} \cdot A\vec{w}$$

$$\text{Hence,} \quad \boxed{A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}}$$

(b) Suppose that $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues λ and μ, respectively. Show that if $\lambda \neq \mu$ then $\vec{v}$ is orthogonal to $\vec{w}$.

A. Using $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$, we get

① $A\vec{v} \cdot \vec{w} = \lambda \vec{v} \cdot \vec{w} = \lambda (\vec{v} \cdot \vec{w})$

② $\vec{v} \cdot A\vec{w} = \vec{v} \cdot (\mu \vec{w}) = \mu (\vec{v} \cdot \vec{w})$

Now apply 1, 2 to equation to get

$$\lambda (\vec{v} \cdot \vec{w}) = \mu (\vec{v} \cdot \vec{w})$$

$$(\lambda - \mu)(\vec{v} \cdot \vec{w}) = 0$$

this cannot be 0 since DISTINCT eig vals,

so $\vec{v} \cdot \vec{w}$ must be 0. Hence,

$\vec{v}$ is ortho to $\vec{w}$.

6. (12 points) Define

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A and state the algebraic multiplicity of each eigenvalue. (No justification needed for this part.)

Eigenvalues: 1 (alg mult 2)

0 (alg mult 2)

(b) Find a basis for each eigenspace.

$E_1 = \ker(A - I) = \ker\left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$

→ 0 vector, all others are L.I basis ✓

$E_0 = \ker(A - 0I) = \ker\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$

2 redundant

2 vectors form basis

$\lambda = 1: E_1$ basis: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 0: E_0$ basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) State the geometric multiplicity of each eigenvalue.

$\lambda = 1$, gemu = 1

$\lambda = 0$, gemu = 2.

(d) Is A diagonalizable? Justify your answer.

No because the sum of gemu's is 3, which less than $n = 4$. (or the vectors combined do not span R^4) form basis of

Page 5

7. (10 points) For this question, it may be helpful to know that

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} .$$

$\underbrace{}_{A} \quad \underbrace{}_{S} \quad \underbrace{}_{\text{Diagonal}} \quad \underbrace{}_{S^{-1} = S^T} \quad S^T S \text{ is orthogonal}$

Find a singular value decomposition $A = U\Sigma V^T$ of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} . \qquad M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Write your answer in the form $\boxed{U = \text{something}, \ \Sigma = \text{something}, \ V = \text{something}}$. $= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Since $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, based on this info

we know $\lambda = 2, 0, 3$ and $\overset{\text{eigenvectors}}{E_2} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \ E_0 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix},$

Hence we know $\sigma = \sqrt{2}, 0, \sqrt{3}$

So $u_1 = \frac{1}{\sqrt{3}} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

u_3 can't exist b/c $\sigma = 0 \rightarrow \frac{1}{0}$

So $\boxed{U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}}$

$\underset{u_1 \ u_2}{} \qquad \underset{\text{singulars}}{} \qquad \underset{\text{ortho}}{}$

8. (20 points) True/False (circle the correct answer). You do not need to justify your answer. Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) Every 5×5 matrix has a real eigenvalue.

conjugate

~~True~~ False

(b) If A is an $n \times n$ matrix such that $\text{im}(A) = \{\vec{0}\}$ then A is invertible.

True ~~False~~ $\hookrightarrow$ ker = 0

(c) If A is symmetric and U is orthogonal then UAU^T is symmetric.

~~True~~ False $(UAU^T)^T$
$(U^T)^T(UA)^T = UA^TU^T$
$= UAU^T$

(d) If A is a symmetric matrix and λ is one of its eigenvalues, then λ is also one of the singular values of A.

$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

True ~~False~~ $\hookrightarrow$ absolute value?
absolute value

(e) The matrices $\begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 4 & 8 & 1 \\ 5 & 1 & 6 \\ 23 & 4 & 2 \end{bmatrix}$ are similar.   trace

True ~~False~~

(f) If $\ker(A^3) = \ker(A^4)$ and $\vec{v}$ is a vector in $\ker(A^5)$, then $\vec{v}$ is automatically in $\ker(A^4)$.

$A^3 v = 0$ $A^3 v = A^4 v$ ~~True~~ False $A^5 v$ $A^5 v = A^4 v$
$A^4 v = 0$

$A(A^4 v)$
(g) If A is a 4×7 matrix then $\text{rank}(A) + \dim(\ker(A)) = 4$. $A(A^3 v) = A^4 v$

True ~~False~~

(h) If A is a symmetric $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

~~True~~ False $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(i) If A is an upper triangular $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

True ~~False~~

(j) If A is an $n \times n$ matrix such that $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$ then A is an orthogonal matrix.

~~True~~ False

9. (20 points) Fill in the blanks with the matrices below (just write A, B, etc. in each blank). You may use some items more than once, and there are some items that you will not use at all. **If none of the matrices on this list makes the statement true, write NONE.**

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Suppose that $M\vec{x} = \vec{b}$ is a linear system with 3 equations and 3 unknowns. If the system has a unique solution then the reduced row echelon form of M equals ___J___.

(b) Suppose that R is a 3×2 matrix and that S is a 2×3 matrix. If $\text{im}(R) = \ker(S)$ then $SR = $ ___A___.

$2 \times 3 \qquad 3 \times 2 \quad = 2 \times 2 \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) If M is a 2×2 rotation by $\pi/6$ radians counterclockwise, then $M^{18} = $ ___F___. $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$18\left(\dfrac{\pi}{6}\right) = 3\pi$ counter 180 rotate

(d) $M = $ ___B___ is a 2×2 matrix such that $\text{im}(M) = \ker(M)$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(e) $M = $ ___NONE___ is a 3×3 matrix such that $\text{im}(M) = \ker(M)$.

(f) $\lambda = 0$ is the only eigenvalue of ___H___. Furthermore, the algebraic multiplicity of λ is 3 and the geometric multiplicity of λ is 1.   G or H

(g) $\text{tr}($ ___E___ $) = 1$.   E

(h) $M = $ ___H___ is a matrix with rank 2 whose kernel is not $\{\vec{0}\}$. 3×3 H or K

$M^{-1}M$

(i) If M is an orthogonal 3×3 matrix then $M^T M = M M^T = $ ___J___.   Identity

(j) ___D___ has complex (non-real) eigenvalues. $\lambda^2 - (a+b) + (ab - cd) = 0$

$\lambda^2 + 1 = 0$ $a^2 + 2ab + b^2$ $\text{trace}^2 - 4\det < 0$

$A = \begin{bmatrix} 9-\lambda & -3 & 2 \\ -2 & 3-\lambda & 2 \\ 5 & -5 & 7-\lambda \end{bmatrix}$

$(\lambda-1)(\lambda-9)(\lambda-6)$

$(\lambda^2-5\lambda+9)(\lambda-6)$

$\lambda^3-5\lambda^2+9\lambda$

$\underline{-6\lambda^2+30\lambda-29}$

$\lambda^3-11\lambda^2+39\lambda-29$

$(9-\lambda)(\lambda^2-7\lambda+12+10)$ 88

$3(2\lambda-8-10) \rightarrow 6\lambda-54$

$2(10-15+5\lambda)$ $10\lambda-30$ -5

$4(12+16)+3(-8-10)+2(10-15)$

$= 88+54-10 = 88-64 = 29$

$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$ $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$V^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$