Math 33A Final Exam

Ashwin Shrikant Ranade

TOTAL POINTS

96 / 100

QUESTION 1

1 Vector in kernel **5 / 5**

✓ **- 0 pts** **Correct**

- 5 pts Blank or completely incorrect

- 1 pts Computation mistakes or incorrect form

QUESTION 2

2 Im and Ker of orthogonal projection **5 / 5**

✓ **- 0 pts** **Correct**

- 5 pts Blank or completely incorrect

- 1 pts Computation mistakes or wrong dimension

- 1 pts Incorrect form (eg. basis not span)

- 3 pts Show some efforts but not enough (eg. unfamiliar with the definitions or did not find a basis for the plane)

QUESTION 3

Diagonal or rotation-scaling 8 pts

3.1 (a) **4 / 4**

✓ **+ 4 pts** **Correct**

+ 2 pts Correct rotation-scaling with real entries based on given complex eigenvalues (or diagonal with complex eigenvalues)

+ 1 pts Rotation-scaling with real entries (or diagonal with complex eigenvalues)

+ 2 pts Eigenvalues 4-i, 4+i

+ 1 pts Used characteristic equation but incorrect or missing eigenvalues

+ 0 pts Incorrect

3.2 (b) **4 / 4**

✓ **+ 4 pts** **Correct**

+ 2 pts Correct matrix based on computed eigenvalues

+ 2 pts Eigenvalues 1 and 3

+ 1 pts Used characteristic equation but incorrect or missing eigenvalues

+ 0 pts Incorrect

QUESTION 4

Change of basis 10 pts

4.1 (a) **6 / 6**

✓ **+ 6 pts** **Correct**

+ 5 pts Minor error (matrix off by one or two entries)

+ 3 pts Stated B = S^-1 A S

+ 2 pts Computed S^-1

+ 3 pts Stated columns = [A v_i]_B

+ 2 pts B-coordinate vector computation

+ 1 pts Computed AS or S^-1 A

+ 0 pts Incorrect

4.2 (b) **4 / 4**

✓ **+ 4 pts** **Correct (diagonal entries of upper triangular B from part (a))**

+ 2 pts Characteristic equation of A or B

+ 0 pts Incorrect

QUESTION 5

Eigenspaces of symmetric matrices are orthogonal 10 pts

5.1 (a) **4 / 4**

✓ **- 0 pts** **Correct**

- 4 pts Blank or completely incorrect

- 3 pts Show efforts but not correct (eg. mistakenly took the transpose or unfamiliar with the definition of symmetry matrix)

- 2 pts Incomplete solution (eg. not using A is symmetric)

5.2 (b) 6 / 6

- ✓ - **0 pts** Correct
- - **6 pts** Blank or completely incorrect
- - **1 pts** Not rigorous for some details (eg. what if one the eigenvalues is 0 or one is the additive inverse of the other one) or missing conclusion
- - **4 pts** Show efforts but not enough/correct (eg. use spectral theorem directly instead of proving from part a)
- - **5 pts** Try something but not complete or incorrect (eg. A is not necessarily invertible or orthogonal)

QUESTION 6

Eigenstuff 12 pts

6.1 (a) 3 / 3

- ✓ - **0 pts** Correct
- - **2 pts** Missing eigenvalue
- - **1 pts** One incorrect multiplicity
- - **3 pts** Incorrect

6.2 (b) 5 / 5

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect basis for eigenvalue 1
- - **2 pts** Incorrect basis for eigenvalue 0
- - **5 pts** Incorrect
- - **1 pts** Incorrect notation

6.3 (c) 2 / 2

- ✓ - **0 pts** Correct
- - **2 pts** Incorrect, based on part b

6.4 (d) 2 / 2

- ✓ - **0 pts** Correct
- - **1 pts** Incorrect justification
- - **2 pts** Incorrect, based on previous parts

QUESTION 7

7 SVD 10 / 10

- ✓ - **0 pts** Correct
- - **8 pts** Serious mistakes; serious conceptual misunderstanding; perhaps U or V isn't orthogonal or the dimensions are wrong.
- - **3 pts** A computational mistake
- - **10 pts** Incorrect
- - **5 pts** Multiple computational mistakes
- - **7 pts** Wrong dimensions but otherwise close

QUESTION 8

TF 20 pts

8.1 (a) 2 / 2

- ✓ - **0 pts** True
- - **2 pts** False

8.2 (b) 2 / 2

- - **2 pts** True
- ✓ - **0 pts** False

8.3 (c) 2 / 2

- ✓ - **0 pts** True
- - **2 pts** False
- - **2 pts** No answer

8.4 (d) 0 / 2

- ✓ - **2 pts** True
- - **0 pts** False

8.5 (e) 2 / 2

- - **2 pts** True
- ✓ - **0 pts** False

8.6 (f) 2 / 2

- ✓ - **0 pts** True
- - **2 pts** False

8.7 (g) 2 / 2

- - **2 pts** True
- ✓ - **0 pts** False

8.8 (h) 2 / 2

- ✓ - **0 pts** True
- - **2 pts** False

8.9 (i) **2 / 2**

- **- 2 pts** True
- ✓ **- 0 pts False**

8.10 (j) **2 / 2**

- ✓ **- 0 pts True**
- **- 2 pts** False

9.9 (i) **2 / 2**

- ✓ **- 0 pts J**
- **- 2 pts** not J

9.10 (j) **2 / 2**

- ✓ **- 0 pts D**
- **- 2 pts** not D

QUESTION 9

Fill in the blanks 20 pts

9.1 (a) **2 / 2**

- ✓ **- 0 pts J**
- **- 2 pts** Not J

9.2 (b) **0 / 2**

- **- 0 pts** A
- ✓ **- 2 pts not A**

9.3 (c) **2 / 2**

- ✓ **- 0 pts F**
- **- 2 pts** not F

9.4 (d) **2 / 2**

- ✓ **- 0 pts B**
- **- 2 pts** not B

9.5 (e) **2 / 2**

- ✓ **- 0 pts None**
- **- 2 pts** not None

9.6 (f) **2 / 2**

- ✓ **- 0 pts H**
- **- 2 pts** not H

9.7 (g) **2 / 2**

- ✓ **- 0 pts E**
- **- 2 pts** not E

9.8 (h) **2 / 2**

- ✓ **- 0 pts H and/or K**
- **- 2 pts** neither H nor K

MATH 33A Final Exam Spring 2019

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2C	Albert Zheng	T	ROLFE 3121
2D		R	ROYCE 162
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Page:	1	2	3	4	5	6	7	8	Total
Points:	10	8	10	10	12	10	20	20	100
Score:									

1. (5 points) Suppose that A is a 5×4 matrix of the form

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

Given that the vector $\begin{bmatrix} 7 \\ 3 \\ -2 \\ 8 \end{bmatrix}$ is in the kernel of A, write $\vec{v}_4$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Box your answer.

$$7\vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3 + 8\vec{v}_4 = 0$$

$$8\vec{v}_4 = -7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$$

$$\boxed{\vec{v}_4 = \frac{1}{8}(-7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3)}$$

2. (5 points) Suppose that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto the plane $3x_1 + x_2 - 2x_3 = 0$. Find a basis for $\mathrm{im}(T)$ and a basis for $\ker(T)$. Box your answers.

$x_2 = -3x_1 + 2x_3$ let $x_1 = s$, $x_3 = t$

$x_2 = -3s + 2t$

$$\begin{bmatrix} s \\ -3s+2t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Basis for plane: $\boxed{\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \text{basis for } \mathrm{im}(T)}$

~~Find orthonormal basis.~~

~~$\vec{u}_1 = \frac{\langle 1,-3,0\rangle}{2}$ $\quad \vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1$~~

~~$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \langle 1,-3,0\rangle \cdot \langle 0,2,1\rangle$~~

ker(T) = all vectors ⊥ to plane

$\langle 1,-3,0\rangle \cdot \langle x_1, x_2, x_3\rangle = 0$

$\langle 0,2,1\rangle \cdot \langle x_1, x_2, x_3\rangle = 0$

$x_1 - 3x_2 = 0$

$2x_2 + x_3 = 0$

(rref)

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

x_3 = free variable $= t$

$x_1 + \frac{3}{2}t = 0$

$x_2 + \frac{1}{2}t = 0$

$$\begin{bmatrix} -3/2\,t \\ -1/2\,t \\ t \end{bmatrix} = \text{span}\begin{bmatrix} -3/2 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$\boxed{\text{So basis for } \ker(T) = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}}$

check: $\langle 1,-3,0\rangle \cdot \langle -3,-1,2\rangle$

$= -3+3 = 0$ ✓

$\langle 0,2,1\rangle \cdot \langle -3,-1,2\rangle$

$= -2+2 = 0$ ✓

$A = SDS^{-1}$

3. (8 points) For each of the 2×2 matrices A below, there is an invertible matrix S such that $B = S^{-1}AS$ is either a diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or a rotation-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Find B in each case (you do not have to find S).

(a) $A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$

Find eigenvalues.

characteristic polynomial:

$\lambda^2 - \operatorname{tr}A\,\lambda + \det A = 0$

$\lambda^2 - 8\lambda + (15+2) = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\lambda = \dfrac{8 \pm \sqrt{64 - 4\cdot 1\cdot 17}}{2} = \dfrac{8 \pm \sqrt{-4}}{2} = \dfrac{8 \pm 2i}{2}$

$\lambda = 4 \pm i$

$17 \times 4 = 68$

Find eigenspace of $4+i$.

$\mathcal{E}_{4+i} = \ker(A - \lambda I_2) = \ker(A - (4+i)I_2)$

$= \ker \begin{bmatrix} 3-4-i & 1 \\ -2 & 5-4-i \end{bmatrix} = \ker \begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix}$ (multiply by conjugate)

$\begin{bmatrix} -1-i \\ -2 \end{bmatrix}(-1+i) = \begin{bmatrix} (-1)^2 - (i)^2 \\ -2(-1+i) \end{bmatrix} = \begin{bmatrix} 1+(+1) \\ 2-2i \end{bmatrix} = \begin{bmatrix} 2 \\ 2-2i \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

So, one eigenvector is $\begin{bmatrix} -1+i \\ -2 \end{bmatrix} = i\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{w}} + \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{\vec{v}}$

$S = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$ ($\vec{w}$, $\vec{v}$)

$B = S^{-1}AS = \dfrac{1}{-2}\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = -\dfrac{1}{2}\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -2 & -8 \end{bmatrix} = -\dfrac{1}{2}\begin{bmatrix} -8 & 2 \\ -2 & -8 \end{bmatrix}$

$\boxed{\text{Hence, } B = \begin{bmatrix} +4 & -1 \\ +1 & +4 \end{bmatrix}}$

check: $\operatorname{tr}A = \operatorname{tr}B$

$\det(A) = \det(B)$

(b) $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

characteristic polynomial:

$\lambda^2 - \operatorname{tr}A\,\lambda + \det A = 0$

$\lambda^2 - (4)\lambda + (-5+8) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda_1 = 3, \lambda_2 = 1$

So $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}$

check: $\operatorname{tr}A = \operatorname{tr}B$

$\det A = -5+8 = 3 = \det B$

4. (10 points) Let $T(\vec{x}) = A\vec{x}$ be the linear transformation with matrix

$$A = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}.$$

(a) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by

$B = S^{-1} AS$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}. \qquad \begin{matrix} 3 \\ -3 \\ 5 \end{matrix}$$

Find the matrix of T in the basis $\mathcal{B}$.
(Another way to say this is: find a matrix B such that $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$.)

$$B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_\beta & \dots & [T(\vec{v}_3)]_\beta \\ | & & | \end{bmatrix}$$

$$T(\vec{v}_1) = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-3 \\ -2+3 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(\vec{v}_1)]_\beta = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_2) = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}_\beta = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_3) = \begin{bmatrix} +4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}_\beta = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

So $\boxed{B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}$

check: $\mathrm{tr} A = 11$
$\mathrm{tr} B = 11$

(b) Find the eigenvalues of A, repeating any eigenvalues according to their algebraic multiplicities. (So if $\lambda_2 = 17$ has algebraic multiplicity 2, list it twice.)
Hint: use your answer from part (a), which should be nice enough that you don't need to compute a determinant for this part. (since the 2 matrices are similar)

The 2 matrices are similar, so same eigenvalues.

B is upper triangular, so just look @ diagonal.

$$\boxed{\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6}$$

5. (10 points) Make sure to fully justify your answers on this page.

Suppose that A is a symmetric $n \times n$ matrix (this is an assumption for both (a) and (b) below).

(a) Show that $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$ for any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$.
Hint: remember that another way to write the dot product is $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$.

$A\vec{v} \cdot \vec{w} = (A\vec{v})^T \vec{w} = \vec{v}^T \vec{A}^T \vec{w}$. Note A is symmetric so $A^T = A$.

Then $\vec{v}^T A^T \vec{w} = \vec{v}^T A \vec{w} = \vec{v} \cdot (A\vec{w})$

Hence $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$ (without loss of generalarity)

(b) Suppose that $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues λ and μ, respectively. Show that if $\lambda \neq \mu$ then $\vec{v}$ is orthogonal to $\vec{w}$.

$A\vec{v} = \lambda\vec{v}$ $\quad A\vec{w} = \mu\vec{w}$

$\vec{w} \cdot (A\vec{v}) = \vec{w} \cdot (\lambda\vec{v})$ $\quad \vec{v} \cdot (A\vec{w}) = \vec{v} \cdot (\mu\vec{w})$

Note $\vec{w} \cdot (A\vec{v}) = \vec{v} \cdot (A\vec{w})$, by (a). So :

$\vec{w} \cdot (A\vec{v}) = \vec{v} \cdot (A\vec{w})$

$\vec{w} \cdot (\lambda\vec{v}) = \vec{v} \cdot (\mu\vec{w})$

$\lambda (\vec{w} \cdot \vec{v}) = \mu (\vec{w} \cdot \vec{v})$

$0 = \mu(\vec{w} \cdot \vec{v}) - \lambda(\vec{w} \cdot \vec{v})$

$0 = (\mu - \lambda)(\vec{w} \cdot \vec{v})$

Since $\lambda \neq \mu$, $\mu - \lambda \neq 0$, so $\vec{w} \cdot \vec{v} = 0$

$\boxed{\text{Hence, } \vec{w} \text{ and } \vec{v} \text{ are orthogonal.}}$

6. (12 points) Define

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A and state the algebraic multiplicity of each eigenvalue. (No justification needed for this part.)

$\lambda_1 = 0$ (algebraic multiplicity 2)

$\lambda_2 = 1$ (algebraic multiplicity 2)

(b) Find a basis for each eigenspace.

$\mathcal{E}_1 = \ker(A - \lambda I_4) = \ker(A)$. Rank $(A) = 2$, so $\dim(\ker(A)) = 2 = \dim(\mathcal{E}_1)$

$\text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Let $x_1 = s$, $x_3 = t$, b/c free variables. Then $x_2 = 0$, $x_4 = 0$, $x_1 = s$, $x_3 = t$

$\ker(A) = \begin{bmatrix} s \\ 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\text{So a basis for } \mathcal{E}_1 \text{, (where } \lambda_1 = 0\text{), is } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$

$\mathcal{E}_2 = \ker(A - \lambda I_4) = \ker(A - I_4) = \ker \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank $(A - I_4) = 3$, so $\dim(\ker(A - I_4)) = \dim(\mathcal{E}_2) = 1$

Trivially, $\ker(A - I_4) = \text{span}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

$\boxed{\text{So basis for } \mathcal{E}_2 \text{, where } \lambda_2 = 1\text{, is } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$

(c) State the geometric multiplicity of each eigenvalue.

gemu $(\lambda_1) = 2$ (eigenvalue $= 0$)

gemu $(\lambda_2) = 1$ (eigenvalue $= 1$)

(d) Is A diagonalizable? Justify your answer.

NO, since $\sum \text{gemu} = 2 + 1 < 4 = n$. The combined dimensions of our eigenspaces is $2+1=3$, which is less than the 4 needed to span $\mathbb{R}^4$, the dimension we are living in for the problem.

$$\begin{matrix} 2 & 1 & 0 \\ -2 & -2 & -2 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 0 \end{matrix}$$

Σ must be in increasing order

7. (10 points) For this question, it may be helpful to know that

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$

Find a singular value decomposition $A = U\Sigma V^T$ of the matrix

$\underset{2\times 3}{A} = U \; \Sigma \; V^T$
$= (2\times 2)(2\times 3)\cancel{(3\times 3)}$

$\underset{n\times m}{\Sigma}$ $\quad A = SDS^{-1}$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Write your answer in the form $\boxed{U = \text{something}, \Sigma = \text{something}, V = \text{something}}$.

$A^T A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad \det(A^TA - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$

Use cofactor expansion. $\det(A^TA - \lambda I_3) = 0 + 1(-1)^5 \det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix} + (2-\lambda)\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$

$0 = -(2-\lambda) + (2-\lambda)((2-\lambda)(1-\lambda) - 1)$

$0 = -2 + \lambda + (2-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 1)$

$= -2 + \lambda + (2-\lambda)(\lambda^2 - 3\lambda + 1) = \lambda - 2 + 2\lambda^2 - 6\lambda + 2 - \lambda^3 + 3\lambda^2 - \lambda$

$0 = +\lambda^3 + -5\lambda^2 + 6\lambda$ $\quad \lambda(\lambda^2 - 5\lambda + 6) = 0$ $\quad \lambda = 0, 2, 3$

$\lambda_1 = 3 \quad \sigma_1 = \sqrt{3}$
$\lambda_2 = 2 \quad \sigma_2 = \sqrt{2}$
$\lambda_3 = 0 \quad \sigma_3 = 0$

Find orthonormal eigenbasis.

$\mathcal{E}_1 = \ker(A^TA - 3I_3) = \ker \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\mathcal{E}_2 = \ker(A^TA - 2I_3) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\mathcal{E}_3 = \ker(A^TA) = \ker \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}}$$

$$\boxed{V = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}}$$

Orthonormal eigenbasis: $\frac{\langle 1,1,1 \rangle}{\sqrt{3}}, \frac{\langle 1,0,-1 \rangle}{\sqrt{2}}, \frac{\langle 1,-2,1 \rangle}{\sqrt{6}}$

$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$

$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\cancel{\sigma_3} \vec{0}\, \vec{u}_3 = A\vec{v}_3$ (we don't have space for this vector, ignore)

$$\boxed{U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

d) $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ ... $\lambda^2 - 3\lambda + 1 = 0$

λ

8. (20 points) True/False (circle the correct answer). You do not need to justify your answer. Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) Every 5×5 matrix has a real eigenvalue.

True False

(b) If A is an $n \times n$ matrix such that $\text{im}(A) = \{\vec{0}\}$ then A is invertible.

True False

(c) If A is symmetric and U is orthogonal then UAU^T is symmetric. (I think)

$(UAU^T)^T = U^{T^T} A^T U^T$

$= U A U^T$

True False

(d) If A is a symmetric matrix and λ is one of its eigenvalues, then λ is also one of the singular values of A.

$A\vec{v} = \lambda \vec{v}$

$A^2 \vec{v} = \lambda^2 \vec{v}$

True False

(e) The matrices $\begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 4 & 8 & 1 \\ 5 & 1 & 6 \\ 23 & 4 & 2 \end{bmatrix}$ are similar.

True False

(f) If $\ker(A^3) = \ker(A^4)$ and $\vec{v}$ is a vector in $\ker(A^5)$, then $\vec{v}$ is automatically in $\ker(A^4)$.

True False

(g) If A is a 4×7 matrix then $\text{rank}(A) + \dim(\ker(A)) = 4$.

$\underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{7 \text{ variables}}$

True False

(h) If A is a symmetric $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

$A = SDS^{-1}$

$A^n = SD\cancel{S^{-1}} \ldots \cancel{S}DS^{-1}$

$= SD^nS^{-1}$

$D = 0$

True False

(i) If A is an upper triangular $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

True False

(j) If A is an $n \times n$ matrix such that $||A\vec{x}|| = ||\vec{x}||$ for all $\vec{x}$ in $\mathbb{R}^n$ then A is an orthogonal matrix.

True False

9. (20 points) Fill in the blanks with the matrices below (just write A, B, etc. in each blank). You may use some items more than once, and there are some items that you will not use at all. **If none of the matrices on this list makes the statement true, write NONE.**

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(2 correct answers)

(a) Suppose that $M\vec{x} = \vec{b}$ is a linear system with 3 equations and 3 unknowns. If the system has a unique solution then the reduced row echelon form of M equals ___J___.

(b) Suppose that R is a 3×2 matrix and that S is a 2×3 matrix. If $\text{im}(R) = \ker(S)$ then $SR =$ ___G___.

$R: \mathbb{R}^2 \to \mathbb{R}^3 \quad S: \mathbb{R}^3 \to \mathbb{R}^2$

(c) If M is a 2×2 rotation by $\pi/6$ radians counterclockwise, then $M^{18} =$ ___F___.

$\left(\frac{\pi}{6}\right) 18 = 3\pi = \pi$ = rotation by 180°

(d) $M =$ ___B___ is a 2×2 matrix such that $\text{im}(M) = \ker(M)$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(e) $M =$ ___None___ is a 3×3 matrix such that $\text{im}(M) = \ker(M)$.

H (f) $\lambda = 0$ is the only eigenvalue of ___H___. Furthermore, the algebraic multiplicity of λ is 3 and the geometric multiplicity of λ is 1.

(g) tr(___E___) = 1.

(h) $M =$ ___H___ is a matrix with rank 2 whose kernel is not $\{\vec{0}\}$.

(i) If M is an orthogonal 3×3 matrix then $M^T M = M M^T =$ ___J___. I_3

$\lambda^2 + 1 = 0$

(j) ___D___ has complex (non-real) eigenvalues. (D ?)

You may use this page for scratch work.