MATH 33A Exam 2 [v2] Spring 2019

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Page:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						

You may use this page for scratch work.

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \vec{V}_3 = \begin{bmatrix} 9 \\ -4 \\ 3 \\ 2 \end{bmatrix} \quad \vec{V}_3^{\perp} = \begin{bmatrix} 9 \\ -4 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 6 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 12/2 \\ -2/2 \\ 12/2 \\ -2/2 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -7/2 \\ 7/2 \\ -7/2 \end{bmatrix}$$

$$\vec{V}_3 \cdot \vec{u}_1 = \frac{9}{2} + 2 - \frac{3}{2} + 1 = 6.$$

$$\hookrightarrow \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix}$$

$$\vec{V}_3 \cdot \vec{u}_2 = \frac{9}{2} - 2 + \frac{3}{2} + 1 = 5 \rightarrow \begin{bmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}$$

$-\frac{1}{4} + \frac{1}{4} - \frac{1}{4}$ $\quad$ $-\frac{1}{4} + \frac{1}{4} - \frac{1}{4}$

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 3/4 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & 3/4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 3/4 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$-\frac{3}{16} - \frac{3}{16} + \frac{1}{16} + \frac{1}{16}$

$-\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$\frac{1}{4} + \frac{1}{4} -\frac{1}{4}$ $\quad$ $\frac{1}{4} + \frac{1}{4} +$ $\quad$ $\frac{-1}{4} + \frac{1}{4} - \frac{1}{4}$

$-\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-11/2$

1. (10 points) (a) Which of the following is the QR-factorization of the matrix

$$\begin{bmatrix} \cancel{9/2} \\ -4 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 6/2 \\ -3 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix} \quad 7/2$$

$$A = \begin{bmatrix} 1 & 1 & 9 \\ -1 & 2 & -4 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}?$$

$$\vec{V}_3 \cdot \vec{u}_1 = \left(\frac{9}{2}+2-\frac{3}{2}+1\right) = \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix}$$

$$\vec{V}_3 \cdot \vec{u}_2 = \left(\frac{9}{2}-2+\frac{3}{2}+1\right) = \begin{bmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}$$

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Circle your answer.

$$\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 12 \\ 0 & 12 & 18 \\ 0 & 0 & 6\sqrt{2} \end{bmatrix}$$

$3+\frac{5}{2}+\frac{7}{2}=9$ $\quad -\frac{1}{4}+\frac{1}{4}-\frac{1}{4}$ $\quad \frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}$

$$\begin{bmatrix} 1 & -5 & 44 \\ 1 & -4 & 22 \\ 1 & -4 & 29 \\ 1 & -5 & 37 \end{bmatrix} \begin{bmatrix} 1 & 6 & 10 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

$3+\frac{5}{2}+\frac{7}{2}=9$

(b) Write down the result of the Gram-Schmidt process on the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 9 \\ -4 \\ 3 \\ 2 \end{bmatrix}.$$

You do not have to show your work (if you need more room, use Page 0).

Box your answer

basis:
$$\boxed{\left(\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}\right)}$$

$3+\frac{5}{2}+\frac{7}{2}=9.$

$-3+\frac{5}{2}-\frac{7}{2}=-4$

$-3+\frac{5}{2}+\frac{7}{2}=3,$

$3+\frac{5}{2}-\frac{7}{2}=2.$ ✓

$3+\frac{5}{2}+\frac{7}{2}=9.$

$-3+\frac{5}{2}-\frac{7}{2}=-4.$

$-3+\frac{5}{2}+\frac{7}{2}=3.$

$3+\frac{5}{2}-\frac{7}{2}=2$ ✓

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

(a) Suppose that $\vec{v}$ is *any* nonzero vector in $\mathbb{R}^2$. Explain why the vectors $\vec{v}$, $A\vec{v}$, and $A^2\vec{v}$ must be linearly dependent. (*Note:* do not use any numerical examples in your answer; your reasoning must be valid no matter what $\vec{v}$ is.)

They must be linearly dependent because in $\mathbb{R}^2$, only 2 linearly independent vectors are needed to form a basis in $\mathbb{R}^2$. A third vector would always be some linear combination of the first 2 vectors, & therefore would be linearly dependent.

(b) Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Part (a) shows that there are scalars c_0, c_1, and c_2 such that

$$c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} = \vec{0}. \tag{1}$$

Explain why this shows that the matrix $c_0 I + c_1 A + c_2 A^2$ is not invertible.

For a matrix to be invertible, the only vector in its kernel must be $\vec{0}$. If there is any other vector in its kernel, then it isn't invertible. Because C_0, C_1, C_2 are nonzero, the vector $\begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$ is also in the kernel, so therefore $C_0 I + C_1 A + C_2 A^2$ is not invertible.

(c) Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find scalars c_0, c_1, and c_2 such that (1) holds. Box your answer

$A\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad A^2\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 1 & 0 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \vec{X} = \begin{bmatrix} -6X_3 \\ -X_3 \\ X_3 \end{bmatrix} = X_3 \begin{bmatrix} -6 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{C_0 = -6, \quad C_1 = -1, \quad C_2 = 1}$$

$$\begin{bmatrix} 0 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = 0 \checkmark$$

3. (10 points) $\boxed{\text{You do not need to show work on this page.}}$ Consider the matrix

$$A = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & \cancel{0} & 0 & \cancel{0} \\ 1 & -3 & \cancel{3} & 0 & \cancel{0} \\ 1 & 1 & \cancel{-1} & 4 & \cancel{-2} \end{bmatrix} \end{array}.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

(a) Find a basis for the image of A. $\boxed{\text{Box your answer}}$

$$\boxed{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right)}$$

$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$

(b) What does the rank-nullity theorem say in this case?
(Your answer should be a simple equation of the form $1 + 1 = 2$).

$3 + 2 = 5.$

(c) Find a basis for the kernel of A. $\boxed{\text{Box your answer}}$

$V_3 = -V_2. \to V_3 + V_2 = 0$

$V_5 = -\frac{1}{2}V_4 \to V_5 + \frac{1}{2}V_4 = 0$

$$\boxed{\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right)}$$

4. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

Remember that True means **always true.** If a statement is sometimes true, but sometimes false, mark it False.

(a) If A is an invertible matrix then $\overset{=\vec{0}}{\ker(A)} = \overset{=\vec{0}}{\ker(A^{-1})}$.

$T: \mathbb{R}^n \to \mathbb{R}^n$

True False

$\ker(A) \in \mathbb{R}^n$

$\ker(A^{-1}) \in \mathbb{R}^m$

(b) If A is an orthogonal matrix then A^TA is also an orthogonal matrix. non sq?

$A^{-1} = A^T$

True False

$A^TA = I$

$V^{\perp} = \ker(\mathrm{proj}_V v)$

(c) There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^{\perp})$, where $V^{\perp}$ denotes the orthogonal complement of V.

True False

(d) If V is a two-dimensional subspace of $\mathbb{R}^3$ (a plane) then there is a basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of the linear transformation $T = \mathrm{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ has the form

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & T\vec{x} \\ S\uparrow & & \uparrow S \\ [\vec{x}]_{\mathcal{B}} & & [T(x)]_{\mathcal{B}} \end{array}$$

$B = S^{-1}AS$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

projects on $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

True False

(e) If A is a symmetric $n \times n$ matrix and S is an orthogonal $n \times n$ matrix then the matrix $S^{-1}AS$ is symmetric.

$S^T = S^{-1}$

True False

$A^T = A.$

$$(S^{-1}AS)^T = S^TA^T(S^{-1})^T = S^{-1}A(S^T)^T = S^{-1}AS$$

$$(S^{-1}AS)^T = S^TA^T(S^{-1})^T = S^{-1}AS$$

5. (10 points) The following statements are the possible answers to questions (a) and (b) below.

(A) The linear system $A\vec{x} = \vec{0}$ has a unique solution

(B) $\dim(\operatorname{im} A) + \dim(\ker A) = n$

(C) $\operatorname{rank}(A) = n$

(D) A is upper triangular

(E) The column vectors of A span $\mathbb{R}^n$

(F) $A^T A = I_n$

(G) $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

(H) $A^n = 0$

(I) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$

(a) Which of the statements above implies that the $n \times n$ matrix A is invertible?
There are five correct answers.

$\boxed{A}$ $\boxed{C}$ $\boxed{E}$ $\boxed{D}$ $\boxed{F}$

(b) Suppose that A is an invertible $n \times n$ matrix. Which of the properties above does A satisfy?
There are five correct answers.

$\boxed{A}$ $\boxed{B}$ $\boxed{C}$ $\boxed{E}$ $\boxed{G}$