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2A	Redmond McNamara	T	GEOLOGY 6704
2B		R	ROYCE 154
2C	Albert Zheng	T	ROLFE 3121
2D		R	ROYCE 162
2E	Weiyi Liu	T	BOELTER 5280
2F		R	BOELTER 5436

Section	2	B

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Page:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						

You may use this page for scratch work.

1. (10 points) (a) Which of the following is the QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 2 & -4 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}?$$

$$\begin{matrix} v_1 & v_2 & v_3 \end{matrix}$$

Circle your answer.

$$\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -6 \\ 0 & 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 12 \\ 0 & 12 & 18 \\ 0 & 0 & 6\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 44 \\ 1 & -4 & 22 \\ 1 & -4 & 29 \\ 1 & -5 & 37 \end{bmatrix} \begin{bmatrix} 1 & 6 & 10 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 6 \\ 0 & 0 & 7 \end{bmatrix} \times$$

(b) Write down the result of the Gram-Schmidt process on the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 9 \\ -4 \\ 3 \\ 2 \end{bmatrix}.$$

You do not have to show your work (if you need more room, use Page 0).

Box your answer

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2^{\perp} = v_2 - (v_2 \cdot u_1)\, u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - (3)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}$$

$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}$$

$v_1 \quad v_2$

(a) Suppose that $\vec{v}$ is *any* nonzero vector in $\mathbb{R}^2$. Explain why the vectors $\vec{v}$, $A\vec{v}$, and $A^2\vec{v}$ must be linearly dependent. (*Note:* do not use any numerical examples in your answer; your reasoning must be valid no matter what $\vec{v}$ is.)

$v = \begin{bmatrix} a \\ b \end{bmatrix} \qquad A\vec{v} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a+b \\ 3b-a \end{bmatrix} \qquad A^2 v = a\begin{bmatrix} 0 \\ -4 \end{bmatrix} + b\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 4b \\ -4a+8b \end{bmatrix}$

$c_0\begin{bmatrix} a \\ b \end{bmatrix} + c_1\begin{bmatrix} a+b \\ 3b-a \end{bmatrix} + c_2\begin{bmatrix} 4b \\ -4a+8b \end{bmatrix} = 0 \qquad \begin{bmatrix} ac_0 + ac_1 + bc_1 + 4bc_2 \\ bc_0 - ac_1 + 3bc_1 - 4ac_2 + 8bc_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

There exist nonzero scalars c_0, c_1, and c_2 such that $c_0 v + c_1 Av$, $c_2 A^2 v = 0$ ~~holds true~~

(b) Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Part (a) shows that there are scalars c_0, c_1, and c_2 such that

$$c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} = \vec{0}. \tag{1}$$

Explain why this shows that the matrix $c_0 I + c_1 A + c_2 A^2$ is not invertible.

In order for the matrix to be invertible, the columns must be linearly independent. However, since $c_0 v + c_1 Av + c_2 A^2 v = 0$ refers to a linear combination of the first columns of I, A and A^2 and this combination is linearly dependent, then the matrix $c_0 I + c_1 A + c_2 A^2$ must also be linearly dependent and this not invertible

(c) Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find scalars c_0, c_1, and c_2 such that (1) holds. Box your answer

$a = 1$

$b = 0$

$\begin{bmatrix} c_0 + c_1 \\ -c_1 - 4c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{c_0 = -1 \quad c_1 = 1 \quad c_2 = -\frac{1}{4}}$

$c_0 = -c_1$

$c_2 = -\frac{1}{4} c_1$

3. (10 points) $\boxed{\text{You do not need to show work on this page.}}$ Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -2 & 2 & -2 & 4 \end{bmatrix}.$$

(a) Find a basis for the image of A. $\boxed{\text{Box your answer}}$

$$\boxed{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)}$$

(b) What does the rank-nullity theorem say in this case?
(Your answer should be a simple equation of the form $1 + 1 = 2$).

ran (A) + nullity (A) = m

$$\boxed{3 + 2 = 5}$$

(c) Find a basis for the kernel of A. $\boxed{\text{Box your answer}}$

$$\boxed{\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right)}$$

4. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

True False

$\dim(V) + \dim(V^\perp) = n$

(b) If A is an invertible matrix then $\ker(A) = \ker(A^{-1})$.

$\ker(A)$ $\ker(A^{-1})$ True False

(c) If A is an orthogonal matrix then $\overset{I}{AA^T}$ is also an orthogonal matrix.

True False

AA^T

(d) If A is a symmetric $n \times n$ matrix and S is an orthogonal $n \times n$ matrix then the matrix $S^{-1}AS$ is symmetric.

True False

$(S^{-1}AS)^T = S^T A^T S^{-1T} = S^{-1}AS$

(e) If V is a two-dimensional subspace of $\mathbb{R}^3$ (a plane) then there is a basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix of the linear transformation $T = \text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ has the form

$\boxed{\text{True}}$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. (10 points) The following statements are the possible answers to questions (a) and (b) below.

(A) The column vectors of A span $\mathbb{R}^n$

(B) $A^T A = I_n$

(C) $\text{rank}(A) = \text{rank}(A^T)$

(D) $A^n = 0$

(E) $||A\vec{x}|| = ||\vec{x}||$ for all $\vec{x}$ in $\mathbb{R}^n$

(F) The linear system $A\vec{x} = \vec{0}$ has a unique solution

(G) $\dim(\text{im}\, A) + \dim(\ker A) = n$

(H) $\text{rank}(A) = n$

(I) A is upper triangular

(a) Which of the statements above implies that the $n \times n$ matrix A is invertible? There are five correct answers.

B	F	H	A	E

(b) Suppose that A is an invertible $n \times n$ matrix. Which of the properties above does A satisfy? There are five correct answers.

G	F	B	A	H