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Section	2	B

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Page:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						

You may use this page for scratch work.

1. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

   Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

   (a) Every vector in $\mathbb{R}^4$ is a linear combination of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{e}_4$.

   $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   True   False

   (b) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if rref(A) contains a row of zeros.

   $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \end{array}\right]$$

   True   False

   $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   $x_1 = a$ $\quad x_3 = c$

   $x_2 = b$ $\quad x_4 = t$

   (c) If A is a 3×4 matrix of rank 3 then the system $A\vec{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ has infinitely many solutions.

   $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

   True   False

   $x_1 = -1$

   $x_2 = 1$

   $x_3 = -1$

   ~~$ax_1 + bx_2 + cx_3 + dx_4$~~

   $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & -a \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & - \\ 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & - \end{array}\right]$$

   (d) There exists a nonzero lower triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   $$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   True   False

   (e) If A and B are invertible $n \times n$ matrices then $A^{-1}B^{-1}AB = I_n$.

   $(AB)^{-1} = B^{-1}A^{-1}$

   True   False

2. (10 points) Show your work.

(a) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_2 \end{bmatrix}. \quad \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4×3 3×1 4×1

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$.

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & \frac{1}{2} & 0 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -3x_1 - x_3 \\ x_2 \\ -5x_1 + \frac{1}{2}x_2 \end{bmatrix} \checkmark$$

✓

(b) If A is a 4×4 matrix such that

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

then what is

$$A\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}?$$

$$A = \begin{bmatrix} 0 & 1 & \\ 0 & -1 & \\ -1 & 0 & \\ 1 & 0 & \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 & \\ 0 & -1 & \\ -1 & 0 & \\ 1 & 0 & \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 0-2 \\ 1+0 \\ -1+0 \end{bmatrix}$$

3. (10 points) For each of the matrices below, either find the inverse or explain why no inverse exists **without using determinants**. Show your work.

(a) $\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{swap rows around}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

Inverse matrix = $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ✓

$-4 \quad 20 \quad 4$

(b) $\left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 1 & 8 & -12 & 0 & 1 & 0 \\ 4 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-4\times R1]{R2-R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 9 & -17 & -1 & 1 & 0 \\ 0 & 9 & -17 & -4 & 0 & 1 \end{array}\right] \xrightarrow[R3-R2]{}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 9 & -17 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & -1 & 1 \end{array}\right]$$

No inverse b/c this matrix cannot row reduce to the identity matrix.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 1 & 8 & -12 & 0 & 1 & 0 \\ 4 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 9 & -17 & -1 & 1 & 0 \\ 0 & 9 & -17 & -4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 9 & -17 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & -1 & 1 \end{array}\right]$$

4. (10 points) Find all lower triangular 2×2 matrices of the form

$$X = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$$

such that X commutes with **every** 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with a, b, c, d all nonzero. Show your work!

$XA = AX$

$$\overset{X}{\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}} \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \begin{bmatrix} ax & bx \\ ay+cz & by+dz \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{X}{\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}} = \begin{bmatrix} ax+by & bz \\ cx+dy & dz \end{bmatrix}$$

$ax = ax + by \rightarrow 0 = by \rightarrow y = 0.$

$bx = bz \rightarrow x = z$

$ay + cz = cx + dy \rightarrow 0 + cz = cx + 0 \rightarrow z = x.$

$by + dz = dz \rightarrow by = 0 \rightarrow y = 0.$

$$\boxed{X = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}}$$

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix} \checkmark$$

5. (10 points) Multiple Choice. Write your answers (A, B, C, etc.) in the boxes provided.

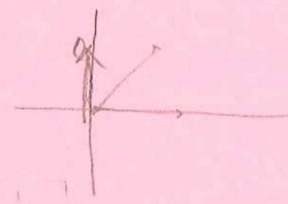
Each of the linear transformations listed below corresponds to one (and only one) of the matrices A through J. Match them up.

$\boxed{I}$ scaling	$\boxed{A}$ reflection	$\boxed{H}$ shear
$\boxed{J}$ rotation	$\boxed{G}$ orthogonal projection	

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad J = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

rotation: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \rightarrow a^2+b^2=1.$



proj: $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \rightarrow$ G, E

$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ref: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$