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Section	2	B

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Page:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						

You may use this page for scratch work.

1. (10 points) True/False (circle the correct answer). You do not need to justify your answer.

Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) If A and B are invertible $n \times n$ matrices then $ABA^{-1}B^{-1} = I_n$.

$(AB)(A^{-1}B^{-1}) = I_n$

True False

(b) Every vector in $\mathbb{R}^4$ is a linear combination of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{e}_4$.

True False

(c) If A is a 3×4 matrix of rank 3 then the system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has infinitely many solutions.

rank = 3

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$

True False

rank $\neq$ m

so cannot have 1 solution

$x_3 + 2x_4 = 3$

$x_2 + x_4 = 2$

rank = n

at least 1

(d) There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

True False

$AA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} a^2 & ab-bc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $a = 0$, $c = 0$ $b \neq 0$

(e) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if rref(A) contains a row of zeros.

True False

2. (10 points) Show your work.

(a) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 - x_3 \\ x_3 \\ 0 \\ \frac{1}{2}x_1 + x_2 \end{bmatrix}.$$

pull out coefficients

4×3 3×1 4×1

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$.

$$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_3 \\ x_3 \\ 0 \\ \frac{1}{2}x_1 + x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0x_1 + 2x_2 - 1x_3 \\ 0x_1 + 0x_2 + 1x_3 \\ 0x_1 + 0x_2 + 0x_3 \\ \frac{1}{2}x_1 + 1x_2 + 0x_3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}}$$

(b) If A is a 4×4 matrix such that

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

then what is

$$A\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}?$$

The basis vectors give use the coefficients in the 2nd and 3rd columns

corresponds to second column

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} & 0 & 1 & \\ & 1 & 0 & \\ & -1 & 0 & \\ & 0 & -1 & \end{bmatrix}$$

4×4

$$A\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}}$$

4×1

3. (10 points) For each of the matrices below, either find the inverse or explain why no inverse exists **without using determinants**. Show your work.

(a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Switch III to I, I to IV, IV to III

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

(b) $\begin{bmatrix} 1 & -1 & 5 \\ 3 & 4 & 2 \\ 1 & 6 & -8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 1 & 6 & -8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (\mathrm{II})-3(\mathrm{I}) \\ (\mathrm{III})-(\mathrm{I}) \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 7 & -13 & -3 & 1 & 0 \\ 0 & 7 & -13 & -1 & 0 & 1 \end{array}\right] (\mathrm{III})-(\mathrm{II})$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 7 & -13 & -3 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right]$$

The matrix has no inverse because it cannot be reduced to the identity matrix

4. (10 points) Find all upper triangular 2×2 matrices of the form

$$X = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

such that X commutes with **every** 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with a, b, c, d all nonzero. Show your work!

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax & ay + bz \\ cx & cy + dz \end{bmatrix}$$

$$XA = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax + cy & bx + dy \\ cz & dz \end{bmatrix}$$

$AX = XA$

$y = 0$
$x = z$

$\cancel{c}x = \cancel{c}z \rightarrow x = z$
$cy + \cancel{dz} = \cancel{dz} \Rightarrow cy = 0, \quad c \neq 0 \quad \text{so } y = 0$

$\cancel{ay} + \cancel{b}z = \cancel{b}x + \cancel{dy}$ $\checkmark$
$x = z$

$$\boxed{X = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \text{ where } n = \text{any number}}$$

5. (10 points) Multiple Choice. Write your answers (A, B, C, etc.) in the boxes provided.

Each of the linear transformations listed below corresponds to one (and only one) of the matrices A through J. Match them up.

I	scaling	D	reflection	E	shear
C	rotation	G	orthogonal projection		

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix} \quad D = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad J = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

$u_1^2 + u_2^2 = 1$

scale $= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ proj $= \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$

shear $= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ ref $= 2\,\text{proj} - x = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

rot $= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $a^2 + b^2 = 1$

$\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ $2u_1^2 - 1 = 0.6$

$u_1^2 = 0.8$

$2u_2^2 - 1 = -0.6$

$u_2^2 = 0.2$