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Section	2	$\mathcal{B}$

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- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
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$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}^{-1} A \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

$\vec{v} + \vec{w} i$ eigenvector

$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i$

$V^\perp = \ker(\text{proj}_V)$

Page:	1	2	3	4	5	6	7	8	Total
Points:	10	8	10	10	12	10	20	20	100
Score:									

1. (5 points) Suppose that A is a 5×4 matrix of the form

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

Given that the vector $\begin{bmatrix} 7 \\ 3 \\ -2 \\ 8 \end{bmatrix}$ is in the kernel of A, write $\vec{v}_4$ as a linear combination of the vectors

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. $\boxed{\text{Box your answer.}}$

$7\vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3 + 8\vec{v}_4 = 0$
$8\vec{v}_4 = -7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$
$\vec{v}_4 = -\frac{7}{8}\vec{v}_1 - \frac{3}{8}\vec{v}_2 - \frac{3}{8}\vec{v}_2 + \frac{1}{4}\vec{v}_2$

$A\vec{v} = \vec{0}$

$$\begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ -2 \\ 8 \end{bmatrix} = \vec{0}$$

$7\vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3 + 8\vec{v}_4 = 0$

$8\vec{v}_4 = -7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$

$$\boxed{\vec{v}_4 = -\frac{7}{8}\vec{v}_1 - \frac{3}{8}\vec{v}_2 + \frac{1}{4}\vec{v}_3}$$

$7\vec{v}_1 + 3\vec{v}_2 - 2\vec{v}_3 + 8\vec{v}_4 = 0$
$8\vec{v}_4 = -7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$
$\vec{v}_4 = -\frac{7}{8}\vec{v}_1 - \frac{3}{8}\vec{v}_2 + \frac{1}{4}\vec{v}_3$ ✓

2. (5 points) Suppose that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the orthogonal projection onto the plane $3x_1 + x_2 - 2x_3 = 0$. Find a basis for $\text{im}(T)$ and a basis for $\text{ker}(T)$. $\boxed{\text{Box your answers.}}$

$\langle 3, 1, -2 \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$.

$\text{im}(T)$ is the plane $3x_1 + x_2 - 2x_3 = 0$
$3x_1 = -x_2 + 2x_3$
$x_1 = -\frac{1}{3}x_2 + \frac{2}{3}x_3 \longrightarrow \vec{x} = \begin{bmatrix} -\frac{1}{3}x_2 + \frac{2}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$

$\text{im}(T) = \text{span}\left(\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} \right)$.

$\boxed{\text{Basis for } \text{im}(T) = \left(\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} \right)}$

$\text{ker}(T) = $ vectors $\perp$ to plane.

$\boxed{\text{basis for } \text{ker}(T) = \left(\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right)}$

$\vec{n} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ is normal to plane. (1)

Page 1

$\lambda^2 - 8\lambda + 17 = 0$

3. (8 points) For each of the 2×2 matrices A below, there is an invertible matrix S such that $B = S^{-1}AS$ is either a diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or a rotation-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Find B in each case (you do not have to find S).

(a) $A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - tr(A)\lambda + \det(A)$

$$= \lambda^2 - 8\lambda + (15+2) = \lambda^2 - 8\lambda + 17 = 0.$$

quadratic formula: $\lambda = \dfrac{8 \pm \sqrt{64 - 4(17)}}{2} = 4 \pm \frac{1}{2}\sqrt{-4}$

$$= 4 \pm \frac{1}{2}(2i)$$
$$= 4 \pm i$$

$$\boxed{B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}}$$

a $b=1$ $\vec{v}$ $\vec{w}$

$E_{4+i} = \ker\left(\begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 1+i \end{bmatrix}\right)$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_2 = (1+i)x_1$

$S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$

$\det(A) = \lambda^2 - 8\lambda + 17 = 0$ $17 \cdot 4 = 40 + 28 = 68$

$\lambda = \dfrac{8 \pm \sqrt{64 - 4(17)}}{2} = 4 \pm \frac{1}{2}\sqrt{-4} = 4 \pm \frac{1}{2}(2i) = 4 \pm i$

$a \pm bi$

$a = 4, \ b = 1$

$S^{-1}AS$

$\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ ✓

(b) $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - 4\lambda + (-5+8)$

$$= \lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda - 3)(\lambda - 1) = 0$$
$$\lambda_1 = 3, \ \lambda_2 = 1.$$

$$\boxed{B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}$$

$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 3)(\lambda - 1) = 0$
$\lambda = 3, 1$

$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 3)(\lambda - 1) = 0$
$\lambda_1 = 3, \ \lambda_2 = 1.$

$E_3 = \ker\left(\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\right) = span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$E_1 = \ker\left(\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

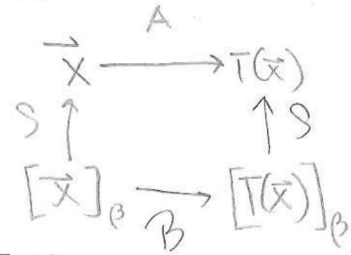$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$D = S^{-1}AS$

$\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

4. (10 points) Let $T(\vec{x}) = A\vec{x}$ be the linear transformation with matrix

$$A = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}.$$

(a) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Find the matrix of T in the basis $\mathcal{B}$.
(Another way to say this is: find a matrix B such that $[T(\vec{x})]_\mathcal{B} = B[\vec{x}]_\mathcal{B}$.)

(handwritten work at top right):

$$\vec{x} \xrightarrow{A} T(\vec{x})$$
$$S\uparrow \qquad \uparrow S$$
$$[\vec{x}]_\mathcal{B} \xrightarrow{B} [T(\vec{x})]_\mathcal{B}$$

$$B = S^{-1}AS$$

$$B = \begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_\mathcal{B} & [T(\vec{v}_2)]_\mathcal{B} & [T(\vec{v}_3)]_\mathcal{B} \\ | & | & | \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$A\vec{v}_2 = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}. \qquad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}_\mathcal{B} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}.$$

$$A\vec{v}_3 = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}. \qquad \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}_\mathcal{B} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} \checkmark$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(b) Find the eigenvalues of A, repeating any eigenvalues according to their algebraic multiplicities. (So if $\lambda_2 = 17$ has algebraic multiplicity 2, list it twice.)

Hint: use your answer from part (a), which should be nice enough that you don't need to compute a determinant for this part.

(handwritten): $B = S^{-1}AS \longrightarrow B, A$ are similar $\to$ have same eigenvalues!

$$\boxed{\lambda_1 = 1 \qquad \lambda_2 = 4 \qquad \lambda_3 = 6}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad [\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad B[\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+6-3 \\ 12-5 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$

$$A\vec{x} = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4-6+6 \\ -2+6+6 \\ 5-10+12 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 7 \end{bmatrix}. \qquad \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \checkmark$$

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5. (10 points) Make sure to fully justify your answers on this page.

Suppose that A is a symmetric $n \times n$ matrix (this is an assumption for both (a) and (b) below).

(a) Show that $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$ for any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$.

Hint: remember that another way to write the dot product is $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$.

$$(A\vec{v})^T \vec{w} = \vec{v}^T A^T \vec{w}$$

$$A = A^T \text{ b/c symmetric}$$

$$= \vec{v}^T (A\vec{w})$$

$$= \vec{v} \cdot A\vec{w} \checkmark$$

(b) Suppose that $\vec{v}$ and $\vec{w}$ are eigenvectors of A with eigenvalues λ and μ, respectively. Show that if $\lambda \neq \mu$ then $\vec{v}$ is orthogonal to $\vec{w}$.

$$A\vec{v} = \lambda\vec{v} \longrightarrow \vec{w}^T A\vec{v} = \vec{w}^T \lambda\vec{v}$$

$$A\vec{w} = \mu\vec{w} \qquad \vec{w}^T A\vec{v} = \lambda(\vec{w}^T \vec{v})$$

$$\vec{v}^T A\vec{w} = \mu(\vec{v}^T \vec{w})$$

$$(\vec{v}^T A\vec{w})^T = \mu(\vec{v}^T \vec{w})^T \qquad \swarrow \text{ take transpose of both sides}$$

$$\vec{w}^T A^T \vec{v} = \mu\vec{w}^T \vec{v}$$

$$A^T = A \quad \text{b/c symmetric}$$

$$\vec{w}^T A\vec{v} = \mu(\vec{w}^T \vec{v})$$

$$\overrightarrow{\left(\vec{w}^T A\vec{v} = \lambda(\vec{w}^T \vec{v}) \right)}$$

_____

$$0 = (\mu - \lambda)(\vec{w}^T \vec{v})$$

$$0 = (\mu - \lambda)(\vec{w} \cdot \vec{v})$$

$$\mu - \lambda \neq 0, \text{ so } \vec{w} \cdot \vec{v} = 0$$

$$\downarrow$$

$$\vec{w} \& \vec{v}$$

are orthogonal!

6. (12 points) Define

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A and state the algebraic multiplicity of each eigenvalue. (No justification needed for this part.) triangular.

$\lambda_1 = 0 \rightarrow$ algmult = 2

$\lambda_2 = 1 \rightarrow$ algmult = 2.

(b) Find a basis for each eigenspace.

$\lambda_1 = 0 \rightarrow E_0 = \ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$

basis for $E_0 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$

$\lambda_2 = 1 \rightarrow E_1 = \ker(A-I) = \ker \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$

basis for $E_1 = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$

(c) State the geometric multiplicity of each eigenvalue.

$\lambda_1 = 0 \rightarrow$ geomult = 2

$\lambda_2 = 1 \rightarrow$ geomult = 1.

(d) Is A diagonalizable? Justify your answer.

No. For A to be diagonalizable, the geometric multiplicities of all the eigenvalues must add up to 4 b/c A is a 4x4. Here, the geometric multiplicities only add up to 3, so A is not diagonalizable.

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$A = U \Sigma V$
$2\times2 \quad 2\times3 \quad 3\times3$

7. (10 points) For this question, it may be helpful to know that

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$

($\vec{v_2}$, $\vec{v_3}$, $\vec{v_1}$ labeled above; M, S, D, S^T labeled below)

Find a singular value decomposition $A = U\Sigma V^T$ of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Write your answer in the form $\boxed{U = \text{something}, \ \Sigma = \text{something}, \ V = \text{something}}$.

M symmetric, S is orthogonal

$M = A^T A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$

$D = S^{-1} M S$
$M = S D S^{-1}$
$M = S D S^T$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 0$
$\sigma_1 = \sqrt{3} \quad \sigma_2 = \sqrt{2} \quad \sigma_3 = 0.$

$V = S$ but rolled around.

$U: \quad \vec{u_1} = \frac{1}{\sigma_1} A \vec{v_1} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

$\vec{u_2} = \frac{1}{\sigma_2} A \vec{v_2} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$

$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$

$V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$U\Sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}$

$U\Sigma V^T = \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \checkmark$

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8. (20 points) True/False (circle the correct answer). You do not need to justify your answer.

Remember that True means **always true**. If a statement is sometimes true, but sometimes false, mark it False.

(a) Every 5×5 matrix has a real eigenvalue.

complex ones in pairs.

(True)   False

(b) If A is an $n \times n$ matrix such that $\text{im}(A) = \{\vec{0}\}$ then A is invertible.

$\text{im}(A) = \vec{0}$ $\ker(A) = \mathbb{R}^n$

$A\vec{v} = \vec{0}$

True   (False)

(c) If A is symmetric and U is orthogonal then UAU^T is symmetric.

$(UAU^T)^T = UA^TU^T$
$= UAU^T$

$(UAU^T)^T = UA^TU^T$
UAU^T

(True)   False

(d) If A is a symmetric matrix and λ is one of its eigenvalues, then λ is also one of the singular values of A.

True   (False) $\sigma = |\lambda|$ positive!

(e) The matrices $\begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 4 & 8 & 1 \\ 5 & 1 & 6 \\ 23 & 4 & 2 \end{bmatrix}$ are similar.   since trace, X

$\text{tr} = 12$ $\text{tr} = 7$ True   (False)

(f) If $\ker(A^3) = \ker(A^4)$ and $\vec{v}$ is a vector in $\ker(A^5)$, then $\vec{v}$ is automatically in $\ker(A^4)$.

$A^3\vec{v}$

(True)   False

$A^5\vec{v} = \vec{0}$ $A\vec{v} \in \ker(A^4)$
$A^4 \cdot (A\vec{v}) = 0$ $A\vec{v} \in \ker(A^3)$
$A^3 \cdot A\vec{v} = 0$
$A^4\vec{v} = 0$

(g) If A is a 4×7 matrix then $\text{rank}(A) + \dim(\ker(A)) = 4$.

$n \times m$ $\text{null}(A)$

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ add to 7.   4   +3

True   (False)

(h) If A is a symmetric $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

diagonalizable.   at u 0

$D = S^{-1}AS \to A = SDS^{-1} \to A^n = SD^nS^{-1} = 0$

(True)   False

$D = S^{-1}AS$
$A = SDS^{-1}$
$A^n = SD^nS^{-1} = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) If A is an upper triangular $n \times n$ matrix such that $A^n = 0$ then $A = 0$.

True   (False)

(j) If A is an $n \times n$ matrix such that $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$ then A is an orthogonal matrix.

(True)   False

9. (20 points) Fill in the blanks with the matrices below (just write A, B, etc. in each blank). You may use some items more than once, and there are some items that you will not use at all. **If none of the matrices on this list makes the statement true, write NONE.**

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Suppose that $M\vec{x} = \vec{b}$ is a linear system with 3 equations and 3 unknowns. If the system has a unique solution then the reduced row echelon form of M equals ___J___. I_3

(b) Suppose that R is a 3×2 matrix and that S is a 2×3 matrix. If $\text{im}(R) = \text{ker}(S)$ then
$SR =$ ___A___.

$SR\vec{v}$

$S(R\vec{v}) = \vec{0}$

SR 2×3 3×2 $= 2\times2$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) If M is a 2×2 rotation by $\pi/6$ radians counterclockwise, then $M^{18} =$ ___F___.

$\frac{\pi}{6} \cdot 18 = 3\pi$ rotated $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(d) $M =$ ___B___ is a 2×2 matrix such that $\text{im}(M) = \text{ker}(M)$.

$\text{im}(B) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\text{ker}(B) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(e) $M =$ ___NONE___ is a 3×3 matrix such that $\text{im}(M) = \text{ker}(M)$.

$\text{rank}(M) + \text{null}(M) = 3$

not possible!

(f) $\lambda = 0$ is the only eigenvalue of ___H___. Furthermore, the algebraic multiplicity of λ is 3 and the geometric multiplicity of λ is 1.

(g) $\text{tr}($ ___E___ $) = 1$.

(h) $M =$ ___K___ is a matrix with rank 2 whose kernel is not $\{\vec{0}\}$.

2 linearly independent. $\text{ker}(K) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right)$

(i) If M is an orthogonal 3×3 matrix then $M^T M = M M^T =$ ___J___.

$M^{-1} = M^T$ $M^{-1}M = I_3$

(j) ___D___ has complex (non-real) eigenvalues.

$\lambda^2 + 2 = 0$

$D: \lambda^2 + 2 = 0.$

$\lambda^2 - \text{tr}(0)\lambda + \det(0)$

$\lambda^2 - 0\lambda + (0+2) = \lambda^2 + 2 = 0$

$$A = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \\ | & | & | \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 4 & -3 & 2 \\ -2 & 3 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$A\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}_\beta = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}.$$

$$A\vec{v}_3 = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}_\beta = \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}. \quad \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} \checkmark$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

8)d). $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda = -1$

$$A^T A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \lambda = 1 \quad \sigma = 1$$

$S^{-1}AS$

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$$

$A^5 \vec{v} = \vec{0}$ $\ker(A^5) = \ker(A^4)$

$A^4(A\vec{v}) = \vec{0}$

$A\vec{v} \in \ker(A^4)$

$A\vec{v} \in \ker(A^3)$

$A^3 \cdot A\vec{v} = \vec{0} \to A^4\vec{v} = 0$

$A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$D = S^{-1}AS$

$$AS = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \checkmark$$