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Prof. Alan J. Laub February 7, 2014

# Math 33A/2 – MIDTERM EXAMINATION
## Winter 2014

**Instructions:**

(a) The exam is closed-book except for one page, written both sides, of notes. It will last 50 minutes. No cell phones or similar electronic devices are allowed at any time. The sole exception is a calculator and then only if your brain has atrophied to the point where you are unable to perform simple arithmetic. However, the use of a calculator is NOT required.

(b) Notation will conform as closely as possible to the standard notation used in the lectures, not the textbook.

(c) Do all 3 problems. Problem 1 and 2 are worth 30 points, problem 3 is worth 40 points. Hand in the exam with ALL appropriate work shown. Some partial credit may be assigned if warranted. Label clearly the problem number and the material you wish to be graded.

(d) You must enter your discussion section above. Failure to provide the correct discussion section will result in a loss of 5 points.

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1. (a) Consider the (symmetric) matrix $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -7 \\ -2 & -7 & 7 \end{bmatrix}$. Perform Gaussian elimination on $A$ to reduce it to upper triangular form $U$.

(b) Write $A$ in the form $LU$ where $L$ is unit lower triangular, i.e., find $L$ explicitly.

(c) Write $A$ in the form $LD\hat{U}$ where $L$ is unit lower triangular, $D$ is diagonal, and $\hat{U}$ is unit upper triangular.

(d) Compute $\det(A)$ directly from either its $LU$ factorization in (b) or its $LD\hat{U}$ factorization in (c).

a) $A = \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -7 & 0 & 1 & 0 \\ -2 & -7 & 7 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -3 & 5 & 1 & 0 & 1 \end{array}\right]$

$A = \left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -4 & 1 & 3 & 1 \end{array}\right]$ $L^{-1}$

$\boxed{U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix}}$ ✓

b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 1 \end{array}\right]$

$A = LU$ L

$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix}}$ ✓

c) $U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \to D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\hat{U} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$A = LD\hat{U}$

$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}$ ✓

d) $\det(A) = \det(L) \cdot \det(U)$

$= (1 \cdot 1 \cdot 1) \cdot (2 \cdot 1 \cdot -4)$

$\boxed{\det(A) = -8}$ ✓

2. Write all solutions of the equations $Ax = b$ where $A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 8 & 7 & 6 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the form $x_p + x_n$ where $x_p$ denotes the particular solution which solves $Ax_p = b$ and the special solutions $x_n$ solve $Ax_n = 0$.

$$A = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 8 & 1 \\ 8 & 7 & 6 & 5 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 8 & 1 \\ 0 & -9 & -18 & -27 & -2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -2 & -4 & \frac{1}{9} \\ 0 & -9 & -18 & -27 & -2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & \frac{1}{18} \\ 0 & 1 & 2 & 3 & \frac{2}{9} \end{array}\right]$$

$x_1 - x_3 - 2x_4 = \frac{1}{18}$

$x_2 + \cancel{2x_3} + \cancel{3x_4} = \frac{2}{9}$

set $x_3 = 0$, $x_4 = 0$

$x_1 = \frac{1}{18}$ $x_2 = \frac{2}{9}$

$$X_p = \begin{bmatrix} \frac{1}{18} \\ \frac{2}{9} \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_3 - 2x_4 = 0$

$x_2 + 2x_3 + 3x_4 = 0$

$x_3 = 1 \Rightarrow x_2 = -1, x_1 = 1$
$x_4 = 0$

$$x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$x_3 = 0 \Rightarrow x_2 = -3$
$x_4 = 1$ $x_1 = 2$

$$x_4 \begin{bmatrix} 2 \\ -3 \\ 9 \end{bmatrix}$$

$X = X_p + X_n$

$$\boxed{X = \begin{bmatrix} \frac{1}{18} \\ \frac{2}{9} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

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3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \\ 8 & 6 & 4 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 4}.$$

(a) Put $A$ in reduced row echelon form (rref) $R$.

(b) Determine rank$(A)$ and state why you have given the answer you have given. Determine rank$(A^T)$ and state why you have given the answer you have given.

(c) Find a basis for $\mathcal{N}(A)$ (the nullspace of $A$) from $R = \text{rref}(A)$.

(d) Find a basis for $\mathcal{R}(A) = \mathcal{C}(\mathcal{A})$ (the range of $A$ = the columnspace of $A$) from $R = \text{rref}(A)$.

(e) Find a basis for $\mathcal{R}(A^T) = \mathcal{C}(\mathcal{A}^T)$ from $R = \text{rref}(A)$.

(f) The matrix $E = \begin{bmatrix} -\frac{4}{5} & \frac{1}{5} & 0 \\ \frac{9}{10} & -\frac{1}{10} & 0 \\ 1 & -1 & 1 \end{bmatrix}$ puts $A$ in rref. Using this $E$, find a basis for $\mathcal{N}(A^T)$.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \\ 8 & 6 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -10 & -20 & -30 \\ 0 & -10 & -20 & -30 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -10 & -20 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -10 & -20 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \boxed{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R}$

b) $\boxed{\text{rank}(A) = 2}$ ← I got that because there are 2 pivot var. in R

$\boxed{\text{rank}(A^T) = 2}$ because I know rank(A) = rank(A^T)

c). $x_1 - x_3 - 2x_4 = 0$ at $x_3 = 1$, $x_4 = 0$ $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ at $x_3 = 0$, $x_4 = 1$ $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$x_2 + 2x_3 + 3x_4 = 0$

basis ⇒ $\mathcal{N}(A) = \boxed{Sp\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right)}$

d) $\mathcal{R}(A) = \mathcal{C}(A) = \boxed{Sp\left(\begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 6 \end{bmatrix}\right)}$

e) $\mathcal{R}(A^T) = \mathcal{C}(A^T)$

$= \boxed{Sp\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}\right)}$

f) $\boxed{\mathcal{N}(A^T) = Sp\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)}$

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