1. Are the following matrices are in reduced row echelon form? If not, find the reduced row echelon form.

(a) (2 points) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

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Yes. ~~since~~ A is in rref.

(b) (2 points) $B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 2 \end{bmatrix}$

2

B is not in reduced row echelon form.

$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 2 \end{bmatrix} \div 3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & \frac{2}{3} \end{bmatrix} \begin{matrix} (ii) \\ (iii) \end{matrix}$ swap row (ii) with row (iii)

$rref(B) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix}$

2. (4 points) Suppose the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ \frac{1}{2}x_1 + x_2 \\ x_2 + \frac{1}{3}x_3 \\ \frac{1}{4}x_3 - x_1 \end{bmatrix}$$

Find a matrix A such that $A\vec{x} = T(\vec{x})$ for all $\vec{x} \in \mathbb{R}^3$

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$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x \\ \frac{1}{2}x_1 + x_2 \\ x_2 + \frac{1}{3}x_3 \\ \frac{1}{4}x_3 - x_1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ -1 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ -1 & 0 & \frac{1}{4} \end{bmatrix}$$

3. (3 points) How many solutions does the following system of linear equations have? Explain why.

$$x_1 + \sqrt{17}x_3 - x_4 = 2^{2^e}$$
$$3490x_2 - x_4 = \sqrt{48}$$
$$e^7x_1 - 14x_3 + x_{\underline{\underline{6}}} = 1$$

Hint: Write down the augmented matrix associated to this system of equations.

The augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & \sqrt{17} & -1 & 2^{2e} \\ 0 & 3490 & 0 & -1 & \sqrt{48} \\ e^7 & 0 & -14 & 1 & 1 \end{array}\right] - e^7\cdot(i)$$

$1+e^7$

$1+(e^7\cdot -1)$

$$= \left[\begin{array}{cccc|c} 1 & 0 & \sqrt{13} & -1 & 2^{2e} \\ 0 & 3490 & 0 & -1 & \sqrt{48} \\ 0 & 0 & -14\cdot e^7\sqrt{17} & 1+e^7 & 1-e^7\cdot 2^{2e} \end{array}\right]$$

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augument matrix

$$A = \left[\begin{array}{cccccc|c} 1 & 0 & \sqrt{17} & -1 & 0 & 0 & 2^{2e} \\ 0 & 3490 & 0 & -1 & 0 & 0 & \sqrt{48} \\ e^7 & 0 & -14 & 0 & 0 & 1 & 1 \end{array}\right] - e^7(i)$$

Since there are only 3 equations versus 6 unknowns

rank(A) < = 3 n=3, m=6

~~omit the last pat.~~

$$A = \left[\begin{array}{cccccc|c} 1 & 0 & \sqrt{17} & -1 & 0 & 0 & 2^{2e} \\ 0 & 3490 & 0 & -1 & 0 & 0 & \sqrt{48} \\ 0 & 0 & (-14-\sqrt{17}e^7) & e^7 & 0 & 1 & 1-e^7\cdot 2^{2e} \end{array}\right] \begin{array}{l} \\ \div 3490 \\ \div(-14-\sqrt{17}e^7) \end{array}$$

$$= \left[\begin{array}{cccccc|c} 1 & 0 & \sqrt{17} & -1 & 0 & 0 & 2^{2e} \\ 0 & 1 & 0 & -\frac{1}{3490} & 0 & 0 & \frac{\sqrt{48}}{3490} \\ 0 & 0 & 1 & \frac{e^7}{-14-\sqrt{17}e^7} & 0 & \frac{1}{-14-\sqrt{17}e^7} & \frac{1-e^7\cdot 2^{2e}}{-14-\sqrt{17}e^7} \end{array}\right] - \sqrt{17}(iii)$$

The rref of it coefficient matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1+\frac{\sqrt{17}}{14+\sqrt{17}e^7} & 0 & \frac{\sqrt{17}}{14+\sqrt{17}e^7} \\ 0 & 1 & 0 & -\frac{1}{3490} & 0 & 0 \\ 0 & 0 & 1 & \frac{-e^7}{14+\sqrt{17}e^7} & 0 & \frac{1}{-14-\sqrt{17}e^7} \end{bmatrix}$$

Therefore, the rank is 3.

rank(A), as from it rref, is 3. therefore there are infinitely many solutions to this system as ~~rank < n~~ rank = n and rank < m

For rank = n, the system is consistent and for rank < m, it has either infinitely many solutions or none solutions.

~~$$\begin{bmatrix} 1 & 0 & 0 & -1+\frac{e^7}{14+\sqrt{17}e^7} & 0 & 0 & \frac{\sqrt{17}}{14+\sqrt{17}e^7} \\ 0 & 1 & 0 & -\frac{1}{3490} & & & \end{bmatrix}$$~~

Therefore, this linear system has infinitely many solutions for it has 3 free variables.

3

4. Are the following statements true or false? No justification is needed.

(a) (1 point) If A is a 3×3 matrix and $A\vec{x} = \vec{0}$ has a unique solution, then $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^3$.

~~Yes,~~ True

(b) (1 point) If A is a 6×5 matrix and B is a 5×6 matrix, then $AB = BA$.

~~No.~~ False.

a b c d e

(c) (1 point) The vector $\vec{x} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of $\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

True.

$$\begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4c_1 \\ 2c_1 \\ 5c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ -c_2 \end{bmatrix}$$

$c_1 = 1$

$2c_1 + c_2 = 4 \quad \rightarrow \quad c_1 = 1$

$\qquad\qquad\qquad\qquad c_2 = 2$

$3 = 5c_1 - c_2$

3

5. (5 points) Two vectors $\vec{v}, \vec{w}$ are *orthogonal* if $\vec{v} \cdot \vec{w} = 0$. Find a matrix equation whose solutions are precisely those vectors orthogonal to both $\vec{w}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Then, find all solutions.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

If $\vec{x} \cdot \vec{w}_1 = 0$,

$x_1 \cdot (-1) + x_2 \cdot 3 + x_3 \cdot 0 + x_4 \cdot 1 = 0$

$-x_1 + 3x_2 + x_4 = 0$ ①

$A\vec{x} = \vec{0}$

If $\vec{x} \cdot \vec{w_2} = 0$

$x_1 \cdot 0 + x_2 \cdot 1 + x_3 \cdot 0 + x_4 \cdot 0 = 0$

$x_2 = 0$ ②

$A = \begin{bmatrix} -1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Subsitute x_2 to equation one

$-x_1 + x_4 = 0$

$x_1 = x_4$ assume x_1 equals to arbitrary number a
x_3 equals to arbitrary number t.

$$\vec{x} = \begin{bmatrix} a \\ 0 \\ t \\ a \end{bmatrix}$$

4

6. For each of the matrices below, either find an inverse or explain why no inverse exists.

(a) (3 points) $A = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 5 & 4 \\ 1 & 3 & -5 \end{bmatrix}$

some matrix ~~≠~~ In

5-2(2)

4-18

-5-9

3

If A is inveritble, AB = In   Rewrite as [A|In]

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 9 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2(i) \\ -(i) \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 9 & 1 & 0 & 0 \\ 0 & 1 & -14 & -2 & 1 & 0 \\ 0 & 1 & -14 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ ② \\ -(ii)\ ③ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 9 & 1 & 0 & 0 \\ 0 & 1 & -14 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right].$$

~~where [0 0 0 : 1 -1 1]~~

Since there is an inconsistency in row (iii), this matrix is not invertible as it could not give unique solution.

or, for row ~~②③~~ and

The lefthand side cannot be a ~~rref~~ In

(b) (3 points) $B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

[A | In]

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \therefore \text{ rewrite as } \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} (i) \\ (ii) \\ (iii) \\ (iv) \end{array}$$

$$① \text{ swap } (i)(ii) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} (ii) \\ (i) \\ (iii) \\ (iv) \end{array}$$

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② swap (i)(iii)

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore \text{ The inverse of B is } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$