M33A Midterm 1: Version B

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Section: 3F

DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.

Question	Points	Score
1	15	15
2	10	10
3	20	15
4	10	7
5	15	15
6	15	15
7	15	15
Total:	100	92

IF YOU CAN READ THIS AND YOU HAVE NOT BEEN TOLD TO START, YOU'RE IN TROUBLE

1. (15 points) Are the vectors $\begin{bmatrix}2\\2\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\3\\3\end{bmatrix}$ linearly independent?

$$\begin{bmatrix}2&1&0\\2&0&3\\0&1&3\end{bmatrix} \xRightarrow{\div 2} \begin{bmatrix}1&\frac{1}{2}&0\\2&0&3\\0&1&3\end{bmatrix} -2\times \text{I} \Rightarrow \begin{bmatrix}1&\frac{1}{2}&0\\0&-1&3\\0&1&3\end{bmatrix} \times(-1) \Rightarrow \begin{bmatrix}1&\frac{1}{2}&0\\0&1&-3\\0&1&3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&0&\frac{3}{2}\\0&1&-3\\0&0&6\end{bmatrix} \xRightarrow[\div 6]{} \begin{bmatrix}1&0&\frac{3}{2}\\0&1&-3\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3$$

$\because$ rref $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = I_3$, $\therefore$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent

15

2. Are the following matrices are in reduced row echelon form? If not, find the reduced row echelon form.

(a) (5 points) $\begin{bmatrix}1&0\\0&1\\0&1\\0&0\end{bmatrix} - \text{II}$

No, it's not

it's rref is $\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$

5

(b) (5 points) $\begin{bmatrix}1&2&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$

Yes it is

5

$m \times 1$

3. (a) (5 points) Define the rank of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

For a linear transformation $T: R^n \to R^m$ ~~with~~ represented by matrix $A_{(m\times n)}$, rank is the dimension of the kernel of A

~~8~~ 2.5

(b) (5 points) Define the nullity of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

For a linear transformation $T: R^n \to R^m$ prepresented by matrix $A_{(m\times n)}$, nullity is the dimension of the image of A

2.5

~~8~~ Had it right in d

(c) (5 points) State the Rank-Nullity Theorem.

For a linear transformation $T: R^n \to R^m$, the rank of it plus the nullity of it is equal to n

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(d) (5 points) Suppose $A, B \in M_{n\times n}(\mathbb{R})$ and that $\operatorname{im}(AB) = \mathbb{R}^n$. What is the nullity of A?

AB is a $n\times n$ matrix

$\text{rank}(AB) + \text{nullity}(AB) = n$

$\dim(\operatorname{im} AB) + \text{nullity}(AB) = n$

$\because \operatorname{im}(AB) = R^n \therefore \dim(\operatorname{im} AB) = n \therefore \text{nullity}(AB) = 0$

$\therefore \ker(AB) = \{\vec{0}\}$

$\because \ker(A) \subseteq \ker(AB)$

$\therefore \ker(A) = \{\vec{0}\}$

$\operatorname{im} A > \operatorname{im}(AB) = R^n$

$\operatorname{im} A \ll R^n$

$\therefore \operatorname{im} A = R^n$

$\text{nullity}(A) = 0$

5

4. (10 points) How many solutions does the following system of linear equations have?

$$x_1 + \sin(5)x_3 - x_4 = 12.3456789$$
$$e^2x_1 + 5x_5 + x_6 = 1$$
$$14000000x_2 + 2\pi x_4 = \sqrt{17}$$

Hint: Write down the matrix associated to this system of equations.

$$\left(\begin{array}{cccccc|c} 1 & 0 & \sin(5) & -1 & 0 & 0 & 12.3456789 \\ e^2 & 0 & 0 & 0 & 5 & 1 & 1 \\ 0 & 14000000 & 0 & 2\pi & 0 & 0 & \sqrt{17} \end{array}\right)$$

$\uparrow \quad \uparrow \uparrow$

there are three redundant column vectors

$\therefore \dim(\operatorname{im}(A)) = 3 \Rightarrow \text{ran}(A) = 3 < 6$

$\therefore$ There are infinitely many solutions

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why consistent?

用rref做一遍

每~~行~~有pivot

列

5. Are the following statements true or false? No justification is needed.

(a) (5 points) If a system of m linear equations in n unknowns is consistent, $m \geq n$.

(b) (5 points) If A is a 3×5 matrix and B is a 5×3 matrix, then $AB = BA$.

False

(c) (5 points) Any surjective linear transformation $\mathbb{R}^7 \to \mathbb{R}^7$ is invertible.

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True

6. (15 points) Is the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ invertible? If it is, find its inverse. If not, find a basis for its kernel.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \overset{\times \frac{1}{2}}{\Rightarrow} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \overset{-\frac{1}{2}\,II}{\Rightarrow} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \div 3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{-\frac{1}{2} \times III}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

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$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is invertible

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{bmatrix} \therefore A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

7. (15 points) Recall that the dot product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is given by the formula

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n.$$

Two vectors $\vec{v}, \vec{w}$ are *orthogonal* if $\vec{v} \cdot \vec{w} = 0$. Find a basis for the subspace V of $\mathbb{R}^4$ R^4 consisting of all vectors orthogonal to both $\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *Hint: Write the conditions $\vec{v} \cdot \vec{w}_i = 0$ as a system of linear equations, then solve this system.*

$\vec{v}\vec{w}_1 = 0, \vec{v}\,\vec{w}_2 = 0 \Rightarrow \vec{w}_1 \vec{v} = 0, \vec{w}_2 \vec{v} = 0$

$\therefore \begin{bmatrix} \leftarrow \vec{w}_1 \rightarrow \\ \leftarrow \vec{w}_2 \rightarrow \end{bmatrix} \vec{v} = \begin{bmatrix} \vec{w}_1 \vec{v} \\ \vec{w}_2 \vec{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

we are trying to find the kernel of

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\quad\ \ \vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3 \ \ \vec{v}_4$$

$\vec{v}_3 = 0\vec{v}_1 + 0\vec{v}_2$

$\vec{v}_4 = 0\vec{v}_2$

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V_3 and V_4 are redundant

$\therefore$ basis of $\ker A$: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

SCRATCH WORK

$\vec{0}$

$(AB)\vec{x} = \vec{0}$

$A(B\vec{x}) = \vec{0}$

$\vec{x}$

$A\vec{x} = \vec{0}$

$AB\vec{x} = \vec{0}$