22W-MATH-32BH-LEC-1 Midterm 1

Student ZEK6 TAQ2

TOTAL POINTS

98 / 100

QUESTION 1

Question 1 25 pts

1.1 (a) **10 / 10**

✓ + **10 pts** Correct

+ **5 pts** Attempted change of variables - did not correctly verify injectivity, and did not correctly change the domain of integration.

+ **0 pts** Incorrect or incomplete work

1.2 (b) **9 / 10**

✓ + **10 pts** Correct

+ **5 pts** Attempted change of variables - did not correctly verify injectivity, and did not correctly change the domain of integration.

+ **0 pts** Incorrect or incomplete work

+ **8 pts** Mostly correct integration; incorrect evaluation at the end

- 1 Point adjustment

off by a sign

1 this should be -y

1.3 (c) **5 / 5**

✓ + **5 pts** **Correct. Observed the function is not bounded, and hence not integrable.**

+ **0 pts** Incorrect. Fubini's theorem does not apply to this function.

+ **4 pts** Observed the function is not continuous - however, we saw in class that Fubini's theorem holds for integrable functions

+ **2 pts** Correctly stated a version of Fubini's theorem.

QUESTION 2

Question 2 25 pts

2.1 (a) **10 / 10**

✓ + **10 pts** Correct

+ **0 pts** Incorrect or invalid proof.

2.2 (b) **10 / 10**

✓ + **10 pts** Correct

+ **9 pts** Correct change of variables, incorrect evaluation of the integral

+ **0 pts** Incorrect or invalid change of variables.

2.3 (c) **5 / 5**

✓ + **5 pts** Correct

+ **4 pts** Did not justify taking positive square root

+ **0 pts** Incorrect

QUESTION 3

Question 3 25 pts

3.1 (a) **10 / 10**

✓ + **10 pts** Correct

+ **6 pts** Good attempt

+ **3 pts** Did not attempt

- **1.5 pts** Lack of justification for $$osc(|f|) \leq osc(f)$$ (or equivalent forms using $$M$$ and $$m$$) (if such a method/definition is used).

- **2 pts** Overall minor lack of justification

3.2 (b) **9 / 10**

✓ + **10 pts** Correct

+ **4 pts** Attempt

✓ **- 0.5 pts** **Did not justify pulling out the $$-1$$ from the integral (if a direct computational proof was done).**

✓ **- 0.5 pts** **Did not justify the ability to take $$\int$$**

on both sides of the inequality.

- **0.5 pts** Did not justify the ability to decompose $$\int_{A \cup B} = \int_A + \int_B$$ (ie check intersection of domains etc).
- **1.5 pts** Minor additional justification errors
- **4 pts** Significant conceptual/justification errors

2 Pulling -1 outside of the integral requires justification

3 To deduce this, you used the fact that f+ and f- >= 0 and thus their integrals are >= 0. This is equivalent to taking integral on both sides of the inequality, which requires justification.

3.3 (c) **5 / 5**

✓ **+ 5 pts** **Correct**

+ **2 pts** Attempt
- **1 pts** Insufficient details
- **1.5 pts** Did not specify the corresponding domain of $$f$$ (which the integral thus diverges on $$\mathbb{R}^2$$)

QUESTION 4

Question 4 25 pts

4.1 (a) **5 / 5**

✓ **+ 5 pts** **Correct**

- **0.5 pts** Did not shade the region to distinguish between the inner/outer region

4.2 (b) **5 / 5**

✓ **+ 5 pts** **Correct**

- **1 pts** Imprecise specification.

4.3 (c) **15 / 15**

✓ **+ 15 pts** **Correct.**

+ **12 pts** Right domain + Incorrect integration techniques (missing $$r$$ in polar Jacobian etc)
+ **10 pts** Right domain + Incorrect Decomposition of domain + Right integration techniques
+ **7 pts** Incorrect domain + Incorrect Decomposition of domain + Right integration techniques.
+ **5 pts** Good Attempt
- **1 pts** Additional computational errors

1. Consider the function

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (10 points) Use the substitution $u = x^2 + y^2$ to compute the iterated integral

$$\int_0^1 \int_0^2 f(x,y)\, dy\, dx$$

$x^2 - y^2 = -u + 2x^2$

$\frac{\partial u}{\partial y} = 2y$

$v = x^2 + 4$

$\frac{dv}{dx} = 2x$

$= \frac{1}{2}\int_0^1 \int_{x^2+0}^{x^2+4} \frac{x(-u+2x^2)}{u^3}\, du\, dx$

$= \frac{1}{2}\int_0^1 \int_{x^2}^{x^2+4} \left(-\frac{x}{u^2} + \frac{2x^3}{u^3}\right) du\, dx$

$= \frac{1}{2}\int_0^1 \left(\frac{x}{u} - \frac{x^3}{u^2}\right)\Big|_{u=x^2}^{u=x^2+4} dx$

$= \frac{1}{2}\int_0^1 \left(\frac{x}{x^2+4} - \frac{x^3}{(x^2+4)^2} - \frac{x}{x^2} + \frac{x^3}{x^4}\right) dx$

$= \frac{1}{2}\int_0^1 \left(\frac{(x^3+4x)-x^3}{(x^2+4)^2}\right) dx$

$= \frac{1}{2}\int_0^1 \frac{4x}{(x^2+4)^2}\, dx$

$= \int_4^5 \frac{1}{v^2}\, dv$

$= \left(-\frac{1}{v}\right)\Big|_4^5$

$= \frac{1}{4} - \frac{1}{5} = \boxed{\frac{1}{20}}$

(b) (10 points) Use the substitution $u = x^2 + y^2$ to compute the iterated integral

$$\int_0^2 \int_0^1 f(x,y)\, dx\, dy$$

$x^2 - y^2 = u - 2y^2$

$\frac{\partial u}{\partial x} = 2x$

$v = y^2 + 1$

$\frac{dv}{dy} = 2y$

$= \frac{1}{2}\int_0^2 \int_{y^2+0}^{y^2+1} \frac{y(u-2y^2)}{u^3}\, du\, dy$

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$= \frac{1}{4}\int_1^5 \frac{1}{v^2}\, dv$

$= \frac{1}{4}\left(-\frac{1}{v}\right)\Big|_1^5$

$= \frac{1}{4}\left(1 - \frac{1}{5}\right) = \boxed{\frac{1}{5}}$

(c) (5 points) How do your answers in (a) and (b) relate to Fubini's theorem?

$\frac{1}{20} \neq \frac{1}{5}$, indicating that Fubini's theorem allowing iterated integrals to be taken in any order does not apply. While f is mostly continuous and has bounded support, f is not bounded.

1.1 (a) **10 / 10**

✓ **+ 10 pts** Correct

+ 5 pts Attempted change of variables - did not correctly verify injectivity, and did not correctly change the domain of integration.

+ 0 pts Incorrect or incomplete work

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1.3 (c) **5 / 5**

✓ **+ 5 pts** **Correct. Observed the function is not bounded, and hence not integrable.**

+ 0 pts Incorrect. Fubini's theorem does not apply to this function.

+ 4 pts Observed the function is not continuous - however, we saw in class that Fubini's theorem holds for integrable functions

+ 2 pts Correctly stated a version of Fubini's theorem.

2. Consider the single-variable improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx := \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b e^{-x^2}\, dx$$

You may assume that the improper integrals I and J converge (that is, the limits for I and J exist and are finite). You can freely use the theorems in the limits and continuity supplement.

(a) (10 points) Prove that $I^2 = J$, where

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy := \lim_{\substack{a,c\to-\infty \\ b,d\to\infty}} \int_c^d \int_a^b e^{-x^2-y^2}\, dx\, dy$$

Let $f(x) = e^{-x^2}$, $g(y) = e^{-y^2}$, and $h(x,y) = f(x)g(y) = e^{-x^2-y^2}$.

Then, $\int_{\mathbb{R}^2} h\,dxdy = \left(\int_{\mathbb{R}} f\,dx\right)\left(\int_{\mathbb{R}} g\,dy\right)$ since x, y are distinct.

$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy = \left(\int_{\mathbb{R}} e^{-x^2}dx\right)\left(\int_{\mathbb{R}} e^{-y^2}dy\right)$

And, $\lim_{\substack{a,c\to-\infty \\ b,d\to\infty}} \int_c^d \int_a^b e^{-x^2-y^2}\,dxdy = \lim_{\substack{a,c\to-\infty \\ b,d\to\infty}} \left(\left(\int_a^b e^{-x^2}dx\right)\left(\int_c^d e^{-y^2}dy\right)\right)$

$= \left(\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b e^{-x^2}dx\right)^2$ by the Product Law.

x and y are arbitrary names, so $\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b e^{-x^2}dx = \lim_{\substack{c\to-\infty \\ d\to\infty}} \int_c^d e^{-y^2}dy$,

So, $I^2 = J$. $\square$

(b) (10 points) Rewrite J as a limit in terms of polar coordinates, and evaluate J. (You may freely evaluate at infinity as in single-variable calculus; you do not need to rigorously prove that $\lim_{R\to\infty} f(R) = L$)

$x^2+y^2=r^2$

$r \geq 0$

$0 \leq \theta \leq 2\pi$

$u = -r^2$

$\frac{du}{dr} = -2r$

$$\boxed{J = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta}$$

$\lim_{b\to\infty} \int_0^b e^{-r^2} r\,dr$

$= -\frac{1}{2}\lim_{b\to\infty} \int_0^{-b^2} e^u\,du$

$= -\frac{1}{2}\lim_{b\to\infty} (e^{-b^2} - 1) = \frac{1}{2}$

$\int_0^{2\pi} \frac{1}{2}\,d\theta$

$= \left(\frac{1}{2}r\right)_0^{2\pi}$

$= \pi$

$\boxed{J = \pi}$

(c) (5 points) Deduce the value of I from parts (a) and (b).

$I^2 = J = \pi$, so $\boxed{I = \sqrt{\pi}}$

$e^{\cdots}$ always positive

2.1 (a) **10 / 10**

✓ **+ 10 pts** Correct

+ 0 pts Incorrect or invalid proof.

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So, $I^2 = J$. □

(b) (10 points) Rewrite J as a limit in terms of polar coordinates, and evaluate J. (You may freely evaluate at infinity as in single-variable calculus; you do not need to rigorously prove that $\lim_{R\to\infty} f(R) = L$)

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$J = \pi$

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$I^2 = J = \pi$, so $I = \sqrt{\pi}$

$e^{\cdots}$ always positive

2.2 (b) **10 / 10**

✓ **+ 10 pts** Correct

+ 9 pts Correct change of variables, incorrect evaluation of the integral

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So, $I^2 \stackrel{?}{=} J$. □

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e^{anything} always positive

2.3 (c) **5 / 5**

- ✓ **+ 5 pts** Correct
- **+ 4 pts** Did not justify taking positive square root
- **+ 0 pts** Incorrect

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function.

(a) (10 points) Prove that the function $|f|(\boldsymbol{x}) := |f(\boldsymbol{x})|$ is integrable.

f is integrable so f is bounded with bounded support. $\Rightarrow |f|(\vec{x})$ has bounded support.

$$|f|(\vec{x}) := \begin{cases} f(\vec{x}), & f(\vec{x}) \geq 0 \\ -f(\vec{x}), & f(\vec{x}) < 0 \end{cases}$$

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Then, $|f|(\vec{x})$ is bounded by lower bound 0 and upper bound $\max(|a|, |b|)$.

For all $\varepsilon > 0$, there exists N such that

$$\sum_{\{C \in D_N \mid osc_C(f) > \varepsilon\}} vol_n C < \varepsilon.$$

If the infimum and supremum of f on a dyadic cube are both nonnegative or both nonpositive, $osc_C(f) = osc_C(|f|)$.

If they are opposite signs, $osc_C(f) > osc_C(|f|)$ (Let a, b be positive such that $m = -a$, $M = b$. $b - (-a) = b + a > b - a$.). Thus, for all $\varepsilon > 0$,

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So, $|f|(\vec{x})$ is integrable. $\square$

(b) (10 points) Prove that $\left|\int_{\mathbb{R}^n} f \, dV\right| \leq \int_{\mathbb{R}^n} |f| \, dV$

Let f be the sum of its nonnegative and nonpositive components: $f = f^+ - f^-$, where f^+ and f^- are nonnegative, defined like so:

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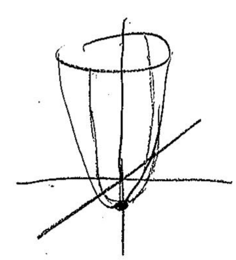
(c) (5 points) Give an example of an integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\left|\int_{\mathbb{R}^2} f \, dV\right| < \int_{\mathbb{R}^2} |f| \, dV$

Example: $$f(x,y) := \begin{cases} x^2 + y^2 - 1, & x^2 + y^2 \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



$0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

continuous almost everywhere (except $x^2 + y^2 = 4$)

Verify: $$\left|\int_0^{2\pi}\int_0^2 (r^2 - 1) r\,dr\,d\theta\right| = \left|\int_0^{2\pi} \left(\tfrac{1}{4}r^4 - \tfrac{1}{2}r^2\right)\Big|_0^2 d\theta\right| = \left|\int_0^{2\pi} 2\,d\theta\right| = 4\pi$$

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$$= 5\pi$$

$5\pi > 4\pi$

3.1 (a) **10 / 10**

- ✓ **+ 10 pts** **Correct**
- **+ 6 pts** Good attempt
- **+ 3 pts** Did not attempt
- **- 1.5 pts** Lack of justification for $$osc(|f|) \leq osc(f)$$ (or equivalent forms using $$M$$ and $$m$$) (if such a method/definition is used).
- **- 2 pts** Overall minor lack of justification

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function.

(a) (10 points) Prove that the function $|f|(\boldsymbol{x}) := |f(\boldsymbol{x})|$ is integrable.

f is integrable so f is bounded with bounded support. $\Rightarrow |f|(\vec{x})$ has bounded support.

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For all $\varepsilon > 0$, there exists N such that

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So, $\left|\int_{\mathbb{R}^n} f \, dV\right| \leq \int_{\mathbb{R}^n} |f| \, dV$. $\square$

(c) (5 points) Give an example of an integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\left|\int_{\mathbb{R}^2} f \, dV\right| < \int_{\mathbb{R}^2} |f| \, dV$

Example:
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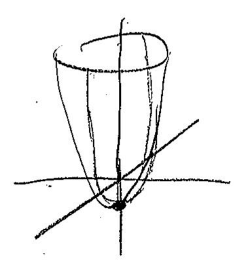

$0 \leq r \leq 2$
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continuous almost everywhere (except $x^2 + y^2 = 4$)

Verify: $$\left|\int_0^{2\pi}\int_0^2 (r^2 - 1) r \, dr \, d\theta\right| = \left|\int_0^{2\pi} \left(\tfrac{1}{4}r^4 - \tfrac{1}{2}r^2\right)\Big|_0^2 d\theta\right| = \left|\int_0^{2\pi} 2 \, d\theta\right| = 4\pi$$

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$$= 5\pi$$

$5\pi > 4\pi$

3.2 (b) **9 / 10**

✓ **+ 10 pts** Correct

+ 4 pts Attempt

✓ **- 0.5 pts Did not justify pulling out the $$-1$$ from the integral (if a direct computational proof was done).**

✓ **- 0.5 pts Did not justify the ability to take $$\int$$ on both sides of the inequality.**

- 0.5 pts Did not justify the ability to decompose $$\int_{A \cup B} = \int_A + \int_B$$ (ie check intersection of domains etc).

- 1.5 pts Minor additional justification errors

- 4 pts Significant conceptual/justification errors

2 Pulling -1 outside of the integral requires justification

3 To deduce this, you used the fact that f+ and f- >= 0 and thus their integrals are >= 0. This is equivalent to taking integral on both sides of the inequality, which requires justification.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function.

(a) (10 points) Prove that the function $|f|(\boldsymbol{x}) := |f(\boldsymbol{x})|$ is integrable.

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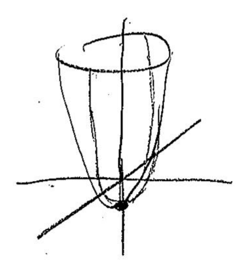

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$$= 5\pi$$

$5\pi > 4\pi$

3.3 (c) **5 / 5**

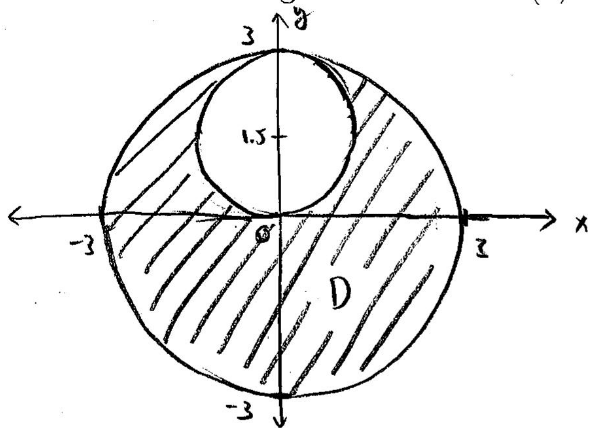
✓ **+ 5 pts** **Correct**

+ 2 pts Attempt

- 1 pts Insufficient details

- 1.5 pts Did not specify the corresponding domain of $$f$$ (which the integral thus diverges on $$\mathbb{R}^2$$)

4. (a) (5 points) Sketch the region D in $\mathbb{R}^2$, which is bounded by a circle of radius 3, centered at the origin, and outside the circle of radius 1.5, centered at the point with rectangular coordinates $(0, 1.5)$.



(b) (5 points) Use equations to describe the boundary of the region D. You may use any coordinate system.

$0 \le r \le 3$,
$r \ge 3\sin\theta$

in polar coordinates:

$D = \{(r,\theta) \mid 3\sin\theta \le r \le 3\}$

$\partial D = \{(r,\theta) \mid r = 3 \text{ or } r = 3\sin\theta\}$

(c) (15 points) Compute the integral $\iint_D \sqrt{x^2+y^2}\, dA$.

$x^2+y^2 = r^2$
$r \ge 0$
$0 \le \theta \le 2\pi$

Note: $3\sin\theta$ from 0 to π

then $\iint_D \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_0^3 r^2\,dr\,d\theta - \int_0^{\pi}\int_0^{3\sin\theta} r^2\,dr\,d\theta$

$= \int_0^{2\pi} \left(\frac{1}{3}r^3\right)\Big|_0^3 d\theta - \int_0^{\pi}\left(\frac{1}{3}r^3\right)\Big|_0^{3\sin\theta} d\theta$

$= \int_0^{2\pi} 9\, d\theta + \int_0^{\pi} -9\sin^3\theta\, d\theta$

$= (9\theta)\Big|_0^{2\pi} + (9u - 3u^3)\Big|_1^{-1}$

$= 18\pi - 9 + 3 - 9 + 3$

$= \boxed{18\pi - 12}$

$\int(-9\sin^3\theta)\,d\theta$

$= \int(-9\sin\theta(1-\cos^2\theta))\,d\theta$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

$= \int 9(1-u^2)\,du$

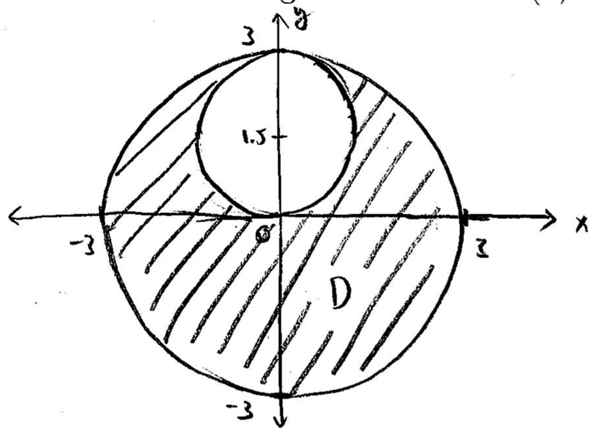
$= 9u - 3u^3$

4.1 (a) **5 / 5**

✓ **+ 5 pts** Correct

- 0.5 pts Did not shade the region to distinguish between the inner/outer region

4. (a) (5 points) Sketch the region D in $\mathbb{R}^2$, which is bounded by a circle of radius 3, centered at the origin, and outside the circle of radius 1.5, centered at the point with rectangular coordinates $(0, 1.5)$.



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$u = \cos\theta$
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$= \int 9(1-u^2)\,du$

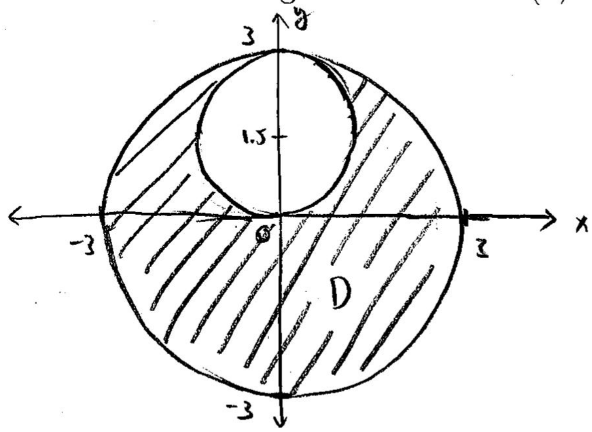
$= 9u - 3u^3$

4.2 (b) **5 / 5**

✓ **+ 5 pts** Correct

- 1 pts Imprecise specification.

4. (a) (5 points) Sketch the region D in $\mathbb{R}^2$, which is bounded by a circle of radius 3, centered at the origin, and outside the circle of radius 1.5, centered at the point with rectangular coordinates $(0, 1.5)$.



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$= \int(-9\sin\theta(1-\cos^2\theta))\,d\theta$

$u = \cos\theta$

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$= 9u - 3u^3$

4.3 (c) **15 / 15**

- ✓ **+ 15 pts** **Correct.**
- **+ 12 pts** Right domain + Incorrect integration techniques (missing $$r$$ in polar Jacobian etc)
- **+ 10 pts** Right domain + Incorrect Decomposition of domain + Right integration techniques
- **+ 7 pts** Incorrect domain + Incorrect Decomposition of domain + Right integration techniques.
- **+ 5 pts** Good Attempt
- **- 1 pts** Additional computational errors