University of California, Los Angeles
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Mathematics 32BH

Final Exam

- This is a take-home exam. Issued: 3/19/2020 11:59 PM. Due 3/20/2020 11:59 PM.
- You have 24 hours to complete it.
- Upload your work to each problem on gradescope in CCLE. Make sure the images are readable and legible.
- This is open book and open notes. You may use your textbook, class notes, homework, shorts notes for the course.
- There are 12 problems, some with multiple parts.
- You must justify your work. You may cite theorems, results, problems, exercises seen in class, homework, notes, or textbook.
- You may use your own sheets of paper to work on. Try to start a new sheet of paper for a new problem if you can. **Write neatly and be organized**.
- You should not need any electronics except to look up those notes or texts.
- Follow the honor code and submit your own work.
- You can do it! Do your best and good luck!

Problem 1.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, with $f(0,0,0) = 0$. Show if $\nabla f(0,0,0) \neq (0,0,0)$, then there exists a smooth 2-surface $\Phi(u,v) : [0,1]^2 \to \mathbb{R}^3$ with $\Phi(0,0) = (0,0,0)$ and that $f \circ \Phi = 0$.

Problem 2.
Consider the set

$$C = \{(x, y) : x^2 + y^2 = 1\},$$

the 1-surface

$$\gamma : [0, 1] \to \mathbb{R}^2, \gamma(t) = (\cos 2\pi t, \sin 2\pi t),$$

and the 1-surface

$$\lambda : [0, 1] \to \mathbb{R}^2, \lambda(t) = (\cos 2\pi t, -\sin 2\pi t).$$

Given C, γ, and λ above, decide true or false for each of the following three statements. If it is true, briefly justify it. If it is false, provide a counterexample.

(A) For every smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int_C f = \int_\gamma f$$

where the left hand side is the Riemann integral of f over the set C, while the right hand side is the integral of f over the 1-surface γ.

(B) For every smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int_\gamma f = \int_\lambda f.$$

(C) For every smooth vector fields $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, the work integrals

$$\int_\gamma \vec{F} \cdot \widehat{T} = \int_\lambda \vec{F} \cdot \widehat{T}$$

are the same.

Problem 3.
Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions.
(A) Prove Cauchy-Schwarz inequality for integrals:

$$\left(\int_a^b fg\right)^2 \leqslant \left(\int_a^b f^2\right)\left(\int_a^b g^2\right),$$

(B) Prove further that equality holds if and only if one function is a scalar multiple of the other, namely there exists some real constant $c \in \mathbb{R}$ such that $g = cf$ or $f = cg$.
Hint: First observe we have the double integral

$$\int_a^b \int_a^b [f(x)g(y) - f(y)g(x)]^2 \, dxdy \geqslant 0.$$

You should indicate relevant theorems or results used in your proof.

Problem 4.

Let the n-cube $\beta_n : [0,1]^n \to \mathbb{R}^n$ be given by

$$\beta_n(x_1, \ldots, x_n) = (x_1, 2x_2^2, 3x_3^3, \ldots, nx_n^n)$$

(so the i-th entry of β_n is ix_i^i)

Consider the $n-1$ form

$$\omega = \sum_{i=1}^{n} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{dx_i}$ means omit the dx_i term.

Compute

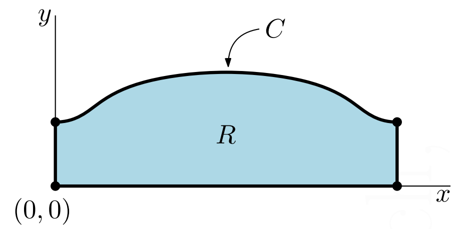
$$\int_{\partial \beta_n} \omega.$$

Problem 5.

Let $0 < b < a$. Consider a curve C in the xy-plane parameterized by

$$\gamma(t) = \begin{pmatrix} at - b\sin t \\ a - b\cos t \end{pmatrix} : [0, 2\pi] \to \mathbb{R}^2$$

Find the area of the region R bounded by this curve C, the x-axis, and the two vertical segments each meeting an end point of the curve C. A diagram is indicated below



(Note: Identify the points in the diagram carefully!)

Problem 6

Consider the 1-form

$$\omega = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy$$

on $A = \mathbb{R}^2 \backslash \{(0,0)\}$.

Let $\gamma : [0,1] \to A$ be any smooth curve (a 1-surface) such that $\gamma(0) = (-1,-1)$ and $\gamma(1) = (2,2)$. Note this curve γ never goes through $(0,0)$.

Compute

$$\int_\gamma \omega.$$

Problem 7.

Let $r : A \to B$ be a smooth function between open sets A, B.

(A) Show if $\sigma \in C_k(A)$ is a cycle, then $r_*\sigma$ is also a cycle.

(B) Is it true that if $\sigma \in C_k(A)$ is a boundary, then $r_*\sigma$ is also a boundary?

(C) Suppose further that $r : A \to B$ is smooth and bijective, with smooth inverse $r^{-1} : B \to A$. Show if every closed form on A is exact then every closed form on B is exact.

Problem 8.

Consider the 1-forms $\lambda_1, \ldots, \lambda_n \in \Lambda^1(\mathbb{R}^n)$ given by

$$\begin{aligned}
\lambda_1 &= dx_1 \\
\lambda_2 &= dx_1 + dx_2 \\
&\vdots \\
\lambda_n &= dx_1 + \cdots + dx_n
\end{aligned}$$

namely, $\lambda_k = \sum_{i=1}^{k} dx_i$.

(A) Show each λ_k is exact by finding an anti-derivative f_k, where $\lambda_k = df_k$. Is the anti-derivative you found for λ_k unique? Why or why not?

(B) Consider the product

$$\omega = \lambda_n \wedge \lambda_{n-1} \wedge \cdots \wedge \lambda_2 \wedge \lambda_1 = \bigwedge_{k=0}^{n-1} \lambda_{n-k}$$

namely, the wedge product of above 1-forms in reverse order. Here ω is an n-form in $\mathbb{R}^n$, so it can be expressed as a product with increasing indexing of dx_i as follows

$$\omega = \epsilon(n) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

for some scalar $\epsilon(n)$. Compute $\epsilon(n)$. Note, your answer depends on n.

Problem 9.

Consider the 2-cube $c \in C_2(\mathbb{R}^3)$ given by

$$c(u,v) = \begin{pmatrix} \cos 2\pi u \\ v \\ \sin 2\pi u \end{pmatrix} : [0,1]^2 \to \mathbb{R}^3$$

and the 1-cubes $\gamma_0, \gamma_1 \in C_1(\mathbb{R}^3)$ given by

$$\gamma_0(t) = \begin{pmatrix} \cos 2\pi t \\ 0 \\ \sin 2\pi t \end{pmatrix} : [0,1] \to \mathbb{R}^3 \quad \text{and} \quad \gamma_1(t) = \begin{pmatrix} \cos 2\pi t \\ 1 \\ \sin 2\pi t \end{pmatrix} : [0,1] \to \mathbb{R}^3$$

Consider a 1-form $\lambda \in \Lambda^1(\mathbb{R}^3)$.

(A) Prove that if $\displaystyle\int_c d\lambda > 0$ then $\displaystyle\int_{\gamma_0} \lambda > \int_{\gamma_1} \lambda$.

(B) What can you say about $\displaystyle\int_{\gamma_0} \lambda$ and $\displaystyle\int_{\gamma_1} \lambda$ if $\displaystyle\int_c d\lambda = 0$?

(C) Give an example of $\lambda \in \Lambda^1(\mathbb{R}^3)$ such that $\displaystyle\int_c d\lambda > 0$.

Problem 10.

Consider a smooth k-form $\omega \in \Lambda^k(A)$, where A is an open set in $\mathbb{R}^n$. Suppose $\omega \neq 0$, namely when we express $\omega = \sum_I f_I dx_I$ uniquely with only increasing indexing, some $f_{I'}$ is not the identically zero function on A for some increasing indexing I'. Show there exists a k-chain $\sigma \in C_k(A)$ such that $\int_\sigma \omega \neq 0$. (In fact σ can be taken to be a k-cube)

Hint: If a smooth scalar function f is not identically zero on A, then there exists a small box B in A such that f is not zero on B.

Problem 11.

Explain and make sense of the following statement:

"Take a cylinder cup and fill it with water, and consider the water molecules in the cup. At any two times t_1 and t_2, there exists a water molecule whose position at time t_1 is the same at time t_2"

You may add additional mathematical assumptions to say something meaningful mathematically.

Problem 12.

Let A be an open set in $\mathbb{R}^n$ such that we can write $A = A_1 \cup A_2 \cup \cdots \cup A_k$, where A_i are pairwise disjoint path-connected open sets (namely A has k many connected components). Consider $\Lambda^0(A)$, the set of all smooth 0-forms on A. Show that if $f \in \Lambda^0(A)$ such that f is closed, then f is a function that is constant on each A_i, namely

$$f(x) = \begin{cases} c_1 & \text{if } x \in A_1 \\ c_2 & \text{if } x \in A_2 \\ \vdots \\ c_k & \text{if } x \in A_k \end{cases}$$

for some constants $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Hence there is a bijection between the sets

$$\left\{ f \in \Lambda^0(A) : df = 0 \right\} \leftrightarrow \mathbb{R}^k.$$

(Remark. By doing above, you have essentially computed the 0-th de Rham cohomology group of A, and it tells us the number of connected components of A!)