Discussion Session: A B C D

Math 32B
Winter 2016
Midterm I
01/29/16
Time Limit: 50 mins(12:00–12:50pm)

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This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may only use your cheating sheet and an non-graphic calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	16	15
2	24	24
3	24	22
4	24	14
5	12	8
Total:	100	83

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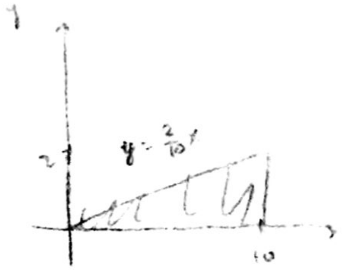
1. (16 points) Evaluate the double integral $\iint_D e^{-x^2}\,dA$, where D is the triangle on the xy-plane with vertices $(0,0)$, $(10,0)$ and $(10,2)$. (Give the answer in terms of e, do not insert a numerical value for e.)



$$\int_{x=0}^{10}\int_{y=0}^{\frac{1}{5}x} e^{x^2}\,dy\,dx$$

$$\int_{x=0}^{10}\left[e^{-x^2}y\right]_{y=0}^{\frac{1}{5}x}dx$$

$$\int_{x=0}^{10}\frac{1}{5}xe^{-x^2}\,dx \qquad u=-x^2 \qquad du=-2x\,dx$$

$$-\frac{1}{10}\int_0^{-100} e^u\,du$$

$$-\frac{1}{10}e^u\Big|_0^{-100} = -10\left(e^{-100}-e^0\right)=\boxed{-10\left(e^{-100}-1\right)}$$

$$\boxed{-10\left(\frac{1}{e^{100}}-1\right)}$$

2. (a) (6 points) Sketch the two-dimension region D in the first quadrant bounded by the line $x = 0$ and the two curves $x^2 + y^2 = 4y$ and $x^2 + y^2 = 2y$.

$x^2+y^2=4y$

$x^2+y^2-4y=0$

$x^2+(y^2-4y+4)=4$

$x^2+(y-2)^2=4$

$(0,2)$ $r=2$

$x^2+y^2-2y=0$

$x^2+(y^2-2y+1)=1$

$x^2+(y-1)^2=1$

$(0,1)$ $r=1$

$x^2+y^2=4y$

$r^2=4r\sin\theta$

$r=4\sin\theta = 2(2\sin\theta)$

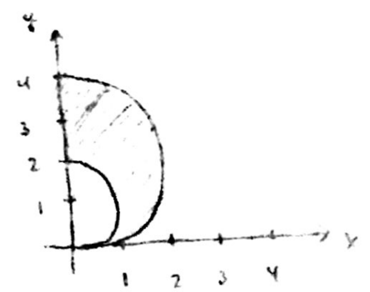
centered @ $(0,2)$, radius 2

$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$

centered at $(0,1)$, radius 1



(b) (18 points) Compute the double integral $\iint_D \sqrt{x^2+y^2}\,dA$, where D is the region described in part (a).

$r=2\sin\theta \quad r=4\sin\theta$

$D: \{0 \le \theta \le \frac{\pi}{2},\ 2\sin\theta \le r \le 4\sin\theta\}$

$$\iint_D \sqrt{x^2+y^2}\,dA = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2\sin\theta}^{4\sin\theta} \sqrt{r^2}\, r\,dr\,d\theta = \int_{\theta=0}^{\pi/2} \int_{r=2\sin\theta}^{4\sin\theta} r^2\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^3}{3}\right]_{r=2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{3}\int_{\theta=0}^{\pi/2} (64\sin^3\theta - 8\sin^3\theta)\,d\theta = \frac{56}{3}\int_{\theta=0}^{\pi/2} \sin^3\theta\,d\theta$$

$$= \frac{56}{3}\int_{\theta=0}^{\pi/2} \sin\theta(1-\cos^2\theta)\,d\theta \qquad u=\cos\theta \quad du=-\sin\theta\,d\theta$$

$$-\frac{56}{3}\int_{u=1}^{0} (1-u^2)\,du = -\frac{56}{3}\left[u-\frac{u^3}{3}\right]_{u=1}^{0} = -\frac{56}{3}\left[-\left(1-\frac{1}{3}\right)\right] = -\frac{56}{3}\left(-\frac{2}{3}\right) = \boxed{\frac{112}{9}}$$

3. (24 points) Compute the triple integral $\iiint_E xy\,dV$, where E is the solid bounded by:

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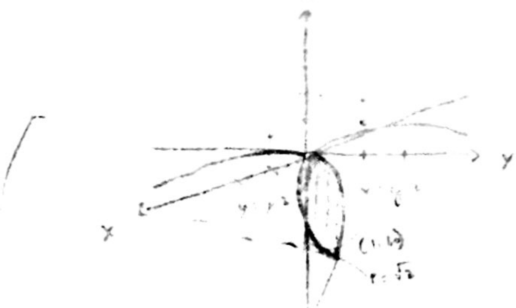
- the two parabolic cylinders $y = x^2$ and $x = y^2$
- the two planes $z = 0$ and $x + y - z = 0$.

$y = x^2$ $\quad r\sin\theta = r^2\cos^2\theta \quad \sin\theta = r\cos^2\theta \quad r = \frac{\sin\theta}{\cos^2\theta}$

$x = y^2$ $\quad r\cos\theta = r^2\sin^2\theta \quad r\sin\theta = \cos\theta \quad r = \frac{\cos\theta}{\sin^2\theta}$

$z = x + y$



D: {

$\frac{\sin\theta}{\cos^2\theta} \quad \frac{\cos\theta}{\sin^2\theta}$

$\frac{1}{\cos\theta} = \frac{1}{\sin\theta}$ $\quad \cos\theta = \sin\theta \to \pi/4$

$D: \{ 0 \le x \le 1,\ \sqrt{x} \le y \le x^2,\ 0 \le z \le x+y \}$

$$\int_0^1 \int_{\sqrt{x}}^{x^2} \int_0^{x+y} xy\, dz\,dy\,dx = \int_{x=0}^1 \int_{y=\sqrt{x}}^{x^2} \left[xyz \Big|_0^{x+y} \right] dy\,dx = \int_{x=0}^1 \int_{y=\sqrt{x}}^{x^2} xy(x+y)\,dy\,dx$$

$$= \int_{x=0}^1 \int_{y=\sqrt{x}}^{x^2} (x^2y + xy^2)\,dy\,dx = \int_{x=0}^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=\sqrt{x}}^{x^2} dx = \int_{x=0}^1 \left[\frac{x^6}{2} + \frac{x^7}{3} - \frac{x^3}{2} - \frac{x^{5/2}}{3} \right] dx$$

$$= \left[\frac{x^7}{14} + \frac{x^8}{24} - \frac{x^4}{8} - \frac{2x^{7/2}}{21} \right]_{x=0}^1 = \boxed{\frac{1}{14} + \frac{1}{24} - \frac{1}{8} - \frac{2}{21}} = \cancel{\frac{24 + 16 - 30}{240} - \frac{2}{21} = \frac{4}{240} - \frac{2}{21}}$$

$$= \cancel{\frac{1}{60} - \frac{2}{21} = \frac{21 - 120}{1260}}$$

4. (a) (8 points) Let $a > 0$ be a constant. Describe the surface $z = \frac{1}{a}\sqrt{x^2+y^2}$ in spherical coordinates (ρ, ϕ, θ).

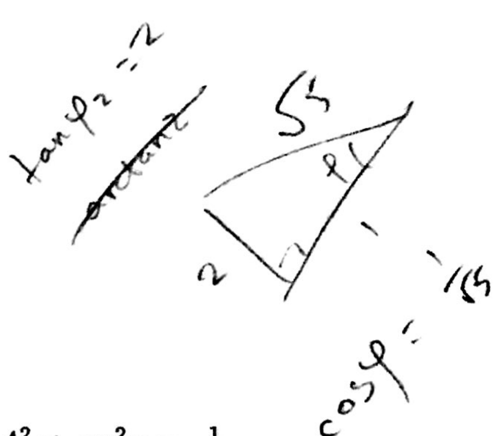
$z = \frac{1}{a}\sqrt{x^2+y^2}$ $\qquad z = \frac{1}{a} r$

$\rho\cos\phi = \frac{1}{a}\sqrt{r^2}$ $\qquad \rho$

$\rho\cos\phi = \frac{1}{a}\rho\sin\phi$

$\cos\phi = \frac{\sin\phi}{a}$

$\boxed{a = \tan\phi}$



(b) (2 points) Prove that, for arbitrary α and A, $tan^2\alpha = A^2 \Rightarrow cos^2\alpha = \frac{1}{1+A^2}$.

$\tan^2\alpha = A^2$

$\cos^2\alpha = \frac{1}{1+A^2}$

$\tan^2\alpha = A^2$

$\frac{\sin^2\alpha}{\cos^2\alpha} = A^2$

$\cos^2\alpha = \frac{\sin^2\alpha}{A^2}$

$\cos^2\alpha = \frac{1-\cos^2\alpha}{A^2}$

$\cos^2\alpha = \frac{1}{A^2} - \frac{\cos^2\alpha}{A^2}$

$\cos^2\alpha + \frac{\cos^2\alpha}{A^2} = \frac{1}{A^2}$

$\cos^2\alpha\left(1 + \frac{1}{A^2}\right) = \frac{1}{A^2}$

$\cos^2\alpha\left(\frac{A^2+1}{A^2}\right) = \frac{1}{A^2}$

$\cos^2\alpha = \frac{1}{\cancel{A^2}} \cdot \frac{\cancel{A^2}}{1+A^2}$

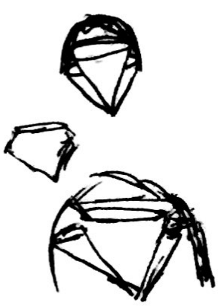
$\boxed{\cos^2\alpha = \frac{1}{1+A^2}}$

(c) (14 points) Compute the volume of the solid bounded by:

- the sphere $x^2 + y^2 + z^2 = 1$,
- the surface $z = \frac{1}{\sqrt{2}}\sqrt{x^2 + y^2}$,
- the surface $z = \frac{1}{2}\sqrt{x^2 + y^2}$.

Hint: the result from part (b) may help.

$\tan^2\alpha = A^2 \Rightarrow \cos^2\alpha = \frac{1}{1+A^2}$



$x^2+y^2+z^2=1$

$D\ \{0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

$\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \rho^2\sin\phi\, d\phi\, d\theta\, d\rho = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\left[-\rho^2\cos\phi\right]_{\phi=0}^{\pi} = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} -\rho^2(-1-1)\, d\theta\, d\rho$

$\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} 2\rho^2\, d\theta\, d\rho = \int_{\rho=0}^{1} 4\pi\rho^2\, d\rho = 4\pi\frac{\rho^3}{3}\Big|_{\rho=0}^{1} = \boxed{\frac{4}{3}\pi}$

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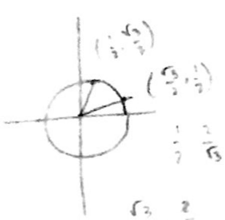
$z = \frac{1}{\sqrt{2}}\sqrt{x^2+y^2}$, $a = \sqrt{2}$



$\sqrt{2} = \tan\phi$

$D\ \{0 \le \phi \le \tan^{-1}(\sqrt{2})\}$

~~$\phi = \tan^{-1}(\sqrt{2})$~~

$\tan^2\phi = (\sqrt{2})^2$

$\cos^2\phi = \frac{1}{1+2}$

$\cos^2\phi = \frac{1}{3}$

$\cos\phi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\phi = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$\boxed{\rho = 1}$

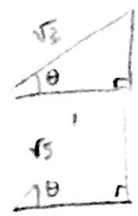
$z = \frac{1}{a}\sqrt{x^2+y^2} \rightarrow a = \tan\varphi$

$\boxed{\sqrt{2} = \tan\varphi} \rightarrow \tan^2\varphi = 2 \rightarrow \cos^2\varphi = \frac{1}{1+2} = \frac{1}{3}$

$\boxed{2 = \tan\varphi} \rightarrow \tan^2\varphi = 4 \rightarrow \cos^2\varphi = \frac{1}{5}$

$\varphi_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\varphi_2 = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$



$\varphi_2 > \varphi_1$

$\int_{\theta=0}^{2\pi}\int_{\varphi=\varphi_1}^{\varphi=\varphi_2}\int_{\rho=0}^{1} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_{\varphi=\varphi_1}^{\varphi=\varphi_2} \frac{\rho^3}{3}\sin\varphi\Big|_{\rho=0}^{1} d\varphi$

$= \frac{2\pi}{3}\int_{\varphi_1}^{\varphi_2} \sin\varphi\, d\varphi = -\frac{2\pi}{3}\cos\varphi\Big|_{\varphi_1}^{\varphi_2}$

$= -\frac{2\pi}{3}(\cos\varphi_2 - \cos\varphi_1)$

$= -\frac{2\pi}{3}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{3}}\right) = \boxed{\frac{2\pi}{3}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{3}}\right)}$

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5. (12 points) (Multiple choice questions). Circle the right answer for each following statement; no justification is required for your answers. For each answer you will receive: 4 points if your answer is correct; 0 points if it is wrong or if you provide no answer.

(1). Which of the following options is true about the three statements below? **(circle one)**

(a) Only statement (iii) is correct;
(b) Only statements (ii) and (iii) are correct;
(c) Only statements (i) and (iii) are correct;
(d) All statements are wrong;

Statements:

6 (i) An iterated integral over the box $R = \{(x, y, z)|x \in [0, a], y \in [0, b], z \in [0, c]\}$ can be express in eight different ways.

(ii) The integral of f over $D = \{(x, y, z)|0 \le x \le 1, 3x - 3 \le y \le 0, 0 \le z \le 5\}$ can be written as:

$$\iiint_D f(x, y, z)dV = \int_{y=3x-3}^{y=0} \int_{x=0}^{x=1} \int_{z=0}^{z=5} f(x, y, z)dzdxdy.$$

✓ (iii) $\int_{z=0}^{z=\pi} \int_{y=0}^{y=z} \int_{x=y}^{x=z} f(x, y, z)dxdydz = \int_{z=0}^{z=\pi} \int_{x=0}^{x=z} \int_{y=0}^{y=x} f(x, y, z)dydxdz.$

dxdydz

(2) Which of the following represents the integral of the function $f(x, y, z)$ over the inside of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first quadrant? **(circle one)**

(a) $\int_{z=0}^{z=1} \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-x^2-y^2}} f(x, y, z)dxdydz;$

(b) $\int_{z=0}^{z=1} \int_{y=0}^{y=\sqrt{1-z^2}} \int_{x=0}^{x=\sqrt{1-y^2-z^2}} f(x, y, z)dxdydz;$

(c) $\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{z=\sqrt{1-y^2}} f(x, y, z)dzdydx;$

(d) $\int_{x=0}^{x=1} \int_{z=0}^{z=\sqrt{1-x^2-y^2}} \int_{y=0}^{y=\sqrt{1-z^2}} f(x, y, z)dydzdx.$

(3) Which polar curve is represented in Figure 1? **(circle one)**

(a) $r = 1 + cos2\theta$, (b) $r = 1 - 2cos\theta$ (c) $r = 1 + cos\theta$ (d) $r = 1 + 2cos\theta$.

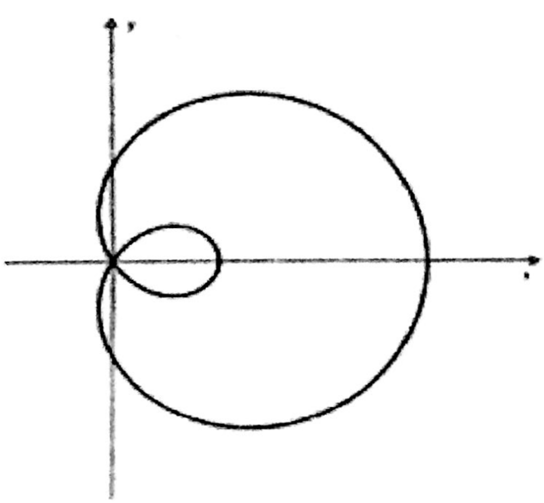

Figure 1.

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