Math 32B-2 Yeliussizov. Midterm 2

Exam time: 6:00-7:30 PM, February 27, 2017

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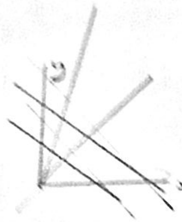
Discussion section: Tianqi 2A Tue, 2B Thu; Christian 2C Tue, 2D Thu; Kalyanswamy 2E Tue, 2F Thu

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (15)	P 2 (15)	P 3 (20)	P 4 (20)	P 5 (20)	Total (90 pt)
15	15	20	20	20	90

Problem 1. (15 points) Let D be the region enclosed by $y = x$, $y = 3x$, $y = 1 - x$, $y = 2 - x$.
(a) (5 points) Find a map $F(x, y)$ whose image $F(D)$ is a rectangle (i.e., maps D to a rectangle)
(b) (10 points) Evaluate $\iint_D \frac{y(x+y)}{x^3}\, dxdy$ using change of variables from F.

a.



$y \geq x,\ y \leq 3x,\ y \geq 1-x,\ y \leq 2-x$

$x \leq y \leq 3x \quad 1 \leq y+x \leq 2$

$1 \leq \frac{y}{x} \leq 3 \quad 1 \leq y+x \leq 2$, thus $u = \frac{y}{x}$, $v = x+y$

$[1,3] \quad [1,2]$

$y \leq x$

$x \leq y$

$y+x \leq 2$

$y+x \geq 1$

$v = y+x$ $(1 \leq v \leq 2)$

$x \leq y \leq 3x$

$1 \leq \frac{y}{x} \leq 3$

$u = \frac{y}{x}$

$F(x,y) = \left(\frac{y}{x}, y+x\right)\ [1,3] \times [1,2]$

$yx^{-1} \to -yx^{-2}$

b. $\mathrm{Jac}(F)(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{vmatrix} = -\frac{y}{x^2} - \frac{1}{x} = \frac{-y-x}{x^2} = -\frac{(y+x)}{x^2}$

$\mathrm{Jac}(G)(u,v) = \mathrm{Jac}(F)(x,y)^{-1} = -\frac{x^2}{y+x}$

$\frac{y(x+y)}{x^3} \left| -\frac{x^2}{y+x} \right| = \frac{y}{x} = u$ (from map)

$\int_1^3 \int_1^2 u\, dv\, du = \int_1^3 [uv]_1^2\, du = \int_1^3 u\, du = \left[\frac{u^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = \boxed{4}$ ✓

Problem 2. (15 points) Let a, b, c be real constants. Show that $\operatorname{div}(\mathbf{F} \times \langle a, b, c\rangle) = 0$ if $\mathbf{F}$ is a conservative vector field.

$\operatorname{div}(F \times \langle a,b,c\rangle) = 0$ Note: $\bar{F} = \langle F_1, F_2, F_3 \rangle$

$(F_1, F_2, F_3) \times$ ← cross product

$(a, b, c) = (cF_2 - F_3 b, F_3 a - F_1 c, F_1 b - F_2 a)$

$\operatorname{div}(\ ) = (cF_2 - F_3 b)_x + (F_3 a - F_1 c)_y + (F_1 b - F_2 a)_z$

eq ② $= cF_{2x} - bF_{3x} + aF_{3y} - cF_{1y} + bF_{1z} - aF_{2z}$

$(\partial x, \partial y, \partial z) \times$

eq ① $(F_1, F_2, F_3) = (F_{3y} - F_{2z}, -F_{3x} + F_{1z}, F_{2x} - F_{1y}) = 0$, for it's conservative.

The curl$(\bar{F}) = 0$ for it's conservative. Then $F_{3y} = F_{2z}$, $F_{3x} = F_{1z}$, $F_{2x} = F_{1y}$, according to eq ① (the components have to be equal to 0). So, in eq ②, we can see how cF_{2x} and $-cF_{1y}$ would cancel;

$-bF_{3x}$ and bF_{1y} would cancel; and aF_{3y} and $-aF_{2z}$ would cancel. Because that is the entirety of eq ②, it must mean that eq ② is equal to 0 if $\bar{F}$ is conservative.

Problem 3. (20 points) Consider the path C parametrized by $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ for $0 \le t \le 1$.
(a) (10 points) Evaluate the length of C.

(b) (10 points) Evaluate the vector line integral $\int_C \mathbf{F}\, d\mathbf{r}$, where $\mathbf{F} = \langle -y, x, z \rangle$.

a. $\int_C 1 ds = \int_0^1 \|r'(t)\| dt =$ $\quad r'(t) = \langle -2\sin 2t, 2\cos 2t, 1 \rangle$, $\|r'(t)\|^2 = 4\sin^2 2t + 4\cos^2 2t + 1$

$\|r'(t)\| = \sqrt{4+1} = \sqrt{5}$

$\int_0^1 \sqrt{5} dt = [\sqrt{5}t]_0^1 = \sqrt{5}$

b. $\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t) dt = \int_0^1 (-\sin 2t, \cos 2t, t) \cdot (-2\sin 2t, 2\cos 2t, 1) dt = \int_0^1 2\sin^2 2t + 2\cos^2 2t + t \, dt$

$= \int_0^1 2 + t dt = [2t + \frac{1}{2}t^2]_0^1 = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$

Problem 4. (20 points) Let C be a path from $(2,0)$ to $(0,1)$ along the ellipse $x^2+4y^2=4$ in the first quadrant, oriented counterclockwise.

(a) (10 points) Let $\mathbf{F} = \langle y - y\sin x, x+\cos x\rangle$. Show that $\mathbf{F}$ is conservative, find a potential function $f(x,y)$ so that $\mathbf{F} = \nabla f$, and evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

(b) (10 points) Let $\mathbf{F} = \langle -y, x\rangle$. Is it conservative? Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

a. $F_{1y} = F_{2x}$

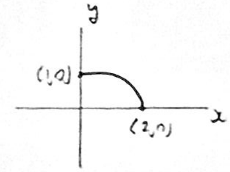
$1 - \sin x = 1 - \sin x$

It's conservative

$\boxed{f = y\cos x + xy}$

$\nabla f = (f_x, f_y) = (y - y\sin x, x + \cos x)$

Thus $F = \nabla f$.



$$\int_C F\,dr = f(P) - f(Q) = f(0,1) - f(2,0) = 1 - 0 = 1$$

(fundamental theorem of consr. vec field).

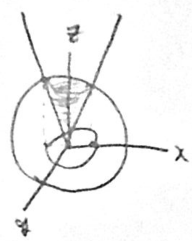
b. $F_{1y} = F_{2x}$

$-1 = 1$, it is <u>not</u> conservtive.

$$x^2+4y^2=4 \to r(t) = (2\cos t, \sin t) \quad r'(t) = (-2\sin t, \cos t) \quad 0 \le t \le \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} F(r(t))\cdot r'(t)\,dt = \int_0^{\frac{\pi}{2}} (-\sin t, 2\cos t)\cdot(-2\sin t, \cos t)\,dt = \int_0^{\frac{\pi}{2}} 2\sin^2 t + 2\cos^2 t\,dt = \int_0^{\frac{\pi}{2}} 2\,dt = \left[2t\right]_0^{\frac{\pi}{2}} = \pi$$

Problem 5. (20 points) Compute the area of the surface enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$ from above.



into sphere

$z^2 = x^2+y^2 \rightarrow 2z^2 = 1,\ z^2 = \frac{1}{2}\ \ z = \frac{\sqrt{2}}{2}$

around the circle $\frac{1}{2} = x^2+y^2$ (this is the circle).

$G(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$

$T_\theta = (-\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0)\ \times$

$T_\phi = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi) = (-\cos\theta\sin^2\phi,\ -\sin\theta\sin^2\phi,\ -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\cos\phi\sin\phi)$

$= -(\cos\theta\sin^2\phi,\ \sin\theta\sin^2\phi,\ \sin\phi\cos\phi)$

$\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi \leq \frac{1}{2}$

$\|T_\theta \times T_\phi\| = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$

$\sin^2\phi \leq \frac{1}{2}$

Around, so $0 \leq \theta \leq 2\pi$

$= \sin\phi = \|N\|$

$\sin\phi \leq \frac{\sqrt{2}}{2}$

$0 \leq \phi \leq \frac{\pi}{4}$

Sphere's Area $= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \sin\phi \, d\theta d\phi = \int_0^{\frac{\pi}{4}} \left[\theta\sin\phi\right]_0^{2\pi} d\phi = \int_0^{\frac{\pi}{4}} 2\pi\sin\phi\, d\phi = \left[-2\pi\cos\phi\right]_0^{\frac{\pi}{4}} = -2\pi\left(\frac{\sqrt{2}}{2}\right) + 2\pi$

$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)$ (sphere part)

Cone's Area $G(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r)$ $0 \leq \theta \leq 2\pi$ $0 \leq r \leq \sqrt{\frac{1}{2}}$, radius of circle

$\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}} r\sqrt{2}\, dr d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\sqrt{2}\right]_0^{\frac{\sqrt{2}}{2}} d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{4}\, d\theta = \frac{\pi\sqrt{2}}{2}$

$T_r = (\cos\theta,\ \sin\theta,\ 1)\ \times$

$T_\theta = (-r\sin\theta,\ r\cos\theta,\ 0) = (-r\cos\theta,\ -r\sin\theta,\ r\cos^2\theta + r\sin^2\theta)$

$= (-r\cos\theta,\ -r\sin\theta,\ r)$

$\|T_r \times T_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = r\sqrt{2} = \|N\|$

Total Surface Area: $\boxed{2\pi\left(1-\frac{\sqrt{2}}{2}\right) + \frac{\pi\sqrt{2}}{2}} = 2\pi - \frac{2\pi\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{2} = 2\pi - \frac{\pi\sqrt{2}}{2}$