Problem 1. (15 points) Evaluate the double integral over the given rectangular domain in the xy-plane

$$\iint_R (1+y+xe^{xy})\,dA, \qquad R=[0,2]\times[-1,1].$$

$1+\cancel{y}+e^{x}+1\ \cancel{y}-e^{-x}$

$2-e^{-x}+e^{x}$

$$\int_0^2\int_{-1}^1 1+y+xe^{xy}\,dy\,dx$$

$$=\int_0^2 y+\frac{1}{2}y^2+e^{xy}\Big|_{-1}^{1}\,dx$$

$$=\int_0^2 1+\frac{1}{2}+e^{x}-(-1+\frac{1}{2}+e^{-x})\,dx$$

$$=\int_0^2 2-e^{-x}+e^{x}\,dx$$

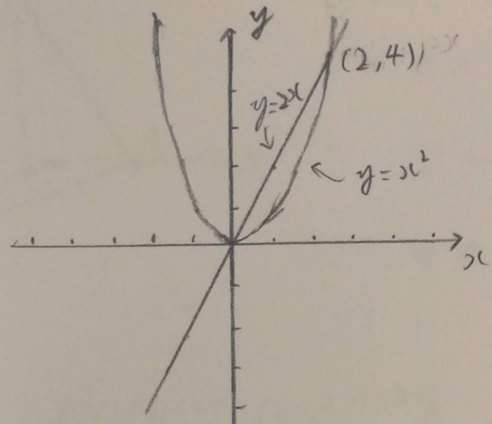
$$=2x+e^{-x}+e^{x}\Big|_0^2$$

$$=4+e^{-2}+e^{2}-0-1-1$$

$$=2+e^{2}+e^{-2}$$

Problem 2. (15 points) Let D be the region bounded by $y = 2x$ and $y = x^2$.
(a) (5 points) Sketch the region D in the xy-plane.
(b) (10 points) Compute the double integral of $f(x,y) = \dfrac{x}{4-y}$ over the domain D. (Choose the order of integration that enables you to evaluate the integral.)

(a)



(b)

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} \frac{x}{4-y}\, dx\, dy$$

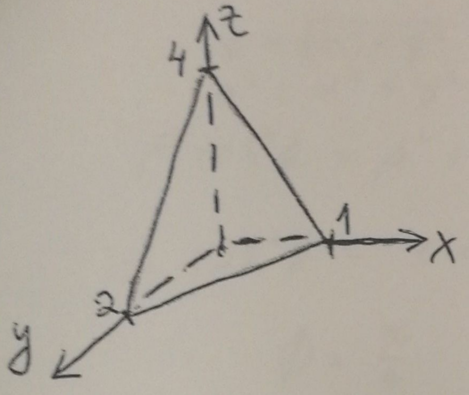
$$= \int_0^4 \frac{1}{4-y} \frac{1}{2} x^2 \Big|_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{y}{8-2y} - \frac{y^2}{4(8-2y)}\, dy$$

$$= \int_0^4 \frac{1}{8} y\, dy$$

$$= \frac{1}{16} y^2 \Big|_0^4 = 1$$

Problem 3. (20 points) Let W be the tetrahedron in the first octant $x, y, z \geq 0$ with vertices at the points $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$ (see the figure). Evaluate the triple integral of the function $f(x,y,z) = 1/(1-x)$ over W.



equation of plan

$4 = -a$

$z = c + ax + by$

$c = 4 \quad b = -2 \quad a = -4$

$z = 4 - 4x - 2y$

$$\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} \frac{1}{1-x} \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{2-2x} \frac{4-4x-2y}{1-x} \, dy \, dx$$

$$= 4\int_0^1 1-x \, dx$$

$$= 4\left(x - \frac{1}{2}x^2 \Big|_0^1\right)$$

$$= 4\left(1 - \frac{1}{2}\right) = 2$$

Problem 4. (20 points) Let W be the region bounded by the sphere $x^2+y^2+z^2=9$ and (above) the cone $z=\sqrt{x^2+y^2}$. Find the volume of W using spherical coordinates.

$$z=\sqrt{x^2+y^2}$$

$$z^2=x^2+y^2$$

$$\rho^2\cos^2\phi=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta$$

$$\cos^2\phi=\sin^2\phi$$

$$\phi=\frac{\pi}{4}$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3\sin\phi\Big|_0^3 \, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}} 9\sin\phi\, d\phi\, d\theta$$

$$=\int_0^{2\pi} -9\cos\phi\Big|_0^{\frac{\pi}{4}} d\theta$$

$$=\int_0^{2\pi} \frac{18-9\sqrt{2}}{2}\, d\theta$$

$$=18\pi-9\sqrt{2}\pi$$

Problem 5. (20 points) Let W be the region bounded by the cylinder $x^2+y^2=1$ and two half-planes $x=|z|$.
(a) (10 points) Find the volume of W.
(b) (10 points) Find the centroid of W (i.e. the center of mass assuming the mass density $\delta(x,y,z)=1$) using cylindrical coordinates.

(a) $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{-x}^{x} 1\, dz\, dx\, dy$

$= \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} 2x\, dx\, dy$

$= \int_{-1}^{1} x^2 \Big|_{0}^{\sqrt{1-y^2}} dy$

$= \int_{-1}^{1} 1-y^2\, dy$

$= y-\frac{1}{3}y^3 \Big|_{-1}^{1}$

$= 1-\frac{1}{3}-(-1+\frac{1}{3})$

$= 1-\frac{1}{3}+1-\frac{1}{3}$

$= 2-\frac{2}{3} = \frac{4}{3}$

(b) since $\delta(x,y,z)=1$ and the region is symmetric with respect to x-z plane and x-y plane we only need to find the point on x-axis

$M_{yz} = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{-x}^{x} x\, dz\, dx\, dy$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{-r\cos\theta}^{r\cos\theta} r^2\cos\theta\, dz\, dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} 2r^3\cos^2\theta\, dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta \frac{1}{4}r^4 \Big|_{0}^{1} d\theta$

$r^2\cos^2\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{2}\cdot(\cos 2\theta - 1)\, d\theta$

$= -\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta - 1\, d\theta$

$= -\frac{1}{4}\cdot\left(\frac{1}{2}\sin 2\theta - \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$

$= -\frac{1}{4}\left(\frac{1}{2}\sin\pi - \frac{\pi}{2} - (\frac{1}{2}\sin(-\pi) + \frac{\pi}{2})\right)$

$= -\frac{1}{4}\cdot -\pi = \frac{\pi}{4}$

$\frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16}$

at $(\frac{3\pi}{16}, 0, 0)$