1. (10 points) Consider the curve C parametrized by

$$\mathbf{r}(t) = (t-2)\mathbf{i} + (t^2-5)\mathbf{j} + \pi\sqrt{t-1}\mathbf{k}, \qquad 2 \le t \le 5.$$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = \left\langle y^2 + \frac{2xz}{1+x^2}, z\sin(yz) + 2xy, \ln(1+x^2) + y\sin(yz) \right\rangle.$$

Is $\vec{F}$ conservative? We want an f such that

$$\frac{\partial f}{\partial x} = y^2 + \frac{2xz}{1+x^2} \implies f = \int \left(y^2 + \frac{2xz}{1+x^2}\right)dx = xy^2 + z\ln(1+x^2) + C(y,z)$$

$$\frac{\partial f}{\partial y} = z\sin(yz) + 2xy \implies f = \int \left(z\sin(yz) + 2xy\right)dy = -\cos(yz) + xy^2 + C(x,z)$$

$$\frac{\partial f}{\partial z} = \ln(1+x^2) + y\sin(yz) \implies f = \int \left(\ln(1+x^2) + y\sin(yz)\right)dz = z\ln(1+x^2) - \cos(yz) + C(x,y)$$

It looks like $\underline{f(x,y,z) = xy^2 + z\ln(1+x^2) - \cos(yz)}$ works!

End point (at $t=5$): $\vec{r}(5) = (3, 20, 2\pi)$

Start point (at $t=2$): $\vec{r}(2) = (0, -1, \pi)$

$$\int_C \vec{F} \cdot d\vec{r} = f(3, 20, 2\pi) - f(0, -1, \pi)$$

$$= \left(3 \cdot 20^2 + 2\pi\ln(1+3^2) - \cos(20 \cdot 2\pi)\right) - \left(\cancel{0(1)^2} + \cancel{\pi\ln(1)} - \cos(-\pi)\right)$$

$$= 1200 + 2\pi\ln(10) - 1 + (-1)$$

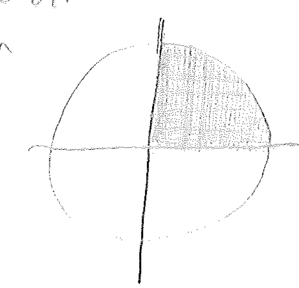
$$= \boxed{1198 + 2\pi\ln(10)}$$

2. (10 points) Let $\mathcal{S}$ be the portion of the cylinder $x^2+z^2=1$ that is *inside* the cylinder $x^2+y^2=1$ and where $x\geq 0$, $y\geq 0$, $z\geq 0$. Compute

$$\iint_{\mathcal{S}} yz\, dS.$$

(Hint: Start with polar/cylindrical coordinates.)

Inside $x^2+y^2=1$, and with $x\geq 0$, $y\geq 0$ means we're talking about the part of the surface lying over



The surface is $x^2+z^2=1$, but where $z\geq 0$ we can say $z=\sqrt{1-x^2}$. So we parametrize this using:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{1-r^2\cos^2\theta} \end{cases} \qquad \begin{array}{l} 0\leq r\leq 1 \\ 0\leq\theta\leq\frac{\pi}{2} \end{array}$$

$$\vec{T}_r\times\vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \frac{-r\cos^2\theta}{\sqrt{1-r^2\cos^2\theta}} \\ -r\sin\theta & r\cos\theta & \frac{r^2\sin\theta\cos\theta}{\sqrt{1-r^2\cos^2\theta}} \end{vmatrix} = \left\langle \frac{r^2\sin^2\theta\cos\theta + r^2\cos^3\theta}{\sqrt{1-r^2\cos^2\theta}},\ \cancel{\frac{r^2\sin\theta\cos^2\theta - r^2\sin\theta\cos^2\theta}{\sqrt{1-r^2\cos^2\theta}}},\ r\cos^2\theta + r\sin^2\theta \right\rangle$$

$$= \left\langle \frac{r\cos\theta}{\sqrt{1-r^2\cos^2\theta}},\ 0,\ r\right\rangle = r\left\langle \frac{r\cos\theta}{\sqrt{1-r^2\cos^2\theta}},\ 0,\ 1\right\rangle \quad \left(= r\left\langle \frac{x}{z},0,1\right\rangle\right)$$

$$\|\vec{T}_r\times\vec{T}_\theta\| = r\sqrt{\frac{r^2\cos^2\theta}{1-r^2\cos^2\theta}+1} = r\sqrt{\frac{\cancel{r^2\cos^2\theta} + 1 - \cancel{r^2\cos^2\theta}}{1-r^2\cos^2\theta}} = \frac{r}{\sqrt{1-r^2\cos^2\theta}} \quad \left(= \frac{r}{z}\right.$$

$$\text{So} \iint_S yz\, dS = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r\sin\theta\cdot\cancel{\sqrt{1-r^2\cos^2\theta}}\cdot\frac{r}{\cancel{\sqrt{1-r^2\cos^2\theta}}}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\sin\theta\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\sin\theta\, d\theta\cdot\int_{r=0}^{1} r^2\, dr$$

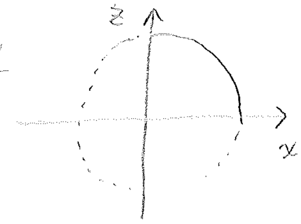
$$= \Big[-\cos\theta\Big]_{\theta=0}^{\frac{\pi}{2}}\cdot\left[\frac{1}{3}r^3\right]_{r=0}^{1} = (0-(-1))\cdot\left(\frac{1}{3}-0\right) = \boxed{\frac{1}{3}}$$

Another (more difficult) way to do #2 (which apparently many people attempted):

Parametrize the cylinder $x^2+z^2=1$ as $\begin{cases} x=\cos\theta \\ y=y \\ z=\sin\theta \end{cases}$ (since it is a cylinder around the y-axis)

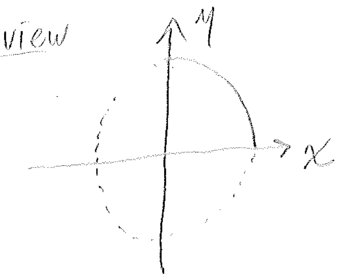
Since we want $x\geq 0, z\geq 0$, θ should go from 0 to $\frac{\pi}{2}$:

Side view



But what are the bounds on y? This is trickier! Since we want $y\geq 0$, y should go from 0 to the intersection of the two cylinders:

Top view



$$\begin{array}{l} x^2+z^2=1 \\ x^2+y^2=1 \end{array} \longrightarrow \begin{array}{l} z^2-y^2=0 \\ y^2=z^2 \\ y=\pm z=\pm\sin\theta \end{array}$$

Since $y\geq 0$, we want $+$ here

So we want $0\leq y\leq \sin\theta$

$$\vec{T}_\theta\times\vec{T}_y=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix}=\langle -\cos\theta, 0, -\sin\theta\rangle, \text{ so } \|\vec{T}_\theta\times\vec{T}_y\|=1$$

(So $dS=1\,dy\,d\theta$)

$$\text{So } \iint_S yz\,dS=\int_{\theta=0}^{\frac{\pi}{2}}\int_{y=0}^{\sin\theta} y\sin\theta\cdot 1\,dy\,d\theta$$

$$=\int_{\theta=0}^{\frac{\pi}{2}}\sin\theta\left[\frac{1}{2}y^2\right]_{y=0}^{\sin\theta}=\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}}\sin^3\theta\,d\theta=\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}}(1-\cos^2\theta)\cdot\sin\theta\,d\theta$$

$u=\cos\theta$
$du=-\sin\theta\,d\theta$

$$=-\frac{1}{2}\left[u-\frac{1}{3}u^3\right]_{u=1}^{u=0}=-\frac{1}{2}\left[0-\frac{2}{3}\right]=\boxed{\frac{1}{3}}$$

3. (10 points) Suppose a magnetic field will impart on some object a force

$$\mathbf{F}(x, y, z) = \left\langle -ye^z, xe^z, e^{x^2} - \cos(z^2) \right\rangle$$

when the object is at the point (x, y, z) in space. Compute the work done by this force in moving the object one full rotation counterclockwise (as viewed from above) around the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

x-radius is 2
y-radius is 4

in the xy-plane.

$$\begin{cases} x = 2\cos\theta \\ y = 4\sin\theta \\ z = 0 \end{cases} \quad 0 \le \theta \le 2\pi \quad \left(\text{So } \vec{r}(t) = \langle 2\cos\theta, 4\sin\theta, 0 \rangle\right)$$

$\vec{r}\,'(t) = \langle -2\sin\theta, 4\cos\theta, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle -4\sin\theta \cdot \cancel{e^0}, 2\cos\theta \cdot \cancel{e^0}, e^{4\cos^2\theta} - \cos(0) \rangle$

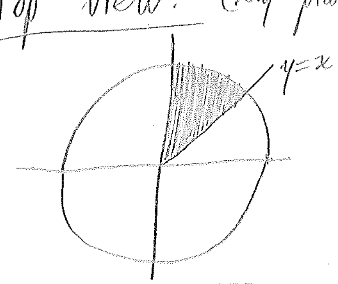
So work $= \int_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} \langle -4\sin\theta, 2\cos\theta, \text{———} \rangle \cdot \langle -2\sin\theta, 4\cos\theta, 0 \rangle \, d\theta$

$= \int_{\theta=0}^{2\pi} (8\sin^2\theta + 8\cos^2\theta + 0) \, d\theta = \int_{\theta=0}^{2\pi} 8 \, d\theta$

$= \boxed{16\pi}$

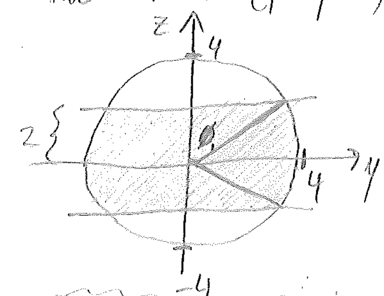
4. (10 points) Let $\mathcal{S}$ be the portion of the sphere $x^2 + y^2 + z^2 = 16$ where $-2 \le z \le 2$ and $0 \le x \le y$. Compute the flux of the vector field $\langle yz, xz, xy \rangle$ through the surface $\mathcal{S}$, oriented with *inward* pointing normal vectors.

Top view: (xy-plane)



($x \ge 0$: right side
$y \ge x$: above $y=x$)

Side view (yz-plane)



$\cos\phi_1 = \frac{2}{4} \rightarrow \phi_1 = \frac{\pi}{3}$

$\cos\phi_2 = \frac{-2}{4} \rightarrow \phi_2 = \frac{2\pi}{3}$

Using spherical coordinates, we would have:

$\rho = 4$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$

The vector $\langle x, y, z \rangle$ is an (outward-pointing) normal vector, so $\langle -x, -y, -z \rangle$ is inward-pointing, and on this sphere the magnitude of this is 4 (the radius!).

So $\vec{e}_N = \left\langle \frac{-x}{4}, \frac{-y}{4}, \frac{-z}{4} \right\rangle$ = inward-pointing unit normal vector.

Thus $\vec{F} \cdot \vec{e}_N = \langle yz, xz, xy \rangle \cdot \frac{\langle -x, -y, -z \rangle}{4} = \frac{-3xyz}{4}$

At this point, since this function has odd symmetry in the z variable (in brief, it's positive above the xy-plane and negative below the xy-plane) and the surface is also symmetric across the xy-plane, the total flux is

$\boxed{0.}$

To see this using integration, we now parametrize the sphere:

$$\begin{cases} x = 4\sin\phi\cos\theta \\ y = 4\sin\phi\sin\theta \\ z = 4\cos\phi \end{cases} \qquad \begin{array}{l} \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \\ \frac{\pi}{3} \le \phi \le \frac{2\pi}{3} \end{array}$$

cont'd...

Compute $dS = \|T_\theta \times T_\phi\| d\theta d\phi = \ldots$ from previous work, we know this will be $\rho^2 \sin\phi = 16\sin\phi$.

So now the flux is

$$\iint_S (\vec{F} \cdot \vec{e_N})\, dS = \iint_S -\frac{3}{4} xyz\, dS$$

$$= -\frac{3}{\cancel{4}} \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\phi=\frac{\pi}{3}}^{\frac{2\pi}{3}} (\cancel{4}\sin\phi\cos\theta)(4\sin\phi\sin\theta)(4\cos\phi) \cdot 16\sin\phi\, d\phi\, d\theta$$

$$= -3 \cdot 16 \cdot 16 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \cdot \int_{\phi=\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^3\phi\cos\phi\, d\phi$$

$$\begin{pmatrix} v = \sin\theta \\ dv = \cos\theta\, d\theta \end{pmatrix} \qquad \begin{pmatrix} u = \sin\phi \\ du = \cos\phi\, d\phi \end{pmatrix}$$

$$= -768 \left[\frac{1}{2}\sin^2\theta \right]_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \left[\frac{1}{4}\sin^4\phi \right]_{\phi=\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= -96\left(1^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right) \cdot \cancel{\left(\left(\frac{\sqrt{3}}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right)}^{0} = \boxed{0}$$