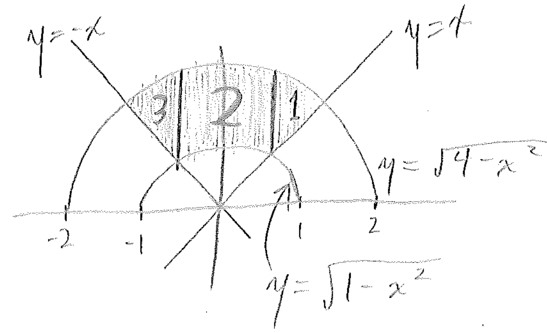
1. (10 points) Compute the following, in any way you like. *(Hint: Draw a picture!)*

$$\underbrace{\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}} y^2\,dy\,dx}_{1} + \underbrace{\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx}_{2} + \overbrace{\int_{-\sqrt{2}}^{-\frac{\sqrt{2}}{2}}\int_{-x}^{\sqrt{4-x^2}} y^2\,dy\,dx}^{3}$$



A much easier way to integrate over this region: polar coords!

$$\int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{r=1}^{2} (r\sin\theta)^2 \cdot r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{r=1}^{2} r^3\sin^2\theta\,dr\,d\theta = \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2\theta\left(\int_{r=1}^{2} r^3\,dr\right)d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2\theta\,d\theta \cdot \int_{r=1}^{2} r^3\,dr$$

$$= \int_{\theta=\frac{\pi}{4}}^{3\pi/4} \frac{1-\cos 2\theta}{2}\,d\theta \cdot \left[\frac{1}{4}r^4\right]_{r=1}^{2}$$

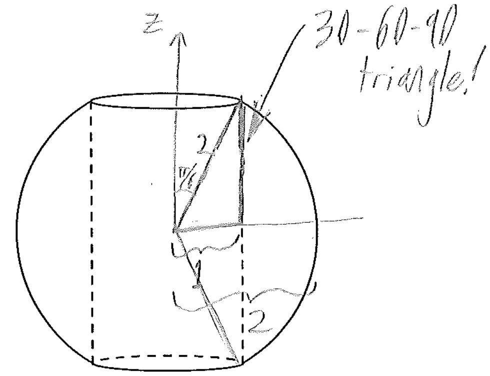
$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\theta=\frac{\pi}{4}}^{3\pi/4} \cdot \left[\frac{1}{4}r^4\right]_{r=1}^{2} = \left[\left(\frac{3\pi}{8} - \frac{1}{4}(-1)\right) - \left(\frac{\pi}{8} - \frac{1}{4}(1)\right)\right]\left[\frac{16}{4} - \frac{1}{4}\right]$$

$$= \left(\frac{\pi}{4} + \frac{2}{4}\right)\cdot\left(\frac{15}{4}\right) = \boxed{\frac{15}{16}(\pi+2)} \quad \text{or} \quad \boxed{\frac{15\pi}{16} + \frac{15}{8}}$$

2. (10 points) A space station is designed as a sphere of radius 2 km with a cylinder of radius 1 km removed from the center of it. The heat density in the air in the space station is given by

$$\frac{1}{x^2+y^2+z^2},$$

where x, y, and z are measured in km, with $(0,0,0)$ located at the center of the sphere. Compute the total amount of heat in the air inside the space station.



We need to compute $\iiint_W \frac{1}{x^2+y^2+z^2}\,dV$

We'll use spherical coordinates...

Sphere is $\rho = 2$

Cylinder: $x^2+y^2=1 \Rightarrow (\rho\sin\phi)^2 = 1$

$\rho\sin\phi = 1$

$\rho = \frac{1}{\sin\phi} = \csc\phi$

Integrand is $\frac{1}{\rho^2}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\rho=\csc\phi}^{2} \frac{1}{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{6}}^{5\pi/6} \sin\phi\cdot(2-\csc\phi)\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{6}}^{5\pi/6}(2\sin\phi-1)\,d\phi\,d\theta = \int_{\theta=0}^{2\pi}\Big[-2\cos\phi-\phi\Big]_{\phi=\frac{\pi}{6}}^{5\pi/6}\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[(-2)\left(-\tfrac{\sqrt{3}}{2}\right)-\tfrac{5\pi}{6}-(2)\left(\tfrac{\sqrt{3}}{2}\right)+\tfrac{\pi}{6}\right]d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(2\sqrt{3}-\tfrac{2\pi}{3}\right)d\theta = \left(2\sqrt{3}-\tfrac{2\pi}{3}\right)\cdot\int_0^{2\pi}d\theta = \boxed{2\pi\left(2\sqrt{3}-\tfrac{2\pi}{3}\right)}$$

3. (10 points) Let $\mathcal{W}$ be the region in the first octant above $z = x^2 + y^2$ and below $z = 2\sqrt{x^2 + y^2}$. Compute

$$\iiint_{\mathcal{W}} x \, dV.$$

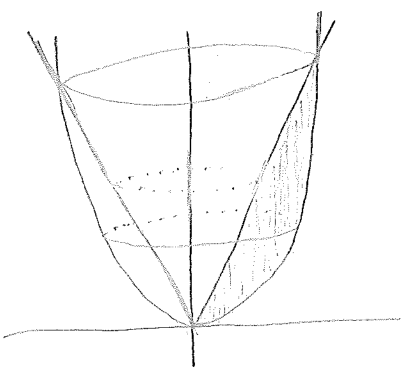
$z = 2r$ (Cone)

$z = r^2$ (Paraboloid)

Paraboloid and cone intersect at: $r^2 = 2r$

$r^2 - 2r = 0$

$r(r-2) = 0 \Longrightarrow r = 0$ (origin) and $r = 2$ (circle of radius 2)



$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} \int_{z=r^2}^{2r} (r\cos\theta) \cdot r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} \int_{z=r^2}^{2r} r^2 \cos\theta \, dz \, dr \, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta \cdot \int_{r=0}^{2} r^2 \cdot \int_{z=r^2}^{2r} dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta \, d\theta \cdot \int_{r=0}^{2} r^2 \cdot \left[2r - r^2 \right] dr$$

$$= \left(\sin\theta \Big|_{\theta=0}^{\frac{\pi}{2}} \right) \cdot \int_{r=0}^{2} (2r^3 - r^4) \, dr = (1 - 0) \cdot \left[\frac{2}{4} r^4 - \frac{1}{5} r^5 \right]_{r=0}^{2}$$

$$= \left(\frac{1}{2} \cdot 16 - \frac{1}{5} \cdot 32 \right) - (0 - 0) = 8 - \frac{32}{5} = \boxed{\frac{8}{5}}$$

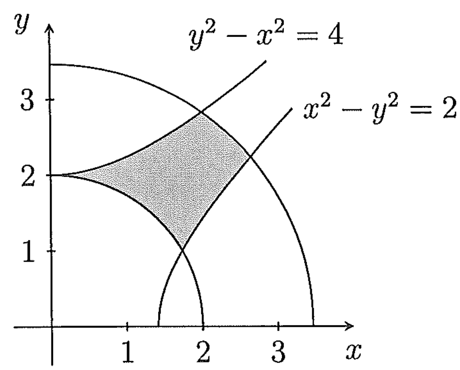
This can also be done in spherical, but it's ... weirder, or, in cartesian, but it's harder and nastier looking. See the last page, if you're interested.

4. (10 points) Let $\mathcal{D}$ be the region in the first quadrant bounded by the circles $x^2+y^2=4$ and $x^2+y^2=12$, and the hyperbolae $x^2-y^2=2$ and $y^2-x^2=4$. Use the change of variables

$$\begin{cases} x=\sqrt{u+v} \\ y=\sqrt{u-v} \end{cases} \qquad \begin{matrix} \Longrightarrow x^2 = u+v \\ \Longrightarrow y^2 = u-v \end{matrix}$$

to compute the integral

$$\iint_{\mathcal{D}} 2xy^3\, dA.$$



$$x^2+y^2=4 \Longrightarrow (u+v)+(u-v)=4$$
$$2u=4$$
$$u=2$$

$$x^2+y^2=12 \Longrightarrow (u+v)+(u-v)=12$$
$$2u=12$$
$$u=6$$

$$x^2-y^2=2 \Longrightarrow (u+v)-(u-v)=2$$
$$2v=2$$
$$v=1$$

$$y^2-x^2=4 \Longrightarrow (u-v)-(u+v)=4 \qquad v=-2$$
$$-2v=4$$

Jacobian determinant: $\left|\det\begin{bmatrix} \frac{1}{2}(u+v)^{-1/2} & \frac{1}{2}(u-v)^{-1/2} \\ \frac{1}{2}(u+v)^{-1/2} & (-1)\cdot\frac{1}{2}(u-v)^{-1/2} \end{bmatrix}\right| = \left|-\frac{1}{4}(u+v)^{-1/2}(u-v)^{-1/2} - \frac{1}{4}(u+v)^{-1/2}(u-v)^{-1/2}\right|$

$$= \frac{1}{2}\frac{1}{\sqrt{u+v}}\cdot\frac{1}{\sqrt{u-v}}$$

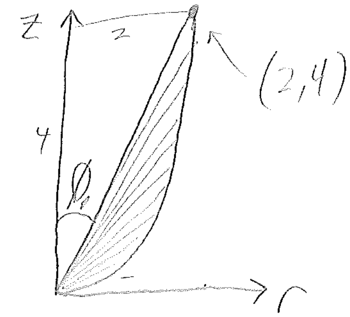
Finally: $\iint_{D} 2xy^3\, dA = \int_{v=-2}^{1}\int_{u=2}^{6} 2\cancel{\sqrt{u+v}}\,(\sqrt{u-v})^{\cancel{3}\,2}\cdot\frac{1}{2}\frac{1}{\cancel{\sqrt{u+v}}}\cdot\frac{1}{\cancel{\sqrt{u-v}}}\,du\,dv$

$$= \int_{v=-2}^{1}\int_{u=2}^{6}(u-v)\,du\,dv \longleftarrow \text{So easy now! ☺}$$

$$= \int_{v=-2}^{1}\left[\frac{1}{2}u^2-uv\right]_{u=2}^{6} dv = \int_{v=-2}^{1}(16-4v)\,dv = 16v-2v^2\Big|_{v=-2}^{1}$$

$$= 14-(-40) = \boxed{54}$$

Problem #3, in spherical coords:



$z = 2r$ intersects $z = r^2$ at $r=2, z=4$:

So the angle ϕ_0 shown here is given by

$\tan\phi_0 = \frac{2}{4} = \frac{1}{2}$ or $\cot\phi_0 = 2$

$\phi_0 = \arctan(\tfrac{1}{2})$ $\phi_0 = \operatorname{arccot}(2)$

So ϕ goes from $\arctan(\tfrac{1}{2})$ to $\frac{\pi}{2}$

ρ goes from 0 (origin) to (the paraboloid). The paraboloid is:

$$z = x^2 + y^2$$

$$\cancel{\rho}\cos\phi = (\rho\sin\phi)^2 = \rho^{\cancel{2}}\sin^2\phi$$

$$\cos\phi = \rho\sin^2\phi \Longrightarrow \rho = \frac{\cos\phi}{\sin^2\phi} = \frac{1}{\sin\phi}\cdot\frac{\cos\phi}{\sin\phi} = \csc\phi\cdot\cot\phi$$

So the integral is: $\csc = \frac{1}{\sin}$!

$$\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=\arctan(\frac{1}{2})}^{\frac{\pi}{2}}\int_{\rho=0}^{\csc\phi\cot\phi} \rho\sin\phi\cos\theta\cdot\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\cos\theta\cdot\int_{\phi=\arctan(\frac{1}{2})}^{\frac{\pi}{2}}\sin^2\phi\cdot\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\csc\phi\cdot\cot\phi} d\phi\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\cos\theta\int_{\phi=\arctan(\frac{1}{2})}^{\frac{\pi}{2}}\tfrac{1}{4}\cancel{\sin^2\phi}\cdot\csc^{\cancel{4}\,2}\phi\cdot\cot^4\phi\, d\phi\, d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\frac{\pi}{2}}\cos\theta\, d\theta\cdot\int_{\phi=\arctan(\frac{1}{2})}^{\frac{\pi}{2}}\csc^2\phi\cdot\cot^4\phi\, d\phi = \frac{1}{4}\left[\sin\theta\right]_{\theta=0}^{\frac{\pi}{2}}\cdot\int_{u=2}^{u=0} -u^4\, du$$

$u = \cot\phi$

$du = -\csc^2\phi\, d\phi$

$$= \frac{1}{4}(1-0)\cdot\left[\frac{-1}{5}u^5\right]_{u=2}^{0} = \frac{-1}{4}\cdot\frac{1}{5}(0-32) = \frac{+32}{4\cdot 5} = \boxed{\frac{8}{5}}$$

Or, in cartesian: $$\int_{y=0}^{2}\int_{x=0}^{\sqrt{4-y^2}}\int_{z=x^2+y^2}^{2\sqrt{x^2+y^2}} x\, dz\, dx\, dy$$

(But why on earth would you want to do it this way?)

$$= \int_{y=0}^{2}\int_{x=0}^{\sqrt{4-y^2}}\Big(\underbrace{2x\sqrt{x^2+y^2}}_{\text{simple u-sub}} - \underbrace{x(x^2+y^2)}_{\text{easy}}\Big)\, dx\, dy$$

$= \ldots$

If you do this as $dy\,dx$ instead of $dx\,dy$, it's much harder!