Sorry the numbers are a bit nasty...
That was supposed to be a 4. Oops.

1. (10 points) Let $\mathcal{S}$ be the part of the cone $x = 2\sqrt{y^2+z^2}$ where $z \geq 0$ and $0 \leq x \leq 3$. Electric charge has accumulated on this surface so that its charge density (per unit area) at each point is

$$\delta(x, y, z) = xz.$$

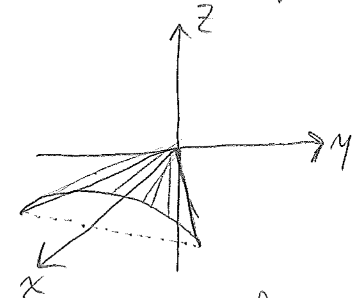
What is the total amount of charge on the surface $\mathcal{S}$?

We want to compute $\iint_S xz\, dS$ (scalar surface integral)

Parametrize S: $x = 3 = 2\sqrt{y^2+z^2}$

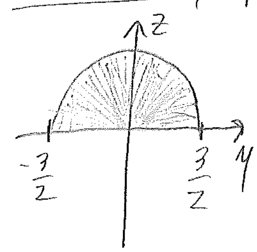
$\frac{3}{2} = \sqrt{y^2+z^2}$

$y^2 + z^2 = \frac{9}{4}$ (Circle in yz-plane)



Let's use polar coordinates in y and z:

View of yz-plane:



$$\begin{cases} x = 2r \\ y = r\cos\theta \\ z = r\sin\theta \end{cases} \qquad 0 \leq r \leq \frac{3}{2}, \quad 0 \leq \theta \leq \pi$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} = \langle r, -2r\cos\theta, -2r\sin\theta \rangle$$

$$= r\langle 1, -2\cos\theta, -2\sin\theta \rangle$$

$$\|\vec{T}_r \times \vec{T}_\theta\| = r\sqrt{1 + 4\cos^2\theta + 4\sin^2\theta} = r\sqrt{5} \quad \text{so} \quad dS = r\sqrt{5}\, dr\, d\theta$$

So $$\iint_S xz\, dS = \int_{\theta=0}^{\pi} \int_{r=0}^{3/2} (2r)(r\sin\theta) \cdot r\sqrt{5}\, dr\, d\theta$$

$$= 2\sqrt{5} \int_{\theta=0}^{\pi} \sin\theta\, d\theta \cdot \int_{r=0}^{3/2} r^3\, dr = 2\sqrt{5}\Big[-\cos\theta\Big]_{\theta=0}^{\pi} \cdot \Big[\frac{1}{4}r^4\Big]_{r=0}^{3/2}$$

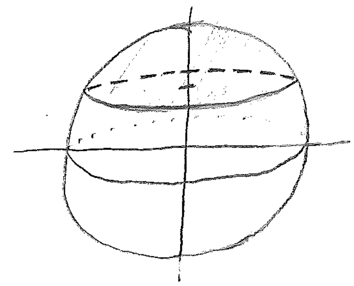
$$= 2\sqrt{5}\big[-(-1) - (1)\big] \cdot \Big[\frac{1}{4}\left(\frac{3}{2}\right)^4 - 0\Big] = \cancel{2}\sqrt{5} \cdot \cancel{2} \cdot \cancel{\frac{1}{4}} \cdot \frac{81}{16} = \boxed{\frac{81}{16}\sqrt{5}}$$

2. (10 points) A street vendor on Tatooine sells you a pallie, a fruit in the shape of a perfectly round ball, with a radius of 2. Assume a coordinate system with the origin at the center of the fruit. You slice the fruit along the plane $z = 1$ and find that the mass density δ at each point inside the fruit is inversely proportional to the distance from the center:

$$\delta(x, y, z) = \frac{3}{\sqrt{x^2 + y^2 + z^2}}. = \frac{3}{\rho}$$

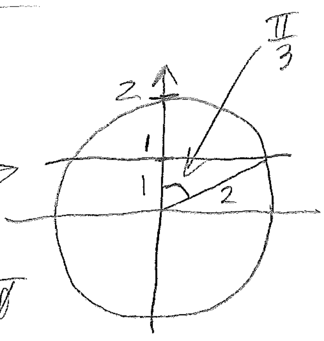
Find the center of mass of the top piece that you sliced off, i.e., the portion of the object above $z = 1$.

First, note that the density function δ is radially symmetric around the z-axis, and so is the top slice we're interested in. So the center of mass will be along the z-axis:



$$x_{CM} = y_{CM} = 0.$$

For the z coordinate: $$z_{CM} = \frac{\iiint_W z\,\delta(x,y,z)\,dV}{\text{Mass}}$$



We'll use spherical coordinates: $z = 1 = \rho\cos\phi \Rightarrow \rho = \frac{1}{\cos\phi}$

So $0 \le \phi \le \frac{\pi}{3}$, $0 \le \theta \le 2\pi$, and $\frac{1}{\cos\phi} \le \rho \le 2$.

$$\text{Mass} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=\frac{1}{\cos\phi}}^{2} \frac{3}{\rho}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 3\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=\frac{1}{\cos\phi}}^{2} \rho\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 3\int_{\theta=0}^{2\pi} d\theta \cdot \int_{\phi=0}^{\frac{\pi}{3}} \sin\phi \cdot \left[\frac{1}{2}\rho^2\right]_{\rho=\frac{1}{\cos\phi}}^{2} d\phi$$

$$= \frac{3}{2}\cdot(2\pi)\cdot\int_{\phi=0}^{\frac{\pi}{3}}\left(4 - \frac{1}{\cos^2\phi}\right)\sin\phi\,d\phi \qquad = 3\pi\int_{u=1}^{\frac{1}{2}} -\left(4 - \frac{1}{u^2}\right)du$$

$u = \cos\phi$, so $du = -\sin\phi\,d\phi$

$$= 3\pi\left[4u + \frac{1}{u}\right]_{u=\frac{1}{2}}^{1} = 3\pi\left[5 - 4\right] = 3\pi = \text{mass!}$$

Swapped and got rid of negative sign

$$\iiint_W z \cdot \delta(x,y,z)\, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=\frac{1}{\cos\phi}}^{2} \rho\cos\phi \cdot \frac{3}{\rho} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi} \cdot \int_{\phi=0}^{\pi/3} \cos\phi \sin\phi \int_{\rho=\frac{1}{\cos\phi}}^{2} 3\rho^2\, d\rho\, d\phi$$

$$= 2\pi \cdot \int_{\phi=0}^{\pi/3} \cos\phi \sin\phi \left[\rho^3\right]_{\rho=\frac{1}{\cos\phi}}^{2} d\phi$$

$$= 2\pi \cdot \int_{\phi=0}^{\pi/3} \cos\phi \sin\phi \left[8 - \frac{1}{\cos^3\phi}\right] d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi/3} \left(8\cos\phi - \frac{1}{\cos^2\phi}\right) \sin\phi\, d\phi \qquad u = \cos\phi,\ du = -\sin\phi\, d\phi$$

$$= 2\pi \int_{u=1}^{u=\frac{1}{2}} -\left(8u - \frac{1}{u^2}\right) du = 2\pi \cdot \left[4u^2 + \frac{1}{u}\right]_{u=\frac{1}{2}}^{1}$$

again, swap bounds and drop the negative.

$$= 2\pi\left[5 - 3\right] = 4\pi$$

So $z_{cm} = \frac{4\pi}{3\pi} = \frac{4}{3}$.

The center of mass is at $\boxed{\left(0, 0, \frac{4}{3}\right)}$

3. (10 points) Let $\mathcal{C}$ be the curve parametrized by

$$\mathbf{r}(t) = (e^t + e^{-t})\mathbf{i} + 2t\mathbf{j} + (e^t - e^{-t})\mathbf{k}, \qquad 0 \le t \le 1.$$

Compute the average value of the function $f(x, y, z) = z$ on $\mathcal{C}$.

$$\text{Average value} = \frac{\int_C z\,ds}{\text{length of } C}$$

Setup: $\vec{r}'(t) = \langle e^t - e^{-t}, 2, e^t + e^{-t} \rangle$

$$\|\vec{r}'(t)\| = \sqrt{(e^t - e^{-t})^2 + 2^2 + (e^t + e^{-t})^2}$$

$$= \sqrt{(e^{2t} - \cancel{2} + e^{-2t}) + 4 + \underbrace{(e^{2t} + \cancel{2} + e^{-2t})}_{}}$$

$$= \sqrt{2e^{2t} + 4 + 2e^{-2t}}$$

same!

$$= \sqrt{2(e^{2t} + 2 + e^{-2t})}$$

$$= \sqrt{2(e^t + e^{-t})^2} = \sqrt{2}(e^t + e^{-t})$$

ds

$$\text{Length of } C = \int_C 1\,ds$$

$$= \int_{t=0}^{1} \sqrt{2}(e^t + e^{-t})\,dt$$

$$= \sqrt{2}\left[e^t - e^{-t}\right]_{t=0}^{1}$$

$$= \sqrt{2}\left[e - e^{-1} - 0\right] = \sqrt{2}(e - e^{-1})$$

$$\int_C z\,ds = \int_{t=0}^{1} (e^t - e^{-t}) \cdot \sqrt{2}(e^t + e^{-t})\,dt = \sqrt{2}\int_{t=0}^{1} (e^{2t} - e^{-2t})\,dt$$

$$= \sqrt{2}\left[\frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}\right]_{t=0}^{1} = \frac{\sqrt{2}}{2}\left[e^2 + e^{-2} - 2\right]$$

So the average value of z on C is

$$\frac{\frac{\cancel{\sqrt{2}}}{2}(e^2 + e^{-2} - 2)}{\cancel{\sqrt{2}}(e - e^{-1})} = \boxed{\frac{e^2 + e^{-2} - 2}{2(e - e^{-1})}}$$

$$= \frac{(e^1 - e^{-1})^{\cancel{2}}}{2\cancel{(e^1 - e^{-1})}} = \boxed{\frac{e^1 - e^{-1}}{2}} = \sinh(1) \quad \smile$$

4. (10 points) Let $\mathcal{D}$ be the region in the first quadrant between the curves $x^4 + y^4 = 1$ and $x^4 + y^4 = 16$. Use the transformation $x = r\sqrt{\cos\theta}$, $y = r\sqrt{\sin\theta}$ to compute the double integral

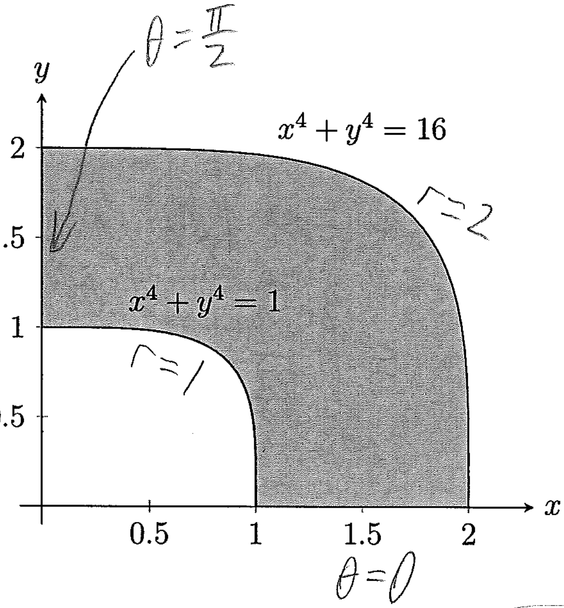
$$\iint_{\mathcal{D}} xy^3\, dA.$$

$x^4 + y^4 = 1 \Longrightarrow (r\sqrt{\cos\theta})^4 + (r\sqrt{\sin\theta})^4 = 1$

$r^4\cos^2\theta + r^4\sin^2\theta = 1$

$r^4 = 1$

$r = 1$



$x^4 + y^4 = 16 \Longrightarrow (r\sqrt{\cos\theta})^4 + (r\sqrt{\sin\theta})^4 = 16$

$r^4\cos^2\theta + r^4\sin^2\theta = 16$

$r^4 = 16 \Longrightarrow r = 2$

$y = 0 \Longrightarrow r\sqrt{\sin\theta} = 0$, and $1 \le r \le 2$

$\Longrightarrow \sqrt{\sin\theta} = 0$

$\sin\theta = 0$

$\theta = 0$

$x = 0 \Longrightarrow r\sqrt{\cos\theta} = 0$, and $1 \le r \le 2 \Longrightarrow \sqrt{\cos\theta} = 0$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

So

$1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

Jacobian: $\left|\det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \sqrt{\cos\theta} & \sqrt{\sin\theta} \\ \frac{1}{2}r\frac{-\sin\theta}{\sqrt{\cos\theta}} & \frac{1}{2}r\frac{\cos\theta}{\sqrt{\sin\theta}} \end{bmatrix}\right|$

$= \left|\frac{1}{2}r\frac{\cos\theta\sqrt{\cos\theta}}{\sqrt{\sin\theta}} + \frac{1}{2}r\frac{\sin\theta\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\right| = \frac{1}{2}r\frac{\cos^2\theta + \sin^2\theta}{\sqrt{\sin\theta\cdot\cos\theta}}$

$= \frac{r}{2\sqrt{\sin\theta\cos\theta}}$

Now $\iint_{\mathcal{D}} xy^3\, dA = \int_{\theta=0}^{\pi/2}\int_{r=1}^{2} r\sqrt{\cancel{\cos\theta}}\cdot(r\sqrt{\sin\theta})^3 \cdot \frac{r}{2\sqrt{\cancel{\sin\theta}\,\cancel{\cos\theta}}}\, dr\, d\theta$

Cont'd $\longrightarrow$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=1}^{2} \frac{1}{2} r^5 \sin\theta \, dr \, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \cdot \int_{r=1}^{2} r^5 \, dr = \frac{1}{2} \Big[-\cos\theta \Big]_{\theta=0}^{\pi/2} \cdot \Big[\frac{1}{6} r^6 \Big]_{r=1}^{2}$$

$$= \frac{1}{2} \Big[0 - (-1) \Big] \cdot \Big[\frac{64}{6} - \frac{1}{6} \Big] = \boxed{\frac{21}{4}}$$

5. (10 points) Let $\mathcal{W}$ be the region inside the cylinder $x^2 + y^2 = 4$ where $x \geq \sqrt{2}$, between the planes $z = 0$ and $z = x$. Let

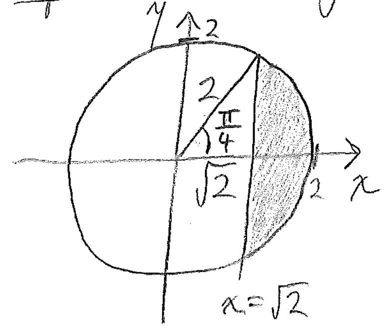
$$\mathbf{F}(x, y, z) = \left\langle \ln x, e^{z^2}, \frac{z}{x} \right\rangle.$$

Compute the flux of $\mathbf{F}$ flowing outward through the boundary of $\mathcal{W}$.

Flux through boundary $= \iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$

By the Divergence Theorem

Top view of region:



$\operatorname{div}(\vec{F}) = \frac{1}{x} + 0 + \frac{1}{x} = \frac{2}{x}$

So we want $\iiint_W \frac{2}{x}\, dV$.

We'll use cylindrical coordinates:

$x = \sqrt{2} \Rightarrow r\cos\theta = \sqrt{2} \Rightarrow r = \frac{\sqrt{2}}{\cos\theta}$

So $\frac{\sqrt{2}}{\cos\theta} \leq r \leq 2$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$0 \leq z \leq x = r\cos\theta$

$$\text{Flux} = \iiint_W \operatorname{div}(\vec{F})\, dV = \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=\frac{\sqrt{2}}{\cos\theta}}^{2} \int_{z=0}^{r\cos\theta} \frac{2}{\cancel{r}\cos\theta} \cdot \cancel{r}\, dz\, dr\, d\theta$$

$$= \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=\frac{\sqrt{2}}{\cos\theta}}^{2} \frac{2}{\cos\theta} \Big[z \Big]_{z=0}^{r\cos\theta} dr\, d\theta = \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=\frac{\sqrt{2}}{\cos\theta}}^{2} \frac{2}{\cancel{\cos\theta}} \cdot r\cancel{\cos\theta}\, dr\, d\theta$$

$$= \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=\frac{\sqrt{2}}{\cos\theta}}^{2} 2r\, dr\, d\theta = \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \Big[r^2 \Big]_{r=\frac{\sqrt{2}}{\cos\theta}}^{2} d\theta$$

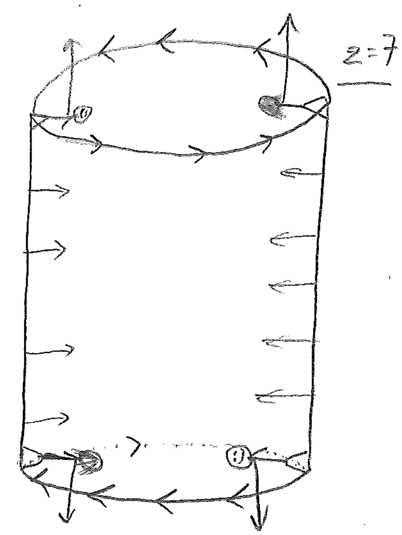
$$= \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(4 - \frac{2}{\cos^2\theta}\right) d\theta = \Big[4\theta - 2\tan\theta \Big]_{\theta=-\frac{\pi}{4}}^{\pi/4} = \Big[(\pi - 2) - (-\pi + 2)\Big] = \boxed{2\pi - 4}$$

6. (10 points) Let $\mathcal{S}$ be the portion of the cylinder $x^2 + y^2 = 9$ where $2 \le z \le 7$, oriented with normal vectors that point in toward the z-axis. (Note this is just the cylinder, without a disk at the top or bottom.) Let

$$\mathbf{F}(x, y, z) = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}.$$

Compute $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

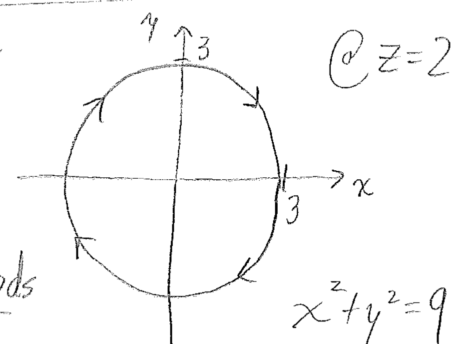
By Stokes' Theorem, $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$



Boundary orientation: CCW for top circle
CW for bottom circle
(viewed from above)

So $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_{\substack{\text{circle @ } z=2 \\ \text{CW}}} \vec{F} \cdot d\vec{r} + \int_{\substack{\text{circle @ } z=7 \\ \text{CCW}}} \vec{F} \cdot d\vec{r}$

Bottom circle: The vector $\langle y, -x, 0 \rangle$ is tangent to the curve, so $\vec{T} = \frac{\langle y, -x, 0 \rangle}{\|\langle y, -x, 0 \rangle\|} = \frac{\langle y, -x, 0 \rangle}{3}$ is the unit tangent vector.



So $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} (\vec{F} \cdot \vec{T})\, ds = \int_{C_1} \langle yz, -xz, xy \rangle \cdot \frac{\langle y, -x, 0 \rangle}{3} ds$

$= \frac{1}{3}\int_{C_1} (x^2 \cancel{z}^{2} + y^2 \cancel{z}^{2})\, ds = \frac{2}{3}\int_{C_1} (\cancel{x^2+y^2})^{9}\, ds = 6\int_{C_1} 1\, ds = 6 \cdot (\text{length of circle})$
$= 6 \cdot 6\pi = 36\pi$

Top circle is similar: $\vec{T} = \frac{\langle -y, x, 0 \rangle}{3}$ now, so

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} (\vec{F} \cdot \vec{T})\, ds = \int_{C_2} \langle yz, -xz, xy \rangle \cdot \frac{\langle -y, x, 0 \rangle}{3} ds = \frac{1}{3}\int_{C_2} -(x^2 z + y^2 z)\, ds$

$= -\frac{7}{3}\int_{C_2} (x^2+y^2)\, ds = -\frac{7}{3}\int_{C_2} 9\, ds = -21 \cdot (\text{length}) = -21 \cdot 6\pi = -126\pi.$

So now $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = 36\pi + -126\pi = \boxed{-90\pi}$

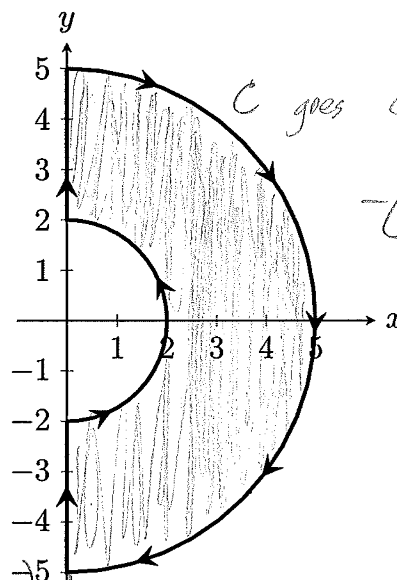
7. (10 points) Let C be the path shown in the figure below. Compute

$$\oint_C (xe^{x^2+y^2} - 2x^3y)\,dx + (ye^{x^2+y^2} + x^2y^2)\,dy$$

By Green's Theorem,

$$\oint_C F_1dx + F_2dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA$$

if C is oriented CCW! Since it's the opposite, we need to negate it...



C goes CW, $-C$ goes CCW.

So $$\int_C F_1dx + F_2dy = -\int_{-C} F_1dx + F_2dy$$

$$\underset{\text{Green's}}{\overset{\text{by}}{=}} -\iint_D \left(\cancel{2xye^{x^2+y^2}} + 2xy^2 - \cancel{2xye^{x^2+y^2}} + 2x^3\right)dA$$

Do this in polar, obviously!

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$2 \le r \le 5$

$$= -\iint_D 2x(x^2+y^2)\,dA$$

$$= -\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=2}^{5} 2r\cos\theta \cdot r^2 \cdot r\,dr\,d\theta$$

$$= -2\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta \cdot \int_{r=2}^{5} r^4\,dr = -2\Big[\sin\theta\Big]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \left[\frac{1}{5}r^5\right]_{r=2}^{5}$$

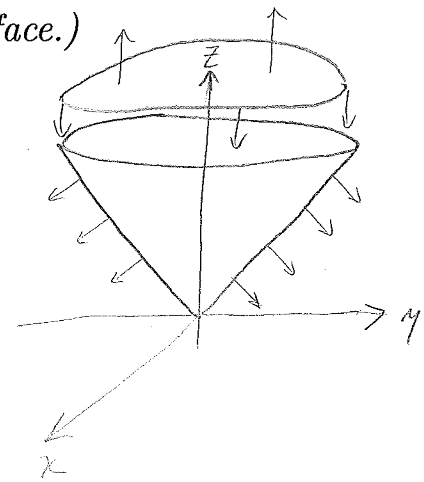
$$= \underbrace{-2[1-(-1)]\cdot\left[\frac{5^5}{5} - \frac{2^5}{5}\right]}_{\text{leaving it in a form like this is fine}} = \underbrace{-4\cdot\left(625 - \frac{32}{5}\right)}_{\text{or this}} = \boxed{\frac{-12372}{5}}$$

$$\text{At } z=4,\ x^2+y^2=16 \Rightarrow \text{radius } 4$$

8. (10 points) Let $\mathcal{S}$ be the portion of the cone $x^2+y^2=z^2$ from $z=0$ to $z=4$, oriented with downward (outward) pointing normal vectors. Compute the flux through $\mathcal{S}$ of the vector field

$$\mathbf{F}(x,y,z)=\langle ye^y, xe^x, 5z\rangle\,.$$

(Hint: Use the Divergence Theorem and another surface.)



Let S_2 be the disk at the top of the cone, with upward normals.
Then $S+S_2$ is a closed surface with outward normals, so by the Divergence Thm,

$$\iint_{S+S_2}\vec{F}\cdot d\vec{S}=\iiint_W \operatorname{div}(\vec{F})\,dV$$ where W is the solid cone.

$$\operatorname{div}(\vec{F})=0+0+5=5,\quad \text{so}\quad \iiint_W \operatorname{div}(\vec{F})\,dV=\iiint_W 5\,dV=5\cdot(\text{volume of } W)$$
$$=5\cdot\tfrac{1}{3}\pi(4)^2\cdot 4=\tfrac{5}{3}\cdot 64\pi$$
$$=\frac{320\pi}{3}$$

Also $\iint_{S+S_2}\vec{F}\cdot d\vec{S}=\iint_S \vec{F}\cdot d\vec{S}+\iint_{S_2}\vec{F}\cdot d\vec{S}$, and the second term here is

$$\iint_{S_2}\vec{F}\cdot d\vec{S}=\iint_{\text{disk at } z=4}(\vec{F}\cdot\vec{e}_n)\,dS=\iint_{\text{disk}}\langle ye^y, xe^x, 5z\rangle\cdot\langle 0,0,1\rangle\,dS$$
$$=\iint_{\text{disk}}5z\,dS=\iint 20\,dS=20\cdot(\text{area of disk})=20\cdot\pi(4)^2$$
$$=320\pi$$

since $z=4$ in this disk

So now $\iint_S \vec{F}\cdot d\vec{S}+320\pi=\frac{320\pi}{3}$, so $\iint_S \vec{F}\cdot d\vec{S}=\frac{320\pi}{3}-320\pi$

$$=\boxed{-\frac{640\pi}{3}}$$

9. Suppose $\mathbf{F}$ is a vector field in $\mathbb{R}^2$ that is undefined at $(-2, 2)$, $(1, -2)$, and $(3, 1)$. Everywhere it is defined, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$. The plot below shows four oriented curves. Suppose you know that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 4, \qquad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 9, \qquad \text{and} \qquad \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = -6.$$

(a) (3 points) Is $\mathbf{F}$ conservative? Why or why not?

No. If $\vec{F}$ were conservative, then for any closed curve C, $\oint_C \vec{F} \cdot d\vec{r}$ would be 0. But for all three of the closed curves C_1, C_2, and C_3, this line integral is not 0. Therefore $\vec{F}$ cannot be conservative.

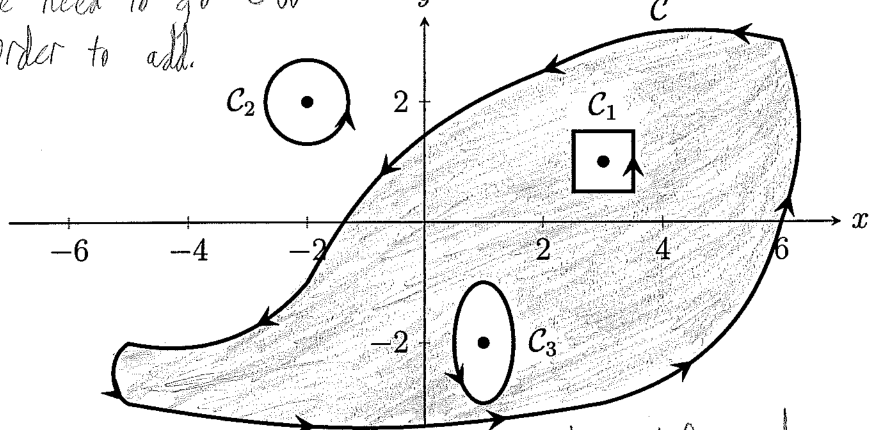
(b) (7 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. For full credit, you must justify your answer sufficiently.

By Green's Theorem, if D is the region inside C but outside C_1 and C_3, then (since $\vec{F}$ is defined everywhere on D!) we have

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{\underbrace{C - C_1 - C_3}} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_3} \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot d\vec{r} - 4 - (-6)$$

subtract because these need to go CW in order to add.



But since $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ everywhere on D, the left side is 0.

So $\int_C \vec{F} \cdot d\vec{r} - 4 - (-6) = 0 \Longrightarrow \int_C \vec{F} \cdot d\vec{r} = 4 + -6 = \boxed{-2}$

10. Define a vector field $\mathbf{F}$ by

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

At every point where this vector field is defined, $\operatorname{div}(\mathbf{F}) = 0.$ In this problem, you will compute the (outward) flux of $\mathbf{F}$ through any closed surface that does not touch the origin. For each of these steps, be sure to justify your answer sufficiently!

(a) (3 points) Suppose $\mathcal{S}$ is any closed surface for which the region enclosed by $\mathcal{S}$ *does not* contain the origin. Orient $\mathcal{S}$ with outward-pointing normal vectors. Compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

Since the only point where $\vec{F}$ is undefined is the origin, in this situation the Divergence Theorem works.

So $$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\,dV = \iiint_W 0\,dV = \boxed{0}$$

W is the region enclosed by S

(b) (3 points) Now suppose $\mathcal{S}$ is a closed surface for which the region enclosed by $\mathcal{S}$ *does* contain the origin, again oriented with outward-pointing normal vectors. Can you use the Divergence Theorem to compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$? Why or why not?

No, we cannot! The hypotheses of the Divergence Theorem require that $\vec{F}$ be defined (and twice continuously differentiable) not only on S, but also everywhere in the region enclosed by S. Here it says this region does contain the origin, where $\vec{F}$ is undefined. So the Divergence Theorem can't be used (directly) here.

Question 10 continues on the next page...

(c) (5 points) For any positive number R, let $\mathcal{S}_R$ be the sphere of radius R centered at the origin:

$$x^2 + y^2 + z^2 = R^2$$

As usual, orient $\mathcal{S}_R$ with outward normals. Compute $\iint_{\mathcal{S}_R} \mathbf{F} \cdot d\mathbf{S}$.

The vector $\langle x,y,z \rangle$ is normal to $\mathcal{S}_R$ and points outward, and $\|\langle x,y,z\rangle\| = \sqrt{x^2+y^2+z^2} = R$ on this surface. So the outward-pointing unit normal vector is

$$\vec{e}_n = \frac{\langle x,y,z\rangle}{\|\langle x,y,z\rangle\|} = \frac{\langle x,y,z\rangle}{R}.$$

Also, on this surface, since $(x^2+y^2+z^2)^{1/2} = R$, $\vec{F}$ simplifies to $\vec{F}(x,y,z) = \left\langle \frac{x}{R^3}, \frac{y}{R^3}, \frac{z}{R^3} \right\rangle = \frac{\langle x,y,z\rangle}{R^3}$.

Thus $\vec{F}\cdot\vec{e}_n = \frac{\langle x,y,z\rangle}{R^3} \cdot \frac{\langle x,y,z\rangle}{R} = \frac{x^2+y^2+z^2}{R^4} = \frac{R^2}{R^4}$

$$= \frac{1}{R^2}.$$

Therefore $\iint_{\mathcal{S}_R} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{S}_R} \vec{F}\cdot\vec{e}_n \, dS = \iint_{\mathcal{S}_R} \frac{1}{R^2}\, dS = \frac{1}{R^2}\iint_{\mathcal{S}_R} 1 \, dS$

$$= \frac{1}{R^2}\cdot(\text{surface area of } \mathcal{S}_R)$$

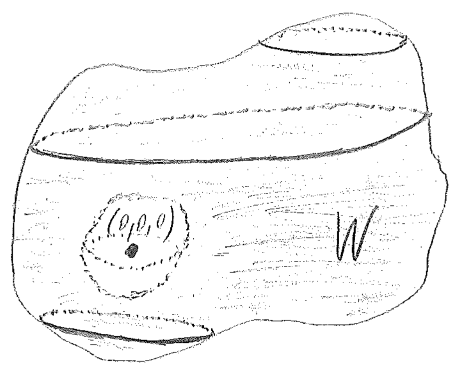
$$= \frac{1}{\cancel{R^2}}\cdot 4\pi \cancel{R^2} = \boxed{4\pi}$$

Rather fascinating that no matter how large or small the sphere, the flux is always 4π...!

Question 10 continues on the next page...

(d) (4 points) Finally, let $\mathcal{S}$ be any closed surface for which the region enclosed by $\mathcal{S}$ *does* contain the origin, oriented with outward normal vectors, exactly as in part (b). Choose a sphere $\mathcal{S}_R$ (as in part (c)) that is small enough that it is entirely inside of $\mathcal{S}$, and let $\mathcal{W}$ be the region outside $\mathcal{S}_R$ but inside $\mathcal{S}$. Apply the Divergence Theorem to the region $\mathcal{W}$ to compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$. *(Hint: This is just like problem 9, but "one dimension higher".)*

Note that the region W described here does not contain the origin (where $\vec{F}$ is undefined), because we have "cut out a hole" around the origin, with S_R.
Therefore we can use the Divergence Theorem on the region W. Its boundary ∂W has two pieces: S, oriented with outward normals, and S_R, but oriented with inward normals.



So the Divergence Theorem says

$$\iiint_W \mathrm{div}(\vec{F})\, dV = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_R} \vec{F}\cdot d\vec{S} \leftarrow \text{inward normal vectors}$$

$$= \iint_S \vec{F}\cdot d\vec{S} + (-4\pi) \leftarrow \text{from part (c), but now reversed.}$$

And since $\mathrm{div}(\vec{F}) = 0$ everywhere in W, the triple integral is 0!

$$0 = \iiint_W \mathrm{div}(\vec{F})\, dV = \iint_S \vec{F}\cdot d\vec{S} - 4\pi, \text{ so}$$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = 4\pi}$$