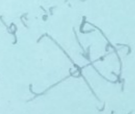
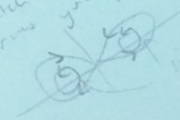
Conclusion:
• Watch out for conservative!!
• always draw line integral curve as it could be not wrapping 0 / i.e. [illegible]
i.e. for origin



• watch out for [illegible] of curves you parametrize



inside curve turn "origin"

Midterm 2

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date: 27 February, 2017

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Austin Guo

ID number: [illegible]

Discussion section: [illegible]

[illegible]	Eun	Robbie
[illegible]	C	3E
[illegible]	D	3F

Question	Points	Score
[illegible]	[illegible]	8
[illegible]	[illegible]	8
[illegible]	[illegible]	4
[illegible]	[illegible]	4
5	11	9
Total:	40	33

+1

34

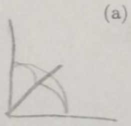
UCLA: Math 32B — Midterm 2 — Winter, 2017

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \le a, \quad x, y, z \ge 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.



sphere in 1st quadrant :. param by

$(\rho \cos\theta \sin\varphi, \rho \sin\theta \sin\varphi, \rho\cos\varphi)$

$0 \le \theta \le \frac{\pi}{2}$

$0 \le \rho \le a$

$0 \le \varphi \le \frac{\pi}{2}$

2

$$\iiint \delta \, dV = \iiint \overset{1}{\delta} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

(b) (2 points) Find the total mass of $\mathcal{E}$.

$$\frac{\cancel{2}\pi}{2} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^3}{3} \right]_0^a \sin\varphi \, d\varphi$$

$$= 2\pi \, \frac{a^3}{3} \left[-\cos\varphi \right]_0^{\frac{\pi}{2}} = 2\pi \frac{a^3}{3}(-0+1)$$

$$= \boxed{\frac{\cancel{2}\pi a^3}{2 \cdot 3}}$$

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

$\iiint (x, y, z)$

Total mass = $\frac{2\pi a^3}{3}$ COM = $\left(\frac{M_{yz}, M_{xz}, M_{xy}}{\text{total mass}}\right)$

COM = $\dfrac{\iiint (x, y, z)\, dx\, dy\, dz}{\left(\frac{2\pi a^3}{3}\right)}$

$$\boxed{\frac{3 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a (\rho \cos\theta \sin\varphi, \rho \sin\theta \sin\varphi, \rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}{2\pi a^3}}$$

3

(d) (2 points) Find the z coordinate of the center of mass.

EeF

2

M_{xy} $\dfrac{\iiint z\,\delta\, dV}{\iiint \delta\, dV}$

M_z

$$\frac{3 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a (\rho \cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}{2\pi a^3}$$

$$= \frac{3 \cancel{(2\pi)} \int_0^{\frac{\pi}{2}} \int_0^a \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi}{\cancel{2\pi}\, a^3}$$

$$= \frac{3 \int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^a \frac{\sin 2\varphi}{2}\, d\varphi}{a^3} = \frac{3 \frac{a^{\cancel{4}}}{4} \left(-\frac{\cos 2\varphi}{2}\right)_0^{\frac{\pi}{2}}}{\cancel{a^3}}$$

$$= \frac{3a}{4} \cdot \frac{1}{2} \left(-(-1) - (-1)\right) = \frac{3a}{\cancel{8}4}(2) = \boxed{\frac{3a}{4}}$$

2. Consider the helix C, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi}t\right) \quad t \in [0, 4\pi],$$

so that C is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of C.

$r'(t) = \langle -\sin t, \cos t, \frac{1}{2\pi} \rangle$ $\qquad \int_C ds = \int_C \|r'(t)\| \, dt$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{4\pi^2}} = \sqrt{1 + \frac{1}{4\pi^2}}$

$\int_0^{4\pi} \sqrt{1 + \frac{1}{4\pi^2}} \, dt$

$\sqrt{1 + \frac{1}{4\pi^2}} \; 4\pi = \sqrt{\frac{4\pi^2 + 1}{4\pi^2}} \; 4\pi$

$\frac{2 \cancel{4\pi} \sqrt{4\pi^2 + 1}}{\cancel{2\pi}}$

$= \boxed{2\sqrt{4\pi^2+1}}$

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x,y,z) = \langle z^2, 2yz^2, 2z(x+y^2) - e^z\rangle$$

on a particle constrained to move on the curve C.

$\int_C F \cdot dr = \int F(r(t)) \cdot dr$

$F(r(t))$ $\frac{1}{4\pi^2}t^2$

$\int \left(\frac{1}{4\pi^2} t^2, \frac{2 \sin t}{4\pi^2} t^2, \frac{2t}{2\pi}(\cos t + \sin^2 t) - e^{\frac{t}{2\pi}} \right) \cdot \left(-\sin t, \cos t, \frac{1}{2\pi}\right)$

$\text{curl} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ F_1 & F_2 & F_3 \end{vmatrix} =$

$= \langle \underbrace{2z(2y) - 2y(2z)}_{=0}, \underbrace{2z - 2z}_{=0}, \underbrace{0 - 0}_{=0} \rangle$

curl $= 0$ conservative vector field

$= \int z^2\,dx = z^2 x + a(y,z)$

$= \int 2yz^2\,dy = y^2z^2 + b(x,z)$

$= \int 2z(x+y^2) - e^z\,dz = z^2x + z^2y^2 - e^z + m(x,y)$

$f = z^2x + z^2y - e^z$

$f(r(4\pi)) - f(r(0)) = f(1,0,2) - f(1,0,0)$

$= 4 + 4\cdot 0 - e^2 - (0 + 0 - e^0)$

$= 4 - e^2 + 1 = \boxed{5 - e^2}$

4/7

3. For this question consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

3

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

No, it is not conservative on the domain described above because the curl $\neq 0$.

curl $= \begin{vmatrix} \partial x & \partial y \\ F_1 & F_2 \end{vmatrix}$

$\partial x \quad \frac{y(x^2+y^2-1)}{x^2+y^2}, \quad \partial y \quad \frac{x(x^2+y^2+1)}{x^2+y^2}$

$=$

$= ?$

1

(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

when $y \geq 0$

0

(c) (2 points) Calculate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

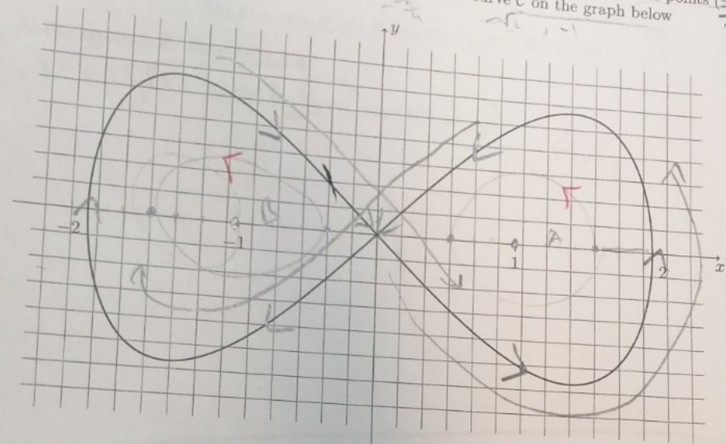
where C is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

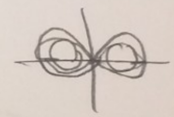
$\int F$ parametrized ellipse:

$(\sqrt{2}\rho\cos\theta + 4, \ \rho\sin\theta,$

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve C on the graph below

curl = 0





1

(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

3

$$\int_{\mathcal{A}} \mathbf{E} \cdot \mathbf{dr} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot \mathbf{dr} = 1.$$

What is $\int_C \mathbf{E} \cdot \mathbf{dr}$? Justify your answer.

$$\int_C E \cdot dr = \int_A E \cdot dr + \int_B E \cdot dr = \boxed{13}$$

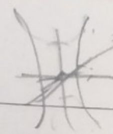
Orientation

1

This is because the vector field is not conservative about the points $(\pm 1, 0)$, so whenever the graph wraps around these points the work done by the vector field / vector line integral is equal to a certain value. In this case, the values for wrapping around $(-1, 0)$ and $(1, 0)$ are given, and this may be used, along with the fact that the curve C wraps around the same points once each, to obtain the answer. With the fact below, compute integral like this

✓

*Fact: $\int_{C_1} F \cdot dr$ along C_1 as horizontal / vertical lines = 0 since $\int F_1 dx + F_2 dy$ either dx or $dy = 0$ and so F_1, F_2 will be 0



$x^2+y^2=1$

$s^2 = 1$

$r=1$

$s^2 = x^2+y^2-1$

s^2-1

$r=1$

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2+y^2-z^2=1$.

(a) (3 points) Find a parameterisation

$$G(s,\theta) = (x(s,\theta), y(s,\theta), z(s,\theta)) \quad (s,\theta) \in \mathbb{R} \times [0, 2\pi]$$

for the hyperboloid. *Hint: Let $z = s$.*

$G(s,\theta) = \langle \sqrt{(s+1)}\cos\theta, \sqrt{(s+1)}\sin\theta, s \rangle$ 2/3

$$\boxed{G(s,\theta) = ((s+1)\cos\theta, (s+1)\sin\theta, s)}$$

(b) (5 points) Express the surface area of the hyperboloid between $z=a$ and $z=-a$ as an iterated integral.

$-a \le s \le a$

$T_s = (\cos\theta, \sin\theta, 1)$

$T_\theta = (-(s+1)\sin\theta, (s+1)\cos\theta, 0)$

$N(s,\theta) = (-(s+1)\cos\theta, -(s+1)\sin\theta, [(s+1)\cos^2\theta + (s+1)\sin^2\theta])$

$N(s,\theta) = (-(s+1)\cos\theta, -(s+1)\sin\theta, (s+1))$

$\|N\| = \sqrt{(s+1)^2 + (s+1)^2} = (s+1)\sqrt{2}$

$$\boxed{\int_0^{2\pi}\int_{-a}^{a} (s+1)\sqrt{2}\, ds\, d\theta}$$

5/5

(extra working room for part (b))

(c) (3 points) Calculate the surface area.

$$\int_0^{2\pi} \int_{-a}^{a} (s+1)\sqrt{2}\, ds\, d\theta$$

$$\sqrt{2}(2\pi)\left[\frac{s^2}{2} + s\right]_{-a}^{a} = 2\sqrt{2}\pi\left(\frac{a^2}{2} - \frac{a^2}{2} + a + a\right)$$

$$= \boxed{4\sqrt{2}\pi a}$$

2/3