Midterm 2

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date: 27 February, 2017

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions.

ID number:

Discussion section (please circle):

Day/TA	Ben	Gyu Eun	Robbie
Tuesday	3A	3C	3E
Thursday	3B	3D	3F

Question	Points	Score
1	9	
2	8	
3	7	
4	5	
5	11	
Total:	40	

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, \mathrm{d}x = \frac{1}{2}\left(x + \cos x \sin x\right)$$

$$\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}\left(x - \cos x \sin x\right)$$

$$\int \sin x \cos x \, \mathrm{d}x = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho\cos\theta\sin\phi \\ y(\rho,\theta,\phi) &= \rho\sin\theta\sin\phi \\ z(\rho,\theta,\phi) &= \rho\cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \leq a, \quad x, y, z \geq 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.

Total mass $= \iiint_{\mathcal{E}} 1 \, dV$. In spherical coords $\mathcal{E}: 0 \leq \rho \leq \sqrt{a}, \; 0 \leq \theta, \phi \leq \frac{\pi}{2}$ since the Jacobian is $\rho^2 \sin\phi$:

$$\iiint_{\mathcal{E}} 1 \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) (2 points) Find the total mass of $\mathcal{E}$.

total mass $=$ total volume $= \frac{1}{8}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{6}\pi a^{3/2}$

OR

$$\text{total mass} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi\right)\left(\int_0^{\sqrt{a}} \rho^2 \, d\rho\right)$$

$$= \frac{\pi}{2} \cdot \Big[-\cos\phi\Big]_0^{\pi/2} \cdot \left[\frac{1}{3}\rho^3\right]_0^{\sqrt{a}}$$

$$= \frac{\pi}{2} \cdot 1 \cdot \frac{1}{3} a^{3/2}$$

$$= \frac{1}{6}\pi a^{3/2}.$$

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

$$CM = \frac{1}{\text{Mass}} \cdot \iiint_{\mathcal{E}} (x, y, z)\, dV$$

$$= \frac{6}{\pi a^{3/2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

(d) (2 points) Find the z coordinate of the center of mass.

$$CM_z = \frac{6}{\pi a^{3/2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^3 \sin\phi\cos\phi\, d\rho\, d\theta\, d\phi$$

$$= \frac{6}{\pi a^{3/2}} \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\phi\cos\phi \cdot \int_0^{\sqrt{a}} \rho^3\, d\rho$$

$$= \frac{6}{\pi a^{3/2}} \frac{\pi}{2} \cdot \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} \cdot \left[\frac{1}{4}\rho^4\right]_0^{\sqrt{a}}$$

$$= \frac{3}{a^{3/2}} \cdot \frac{1}{2} \cdot \frac{1}{4} a^2 = \frac{3\sqrt{a}}{8}$$

2. Consider the helix $\mathcal{C}$, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi}t\right) \quad t \in [0, 4\pi],$$

so that $\mathcal{C}$ is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of $\mathcal{C}$.

$$\text{length} = \int_{\mathcal{C}} 1 \, ds = \int_0^{4\pi} 1 \cdot |\underline{r}'(t)| \, dt$$

$$\underline{r}'(t) = \left(-\sin(t), \cos(t), \frac{1}{2\pi}\right)$$

$$|\underline{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + \frac{1}{4\pi^2}} = \sqrt{1 + \frac{1}{4\pi^2}}$$

$$\text{length} = \int_0^{4\pi} \sqrt{1 + \frac{1}{4\pi^2}} \, dt = 4\pi \sqrt{1 + \frac{1}{4\pi^2}}$$

$$= 2\sqrt{4\pi^2 + 1}$$

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x,y,z) = \langle z^2, 2yz^2, 2z(x+y^2) - e^z \rangle$$

on a particle constrained to move on the curve $\mathcal{C}$.

Note that $\partial_y F_3 - \partial_z F_2 = 4yz - 4yz = 0$

$\partial_x F_3 - \partial_z F_1 = 2z - 2z = 0$

$\partial_x F_2 - \partial_y F_1 = 0 - 0 = 0$

so $\underline{F}$ is conservative (it is def on $\mathbb{R}^2$ which is simply conn.).

If $\underline{F} = \nabla f$ then $f = xz^2 + \alpha(y\ z)$

$= y^2 z^2 + \beta(x\ z)$

$= z^2(x+y^2) - e^z + \gamma(x\ y)$

so we can take $f = z^2(x+y^2) - e^z$

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = f(\underline{r}(4\pi)) - f(\underline{r}(0))$$

$$= f(1,0,2) - f(1\ 0\ 0)$$

$$= 4 - e^2 - -e^0$$

$$= 5 - e^2$$

3. For this question consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

We check $\int_{\mathcal{C}} \underline{F} \cdot d\underline{r}$ where $\mathcal{C}$ is the unit circle w/ the ↺ orientation.

$$\underline{r}(t) = (\cos t, \sin t) \qquad t \in [0\ 2\pi]$$

$$\underline{r}'(t) = (-\sin t, \cos t)$$

On $\mathcal{C}$, $r = 1$ so $\underline{F} = \langle 0, 2x\rangle = \langle 0, 2\cos t\rangle$

thus

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \underline{F} \cdot \underline{r}'(t)\, dt = \int_0^{2\pi} 2\cos^2 t\, dt$$

$$= \Big[t + \cos t \sin t\Big]_0^{2\pi}$$

$$= 2\pi \neq 0$$

so not conservative.

(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

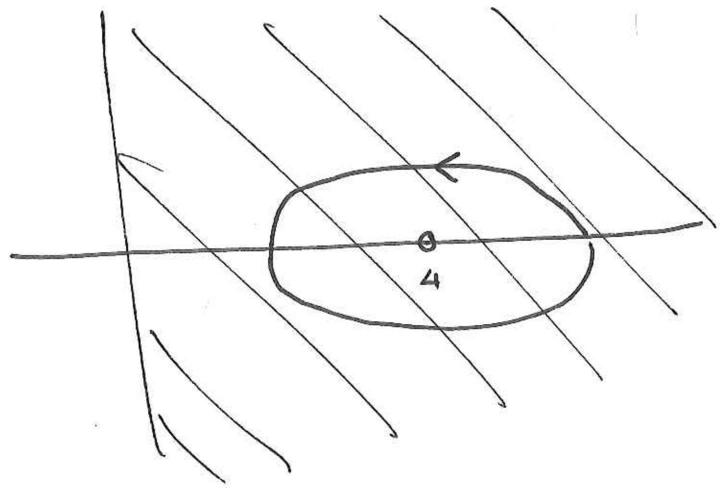
$x > 0$

(c) (2 points) Calculate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

$\mathcal{C}$ lie entirely in the domain $\{x>0\}$



since $\underline{F}$ is conservative on $\{x>0\}$.

$$\oint_{\mathcal{C}} \underline{F} \cdot d\underline{r} = 0$$

Note: $\nabla \times F = \nabla \times \langle y(1-r^{-2}), x(1+r^{-2})\rangle$

$$= (1+r^{-2}) \mp 2 r_x r^{-3} x - (1-r^{-2}) \mp 2 r_y r^{-3} y$$

$$= 2r^{-2} \mp 2r^{-3}(x r_x + y r_y) - 2r^{-2} - 2r^{-3}\left(\frac{r^2}{r}\right) = 0$$

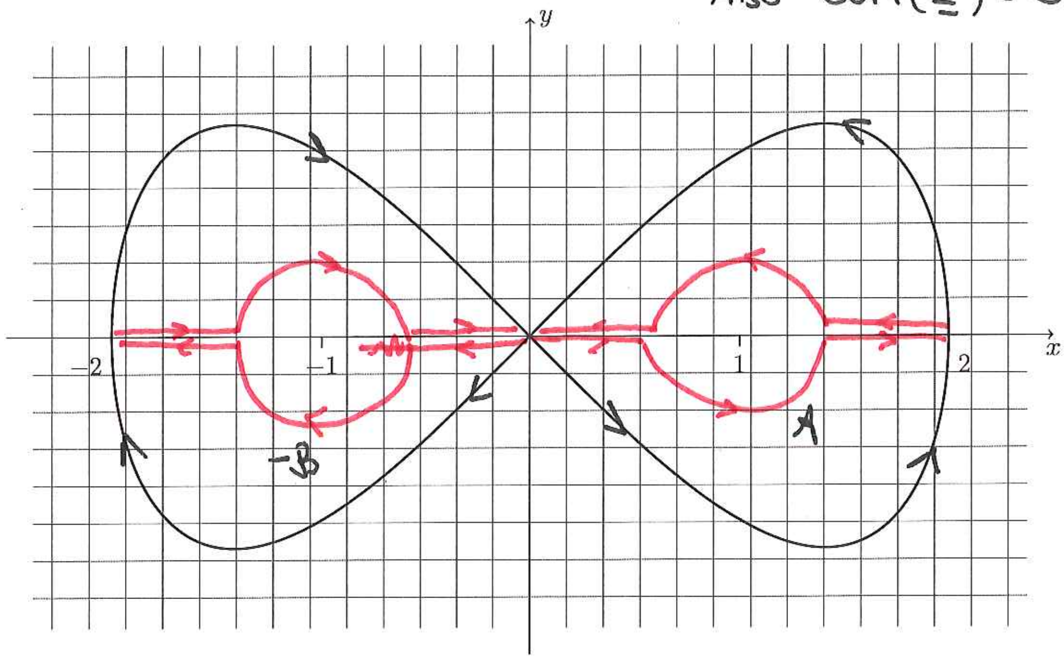
or much more simply: $\underline{F} = \langle -yr^{-2}, xr^{-2}\rangle + \langle y, x\rangle$

$\langle -yr^{2}, xr^{2}\rangle$ = vortex, obviously $\nabla \times \langle y, x\rangle = 0$ so

$$\nabla \times \underline{F} = \nabla \times \text{vortex} + \nabla \times \langle y\, x\rangle = 0$$

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve C on the graph below

Also $\text{Curl}(\underline{E}) = 0$



(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot \mathbf{dr} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot \mathbf{dr} = 1.$$

What is $\int_C \mathbf{E} \cdot \mathbf{dr}$? Justify your answer.

$\underline{E}$ is conservative restricted to any quatrant. By path ind. $\int_{\mathcal{C}} \underline{E} \cdot d\underline{r}$ is the same as $\int_{\mathcal{L}} \underline{E} \cdot d\underline{r}$ where $\mathcal{L}$ is the path shown above. So

$$\int_{\mathcal{C}} \underline{E} \cdot d\underline{r} = \int_{\mathcal{L}} \underline{E} \cdot d\underline{r} = \int_{\mathcal{A}} \underline{E} \cdot d\underline{r} + \int_{-\mathcal{B}} \underline{E} \cdot d\underline{r}$$

op. orientation

$$= \int_{\mathcal{A}} \underline{E} \cdot d\underline{r} - \int_{\mathcal{B}} \underline{E} \cdot dr$$

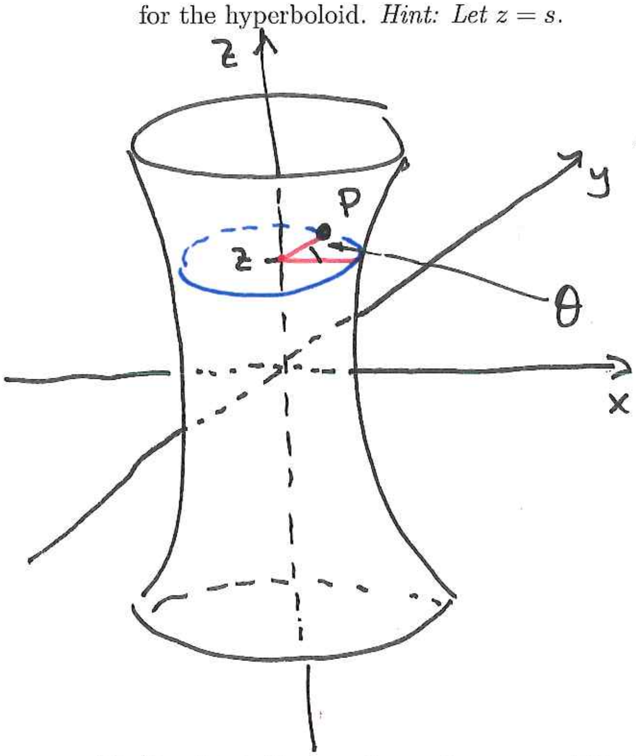
$$= 2 - 1 = 1$$

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

(a) (3 points) Find a parameterisation

$$G(s,\theta) = (x(s,\theta), y(s,\theta), z(s,\theta)) \quad (s,\theta) \in \mathbb{R} \times [0, 2\pi]$$

for the hyperboloid. *Hint: Let $z = s$.*



~~The~~ When $z=s$, P lies on the circle $x^2+y^2=s^2+1$ so

$$P = G(s,\theta) = \left(\sqrt{s^2+1}\cos\theta, \sqrt{s^2+1}\sin\theta, s\right)$$

(b) (5 points) Express the surface area of the hyperboloid between $z = a$ and $z = -a$ as an iterated integral.

$$SA = \iint_S 1\, dS = \iint_D \|N(s\theta)\|\, dA_{s\theta} \quad \text{where } D = [-a,a]\times[0,2\pi]$$

$$\underline{T}_s = \left(\frac{s}{\sqrt{s^2+1}}\cos\theta, \frac{s}{\sqrt{s^2+1}}\sin\theta, 1\right)$$

$$\underline{T}_\theta = \left(-\sqrt{s^2+1}\sin\theta, +\sqrt{s^2+1}\cos\theta, 0\right)$$

$$\underline{N} = \left(-\sqrt{s^2+1}\cos\theta, -\sqrt{s^2+1}\sin\theta, s\cos^2\theta + s\sin^2\theta\right)$$

$$\|\underline{N}\| = \sqrt{(s^2+1)(\cos^2\theta+\sin^2\theta)+s^2} = \sqrt{2s^2+1}$$

$$SA = \int_{-a}^{a}\int_0^{2\pi} \sqrt{2s^2+1}\, d\theta\, ds$$

(extra working room for part (b))

Note I feel bad about giving you this integral to do. On the final I would provide a formula.

(c) (3 points) Calculate the surface area.

$$S.A. = \int_{-a}^{a}\int_{0}^{2\pi}\sqrt{2s^2+1}\,d\theta\,ds = 2\pi\int_{-a}^{a}\sqrt{2s^2+1}\,ds = 4\pi\int_{0}^{a}\sqrt{2s^2+1}\,ds$$

let $\sqrt{2}s = \tan\theta \rightsquigarrow \sqrt{2}\,ds = \sec^2\theta\,d\theta$ so

$$\int_0^a\sqrt{2s^2+1}\,ds = \frac{1}{\sqrt{2}}\int_0^a \sec^2\theta\sqrt{\tan^2\theta+1}\,d\theta = \frac{1}{\sqrt{2}}\int_{s=0}^{s=a}\sec^3\theta\,d\theta = \frac{1}{\sqrt{2}}\int \frac{\cos\theta}{\cos^4\theta}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos\theta\,d\theta}{(1+\sin^2\theta)^2}d\theta = \frac{1}{\sqrt{2}}\int\frac{du}{(1+u^2)^2} = \frac{1}{4\sqrt{2}}\int \frac{1}{1-u}+\frac{1}{(1-u)^2}+\frac{1}{1+u}+\frac{1}{(1+u)^2}\,du$$

part. fractions

$$= \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{1+u}{1-u}\right)+\frac{2u}{1-u^2}\right] = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{1+\sin(\tan^{-1}(\sqrt{2}s))}{1-\sin(\tan^{-1}(\sqrt{2}s))}\right)+\frac{2\sin(\tan^{-1}\sqrt{2}s)}{\cos^2(\tan^{-1}\sqrt{2}s)}\right]_0^a$$

$$= \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{\sqrt{1+2s^2}+\sqrt{2}s}{\sqrt{1+2s^2}-\sqrt{2}s}\right)+\frac{\sqrt{2}s}{(1+2s^2)^{3/2}}\right]_0^a = \frac{1}{4\sqrt{2}}\left(\ln\left(\frac{\sqrt{1+2a^2}+\sqrt{2}a}{\sqrt{1+2a^2}-\sqrt{2}a}\right)+\frac{\sqrt{2}a}{(1+2a^2)^{3/2}}\right)$$

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