Math 32B Lecture 1 Midterm 1

JACK CORDDRY

TOTAL POINTS

97 / 100

QUESTION 1

1 Problem 1 **20 / 20**

✓ **- 0 pts Correct. The answer is $$\pi(e-1)$$.**

- 1 pts Minor arithmetic mistake.

- 2 pts Missing the factor of 1/2 from the change of variables $$rdr = \frac{1}{2} du$$.

- 2 pts The lower bound of the integral in $$r$$ produces a term $$-e^0 = -1$$, which is missing.

- 3 pts You mixed up your bounds in $$u$$ and in $$\theta$$.

- 5 pts Did not change variables correctly. Note $$e^{x^2 + y^2} = e^{r^2}$$.

- 8 pts You set up the integral correctly, but did not evaluate it.

- 15 pts Computed the wrong integral.

- 10 pts You have included an extraneous integral in the variable $$z$$. We are integrating over a two-dimensional region $$D$$, so this problem calls for a double integral.

- 3 pts Incorrect bounds for $$\theta$$. The correct bounds are $$0 \leq \theta \leq 2\pi$$.

QUESTION 2

Problem 2 30 pts

2.1 (a) **9 / 10**

Turned the integral into a manageable form

+ 3 pts Correctly broke up the integral into two parts OR correctly argued by symmetry that the total integral is twice the integral over one region.

✓ **+ 2 pts Said the total integral is twice the integral over the top/bottom half of the region, but there was only partial or flawed justification of this.**

+ 1 pts Said the total integral is twice the integral over one region without symmetry argument or with only a vague "by symmetry".

✓ **+ 3.5 pts Correctly set up the bounds of the integral(s).**

✓ **+ 1.5 pts Used the correct integrand (including the Jacobian!)**

✓ **+ 2 pts Calculation carried out correctly**

+ 0 pts Blank or completely incorrect

(1) You need to note that the region itself is also symmetric about the reflection through the xy plane

2.2 (b) **18 / 20**

$$x$$ coordinate

✓ **+ 6 pts Correctly argued by symmetry that the $$x$$ coordinate of the center of mass is zero OR correctly carried out this calculation**

+ 5 pts Said the $$x$$ coordinate of the COM is zero with only a partial or flawed justification OR correctly calculated the integral, but there were errors in the justification of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

+ 4 pts Said the $$x$$ coordinate of the COM is zero with only a vague "by symmetry" justification OR correctly calculated the integral, but there was no justification (or only a vague "by symmetry" justification) of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

$$z$$ coordinate

✓ **+ 6 pts Correctly argued by symmetry that the $$z$$ coordinate of the center of mass is zero OR correctly carried out this calculation**

+ 5 pts Said the $$z$$ coordinate of the COM is zero with with only a partial or flawed justification.

+ 4 pts Said the $$z$$ coordinate of the COM is zero with no justification or only a vague "by symmetry" justification.

+ **1 pts** Said the integral for the $$z$$ coordinate of the COM is two times the integral over the top/bottom half of the region (this is incorrect!), but computed the integral correctly.

+ **0 pts** Said the integral for the $$z$$ coordinate of the COM is two times the integral over the top/bottom half of the region (this is incorrect!), and there was an error in computing the resulting integral

Setting up the integral(s) for the $$y$$ coordinate.

+ **4 pts** Correctly set up the integral for the $$y$$ coordinate of the center of mass by either breaking it up into two integrals, or correctly arguing by symmetry that the total integral is twice the integral over the top/bottom half of the region.

+ **3 pts** Correctly set up the integral for the $$y$$ coordinate of the center of mass, but there were errors in the justification of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

✓ + **2 pts** **Correctly set up the integral for the $$y$$ coordinate of the center of mass, but there was no justification (or only a vague "by symmetry") of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).**

✓ + **3 pts** **Correctly carried out the integration for the $$y$$ coordinate of the center of mass.**

✓ + **1 pts** **Divided by the total mass.**

+ **0 pts** Blank or completely incorrect

2 Again, you need to explain why you can multiply by 2 and integrate over half the region more precisely

QUESTION 3

3 Problem 3 20 / 20

✓ + **20 pts** **Correct**

+ **4 pts** A good choice of change of variables

+ **2 pts** Correct bounds for the new variables

+ **4 pts** Correct Jacobian

+ **5 pts** Formula for Jac(G) from Jac(G^-1)

+ **3 pts** Computing the Integral

+ **2 pts** Absolute Value of the Jacobian in the Integral

- **1 pts** Small arithmetic mistakes

+ **2 pts** Just writing down change of variables formula.

+ **0 pts** No progress

- **2 pts** G has mistakes in the formula

+ **4 pts** Partial progress in polar coordinates.

+ **1 pts** Partial progress on computing the integral

+ **0 pts** Click here to replace this description.

QUESTION 4

Problem 4 30 pts

4.1 (a) 10 / 10

✓ - **0 pts** **Correct**

- **3 pts** Flipped the sketch

- **4 pts** Wrong or no sketch

- **6 pts** Did not properly define the domain (0<=y<=1,y^2<=x<=y)

- **3 pts** Properly defined the functions which give the bounds, but never actually defined the domain.

- **2 pts** Did not give the proper y-bounds of the domain (0<=y<=1)

- **4 pts** Did not give the proper x-bounds of the domain (y^2<=x<y)

- **2 pts** Switched vertically and horizontally simple (points are taken off in both parts).

- **10 pts** No points

4.2 (b) 10 / 10

✓ - **0 pts** **Correct**

- **10 pts** Did not properly define the domain (0<=x<=1,x<=y<=sqrt(x))

- **5 pts** Properly defined the functions which give the bounds, but never actually defined the domain.

- **4 pts** Did not give the proper x-bounds of the domain (0<=x<=1)

- **6 pts** Did not give the proper y-bounds of the domain (x<=y<=sqrt(x))

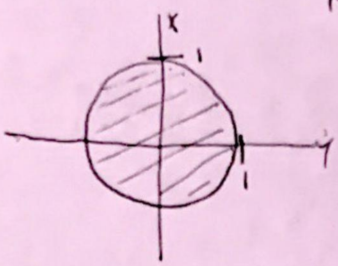
- **4 pts** Flipped the y-bound (wrote sqrt(x)<=y<=x)
- **4 pts** Switched vertically and horizontally simple (points are taken off in both parts).

4.3 (C) **10 / 10**

✓ **- 0 pts** **Correct**

- **0 pts** Correct based on your answer to part (b) (which did not trivialize the problem).
- **1 pts** Basic computation mistake
- **3 pts** Integrated in the order dx dy
- **10 pts** No credit

1. (20 points) Integrate the function $e^{x^2+y^2}$ over the disk D described by the inequality $x^2 + y^2 \leq 1$.



$x = r\cos\theta \quad y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta \leq 1$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$\int_0^{2\pi}\int_0^1 e^{r^2\cos^2\theta + r^2\sin^2\theta} r\,dr\,d\theta$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta$

$u = r^2 \quad du = 2r\,dr$

$\int \frac{1}{2}e^u\,du = \frac{e^u}{2}$

$\frac{1}{2}e^u \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1 d\theta$

$\int_0^{2\pi} \frac{1}{2}e - \frac{1}{2}\,d\theta$

$\frac{e}{2}\theta - \frac{1}{2}\theta \Big|_0^{2\pi}$

$= \frac{e-1}{2}\theta \Big|_0^{2\pi}$

$= 2\pi\left(\frac{e-1}{2}\right)$

$= \pi(e-1)$

$= 5.398$

/20

1 Problem 1 **20 / 20**

✓ **- 0 pts** **Correct. The answer is $$\pi(e-1)$$.**

- 1 pts Minor arithmetic mistake.

- 2 pts Missing the factor of 1/2 from the change of variables $$rdr = \frac{1}{2} du$$.

- 2 pts The lower bound of the integral in $$r$$ produces a term $$-e^0 = -1$$, which is missing.

- 3 pts You mixed up your bounds in $$u$$ and in $$\theta$$.

- 5 pts Did not change variables correctly. Note $$e^{x^2 + y^2} = e^{r^2}$$.

- 8 pts You set up the integral correctly, but did not evaluate it.

- 15 pts Computed the wrong integral.

- 10 pts You have included an extraneous integral in the variable $$z$$. We are integrating over a two-dimensional region $$D$$, so this problem calls for a double integral.

- 3 pts Incorrect bounds for $$\theta$$. The correct bounds are $$0 \leq \theta \leq 2\pi$$.

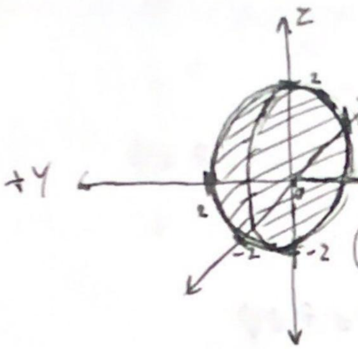
2. Consider the hemispherical region R described by the inequalities

$$x^2 + y^2 + z^2 \leq 4, y \geq 0.$$

(a) (10 points) Let R have mass density function

$$f(x, y, z) = |z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z \leq 0 \end{cases}.$$

Compute the total mass of the solid in the region R.



(drawn w/ nonstandard axes)

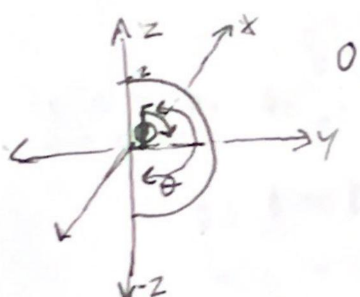
Equal to 2x the integral of the section above the xy plane because $|z|$ is symmetric on either side of $z=0$

$|z|$ above $z=0$ is equal to 0

$r = \sqrt{4} = 2$

$0 \leq \phi \leq \frac{\pi}{2}$

→ Polar $z = \rho \cos\phi$



$0 \leq \theta \leq \pi$

$0 \leq \rho \leq 2$

$$\text{Mass} = 2\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} \rho\cos\phi\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 2\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 2\int_0^{\pi/2}\int_0^{\pi} \tfrac{1}{4}\rho^4 \cos\phi \sin\phi \Big|_0^2 d\theta \, d\phi = 2\int_0^{\pi/2}\int_0^{\pi} 4\cos\phi \sin\phi \, d\theta \, d\phi$$

$$= 2\int_0^{\pi/2} 4\pi \cos\phi \sin\phi \, d\phi$$

$u = \sin\phi \quad du = \cos\phi \, d\phi$

$$= 8\pi \tfrac{1}{2}u^2 \Big|_0^{\pi/2} = 4\pi \sin^2\theta \Big|_0^{\pi/2}$$

$$= 4\pi(1) - 4\pi(0)$$

$$= 4\pi$$

/10

2.1 (a) 9 / 10

Turned the integral into a manageable form

+ **3 pts** Correctly broke up the integral into two parts OR correctly argued by symmetry that the total integral is twice the integral over one region.

✓ **+ 2 pts Said the total integral is twice the integral over the top/bottom half of the region, but there was only partial or flawed justification of this.**

+ **1 pts** Said the total integral is twice the integral over one region without symmetry argument or with only a vague "by symmetry".

✓ **+ 3.5 pts Correctly set up the bounds of the integral(s).**

✓ **+ 1.5 pts Used the correct integrand (including the Jacobian!)**

✓ **+ 2 pts Calculation carried out correctly**

+ **0 pts** Blank or completely incorrect

1. You need to note that the region itself is also symmetric about the reflection through the xy plane

(b) (20 points) Find the coordinates of the center of mass of R.

Because the region and density function are symmetric across the x and z axes,

X_{cm} and Z_{cm} both $= 0$,

as equal mass lies above and below the xy plane, and on either side of the zy plane

$Y_{cm} = \frac{M_{xz}}{M}$ $\quad M_{xz} = \iiint_R y\, \delta(x,y,z)\, dV$

$$= \int_0^{\pi/2}\int_0^{\pi}\int_0^{2} y \cdot 2\rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$y = \rho\sin\phi\sin\theta$

$$= 2\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} \rho^4 \sin^2\phi\cos\phi\sin\theta\, d\rho\, d\theta\, d\phi$$

$$= 2\int_0^{\pi/2}\int_0^{\pi} \frac{2^5}{5}\sin^2\phi\cos\phi\sin\theta\, d\theta\, d\phi$$

$$= \frac{64}{5}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\phi \cdot -\cos\theta \Big|_0^{\pi} d\phi \qquad \cos\pi = -1,\ \cos 0 = 1$$

$$= \frac{-64}{5}\int_0^{\pi/2} -\sin^2\phi\cos\phi - \sin^2\phi\cos\phi\, d\phi$$

$$= \frac{128}{5}\int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi$$

$$= \frac{128}{5}\ \frac{\sin^3\phi}{3}\Big|_0^{\pi/2} \qquad \sin\frac{\pi}{2} = 1$$

$$= \frac{128}{5}\cdot\frac{(1)^3}{3}$$

$$= \frac{128}{15}$$

$Y_{cm} = \frac{M_{xz}}{M}$ $\quad M = 4\pi$ (last page) $\quad Y_{cm} = \frac{128}{60\pi} = 0.679$

/20

2.2 (b) **18 / 20**

$$x$$ coordinate

✓ **+ 6 pts Correctly argued by symmetry that the $$x$$ coordinate of the center of mass is zero OR correctly carried out this calculation**

+ **5 pts** Said the $$x$$ coordinate of the COM is zero with only a partial or flawed justification OR correctly calculated the integral, but there were errors in the justification of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

+ **4 pts** Said the $$x$$ coordinate of the COM is zero with only a vague "by symmetry" justification OR correctly calculated the integral, but there was no justification (or only a vague "by symmetry" justification) of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

$$z$$ coordinate

✓ **+ 6 pts Correctly argued by symmetry that the $$z$$ coordinate of the center of mass is zero OR correctly carried out this calculation**

+ **5 pts** Said the $$z$$ coordinate of the COM is zero with with only a partial or flawed justification.

+ **4 pts** Said the $$z$$ coordinate of the COM is zero with no justification or only a vague "by symmetry" justification.

+ **1 pts** Said the integral for the $$z$$ coordinate of the COM is two times the integral over the top/bottom half of the region (this is incorrect!), but computed the integral correctly.

+ **0 pts** Said the integral for the $$z$$ coordinate of the COM is two times the integral over the top/bottom half of the region (this is incorrect!), and there was an error in computing the resulting integral

Setting up the integral(s) for the $$y$$ coordinate.

+ **4 pts** Correctly set up the integral for the $$y$$ coordinate of the center of mass by either breaking it up into two integrals, or correctly arguing by symmetry that the total integral is twice the integral over the top/bottom half of the region.

+ **3 pts** Correctly set up the integral for the $$y$$ coordinate of the center of mass, but there were errors in the justification of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).

✓ **+ 2 pts Correctly set up the integral for the $$y$$ coordinate of the center of mass, but there was no justification (or only a vague "by symmetry") of why the total integral is twice the integral over the top/bottom half of the region (the symmetry argument requires updating!).**

✓ **+ 3 pts Correctly carried out the integration for the $$y$$ coordinate of the center of mass.**

✓ **+ 1 pts Divided by the total mass.**

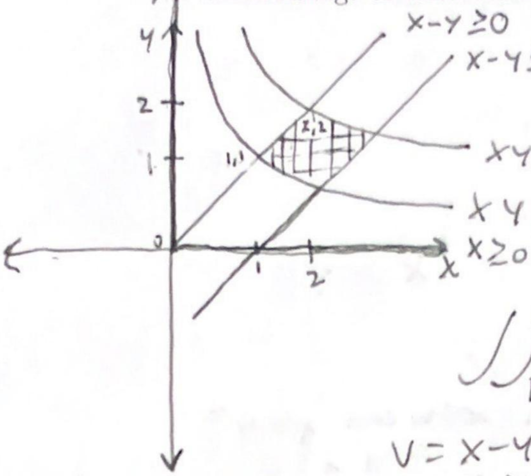
+ **0 pts** Blank or completely incorrect

(2) Again, you need to explain why you can multiply by 2 and integrate over half the region more precisely

3. (20 points) Let D be the domain in the xy-plane described by the inequalities

$$1 \le xy \le 4,\ 0 \le x - y \le 1,\ x, y \ge 0.$$

$y \ge 0$ Use a change of variables to integrate the function $f(x, y) = x + y$ over the domain D.



$xy \ge 1 \quad xy \le 4$

$y \ge \frac{1}{x} \quad y \le \frac{4}{x}$

$\iint_D f(x,y)\,dA = \iint_{D_0} f(|\text{Jac}G|)\,du\,dv$

$v = x - y$ → $1 \le u \le 4$, $0 \le v \le 1$ → $x = \frac{u}{y}$, $y = x - u$

$u = xy$

$(x,y) \to (xy, x-y)$

$\text{Jac}(G) = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix} = |-y - x|$

$= x + y$

$\iint_D x + y\,dx\,dy = \iint_{D_0} (x+y)(x+y)\,du\,dv$

$= \iint_{D_0} x^2 + 2xy + y^2\,du\,dv$

$v^2 = (x-y)^2 = x^2 - 2xy + y^2$

$4u = 4xy$

$v^2 + 4u = x^2 - 2xy + y^2 + 4xy$

$= x^2 + 2xy + y^2$

$\int_1^4 \int_0^1 v^2 + 4u\,dv\,du = \int_1^4 \frac{1}{3}v^3 + 4uv \Big|_0^1 = \int_1^4 \frac{1}{3} + 4u\,du$

$= \frac{1}{3}u + 2u^2 \Big|_1^4 = \left(\frac{1}{3}\cdot 4 + 32\right) - \left(\frac{1}{3} + 2\right) = 33.\overline{3} - 2.\overline{3}$

$\boxed{= 31}$

/20

3 Problem 3 **20 / 20**

✓ **+ 20 pts** **Correct**

+ 4 pts A good choice of change of variables

+ 2 pts Correct bounds for the new variables

+ 4 pts Correct Jacobian

+ 5 pts Formula for Jac(G) from Jac(G^-1)

+ 3 pts Computing the Integral

+ 2 pts Absolute Value of the Jacobian in the Integral

- 1 pts Small arithmetic mistakes

+ 2 pts Just writing down change of variables formula.

+ 0 pts No progress

- 2 pts G has mistakes in the formula

+ 4 pts Partial progress in polar coordinates.

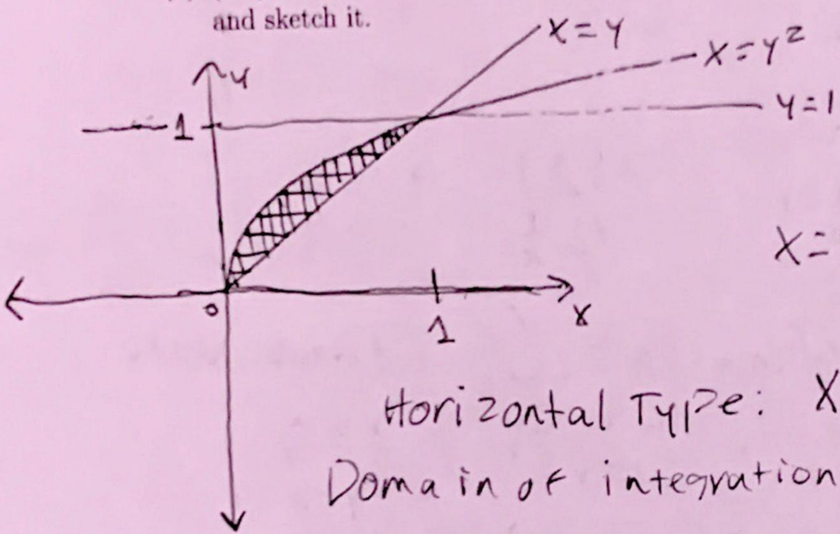
+ 1 pts Partial progress on computing the integral

+ 0 pts Click here to replace this description.

4. Consider the integral

$$\int_0^1 \int_{y^2}^{y} y \, dx \, dy.$$

(a) (10 points) Describe the domain of integration as a simple region of horizontal type and sketch it.



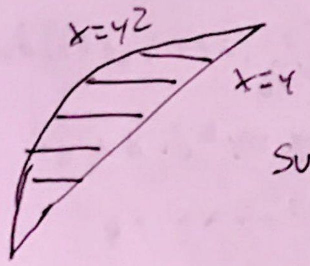
$x=y \quad x=y^2$

Horizontal Type: X between $x=y^2$ and $x=y$

Domain of integration:

$y^2 \leq x \leq y$

$0 \leq y \leq 1$



Sum horizontal segments of the region

/10

4.1 (a) **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 3 pts** Flipped the sketch
- **- 4 pts** Wrong or no sketch
- **- 6 pts** Did not properly define the domain (0<=y<=1,y^2<=x<=y)
- **- 3 pts** Properly defined the functions which give the bounds, but never actually defined the domain.
- **- 2 pts** Did not give the proper y-bounds of the domain (0<=y<=1)
- **- 4 pts** Did not give the proper x-bounds of the domain (y^2<=x<y)
- **- 2 pts** Switched vertically and horizontally simple (points are taken off in both parts).
- **- 10 pts** No points

(b) (10 points) Describe the domain of integration as a simple region of vertical type.

Two boundary lines:

$x = y$ and $x = y^2$

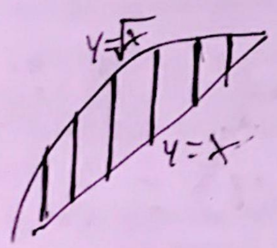
or

$y = x$ and $y = \sqrt{x}$

y between $y = x$ and $y = \sqrt{x}$

$x \leq y \leq \sqrt{x}$

$0 \leq x \leq 1$



sum vertical segments of the region

/10

4.2 (b) **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 10 pts** Did not properly define the domain (0<=x<=1,x<=y<=sqrt(x))
- **- 5 pts** Properly defined the functions which give the bounds, but never actually defined the domain.
- **- 4 pts** Did not give the proper x-bounds of the domain (0<=x<=1)
- **- 6 pts** Did not give the proper y-bounds of the domain (x<=y<=sqrt(x))
- **- 4 pts** Flipped the y-bound (wrote sqrt(x)<=y<=x)
- **- 4 pts** Switched vertically and horizontally simple (points are taken off in both parts).

(c) (10 points) Compute the integral in the order dy dx.

$$\int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx$$

$$\int_0^1 \frac{1}{2} y^2 \Big|_x^{\sqrt{x}} dx$$

$$\int_0^1 \frac{1}{2}\sqrt{x}^2 - \frac{1}{2} x^2 \, dx$$

$$\int_0^1 \frac{x}{2} dx - \int_0^1 \frac{x^2}{2} dx$$

$$\frac{x^2}{4}\Big|_0^1 - \frac{x^3}{6}\Big|_0^1$$

$$\frac{(1)^2}{4} - \frac{(1)^3}{6}$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12}$$

$$= .08\overline{3} = \frac{1}{12}$$

/10

4.3 (c) 10 / 10

- ✓ **- 0 pts** Correct
- **- 0 pts** Correct based on your answer to part (b) (which did not trivialize the problem).
- **- 1 pts** Basic computation mistake
- **- 3 pts** Integrated in the order dx dy
- **- 10 pts** No credit