20W-MATH32B-2 Midterm 1

JEREMY TSAI

TOTAL POINTS

40 / 40

QUESTION 1

1 Question 1.(a) 1 / 1

✓ **- 0 pts** Correct

QUESTION 2

2 Question 1.(b) 1 / 1

✓ **- 0 pts** Correct

QUESTION 3

3 Question 1.(c) 1 / 1

✓ **- 0 pts** Correct

QUESTION 4

4 Question 1.(d) 1 / 1

✓ **- 0 pts** Correct

QUESTION 5

5 Question 1.(e) 1 / 1

✓ **- 0 pts** Correct

QUESTION 6

6 Question 2.(a) 3 / 3

✓ **- 0 pts** Correct

QUESTION 7

7 Question 2.(b) 3 / 3

✓ **- 0 pts** Correct

QUESTION 8

8 Question 2.(c) 5 / 5

✓ **- 0 pts** Correct (Answer: 28/3)

QUESTION 9

9 Question 3.(a) 3 / 3

✓ **+ 3 pts** Fully correct

+ **2 pts** One of theta and r correct and the other one partially correct

+ **1 pts** Both partially correct

+ **0 pts** Nothing correct

+ **1 pts** Missing theta or r

QUESTION 10

10 Question 3.(b) 5 / 5

✓ **+ 2 pts** Correct bounds and set up

+ **1 pts** Partial correct bounds and set up

✓ **+ 1 pts** Inside integral correct

✓ **+ 2 pts** Outside integral correct

+ **1 pts** Outside integral partial correct

+ **0 pts** Nothing correct

- **1 pts** Didn't simplify final result

QUESTION 11

11 Question 4 6 / 6

✓ **+ 6 pts** Correct answer with work

+ **3 pts** Correct bounds

+ **2 pts** Bounds correct for two of three variables

+ **1 pts** Correct work/attempt to solve integral

+ **2 pts** Correct integration factor (if using polar)

+ **0 pts** No correct work

QUESTION 12

12 Question 5.(a) 5 / 5

✓ **- 0 pts** Correct

QUESTION 13

13 Question 5.(b) 5 / 5

✓ **- 0 pts** Correct

Midterm 1

Name: Jeremy Tsai

Student ID: 1 0 5 3 3 5 4 8 4

Section:

Tuesday	Thursday
☐ 2A	☐ 2B
☐ 2C	☑ 2D
☐ 2E	☐ 2F

Instructions:

- Do not open this exam until instructed to do so.
- You have 50 minutes to complete the exam.
- Please print your name and student ID number above and check the box of your discussion section.
- **You may not use calculators**, books, notes, or any other material to help you. Please make sure your **phone is silenced and stowed** where you cannot see it.
- We will only grade your work within the pages that are originally included.
- In each of the questions 2 through 5, you must **show your work** to receive full credit. Please write your solutions in the space below the questions. You must indicate if you go over the pages.

Please do not write below this line.

Question	Points	Score
1	5	
2	11	
3	8	
4	6	
5	10	
Total	40	

1. Each of the following multiple choice questions has exactly one correct answer. Indicate your response by **marking the corresponding box** in each of the questions.

   You do **not** need to show any scratch work, and we will **not** grade your scratch work. No partial credits will be given.

   (a) (1 pt) For any continuous function $f(x, y)$ over $\mathcal{R} = [0, 1] \times [0, 3]$, the integral $\iint_{\mathcal{R}} f(x, y)\, \mathrm{d}A$ is always equal to:

   $\int_0^1 \int_0^3 f(x,y)\, dy\, dx$

   ~~A.~~ $\int_0^3 \left(\int_0^1 f(y, x)\, \mathrm{d}x \right) \mathrm{d}y$

   ~~B.~~ $\int_0^1 \left(\int_0^3 f(x, y)\, \mathrm{d}x \right) \mathrm{d}y$

   C. $\int_0^1 \left(\int_0^3 f(x, y)\, \mathrm{d}y \right) \mathrm{d}x$

   D. $\int_0^3 \left(\int_0^1 f(y, x)\, \mathrm{d}y \right) \mathrm{d}x$

   Your response:

A	B	C	D

   (b) (1 pt) If $\mathcal{R} = [0, 2] \times [0, 2]$, the integral $\iint_{\mathcal{R}} (y - x)\, \mathrm{d}A$ is equal to:

   A. 0

   B. -2

   C. 2

   D. -4

   $\int_0^2 \int_0^2 y - x\, dy\, dx$

   $\int_0^2 \left[\frac{y^2}{2} - xy \right]_0^{2=y} dx$

   $= \int_0^2 2 - 2x\, dx = \left[2x - x^2\right]_0^2 = 0.$

   Your response:

A	B	C	D

   (c) (1 pt) The total mass of the rectangle $0 \le x \le 2$, $0 \le y \le 1$ (in meters) assuming a mass density of $\delta(x, y) = 3x^2 y$ kg/m^2 is equal to

   A. 2 kg

   B. 3 kg

   C. 4 kg

   D. 6 kg

   $\int_0^2 \int_0^1 3x^2 y\, dy\, dx$

   $= \int_0^2 3x^2 dx \int_0^1 y\, dy = \left[x^3\right]_0^2 \left[\frac{y^2}{2}\right]_0^1$

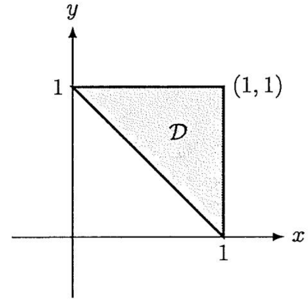
   $= 8 \left(\frac{1}{2}\right) = 4.$

   Your response:

A	B	C	D

(d) (1 pt) The integral $\iint_{\mathcal{D}} 4\,\mathrm{d}A$ over the domain $\mathcal{D}$ defined by $0 \le x \le 1$ and $1 - x \le y \le 1$ is equal to:

A. 1
B. 2
C. 3
D. 4



$\frac{(1*1)}{2}(4) = 2.$

$\int_0^1 \int_{1-x}^1 4\,dy\,dx$

$= \int_0^1 [4y]_{1-x}^{y=1}\,dx = \int_0^1 4 - 4(1-x)\,dx$

$= \int_0^1 4 - 4 + 4x\,dx$

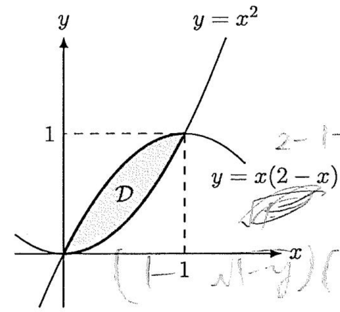
$= [2x^2]_0^1 = 2.$

Your response:

A	B	C	D

(e) (1 pt) If $\mathcal{D}$ is the domain bounded by the curves $y = x^2$ and $y = x(2 - x)$, then $\mathcal{D}$ has the description:

A. $0 \le x \le 1, \quad 0 \le y \le 1$
B. $0 \le y \le 1, \quad 1 - \sqrt{1-y} \le x \le \sqrt{y}$
C. $0 \le y \le 1, \quad \sqrt{y} \le x \le 1 + \sqrt{1-y}$
D. $0 \le x \le 1, \quad x(2-x) \le y \le x^2$



$2 - 1 + \sqrt{1-y}$

$(1 - \sqrt{1-y})(1 + \sqrt{1-y})$

$y = x^2 \quad x = \sqrt{y}. \quad 1 - 1 + y = y$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

$\frac{2 \pm \sqrt{4 - 4y}}{2}$

$= 1 + \sqrt{1-y}$

Your response:

A	B	C	D

2. Let $\mathcal{D}$ be the horizontally simple region described by

$$-1 \leq y \leq 3, \qquad 1 \leq x \leq 1 + \sqrt{y+1}.$$

(a) (3 pts) Sketch the region $\mathcal{D}$ on the graph provided:

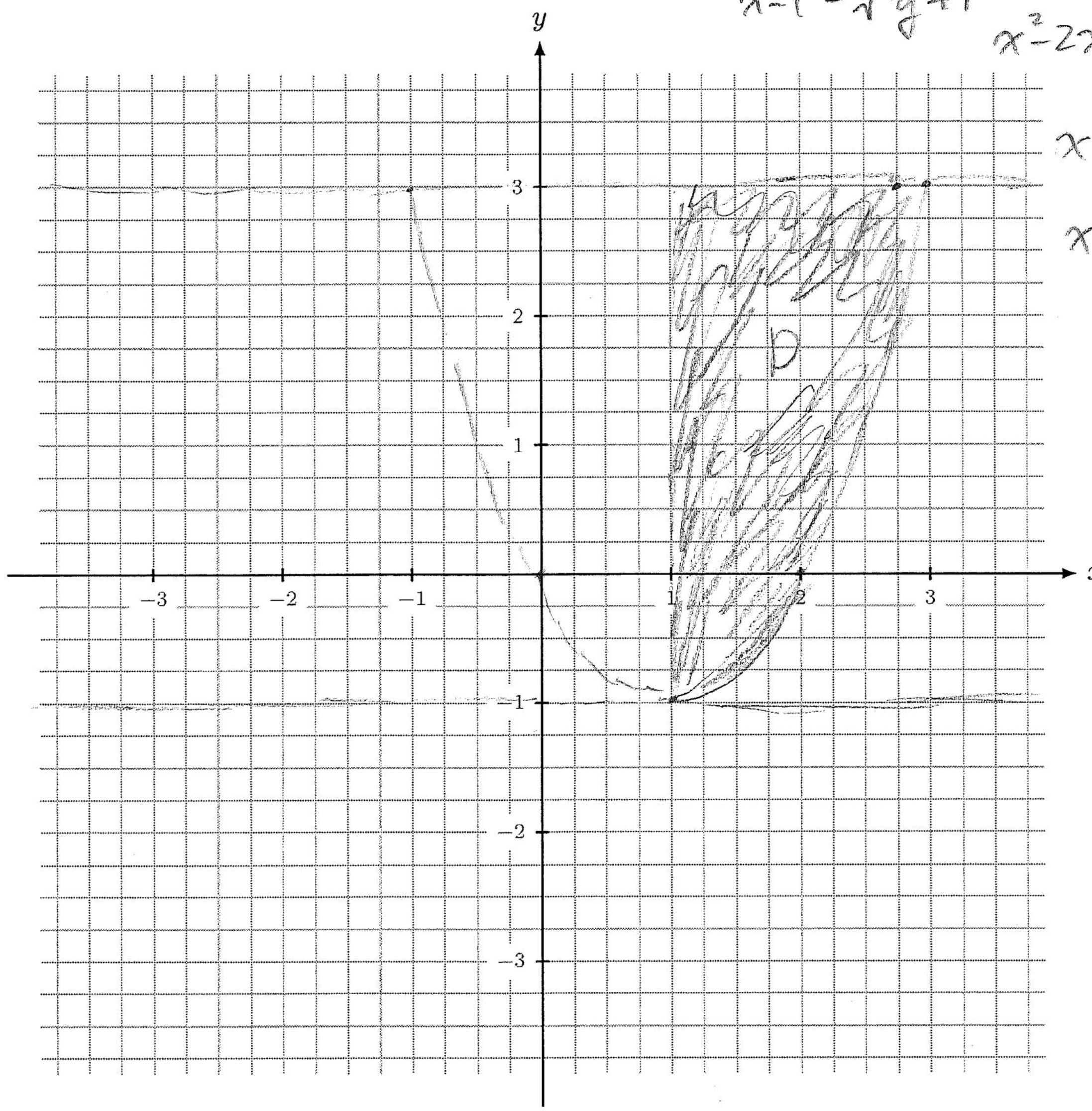
$x = 1 + \sqrt{y+1}$

$x - 1 = \sqrt{y+1}$

$x^2 - 2x + 1 = y + 1$

$x^2 - 2x = y$

$x(x-2) = y$



(b) (3 pts) Express $\mathcal{D}$ as a vertically simple region, i.e., in the form $a \le x \le b,\ g_1(x) \le y \le g_2(x)$.

$$\boxed{\begin{array}{l} 1 \le x \le 3 \\ x^2-2x \le y \le 3. \end{array}}$$

(c) (5 pts) Compute the integral

$$\iint_{\mathcal{D}} x \, dx \, dy$$

$$\iint_D x\,dx\,dy = \int_1^3 \int_{x^2-2x}^{3} x\,dy\,dx = \int_1^3 \left[xy\right]_{y=x^2-2x}^{3} dx$$

$$= \int_1^3 3x - x(x^2-2x)\,dx = \int_1^3 3x - x^3 + 2x^2\,dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^4}{4} + \frac{2}{3}x^3\right]_1^3 = \frac{3}{2}(9) - \frac{81}{4} + \frac{2}{3}(27) - \left(\frac{3}{2} - \frac{1}{4} + \frac{2}{3}\right)$$

$$= \frac{27}{2} - \frac{81}{4} + 18 - \frac{3}{2} + \frac{1}{4} - \frac{2}{3} = \frac{54-81-6+1}{4} - \frac{2}{3} + 18$$

$$= -8 + 18 - \frac{2}{3} = 10 - \frac{2}{3} = \boxed{\frac{28}{3}}$$

3. Consider the region $\mathcal{D}$ in the xy-plane described by the inequalities

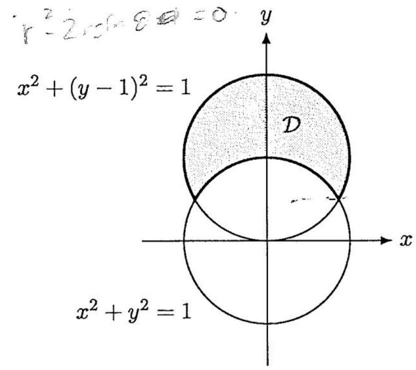
$$x^2 + (y-1)^2 \le 1 \qquad \text{and} \qquad x^2 + y^2 \ge 1.$$

This is the shaded region in the figure below right.

(a) (3 pts) Express $\mathcal{D}$ as a radially simple region, i.e., in the form $\theta_1 \le \theta \le \theta_2,\ r_1(\theta) \le r \le r_2(\theta)$ in polar coordinates.

$4\sin^2\theta - 24\sin$

$r^2 - 2r\sin\theta = 0$



$x^2+y^2-2y+1=1$ and $x^2+y^2=1$

$x^2+y^2-2y=0$ and $x^2+y^2=1$

$1-2y=0 \quad y=\frac{1}{2}$.

Intersect at $y=\frac{1}{2} \to \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$r^2-2r\sin\theta=0 \qquad r=2\sin\theta$

$$\boxed{\frac{\pi}{6} \le \theta \le \frac{5\pi}{6} \quad \text{and} \quad 1 \le r \le 2\sin\theta}$$

(b) (5 pts) Integrate the function $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ over $\mathcal{D}$.

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2\sin\theta} \frac{1}{r}\, r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2\sin\theta} dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2\sin\theta - 1\, d\theta$$

$$= \left[-2\cos\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{-2(-\sqrt{3})}{2} - \frac{5\pi}{6} - \left(-2\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$

$$= \sqrt{3} - \frac{5\pi}{6} + \sqrt{3} + \frac{\pi}{6} = \boxed{2\sqrt{3} - \frac{2\pi}{3}}$$

4. (6 pts) Let $\mathcal{W}$ be the solid region given by $x^2 + y^2 \leq z \leq 4$. Find the volume of $\mathcal{W}$. In other words, compute $\iiint_{\mathcal{W}} 1 \, dV$.

Note: Depending on your approach, the identity $\int \cos^4 \theta \, d\theta = \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C$ *might be helpful.*

$r^2 \leq z \leq 4 \qquad 0 \leq r \leq 2; \quad 0 \leq \theta \leq 2\pi$

$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 1 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left[rz \right]_{r^2}^4 dr \, d\theta$

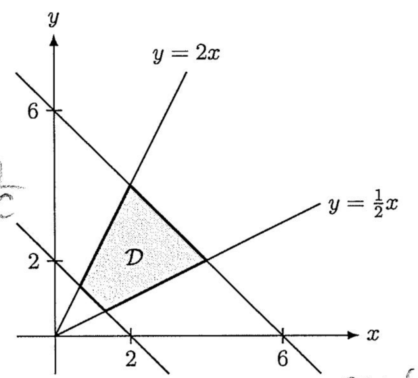
$= \int_0^{2\pi} \int_0^2 4r - r^3 \, dr \, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^2 d\theta$

$= \int_0^{2\pi} 8 - 4 \, d\theta = \boxed{8\pi}$

5. Let $\mathcal{D}$ be the shaded region in the figure below right. We will consider the change of variables

$$(x, y) = G(u, v) = \left(\frac{v}{u(v+1)}, \frac{1}{u(v+1)}\right).$$

(a) (5 pts) Find the region $\mathcal{R}$ in the first quadrant of the uv-plane such that $G(\mathcal{R}) = \mathcal{D}$.



$-x+2$ $\quad$ $-x+6$

$x = \frac{v}{u(v+1)} \quad y = \frac{1}{u(v+1)}$

$u = C \rightarrow x = \frac{v}{c(v+1)}, \; y = \frac{1}{c(v+1)}$

$x + y = \frac{v+1}{c(v+1)} = \frac{1}{c} \Rightarrow y = -x + \frac{1}{c}$

$V = C \rightarrow x = \frac{c}{u(c+1)}, \; y = \frac{1}{u(c+1)}$

$\frac{x}{y} = c \quad y = \frac{x}{c}$

$\frac{1}{2}x \le y \le 2x \rightarrow \boxed{\frac{1}{2} \le v \le 2.}$

$-x+2 \le y \le -x+6 \rightarrow \boxed{\frac{1}{6} \le u \le \frac{1}{2}}$

(b) (5 pts) Apply the Change of Variables Formula to the map G to evaluate the integral

$$\iint_{\mathcal{D}} \frac{1}{x^2}\, dx\, dy.$$

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-v}{(v+1)u^2} & \frac{u(v+1)-vu}{u^2(v+1)^2} \\ \frac{-1}{u^2(v+1)} & \frac{-1}{u(v+1)^2} \end{vmatrix}$$

$$= \frac{-v}{(v+1)u^2}\left(\frac{-1}{u(v+1)^2}\right) - \left(\frac{uv+u-uv}{u^2(v+1)^2}\right)\left(\frac{-1}{u^2(v+1)}\right)$$

$$= \frac{v}{(v+1)^3 u^3} - \left(\frac{\cancel{u}}{u^2(v+1)^2}\right)\left(\frac{-1}{u(v+1)}\right)$$

$$= \frac{v}{(v+1)^3 u^3} + \frac{1}{u^3 (v+1)^3} = \frac{v+1}{(v+1)^3 u^3} = \frac{1}{(v+1)^2 u^3}$$

$$\int_{\frac{1}{6}}^{\frac{1}{2}} \int_{\frac{1}{2}}^{2} \frac{\cancel{u^2(v+1)^2}}{v^2} \frac{1}{\cancel{(v+1)^2} u^3}\, dv\, du = \int_{\frac{1}{6}}^{\frac{1}{2}} \int_{1/2}^{2} \frac{1}{v^2 u}\, dv\, du$$

continued on next page.

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5b cont.) $\int_{\frac{1}{6}}^{1/2} \frac{1}{u}\,du \int_{1/2}^{2} \frac{1}{v^2}\,dv.$

$= \left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{6}\right)\right)\left(\left[\frac{-1}{v}\right]_{1/2}^{2}\right)$

$= (\ln 3)\left(\frac{-1}{2} + 2\right) = \boxed{\frac{3}{2}\ln 3}$

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