32B Lect. 1 Midterm 1 January 31

First Name: [redacted]

Last Name:

Section:

Rules.

- There are **FOUR** problems; ten points per problem.
- There is an extra page at the end. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	Σ
9	4	10	5	28
Killip	Aulla	Khang	Kyung	

(1) Find the center of mass for a homogeneous planar body occupying the region where

$$-1 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq x^4$$

$x_{CM} \dfrac{\iint x \, dA}{\iint 1 \, dA}$ and $y_{CM} = \dfrac{\iint y \, dA}{\iint 1 \, dA}$

$$\iint 1 \, dA = \int_{-1}^{1} \int_{0}^{x^4} 1 \, dy \, dx = \int_{-1}^{1} (y)_0^{x^4} dx = \int_{-1}^{1} x^4 \, dx = \left(\frac{x^5}{5}\right)_{-1}^{1}$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\iint x \, dA = \int_{-1}^{1} \int_{0}^{x^4} x \, dy dx = \int_{-1}^{1} x(x^4) dx = \int_{-1}^{1} x^5 dx = \left(\frac{x^6}{6}\right)_{-1}^{1}$$

$$= \frac{1}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\iint y \, dA = \int_{-1}^{1} \int_{0}^{x^4} y \, dy dx = \int_{-1}^{1} \left(\frac{y^2}{2}\right)_0^{x^4} dx = \int_{-1}^{1} \frac{x^8}{2} dx = \left(\frac{1}{2} \cdot \frac{x^9}{9}\right)_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{9} + \frac{1}{9}\right) = \frac{1}{2}\left(\frac{2}{9}\right) = \frac{1}{9}$$

$$x_{CM} = \frac{5}{2} \cdot \frac{1}{3} = \frac{5}{6}$$

$$y_{CM} = \frac{5}{2} \cdot \frac{1}{9} = \frac{5}{18}$$

$$\boxed{\left(\frac{5}{6}, \frac{5}{18}\right)}$$

Hannah Park

(2) Evaluate the integral

$$\int_0^\infty \int_{\sqrt{y}}^\infty \frac{y}{(x^2+y^2)^2}\,dx\,dy$$

by converting to polar coordinates.

$$\frac{r\sin\theta}{(r^2)^2} = \frac{r\sin\theta}{r^4} = \frac{\sin\theta}{r^3}$$

$0 \le y \le \infty$

$0 \le r\sin\theta \le \infty \to 0 \le r \le \infty$

$\sqrt{y} \le x \le \infty$

$\sqrt{r\sin\theta} \le r\cos\theta \le \infty$

$\frac{r\sin\theta}{r} \le \frac{r^2\cos^2\theta}{r}$

$\sin\theta \le \cos^2\theta\, r$

$\sqrt{y}$, x, θ

$0 \le \theta \le \tan^{-1}\left(\frac{\sqrt{y}}{x}\right)$

$\tan^{-1}\left(\frac{\sqrt{r\sin\theta}}{r\cos\theta}\right)$

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\tan^{-1}\left(\frac{\sqrt{r\sin\theta}}{r\cos\theta}\right)} \frac{\sin\theta}{r^3}\, r\,d\theta\,dr = \int_0^\infty \int_0^{\tan^{-1}\left(\frac{\sqrt{r\sin\theta}}{r\cos\theta}\right)} \frac{\sin\theta}{r^2}\,d\theta\,dr$$

$$= \int_0^\infty -\frac{1}{r^2}\left(\cos\theta\right)\Big|_0^{\tan^{-1}\left(\frac{\sqrt{r\sin\theta}}{r\cos\theta}\right)} dr$$

$y \le x^2$

$\cancel{r\sin\theta \le r^2\cos^2\theta}$

$\cancel{\frac{\sin\theta}{\cos^2\theta} \le r}$

$\sqrt{y} = x$

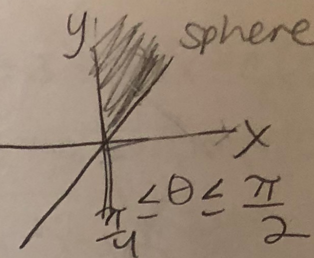
$r\sin\theta = r^2\cos^2\theta$

$r\tan\theta\sec\theta = r$

OR

$$\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\tan\theta\sec\theta} \frac{\sin\theta}{r^2}\,dr\,d\theta$$

4



$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

(3) Determine the volume of the region defined by the following inequalities

$$0 \le x \le y \quad \text{and} \quad x^2 + y^2 + z^2 \le 1$$

$0 \le \rho\sin\phi\cos\theta \le \rho\sin\phi\sin\theta$ $\quad \boxed{0 \le \rho \le 1}$

$0 \le \cos\theta \le \sin\theta$ $\quad \boxed{0 \le \phi \le \pi}$

$0 \le \cot\theta \le 1$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\pi} \left(\frac{\rho^3}{3}\right)\Big|_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\pi} \frac{1}{3} \sin\phi \, d\phi \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{3}(\cos\cancel{\phi})\Big|_0^{\pi} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{3}(\cos\pi - \cos 0) \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{1}{3}(-1-1) \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{3} \, d\theta = \frac{2}{3}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{2}{3}\left(\frac{2\pi}{4} - \frac{\pi}{4}\right) = \frac{\cancel{2}^{1}}{3}\left(\frac{\pi}{\cancel{4}_{2}}\right)$$

$$= \frac{\pi}{6}$$

(4) Consider the region $\mathcal{R}$ defined by the inequalities

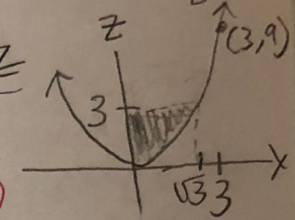
$$0 \le x^2 \le y \le z \le 3$$

(a) Determine the volume of the region $\mathcal{R}$.
(b) Determine the area of the cross-section of $\mathcal{R}$ lying in the plane $y = 1$.

$x^2 \le y \le z$

3 $x^2 \le z$

2



$0 \le x \le \sqrt{3}$

$x^2 \le z \le 3$

a)

$$\text{volume} = \int_{x=0}^{\sqrt{3}} \int_{z=x^2}^{3} \int_{y=x^2}^{z} 1 \, dy\, dz\, dx$$

$$= \int_0^{\sqrt{3}} \int_{x^2}^{3} (z - x^2)\, dz\, dx = \int_0^{\sqrt{3}} \left(\frac{z^2}{2} - x^2 z \right)\Big|_{z=x^2}^{3} dx$$

$$= \int_0^{\sqrt{3}} \left(\frac{9}{2} - 3x^2 - \frac{(x^2)^2}{2} + x^2(x^2) \right) dx = \int_0^{\sqrt{3}} \frac{9}{2} - 3x^2 - \frac{x^4}{2} + x^4 \, dx$$

$$= \int_0^{\sqrt{3}} \frac{9}{2} - 3x^2 + \frac{x^4}{2} \, dx = \left(\frac{9}{2}x - x^3 + \frac{x^5}{10} \right)\Big|_0^{\sqrt{3}}$$

$$\left(\frac{9}{2} \cdot \sqrt{3} - (\sqrt{3})^3 + \frac{(\sqrt{3})^5}{10} \right) = \frac{9\sqrt{3}}{2} - 9\sqrt{3} + \frac{81\sqrt{3}}{10}$$

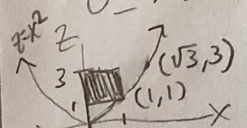
$$= \frac{45\sqrt{3} - 90\sqrt{3} + 81\sqrt{3}}{10} = \frac{-45\sqrt{3} + 81\sqrt{3}}{10} = \frac{36\sqrt{3}}{10} = \boxed{\frac{18\sqrt{3}}{5}}$$

b) $y = 1$, $0 \le x^2 \le 1 \le z \le 3$

$1 \le z \le 3$ $\quad x^2 \le 1 \le z \le 3$

$0 \le x \le 1$

$$\int_{x=0}^{1} \int_{z=1}^{3} 1 \, dz\, dx$$



$$= \int_0^1 (3-1)\, dx = \int_0^1 2\, dx = 2(x)\Big|_0^1 = \boxed{2}$$