Math 32b – Lecture 3 – Winter 2022
Final Exam
Due via Gradescope no later than 8am Pacific daylight time on Thursday Mar/17/2022.

Sign and submit the following honor statement:

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

Signed:

Print name:

This exam contains 10 pages (including this cover page) and 9 problems. There are a total of 135 points available.

- Use extra pages as you need them.
- Attempt all questions.
- The problems are in no particular order.
- The work submitted must be entirely your own: you may not discuss with anyone else, except that...
- You may email the instructor `killip@math.ucla.edu` with any queries about what the questions are asking.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1. (15 points) • Which of the following is a correct expression of the area of a certain triangle?

$$\mathbf{A}: \int_0^x \int_0^1 1\,dy\,dx \qquad \mathbf{B}: \int_0^1 \int_0^x 1\,dy\,dx$$

$$\mathbf{C}: \int_0^1 \int_0^x 1\,dx\,dy \qquad \mathbf{D}: \int_0^x \int_0^1 1\,dx\,dy$$

• Let $\mathcal{R}$ denote the rectangle with vertices

$$(0,0,0), \quad (0,1,0), \quad (3,0,4), \quad \text{and} \quad (3,1,4).$$

What is the value of $\iint_{\mathcal{R}} 1\,dS$?

$\mathbf{A}: \sqrt{3}$ $\quad \mathbf{B}: 2$ $\quad \mathbf{C}: \sqrt{5}$ $\quad \mathbf{D}: 3$ $\quad \mathbf{E}: 4$ $\quad \mathbf{F}: 5$

• The Jacobian for the change of variables $x(u,v) = u - 5v$, $y(u,v) = 2v$ is

$\mathbf{A}: 2$ $\quad \mathbf{B}: 5$ $\quad \mathbf{C}: -1$ $\quad \mathbf{D}: 10$ $\quad \mathbf{E}:$ undefined

2. (15 points) A circular pond is described by the inequalities

$$x^2 + y^2 \leq 1 \quad \text{and} \quad xy - 1 \leq z \leq 0$$

Determine the volume of the pond.

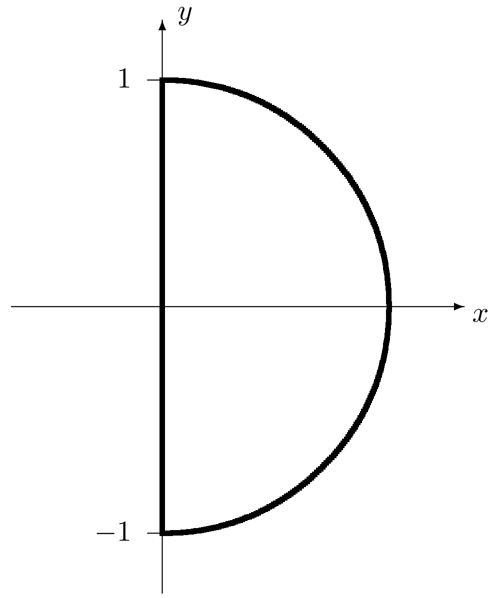
3. (15 points) I traverse a bow-tie shaped path by walking in straight lines between the points

$$(-1,-1), \quad (1,1), \quad (1,-1), \quad (-1,1), \quad \text{and then back to} \quad (-1,-1).$$

Demonstrate the use of Green's Theorem by using it to compute the work I do against the field

$$\vec{F} = \begin{bmatrix} x\cos(y)\sin(y) \\ x^2\cos^2(y) \end{bmatrix}$$

4. (15 points) A fine wire of density 100 g/m is formed into the following D shape comprised of a line segment and a semicircle:



Labels are in meters. Determine the moment of inertia for rotations about the y axis

5. (15 points) An Egyptian-style pyramid $\mathcal{P}$ has corners at the points

$$(-3,-3,0), \quad (-3,+3,0), \quad (+3,+3,0), \quad (+3,-3,0), \quad \text{and} \quad (0,0,7).$$

We orient the faces of the pyramid outwards.

(a) Find limits of integration so as to write $\iiint_{\mathcal{P}} f(x,y,z)\,dV$ as an iterated integral.

(b) We wish to find the flux of the vector field

$$\vec{F} = \begin{bmatrix} 1+z \\ 2+z^2 \\ 3+z^3 \end{bmatrix}$$

through the four triangular faces of the pyramid. Use the divergence theorem to relate this to an integral over the (square) bottom of the pyramid.

(c) Use your result from (b) to compute the sought-after flux.

6. (15 points) For this problem, let $A(\phi) = \dfrac{1}{\sqrt{\sin^2(\phi) - \cos^2(\phi)}}$.

Convert the following integral, given in spherical polar coordinates, to cylindrical coordinates.

$$\int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \sin(\phi)\, d\rho\, d\theta\, d\phi$$

You are **not** expected to evaluate it!

7. (15 points) Consider a pipe parallel with the z-axis given by $x^2 + y^2 \leq 2$. The velocity of a fluid flowing in the pipe is given by

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 4 - x^2 - y^2 \end{bmatrix}$$

(a) Determine $\oint (\nabla \times \vec{v}) \cdot d\vec{r}$ around the loop $x = \cos(\theta)$, $y = \sin(\theta)$, $z = 0$, $0 \leq \theta \leq 2\pi$.

(b) Demonstrate the use of Stokes' Theorem to confirm your answer to part (a).
Be explicit about the surface you use and its orientation!

8. (15 points) Use the change of variables

$$x = e^u \qquad y = ve^u$$

to evaluate the following integral:

$$\int_7^9 \int_x^{x\ln(x)} \frac{y}{x^3}\, dy\, dx$$

9. (15 points) Let $\mathcal{R}$ denote the three-dimensional region where $x^2 + y^2 > 1$ and consider the scalar function

$$f(x, y, z) = \ln(x^2 + y^2) \qquad \text{defined on } \mathcal{R}.$$

(a) Is $\mathcal{R}$ simply connected?

(b) Determine $\vec{F} = \nabla f$

(c) Determine $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$

(d) Compute $\oint \vec{F} \cdot d\vec{r}$ around the circle defined by

$$(x - 3)^2 + y^2 = 1 \quad \text{and} \quad z = 0$$

oriented clockwise when viewed from above (i.e. from a point where $z > 0$).

(e) Is $\vec{F}$ conservative? Justify your answer.

MATH 32B Final

Jason Cheng

March 17, 2022

Honor Statement

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signed: Pinran Cheng

Problem 1

1. Which of the following is a correct expression of the area of a certain triangle?

   Answer: B

2. What is the value of $\iint_R 1 dS$?

   Answer: F

3. The Jacobian for the change of variables $x(u, v) = u - 5v, y(u, v) = 2v$ is

   Answer: A

Problem 2

First, parametrize the volume in cylindrical coordinates:

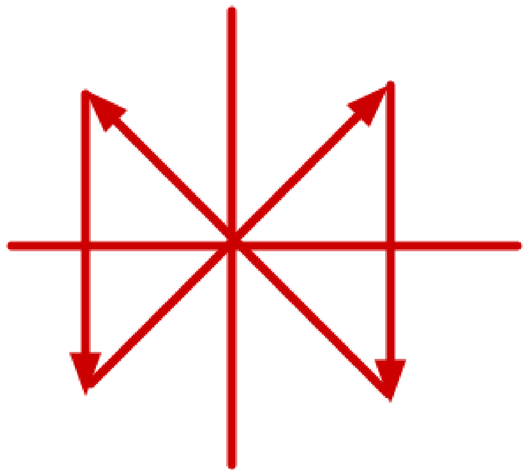
$$x^2 + y^2 \leq 1 \implies r^2 \leq 1 \implies 0 \leq r \leq 1$$
$$xy - 1 \leq z \leq 0 \implies r^2 \sin\theta\cos\theta - 1 \leq z \leq 0$$

Then, integrate using cylindrical coordinates:

$$\begin{aligned}
V &= \iiint_W dV \\
&= \int_0^{2\pi}\int_0^1\int_{r^2\sin\theta\cos\theta-1}^0 r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r\left(1 - r^2\sin\theta\cos\theta\right)dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{2}r^2 - \frac{1}{4}r^2\sin\theta\cos\theta\right]_0^1 d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\sin\theta\cos\theta\right)d\theta \\
&= \pi - \frac{1}{4}\int_0^{2\pi}\sin\theta\, d(\sin\theta) \\
&= \boxed{\pi}
\end{aligned}$$

Problem 3

Path:



We can break this figure down into two regions, D_1 on the left and D_2 on the right. The left loop of the bow-tie is oriented correctly (counterclockwise), so we can directly use ∂D_1 as the loop, while the right loop is clockwise, so we have to subtract ∂D_2.

$$W = \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r}$$

By Green's Theorem, this becomes

$$\begin{aligned} W &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_{D_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA - \iint_{D_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \\ &= \iint_{D_1} x dA - \iint_{D_2} x dA \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x \cos^2 y - x \left(-\sin^2 y + \cos^2 y \right) = x \\ &= \int_{-1}^{0} \int_{x}^{-x} x dy dx - \int_{0}^{1} \int_{-x}^{x} x dy dx \\ &= -\frac{2}{3} - \frac{2}{3} \\ &= -4/3 \end{aligned}$$

This is the work done by the field as you travel along the path, so the work done by you is $\boxed{4/3}$.

Problem 4

We can break the D shape into a vertical part and a semicircular part. The vertical part contributes no rotational inertia since its distance from the y-axis is 0. The circular part can be parametrized by the equation $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $-\pi/2 \leq t \leq \pi/2$.

$$\begin{aligned}
I_y &= \int_l x^2 \rho ds \\
&= \rho \int_{-\pi/2}^{\pi/2} \cos^2 t \cdot ||\mathbf{r}'(t)|| dt \\
&= \rho \int_{-\pi/2}^{\pi/2} \cos^2 t dt && ||\mathbf{r}'(t)|| = ||\langle -\sin t, \cos t \rangle|| = 1 \\
&= \frac{\rho}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2t) dt \\
&= 50 \left[t + \frac{1}{2} \sin 2t \right]_{-\pi/2}^{\pi/2} \\
&= \boxed{50\pi \text{ gm}^2}
\end{aligned}$$

Problem 5

(a) First, the limits of z are easy to find: $0 \le z \le 7$. At each z-coordinate, the cross-section of the pyramid looks like a square centered at $(0,0)$. When $z=0$, the side length is 6, and when $z=7$, the side length is 0. Thus, the side length can be given by a linear function $s = 6 - 6z/7$. Then, the limits of x and y are $-s/2 \le x \le s/2, -s/2 \le y \le s/2 \implies 3z/7 - 3 \le x \le 3 - 3z/7, 3z/7 - 3 \le y \le 3 - 3z/7$.

Limits of integration: $\boxed{0 \le z \le 7, 3z/7 - 3 \le x \le 3 - 3z/7, 3z/7 - 3 \le y \le 3 - 3z/7}$

Iterated integral:

$$\iiint_P f(x,y,z)dV = \int_0^7 \int_{3z/7-3}^{3-3z/7} \int_{3z/7-3}^{3-3z/7} f(x,y,z)dxdydz$$

(b) Let the sides be S_1, S_2, S_3, and S_4, and the bottom be B. We wish to find the flux of the vector field through the four sides, which is

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S}$$

By the Divergence Theorem,

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} + \iint_{B} \mathbf{F} \cdot d\mathbf{S} = \iiint_P \operatorname{div}(\mathbf{F})dV$$

Thus, the flux through the four sides is

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \boxed{\iiint_P \operatorname{div}(\mathbf{F})dV - \iint_{B} \mathbf{F} \cdot d\mathbf{S}}$$

(c) The bottom face can be parametrized by $\mathbf{G}(x,y) = \langle x, y, 0 \rangle, -3 \le x \le 3, -3 \le y \le 3$. Then,

$$\frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} = \langle 0, 0, 1 \rangle$$

This is pointing inside the pyramid, so we use $\mathbf{N}(x,y) = \langle 0, 0, -1 \rangle$ instead.

Furthermore,

$$\operatorname{div}(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 3z^2 = 3z^2$$

$$\begin{aligned}
\iiint_P \operatorname{div}(\mathbf{F})dV - \iint_B \mathbf{F} \cdot d\mathbf{S} &= \int_0^7 \int_{3z/7-3}^{3-3z/7} \int_{3z/7-3}^{3-3z/7} 3z^2 dxdydz - \int_{-3}^3 \int_{-3}^3 \mathbf{F}(\mathbf{G}(x,y)) \cdot \mathbf{N}(x,y)dxdy \\
&= \int_0^7 3z^2 \left(6 - \frac{6}{7}z\right)^2 dz - \int_{-3}^3 \int_{-3}^3 \langle 1, 2, 3 \rangle \cdot \langle 0, 0, -1 \rangle dxdy \\
&= \boxed{6714/5}
\end{aligned}$$

Problem 6

Cylindrical coordinates as a function of spherical coordinates:

$$z = \rho \cos\phi$$
$$r = \rho \sin\phi$$
$$\theta = \theta$$

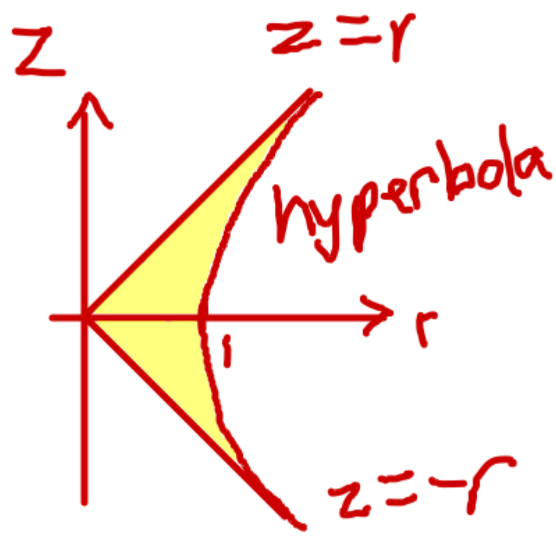
Original integral:

$$\int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \sin\phi d\rho d\theta d\phi = \int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \frac{1}{\rho^2} \cdot \rho^2 \sin\phi d\rho d\theta d\phi$$

Now, we convert the spherical limits into cylindrical ones:

$$0 \le \theta \le 2\pi \text{ remains unchanged.}$$
$$\frac{\pi}{4} \le \phi \le \frac{3\pi}{4} \implies -1 \le \cot\phi \le 1 \implies -1 \le \frac{z}{r} \le 1 \implies -r \le z \le r$$
$$0 \le \rho \le \frac{1}{\sqrt{\sin^2\phi - \cos^2\phi}} \implies \rho^2 \sin^2\phi - \rho^2 \cos^2\phi \le 1 \implies r^2 - z^2 \le 1$$

Sketch of cylindrical limits:



Thus,

$$\int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \sin\phi d\rho d\theta d\phi = \int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \frac{1}{\rho^2} \cdot \rho^2 \sin\phi d\rho d\theta d\phi$$
$$= \boxed{\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{|z|}^{\sqrt{1+z^2}} \frac{1}{r^2 + z^2} \cdot r dr dz d\theta}$$

Problem 7

(a)

$$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle \implies \mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle$$
$$\nabla \times \mathbf{v} = \langle -2y - 0, -(-2x - 0), 0 - 0\rangle = \langle -2y, 2x, 0\rangle = \langle -2\sin\theta, 2\cos\theta, 0\rangle$$

$$\begin{aligned}\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -2\sin\theta, 2\cos\theta, 0\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle d\theta \\ &= \int_0^{2\pi} 2d\theta \\ &= \boxed{4\pi}\end{aligned}$$

(b) Let S be the unit circle in the xy-plane, which is a surface bounded by the loop (call this l) in part (a). Then, by Stokes' Theorem,

$$\oint_l (\nabla \times \mathbf{v}) \cdot d\mathbf{r} = \iint_S \nabla \times (\nabla \times \mathbf{v}) \cdot d\mathbf{S}$$

S is parametrized by $\mathbf{G}(r, \theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$, so

$$\mathbf{N}(r, \theta) = \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = \langle 0, 0, r\rangle$$

Which is correctly oriented since it points in the positive z direction.

Furthermore,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla \times \langle -2y, 2x, 0\rangle = \langle 0, 0, 4\rangle$$

Putting it all together,

$$\begin{aligned}\oint_l (\nabla \times \mathbf{v}) \cdot d\mathbf{r} &= \iint_S \nabla \times (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \\ &= \int_0^{2\pi} \int_0^1 \langle 0, 0, 4\rangle \cdot \langle 0, 0, r\rangle dr d\theta \\ &= 2\pi \left[2r^2\right]_0^1 \\ &= \boxed{4\pi}\end{aligned}$$

Problem 8

Jacobian:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = (e^u)(e^u) - 0 = e^{2u}$$

Limits:

$$\begin{aligned} e^u = 7 &\implies u = \ln 7 \\ e^u = 9 &\implies u = \ln 9 \\ ve^u = e^u &\implies v = 1 \\ ve^u = e^u(\ln e^u) &\implies v = u \end{aligned}$$

$$\begin{aligned} \int_7^9 \int_x^{x \ln x} \frac{y}{x^3} dy dx &= \int_{\ln 7}^{\ln 9} \int_1^u \frac{ve^u}{e^{3u}} \cdot e^{2u} dv du \\ &= \int_{\ln 7}^{\ln 9} \left[\frac{1}{2}v^2\right]_1^u du \\ &= \frac{1}{2}\left[\frac{1}{3}u^3 - u\right]_{\ln 7}^{\ln 9} \\ &= \boxed{\frac{1}{6}(\ln 9)^3 - \frac{1}{6}(\ln 7)^3 - \frac{1}{2}\ln 9 + \frac{1}{2}\ln 7} \end{aligned}$$

Problem 9

(a) R is not simply connected because a loop such as $x^2 + y^2 = 2$ would not be able to be tightened into a point.

(b)

$$\mathbf{F} = \nabla f = \boxed{\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, 0 \right\rangle}$$

(c)

$$\nabla \cdot \mathbf{F} = \boxed{\left\langle \frac{2\left(y^2 - x^2\right)}{\left(x^2+y^2\right)^2}, \frac{2\left(x^2 - y^2\right)}{\left(x^2+y^2\right)^2}, 0 \right\rangle}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle 0, 0, -2y\left(x^2+y^2\right)^{-2} \cdot 2x - \left(-2x\left(x^2+y^2\right)^{-2} \cdot 2y\right) \right\rangle = \boxed{\langle 0,0,0 \rangle}$$

(d) The region bounded by the circle can be parametrized by the surface $\mathbf{G}(r,\theta) = \langle 3 + r\cos\theta, r\sin\theta, 0\rangle, 0 \le r \le 1, 0 \le \theta \le 2\pi$. Then,

$$\mathbf{N}(r,\theta) = \frac{\partial \mathbf{G}}{\partial r} \times \frac{\partial \mathbf{G}}{\partial \theta} = \langle 0, 0, r \rangle$$

However, this is pointing in the positive z direction, which is the wrong orientation, so we use $\mathbf{N}(r,\theta) = \langle 0, 0, -r\rangle$ instead.

By Stokes' Theorem,

$$\begin{aligned} \oint_l \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \\ &= \int_0^{2\pi} \int_0^1 \langle 0,0,0\rangle \cdot \langle 0,0,-r\rangle \, dr\, d\theta \\ &= \boxed{0} \end{aligned}$$

(e) $\mathbf{F}$ is not conservative because its domain is not simply connected.