Math 32B Quiz 6, Thursday Mar 10, 2022

Name: ______________________________ **UID:** ________________________

TA/Section: ______________________________

Clearly denote your answer by putting a box around it.

Many LAs would appreciate hearing your feedback on where and how they improved and/or have more room for improvement. As the first part of this quiz, take a few minutes to fill out this survey about how your LA did this quarter: http://tinyurl.com/W22-LA-Feedback

1. (50 points) Compute the area between the x-axis and the cycloid parametrized by

$$\boldsymbol{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$$

for $0 \leq t \leq 2\pi$.

By the parametrization, we have

$x = t - \sin t$, $dy = \sin t$

By Green's Theorem, the area enclosed by a closed curve is

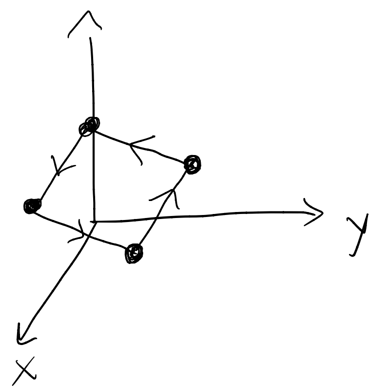
$$\begin{aligned} A &= \oint_r x\,dy \\ &= \int_0^{2\pi} (t - \sin t)\sin t\,dt \\ &= \int_0^{2\pi} (t\sin t - \sin^2 t)\,dt \\ &= \left[-t\cos t + \sin t - \tfrac{1}{2}t + \tfrac{1}{4}\sin(2t)\right]_0^{2\pi} \\ &= -2\pi - \pi \\ &= -3\pi \end{aligned}$$

Notice that $\vec{r}(t)$ parametrizes the cycloid in clockwise orientation, so the above integral gave us the negative of the area.

Thus the area is

$$\boxed{3\pi}$$

2. (25 points) Let C be the square with vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, and $(0,1,1)$, oriented counterclockwise as viewed from above. Let $\boldsymbol{F} = \langle yz, xy, xz \rangle$. Compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$



We will use Stoke's Theorem. First we find curl$(\vec{F})$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xy & xz \end{vmatrix}$$

$$= (0-0)\hat{\imath} - (z-y)\hat{\jmath} + (y-z)\hat{k}$$
$$= \langle 0, y-z, y-z \rangle$$

Let the square be parametrized by $G: \mathbb{R}^2 \mapsto \mathbb{R}^3$ such that

$G(u,v) = (u, v, 1) \quad 0 \le u, v \le 1$

Then

$\vec{T}_u = (1,0,0)$, $\vec{T}_v (0,1,0)$

And

$\vec{N}(u,v) = \vec{T}_u \times \vec{T}_v$
$= \langle 0, 0, 1 \rangle$

Also

$\text{curl}(G(u,v)) = \langle 0, v-1, v-1 \rangle$

Therefore

$\text{curl}(G(u,v)) \cdot \vec{N}(u,v) = \langle 0, v-1, v-1 \rangle \cdot \langle 0, 0, 1 \rangle$
$= v-1$

Finally we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^1 \text{curl}(G(u,v)) \cdot \vec{N}(u,v)\, du\, dv$$
$$= \int_0^1 \int_0^1 (v-1)\, du\, dv$$
$$= \int_0^1 [vu - u]_0^1\, dv$$
$$= \int_0^1 (v-1)\, dv$$
$$= \tfrac{1}{2}v^2 - v \Big]_0^1$$
$$= \tfrac{1}{2} - 1$$
$$\boxed{= -\tfrac{1}{2}}$$

3. (25 points) Let S be the boundary of the box $[0,1]\times[2,4]\times[1,5]$, with inward-pointing normal, and let $\boldsymbol{F} = \langle xy, yz, x^2z + z^2\rangle$. Compute $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$.

$$\iint_S F\cdot dS = \iiint_W \operatorname{div}(F)\, dV$$

$$\begin{aligned}\operatorname{div}(F) &= \nabla\cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle xy, yz, x^2z+z^2\rangle \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(x^2z+z^2) \\ &= y + z + x^2 + 2z \\ &= y + x^2 + 3z\end{aligned}$$

$$\begin{aligned}\iint_S F\cdot ds &= \int_0^1\int_2^4\int_1^5 x^2 + y + 3z\, dz\,dy\,dx \\ &= \int_0^1\int_2^4 x^2z + yz + \frac{3}{2}z^2 \Big|_1^5\, dy\,dx \\ &= \int_0^1\int_2^4 5x^2 + 5y + \frac{3}{2}(5^2) - x^2 + y + \frac{3}{2}\, dy\,dx \\ &= \int_0^1\int_2^4 4x^2 + 4y + 36\, dy\,dx \\ &= \int_0^1 4x^2y + 2y^2 + 36y \Big|_2^4\, dx \\ &= \int_0^1 16x^2 + 32 + 144 - 8x^2 - 8 - 72\, dx \\ &= \int_0^1 8x^2 + 96\, dx \\ &= \frac{8}{3}x^3 + 96x \Big|_0^1 \\ &= \frac{8}{3} + 96 = \frac{296}{3}\end{aligned}$$

Since the normal vectors point inwards, we take the negative and

$$\boxed{\iint_S F\cdot ds = -\frac{296}{3}}$$

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Reflection:

5. For each problem, which of the following describes your quiz experience today?

   (a) I did the problem correctly by myself.

   (b) I thought I did the problem correctly, but working in groups clarified a mistake or misunderstanding.

   (c) I did not know how to do the problem correctly, but I think I understand how to do it now.

   (d) I still don't completely understand how to do the problem.

   (e) Other (feel free to describe below).