Midterm #2

Friday, February 25, 2022 12:16 PM

1. Consider the vector field defined on $\mathbb{R}^2 - \{(1,2)\}$ (that is, everywhere except the point $(1,2)$) given by

$$\boldsymbol{F} = \left\langle \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}}, \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} \right\rangle$$

(a) (10 points) Is $\boldsymbol{F}$ a conservative vector field on the domain $\mathbb{R}^2 - \{(1,2)\}$? Justify your answer.

The cross-partials condition cannot be applied since the domain is not simply connected; there is a hole at $(1,2)$.

Determine a potential function for $\vec{F}$:

$$\int \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}}\,\partial x = \sqrt{(x-1)^2+(y-2)^2} + \cancel{f(y)}\ c$$

$u = x-1$
$\partial u = \partial x$ $\displaystyle\int \frac{u}{\sqrt{u^2+(y-2)^2}}\,\partial u$

$$\int \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}}\,\partial y = \sqrt{(x-1)^2+(y-2)^2} + \cancel{g(x)}\ c$$

$$f = \sqrt{(x-1)^2+(y-2)^2} + C$$

The potential function is defined at $(1,2)$. Furthermore, it is defined everywhere else in $\mathbb{R}^2$. Using this potential function, it can be shown that $\oint_c \vec{F}\cdot\partial\vec{r} = 0$ for any path since

$$f(a,b) - f(a,b) = \sqrt{(a-1)^2+(b-2)^2} - \sqrt{(a-1)^2+(b-2)^2} = 0$$

Since the circulation is zero for every closed path, $\vec{F}$ is indeed a conservative vector field.

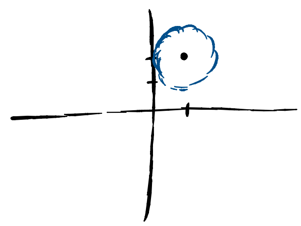
(b) (7 points) Compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for C the straight line path from $(0,0)$ to $(3,3)$.

$$f(3,3)-f(0,0) = \sqrt{(3-1)^2+(3-2)^2} - \sqrt{(0-1)^2+(0-2)^2}$$

$$= \sqrt{4+1} - \sqrt{1+4}$$

$$= \boxed{0}$$

(c) (8 points) Compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for C the circle of radius 1, centered at $(1,2)$, oriented counterclockwise.



Since it was determined that $\vec{F}$ is conservative (despite not being simply connected), $\boxed{\oint_C \vec{F}\cdot d\vec{r} = 0}$

2. Consider the vector field

$$\boldsymbol{F} = \langle yz^2, xz^2, 2xyz\rangle$$

(a) (10 points) Compute curl($\boldsymbol{F}$).

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial z}(xz^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial z}(yz^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial y}(yz^2)\right)$$

$$= \hat{i}(2xz-2xz) - \hat{j}(2yz-2yz) + \hat{k}(z^2-z^2)$$

$$= \boxed{\langle 0,0,0\rangle}$$

(b) (5 points) Compute div(curl($\boldsymbol{F}$)).

$$\operatorname{div}(\operatorname{curl}(\vec{F})) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 0,0,0 \rangle$$

$$= \boxed{0}$$

(c) (10 points) Further consider the vector field

$$\boldsymbol{G} = \langle ye^{xy}, xe^{xy}, 2z \rangle$$

Compute $\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G})$.

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz^2 & xz^2 & 2xyz \\ ye^{xy} & xe^{xy} & 2z \end{vmatrix} = \hat{i}(2xz^3 - 2x^2yze^{xy}) - \hat{j}(2yz^3 - 2y^2xze^{xy}) + \hat{k}(xyz^2e^{xy} - xyz^2e^{xy})$$

$$= \langle 2xz^3 - 2x^2yze^{xy}, 2y^2xze^{xy} - 2yz^3, 0 \rangle$$

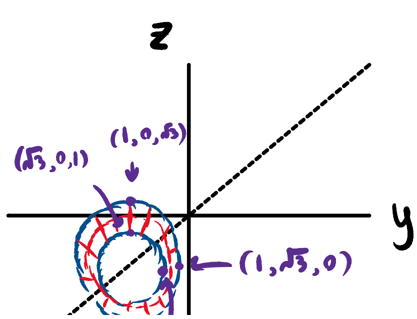
$$\operatorname{div}(\vec{F} \times \vec{G}) = \frac{\partial}{\partial x}(2xz^3 - 2x^2yze^{xy}) + \frac{\partial}{\partial y}(2y^2xze^{xy} - 2yz^3) + \frac{\partial}{\partial z}(0)$$

$$= (\cancel{2z^3} - \cancel{4xyze^{xy}} - \cancel{2x^2y^2ze^{xy}}) + (\cancel{4xyze^{xy}} + \cancel{2x^2y^2ze^{xy}} - \cancel{2z^3})$$

$$= \boxed{0}$$

3. Let S be the surface that is the portion of the sphere $x^2+y^2+z^2 = 4$ where $1 \le y^2+z^2 \le 3$, and $x \ge 0$.

(a) (5 points) Sketch the surface S, and label some points on the surface.



Let $z=0$:

$1 \le y^2 \le 3$

$1 \le y \le \sqrt{3}$

$x^2 + y^2 = 4$

$(\sqrt{3}, \pm 1, 0)$

$(1, \pm\sqrt{3}, 0)$

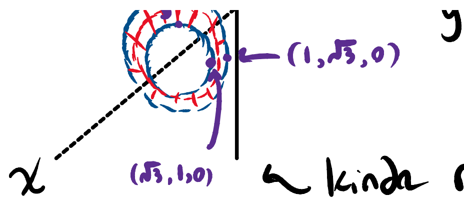
Let $y=0$:

$1 \le z \le \sqrt{3}$

$x^2 + z^2 = 4$

$(\sqrt{3}, 0, \pm 1)$

$(1, 0, \pm\sqrt{3})$



← kinda ring-shaped

$(\sqrt{3}, \pm 1, 0)$
$(1, \pm\sqrt{3}, 0)$

$(1, 0, \pm\sqrt{3})$

(b) (10 points) Find a parametrization of S. What coordinate system does this correspond to?

$$g(\theta,\phi) = \boxed{\langle 2\cos\phi\ ,\ 2\sin\theta\sin\phi, 2\cos\theta\sin\phi \rangle}$$ ← spherical coordinates

$$1 \leq 4\sin^2\theta\sin^2\phi + 4\cos^2\theta\sin^2\phi \leq 3$$

$$1 \leq 4\sin^2\phi \leq 3$$

$$\frac{1}{4} \leq \sin^2\phi \leq \frac{3}{4}$$

$$\frac{1}{2} \leq \sin\phi \leq \frac{\sqrt{3}}{2}$$

$$\boxed{\begin{array}{c} \frac{\pi}{6} \leq \phi \leq \frac{\pi}{3} \\ 0 \leq \theta \leq 2\pi \end{array}}$$ ← $x \geq 0$ ✓

(c) (10 points) Write down an integral expressing the surface area of S, and compute the surface area of S.

$$T_\theta = \langle 0, 2\cos\theta\sin\phi, -2\sin\theta\sin\phi \rangle$$

$$T_\phi = \langle -2\sin\phi, 2\sin\theta\cos\phi, 2\cos\theta\cos\phi \rangle$$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2\cos\theta\sin\phi & -2\sin\theta\sin\phi \\ -2\sin\phi & 2\sin\theta\cos\phi & 2\cos\theta\cos\phi \end{vmatrix}$$

$$= \hat{i}(4\cos\phi\sin\phi) - \hat{j}(-4\sin\theta\sin^2\phi) + \hat{k}(4\cos\theta\sin^2\phi)$$

$$= \langle 4\cos\phi\sin\phi,\ 4\sin\theta\sin^2\phi, 4\cos\theta\sin^2\phi \rangle$$

$$= 4\sin\phi \langle \cos\phi, \sin\theta\sin\phi, \cos\theta\sin\phi \rangle$$

$$\|\vec{N}\| = 4\sin\phi\sqrt{\cos^2\phi \ \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi}$$

$$= 4\sin\phi\sqrt{\cos^2\phi + \sin^2\phi}$$

$$= 4\sin\phi$$

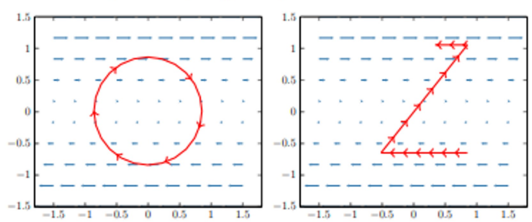
$$SA = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin\phi \, \partial\phi \, \partial\theta$$

$$= 2\pi\left(4\left(-\cos\phi\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\right)$$

$$= 8\pi\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \boxed{8\pi\left(\frac{\sqrt{3}-1}{2}\right)}$$

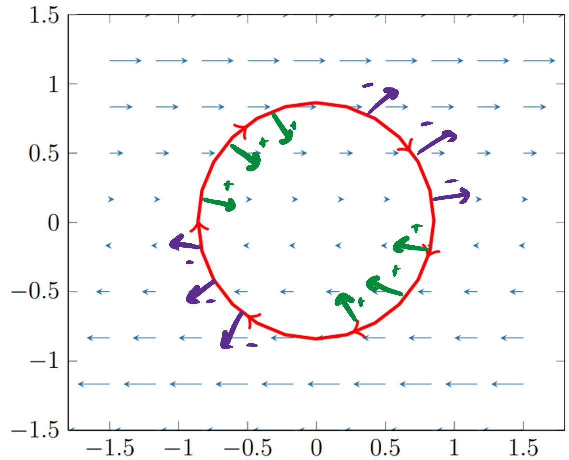
4. **You must explain your reasoning.**

(a) (10 points) Determine whether the **flux** line integrals of the vector fields along the given oriented curves are positive, negative, or zero.

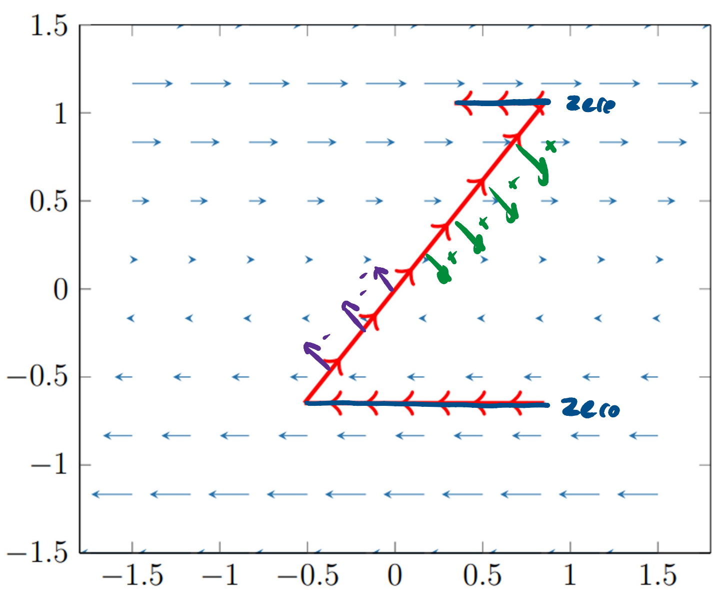


For the left curve, the flux line integral is zero.

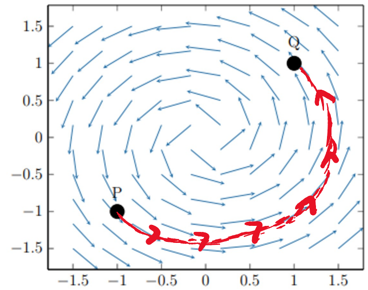
Going along the path, the normal component of the field accumulates but then cancels out with an equal but opposite direction normal component of $\vec{F}$, resulting in an overall flux of zero.



For the right curve, the flux should be positive. The straight horizontal sections are only tangent to the field, so they yield a flux of zero. However, the sloped section has normal components. The section with $y<0$ has the field with normal components opposite to the positive direction, but the section with $y>0$ has normal components in the same direction. The section with $y>0$ is slightly longer than the section with $y<0$, so the flux is positive.

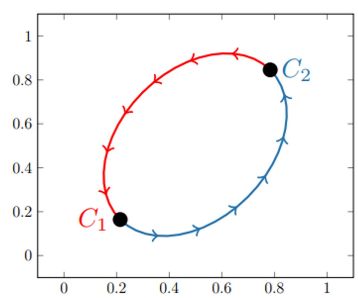


(b) (10 points) Draw an oriented curve C from P to Q such that $\int_C (\boldsymbol{F}\cdot\boldsymbol{n})\,ds = 0$, but $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} > 0$.



The curve is drawn in a circular manner such that the curve is always tangent and in the same direction as the vector field, resulting in $\int_C \vec{F}\cdot\partial\vec{r} > 0$. Since the curve is always tangent, the normal component of $\vec{F}$ along the curve is always zero, so $\int_C (F\cdot\vec{n})\,\partial\vec{s} = 0$

(c) (5 points) Write down a vector field such that for the two curves C_1 (left) and C_2 (right), we have $\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} \neq \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}$.



If $\vec{F}$ is a conservative vector field, then

$\int_{C_1}\vec{F}\cdot\partial\vec{r} + \int_{C_2}\vec{F}\cdot\partial\vec{r} = 0 \rightarrow \int_{C_1}\vec{F}\cdot\partial\vec{r} = -\int_{C_2}\vec{F}\cdot\partial\vec{r}$,

implying that $\int_{C_1}\vec{F}\cdot\partial\vec{r} \neq -\int_{C_2}\vec{F}\cdot\partial\vec{r}$

A constant vector field is conservative, so

$\boxed{\vec{F} = \langle 1, 1 \rangle}$ can be used.