1. Consider the vector field defined on $\mathbb{R}^2 - \{(1,2)\}$ (that is, everywhere except the point $(1,2)$) given by

$$\boldsymbol{F} = \left\langle \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}}, \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} \right\rangle$$

(a) (10 points) Is $\boldsymbol{F}$ a conservative vector field on the domain $\mathbb{R}^2 - \{(1,2)\}$? Justify your answer.

Let $r(x,y) = \sqrt{(x-1)^2+(y-2)^2}$, $r_x(x,y) = \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}} = \frac{x-1}{r(x,y)}$

$r_y(x,y) = \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} = \frac{y-2}{r(x,y)}$

$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x-1}{r(x,y)}\right) = \frac{r(x,y)(0) - (x-1)r_y(x,y)}{(r(x,y))^2} = \frac{1}{r(x,y)} - \frac{(x-1)r_y(x,y)}{r(x,y)^2} = \frac{1}{r(x,y)} - \frac{(x-1)(y-2)}{(r(x,y))^3}$

$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y-2}{r(x,y)}\right) = \frac{r(x,y)(0) - (y-2)r_x(x,y)}{(r(x,y))^2} = \frac{1}{r(x,y)} - \frac{(y-2)r_x(x,y)}{(r(x,y))^2} = \frac{1}{r(x,y)} - \frac{(y-2)(x-1)}{(r(x,y))^3}$.

$\Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow$ F satisfies the cross-partials condition $\Rightarrow$ **F is conservative**

(b) (7 points) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for C the straight line path from $(0,0)$ to $(3,3)$.

Potential Function: $f(x,y) = r(x,y) = \sqrt{(x-1)^2+(y-2)^2}$

$\nabla f = \langle r_x(x,y), r_y(x,y) \rangle = \left\langle \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}}, \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} \right\rangle = F$

$\int_C F \cdot dr = f(Q) - f(P) = f(3,3) - f(0,0) = \sqrt{(3-1)^2+(3-2)^2} - \sqrt{(0-1)^2+(0-2)^2} = \sqrt{4+1} - \sqrt{1+4} = \boxed{0}$

(c) (8 points) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for C the circle of radius 1, centered at $(1,2)$, oriented counterclockwise.

Since F is conservative and C is a closed path,

$\boxed{\int_C F \cdot dr = 0}$

2. Consider the vector field

$f = xyz^2$

$$\boldsymbol{F} = \langle yz^2, xz^2, 2xyz \rangle$$

(a) (10 points) Compute curl($\boldsymbol{F}$).

$$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix} = \langle 2xz - 2xz, 2yz - 2yz, z^2 - z^2 \rangle = \boxed{\langle 0, 0, 0 \rangle}.$$

(b) (5 points) Compute div(curl($\boldsymbol{F}$)).

$$\text{div}(\text{curl}(F)) = \text{div}(\langle 0,0,0 \rangle) = \nabla \cdot \langle 0,0,0 \rangle = \boxed{0}$$

(c) (10 points) Further consider the vector field

$$\boldsymbol{G} = \langle ye^{xy}, xe^{xy}, 2z \rangle$$

Compute div($\boldsymbol{F} \times \boldsymbol{G}$).

$$F \times G = \begin{vmatrix} i & j & k \\ yz^2 & xz^2 & 2xyz \\ ye^{xy} & xe^{xy} & 2z \end{vmatrix} = \langle 2xz^3 - 2x^2yze^{xy}, 2xy^2ze^{xy} - 2yz^3, xyz^2e^{xy} - xyz^2e^{xy} \rangle$$

$$= \langle 2xz^3 - 2x^2yze^{xy}, 2xy^2ze^{xy} - 2yz^3, 0 \rangle$$

$$\text{div}(F \times G) = \nabla \cdot (F \times G) = \frac{\partial}{\partial x}(2xz^3 - 2x^2yze^{xy}) + \frac{\partial}{\partial y}(2xy^2ze^{xy} - 2yz^3) + \frac{\partial}{\partial z}(0)$$

$$= 2z^3 - ((4xyz)e^{xy} + (2x^2yz)e^{xy}) + ((4xyz)e^{xy} + (2xy^2z)e^{xy} - 2z^3) + 0$$

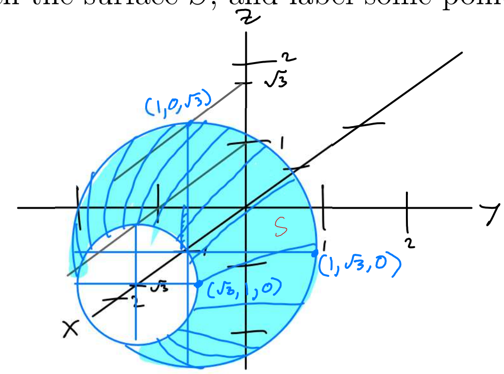
$$= 2z^3 - 4xyze^{xy} - 2x^2yze^{xy} + 4xyze^{xy} + 2xy^2ze^{xy} - 2z^3$$

$$= 2xy^2ze^{xy} - 2x^2yze^{xy}$$

$$= \boxed{(y-x)(2xyze^{xy})}$$

3. Let S be the surface that is the portion of the sphere $x^2+y^2+z^2=4$ where $1 \le y^2+z^2 \le 3$, and $x \ge 0$.

   (a) (5 points) Sketch the surface S, and label some points on the surface.



$x^2+3=4 \Rightarrow x=1$

$x^2+1=4 \Rightarrow x=\sqrt{3}$

   (b) (10 points) Find a parametrization of S. What coordinate system does this correspond to?

Using spherical coordinates: (switching x and y in spherical coordinates lets us pick easier bounds)

with $\rho=\sqrt{4}=2$, $x=2\cos\phi$, $y=2\sin\theta\sin\phi$, $z=2\cos\theta\sin\phi$

$(\sqrt{3},1,0) \Rightarrow 2\cos\phi=\sqrt{3} \Rightarrow \phi=\frac{\pi}{6}$

$(1,\sqrt{3},0) \Rightarrow 2\cos\phi=1 \Rightarrow \phi=\frac{\pi}{3}$

$$\boxed{G(\theta,\phi)=(2\cos\phi,\ 2\sin\theta\sin\phi,\ 2\cos\theta\sin\phi) \text{ for } \mathcal{D}=\begin{cases}0\le\theta\le 2\pi\\ \frac{\pi}{6}\le\phi\le\frac{\pi}{3}\end{cases}}$$

   (c) (10 points) Write down an integral expressing the surface area of S, and compute the surface area of S.

$T_\theta=\langle 0,\ 2\cos\theta\sin\phi,\ -2\sin\theta\sin\phi\rangle$

$T_\phi=\langle -2\sin\phi,\ 2\sin\theta\cos\phi,\ 2\cos\theta\cos\phi\rangle$

$N(\theta,\phi)=T_\theta\times T_\phi$

$=\langle 4\cos^2\theta\cos\phi\sin\phi+4\sin^2\theta\sin\phi\cos\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\theta\sin^2\phi\rangle$

$=\langle 4\cos\phi\sin\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\theta\sin^2\phi\rangle$

$\|N(\theta,\phi)\|=\sqrt{16\cos^2\phi\sin^2\phi+16\sin^2\theta\sin^4\phi+16\cos^2\theta\sin^4\phi}$

$=\sqrt{16\cos^2\phi\sin^2\phi+16\sin^4\phi}$

$=\sqrt{16\sin^2\phi}$

$=4\sin\phi$

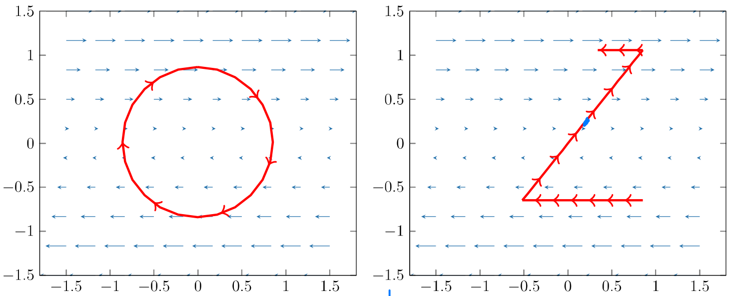
$\text{area}(S)=\iint_S dS=\iint_{\mathcal{D}}\|N(\theta,\phi)\|\,d\theta\,d\phi=\boxed{\int_{\phi=\frac{\pi}{6}}^{\pi/3}\int_{\theta=0}^{2\pi}4\sin\phi\,d\theta\,d\phi}$

$=\int_{\phi=\frac{\pi}{6}}^{\pi/3}8\pi\sin\phi\,d\phi$

$=-8\pi\cos\phi\Big|_{\pi/6}^{\pi/3}=(-4\pi+4\pi\sqrt{3})=\boxed{4\pi(\sqrt{3}-1)}$

4. **You must explain your reasoning.**

(a) (10 points) Determine whether the **flux** line integrals of the vector fields along the given oriented curves are positive, negative, or zero.

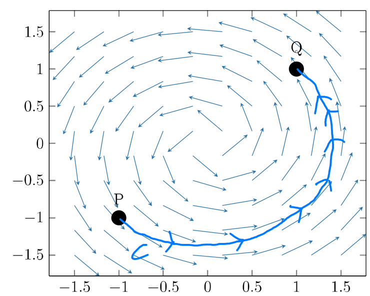


The normal components of the vector field along the top half of the circle evenly cancel out the normal components along the bottom half of the circle. Thus, flux line integral is zero.

The top and bottom of the line are parallel to the vector field, so we must only consider the diagonal portion of the line. The vectors toward the top half appear to be longer than those toward the bottom half, so the flux line integral is positive.

(b) (10 points) Draw an oriented curve C from P to Q such that $\int_C (\boldsymbol{F} \cdot \boldsymbol{n})\, ds = 0$, but $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} > 0$.

C runs parallel to the vectors of F, so the normal component of the field at any point along C is 0. Thus $\int_C (F \cdot n) ds = 0$.



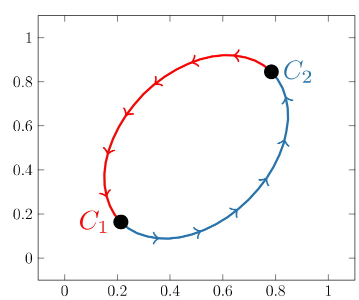
C is oriented such that the tangential component of the field points in the same direction as C. Thus, $\int_C F \cdot dr > 0$.

(c) (5 points) Write down a vector field such that for the two curves C_1 (left) and C_2 (right), we have $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} \neq \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$.

$F = \langle 2xy^2, 2x^2y \rangle$

If F is conservative, then

$\int_{C_1} F \cdot dr = -\int_{C_2} F \cdot dr \neq \int_{C_2} F \cdot dr$.



Observe

$\frac{\partial F_1}{\partial y} = 4xy = \frac{\partial F_2}{\partial x}$. Thus, F is conservative and $\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$.