22W-MATH-32B-LEC-4 Midterm 1

[redacted]

TOTAL POINTS

[redacted]

QUESTION 1

25 pts

1.1 10 / 10

✓ **- 0 pts** Correct

- 1 pts Incorrect bound in one region (while using a 4 region decomposition)

- 2 pts Correctly wrote integrals to describe the area of vertically simple regions; but did not answer the question.

- 4 pts Incorrect bounds in two regions (while using a 4 region decomposition)

- 7.5 pts Correct graph, but incorrect bounds

- 2 pts Swapped horizontally simple and vertically simple with correct bounds

- 2.5 pts Incorrect graph, hence incorrect bounds

1.2 10 / 10

✓ **- 0 pts** Correct

- 1 pts Incorrect bound in one region (while using a 4 region decomposition)

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- 4 pts Incorrect bounds in two regions (while using a 4 region decomposition)

- 7.5 pts Correct graph, but incorrect bounds

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1.3 5 / 5

✓ **- 0 pts** Correct

- 1 pts Incorrect region of integration

- 1 pts Incorrect numerical relationship between integrals

- 2 pts Did not identify symmetry of the function x^2 + y^2.

- 2 pts Described a relationship between the domains, but did not describe a relationship between the integrals.

- 4 pts Identified the region of integration correctly, but did not describe any relationship between the integrals or the domains.

- 5 pts Incorrect interpretation of the integrals.

- 2.5 pts Correctly identified domain of integration, but incorrectly changed the definition of R in the problem.

QUESTION 2

25 pts

2.1 5 / 5

✓ **+ 5 pts** Correct sketch of region

+ 2.5 pts Small mistake when sketching region

+ 0 pts Incorrect

2.2 3.5 / 5

✓ **+ 1 pts** Outer boundary radial description

✓ **+ 1 pts** Outer boundary angular desciption

✓ **+ 1.5 pts** Inner boundary radial description

+ 1.5 pts Inner boundary angular description

- 1 pts Minor description mistake

+ 0 pts Incorrect

(1) The lower bound of $$3\sin(\theta)$$ is only valid for $$0 \leq \theta \leq \pi$$ while the upper bound of $$3$$ is valid for all $$\theta$$.

2.3 10 / 15

+ 5 pts Separated integral into simple regions

✓ **+ 3 pts** Set up integral bounds

✓ **+ 3 pts** Converted to polar coordinates

✓ + **4 pts** **Correct evaluation of integral**

- **2 pts** Minor integration mistake

- **1 pts** Minor arithmetic mistake

(2) For the region $$\pi \leq \theta \leq 2\pi$$, $$r$$ only varies from $$0$$ to $$3$$ because $$3\sin(\theta) < 0$$. So you needed to split this integral into two portions.

QUESTION 3

25 pts

3.1 **0 / 5**

+ **5 pts** Correct bounds in part (a)

✓ + **0 pts** **Wrong bounds**

3.2 **10 / 10**

✓ + **10 pts** **Correct answer using bounds in part (a)**

+ **0 pts** None of the above

3.3 **10 / 10**

✓ + **10 pts** **Correctly set up integral and answer using bounds in part (a) and answer from part (b)**

+ **0 pts** None of the above

QUESTION 4

25 pts

4.1 **7 / 7**

✓ - **0 pts** **Correct**

- **1 pts** Misc. minor error

- **4 pts** Some correct work but wrote down wrong map (e.g. inverse)

- **4 pts** Misread question: mapped $$[0, 1] \times [0, 2\pi]$$ to the ellipse using scaled polar coordinates instead

- **7 pts** Blank / minimal progress

4.2 **8 / 8**

✓ - **0 pts** **Correct ($$ab$$)**

- **0 pts** OK given answer to part (a)

- **2 pts** Difference instead of product

- **2 pts** Jacobian is upside down (map in (a) is correct)

- **3 pts** Some entries wrong in matrix

- **8 pts** Blank / minimal progress

4.3 **10 / 10**

✓ - **0 pts** **Correct**

- **0 pts** OK given (a), (b)

- **1 pts** Minor error

- **3 pts** Wrong bounds of integration

- **3 pts** Incomplete computation

- **5 pts** Wrong integral

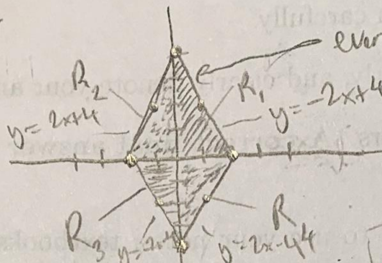
- **10 pts** Blank / minimal progress

1. Consider the region R defined by

$$R = \{(x, y) \mid 2|x| + |y| \leq 4\}$$

(a) (10 points) Express R as a (union of) vertically simple region(s).

$2|x|+|y| \leq 4$



$R_1: \begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq -2x+4 \end{cases}$ $R_3: \begin{cases} -2 \leq x \leq 0 \\ -2x-4 \leq y \leq 0 \end{cases}$

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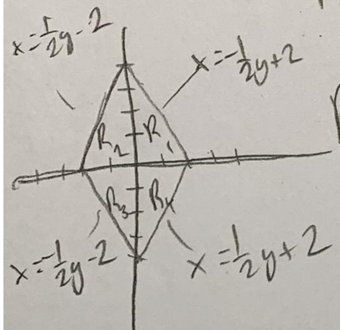
$R = R_1 \cup R_2 \cup R_3 \cup R_4$

(b) (10 points) Express R as a (union of) horizontally simple region(s).



$R_1: \begin{cases} 0 \leq y \leq 4 \\ 0 \leq x \leq -\frac{1}{2}y+2 \end{cases}$ $R_3: \begin{cases} -4 \leq y \leq 0 \\ -\frac{1}{2}y-2 \leq x \leq 0 \end{cases}$

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(c) (5 points) Describe the relationship between

$$\iint_R x^2 + y^2 \, dA \text{ and } \int_{-2}^{0} \int_{0}^{2x+4} x^2 + y^2 \, dy \, dx$$

$\int_{-2}^{0} \int_{0}^{2x+4} x^2+y^2 \, dy\,dx$ is the integral of x^2+y^2 for the region R_2. The region R_2 is 1/4 of the total region R. Since $f(x)=x^2$ is an even function, $(-x)^2=x^2$ and $(-y)^2=y^2$. Therefore, $\int_{-2}^{0}\int_{0}^{2x+4} x^2+y^2 \, dy\,dx$ is exactly one fourth of $\iint_R x^2+y^2 \, dA$.

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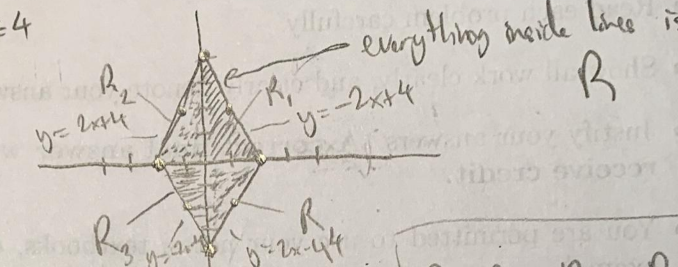

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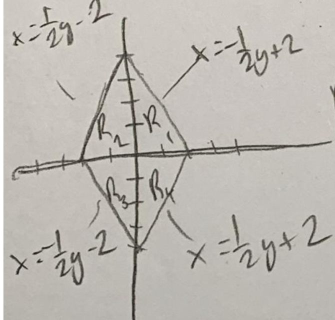

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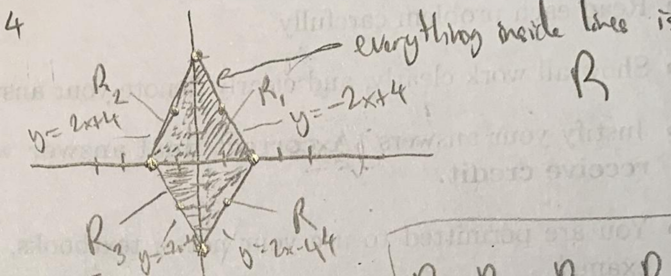

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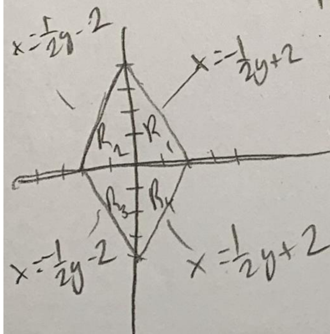

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1.3 **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Incorrect region of integration

- 1 pts Incorrect numerical relationship between integrals

- 2 pts Did not identify symmetry of the function x^2 + y^2.

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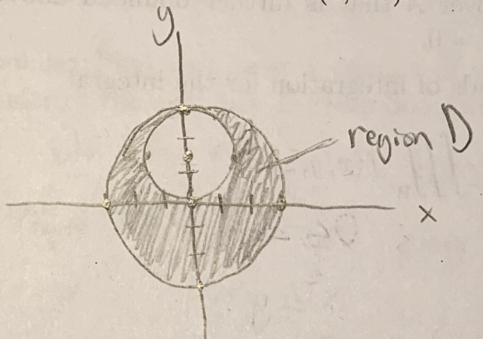
- 5 pts Incorrect interpretation of the integrals.

- 2.5 pts Correctly identified domain of integration, but incorrectly changed the definition of R in the problem.

2. (a) (5 points) Sketch the region D in $\mathbb{R}^2$, which is bounded by a circle of radius 3, centered at the origin, and outside the circle of radius 1.5, centered at the point with rectangular coordinates $(0, 1.5)$.



(b) (5 points) Use equations to describe the boundary of the region D. You may use any coordinate system.

region D in the polar coordinate system

Outer circle: $r = 3$

inner circle: $x^2 + (y-1.5)^2 = 1.5^2$

$r^2\cos^2\theta + r^2\sin^2\theta + 2.25 - 3r\sin\theta = 2.25$

$r^2 = 3r\sin\theta$

$r = 3\sin\theta$

covers all quadrants, so $0 \le \theta \le 2\pi$

So: $D: \begin{cases} 3\sin\theta \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$

1

(c) (15 points) Compute the integral $\iint_D \sqrt{x^2+y^2}\, dA$.

$f(x,y) = \sqrt{x^2+y^2}$

$f(r,\theta) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

$\iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

$\int_0^{2\pi}\int_{3\sin\theta}^{3} r^2\, dr\, d\theta$

2

$\int_0^{2\pi} \left[\frac{r^3}{3}\right]_{3\sin\theta}^{3} d\theta$

$\int_0^{2\pi} (9 - 9\sin^3\theta)\, d\theta$

$9\int_0^{2\pi} 1 - 1\sin^3\theta\, d\theta$

$9\int_0^{2\pi} 1 - (1-\cos^2\theta)\sin\theta\, d\theta$

$9\int_0^{2\pi} 1 - (\sin\theta - \sin\theta\cos^2\theta)\, d\theta$

$9\int_0^{2\pi} 1 - \sin\theta + \sin\theta\cos^2\theta\, d\theta$

$u = \cos\theta$, $du = -\sin\theta\, d\theta$, $u(2\pi) = 1$, $u(0) = 1$

$9\int_0^{2\pi} 1 - 9\int_1^1 1 - u$

So $9[\theta]_0^{2\pi} = 9 \cdot 2\pi = \boxed{18\pi}$

2.1 5 / 5

- ✓ **+ 5 pts** Correct sketch of region
- **+ 2.5 pts** Small mistake when sketching region
- **+ 0 pts** Incorrect

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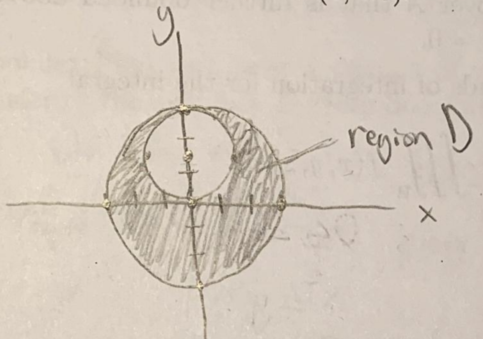

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Covers all quadrants, so $0 \le \theta \le 2\pi$

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2.2 3.5 / 5

- ✓ + **1 pts** **Outer boundary radial description**
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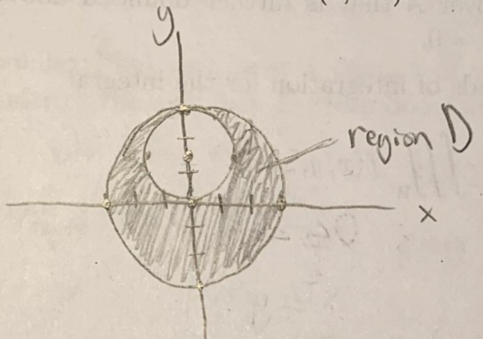

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- ✓ + **3 pts** **Set up integral bounds**
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2 For the region $$\pi \leq \theta \leq 2\pi$$, $$r$$ only varies from $$0$$ to $$3$$ because $$3\sin(\theta) < 0$$. So you needed to split this integral into two portions.

3. Consider the region

$$A = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$$

Let $W \subset \mathbb{R}^3$ be the solid region over A that is further bounded above by the plane $z = x - y$ and below by the plane $z = 0$.

(a) (5 points) Determine the bounds of integration for the integral

$$\iiint_W f(x,y,z)\, dV$$

above by $z = x-y$ and below by $z=0$ means $0 \le z \le x-y$ and integral is z-simple

$x^2 \le y \le 1$

$0 \le x \le 1$

so

$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} f(x,y,z)\, dz\, dy\, dx$$

(b) (10 points) Suppose that the temperature of a point in W is given by $f(x,y,z) = z$. Compute the total tempurature of W.

$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} z\, dz\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \left[\frac{z^2}{2}\right]_0^{x-y} dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \frac{(x-y)^2}{2}\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \frac{x^2 - 2xy + y^2}{2}\, dy\, dx$$

$$\frac{1}{2} \int_0^1 \int_{x^2}^1 x^2 - 2xy + y^2\, dy\, dx$$

$$\frac{1}{2} \int_0^1 \left[x^2 y - xy^2 + \frac{y^3}{3}\right]_{x^2}^1 dx$$

$$\frac{1}{2} \int_0^1 \left(x^2 - x + \frac{1}{3} - x^4 + x^5 - \frac{x^6}{3}\right) dx$$

$$\frac{1}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{3}x - \frac{x^5}{5} + \frac{x^6}{6} - \frac{x^7}{21}\right]_0^1$$

$$= \frac{3}{70}$$

(c) (10 points) The average temperature of a solid region W is given by total temperature of W divided by the volume of W. Compute the average temperature of W.

average temperature: $\frac{1}{V} \iiint_W z\, dV$

$$= \frac{1}{V} \cdot \frac{3}{70}$$

$$V = \iiint_W 1\, dV = \int_0^1 \int_{x^2}^1 \int_0^{x-y} dz\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 (x-y)\, dy\, dx$$

$$\int_0^1 \left[xy - \frac{y^2}{2}\right]_{x^2}^1 dx$$

$$\int_0^1 \left(x - \frac{1}{2} - x^3 + \frac{x^4}{2}\right) dx$$

$$\left[\frac{x^2}{2} - \frac{1}{2}x - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{-3}{20}$$

so,

$$\frac{1}{V} \cdot \iiint_W dV$$

$$= \frac{1}{V} \cdot \frac{3}{70}$$

$$= -\frac{20}{3} \cdot \frac{3}{70} = -\frac{20}{70} = -\frac{2}{7}$$

3.1 0 / 5

+ **5 pts** Correct bounds in part (a)

✓ + **0 pts** Wrong bounds

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(a) (5 points) Determine the bounds of integration for the integral
$$\iiint_W f(x, y, z)\, dV$$

above by $z=x-y$ and below by $z=0$ means $0 \le z \le x-y$ and integral is z-simple
$x^2 \le y \le 1$
$0 \le x \le 1$

so
$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} f(x,y,z)\,dz\,dy\,dx$$

(b) (10 points) Suppose that the temperature of a point in W is given by $f(x, y, z) = z$. Compute the total tempurature of W.

$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} z\,dz\,dy\,dx$$
$$\int_0^1 \int_{x^2}^1 \left[\frac{z^2}{2}\right]_0^{x-y} dy\,dx$$
$$\int_0^1 \int_{x^2}^1 \frac{(x-y)^2}{2}\,dy\,dx$$
$$\int_0^1 \int_{x^2}^1 \frac{x^2-2xy+y^2}{2}\,dy\,dx$$
$$\frac{1}{2}\int_0^1 \int_{x^2}^1 x^2-2xy+y^2\,dy\,dx$$
$$\frac{1}{2}\int_0^1 \left[x^2y - xy^2 + \frac{y^3}{3}\right]_{x^2}^1 dx$$
$$\frac{1}{2}\int_0^1 \left(x^2 - x + \frac{1}{3} - x^4 + x^5 - \frac{x^6}{3}\right) dx$$
$$\frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{3}x - \frac{x^5}{5} + \frac{x^6}{6} - \frac{x^7}{21}\right]_0^1$$
$$= \frac{3}{70}$$

(c) (10 points) The average temperature of a solid region W is given by total temperature of W divided by the volume of W. Compute the average temperature of W.

average temperature: $\frac{1}{V}\iiint_W z\,dV$
$$= \frac{1}{V}\cdot\frac{3}{70}$$
$$V = \iiint_W 1\,dV = \int_0^1 \int_{x^2}^1 \int_0^{x-y} dz\,dy\,dx$$
$$\int_0^1 \int_{x^2}^1 (x-y)\,dy\,dx$$
$$\int_0^1 \left[xy - \frac{y^2}{2}\right]_{x^2}^1 dx$$
$$\int_0^1 \left(x - \frac{1}{2} - x^3 + \frac{x^4}{2}\right) dx$$
$$\left[\frac{x^2}{2} - \frac{1}{2}x - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{-3}{20}$$

so,
$$\frac{1}{V}\cdot\iiint_W dV$$
$$= \frac{1}{V}\cdot\frac{3}{70}$$
$$= -\frac{20}{3}\cdot\frac{3}{70} = -\frac{20}{70} = -\frac{2}{7}$$

3.2 10 / 10

✓ **+ 10 pts** Correct answer using bounds in part (a)

+ 0 pts None of the above

3. Consider the region

$$A = \{(x, y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$$

Let $W \subset \mathbb{R}^3$ be the solid region over A that is further bounded above by the plane $z = x - y$ and below by the plane $z = 0$.

(a) (5 points) Determine the bounds of integration for the integral

$$\iiint_W f(x, y, z)\, dV$$

above by $z = x - y$ and below by $z = 0$ means $0 \le z \le x - y$ and integral is z-simple

$x^2 \le y \le 1$

$0 \le x \le 1$

so

$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} f(x,y,z)\, dz\, dy\, dx$$

(b) (10 points) Suppose that the temperature of a point in W is given by $f(x, y, z) = z$. Compute the total tempurature of W.

$$\int_0^1 \int_{x^2}^1 \int_0^{x-y} z\, dz\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \left[\frac{z^2}{2}\right]_0^{x-y} dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \frac{(x-y)^2}{2}\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 \frac{x^2 - 2xy + y^2}{2}\, dy\, dx$$

$$\frac{1}{2} \int_0^1 \int_{x^2}^1 x^2 - 2xy + y^2\, dy\, dx$$

$$\frac{1}{2} \int_0^1 \left[x^2 y - xy^2 + \frac{y^3}{3}\right]_{x^2}^1 dx$$

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$$= \boxed{\frac{3}{70}}$$

(c) (10 points) The average temperature of a solid region W is given by total temperature of W divided by the volume of W. Compute the average temperature of W.

average temperature: $\frac{1}{V} \iiint_W z\, dV$

$= \frac{1}{V} \cdot \frac{3}{70}$

$$V = \iiint_W 1\, dV = \int_0^1 \int_{x^2}^1 \int_0^{x-y} dz\, dy\, dx$$

$$\int_0^1 \int_{x^2}^1 (x-y)\, dy\, dx$$

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$$\int_0^1 \left(x - \frac{1}{2} - x^3 + \frac{x^4}{2}\right) dx$$

$$\left[\frac{x^2}{2} - \frac{1}{2}x - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{-3}{20}$$

so,

$$\frac{1}{V} \cdot \iiint_W dV$$

$$= \frac{1}{V} \cdot \frac{3}{70}$$

$$= -\frac{20}{3} \cdot \frac{3}{70} = -\frac{20}{70} = \boxed{-\frac{2}{7}}$$

3.3 10 / 10

✓ **+ 10 pts** Correctly set up integral and answer using bounds in part (a) and answer from part (b)

+ 0 pts None of the above

4. Consider the ellipse E, with boundary given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(a) (7 points) Find and draw a change of variables $G(u, v) = (x(u, v), y(u, v))$ that transforms the unit disk D (with boundary the unit circle) into E.

unit disk: $u^2+v^2 \leq 1$

boundary (unit circle): $u^2+v^2=1$

$u = \frac{x}{a}$ $v = \frac{y}{b}$

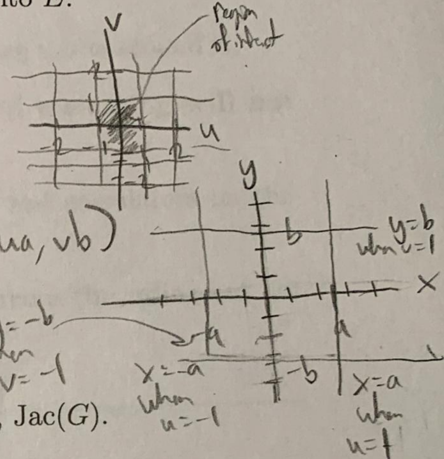
$x = ua$ $y = vb$

so,

$G(u,v) = (x(u,v), y(u,v)) = (ua, vb)$

$-1 \leq u \leq 1$

$-1 \leq v \leq 1$



(b) (8 points) Compute the determinant of the Jacobian matrix, Jac(G).

$x = ua$, $y = vb$

$$\text{Jac}(G): \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab - 0 = \boxed{ab}$$

(c) (10 points) Choose $a \geq b > 0$ such that the area of the ellipse E is 10π. Use the previous parts to show your work.

another change of variables to polar in order to get area of ellipse transformed from disk;

Jacobian for polar is r

$$\int_0^{2\pi}\int_0^1 r\,ab\,dr\,d\theta$$

$$\int_0^{2\pi} ab\left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}ab\,d\theta$$

$$\left[\frac{1}{2}ab\theta\right]_0^{2\pi} = \frac{1}{2}ab\,2\pi = \pi ab$$

if $a=5$ and $b=2$, $a \geq b > 0$ and the area is 10π. So $\boxed{a=5, b=2}$

4.1 7 / 7

✓ **- 0 pts** **Correct**

- 1 pts Misc. minor error

- 4 pts Some correct work but wrote down wrong map (e.g. inverse)

- 4 pts Misread question: mapped $$[0, 1] \times [0, 2\pi]$$ to the ellipse using scaled polar coordinates instead

- 7 pts Blank / minimal progress

4. Consider the ellipse E, with boundary given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

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unit disk: $u^2+v^2 \leq 1$

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$u = \frac{x}{a}$ $v = \frac{y}{b}$

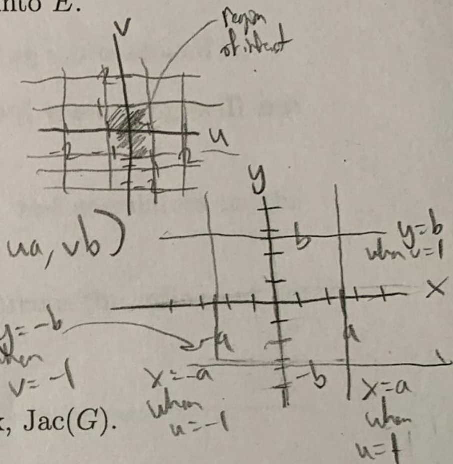
$x = ua$ $y = vb$

so,

$G(u,v) = (x(u,v), y(u,v)) = (ua, vb)$

$-1 \leq u \leq 1$

$-1 \leq v \leq 1$



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if $a=5$ and $b=2$, $a \geq b > 0$ and the area is 10π. So $\boxed{a=5, b=2}$

4.2 8 / 8

✓ **- 0 pts** **Correct ($$ab$$)**

- 0 pts OK given answer to part (a)

- 2 pts Difference instead of product

- 2 pts Jacobian is upside down (map in (a) is correct)

- 3 pts Some entries wrong in matrix

- 8 pts Blank / minimal progress

4. Consider the ellipse E, with boundary given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

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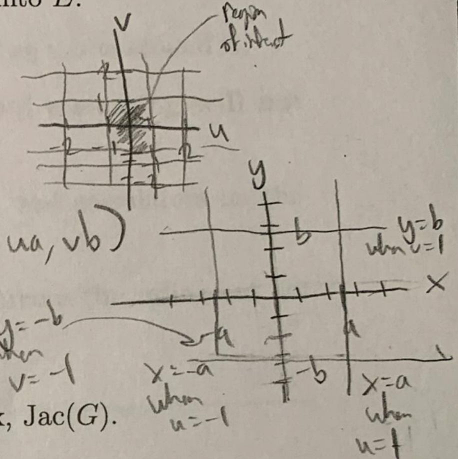

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if $a=5$ and $b=2$, $a \ge b > 0$ and the area is 10π, So $\boxed{a=5, b=2}$

4.3 **10 / 10**

✓ **- 0 pts** **Correct**

- 0 pts OK given (a), (b)

- 1 pts Minor error

- 3 pts Wrong bounds of integration

- 3 pts Incomplete computation

- 5 pts Wrong integral

- 10 pts Blank / minimal progress

Name: Aaryan Dvate **UID:** 205691736

TA/Section: 4B

Instructions:

- Read each problem carefully.
- Show all work clearly, and clearly denote your answer by putting a box around it.
- Justify your answers. **A correct final answer without valid reasoning will not receive credit.**
- You are permitted to use your notes, textbooks, computers, and calculators on this exam.
- **You are not allowed to collaborate or use human resources** (including but not limited to Chegg, Math Stack Exchange, etc.).

Question	Possible Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	