22W-MATH-32B-LEC-4 Final Exam

TOTAL POINTS

177 / 200

QUESTION 1

25 pts

1.1 (a) 7 / 7

✓ **- 0 pts Correct**

- 5 pts described the image of S under G

- 3 pts Only described the corner points

- 7 pts Incorrect or incorrect work

1.2 (b) 8 / 8

✓ **+ 6 pts Correct Jacobian matrix**

+ 4 pts Minor error in Jacobian matrix

✓ **+ 2 pts Correct Jacobian (determinant calculation)**

+ 1 pts Minor calculation error in Jacobian (determinant calculation)

1.3 (c) 10 / 10

✓ **- 0 pts Correct answer (of pi^4/3)**

- 0 pts Correct (given error in part a or part b)

- 2 pts did not take absolute value of Jacobian, so off by sign

- 2.5 pts incorrect or missing Jacobian

- 10 pts Incorrect

QUESTION 2

25 pts

2.1 (a) 5 / 5

✓ **+ 5 pts Correct sketch of $$D$$ (more or less)**

+ 2 pts Correct drawing but wrong region labeled

+ 0 pts Incorrect or blank

2.2 (b) 10 / 10

✓ **+ 10 pts Correct region(s)**

+ 10 pts Used complement (correctly) instead

+ 0 pts Incorrect or blank

+ 5 pts One correct region

+ 3 pts Partial credit (e.g. only complement)

+ 0 pts These are curves, not regions

2.3 (c) 10 / 10

✓ **+ 10 pts Correct answer**

+ 10 pts Incorrect but based on incorrect answer to part (b)

+ 7 pts Computational error(s)

+ 0 pts Incorrect or blank

QUESTION 3

25 pts

3.1 (a) 10 / 10

✓ **+ 10 pts Correct**

+ 8 pts Correct, except used the upward normal

+ 8 pts Partial credit (computation error)

+ 5 pts Partial credit

+ 0 pts Incorrect or blank

3.2 (b) 15 / 15

✓ **+ 15 pts Correct (either parameterization or div. thm.)**

+ 14 pts Wrong sign

+ 11 pts Partial credit (computation)

+ 5 pts Partial credit (parameterization)

+ 3 pts Partial credit

+ 0 pts Incorrect or blank

QUESTION 4

25 pts

4.1 (a) 5 / 5

✓ **+ 5 pts Correct drawing (more or less)**

+ 3 pts Partially correct drawing (e.g. missing a lot of the region, or both paraboloids go through 0)

+ **2 pts** Wrong region labeled

+ **0 pts** Incorrect or blank

4.2 (b) **10 / 10**

✓ + **10 pts** **Correct**

+ **7 pts** Forgot $$r$$

+ **6 pts** Computation errors or bound errors

+ **3 pts** Integrate over square or cylinder

+ **0 pts** Incorrect or blank

+ **10 pts** Wrong W but consistent with (a)

4.3 (c) **10 / 10**

✓ + **10 pts** **Obtain $$8$$ vol $$W$$**

+ **0 pts** Incorrect or blank

+ **7 pts** Error computing divergence

+ **5 pts** Partial credit: use/state divergence theorem

+ **2 pts** Assume field has a vector potential and get 0

QUESTION 5

25 pts

5.1 (a) **10 / 10**

✓ + **10 pts** **Correct**

+ **0 pts** Incorrect (see point adjustment for partial credit)

5.2 (b) **12 / 15**

✓ + **10 pts** **Correct answer (using integral from part (a))**

+ **5 pts** Cited Fubini's theorem to change order of integration

+ **0 pts** None of the above

+ 2 **Point adjustment**

mentioned that region is vertically & horizontally simple

QUESTION 6

25 pts

6.1 (a) **5 / 5**

✓ + **5 pts** **Correct.**

- **2 pts** Minor mistake in form of quotient rule.

- **1 pts** Dropped a negative in derivatives.

+ **0 pts** Incorrect.

6.2 (b) **1 / 5**

+ **1 pts** Parameterized curve

+ **2 pts** Setup integral

+ **2 pts** Evaluated integral correctly.

- **1 pts** Minor mistake

+ **0 pts** Incorrect

✓ + **1 pts** **Set up Green's theorem.**

NOTE: you cannot use Green's theorem because $$F$$ is not smooth at the origin, which is contained in the circle.

ALTERNATE : Show that $$F$$ is conservative.

+ **4 pts** Showed that $$F$$ is conservative by finding potential function.

+ **1 pts** Concluded result.

- **1 pts** Minor mistake in calculation of potential function.

1 Green's theorem cannot be used here because $$F$$ is not smooth at the origin, which is contained in the circle.

6.3 (c) **12 / 15**

Green's Theorem Method

+ **2 pts** Correct general formula for Green's theorem

+ **1 pts** Choose $$R$$ so that $$C_R$$ is within $$Q$$

+ **10 pts** Correct boundary to apply Green's theorem

+ **2 pts** Conclude based on previous part

- **1 pts** Minor mistake

Conservative Method

✓ + **10 pts** **Show that $$F$$ is conservative**

✓ + **5 pts** **Conclude that the integral is $$0$$**

- **4 pts** Significant error in finding potential function

- **1 pts** Small mistake in computing potential function

+ **0 pts** Incorrect or no valid justification

- 3 **Point adjustment**

Claimed that Green's theorem could be applied.

2 Green's theorem cannot be applied here because $$F$$ is not smooth at the origin, which is contained in the curve.

3 This is a correct method

QUESTION 7

25 pts

7.1 (a) **7 / 9**

+ **9 pts** Correct (found a potential function or drew level curves)
+ **0 pts** Incorrect or no explanation

✓ + **7 pts** **Provided evidence that F was path independent on the domain or that all circulation integrals in the domain will be 0.**

+ **7 pts** Showed curl was 0/cross partials condition, **and** tried to determine if the **domain of the vector field** was simply connected or not
+ **4 pts** Showed curl was 0 or checked cross partials condition. This is not sufficient to answer the question.

7.2 (b) **1 / 8**

+ **5 pts** Observed that F is path independent
- **2 pts** Incorrect interpretation of path independence
+ **2 pts** Observed that A & Q, and B&P have the same potential value (observing the distances AB and PQ are the same is not sufficient)

✓ + **1 pts** **Observed that C1 goes from A to B, while C2 goes from Q to P**

+ **0 pts** Incorrect or no work

7.3 (c) **4 / 8**

- **0 pts** Correct
- **8 pts** Incorrect or no reasoning

✓ - **4 pts** **Partially correct reasoning (does not describe sign)**

QUESTION 8

25 pts

8.1 (a) **9 / 9**

✓ - **0 pts** **Correct (F is not conservative)**

- **9 pts** Incorrect or no reasoning.
- **4 pts** Did not prove path dependence/ or did not prove that there is a curve with nonzero circuation
- **4 pts** Invalid or no justification for why curl is non-zero
- **6 pts** incorrectly stated curl = 0 (and did not check simply-connectedness)
- **6 pts** incorrectly stated level curves exist

8.2 (b) **8 / 8**

✓ - **0 pts** **Correct**

- **8 pts** Incorrect or no explanation
+ **4 pts** Incorrect/unexplained answer, but interpreted divergence correctly

8.3 (c) **8 / 8**

✓ - **0 pts** **Correct**

- **1 pts** interpreted CCW circulation correctly, but said negative curl
- **8 pts** Incorrect or no explanation.
+ **4 pts** Incorrect/unexplained answer, but interpreted curl correctly
- **3 pts** correct reasoning but incorrect answer (claimed that F was conservative)
- **2 pts** correct reasoning (but did not (or incorrectly) explain why $$\frac{\partial F_2}{\partial x}$$ is positive)

1. Let S be the diamond in the xy-plane with vertices $A = (\pi, 0)$, $B = (2\pi, \pi)$, $C = (\pi, 2\pi)$, $D = (0, \pi)$, and consider the change of variables

$$G(u, v) = \left\langle \frac{u+v}{2}, \frac{u-v}{2} \right\rangle$$

(a) (7 points) Describe, as a set, the region S_0 in the uv-plane that G **maps to** S.

$A: \frac{u+v}{2} = \pi, \frac{u-v}{2} = 0$

$\Rightarrow u = v, \; u+v = 2\pi$

$\Rightarrow u = v = \pi$

$B: \frac{u+v}{2} = 2\pi, \frac{u-v}{2} = \pi$

$\Rightarrow u+v = 4\pi, u-v = 2\pi$

$\Rightarrow u = 3\pi, v = \pi$

$C: \frac{u+v}{2} = \pi, \frac{u-v}{2} = 2\pi$

$\Rightarrow u+v = 2\pi, u-v = 4\pi$

$\Rightarrow u = 3\pi, v = -\pi$

$D: \frac{u+v}{2} = 0, \frac{u-v}{2} = \pi$

$\Rightarrow u = -v, u-v = 2\pi$

$\Rightarrow 2u = 2\pi$

$\Rightarrow u = \pi, v = -\pi$

$\therefore$ in $G(u,v)$:

$A = (\pi, \pi)$, $B = (3\pi, \pi)$

$C = (3\pi, -\pi)$, $D = (\pi, -\pi)$

$\therefore S_0 = \{(u,v) \mid \pi \le u \le 3\pi, -\pi \le v \le \pi\}$

(b) (8 points) Compute the determinant of the Jacobian matrix, Jac(G).

$$[J_G] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\text{Jac}(G) = \det[J_G] = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

(c) (10 points) Use your work from the previous parts to compute

$$\iint_S (x-y)^2 \sin^2(x+y) \, dx \, dy$$

$$\iint_S (x-y)^2 \sin^2(x+y)\,dx\,dy$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} \left(\frac{u+v}{2} - \frac{u-v}{2}\right)^2 \left(\sin^2\left(\frac{u+v}{2} + \frac{u-v}{2}\right)\right) \left|-\frac{1}{2}\right| du\,dv$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} v^2 (\sin^2 u) \left(\frac{1}{2}\right) du\,dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left[\frac{u}{2} - \frac{\cos u \sin u}{2}\right]_{\pi}^{3\pi} dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left(\frac{3\pi}{2} - \frac{\pi}{2} - 0 + 0\right) dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \pi \, dv$$

$$= \left(\frac{1}{2}\right) \pi \left[\frac{v^3}{3}\right]_{-\pi}^{\pi}$$

$$= \left(\frac{1}{2}\right) \pi \left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \left(\frac{1}{2}\right) \pi \left(\frac{2\pi^3}{3}\right)$$

$$= \frac{\pi^4}{3}$$

1.1 (a) **7 / 7**

- ✓ **- 0 pts** Correct
- **- 5 pts** described the image of S under G
- **- 3 pts** Only described the corner points
- **- 7 pts** Incorrect or incorrect work

1. Let S be the diamond in the xy-plane with vertices $A = (\pi, 0)$, $B = (2\pi, \pi)$, $C = (\pi, 2\pi)$, $D = (0, \pi)$, and consider the change of variables

$$G(u, v) = \left\langle \frac{u+v}{2}, \frac{u-v}{2} \right\rangle$$

(a) (7 points) Describe, as a set, the region S_0 in the uv-plane that G **maps to** S.

A: $\frac{u+v}{2} = \pi$, $\frac{u-v}{2} = 0$

$\Rightarrow u = v$, $u+v = 2\pi$

$\Rightarrow u = v = \pi$

B: $\frac{u+v}{2} = 2\pi$, $\frac{u-v}{2} = \pi$

$\Rightarrow u+v = 4\pi$, $u-v = 2\pi$

$\Rightarrow u = 3\pi$, $v = \pi$

C: $\frac{u+v}{2} = \pi$, $\frac{u-v}{2} = 2\pi$

$\Rightarrow u+v = 2\pi$, $u-v = 4\pi$

$\Rightarrow u = 3\pi$, $v = -\pi$

D: $\frac{u+v}{2} = 0$, $\frac{u-v}{2} = \pi$

$\Rightarrow u = -v$, $u-v = 2\pi$

$\Rightarrow 2u = 2\pi$

$\Rightarrow u = \pi$, $v = -\pi$

$\therefore$ in $G(u,v)$:

$A = (\pi, \pi)$, $B = (3\pi, \pi)$

$C = (3\pi, -\pi)$, $D = (\pi, -\pi)$

$\therefore S_0 = \{(u,v) \mid \pi \le u \le 3\pi, -\pi \le v \le \pi\}$

(b) (8 points) Compute the determinant of the Jacobian matrix, Jac(G).

$$[J_G] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\text{Jac}(G) = \det[J_G] = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

(c) (10 points) Use your work from the previous parts to compute

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy$$

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} \left(\frac{u+v}{2} - \frac{u-v}{2}\right)^2 \left(\sin^2\left(\frac{u+v}{2} + \frac{u-v}{2}\right)\right) \left|-\frac{1}{2}\right| du\, dv$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} v^2 (\sin^2 u) \left(\tfrac{1}{2}\right) du\, dv$$

$$= \left(\tfrac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left[\frac{u}{2} - \frac{\cos u \sin u}{2}\right]_{\pi}^{3\pi} dv$$

$$= \left(\tfrac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left(\frac{3\pi}{2} - \frac{\pi}{2} - 0 + 0\right) dv$$

$$= \left(\tfrac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \pi\, dv$$

$$= \left(\tfrac{1}{2}\right) \pi \left[\frac{v^3}{3}\right]_{-\pi}^{\pi}$$

$$= \left(\tfrac{1}{2}\right) \pi \left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \left(\tfrac{1}{2}\right) \pi \left(\frac{2\pi^3}{3}\right)$$

$$= \frac{\pi^4}{3}$$

1.2 (b) **8 / 8**

- ✓ **+ 6 pts** **Correct Jacobian matrix**
- **+ 4 pts** Minor error in Jacobian matrix
- ✓ **+ 2 pts** **Correct Jacobian (determinant calculation)**
- **+ 1 pts** Minor calculation error in Jacobian (determinant calculation)

1. Let S be the diamond in the xy-plane with vertices $A = (\pi, 0)$, $B = (2\pi, \pi)$, $C = (\pi, 2\pi)$, $D = (0, \pi)$, and consider the change of variables

$$G(u, v) = \left\langle \frac{u+v}{2}, \frac{u-v}{2} \right\rangle$$

(a) (7 points) Describe, as a set, the region S_0 in the uv-plane that G **maps to** S.

A: $\frac{u+v}{2} = \pi$, $\frac{u-v}{2} = 0$
$\Rightarrow u = v$, $u+v = 2\pi$
$\Rightarrow u = v = \pi$

B: $\frac{u+v}{2} = 2\pi$, $\frac{u-v}{2} = \pi$
$\Rightarrow u+v = 4\pi$, $u-v = 2\pi$
$\Rightarrow u = 3\pi$, $v = \pi$

C: $\frac{u+v}{2} = \pi$, $\frac{u-v}{2} = 2\pi$
$\Rightarrow u+v = 2\pi$, $u-v = 4\pi$
$\Rightarrow u = 3\pi$, $v = -\pi$

D: $\frac{u+v}{2} = 0$, $\frac{u-v}{2} = \pi$
$\Rightarrow u = -v$, $u-v = 2\pi$
$\Rightarrow 2u = 2\pi$
$\Rightarrow u = \pi$, $v = -\pi$

$\therefore$ in $G(u,v)$:
$A = (\pi, \pi)$, $B = (3\pi, \pi)$
$C = (3\pi, -\pi)$, $D = (\pi, -\pi)$

$\therefore S_0 = \{(u,v) \mid \pi \le u \le 3\pi, -\pi \le v \le \pi\}$

(b) (8 points) Compute the determinant of the Jacobian matrix, Jac(G).

$$[J_G] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\text{Jac}(G) = \det[J_G] = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

(c) (10 points) Use your work from the previous parts to compute

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy$$

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} \left(\frac{u+v}{2} - \frac{u-v}{2}\right)^2 \left(\sin^2\left(\frac{u+v}{2} + \frac{u-v}{2}\right)\right) \left|-\frac{1}{2}\right| du\, dv$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} v^2 (\sin^2 u) \left(\frac{1}{2}\right) du\, dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left[\frac{u}{2} - \frac{\cos u \sin u}{2}\right]_{\pi}^{3\pi} dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \left(\frac{3\pi}{2} - \frac{\pi}{2} - 0 + 0\right) dv$$

$$= \left(\frac{1}{2}\right) \int_{-\pi}^{\pi} v^2 \pi\, dv$$

$$= \left(\frac{1}{2}\right) \pi \left[\frac{v^3}{3}\right]_{-\pi}^{\pi}$$

$$= \left(\frac{1}{2}\right) \pi \left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \left(\frac{1}{2}\right) \pi \left(\frac{2\pi^3}{3}\right)$$

$$= \frac{\pi^4}{3}$$

1.3 (c) **10 / 10**

✓ **- 0 pts** **Correct answer (of pi^4/3)**

- 0 pts Correct (given error in part a or part b)

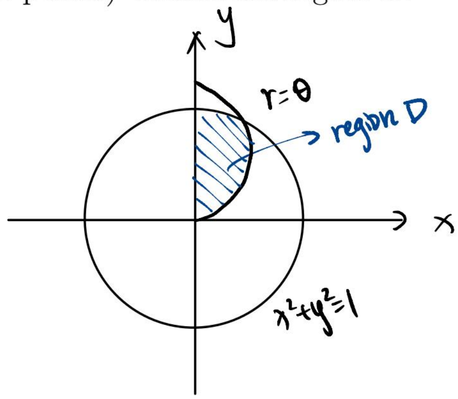
- 2 pts did not take absolute value of Jacobian, so off by sign

- 2.5 pts incorrect or missing Jacobian

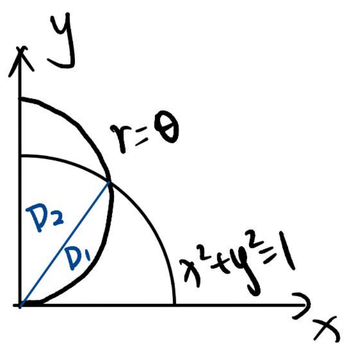
- 10 pts Incorrect

2. Let D be the region in the first quadrant inside the circle of radius 1, and bounded by the polar spiral $r = \theta$ and the y-axis.

   (a) (5 points) Sketch the region D.



   (b) (10 points) Express the region D as a (union of) radially simple region(s).



where $x^2+y^2=1$ and $r=\theta$ intersect:

change $x^2+y^2=1$ into polar coordinates:

$r^2=1$, $r^2=\theta^2 \Rightarrow \theta^2=1$

$\theta=1$ because we are in 1st quadrant

$D_1 = \{(r,\theta) \mid 0 \le \theta \le 1,\ 0 \le r \le \theta\}$

$D_2 = \{(r,\theta) \mid 1 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1\}$

$D = D_1 \cup D_2$

   (c) (10 points) Use your work from the previous parts to compute

$$\iint_D \sqrt{x^2+y^2}\, dA$$

First convert to polar:

$$\iint_D \sqrt{x^2+y^2}\, dA$$

$$= \iint_D \sqrt{r^2}\cdot r\, dA$$

$$= \iint_D r^2\, dA$$

$$= \iint_{D_1} r^2\, dA + \iint_{D_2} r^2\, dA$$

$$= \int_0^1 \int_0^\theta r^2\, dr\, d\theta + \int_1^{\pi/2} \int_0^1 r^2\, dr\, d\theta$$

$$= \int_0^1 \left[\frac{r^3}{3}\right]_0^\theta d\theta + \int_1^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_0^1 \frac{\theta^3}{3}\, d\theta + \int_1^{\pi/2} \frac{1}{3}\, d\theta$$

$$= \left[\frac{\theta^4}{12}\right]_0^1 + \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{3}\cdot 1$$

$$= \frac{1}{12} + \frac{\pi}{6} - \frac{1}{3}$$

$$= \frac{\pi}{6} - \frac{1}{4}$$

2.1 (a) **5 / 5**

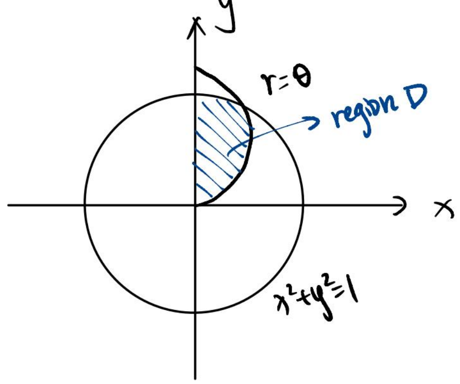
✓ **+ 5 pts** **Correct sketch of $$D$$ (more or less)**

+ 2 pts Correct drawing but wrong region labeled

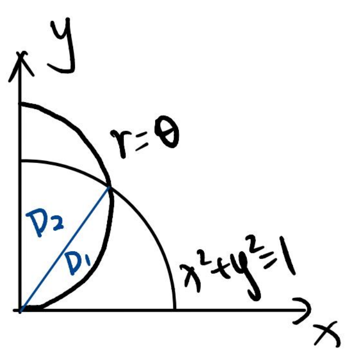
+ 0 pts Incorrect or blank

2. Let D be the region in the first quadrant inside the circle of radius 1, and bounded by the polar spiral $r = \theta$ and the y-axis.

   (a) (5 points) Sketch the region D.



   (b) (10 points) Express the region D as a (union of) radially simple region(s).



where $x^2+y^2=1$ and $r=\theta$ intersect:

change $x^2+y^2=1$ into polar coordinates:

$r^2=1$, $r^2=\theta^2 \Rightarrow \theta^2=1$

$\theta=1$ because we are in 1st quadrant

$D_1 = \{(r,\theta) \mid 0 \le \theta \le 1,\ 0 \le r \le \theta\}$

$D_2 = \{(r,\theta) \mid 1 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1\}$

$D = D_1 \cup D_2$

   (c) (10 points) Use your work from the previous parts to compute

$$\iint_D \sqrt{x^2+y^2}\, dA$$

First convert to polar:

$$\iint_D \sqrt{x^2+y^2}\, dA$$
$$= \iint_D \sqrt{r^2}\cdot r\, dA$$
$$= \iint_D r^2\, dA$$
$$= \iint_{D_1} r^2\, dA + \iint_{D_2} r^2\, dA$$
$$= \int_0^1\int_0^\theta r^2\, dr\, d\theta + \int_1^{\pi/2}\int_0^1 r^2\, dr\, d\theta$$

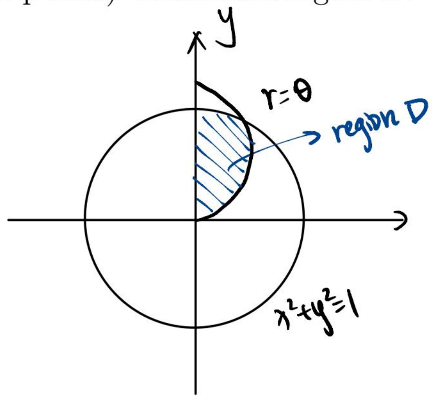
$$= \int_0^1 \left[\frac{r^3}{3}\right]_0^\theta d\theta + \int_1^{\pi/2}\left[\frac{r^3}{3}\right]_0^1 d\theta$$
$$= \int_0^1 \frac{\theta^3}{3}\, d\theta + \int_1^{\pi/2}\frac{1}{3}\, d\theta$$
$$= \left[\frac{\theta^4}{12}\right]_0^1 + \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{3}\cdot 1$$
$$= \frac{1}{12} + \frac{\pi}{6} - \frac{1}{3}$$
$$= \frac{\pi}{6} - \frac{1}{4}$$

2.2 (b) **10 / 10**

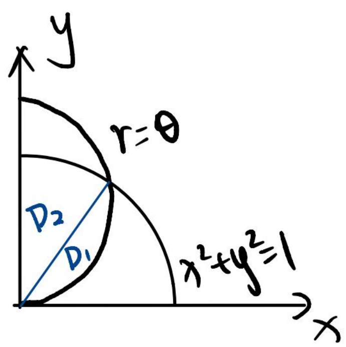
- ✓ **+ 10 pts** **Correct region(s)**
- **+ 10 pts** Used complement (correctly) instead
- **+ 0 pts** Incorrect or blank
- **+ 5 pts** One correct region
- **+ 3 pts** Partial credit (e.g. only complement)
- **+ 0 pts** These are curves, not regions

2. Let D be the region in the first quadrant inside the circle of radius 1, and bounded by the polar spiral $r = \theta$ and the y-axis.

   (a) (5 points) Sketch the region D.



   (b) (10 points) Express the region D as a (union of) radially simple region(s).



where $x^2+y^2=1$ and $r=\theta$ intersect:
change $x^2+y^2=1$ into polar coordinates:
$r^2=1$, $r^2=\theta^2 \Rightarrow \theta^2=1$
$\theta=1$ because we are in 1st quadrant

$D_1 = \{(r,\theta) \mid 0 \le \theta \le 1,\ 0 \le r \le \theta\}$
$D_2 = \{(r,\theta) \mid 1 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1\}$
$D = D_1 \cup D_2$

   (c) (10 points) Use your work from the previous parts to compute

$$\iint_D \sqrt{x^2+y^2}\, dA$$

First convert to polar:

$$\iint_D \sqrt{x^2+y^2}\, dA$$
$$= \iint_D \sqrt{r^2}\cdot r\, dA$$
$$= \iint_D r^2\, dA$$
$$= \iint_{D_1} r^2\, dA + \iint_{D_2} r^2\, dA$$
$$= \int_0^1\int_0^\theta r^2\, dr\, d\theta + \int_1^{\pi/2}\int_0^1 r^2\, dr\, d\theta$$
$$= \int_0^1 \left[\frac{r^3}{3}\right]_0^\theta d\theta + \int_1^{\pi/2}\left[\frac{r^3}{3}\right]_0^1 d\theta$$
$$= \int_0^1 \frac{\theta^3}{3}\, d\theta + \int_1^{\pi/2} \frac{1}{3}\, d\theta$$
$$= \left[\frac{\theta^4}{12}\right]_0^1 + \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{3}\cdot 1$$
$$= \frac{1}{12} + \frac{\pi}{6} - \frac{1}{3}$$
$$= \frac{\pi}{6} - \frac{1}{4}$$

2.3 (c) **10 / 10**

- ✓ **+ 10 pts** **Correct answer**
- **+ 10 pts** Incorrect but based on incorrect answer to part (b)
- **+ 7 pts** Computational error(s)
- **+ 0 pts** Incorrect or blank

3. The velocity vector field of a fluid is given by

$$\boldsymbol{v} = \langle 0, x^2 + y^2, z^2 \rangle$$

(a) (10 points) Find the flow rate of $\boldsymbol{v}$ in the **negative** z-direction through the disk

$$D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$$

parametrize D:

$G(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle$, $0 \leq \theta \leq 2\pi$

$\vec{n}$ is in the negative direction of z, so $\vec{n} = \langle 0, 0, -1 \rangle$

$$\iint_S \vec{v} \cdot \vec{n} \, dS = \int_0^{2\pi} \langle 0, \cos^2\theta + \sin^2\theta, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, d\theta$$

$$= \int_0^{2\pi} 0 \, d\theta$$

$$\boxed{= 0}$$

(b) (15 points) Find the flow rate of $\boldsymbol{v}$ in the **positive** z-direction through the hemisphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$$

Parametrize S:

$S(\phi, \theta) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{2}$

$\frac{\partial S}{\partial \theta} = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$ $\quad \frac{\partial S}{\partial \phi} = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$

$$\vec{N}(\phi, \theta) = \frac{\partial S}{\partial \phi} \times \frac{\partial S}{\partial \theta}$$

$$= \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle \times \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$$

$$= \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \sin^2\theta\cos\phi\sin\phi + \cos^2\theta\sin\phi\cos\phi \rangle$$

$$= \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \cos\phi\sin\phi \rangle$$

$$\vec{v} = \langle 0, x^2+y^2, z^2 \rangle = \langle 0, \sin^2\phi, \cos^2\phi \rangle$$

$$\iint_S \vec{v} \cdot \vec{N} \, dS = \int_0^{\pi/2} \int_0^{2\pi} \langle 0, \sin^2\phi, \cos^2\phi \rangle \cdot \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \cos\phi\sin\phi \rangle \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \sin^4\phi\sin\theta + \cos^3\phi\sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \left[\cos\theta\sin^4\phi + \theta\cos^3\phi\sin\phi \right]_0^{2\pi} d\phi$$

$$= \int_0^{\pi/2} \left[0 + 2\pi\cos^3\phi\sin\phi \right] d\phi$$

$$= 2\pi \left[\frac{-\cos^4\phi}{4} \right]_0^{\pi/2}$$

$$= 2\pi \left[\frac{1}{4} \right] \boxed{= \frac{\pi}{2}}$$

3.1 (a) **10 / 10**

- ✓ **+ 10 pts** Correct
- **+ 8 pts** Correct, except used the upward normal
- **+ 8 pts** Partial credit (computation error)
- **+ 5 pts** Partial credit
- **+ 0 pts** Incorrect or blank

3. The velocity vector field of a fluid is given by

$$\boldsymbol{v} = \langle 0, x^2 + y^2, z^2 \rangle$$

(a) (10 points) Find the flow rate of $\boldsymbol{v}$ in the **negative** z-direction through the disk

$$D = \{(x, y, 0) \mid x^2 + y^2 \le 1\}$$

parametrize D:

$G(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle, \quad 0 \le \theta \le 2\pi$

$\vec{n}$ is in the negative direction of z, so $\vec{n} = \langle 0, 0, -1 \rangle$

$$\iint_S \vec{v} \cdot \vec{n}\, dS = \int_0^{2\pi} \langle 0, \cos^2\theta + \sin^2\theta, 0 \rangle \cdot \langle 0, 0, -1 \rangle\, d\theta$$

$$= \int_0^{2\pi} 0\, d\theta$$

$$\boxed{= 0}$$

(b) (15 points) Find the flow rate of $\boldsymbol{v}$ in the **positive** z-direction through the hemisphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \ge 0\}$$

Parametrize S:

$S(\phi, \theta) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$

$\frac{\partial S}{\partial \theta} = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle \qquad \frac{\partial S}{\partial \phi} = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$

$\vec{N}(\phi, \theta) = \frac{\partial S}{\partial \phi} \times \frac{\partial S}{\partial \theta}$

$= \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle \times \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$

$= \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \sin^2\theta\cos\phi\sin\phi + \cos^2\theta\sin\phi\cos\phi \rangle$

$= \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \cos\phi\sin\phi \rangle$

$\vec{v} = \langle 0, x^2 + y^2, z^2 \rangle = \langle 0, \sin^2\phi, \cos^2\phi \rangle$

$$\iint_S \vec{v} \cdot \vec{N}\, dS = \int_0^{\pi/2} \int_0^{2\pi} \langle 0, \sin^2\phi, \cos^2\phi \rangle \cdot \langle \sin^2\phi\cos\theta, \sin\theta\sin^2\phi, \cos\phi\sin\phi \rangle\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \sin^4\phi\sin\theta + \cos^3\phi\sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \left[\cos\theta\sin^4\phi + \theta\cos^3\phi\sin\phi \right]_0^{2\pi} d\phi$$

$$= \int_0^{\pi/2} [0 + 2\pi\cos^3\phi\sin\phi]\, d\phi$$

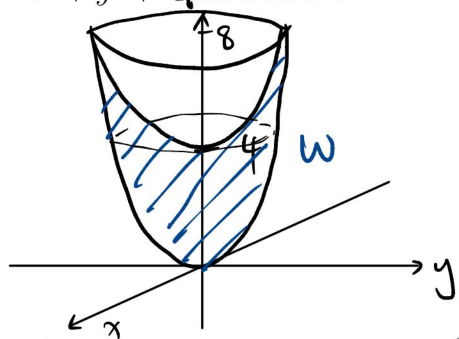
$$= 2\pi \left[\frac{-\cos^4\phi}{4} \right]_0^{\pi/2}$$

$$= 2\pi \left[\frac{1}{4} \right] \boxed{= \frac{\pi}{2}}$$

3.2 (b) **15 / 15**

- ✓ **+ 15 pts** **Correct (either parameterization or div. thm.)**
- **+ 14 pts** Wrong sign
- **+ 11 pts** Partial credit (computation)
- **+ 5 pts** Partial credit (parameterization)
- **+ 3 pts** Partial credit
- **+ 0 pts** Incorrect or blank

4. (a) (5 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Sketch W.



When the two paraboloids intersect:

$2x^2 + 2y^2 = x^2 + y^2 + 4$

$x^2 + y^2 = 4$, $z = 8$

(b) (10 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Find the volume of W.

Using cylindrical coordinates:

$r^2 = x^2 + y^2$

$$W_{volume} = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^2+4-2r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(8 - \frac{16}{4}\right) d\theta$$

$$= 4 \cdot 2\pi \boxed{= 8\pi}$$

(c) (10 points) Let ∂W denote the boundary of W, oriented by outward-pointing normal vectors. Compute

$$\iint_{\partial W} \left\langle x + xy + \sin(z), x^3 + 3y - \frac{y^2}{2}, 4z + \cos(y) \right\rangle \cdot d\mathbf{S}$$

By the divergence theorem:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \text{div}(\vec{F}) \, dV$$

$\text{div}(\vec{F}) = 1 + y + 3 - y + 4 = 8$

$$\iiint_W \text{div}(\vec{F}) \, dV = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} 8r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 8r(r^2+4-2r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 -8r^3 + 32r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[-2r^4 + 16r^2\right]_0^2 d\theta$$

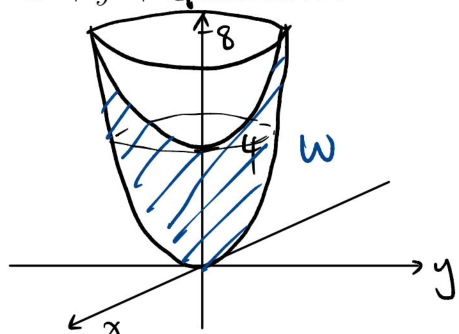
$$= \int_0^{2\pi} -32 + 64 \, d\theta$$

$$\boxed{= 64\pi}$$

4.1 (a) **5 / 5**

- ✓ **+ 5 pts** Correct drawing (more or less)
- **+ 3 pts** Partially correct drawing (e.g. missing a lot of the region, or both paraboloids go through 0)
- **+ 2 pts** Wrong region labeled
- **+ 0 pts** Incorrect or blank

4. (a) (5 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Sketch W.



When the two paraboloids intersect:
$2x^2+2y^2 = x^2+y^2+4$
$x^2+y^2=4$, $z=8$

(b) (10 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Find the volume of W.

Using cylindrical coordinates:
$r^2 = x^2+y^2$

$$W_{\text{volume}} = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (r^2+4-2r^2)r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 4r - r^3\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$
$$= \int_0^{2\pi}\left(8 - \frac{16}{4}\right)d\theta$$
$$= 4\cdot 2\pi \ \boxed{= 8\pi}$$

(c) (10 points) Let ∂W denote the boundary of W, oriented by outward-pointing normal vectors. Compute

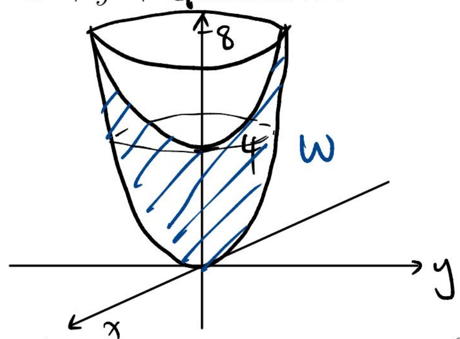
$$\iint_{\partial W}\left\langle x + xy + \sin(z), x^3 + 3y - \frac{y^2}{2}, 4z + \cos(y)\right\rangle \cdot d\boldsymbol{S}$$

By the divergence theorem:
$$\iint_{\partial W}\vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\,dV$$
$\operatorname{div}(\vec{F}) = 1+y+3-y+4 = 8$
$$\iiint_W \operatorname{div}(\vec{F})\,dV = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} 8r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 8r(r^2+4-2r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 -8r^3 + 32r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[-2r^4 + 16r^2\right]_0^2 d\theta$$
$$= \int_0^{2\pi} -32 + 64\,d\theta$$
$$\boxed{= 64\pi}$$

4.2 (b) **10 / 10**

- ✓ **+ 10 pts** **Correct**
- **+ 7 pts** Forgot $$r$$
- **+ 6 pts** Computation errors or bound errors
- **+ 3 pts** Integrate over square or cylinder
- **+ 0 pts** Incorrect or blank
- **+ 10 pts** Wrong W but consistent with (a)

4. (a) (5 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Sketch W.



when the two paraboloids intersect:

$2x^2+2y^2 = x^2+y^2+4$

$x^2+y^2=4,\ z=8$

(b) (10 points) Let W be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Find the volume of W.

Using cylindrical coordinates:

$r^2 = x^2+y^2$

$$W_{\text{volume}} = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^2+4-2r^2)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r - r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(8 - \frac{16}{4}\right) d\theta$$

$$= 4 \cdot 2\pi \boxed{= 8\pi}$$

(c) (10 points) Let ∂W denote the boundary of W, oriented by outward-pointing normal vectors. Compute

$$\iint_{\partial W} \left\langle x + xy + \sin(z), x^3 + 3y - \frac{y^2}{2}, 4z + \cos(y) \right\rangle \cdot d\mathbf{S}$$

By the divergence theorem:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$$

$\operatorname{div}(\vec{F}) = 1+y+3-y+4 = 8$

$$\iiint_W \operatorname{div}(\vec{F})\, dV = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} 8r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 8r(r^2+4-2r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 -8r^3 + 32r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-2r^4 + 16r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi} -32 + 64\, d\theta$$

$$\boxed{= 64\pi}$$

4.3 (c) 10 / 10

✓ + **10 pts** **Obtain $$8$$ vol $$W$$**

+ **0 pts** Incorrect or blank

+ **7 pts** Error computing divergence

+ **5 pts** Partial credit: use/state divergence theorem

+ **2 pts** Assume field has a vector potential and get 0

5. (a) (10 points) Let D be the region in $\mathbb{R}^2$ given by

$$D = \{(x,y) \mid 0 \le x \le 1, x^{2/3} \le y \le 1\}$$

Let ∂D denote the boundary of D, with the boundary orientation. Use Green's theorem to rewrite the following circulation integral as a double integral over D.

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x} 3\cos x + 2x + \frac{x^2 e^{y^4}}{2} = -3\sin x + 2 + xe^{y^4}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} x^3 - 3y\sin x - y = -3\sin x - 1$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3\sin x + 2 + xe^{y^4} + 3\sin x + 1 = 3 + xe^{y^4}$$

By Green's Theorem:

$$\oint_{\partial D} \langle x^3 - 3y\sin x - y, 3\cos x + 2x + \frac{x^2 e^{y^4}}{2} \rangle \cdot dr$$

$$\boxed{= \iint_D 3 + xe^{y^4} \, dA}$$

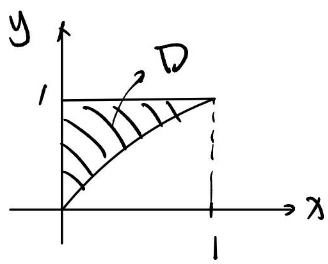
(b) (15 points) Use your work from the previous part to compute

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$

$$\oint_{\partial D} \langle x^3 - 3y\sin x - y, 3\cos x + 2x + \frac{x^2 e^{y^4}}{2} \rangle \cdot dr$$

$$= \iint_D 3 + xe^{y^4} \, dA$$

$$= \int_0^1 \int_{x^{2/3}}^1 3 + xe^{y^4} \, dy \, dx$$



$$y = x^{2/3} \Rightarrow x = y^{3/2}$$

since the region is both horizontally and vertically simple:

$$\int_0^1 \int_{x^{2/3}}^1 3 + xe^{y^4} \, dy \, dx$$

$$= \int_0^1 \int_0^{y^{3/2}} 3 + xe^{y^4} \, dx \, dy$$

$$= \int_0^1 \left[3x + \frac{x^2}{2} e^{y^4}\right]_0^{y^{3/2}} dy$$

$$= \int_0^1 3y^{3/2} + \frac{y^3}{2} e^{y^4} \, dy$$

$$= \left[3 \cdot \frac{2}{5} y^{5/2} + \frac{1}{8} e^{y^4}\right]_0^1$$

$$= \frac{6}{5} + \frac{1}{8}e - \frac{1}{8} = \frac{e}{8} + \frac{48-5}{40} \boxed{= \frac{e}{8} - \frac{43}{40}}$$

5.1 (a) **10 / 10**

✓ **+ 10 pts** Correct

+ 0 pts Incorrect (see point adjustment for partial credit)

5. (a) (10 points) Let D be the region in $\mathbb{R}^2$ given by

$$D = \{(x, y) \mid 0 \leq x \leq 1, x^{2/3} \leq y \leq 1\}$$

Let ∂D denote the boundary of D, with the boundary orientation. Use Green's theorem to rewrite the following circulation integral as a double integral over D.

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x} 3\cos x + 2x + \frac{x^2 e^{y^4}}{2}$$

$$= -3\sin x + 2 + x e^{y^4}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} x^3 - 3y\sin x - y$$

$$= -3\sin x - 1$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3\sin x + 2 + x e^{y^4} + 3\sin x + 1$$

$$= 3 + x e^{y^4}$$

By Green's Theorem:

$$\oint_{\partial D} \langle x^3 - 3y\sin x - y,\ 3\cos x + 2x + \frac{x^2 e^{y^4}}{2} \rangle \cdot dr$$

$$\boxed{= \iint_D 3 + x e^{y^4} \, dA}$$

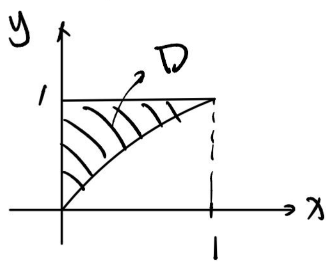
(b) (15 points) Use your work from the previous part to compute

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$

$$\oint_{\partial D} \langle x^3 - 3y\sin x - y,\ 3\cos x + 2x + \frac{x^2 e^{y^4}}{2} \rangle \cdot dr$$

$$= \iint_D 3 + x e^{y^4} \, dA$$

$$= \int_0^1 \int_{x^{2/3}}^1 3 + x e^{y^4} \, dy \, dx$$



$$y = x^{2/3} \Rightarrow x = y^{3/2}$$

since the region is both horizontally and vertically simple:

$$\int_0^1 \int_{x^{2/3}}^1 3 + x e^{y^4} \, dy \, dx$$

$$= \int_0^1 \int_0^{y^{3/2}} 3 + x e^{y^4} \, dx \, dy$$

$$= \int_0^1 \left[3x + \frac{x^2}{2} e^{y^4} \right]_0^{y^{3/2}} dy$$

$$= \int_0^1 3y^{3/2} + \frac{y^3}{2} e^{y^4} \, dy$$

$$= \left[3 \cdot \frac{2}{5} y^{5/2} + \frac{1}{8} e^{y^4} \right]_0^1$$

$$= \frac{6}{5} + \frac{1}{8}e - \frac{1}{8} = \frac{e}{8} + \frac{48-5}{40} \boxed{= \frac{e}{8} - \frac{43}{40}}$$

5.2 (b) **12 / 15**

- ✓ **+ 10 pts** **Correct answer (using integral from part (a))**
- **+ 5 pts** Cited Fubini's theorem to change order of integration
- **+ 0 pts** None of the above

+ 2 Point adjustment

mentioned that region is vertically & horizontally simple

6. Consider the vector field on $\mathbb{R}^2 - \{(0,0)\}$ given by

$$\boldsymbol{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

(a) (5 points) Compute $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\frac{\partial F_2}{\partial x} = \frac{0 - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} - \left(-\frac{2xy}{(x^2+y^2)^2}\right)$$

$$\boxed{= 0}$$

(b) (5 points) Let C be the circle of radius R centered at the origin, oriented counter-clockwise. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Since C is a simple closed curve,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \int_0^{2\pi}\int_0^R 0 \cdot r \, dr \, d\theta$$

$$\boxed{= 0}$$

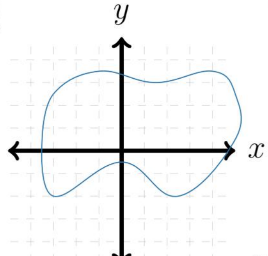
(c) (15 points) Let Q be the closed curve below, oriented counterclockwise. Compute $\int_Q \boldsymbol{F} \cdot d\boldsymbol{r}$.

Since Q is a simple closed curve, we can use Green's Theorem:

$$\int_Q \vec{F} \cdot d\vec{r} = \iint_{\partial Q} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \iint_{\partial Q} 0 \, dA$$

$$\boxed{= 0}$$



We also try to find a potential function:

$$\int \frac{x}{x^2+y^2} dx = \frac{\ln(x^2+y^2)}{2} + C$$

$$\int \frac{y}{x^2+y^2} dy = \frac{\ln(x^2+y^2)}{2} + C$$

Since we can find a potential function for $\vec{F}$ and Q is a simple closed curve,

$$\int_Q \vec{F} \cdot d\vec{r} \boxed{= 0.}$$

6.1 (a) **5 / 5**

- ✓ **+ 5 pts** **Correct.**
- **- 2 pts** Minor mistake in form of quotient rule.
- **- 1 pts** Dropped a negative in derivatives.
- **+ 0 pts** Incorrect.

6. Consider the vector field on $\mathbb{R}^2 - \{(0,0)\}$ given by

$$\boldsymbol{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

(a) (5 points) Compute $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\frac{\partial F_2}{\partial x} = \frac{0 - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} - \left(-\frac{2xy}{(x^2+y^2)^2}\right)$$

$$\boxed{= 0}$$

(b) (5 points) Let C be the circle of radius R centered at the origin, oriented counter-clockwise. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

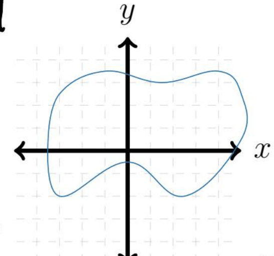
Since C is a simple closed curve,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \int_0^{2\pi}\int_0^R 0 \cdot r \, dr \, d\theta$$

$$\boxed{= 0}$$

(c) (15 points) Let Q be the closed curve below, oriented counterclockwise. Compute $\int_Q \boldsymbol{F} \cdot d\boldsymbol{r}$.



Since Q is a simple closed curve, we can use Green's Theorem:

$$\int_Q \vec{F} \cdot d\vec{r} = \iint_{\partial Q} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \iint_{\partial Q} 0 \, dA$$

$$\boxed{= 0}$$

We also try to find a potential function:

$$\int \frac{x}{x^2+y^2} dx = \frac{\ln(x^2+y^2)}{2} + C$$

$$\int \frac{y}{x^2+y^2} dy = \frac{\ln(x^2+y^2)}{2} + C$$

Since we can find a potential function for $\vec{F}$ and Q is a simple closed curve,

$$\int_Q \vec{F} \cdot d\vec{r} \boxed{= 0.}$$

6.2 (b) **1 / 5**

+ **1 pts** Parameterized curve
+ **2 pts** Setup integral
+ **2 pts** Evaluated integral correctly.
- **1 pts** Minor mistake
+ **0 pts** Incorrect

✓ + **1 pts** **Set up Green's theorem.**

NOTE: you cannot use Green's theorem because $$F$$ is not smooth at the origin, which is contained in the circle.

ALTERNATE : Show that $$F$$ is conservative.

+ **4 pts** Showed that $$F$$ is conservative by finding potential function.
+ **1 pts** Concluded result.
- **1 pts** Minor mistake in calculation of potential function.

1 Green's theorem cannot be used here because $$F$$ is not smooth at the origin, which is contained in the circle.

6. Consider the vector field on $\mathbb{R}^2 - \{(0,0)\}$ given by

$$\boldsymbol{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

(a) (5 points) Compute $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\frac{\partial F_2}{\partial x} = \frac{0 - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} - \left(-\frac{2xy}{(x^2+y^2)^2}\right)$$

$$\boxed{= 0}$$

(b) (5 points) Let C be the circle of radius R centered at the origin, oriented counterclockwise. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Since C is a simple closed curve,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \int_0^{2\pi}\int_0^R 0 \cdot r \, dr \, d\theta$$

$$\boxed{= 0}$$

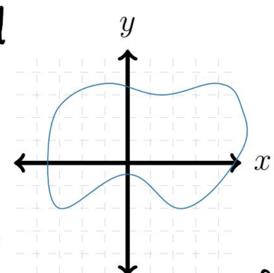
(c) (15 points) Let Q be the closed curve below, oriented counterclockwise. Compute $\int_Q \boldsymbol{F} \cdot d\boldsymbol{r}$.

Since Q is a simple closed curve, we can use Green's Theorem:

$$\int_Q \vec{F} \cdot d\vec{r} = \iint_{\partial Q} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \iint_{\partial Q} 0 \, dA$$

$$\boxed{= 0}$$



We also try to find a potential function:

$$\int \frac{x}{x^2+y^2} \, dx = \frac{\ln(x^2+y^2)}{2} + C$$

$$\int \frac{y}{x^2+y^2} \, dy = \frac{\ln(x^2+y^2)}{2} + C$$

Since we can find a potential function for $\vec{F}$ and Q is a simple closed curve,

$$\int_Q \vec{F} \cdot d\vec{r} \boxed{= 0.}$$

6.3 (c) **12 / 15**

Green's Theorem Method

+ **2 pts** Correct general formula for Green's theorem
+ **1 pts** Choose $$R$$ so that $$C_R$$ is within $$Q$$
+ **10 pts** Correct boundary to apply Green's theorem
+ **2 pts** Conclude based on previous part
- **1 pts** Minor mistake

Conservative Method

✓ + **10 pts Show that $$F$$ is conservative**

✓ + **5 pts Conclude that the integral is $$0$$**

- **4 pts** Significant error in finding potential function
- **1 pts** Small mistake in computing potential function

+ **0 pts** Incorrect or no valid justification

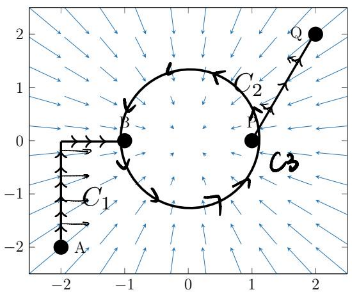
- 3 Point adjustment

Claimed that Green's theorem could be applied.

2 Green's theorem cannot be applied here because $$F$$ is not smooth at the origin, which is contained in the curve.

3 This is a correct method

7. **You must explain your reasoning.** Consider the following radial vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the circular path C_3 drawn above in the vector field. Since the vector field is perpendicular to the path everywhere, there is no tangential component of the vector field along C_3. Therefore, $\oint_{C_3} \vec{F} \cdot d\vec{r} = 0$. This is the case when C_3 has any radius, which means $\boxed{\vec{F} \text{ is conservative}}$

(b) (8 points) Suppose that for the two curves C_1 (left) and C_2 (right), we have

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 5$$

What is $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$?

Since $\vec{F}$ is conservative, $\int_{C_2} \vec{F} \cdot d\vec{r}$ can be seen as $\int_{C_1} \vec{F} \cdot d\vec{r}$ rotated 180° in the vector field. Their start and end points have the same distance between them. $\vec{F}$ is in opposite directions but have the same magnitudes.

$$\therefore \int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} \boxed{= -5}$$

(c) (8 points) Which is greater, $\int_{C_1} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$ or $\int_{C_2} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$? Or are they equal?

$\int_C \vec{F} \cdot \vec{n}\, ds$ is the integral of the normal component of $\vec{F}$ along C.
By inspection, we see the horizontal part of C_1 does not contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$, but vectors with larger magnitudes contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$ along the diagonal part of C_1 than that contribute to $\int_{C_2} \vec{F} \cdot \vec{n}\, ds$. Therefore, even though C_2 is longer than C_1, $\boxed{\int_{C_1} \vec{F} \cdot \vec{n}\, ds > \int_{C_2} \vec{F} \cdot \vec{n}\, ds.}$

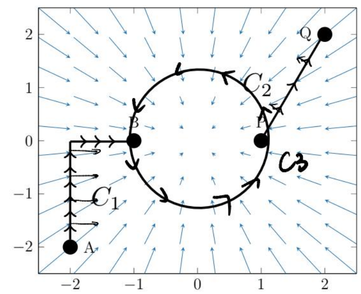
7.1 (a) **7 / 9**

+ **9 pts** Correct (found a potential function or drew level curves)
+ **0 pts** Incorrect or no explanation

✓ + **7 pts** **Provided evidence that F was path independent on the domain or that all circulation integrals in the domain will be 0.**

+ **7 pts** Showed curl was 0/cross partials condition, **and** tried to determine if the **domain of the vector field** was simply connected or not
+ **4 pts** Showed curl was 0 or checked cross partials condition. This is not sufficient to answer the question.

7. **You must explain your reasoning.** Consider the following radial vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the circular path C_3 drawn above in the vector field. Since the vector field is perpendicular to the path everywhere, there is no tangential component of the vector field along C_3. Therefore, $\oint_{C_3} \vec{F} \cdot d\vec{r} = 0$. This is the case when C_3 has any radius, which means $\boxed{\vec{F} \text{ is conservative}}$

(b) (8 points) Suppose that for the two curves C_1 (left) and C_2 (right), we have

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 5$$

What is $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$?

Since $\vec{F}$ is conservative, $\int_{C_2} \vec{F} \cdot d\vec{r}$ can be seen as $\int_{C_1} \vec{F} \cdot d\vec{r}$ rotated 180° in the vector field. Their start and end points have the same distance between them. $\vec{F}$ is in opposite directions but have the same magnitudes.

$$\therefore \int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} \boxed{= -5}$$

(c) (8 points) Which is greater, $\int_{C_1} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$ or $\int_{C_2} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$? Or are they equal?

$\int_C \vec{F} \cdot \vec{n}\, ds$ is the integral of the normal component of $\vec{F}$ along C.
By inspection, we see the horizontal part of C_1 does not contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$, but vectors with larger magnitudes contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$ along the diagonal part of C_1 than that contribute to $\int_{C_2} \vec{F} \cdot \vec{n}\, ds$. Therefore, even though C_2 is longer than C_1, $\boxed{\int_{C_1} \vec{F} \cdot \vec{n}\, ds > \int_{C_2} \vec{F} \cdot \vec{n}\, ds.}$

7.2 (b) **1 / 8**

+ **5 pts** Observed that F is path independent

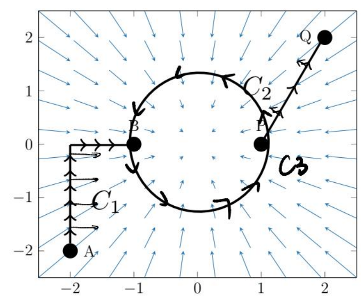
- **2 pts** Incorrect interpretation of path independence

+ **2 pts** Observed that A & Q, and B&P have the same potential value (observing the distances AB and PQ are the same is not sufficient)

✓ + **1 pts** **Observed that C1 goes from A to B, while C2 goes from Q to P**

+ **0 pts** Incorrect or no work

7. **You must explain your reasoning.** Consider the following radial vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the circular path C_3 drawn above in the vector field. Since the vector field is perpendicular to the path everywhere, there is no tangential component of the vector field along C_3. Therefore, $\oint_{C_3} \vec{F} \cdot d\vec{r} = 0$. This is the case when C_3 has any radius, which means $\boxed{\vec{F} \text{ is conservative}}$

(b) (8 points) Suppose that for the two curves C_1 (left) and C_2 (right), we have

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 5$$

What is $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$?

Since $\vec{F}$ is conservative, $\int_{C_2} \vec{F} \cdot d\vec{r}$ can be seen as $\int_{C_1} \vec{F} \cdot d\vec{r}$ rotated 180° in the vector field. Their start and end points have the same distance between them. $\vec{F}$ is in opposite directions but have the same magnitudes.

$$\therefore \int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} \boxed{= -5}$$

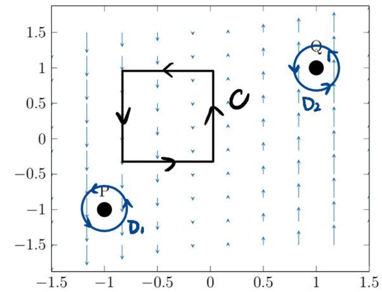
(c) (8 points) Which is greater, $\int_{C_1} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$ or $\int_{C_2} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$? Or are they equal?

$\int_C \vec{F} \cdot \vec{n}\, ds$ is the integral of the normal component of $\vec{F}$ along C.
By inspection, we see the horizontal part of C_1 does not contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$, but vectors with larger magnitudes contribute to $\int_{C_1} \vec{F} \cdot \vec{n}\, ds$ along the diagonal part of C_1 than that contribute to $\int_{C_2} \vec{F} \cdot \vec{n}\, ds$. Therefore, even though C_2 is longer than C_1, $\boxed{\int_{C_1} \vec{F} \cdot \vec{n}\, ds > \int_{C_2} \vec{F} \cdot \vec{n}\, ds.}$

7.3 (c) **4 / 8**

- **- 0 pts** Correct
- **- 8 pts** Incorrect or no reasoning
- ✓ **- 4 pts Partially correct reasoning (does not describe sign)**

8. **You must explain your reasoning.** Consider the following vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the square C drawn in the vector field above.

$\int \vec{F} \cdot d\vec{r}$ of the two horizontal parts cancel out because the curves are in different directions. $\int \vec{F} \cdot d\vec{r}$ of the right vertical part is zero because the vector field does not exist there. $\int \vec{F} \cdot d\vec{r}$ of the left vertical part is positive because the curve goes in the same direction as the vector field. therefore $\oint_{\text{closed square}} \vec{F} \cdot d\vec{r} \neq 0$, which means $\vec{F}$ is not conservative

(b) (8 points) Is divergence of $\boldsymbol{F}$ at $P = (-1, -1)$ positive, negative, or zero?

Let there be a small disk D_1 around P, by Green's Theorem:

$$\operatorname{div}(\vec{F})(P) \approx \frac{1}{\text{area}(D_1)} \oint_{\partial D_1} \vec{F} \cdot \vec{n} \, ds$$

Since all flux into D_1 leaves D_1, the net outward flux of $\vec{F}$ through $D_1 = 0$. Therefore, $\operatorname{div}(\vec{F})$ at $P = (-1, -1)$ is 0

(c) (8 points) Is $\operatorname{curl}_z \boldsymbol{F}$ at $Q = (1, 1)$ positive, negative, or zero?

Let there be a small disk D_2 around Q, by Green's Theorem:

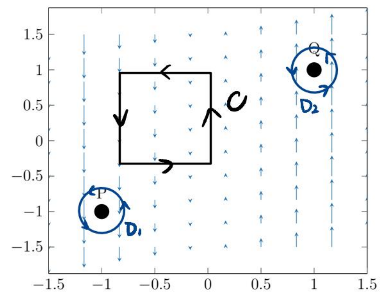
$$\operatorname{curl}_z(\vec{F})(Q) \approx \frac{1}{\text{area}(D_2)} \oint_{\partial D_2} \vec{F} \cdot d\vec{r}$$

Since ∂D_2 is a circle oriented counterclockwise and the vector field $\vec{F}$ is in has a greater magnitude on the right with an upward direction, $\oint_{\partial D_2} \vec{F} \cdot d\vec{r}$ should be positive. Therefore, $\operatorname{curl}_z(\vec{F})$ at $Q = (1, 1)$ should be positive

8.1 (a) **9 / 9**

- ✓ **- 0 pts** **Correct (F is not conservative)**
- **- 9 pts** Incorrect or no reasoning.
- **- 4 pts** Did not prove path dependence/ or did not prove that there is a curve with nonzero circuation
- **- 4 pts** Invalid or no justification for why curl is non-zero
- **- 6 pts** incorrectly stated curl = 0 (and did not check simply-connectedness)
- **- 6 pts** incorrectly stated level curves exist

8. **You must explain your reasoning.** Consider the following vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the square C drawn in the vector field above.

$\int \vec{F}\cdot d\vec{r}$ of the two horizontal parts cancel out because the curves are in different directions. $\int \vec{F}\cdot d\vec{r}$ of the right vertical part is zero because the vector field does not exist there. $\int \vec{F}\cdot d\vec{r}$ of the left vertical part is positive because the curve goes in the same direction as the vector field. therefore $\oint_{\text{closed square}} \vec{F}\cdot d\vec{r} \neq 0$, which means

$\vec{F}$ is not conservative

(b) (8 points) Is divergence of $\boldsymbol{F}$ at $P = (-1,-1)$ positive, negative, or zero?

Let there be a small disk D_1 around P, by Green's Theorem:

$$\text{div}(\vec{F})(P) \approx \frac{1}{\text{area}(D_1)} \oint_{\partial D_1} \vec{F}\cdot\vec{n}\, ds$$

Since all flux into D_1 leaves D_1, the net outward flux of $\vec{F}$ through $D_1 = 0$. Therefore, div($\vec{F}$) at $P=(-1,-1)$ is 0

(c) (8 points) Is $\text{curl}_z\boldsymbol{F}$ at $Q = (1, 1)$ positive, negative, or zero?

Let there be a small disk D_2 around Q, by Green's Theorem:

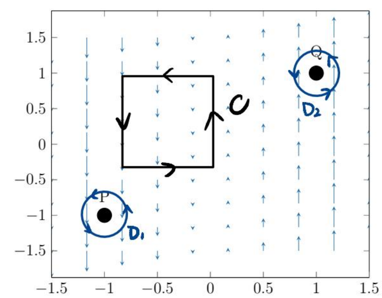
$$\text{curl}_z(\vec{F})(Q) \approx \frac{1}{\text{area}(D_2)} \oint_{\partial D_2} \vec{F}\cdot d\vec{r}$$

Since ∂D_2 is a circle oriented counterclockwise and the vector field $\vec{F}$ is in has a greater magnitude on the right with an upward direction, $\oint_{\partial D_2} \vec{F}\cdot d\vec{r}$ should be positive. Therefore, $\text{curl}_z(\vec{F})$ at $Q=(1,1)$ should be positive

8.2 (b) **8 / 8**

- ✓ **- 0 pts** Correct
- **- 8 pts** Incorrect or no explanation
- **+ 4 pts** Incorrect/unexplained answer, but interpreted divergence correctly

8. **You must explain your reasoning.** Consider the following vector field $\boldsymbol{F}$:



(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

Refer to the square C drawn in the vector field above.

$\int \vec{F}\cdot d\vec{r}$ of the two horizontal parts cancel out because the curves are in different directions. $\int \vec{F}\cdot d\vec{r}$ of the right vertical part is zero because the vector field does not exist there. $\int \vec{F}\cdot d\vec{r}$ of the left vertical part is positive because the curve goes in the same direction as the vector field. Therefore $\oint_{\text{closed square}} \vec{F}\cdot d\vec{r} \neq 0$, which means $\vec{F}$ is not conservative

(b) (8 points) Is divergence of $\boldsymbol{F}$ at $P = (-1, -1)$ positive, negative, or zero?

Let there be a small disk D_1 around P, by Green's Theorem:

$$\operatorname{div}(\vec{F})(P) \approx \frac{1}{\operatorname{area}(D_1)} \oint_{\partial D_1} \vec{F}\cdot\vec{n}\, ds$$

Since all flux into D_1 leaves D_1, the net outward flux of $\vec{F}$ through $D_1 = 0$. Therefore, $\operatorname{div}(\vec{F})$ at $P = (-1,-1)$ is 0

(c) (8 points) Is $\operatorname{curl}_z \boldsymbol{F}$ at $Q = (1, 1)$ positive, negative, or zero?

Let there be a small disk D_2 around Q, by Green's Theorem:

$$\operatorname{curl}_z(\vec{F})(Q) \approx \frac{1}{\operatorname{area}(D_2)} \oint_{\partial D_2} \vec{F}\cdot d\vec{r}$$

Since ∂D_2 is a circle oriented counterclockwise and the vector field $\vec{F}$ is in has a greater magnitude on the right with an upward direction, $\oint_{\partial D_2} \vec{F}\cdot d\vec{r}$ should be positive. Therefore, $\operatorname{curl}_z(\vec{F})$ at $Q = (1,1)$ should be positive

8.3 (c) **8 / 8**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** interpreted CCW circulation correctly, but said negative curl
- **- 8 pts** Incorrect or no explanation.
- **+ 4 pts** Incorrect/unexplained answer, but interpreted curl correctly
- **- 3 pts** correct reasoning but incorrect answer (claimed that F was conservative)
- **- 2 pts** correct reasoning (but did not (or incorrectly) explain why $$\frac{\partial F_2}{\partial x}$$ is positive)