# Final Exam

Monday, March 14, 2022 12:08 PM

1. Let $S$ be the diamond in the $xy$-plane with vertices $A = (\pi, 0)$, $B = (2\pi, \pi)$, $C = (\pi, 2\pi)$, $D = (0, \pi)$, and consider the change of variables

$$G(u, v) = \left\langle \frac{u+v}{2}, \frac{u-v}{2} \right\rangle$$

(a) (7 points) Describe, as a set, the region $S_0$ in the $uv$-plane that $G$ **maps to** $S$.

$$\left( \frac{u+v}{2}, \frac{u-v}{2} \right) = (\pi, 0)$$

$$\frac{u+v}{2} = \pi \rightarrow \frac{2u}{2} = \pi \rightarrow u = \pi, \ v = \pi$$

$$\frac{u-v}{2} = 0 \rightarrow u = v$$

$$\left( \frac{u+v}{2}, \frac{u-v}{2} \right) = (2\pi, \pi)$$

$$\frac{u+v}{2} = 2\pi \rightarrow u + \cancel{v} = 4\pi$$

$$\frac{u-v}{2} = \pi \qquad + \ u - \cancel{v} = 2\pi$$

$$2u = 6\pi \rightarrow u = 3\pi, \ v = \pi$$

$$\left( \frac{u+v}{2}, \frac{u-v}{2} \right) = (\pi, 2\pi)$$

$$\frac{u+v}{2} = \pi \rightarrow u + v = 2\pi$$

$$\frac{u-v}{2} = 2\pi \rightarrow u - v = 4\pi$$

$$2u = 6\pi \rightarrow u = 3\pi, \ v = -\pi$$

$$\left( \frac{u+v}{2}, \frac{u-v}{2} \right) = (0, \pi)$$

$$\frac{u+v}{2} = 0 \rightarrow u = -v$$

$$\frac{u-v}{2} = \pi \rightarrow -v - v = 2\pi \rightarrow v = -\pi$$

$$\frac{u-v}{2} = \pi \rightarrow -v-v = 2\pi \rightarrow v = -\pi$$

$$u = \pi$$

$$\boxed{S_0 : \{(u,v) \mid \pi \le u \le 3\pi,\ -\pi \le v \le \pi\}}$$

(b) (8 points) Compute the determinant of the Jacobian matrix, Jac($G$).

$$\text{Jac}(G) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left| -\frac{1}{4} - \frac{1}{4} \right| = \boxed{\frac{1}{2}}$$

(c) (10 points) Use your work from the previous parts to compute

$$\iint_S (x-y)^2 \sin^2(x+y)\, dx\, dy$$

$$= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} \left(\frac{u+v}{2} - \frac{u-v}{2}\right)^2 \sin^2\left(\frac{u+v}{2} + \frac{u-v}{2}\right) \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} v^2 \sin^2 u\, du\, dv$$

$$= \frac{1}{2} \left(\int_{-\pi}^{\pi} v^2\, dv\right) \left(\int_{\pi}^{3\pi} \sin^2 u\, du\right)$$

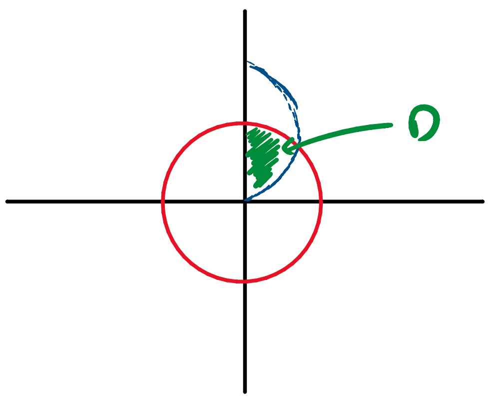
$$= \frac{1}{2} \left(\frac{v^3}{3}\right)\Big|_{-\pi}^{\pi} \left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)\Big|_{\pi}^{3\pi}$$

$$= \frac{1}{2}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right)\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi^3}{3}\right)(\pi) = \boxed{\frac{\pi^4}{3}}$$

2. Let $D$ be the region in the first quadrant inside the circle of radius 1, and bounded by the polar spiral $r = \theta$ and the $y$-axis.

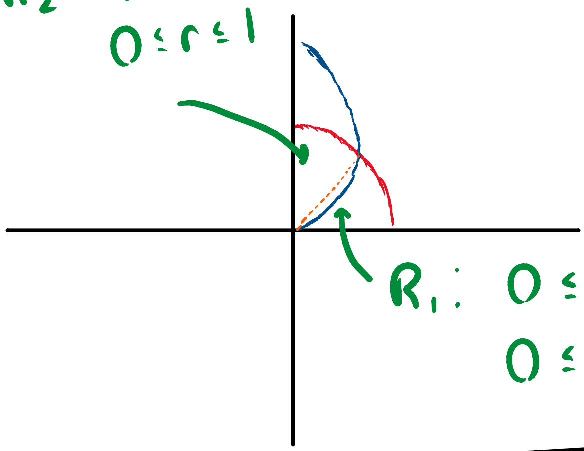
   (a) (5 points) Sketch the region D.




(b) (10 points) Express the region $D$ as a (union of) radially simple region(s).

$R_2$: $1 \le \theta \le \frac{\pi}{2}$
$0 \le r \le 1$

$r = 1 = \theta$



$R_1$: $0 \le \theta \le 1$
$0 \le r \le \theta$

$$\boxed{\begin{aligned} D &= R_1 \cup R_2, \text{ where} \\ R_1 &= \{(r,\theta) \mid 0 \le r \le \theta,\ 0 \le \theta \le 1\} \\ R_2 &= \{(r,\theta) \mid 0 \le r \le 1,\ 1 \le \theta \le \tfrac{\pi}{2}\} \end{aligned}}$$

(c) (10 points) Use your work from the previous parts to compute

$$\iint_D \sqrt{x^2+y^2}\, dA$$

$R_1$: $\int_0^1 \int_0^\theta \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \quad r\, \partial r \partial\theta$

$= \int_0^1 \int_0^\theta r^2 \partial r \partial\theta$

$= \int_0^1 \left(\frac{\theta^3}{3}\right) \partial\theta = \frac{1}{12}$

$R_2$: $\int_1^{\frac{\pi}{2}} \int_0^1 r^2 \partial r \partial\theta = \left(\frac{\pi}{2} - 1\right)\left(\frac{1}{3}\right) = \frac{\pi - 2}{6}$

$\frac{1}{12} + \frac{\pi-2}{6} = \frac{1+2\pi-4}{12} = \boxed{\frac{2\pi-3}{12}}$

3. The velocity vector field of a fluid is given by

$$\boldsymbol{v} = \langle 0, x^2+y^2, z^2 \rangle$$

(a) (10 points) Find the flow rate of $\boldsymbol{v}$ in the **negative** $z$-direction through the disk

$$D = \{(x, y, 0) \mid x^2+y^2 \le 1\}$$

$G(x,y) = \langle x, y, 0 \rangle$

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle \quad 0 \le \theta \le 2\pi, \; 0 \le r \le 1$

$\vec{N} = \langle 0, 0, -1 \rangle$

$$\int_0^{2\pi}\int_0^1 \langle 0, r^2, 0\rangle \cdot \langle 0,0,-1\rangle \, \partial r \partial \theta$$

$$= \boxed{0}$$

(b) (15 points) Find the flow rate of $\boldsymbol{v}$ in the **positive** $z$-direction through the hemisphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$$

$$G(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \frac{\pi}{2}$$

$$T_\theta = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0 \rangle$$
$$T_\phi = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi \rangle$$

$$N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta\sin\phi & \cos\theta \sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix}$$

$$= \hat{i}(-\cos\theta \sin^2\phi) + \hat{j}(-\sin\theta \sin^2\phi) + \hat{k}(-\sin\phi\cos\phi)$$

$$= -\sin\phi \langle \cos\theta, \sin\theta \sin\phi, \cos\phi \rangle$$

Outward Pointing $\vec{N} = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \langle 0, \sin^2\phi, \cos^2\phi \rangle \cdot \langle \cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi \rangle \sin\phi \, \partial\phi \partial\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (\sin^4\phi \sin\theta + \cos^3\phi \sin\phi) \, \partial\phi \partial\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^4\phi) \, \partial\phi \underbrace{\int_0^{2\pi} \sin\theta \, \partial\theta}_{\to 0} + \left(\int_0^{\frac{\pi}{2}} \cos^3\phi \sin\phi \, \partial\phi\right)\left(\int_0^{2\pi} \partial\theta\right)$$

$$= \left(\frac{1}{4}\right)(2\pi) = \boxed{\frac{\pi}{2}}$$

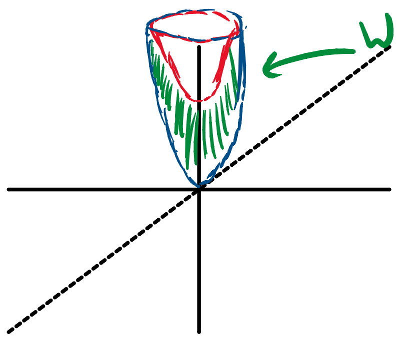
$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$-\int u^3 du$

$= \left(-\frac{\cos^4\phi}{4}\right)\Big|_0^{\frac{\pi}{2}}$

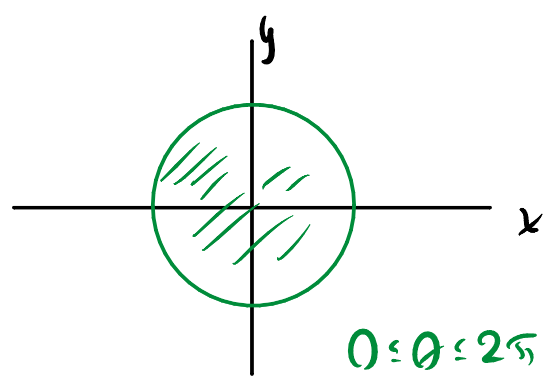
4. (a) (5 points) Let $W$ be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Sketch $W$.




(b) (10 points) Let $W$ be a solid region in $\mathbb{R}^3$ bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 4$. Find the volume of $W$.

$$2x^2 + 2y^2 = x^2 + y^2 + 4$$

$$x^2 + y^2 = 4$$




$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$W = \int_0^{2\pi}\int_0^2\int_{2r^2}^{r^2+4} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(r^2 + 4 - 2r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r - r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 d\theta$$

$$= 2\pi(8-4) = \boxed{8\pi}$$

(c) (10 points) Let $\partial W$ denote the boundary of $W$, oriented by outward-pointing normal vectors. Compute

$$\iint_{\partial W} \left\langle x + xy + \sin(z), x^3 + 3y - \frac{y^2}{2}, 4z + \cos(y) \right\rangle \cdot d\boldsymbol{S}$$

$$\vec{F} = \langle x + xy + \sin(z), x^3 + 3y - \frac{y^2}{2}, 4z + \cos(y) \rangle$$

$$\partial iv(\vec{F}) = (1+y) + (3-y) + (4)$$

$$= 8$$

$$\iiint_W 8\, \partial V = 8 \iiint_W \partial V = 8(8\pi) = \boxed{64\pi}$$

5. (a) (10 points) Let $D$ be the region in $\mathbb{R}^2$ given by

$$D = \{(x, y) \mid 0 \leq x \leq 1, x^{2/3} \leq y \leq 1\}$$

Let $\partial D$ denote the boundary of $D$, with the boundary orientation. Use Green's theorem to rewrite the following circulation integral as a double integral over $D$.

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$

$$\vec{F} = \langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \rangle$$

$$\frac{\partial F_2}{\partial x} = -3\sin(x) + 2 + xe^{y^4}$$

$$\frac{\partial F_1}{\partial y} = -3\sin(x) - 1$$

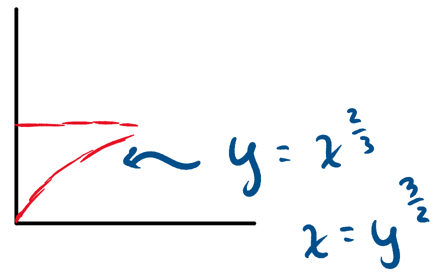
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3\sin(x) + 2 + xe^{y^4} + 3\sin(x) + 1$$

$$= 3 + xe^{y^4}$$

$$\boxed{\int_0^1 \int_{x^{2/3}}^1 \left(3 + x e^{y^4}\right) \partial y \, \partial x}$$

(b) (15 points) Use your work from the previous part to compute

$$\oint_{\partial D} \left\langle x^3 - 3y\sin(x) - y, 3\cos(x) + 2x + \frac{x^2 e^{y^4}}{2} \right\rangle \cdot d\boldsymbol{r}$$




$$\int_0^1 \int_{x^{2/3}}^1 \left(3 + x e^{y^4}\right) \partial y \, \partial x$$

$$= \int_0^1 \int_0^{y^{\frac{3}{2}}} \left(3 + x e^{y^4}\right) \partial x \, \partial y$$

$$= \int_0^1 \left(3x + \frac{x^2}{2} e^{y^4}\right)\Big|_0^{y^{\frac{3}{2}}} \partial y$$

$$= \int_0^1 \left(3y^{\frac{3}{2}} + \frac{y^3}{2} e^{y^4}\right) \partial y$$

$$= \left(\frac{6y^{\frac{5}{2}}}{5} + \frac{1}{8} e^{y^4}\right)\Big|_0^1$$

$$= \left(\frac{6}{5} + \frac{1}{8} e - \frac{1}{8}\right)$$

$$= \frac{43}{40} + \frac{5e}{40} = \boxed{\frac{43 + 5e}{40}}$$

6. Consider the vector field on $\mathbb{R}^2 - \{(0,0)\}$ given by

$$\boldsymbol{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

(a) (5 points) Compute $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\frac{\partial F_2}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{-x(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2xy+2xy}{(x^2+y^2)^2} = \boxed{0}$$

(b) (5 points) Let $C$ be the circle of radius $R$ centered at the origin, oriented counter-clockwise. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

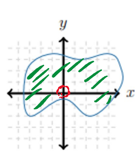
$$\vec{r}(t) = \langle R\cos t, R\sin t \rangle \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -R\sin t, R\cos t \rangle$$

$$\int_0^{2\pi} \left\langle \frac{R\cos t}{R^2}, \frac{R\sin t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle \, \partial t$$

$$= \int_0^{2\pi} (\cos t \sin t - \cos t \sin t)\, \partial t = \boxed{0}$$

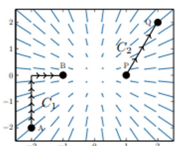
(c) (15 points) Let $Q$ be the closed curve below, oriented counterclockwise. Compute $\int_Q \boldsymbol{F} \cdot d\boldsymbol{r}$.




Since $(0,0)$ is in the region, it must be split; let there be a small circle of radius $R$

centered at $(0,0)$.

The circulation of the region excluding this circle is zero since $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ and $(0,0)$ is excluded from the region. The circle containing $(0,0)$ was computed to possess a circulation of zero, so $\int_Q \vec{F} \cdot d\vec{r} = 0 + 0 = \boxed{0}$




(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

A vector field is conservative if its curl is zero. If $(0,0)$ is defined on $\vec{F}$'s domain, then $\vec{F}$ is conservative since, about any point, the curl (circulation per unit area) is zero.

(b) (8 points) Suppose that for the two curves $C_1$ (left) and $C_2$ (right), we have

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 5$$

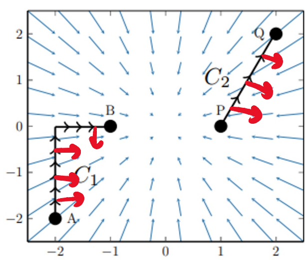
What is $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$?

Given that $\vec{F}$ is conservative, its line integrals are path independent. Moreso, $\vec{F}$ is symmetric since it is a radial vector field. The curves $C_1$ and $C_2$ start and end at the same distance from the origin, so they will have the same magnitude. Since $C_1$ goes in an opposite direction from $C_2$ though, they will differ in sign!

$$\boxed{\int_{C_2} \vec{F}\cdot d\vec{r} = -5}$$

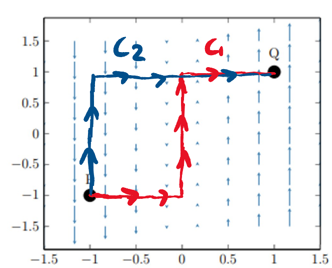
(c) (8 points) Which is greater, $\int_{C_1} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$ or $\int_{C_2} (\boldsymbol{F} \cdot \boldsymbol{n})\, ds$? Or are they equal?

$\int_{C_1} (\vec{F}\cdot\vec{n})\, ds$ is greater.




The flux for $C_2$ is negative since $\vec{F}$ points in the opposite direction from positive orientation. For $C_1$, $\vec{F}$ points in the same direction, so its flux is positive.

8. **You must explain your reasoning.** Consider the following vector field $\boldsymbol{F}$:




(a) (9 points) Is $\boldsymbol{F}$ a conservative vector field?

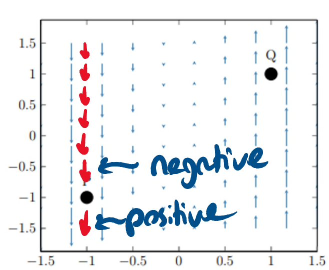
$\vec{F}$ is not conservative since its line integral is path dependent.

The line integral for $C_1$ is zero (it is normal to $\vec{F}$ for the two horizontal sections and for the vertical section $\vec{F} = \langle 0,0 \rangle$).

However, the line integral for $C_2$ is nonzero (it is normal to $\vec{F}$ for the horizontal section but the vertical section has a tangential component that contributes to a negative value).

(b) (8 points) Is divergence of $\boldsymbol{F}$ at $P = (-1,-1)$ positive, negative, or zero?

The divergence is zero; it can be interpreted as the outward flux near $P$ but the fluxes in opposite directions cancel out.




(c) (8 points) Is $\text{curl}_z \boldsymbol{F}$ at $Q = (1,1)$ positive, negative, or zero?

$\text{curl}_z \vec{F}$ should be positive since it represents the circulation about $Q$; the tangential component in the counterclockwise direction is greater than the tangential component in the clockwise direction, so it is positive.