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Midterm Exam II – Mathematics 32B
February 28, 2014

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1. (10 pts.) Compute $\int_C f\,ds$ when C is traced by $\vec{c}(t) = \langle t - \cos t, 1 - \sqrt{2}\sin t, 1 - \cos t\rangle$, $0 \le t \le \pi/2$, and $f(x,y,z) = 1 - z$.

$c'(t) = \langle 1 + \sin t, -\sqrt{2}\cos t, \sin t\rangle$

$\|c'(t)\| = \sqrt{(1+\sin t)^2 + (-\sqrt{2}\cos t)^2 + (-\sin t)^2}$

$= \sqrt{1 + 2\sin t + \sin^2 t + 2\cos^2 t + \sin^2 t}$

$= \sqrt{1 + 2\sin t + 2} = \sqrt{3 + 2\sin t}$ ✓

$\int_C f\,ds = \int_0^{\pi/2} f(\vec{c}(t))\,\|c'(t)\|\,dt$

$f(\vec{c}(t)) = 1 - (1 - \cos t) = \cos t$ ✓

$\int_0^{\pi/2} (\cos t)\sqrt{3 + 2\sin t}\,dt$ ✓

$u = 3 + 2\sin t \qquad \begin{matrix} t & & u \\ \pi/2 & \to & 5 \\ 0 & \to & 3 \end{matrix}$

$du = 2\cos t$

$\int_3^5 \frac{1}{2} u^{1/2}\,du = \frac{1}{3} u^{3/2} \Big|_3^5 = \frac{5\sqrt{5} - 3\sqrt{3}}{3}$ ✓

2. (10 pts.) For each of the following vector fields either *show* that it is not conservative or find a potential function for it

$$\vec{F} = \langle z^2y + 1, z^2x, 2xyz + 1\rangle \qquad \vec{G} = \langle z^2y, z^2x + 1, 2xy\rangle$$

$\int F_1 dx \stackrel{?}{=} \int F_2 dy \stackrel{?}{=} \int F_3 dz$

$\int z^2y + 1\, dx = z^2xy + x + f(y,z)$

$\int z^2x\, dy = z^2xy + g(x,z)$

$\int 2xyz + 1\, dz = z^2xy + z + h(x,y)$

$\vec{F} = \nabla V$ for $V = z^2xy + x + z$

$\int G_1 dx \stackrel{?}{=} \int G_2 dy \stackrel{?}{=} \int G_3 dz$

$\int z^2y\, dx = z^2xy + f(y,z)$

$\int z^2x + 1\, dy = z^2xy + y + g(x,z)$

$\int 2xy\, dz = 2xyz + h(x,y)$

$z^2xy + f(y,z) \neq 2xyz + h(x,y)$

for any $f(y,z)$, $h(x,y)$.

So $\vec{G}$ is not conservative.

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3. Let Σ be the surface parameterized by

$$G(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u)$$

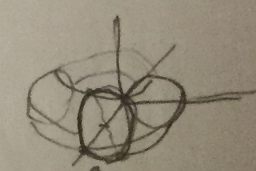
with $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

(a) (2 pts.) What is this thing? What is the curve traced by $G(u, 0)$ as u goes from 0 to 2π?

$G(u,0) = (2+\cos u,\ 0,\ \sin u)$

traces a circle in the xz plane.

This shape is (I think) a doughnut of sorts with no inner hole.



↑ $G(u,0)$

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(b) (4 pts.) Find a unit normal to this surface at $G(u, v)$.

normal = $G_u \times G_v$

$G_u = \langle -\sin u \cos v, -\sin u \sin v, \cos u \rangle$

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$G_v = \langle -(2+\cos u)\sin v, (2+\cos u)\cos v, 0 \rangle$

$\vec{n} = G_u \times G_v = \langle -\cos u(2+\cos u)\cos v, -\cos u(2+\cos u)\sin v, -\sin u\cos^2 v(2+\cos u) - \sin u \sin^2 v(2+\cos u) \rangle$

$\|\vec{n}\| = \sqrt{((2\cos u+\cos^2 u)\cos v)^2 + ((-2\cos u-\cos^2 u)\sin v)^2 + (-2\sin u-\sin u\cos u)^2}$

$= \sqrt{(2\cos u+\cos^2 u)^2 + (-2\sin u-\sin u\cos u)^2}$

$= \sqrt{4\cos^2 u + 2\cos^3 u + \cos^4 u + 4\sin^2 u + 2\sin^2 u\cos u + \sin^2 u\cos^2 u}$

$= \sqrt{4 + 2\cos^3 u + \cos^4 u + 2\sin^2 u\cos u + \sin^2 u\cos^2 u}$

unit normal = $\dfrac{\text{this}}{\text{this}}$ how ugly...

(c) (4 pts.) Find the area of this surface.

$\iint_S 1\,dS = \int_0^{2\pi}\int_0^{2\pi} \|\vec{n}\|\,du\,dv = 2\pi\int_0^{2\pi} \sqrt{4+2\cos^3 u+\cos^4 u+2\sin^2 u\cos u+\sin^2 u\cos^2 u}\,du$

uhh...

Surface Area of Doughnut

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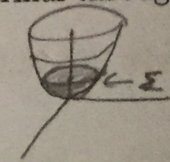
4. (10 pts.) Suppose that $\vec{F} = \langle x, 0, z\rangle$ is the velocity vector field for a fluid. Compute the flux for this vector field with respect to the upward normal through the portion Σ of the surface $z = x^2 + y^2$ where $z \leq 1$.



$z=1 \quad z=x^2$

$G(x,y) = \langle x, y, x^2+y^2\rangle$

$\vec{n} = \langle -2x, -2y, 1\rangle$

$$\text{Flux} = \iint_\Sigma \vec{F}\cdot\vec{N}\,dS = \iint_{x^2+y^2\leq 1} \langle x, 0, x^2+y^2\rangle\cdot\langle -2x,-2y,1\rangle\,dA$$

$$= \iint_{x^2+y^2\leq 1} -2x^2+x^2+y^2\,dy\,dx = \iint_{x^2+y^2\leq 1} y^2-x^2\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (r^2\sin^2\theta - r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^3(\cos^2\theta-\sin^2\theta)\,dr\,d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ I think

$$= \int_0^{2\pi}\int_0^1 -r^3(\cos 2\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{r^4}{4}\Big|_0^1 \cos 2\theta\,d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi}\cos 2\theta\,d\theta$$

$$= -\frac{1}{4}\cdot\frac{1}{2}\sin 2\theta\Big|_0^{2\pi} = -\frac{1}{8}(\sin 4\pi - \sin 0) = 0$$ that can't be right.

5. Consider the vector field $\vec{F} = \langle x/\rho, y/\rho, z/\rho \rangle$ where ρ is the spherical coordinate, $\rho = (x^2+y^2+z^2)^{1/2}$.

(a) (5 pts.) If C the straight line starting at (1,1,1) and ending at (10,10,10), what is the value of $\int_C \vec{F}\cdot\vec{T}ds$, where $\vec{T}$ is the unit tangent vector to C?

$\vec{c}(t) = \langle 1,1,1\rangle + \langle 9t, 9t, 9t\rangle \quad 0 \le t \le 1$

$\vec{c}(t) = \langle 9t+1, 9t+1, 9t+1\rangle$

$\vec{c}\,'(t) = \langle 9,9,9\rangle$

$\int_C \vec{F}\cdot\vec{T}ds = \int_0^1 \left\langle \frac{9t+1}{\rho_a}, \frac{9t+1}{\rho_a}, \frac{9t+1}{\rho_a}\right\rangle\cdot\langle 9,9,9\rangle dt$

where $\rho_a = \sqrt{3(9t+1)^2}$

$= \int_0^1 27\left(\frac{9t+1}{\rho_a}\right)dt = 27\int_0^1 \frac{9t+1}{\sqrt{3(9t+1)^2}}dt = \frac{27}{\sqrt{3}}\int_0^1 1dt = \frac{27}{\sqrt{3}} = \frac{27\sqrt{3}}{3} = 9\sqrt{3}$

or, $\vec{F} = \vec{\nabla}V$

$\int \frac{x}{\sqrt{x^2+y^2+z^2}}dx = \frac{1}{2}\int u^{-1/2}du = \frac{2}{2}u^{1/2} = \sqrt{x^2+y^2+z^2}$

$u = x^2+y^2+z^2$

$du = 2x\,dx$

$\int_C \vec{F}\cdot\vec{T}ds = V(\vec{c}(1)) - V(\vec{c}(0))$

$= \sqrt{10^2+10^2+10^2} - \sqrt{1^2+1^2+1^2}$

$= 10\sqrt{3} - \sqrt{3} = 9\sqrt{3}$ ok good they match...

(b) (5 pts.) Suppose now that instead of the curve C in part (a), C_1 is the curve traced by $\vec{c}(t) = \langle t, t^2/10, t^3/100\rangle$ as t goes from 1 to 10. If $\vec{F}$ remains the same, what is $\int_{C_1}\vec{F}\cdot\vec{T}ds$?

$\vec{c}(10) = \langle 10, 10, 10\rangle \quad \vec{c}(1) = \langle 1, 1/10, 1/100\rangle$

$\vec{F} = \vec{\nabla}V$

$\int_{C_1} \vec{F}\cdot\vec{T}ds = V(\vec{c}(10)) - V(\vec{c}(1)) = \sqrt{3(10^2)} - \sqrt{1+\frac{1}{100}+\frac{1}{100^2}}$

$\frac{100^2+100+1}{100^2}$

$= 10\sqrt{3} - \sqrt{1+\frac{1}{100}+\frac{1}{100^2}}$

$= 10\sqrt{3} - \frac{\sqrt{100^2+100+1}}{100}$

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