MATH 32B
SECOND MIDTERM EXAMINATION: SOLUTIONS

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:______________________________ Section:____________________

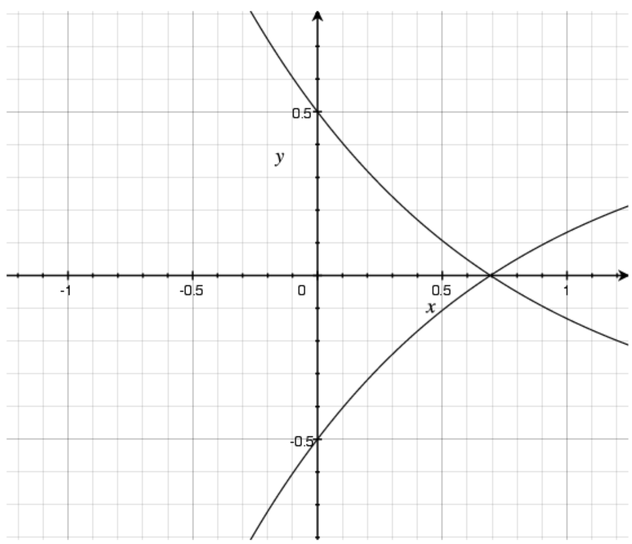
#1	#2	#3	#4	#5		Total

Problem 1. Let D be a constant density lamina (flat plate) on the xy-plane, bounded by the y-axis and the curves

$$\begin{aligned} y - e^{-x} &= -\frac{1}{2}, \\ y + e^{-x} &= \frac{1}{2}. \end{aligned}$$

for $0 \leq x \leq 1$. The density of the lamina is $\delta(x, y) = 1$ for all (x, y). Find the moments M_x and M_y of this lamina with respect to both axes.

Solution. The two curves are given by $y = \pm(e^{-x} - \frac{1}{2})$ and are reflections of one another about the x-axis. They intersect at $e^{-x} = \frac{1}{2}$, i.e., $x = \log 2$. The region bounded by the curves is shown below.



Since the region is clearly symmetric about the x-axis,

$$M_x = 0.$$

The other moment is given by:

$$\begin{aligned} M_y &= \iint_D x dA = \\ &= \int_0^{\log 2} \int_{-e^{-x}+1/2}^{e^{-x}\text{-}1/2} x dy dx = \\ &= \int_0^{\log 2} (2xe^{-x} - x) dx = \\ &= \int_0^{\log 2} ((2xe^{-x} - 2e^{-x}) + 2e^{-x} - x) dx = \\ &= (-2xe^{-x} - 2e^{-x} - \frac{1}{2}x^2)\Big|_0^{\log 2} = 1 - \log 2 - \frac{1}{2}(\log 2)^2. \end{aligned}$$

Problem 2. Use the change of variables

$$\begin{aligned} u &= x - y \\ v &= x + y \end{aligned}$$

to evaluate the integral

$$\iint \sqrt{\frac{x-y}{x+y+1}}\, dA,$$

over region D, where D is the square with the vertices $(0,0)$, $(1,-1)$, $(2,0)$ and $(1,1)$.

Solution. We first find coordinate description of D in the new coordinates u, v. Since the transformation from x, y to u, v is linear, lines are taken to lines and parallel lines are taken to parallel lines. Therefore D will be taken to some parallelogram, which is completely determined by its vertices. We therefore proceed to find the u, v coordinates of the vertices of D.

$x = 0, y = 0$ corresponds to $u = 0$, $v = 0$
$x = 1$, $y = -1$ corresponds to $u = 2$, $v = 0$
$x = 2$, $y = 0$ corresponds to $u = 2$, $v = 2$
$x = 1$, $y = 1$ corresponds to $u = 0$, $v = 1$.

Thus the image of D in the (u, v) coordinate plane is a square with vertices $(0,0)$, $(0,2)$, $(2,0)$ and $(2,2)$.

Next, we find

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \left(\det\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}\right)^{-1} = \left(\det\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\right)^{-1} = \frac{1}{2}.$$

It follows that

$$\begin{aligned} \iint \sqrt{\frac{x-y}{x+y+1}}\, dA &= \int_0^2 \int_0^2 \sqrt{\frac{u}{v+1}}\, \frac{1}{2} du dv \\ &= \frac{1}{2}\int_0^2 \sqrt{u} du \cdot \int_0^2 \frac{1}{\sqrt{v+1}} dv \\ &= \frac{1}{3} u^{3/2}\Big|_0^2 \cdot 2\sqrt{v+1}\Big|_0^2 \\ &= \frac{2}{3}\sqrt{8}(\sqrt{3}-1). \end{aligned}$$

Problem 3. Consider the unit radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|},$$

where $\mathbf{r}= \langle x, y\rangle$.

(a) Compute the divergence of this vector field. Write down the final answer in terms of $\mathbf{r}$ and $|\mathbf{r}|$ only.

Solution. We have

$$F = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\rangle.$$

Thus

$$\frac{\partial}{\partial x}F_x = \frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}\frac{x\cdot 2x}{(x^2+y^2)^{3/2}} = \frac{1}{|\mathbf{r}|} - \frac{x^2}{|\mathbf{r}|^3}.$$

Similarly,

$$\frac{\partial}{\partial y}F_y = \frac{1}{|\mathbf{r}|} - \frac{y^2}{|\mathbf{r}|^3}.$$

Thus

$$\begin{aligned}
\mathrm{div}(\overrightarrow{F}) = \nabla\cdot F &= \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y \\
&= \frac{2}{|\mathbf{r}|} - \frac{x^2+y^2}{|\mathbf{r}|^3} = \frac{2}{|\mathbf{r}|} - \frac{|\mathbf{r}|^2}{|\mathbf{r}|^3} \\
&= \frac{1}{|\mathbf{r}|}.
\end{aligned}$$

(b) Compute the flux of this vector field accross the circle of radius R centered at the origin.

Solution. The unit normal vector to the circle of radius R centered at the origin is given by

$$\mathbf{n} = \frac{\langle x, y\rangle}{|\mathbf{r}|} = \frac{\mathbf{r}}{|\mathbf{r}|}.$$

Thus, the $\mathbf{n} = \mathbf{F}$ and $\mathbf{F}\cdot\mathbf{n} = 1$. Parametrize the circle by $\mathbf{r}(t) = R\langle \mathrm{cos}t, \mathrm{sin}t\rangle$. The flux across the circle is given by the integral

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^{2\pi} 1\cdot R dt = 2\pi R.$$

Problem 4. Let $\mathbf{F} = \langle -y, x \rangle$ be a vector field on the plane. Let C be the quarter of the unit circle on the plane lying in the first quadrant and traced in the counterclockwise direction (i.e., from point $(1, 0)$ to point $(0, 1)$).

(a) Set up and evaluate the line integral of $\mathbf{F}$ along C.

Solution. Let us parameterize C by $\mathbf{r}(t)\langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi/2$. The unit tangent vector to the circle is then given by $\mathbf{T}(t) = \langle -\sin t, \cos t \rangle$. Thus the line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{\pi/2} (\sin^2 t + \cos^2 t) dt \\ &= \int_0^{\pi/2} dt = \pi/2. \end{aligned}$$

(b) Let C' be any other path connecting the same two points. Can we expect that the value of the integral of $\mathbf{F}$ along this path will be the same? Explain your reasoning.

Solution. Let us first check to see if F is conservative. We compute curl. The only component which could be non-zero is the 3rd (z-component). It is equal to

$$\partial_x F_y - \partial_y F_x = 1 + 1 = 2 \neq 0.$$

Thus F is not conservative and its integral is not path-independent. Thus, we can not expect the value of the integral along a different path connecting the same two points to be the same.

Problem 5. Multiple-choice questions:

(1) The vector field $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$, where $r = \sqrt{x^2 + y^2 + z^2}$, is

(a) not conservative because $\text{curl}(\mathbf{F}) \neq 0$; **FALSE, see (d) or compute explicitly**

(b) not conservative because there is no function $f = f(x, y, z)$ such that $\nabla f = \mathbf{F}$; **FALSE, see (d)**

(c) is conservative with potential function $f(x, y, z) = r$; **TRUE**

(d) is conservative with potential function $f(x, y, z) = \frac{1}{r}$; **FALSE**

(2) Which of the following statements is *not* true:

(a) The value of a scalar line integral along a curve does not depend on a parametrization of the curve; **TRUE**

(b) Any vector field of the form $\mathbf{F} = \langle f(x), g(y), h(z) \rangle$ is conservative. **TRUE, it will be the gradient of $F(x) + G(y) + H(z)$ where F, G, H are antiderivatives of f, g, h respectively**

(c) Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a vector field on the plane. Then

$$\text{curl}(x \cdot \mathbf{F}) = \left\langle 0, 0,\ F_2 + x\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\rangle.$$

TRUE: If $G = \langle G_1, G_2 \rangle$ **then** $curl(G) = \langle 0, 0, \partial_x G_2 - \partial_y G_1 \rangle$ **and** $x \cdot F = \langle xF_1, xF_2 \rangle$; use the formula for $curl$ and the product rule.

(d) The circulation of the rotation vector field $\mathbf{F} =< -y, x >$ around the unit circle is equal to 0. **FALSE. The circulation is** $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$. **(The computation is essentially the same as in Problem 2a).**

(3) Circle the statement which is correct:

(a) Let T be a linear transformation on the plane such that the Jacobian of T is identically equal to 5. Let R be a rectangle of area 10. Then the area of the image $T(R)$ of rectangle R under the transformation T equals to 2. **FALSE: the area will be equal to** 50.

(b) Let T be the transformation that changes cylindrical coordinates into spherical. I.e., $T : (r, \theta, z) \mapsto (\rho, \theta, \varphi)$. Then the Jacobian of this transformation equals to

$\frac{4z}{z^2+r^2}$. **FALSE: either compute directly or write** $\frac{\partial(\rho,\theta,\phi)}{\partial(r,\theta,z)} = \frac{\partial(\rho,\theta,\phi)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \rho^2 \sin\theta \cdot \frac{1}{r} = \frac{(r^2+z^2)\sin\theta}{r}$.

(c) If a region on the plane is symmetric with respect to the y-axis and has uniform density, the center of mass lies on the y-axis. **TRUE: By symmetry, the integral representing M_y is zero.**

(d) The center of mass for a region on the plane with uniform density distribution always lies in this region. **FALSE: this is only true of the region is *convex*. As an exampe, consider a U-shaped region. It center of mass clearly does not lie inside of that region.**

(4) Let $\mathbf{F}(x,y) = \frac{\mathbf{i}+\mathbf{j}}{x^2+y^2}$ be a vector field. For each of the properties listed below mark whether it is true for this vector field or not:

(a) **T / F** All vectors point directly away from the origin (i.e., a vector attached to a point is parallel to the position vector for this point); **FALSE: all vector point in the direction** $\langle 1,1 \rangle$

(b) **T / F** The vector field has rotational symmetry about the origin; **FALSE – see (a)**

(c) **T / F** All vectors that start on a circle centered at the origin have the same length; **TRUE: the length depends only on x^2+y^2 which is constant along any circle centered at** $(0,0)$

(d) **T / F** None of the vectors are pointing in the North Western direction. **TRUE: all vectors point in the North-Eastern direction.**

(5) For the vector field $\overrightarrow{F} = \frac{\overrightarrow{r}}{r^3}$, where $\overrightarrow{r} = \langle x,y,z \rangle$ and $r = |\overrightarrow{r}|$, which of the following statements are true:
A: $\nabla \times \overrightarrow{F} = 0$ **TRUE**
B: $\nabla \cdot \overrightarrow{F} = 0$ **TRUE**

(a) **TRUE** both **A** and **B** are true;
(b) **FALSE** **A** is true and **B** is not true;
(c) **FALSE** **B** is true and **A** is not true;
(d) **FALSE** both **A** and **B** are not true;