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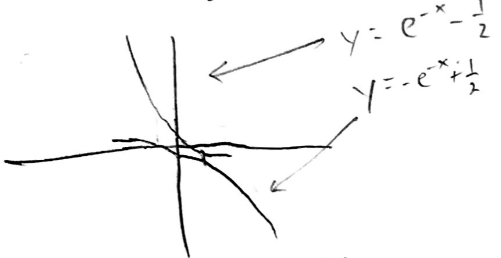
Problem 1. Let D be a constant density lamina (flat plate) on the xy-plane, bounded by the y-axis and the curves

$$y - e^{-x} = -\frac{1}{2},$$
$$y + e^{-x} = \frac{1}{2}.$$

for $0 \le x \le 1$. The density of the lamina is $\delta(x, y) = 1$ for all (x, y). Find the moments M_x and M_y of this lamina with respect to both axes.



$$M_x = \int_0^1 \int_{-e^{-x}+\frac{1}{2}}^{e^{-x}-\frac{1}{2}} y \, dy \, dx$$
$$= \int_0^1 \left[\frac{y^2}{2}\right]_{-e^{-x}+\frac{1}{2}}^{e^{-x}-\frac{1}{2}} dx$$
$$= \int_0^1 \frac{(e^{-x}-\frac{1}{2})^2}{2} - \frac{(-e^{-x}+\frac{1}{2})^2}{2} \, dx$$
$$= \int_0^1 \frac{e^{-2x} - e^{-x} + \frac{1}{4} - (e^{-2x} - e^{-x} + \frac{1}{4})}{2} \, dx$$
$$= \int_0^1 0 \, dx$$
$$= \boxed{0}$$

$$M_y = \int_0^1 \int_{-e^{-x}+\frac{1}{2}}^{e^{-x}-\frac{1}{2}} x \, dy \, dx$$
$$= \int_0^1 x\left(e^{-x} - \frac{1}{2} - \left(-e^{-x} + \frac{1}{2}\right)\right) dx$$
$$= \int_0^1 x(2e^{-x} - 1) \, dx$$

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$$= \int_0^1 2xe^{-x} - x \, dx$$

$u = 2x \quad dv = e^{-x}dx$
$du = 2dx \quad v = -e^{-x}$

$$= -2xe^{-x} - 2e^{-x} - \frac{x^2}{2}\Big]_0^1$$
$$= -2e^{-1} - 2e^{-1} - \frac{1}{2} + 2$$
$$= \boxed{-4e^{-1} + \frac{3}{2}}$$

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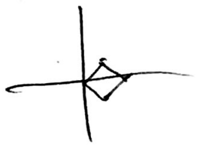
Problem 2. Use the change of variables

$$\begin{aligned} u &= x - y \\ v &= x + y \end{aligned}$$

to evaluate the integral

$$\iint \sqrt{\frac{x-y}{x+y+1}}\, dA,$$

over region D, where D is the square with the vertices $(0,0)$, $(1,-1)$, $(2,0)$ and $(1,1)$.



$\text{Jac } G^{-1} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

$\text{Jac } G = \frac{1}{2}$

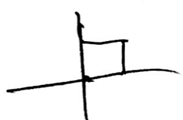
$(x,y) \quad (u,v)$

$(0,0) \to (0,0)$

$(1,-1) \to (2,0)$

$(2,0) \to (2,2)$

$(1,1) \to (0,2)$



$0 \le u \le 2$

$0 \le v \le 2$

$$\int_0^2 \int_0^2 \sqrt{\frac{u}{v+1}} \left(\frac{1}{2}\right) du\, dv$$

$$= \frac{1}{2}\int_0^2 \int_0^2 \frac{\sqrt{u}}{\sqrt{v+1}}\, du\, dv = \frac{1}{3}\int_0^2 \left[\frac{u^{3/2}}{\sqrt{v+1}}\right]_0^2 dv$$

$$= \frac{1}{3}\int_0^2 \frac{2^{3/2}}{\sqrt{v+1}}\, dv = \frac{2^{3/2}}{3}\int_0^2 (v+1)^{-\frac{1}{2}}\, dv$$

$$= \frac{4}{3}\left[(v+1)^{\frac{1}{2}}\right]_0^2 = \frac{4}{3}(3)^{\frac{1}{2}} - \frac{4}{3}(1) = \boxed{\frac{4}{3}(\sqrt{3}-1)}$$

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Problem 3. Consider the unit radial vector field 2D

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|},$$

where $\mathbf{r} = \langle x, y, \cancel{z}\rangle$.

(a) Compute the divergence of this vector field. Write down the final answer in terms of $\mathbf{r}$ and $|\mathbf{r}|$ only.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{|r|}\right) + \frac{\partial}{\partial y}\left(\frac{y}{|r|}\right)$$
$$= \frac{1}{|r|}$$

$$\frac{x}{\sqrt{x^2+y^2}}$$

$$x(x^2+y^2)^{-\frac{1}{2}}$$
$$(x^2+y^2)^{-\frac{1}{2}} + x(2x)(-\tfrac{1}{2})(x^2+y^2)^{-3/2}$$
$$(x^2+y^2)^{-\frac{1}{2}} - x^2(x^2+y^2)^{-3/2} + (x^2+y^2)^{-\frac{1}{2}}$$
$$-y^2(x^2+y^2)^{-3/2}$$
$$2(x^2+y^2)^{-\frac{1}{2}} - (x^2+y^2)(x^2+y^2)^{-3/2}$$
$$2(x^2+y^2)^{-\frac{1}{2}} - (x^2+y^2)^{-\frac{1}{2}}$$
$$(x^2+y^2)^{-\frac{1}{2}}$$

(b) Compute the flux of this vector field accross the circle of radius R centered at the origin.

$\vec{r}(t) = \langle R\cos t, R\sin t\rangle \quad 0 \le t < 2\pi$ ✓

$\vec{F}(\vec{r}(t)) = \langle \cos t, \sin t\rangle$ ✓

$\vec{r}\,'(t) = \langle -R\sin t, R\cos t\rangle$

$\vec{N}(t) = \langle -R\cos t, R\sin t\rangle$ –

check signs.

$$\int_0^{2\pi} \langle \cos t, \sin t\rangle \cdot \langle -R\cos t, R\sin t\rangle\, dt$$
$$= \int_0^{2\pi} -R\cos^2 t + R\sin^2 t\, dt \quad ✓$$
$$= -\int_0^{2\pi} \cos 2t\, dt$$
$$= -\tfrac{1}{2}\sin 2t\Big]_0^{2\pi}$$
$$= -\tfrac{1}{2}\sin 4\pi - 0$$
$$= 0$$

3.

Problem 4. Let $\mathbf{F} = \langle -y, x\rangle$ be a vector field on the plane. Let C be the quarter of the unit circle on the plane lying in the first quadrant and traced in the counterclockwise direction (i.e., from point $(1, 0)$ to point $(0, 1)$).

(a) Set up and evaluate the line integral of $\mathbf{F}$ along C.

$\vec{r}(t) = \langle \cos t, \sin t\rangle \quad 0 \le t \le \frac{\pi}{2}$

$\vec{F}(\vec{r}(t)) = \langle -\sin t, \cos t\rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t\rangle$

$\int_0^{\frac{\pi}{2}} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt$

$= \int_0^{\frac{\pi}{2}} 1 \, dt = \boxed{\frac{\pi}{2}}$

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(b) Let C' be any other path connecting the same two points. Can we expect that the value of the integral of $\mathbf{F}$ along this path will be the same? Explain your reasoning.

$\frac{\partial}{\partial x}(F_2) = 1$

$\frac{\partial}{\partial y}(F_1) = -1$

Since the cross partials condition is not satisfied, $\vec{F}$ is not a conservative vector field and so it does not have the path independence property so no.

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Problem 5. Multiple-choice questions:

(1) The vector field $\mathbf{F} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$, where $r = \sqrt{x^2+y^2+z^2}$, is $(\frac{1}{2}) 2x (x^2+y^2+z^2)^{-\frac{1}{2}}$

(a) not conservative because curl($\mathbf{F}$) $\neq 0$;

(b) not conservative because there is no function $f = f(x,y,z)$ such that $\nabla f = \mathbf{F}$;

2 (c) is conservative with potential function $f(x,y,z) = r$;

(d) is conservative with potential function $f(x,y,z) = \frac{1}{r}$;

(2) Which of the following statements is *not* true:

(a) The value of a scalar line integral along a curve does not depend on a parametrization of the curve;

(b) Any vector field of the form $\mathbf{F} = \langle f(x), g(y), h(z) \rangle$ is conservative.

(c) Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a vector field on the plane. Then

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$$\mathrm{curl}(x \cdot \mathbf{F}) = \left\langle 0, 0,\ F_2 + x\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\rangle.$$

(d) The circulation of the rotation vector field $\mathbf{F} =< -y, x >$ around the unit circle is equal to 0.

(3) Circle the statement which is correct:

(a) Let T be a linear transformation on the plane such that the Jacobian of T is identically equal to 5. Let R be a rectangle of area 10. Then the area of the image $T(R)$ of rectangle R under the transformation T equals to 2.

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(b) Let T be the transformation that changes cylindrical coordinates into spherical. I.e., $T : (r, \theta, z) \mapsto (\rho, \theta, \varphi)$. Then the Jacobin of this transformation equals to $\frac{4z}{z^2+r^2}$.

(c) If a region on the plane is symmetric with respect to the y-axis and has uniform density, the center of mass lies on the y-axis.

(d) Center of mass for a region on the plane with uniform density distribution always lies in this region.

(4) Let $\mathbf{F}(x, y) = \frac{\mathbf{i}+\mathbf{j}}{x^2+y^2}$ be a vector field. For each of the properties listed below mark whether it is true for this vector field or not:

$\left\langle \frac{1}{x^2+y^2}, \frac{1}{x^2+y^2} \right\rangle$

(a) **T** / **F** All vectors point directly away from the origin (i.e., a vector attached to a point is parallel to the position vector for this point);

(b) **T** / **F** The vector field has rotational symmetry about the origin;

0.5

(c) **T** / **F** All vectors that start on a circle centered at the origin have the same length;

(d) **T** / **F** None of the vectors are pointing in the North Western direction.

(5) For the vector field $\vec{F} = \frac{\vec{r}}{r^3}$, where $\vec{r} = \langle x, y, z \rangle$ and $r = |\vec{r}|$, which of the following statements are true:

A: $\nabla \times \vec{F} = 0$

B: $\nabla \cdot \vec{F} = 0$

(a) both **A** and **B** are true;

(b) **A** is true and **B** is not true;

(c) **B** is true and **A** is not true;

(d) both **A** and **B** are not true;