MATH 32B
SECOND MIDTERM EXAMINATION

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

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#1	#2	#3	#4	#5		Total
8	10	6	10	8		42

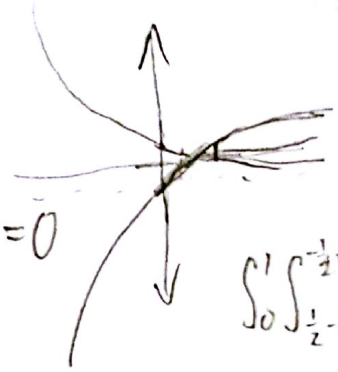
8

Problem 1. Let D be a constant density lamina (flat plate) on the xy-plane, bounded by the y-axis and the curves

$$y - e^{-x} = -\frac{1}{2}, \qquad y = -\tfrac{1}{2} + e^{-x}$$
$$y + e^{-x} = \frac{1}{2}. \qquad y = \tfrac{1}{2} - e^{-x}$$

for $0 \le x \le 1$. The density of the lamina is $\delta(x, y) = 1$ for all (x, y). Find the moments M_x and M_y of this lamina with respect to both axes.

$$M_x = \iint y \cdot \delta(x,y)\,dA = \int_0^1 \int_{\frac{1}{2}-e^{-x}}^{-\frac{1}{2}+e^{-x}} y \cdot 1\,dy\,dx =$$



$$\int_0^1 \int_{\frac{1}{2}-e^{-x}}^{-\frac{1}{2}+e^{-x}} f(x,y)\,dy\,dx$$

$$\frac{1}{2}\int_0^1 \left(\tfrac{1}{4} - e^{-x} + e^{-2x}\right) - \left(\tfrac{1}{4} - e^{-x} + e^{-2x}\right) dx = \frac{1}{2}\int_0^1 0\,dx = 0$$

$$M_x = 0$$

x

$$M_y = \iint x \cdot \delta(x,y)\,dA = \int_0^1 \int_{\frac{1}{2}-e^{-x}}^{-\frac{1}{2}+e^{-x}} x\,dy\,dx = \int_0^1 x(-1+2e^{-x})\,dx = \int_0^1 -x + 2xe^{-x}\,dx =$$

$$-\frac{x^2}{2}\Big]_0^1 + 2\int_0^1 xe^{-x}\,dx = -\frac{1}{2} + 2\int_0^1 xe^{-x}\,dx = -\frac{1}{2} + 2\left(xe^{-x}\Big]_0^1 + \int_0^1 e^{-x}\,dx\right) = -\frac{1}{2} + 2e^{-1}\left(-e^{-x}\right)\Big]_0^1 = -\frac{1}{2} + 2e^{-1} - e^{-1} + 1 =$$

$u = x \quad v = -e^{-x}$
$du = dx \quad dv = e^{-x}dx$

3

$$-\frac{1}{2} + 2e^{-1} - e^{-1} + 1 = \boxed{\frac{1}{2} + e^{-1} = M_y}$$

$$\boxed{0 = M_x}$$ ✓ 5

Problem 2. Use the change of variables

$$u = x - y$$
$$v = x + y$$

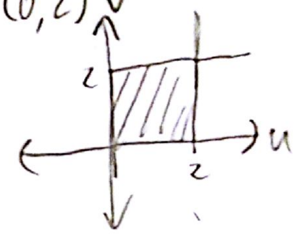
to evaluate the integral

$$\iint \sqrt{\frac{x-y}{x+y+1}}\, dA. \quad = \iint \sqrt{\frac{u}{v+1}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

over region D, where D is the square with the vertices $(0,0)$, $(1,-1)$, $(2,0)$ and $(1,1)$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2$$

$(0,0)$ $(2,0)$, $(2,2)$, $(0,2)$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{2}$$

$0 \le u \le 2$

$0 \le v \le 2$

$$\int_0^2 \int_0^2 \sqrt{\frac{u}{v+1}} \cdot \frac{1}{2}\, du\, dv =$$

$$\frac{1}{2} \int_0^2 \sqrt{\frac{1}{v+1}}\, dv \int_0^2 \sqrt{u}\, du =$$

$$\int_0^2 \sqrt{\frac{1}{v+1}}\, dv = \boxed{u = v+1,\ du = dv}$$

$$\int_1^3 \sqrt{\frac{1}{u}}\, du = \int_1^3 u^{-1/2}\, du = \left(2u^{1/2}\right)\Big]_1^3 = 2(\sqrt{3}-1)$$

$$\int_0^2 \sqrt{u}\, dv = \left(\frac{2}{3} u^{3/2}\right)\Big]_0^2 = \frac{2}{3}(2)^{3/2} = \frac{2}{3} \cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3}$$

$$\iint \sqrt{\frac{x-y}{x+y+1}}\, dA = \frac{1}{2}\left(2(\sqrt{3}-1)\right)\left(\frac{4\sqrt{2}}{3}\right) = \frac{4\sqrt{2}}{3}(\sqrt{3}-1)$$

10

Problem 3. Consider the unit radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|},$$

where $\mathbf{r}$= (x,y) ~~$\langle x, y, z\rangle$~~.

(a) Compute the divergence of this vector field. Write down the final answer in terms of $\mathbf{r}$ and $|\mathbf{r}|$ only.

$\cancel{\operatorname{div} F = \nabla\cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{|r|}, \frac{y}{|r|}, \frac{z}{|r|}\right) = \frac{1}{|r|}(1+1+1) = \frac{3}{|r|}}$

$$\operatorname{div} F = \nabla\cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot\left(\frac{x}{|r|}, \frac{y}{|r|}\right) = \frac{1}{|r|}(1+1) = \boxed{\frac{2}{|r|}}$$

2

No — $\frac{\partial}{\partial x}\left(\frac{x}{|r|}\right) \neq \frac{1}{|r|}$

because $|r|$ depends on x: $|r| = \sqrt{x^2+y^2}$.

(b) Compute the flux of this vector field accross the circle of radius R centered at the origin. ~~In this problem, we are working in the xy plane for z=0~~

$r = (R\cos\theta, R\sin\theta) \quad 0 \le \theta < 2\pi$ $\qquad F = \frac{r}{|r|}, \quad |r| = R$

$r' = (-R\sin\theta, R\cos\theta)$ $\qquad F = \frac{r}{R} = \frac{(R\cos\theta, R\sin\theta)}{R} = (\cos\theta, \sin\theta)$

$n = (y'(\theta), -x'(\theta)) = (R\cos\theta, R\sin\theta)$ ~~$\cos\theta, \sin\theta$, $F\cdot(R\cos\theta, R\sin\theta)$~~

y-component of r'

$$\text{Flux} = \int_C \vec{F}\cdot\vec{n}\,ds = \int_{\theta_1}^{\theta_2} F\cdot n\,d\theta = \int_0^{2\pi} (\cos\theta, \sin\theta)\cdot(R\cos\theta, R\sin\theta)\,d\theta =$$

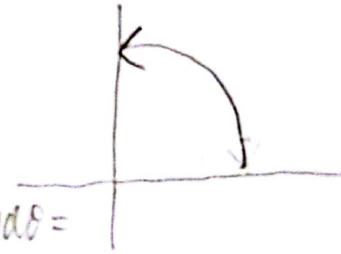
$$\int_0^{2\pi} R\cos^2\theta + R\sin^2\theta\,d\theta = \int_0^{2\pi} R\,d\theta = R\theta\Big]_0^{2\pi} = 2\pi R$$

~~4~~ 4

6

Problem 4. Let $\mathbf{F} = \langle -y, x\rangle$ be a vector field on the plane. Let C be the quarter of the unit circle on the plane lying in the first quadrant and traced in the counterclockwise direction (i.e., from point $(1,0)$ to point $(0,1)$).

(a) Set up and evaluate the line integral of $\mathbf{F}$ along C.



$r(\theta) = (\cos(\theta), \sin(\theta))$ $\quad F(r(\theta)) = (-\sin(\theta), \cos\theta)$

$r'(\theta) = (-\sin\theta, \cos\theta)$ $\quad 0 \le \theta < \pi/2$

$\int_C \vec{F}\cdot d\vec{r} = \int_{\theta_1}^{\theta_2} F(r(\theta))\cdot r'(\theta)\,d\theta = \int_0^{\pi/2} (-\sin\theta, \cos\theta)\cdot(-\sin\theta, \cos\theta)\,d\theta =$

$\int_0^{\pi/2} \sin^2\theta + \cos^2\theta\,d\theta = \int_0^{\pi/2} d\theta = \theta\Big]_0^{\pi/2} = \boxed{\frac{\pi}{2}}$

5

(b) Let C' be any other path connecting the same two points. Can we expect that the value of the integral of $\mathbf{F}$ along this path will be the same? Explain your reasoning.

$\frac{\partial F_1}{\partial y} = -1, \quad \frac{\partial F_2}{\partial x} = 1$

5

Because $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, F is not conservative and so C' would not result in the same value of the line integral. Functions are only path independent if and only if they are conservative.

8

Problem 5. Multiple-choice questions:

(1) The vector field $\mathbf{F} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$, where $r = \sqrt{x^2 + y^2 + z^2}$, is

(a) not conservative because $\text{curl}(\mathbf{F}) \neq 0$;

(b) not conservative because there is no function $f = f(x, y, z)$ such that $\nabla f = \mathbf{F}$;

2

(c) is conservative with potential function $f(x, y, z) = r$;

(d) is conservative with potential function $f(x, y, z) = \frac{1}{r}$;

(2) Which of the following statements is *not* true:

(a) The value of a scalar line integral along a curve does not depend on a parametrization of the curve;

(b) Any vector field of the form $\mathbf{F} = \langle f(x), g(y), h(z) \rangle$ is conservative.

(c) Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a vector field on the plane. Then

2

$$\text{curl}(x \cdot \mathbf{F}) = \left\langle 0, 0, \; F_2 + x\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\rangle.$$

(d) The circulation of the rotation vector field $\mathbf{F} = < -y, x >$ around the unit circle is equal to 0.

$(\rho \sin\varphi\cos\theta)^2 + ($[illegible]$)^2$

$\rho^4 \sin$ [illegible]

(3) Circle the statement which is correct:

(a) Let T be a linear transformation on the plane such that the Jacobian of T is identically equal to 5. Let R be a rectangle of area 10. Then the area of the image $T(R)$ of rectangle R under the transformation T equals to 2.

2

(b) Let T be the transformation that changes cylindrical coordinates into spherical. I.e., $T : (r, \theta, z) \mapsto (\rho, \theta, \varphi)$. Then the Jacobin of this transformation equals to $\frac{4z}{z^2+r^2}$.

(c) If a region on the plane is symmetric with respect to the y-axis and has uniform density, the center of mass lies on the y-axis.

(d) Center of mass for a region on the plane with uniform density distribution always lies in this region.

(4) Let $\mathbf{F}(x, y) = \frac{\mathbf{i}+\mathbf{j}}{x^2+y^2}$ be a vector field. For each of the properties listed below mark whether it is true for this vector field or not:

(a) **T** / **F** All vectors point directly away from the origin (i.e., a vector attached to a point is parallel to the position vector for this point);

(b) **T** / **F** The vector field has rotational symmetry about the origin;

2

(c) **T** / **F** All vectors that start on a circle centered at the origin have the same length;

(d) **T** / **F** None of the vectors are pointing in the North Western direction.

(5) For the vector field $\vec{F} = \frac{\vec{r}}{r^3}$, where $\vec{r} = \langle x, y, z \rangle$ and $r = |\vec{r}|$, which of the following statements are true:

A: $\nabla \times \vec{F} = 0$

B: $\nabla \cdot \vec{F} = 0$

(a) both **A** and **B** are true;

(b) **A** is true and **B** is not true;

(c) **B** is true and **A** is not true;

(d) both **A** and **B** are not true;