MATH 32B
FIRST MIDTERM EXAMINATION

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name: [redacted] Section: [redacted]

[redacted]

#1	#2	#3	#4	#5		Total
10	10	8	4	10		42

10

Problem 1. Evaluate the integral

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{\pi}} xy\sin(x^2y)\,dy\,dx.$$

$$\int_0^{\frac{2}{\pi}}\int_0^{\pi/2} xy\sin(x^2y)\,dx\,dy \qquad u=x^2y \quad du=2xy\,dx$$

$$=\frac{1}{2}\int_0^{\frac{2}{\pi}}\int \sin(u)\,du\,dy = \frac{1}{2}\int_0^{\frac{2}{\pi}}\Big[-\cos(x^2y)\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\int_0^{\frac{2}{\pi}}\left(-\cos\left(\frac{\pi^2}{4}y\right)+1\right)dy$$

$$=\frac{1}{2}\left[\frac{-4}{\pi^2}\sin\left(\frac{\pi^2}{4}y\right)+y\right]_0^{\frac{2}{\pi}} = \frac{1}{2}\left[\left(-\frac{4}{\pi^2}+\frac{2}{\pi}\right)-(0+0)\right]$$

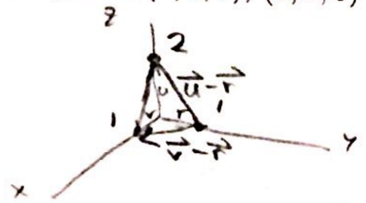
$$=\frac{1}{2}\left(\frac{-4+2\pi}{\pi^2}\right)=\boxed{\frac{-2+\pi}{\pi^2}}$$

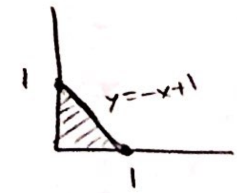
10/10

Problem 2. Use a triple integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz\,dy\,dx$$

to find the volume of the solid bounded by the coordinate planes and the plane going through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$.





$0 \le x \le 1$

$0 \le y \le -x+1$

$0 \le z \le 2-2x-2y$

$\vec{v} = \langle 1,0,0 \rangle$

$\vec{r} = \langle 0,1,0 \rangle$

$\vec{u} = \langle 0,0,2 \rangle$

$(\vec{v}-\vec{r}) \times (\vec{u}-\vec{r})$

$\langle 1,-1,0 \rangle \times \langle 0,1,-2 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{vmatrix} = (2)\hat{i} - (-2)\hat{j} + (1)k$$

$\vec{n} = \langle 2,2,1 \rangle$

$2(x-1) + 2y + z = 0$

$2x + 2y + z = 2$

$z = 2-2x-2y$

$$\int_0^1 \int_0^{-x+1} \int_0^{2-2x-2y} 1\, dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (2-2x-2y)\,dy\,dx$$

$$= \int_0^1 \left[2y-2xy-y^2\right]_0^{1-x} dx$$

$(1-x)(1-x)$

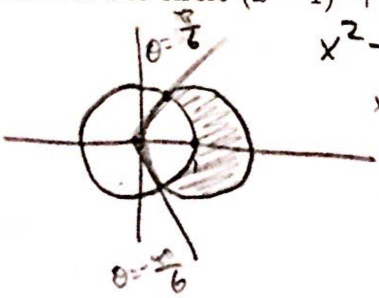
$1-2x+x^2$

$$= \int_0^1 \left[2(1-x) - 2x(1-x) - (1-x)^2\right] dx$$

$$= \int_0^1 \left[2-2x-\cancel{2x}+2x^2-(1+\cancel{2x}-x^2\right] dx$$

$$= \int_0^1 \left[1-2x+x^2\right] dx = \left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \cancel{1}-\cancel{1}+\frac{1}{3} = \boxed{\frac{1}{3}}$$

Problem 3. Use integration in polar coordinates to find the area of the region that lies inside of the circle $(x-1)^2+y^2=1$ and outside of the circle $x^2+y^2=1$.



$x^2-2x+1+y^2=1$

$x^2+y^2=1$

$1-2x+\cancel{1}=\cancel{1}$

$1-2x=0$

$x=\frac{1}{2} \quad y=\pm\frac{\sqrt{3}}{2}$

$\Rightarrow \theta=\pm\frac{\pi}{6}$

$r^2-2r\cos\theta+\cancel{1}=\cancel{1}$

$r^2=2r\cos\theta$

$r=2\cos\theta$

$-\frac{\pi}{6}\le\theta\le\frac{\pi}{6}$

$1\le r\le 2\cos\theta$

$\left(\frac{240}{360}\right)1$

$\frac{24}{36}=\frac{4}{6}=\frac{2}{3}$

$\frac{\pi}{3}$

$$\int_{-\pi/6}^{\pi/6}\int_{1}^{2\cos\theta} r\,dr\,d\theta=\int_{-\pi/6}^{\pi/6}\left[\frac{r^2}{2}\right]_1^{2\cos\theta}d\theta$$

-2

$$=\int_{-\pi/6}^{\pi/6}\left(2\cos^2\theta-\frac{1}{2}\right)d\theta=\int_{-\pi/6}^{\pi/6}\left[(1+\cos2\theta)-\frac{1}{2}\right]d\theta$$

$$=\left[\frac{\theta}{2}+\frac{\sin2\theta}{2}\right]_{-\pi/6}^{\pi/6}=\left[\left(\frac{\pi}{12}+\frac{\sqrt{3}}{4}\right)+\left(+\frac{\pi}{12}++\frac{\sqrt{3}}{4}\right)\right]$$

$$=\boxed{\frac{\pi}{6}+\frac{\sqrt{3}}{2}}$$

Problem 4. Set up (but do not evaluate) triple integral in cylindrical coordinates that represents the volume of the region bounded by the xy-plane and the surfaces

$$z = 1-(x^2+y^2)$$
$$x = x^2+y^2$$

$z = 1 - r^2$

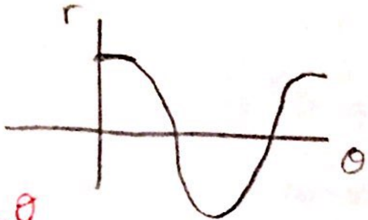
$r\cos\theta = r^2$

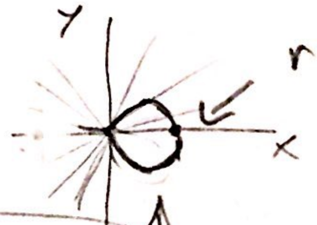
$\cos\theta = r$

$0 \le z \le \sin^2\theta$

$z = 1-\cos^2\theta$

$z = \sin^2\theta$





r max at 1
min @ 0

$0 \le r \le 1$

graph demonstrates
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int_0^{\cancel{1}\ \cos\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cancel{\sin^2\theta}\ 1-r^2} r\, dz\, d\theta\, dr$$

Problem 5. Set up (but do not evaluate) triple integral in spherical coordinates representing the volume of the region lying in the first octant (i.e., $x \geq 0$, $y \geq 0$, $z \geq 0$) and

- above the surface
$$z = \sqrt{3(x^2 + y^2)}$$
$z^2 = 3x^2 + 3y^2$ cone
- below the surface
$$z = \sqrt{4 - (x^2 + y^2)}$$
$z^2 = 4 - x^2 - y^2$
- between the planes
$$y = \frac{x}{\sqrt{3}}$$
$$y = x.$$

$\sin\phi = \frac{r}{\rho}$

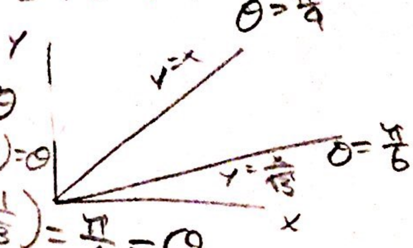
$r = \rho\sin\phi$

$m = \tan\theta$

$\arctan(m) = \theta$

$\theta = \frac{\pi}{4}$

$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} = \theta$



(*Hint:* Draw pictures on several coordinate planes).

$\rho\cos\phi = \sqrt{3\rho^2\sin^2\phi}$

$\rho^2\cos^2\phi = 3\rho^2\sin^2\phi$

$\frac{1}{3} = \tan^2\phi$

$\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$

$\frac{\pi}{6}$

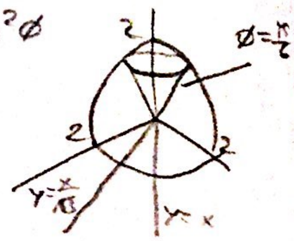
$\frac{\sqrt{3}}{3} = \tan\phi$

$\phi = \frac{\pi}{6}$

$\rho^2\cos^2\phi = 4 - \rho^2\sin^2\phi$

$\rho^2 = 4$

$\rho = 2$



$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}$

$0 \leq \rho \leq 2$

$0 \leq \phi \leq \frac{\pi}{6}$

$$\boxed{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{0}^{\frac{\pi}{6}} \int_{0}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$