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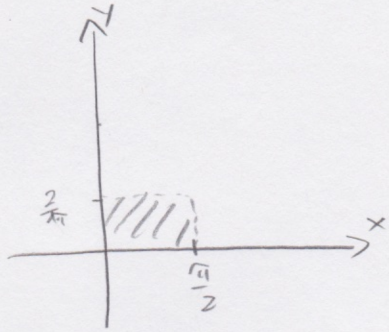
Problem 1. Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\pi}} xy \sin(x^2 y)\, dy\, dx.$$



By Fubini's theorem,

$$\int_0^{\pi/2}\int_0^{\frac{2}{\pi}} xy\sin(x^2y)\,dy\,dx$$

$$= \int_0^{\frac{2}{\pi}}\int_0^{\pi/2} xy\sin(x^2y)\,dx\,dy$$

$u = x^2 y$

$du = 2xy\,dx$

$\frac{du}{2} = xy\,dx$

$\int \sin(u)\,du$

$= -\cos(u)$

$\cos(0) = 1$

$$= -\frac{1}{2}\int_0^{\frac{2}{\pi}} \cos(x^2 y)\Big|_{x=0}^{x=\pi/2} dy$$

$$= -\frac{1}{2}\int_0^{\frac{2}{\pi}} \cos\left(\frac{\pi^2}{4}y\right) - 1\, dy$$

$$= -\frac{1}{2}\left[\left(\frac{4}{\pi^2}\right)\sin\left(\frac{\pi^2}{4}y\right) - y\right]_0^{2/\pi}$$

$$= -\frac{1}{2}\left(\left[\left(\frac{4}{\pi^2}\right)\sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\right] - \left[\frac{4}{\pi^2}\sin(0) - 0\right]\right)$$

$$= -\frac{1}{2}\left(\frac{4}{\pi^2} - \frac{2}{\pi}\right)$$

$$= -\frac{1}{2}\left(\frac{4-2\pi}{\pi^2}\right) = \frac{2\pi - 4}{2\pi^2} = \boxed{\frac{\pi - 2}{\pi^2}}$$

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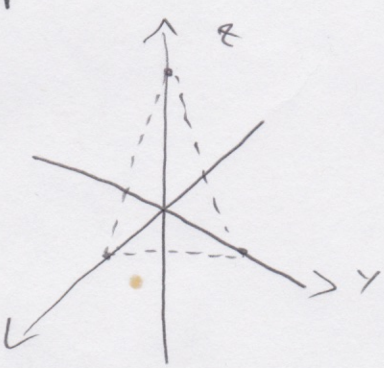
Problem 2. Use a triple integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, \underline{dzdydx}$$

to find the volume of the solid bounded by the coordinate planes and the plane going through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$.

$$\text{volume}(w) = \iiint_w 1\, dV$$

$$w \begin{cases} \frac{x}{1} + \frac{y}{1} + \frac{z}{2} = 1 \\ \to 2x + 2y + z = 2 \\ x, y, z \geq 0 \end{cases}$$



$$\text{volume of integration} \begin{cases} 0 \leq z \leq 2 - 2x - 2y \\ 0 \leq y \leq 1 - x \\ 0 \leq x \leq 1 \end{cases} \checkmark$$

$$\int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} 1\, dz\, dy\, dx$$

$z = 2 - 2x - 2y$

$$= \int_0^1 \int_0^{1-x} 2 - 2x - 2y\, dy\, dx$$

$$= \int_0^1 2y - 2xy - y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 2(1-x) - 2x(1-x) - (1-x)^2\, dx$$

$$= \int_0^1 2 - 2x - 2x + 2x^2 - [1 - 2x + x^2]\, dx$$

$$= \int_0^1 2 - 2x - 2x + 2x^2 - 1 + 2x - x^2\, dx$$

$$\to \int_0^1 1 - 2x + x^2\, dx$$

$$= \left[x - x^2 + \frac{x^3}{3} \right]_0^1$$

$$= 1 - 1 + \frac{1}{3} = \boxed{\frac{1}{3}}$$

Problem 3. Use integration in polar coordinates to find the area of the region that lies inside of the circle $(x-1)^2 + y^2 = 1$ and outside of the circle $x^2 + y^2 = 1$.

$(x-1)^2 + y^2 = 1$

$x^2 - 2x + 1 + y^2 = 1$

$x^2 - 2x + y^2 = 0$

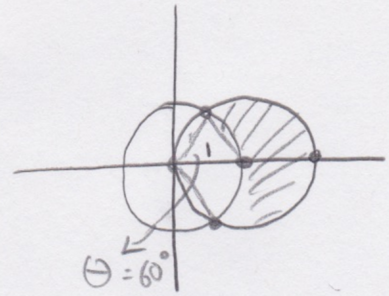
$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$\boxed{r = 2\cos\theta}$

$x^2 + y^2 = 1$

$\boxed{r = 1}$



$$\begin{cases} 1 \le r \le 2\cos\theta \\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \end{cases}$$

$2\cos(0) = 2$

$2\cos(\frac{\pi}{2}) = 0$

$2\cos(\frac{\pi}{3}) = 1$

$2\cos(-\frac{\pi}{3}) = 1$

$\text{Area}(D) = \iint_D 1\, dA$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} 1 \cdot r\, dr\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} r^2 \Big|_1^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2\theta - 1\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2(1+\cos 2\theta) - 1\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1 + 2\cos 2\theta\, d\theta$$

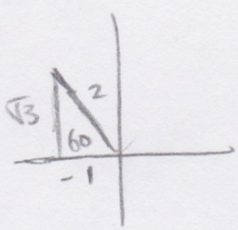
$$\frac{1}{2}\Big[\theta + \sin 2\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

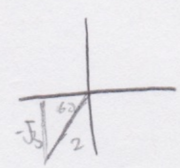
$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{3} + \sqrt{3}\right]$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$





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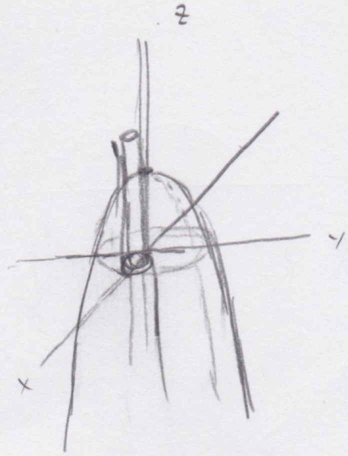
$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

Problem 4. Set up (but do not evaluate) triple integral in cylindrical coordinates that represents the volume of the region bounded by the xy-plane and the surfaces

$$z = 1 - (x^2 + y^2)$$
$$x = x^2 + y^2$$



$z = 1 - r^2$

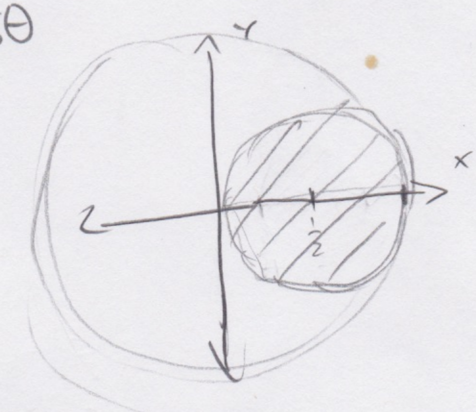
$r^2 = r\cos\theta$

$\to r = \cos\theta$

$0 \le z \le 1 - r^2$

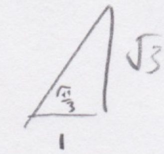
$0 \le r \le \cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



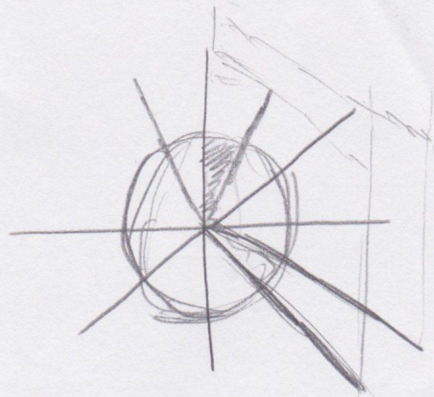
$\text{volume}(w) = \iiint_w 1\,dV$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{1-r^2} 1 \cdot r\,dz\,dr\,d\theta}$$



Problem 5. Set up (but do not evaluate) triple integral in spherical coordinates representing the volume of the region lying in the first octant (i.e., $x \geq 0$, $y \geq 0$, $z \geq 0$) and

- above the surface
$$z = \sqrt{3(x^2 + y^2)}$$
- below the surface
$$z = \sqrt{4 - (x^2 + y^2)}$$
- between the planes
$$y = \frac{x}{\sqrt{3}}$$
$$y = x.$$



$z^2 = 3(x^2+y^2)$

$z^2+x^2+y^2 = 4$

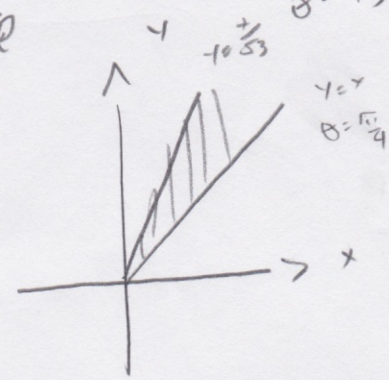
(*Hint:* Draw pictures on several coordinate planes).

$\rho\cos\varphi = \sqrt{3\rho^2\sin^2\varphi}$ $\qquad$ $\rho\cos\varphi = \sqrt{4-\rho^2\sin^2\varphi}$

$\frac{1}{\sqrt{3}} = \tan\varphi$ $\qquad$ $\rho^2\cos^2\varphi = 4 - \rho^2\sin^2\varphi$

$\varphi = \frac{\pi}{3}$ $\qquad$ $\rho^2 = 4$

$\rho = 2$



$\frac{y}{x} = \frac{1}{\sqrt{3}}$ $\qquad$ $y = x$

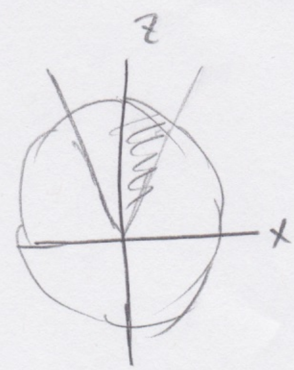
$\tan\theta = \frac{1}{\sqrt{3}}$ $\qquad$ $\tan\theta = 1$

$\theta = \frac{\pi}{3}$ $\qquad$ $\theta = \frac{\pi}{4}$

$0 \leq \rho \leq 2$

$0 \leq \varphi \leq \frac{\pi}{3}$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}$



$\text{volume}(W) = \iiint_W 1\, dV$

$$\boxed{\int_{\pi/4}^{\pi/3}\int_0^{\cancel{\pi/3}\ \pi/6}\int_0^2 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$