MATH 32B
FIRST MIDTERM EXAMINATION

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:______________________________ Section:____________________

#1	#2	#3	#4	#5		Total

Problem 1. Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\pi}} xy\sin(x^2y)\ dy\ dx.$$

SOLUTION: The region of integration is a rectangle $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{2}{\pi}\right]$ on the xy-plane. Switch the order of integration to get

$$\int_0^{\frac{2}{\pi}} y \int_0^{\frac{\pi}{2}} x\sin(x^2y)\ dx\ dy.$$

Use the substitution $u = yx^2$ to evaluation the internal integral. Then $du = 2yxdx$ and $u(0) = 0$, $u\left(\frac{\pi}{2}\right) = y \cdot \frac{\pi^2}{4}$

$$\int_0^{\frac{\pi}{2}} x\sin(x^2y)dx \quad = \quad \frac{1}{2y}\int_0^{y\cdot\frac{\pi^2}{4}} \sin(u)du = \frac{1}{2y}\left(1 - \cos\left(y\cdot\frac{\pi^2}{4}\right)\right).$$

The outside integral becomes

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{2}{\pi}} \left(1 - \cos\left(y\cdot\frac{\pi^2}{4}\right)\right) dy \quad &= \quad \frac{1}{\pi} - \frac{1}{2}\cdot\frac{4}{\pi^2}\sin\left(y\cdot\frac{\pi^2}{4}\right)\Big|_0^{\frac{2}{\pi}} = \\ &= \quad \frac{1}{\pi} - \frac{2}{\pi^2}.\end{aligned}$$

Thus, the answer is $\frac{1}{\pi} - \frac{2}{\pi^2}$.

Problem 2. Use a triple integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dzdydx$$

to find the volume of the solid bounded by the plane going through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$ and the coordinate planes.

SOLUTION: The region of integration on the (x,y) is the triangle with the vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$, i.e.,

$$D = \{(x,y)|\ 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

The plane through the given points has an equation

$$2x + 2y + z = 2,$$

so that $z = 2 - 2x - 2y$. The integral representing the volume is

$$\begin{aligned} \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} dzdydx &= \int_0^1 \int_0^{1-x} (2-2x-2y)dydx = \\ \int_0^1 (2-2x)(1-x) - (1-x)^2 dx &= \int_0^1 (1-x)^2 dx = -\int_1^0 u^2 du, \end{aligned}$$

where $u = 1 - x$. The last integral equals to $\frac{1}{3}$. Thus, the volume is equal to $\frac{1}{3}$.

Problem 3. Use integration in polar coordinates to find the area of the region that lies inside of the circle $(x-1)^2+y^2=1$ and outside of the circle $x^2+y^2=1$.

SOLUTION: Using polar coordinates, rewrite the equations of both of the circles. For the first circle, we get:

$$\begin{aligned}(x-1)^2+y^2=1 \quad &\Leftrightarrow \quad x^2+y^2-2x=0\\ r^2=2r\cos\theta \quad &\Rightarrow \quad r=2\cos\theta.\end{aligned}$$

For the second circle, the equation is $r=1$. The two circles intersect at the following points:

$$2\cos\theta=1 \Leftrightarrow \theta=\pm\frac{\pi}{3}.$$

The integral becomes

$$\begin{aligned}\text{Area}=\int_D 1\cdot dA \quad &= \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} rdrd\theta=\\ =\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} rdr\ d\theta \quad &= \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(2\cos^2\theta-\frac{1}{2}\right)d\theta=\\ &= \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(1+2\cos 2\theta-\frac{1}{2}\right)d\theta=\\ &= \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(\frac{1}{2}+2\cos 2\theta\right)d\theta=\\ &= \quad \frac{\pi}{3}+\frac{\sqrt{3}}{2}.\end{aligned}$$

Problem 4. Set up (but do not evaluate) triple integral in cylindrical coordinates that represents the volume of the region bounded by the xy-plane and the surfaces

$$\begin{aligned} z &= 1-(x^2+y^2) \\ x &= x^2+y^2 \end{aligned}$$

SOLUTION: The first surface is a parabaloid with equation $z = 1 - r^2$ in cylindrical coordinates.

The second surface is a cylinder with equation in cylindrical coordinates given by:

$$\begin{aligned} r^2 &= r\cos\theta \\ r &= \cos\theta. \end{aligned}$$

The disk bounded by this circle corresponds to $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $0 \le r \le \cos\theta$.

Thus, the volume is given by the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{1-r^2} r dz dr d\theta.$$

Problem 5. Set up (but do not evaluate) triple integral in spherical coordinates representing the volume of the region lying lying in the first octant (i.e., $x \geq 0$, $y \geq 0$, $z \geq 0$) and

- above the surface
$$z = \sqrt{3(x^2 + y^2)}$$
- below the surface
$$z = \sqrt{4 - (x^2 + y^2)}$$
- between the planes
$$\begin{aligned} y &= \frac{x}{\sqrt{3}} \\ y &= x. \end{aligned}$$

Problem 6. Solution: The first surface is a cone with the equation $z = \sqrt{3}r$ in cylindrical coordinates. The tangent of the angle corresponding to this cone is $\frac{r}{z} = \frac{1}{\sqrt{3}}$. Thus, the angle is $\frac{\pi}{6}$. The equation of the top part of the cone in the spherical coordinates is $\varphi = \frac{\pi}{6}$, $z \geq 0$.

The second surface is a hemi-sphere with equation $x^2 + y^2 + z^2 = 4$, $z \geq 0$. In spherical coordinates, we have $\rho = 2$.

The region bounded by these two surfaces has the shape of an ice cream cone.

The planes have equations $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$ in spherical cooridnates. Thus, the integral representing the volume of the region is
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin\varphi d\rho d\varphi . d\theta$$