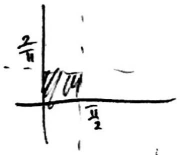
Problem 1. Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\pi}} xy \sin(x^2 y)\, dy\, dx.$$



$$= \int_0^{\frac{2}{\pi}} \int_0^{\pi/2} xy\sin(x^2y)\, dx\, dy$$

$u = x^2y$
$du = 2xy\,dx$
$\frac{du}{2} = xy\,dx$

$$= \int_0^{\frac{2}{\pi}} -\frac{\cos(x^2y)}{2}\Big]_0^{\pi/2} dy$$

$$= \int_0^{\frac{2}{\pi}} -\frac{\cos\left(\left(\frac{\pi}{2}\right)^2 y\right)}{2} + \frac{1}{2}\, dy$$

$u = \left(\frac{\pi}{2}\right)^2 y$
$du = \left(\frac{\pi}{2}\right)^2 dy$
$\frac{du}{\left(\frac{\pi}{2}\right)^2} = dy$

$$= -\frac{\sin\left(\left(\frac{\pi}{2}\right)^2 y\right)}{2\left(\frac{\pi}{2}\right)^2} + \frac{1}{2}y\Big]_0^{\frac{2}{\pi}}$$

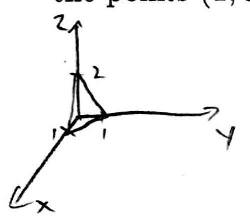
$$= -\frac{1}{\frac{\pi^2}{2}} + \frac{1}{2}\left(\frac{2}{\pi}\right) + 0$$

$$= -\frac{2}{\pi^2} + \frac{1}{\pi} = -\frac{2}{\pi^2} + \frac{\pi}{\pi^2} = \boxed{\frac{\pi - 2}{\pi^2}}$$

Problem 2. Use a triple integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dzdydx$$

to find the volume of the solid bounded by the coordinate planes and the plane going through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$.



$$\int_0^1\int_0^{1-x}\int_0^{2-2x-2y} 1\, dzdydx = \int_0^1\int_0^{1-x} z\Big]_0^{2-2x-2y} dydx$$

$$= \int_0^1\int_0^{1-x} 2-2x-2y\, dydx$$

$$= \int_0^1 2y-2xy-y^2\Big]_0^{1-x} dx$$

$$= \int_0^1 2-2x-2x+2x^2-1+2x-x^2\, dx$$

$$= \int_0^1 1-2x+x^2\, dx$$

$$= x-x^2+\frac{x^3}{3}\Big]_0^1 = 1-1+\frac{1}{3}$$

$$= \boxed{\frac{1}{3}}$$

$y=1-x$

$2x+2y+z=2 \rightarrow z=2-2x-2y$

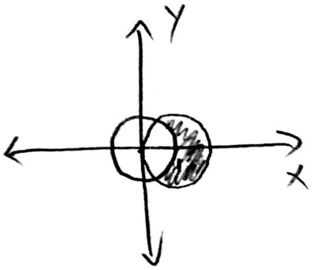
equation of plane

when $y=z=0$, $x=1$

when $x=z=0$, $y=1$

when $x=y=0$, $z=2$

Problem 3. Use integration in polar coordinates to find the area of the region that lies inside of the circle $(x-1)^2 + y^2 = 1$ and outside of the circle $x^2 + y^2 = 1$.



$(x-1)^2+y^2=1 \downarrow r=2\cos\theta$

$x^2+y^2=1 \downarrow r=1$

To find intersection

$1=2\cos\theta$

$\cos\theta=\frac{1}{2}$

$\theta=\frac{\pi}{3}$ and $-\frac{\pi}{3}$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\Big]_1^{2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2\theta - \frac{1}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1+\cos 2\theta - \frac{1}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos 2\theta + \frac{1}{2}\, d\theta$$

$$= \frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{3}\right)$$

$$= \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$$

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Problem 4. Set up (but do not evaluate) triple integral in cylindrical coordinates that represents the volume of the region bounded by the xy-plane and the surfaces

$$z = 1-(x^2+y^2) \quad \rightarrow z = 1-r^2$$
$$x = x^2+y^2 \quad \rightarrow x = r^2$$

$\downarrow$

$r\cos\theta = r^2$

$r = \cos\theta$

xy-plane

$z=0$

$$\cancel{\int_{0}^{\pi}} \int_0^{\cos\theta} \int_0^{1-r^2} r\,dz\,dr\,d\theta$$

Problem 5. Set up (but do not evaluate) triple integral in spherical coordinates representing the volume of the region lying in the first octant (i.e., $x \geq 0,\ y \geq 0,\ z \geq 0$) and

- above the surface
$$z = \sqrt{3(x^2 + y^2)}$$
- below the surface
$$z = \sqrt{4 - (x^2 + y^2)}$$
- between the planes
$$y = \frac{x}{\sqrt{3}}$$
$$y = x.$$

(*Hint:* Draw pictures on several coordinate planes).

$z = \sqrt{3(x^2+y^2)}$

$\rho\cos\phi = \sqrt{3}\,\rho\sin\phi$

$\cos\phi = \sqrt{3}\sin\phi$

$\sqrt{3} = \tan\phi$

$\phi = \frac{\pi}{3}$

$z = \sqrt{4-(x^2+y^2)}$

$\rho\cos\phi = \sqrt{4-\rho^2\sin^2\phi}$

$\sqrt{3(x^2+y^2)} = \sqrt{4-(x^2+y^2)}$

$\sqrt{3\rho^2\sin^2\phi} = \sqrt{4-\rho^2\sin^2\phi}$

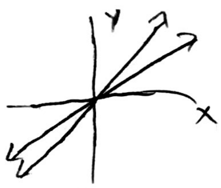
$3\rho^2\sin^2\phi = 4-\rho^2\sin^2\phi$

$4\rho^2\sin^2\phi = 4$

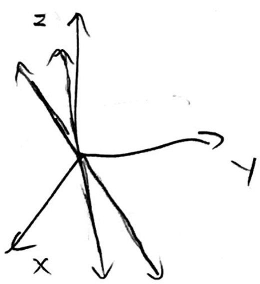
$\rho^2\sin^2\phi = 1$

$\rho\sin\phi = 1$

$\rho = \frac{1}{\sin\phi}$



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{\cancel{\frac{\pi}{3}}\,\frac{\pi}{6}} \int_0^{\cancel{\csc\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$\tan^{-1}(1) = \frac{\pi}{4}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$