1 PRECALCULUS REVIEW

1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a + b| = |-3 + 1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4.$$

Thus, $|a + b| < |a| + |b|$.

4. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

5. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

6. What is the radius of the circle with equation $(x - 9)^2 + (y - 9)^2 = 9$?

SOLUTION The circle with equation $(x - 9)^2 + (y - 9)^2 = 9$ has radius 3.

7. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

8. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

Exercises

1. Use a calculator to find a rational number r such that $|r - \pi^2| < 10^{-4}$.

SOLUTION r must satisfy $\pi^2 - 10^{-4} < r < \pi^2 + 10^{-4}$, or $9.869504 < r < 9.869705$. $r = 9.8696 = \frac{12337}{1250}$ would be one such number.

2. Which of (a)–(f) are true for $a = -3$ and $b = 2$?

(a) $a < b$ **(b)** $|a| < |b|$ **(c)** $ab > 0$

(d) $3a < 3b$ **(e)** $-4a < -4b$ **(f)** $\frac{1}{a} < \frac{1}{b}$

SOLUTION

(a) True.
(b) False, $|a| = 3 > 2 = |b|$.
(c) False, $(-3)(2) = -6 < 0$.
(d) True.
(e) False, $(-4)(-3) = 12 > -8 = (-4)(2)$.
(f) True.

In Exercises 3–8, express the interval in terms of an inequality involving absolute value.

3. $[-2, 2]$

SOLUTION $|x| \le 2$

4. $(-4, 4)$

SOLUTION $|x| < 4$

5. $(0, 4)$

SOLUTION The midpoint of the interval is $c = (0 + 4)/2 = 2$, and the radius is $r = (4 - 0)/2 = 2$; therefore, $(0, 4)$ can be expressed as $|x - 2| < 2$.

6. $[-4, 0]$

SOLUTION The midpoint of the interval is $c = (-4 + 0)/2 = -2$, and the radius is $r = (0 - (-4))/2 = 2$; therefore, the interval $[-4, 0]$ can be expressed as $|x + 2| \le 2$.

7. $[1, 5]$

SOLUTION The midpoint of the interval is $c = (1 + 5)/2 = 3$, and the radius is $r = (5 - 1)/2 = 2$; therefore, the interval $[1, 5]$ can be expressed as $|x - 3| \le 2$.

8. $(-2, 8)$

SOLUTION The midpoint of the interval is $c = (8 - 2)/2 = 3$, and the radius is $r = (8 - (-2))/2 = 5$; therefore, the interval $(-2, 8)$ can be expressed as $|x - 3| < 5$

In Exercises 9–12, write the inequality in the form $a < x < b$.

9. $|x| < 8$

SOLUTION $-8 < x < 8$

10. $|x - 12| < 8$

SOLUTION $-8 < x - 12 < 8$ so $4 < x < 20$

11. $|2x + 1| < 5$

SOLUTION $-5 < 2x + 1 < 5$ so $-6 < 2x < 4$ and $-3 < x < 2$

12. $|3x - 4| < 2$

SOLUTION $-2 < 3x - 4 < 2$ so $2 < 3x < 6$ and $\frac{2}{3} < x < 2$

In Exercises 13–18, express the set of numbers x satisfying the given condition as an interval.

13. $|x| < 4$

SOLUTION $(-4, 4)$

14. $|x| \le 9$

SOLUTION $[-9, 9]$

15. $|x - 4| < 2$

SOLUTION The expression $|x - 4| < 2$ is equivalent to $-2 < x - 4 < 2$. Therefore, $2 < x < 6$, which represents the interval $(2, 6)$.

16. $|x + 7| < 2$

SOLUTION The expression $|x + 7| < 2$ is equivalent to $-2 < x + 7 < 2$. Therefore, $-9 < x < -5$, which represents the interval $(-9, -5)$.

17. $|4x - 1| \leq 8$

SOLUTION The expression $|4x - 1| \leq 8$ is equivalent to $-8 \leq 4x - 1 \leq 8$ or $-7 \leq 4x \leq 9$. Therefore, $-\frac{7}{4} \leq x \leq \frac{9}{4}$, which represents the interval $[-\frac{7}{4}, \frac{9}{4}]$.

18. $|3x + 5| < 1$

SOLUTION The expression $|3x + 5| < 1$ is equivalent to $-1 < 3x + 5 < 1$ or $-6 < 3x < -4$. Therefore, $-2 < x < -\frac{4}{3}$ which represents the interval $(-2, -\frac{4}{3})$

In Exercises 19–22, describe the set as a union of finite or infinite intervals.

19. $\{x : |x - 4| > 2\}$

SOLUTION $x - 4 > 2$ or $x - 4 < -2 \Rightarrow x > 6$ or $x < 2 \Rightarrow (-\infty, 2) \cup (6, \infty)$

20. $\{x : |2x + 4| > 3\}$

SOLUTION $2x + 4 > 3$ or $2x + 4 < -3 \Rightarrow 2x > -1$ or $2x < -7 \Rightarrow (-\infty, -\frac{7}{2}) \cup (-\frac{1}{2}, \infty)$

21. $\{x : |x^2 - 1| > 2\}$

SOLUTION $x^2 - 1 > 2$ or $x^2 - 1 < -2 \Rightarrow x^2 > 3$ or $x^2 < -1$ (this will never happen) $\Rightarrow x > \sqrt{3}$ or $x < -\sqrt{3} \Rightarrow$ $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

22. $\{x : |x^2 + 2x| > 2\}$

SOLUTION $x^2 + 2x > 2$ or $x^2 + 2x < -2 \Rightarrow x^2 + 2x - 2 > 0$ or $x^2 + 2x + 2 < 0$. For the first case, the zeroes are

$$x = -1 \pm \sqrt{3} \Rightarrow (-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty).$$

For the second case, note there are no real zeros. Because the parabola opens upward and its vertex is located above the x-axis, there are no values of x for which $x^2 + 2x + 2 < 0$. Hence, the solution set is $(-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty)$.

23. Match (a)–(f) with (i)–(vi).

(a) $a > 3$

(b) $|a - 5| < \dfrac{1}{3}$

(c) $\left|a - \dfrac{1}{3}\right| < 5$

(d) $|a| > 5$

(e) $|a - 4| < 3$

(f) $1 \leq a \leq 5$

(i) a lies to the right of 3.
(ii) a lies between 1 and 7.
(iii) The distance from a to 5 is less than $\frac{1}{3}$.
(iv) The distance from a to 3 is at most 2.
(v) a is less than 5 units from $\frac{1}{3}$.
(vi) a lies either to the left of -5 or to the right of 5.

SOLUTION

(a) On the number line, numbers greater than 3 appear to the right; hence, $a > 3$ is equivalent to the numbers to the right of 3: **(i)**.

(b) $|a - 5|$ measures the distance from a to 5; hence, $|a - 5| < \frac{1}{3}$ is satisfied by those numbers less than $\frac{1}{3}$ of a unit from 5: **(iii)**.

(c) $|a - \frac{1}{3}|$ measures the distance from a to $\frac{1}{3}$; hence, $|a - \frac{1}{3}| < 5$ is satisfied by those numbers less than 5 units from $\frac{1}{3}$: **(v)**.

(d) The inequality $|a| > 5$ is equivalent to $a > 5$ or $a < -5$; that is, either a lies to the right of 5 or to the left of -5: **(vi)**.

(e) The interval described by the inequality $|a - 4| < 3$ has a center at 4 and a radius of 3; that is, the interval consists of those numbers between 1 and 7: **(ii)**.

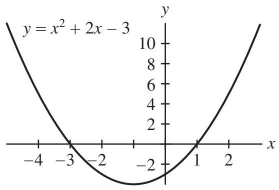
(f) The interval described by the inequality $1 < x < 5$ has a center at 3 and a radius of 2; that is, the interval consists of those numbers less than 2 units from 3: **(iv)**.

24. Describe $\left\{x : \dfrac{x}{x+1} < 0\right\}$ as an interval.

SOLUTION Case 1: $x < 0$ and $x + 1 > 0$. This implies that $x < 0$ and $x > -1 \Rightarrow -1 < x < 0$.
Case 2: $x > 0$ and $x < -1$ for which there is no such x. Thus, solution set is therefore $(-1, 0)$.

25. Describe $\{x : x^2 + 2x < 3\}$ as an interval. *Hint:* Plot $y = x^2 + 2x - 3$.

SOLUTION The inequality $x^2 + 2x < 3$ is equivalent to $x^2 + 2x - 3 < 0$. In the figure below, we see that the graph of $y = x^2 + 2x - 3$ falls below the x-axis for $-3 < x < 1$. Thus, the set $\{x : x^2 + 2x < 3\}$ corresponds to the interval $-3 < x < 1$.



26. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

SOLUTION We will break the problem into three cases: $x \ge 3$, $2 \le x < 3$ and $x < 2$. For $x \ge 3$, both $x - 3$ and $x - 2$ are greater than or equal to 0, so $|x - 3| = x - 3$ and $|x - 2| = x - 2$. The equation $|x - 3| = |x - 2| + 1$ then becomes $x - 3 = x - 2 + 1$, which is equivalent to $-1 = 1$. Thus, for $x \ge 3$, there are no solutions. Next, we consider $2 \le x < 3$. Now, $x - 3 < 0$, so $|x - 3| = 3 - x$, but $x - 2 \ge 0$, so $|x - 2| = x - 2$. The equation $|x - 3| = |x - 2| + 1$ then becomes $3 - x = x - 2 + 1$, which is equivalent to $x = 2$. Thus, $x = 2$ is a solution. Finally, consider $x < 2$. Both $x - 3$ and $x - 2$ are negative, so $|x - 3| = 3 - x$ and $|x - 2| = 2 - x$. The equation $|x - 3| = |x - 2| + 1$ then becomes $3 - x = 2 - x + 1$, which is equivalent to $1 = 1$. Hence, every $x < 2$ is a solution. Bringing all three cases together, it follows that $|x - 3| = |x - 2| + 1$ is satisfied for all $x \le 2$, or for all x on the half-infinite interval $(-\infty, 2]$.

27. Show that if $a > b$, then $b^{-1} > a^{-1}$, provided that a and b have the same sign. What happens if $a > 0$ and $b < 0$?

SOLUTION Case 1a: If a and b are both positive, then $a > b \Rightarrow 1 > \frac{b}{a} \Rightarrow \frac{1}{b} > \frac{1}{a}$.

Case 1b: If a and b are both negative, then $a > b \Rightarrow 1 < \frac{b}{a}$ (since a is negative) $\Rightarrow \frac{1}{b} > \frac{1}{a}$ (again, since b is negative).

Case 2: If $a > 0$ and $b < 0$, then $\frac{1}{a} > 0$ and $\frac{1}{b} < 0$ so $\frac{1}{b} < \frac{1}{a}$. (See Exercise 2f for an example of this).

28. Which x satisfy both $|x - 3| < 2$ and $|x - 5| < 1$?

SOLUTION $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 1 < x < 5$. Also $|x - 5| < 1 \Rightarrow 4 < x < 6$. Since we want an x that satisfies both of these, we need the intersection of the two solution sets, that is, $4 < x < 5$.

29. Show that if $|a - 5| < \frac{1}{2}$ and $|b - 8| < \frac{1}{2}$, then $|(a + b) - 13| < 1$. *Hint:* Use the triangle inequality.

SOLUTION

$$\begin{aligned} |a + b - 13| &= |(a - 5) + (b - 8)| \\ &\le |a - 5| + |b - 8| \quad \text{(by the triangle inequality)} \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

30. Suppose that $|x - 4| \le 1$.

(a) What is the maximum possible value of $|x + 4|$?

(b) Show that $|x^2 - 16| \le 9$.

SOLUTION

(a) $|x - 4| \le 1$ guarantees $3 \le x \le 5$. Thus, $7 \le x + 4 \le 9$, so $|x + 4| \le 9$.

(b) $|x^2 - 16| = |x - 4| \cdot |x + 4| \le 1 \cdot 9 = 9$.

31. Suppose that $|a - 6| \le 2$ and $|b| \le 3$.

(a) What is the largest possible value of $|a + b|$?

(b) What is the smallest possible value of $|a + b|$?

SOLUTION $|a - 6| \le 2$ guarantees that $4 \le a \le 8$, while $|b| \le 3$ guarantees that $-3 \le b \le 3$. Therefore $1 \le a + b \le 11$. It follows that

(a) the largest possible value of $|a + b|$ is 11; and

(b) the smallest possible value of $|a + b|$ is 1.

32. Prove that $|x| - |y| \le |x - y|$. *Hint:* Apply the triangle inequality to y and $x - y$.

SOLUTION First note

$$|x| = |x - y + y| \le |x - y| + |y|$$

by the triangle inequality. Subtracting $|y|$ from both sides of this inequality yields

$$|x| - |y| \le |x - y|.$$

33. Express $r_1 = 0.\overline{27}$ as a fraction. *Hint:* $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

SOLUTION Let $r_1 = .\overline{27}$. We observe that $100r_1 = 27.\overline{27}$. Therefore, $100r_1 - r_1 = 27.\overline{27} - .\overline{27} = 27$ and

$$r_1 = \frac{27}{99} = \frac{3}{11}.$$

Now, let $r_2 = 0.2\overline{666}$. Then $10r_2 = 2.\overline{666}$ and $100r_2 = 26.\overline{666}$. Therefore, $100r_2 - 10r_2 = 26.\overline{666} - 2.\overline{666} = 24$ and

$$r_2 = \frac{24}{90} = \frac{4}{15}.$$

34. Represent $1/7$ and $4/27$ as repeating decimals.

SOLUTION $\frac{1}{7} = 0.\overline{142857}$; $\frac{4}{27} = 0.\overline{148}$

35. The text states: *If the decimal expansions of numbers a and b agree to k places, then* $|a - b| \le 10^{-k}$. Show that the converse is false: For all k there are numbers a and b whose decimal expansions *do not agree at all* but $|a - b| \le 10^{-k}$.

SOLUTION Let $a = 1$ and $b = 0.\overline{9}$ (see the discussion before Example 1). The decimal expansions of a and b do not agree, but $|1 - 0.\overline{9}| < 10^{-k}$ for all k.

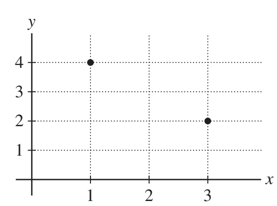
36. Plot each pair of points and compute the distance between them:

(a) $(1, 4)$ and $(3, 2)$
(b) $(2, 1)$ and $(2, 4)$
(c) $(0, 0)$ and $(-2, 3)$
(d) $(-3, -3)$ and $(-2, 3)$

SOLUTION

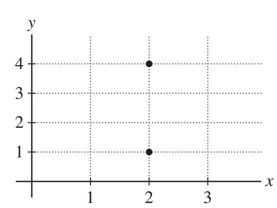
(a) The points $(1, 4)$ and $(3, 2)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(3-1)^2 + (2-4)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}.$$



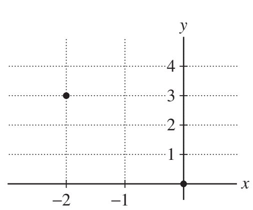
(b) The points $(2, 1)$ and $(2, 4)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(2-2)^2 + (4-1)^2} = \sqrt{9} = 3.$$



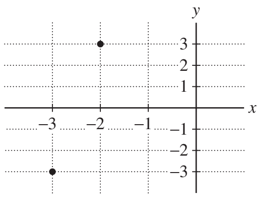
(c) The points $(0, 0)$ and $(-2, 3)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}.$$



(d) The points $(-3, -3)$ and $(-2, 3)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(-3 - (-2))^2 + (-3 - 3)^2} = \sqrt{1 + 36} = \sqrt{37}.$$



37. Find the equation of the circle with center $(2, 4)$:

(a) with radius $r = 3$.

(b) that passes through $(1, -1)$.

SOLUTION

(a) The equation of the indicated circle is $(x - 2)^2 + (y - 4)^2 = 3^2 = 9$.

(b) First determine the radius as the distance from the center to the indicated point on the circle:

$$r = \sqrt{(2 - 1)^2 + (4 - (-1))^2} = \sqrt{26}.$$

Thus, the equation of the circle is $(x - 2)^2 + (y - 4)^2 = 26$.

38. Find all points with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from $(2, 3)$.

SOLUTION

- To be located a distance 5 from the origin, the points must lie on the circle $x^2 + y^2 = 25$. This leads to 12 points with integer coordinates:

$(5, 0)$	$(-5, 0)$	$(0, 5)$	$(0, -5)$
$(3, 4)$	$(-3, 4)$	$(3, -4)$	$(-3, -4)$
$(4, 3)$	$(-4, 3)$	$(4, -3)$	$(-4, -3)$

- To be located a distance 5 from the point $(2, 3)$, the points must lie on the circle $(x - 2)^2 + (y - 3)^2 = 25$, which implies that we must shift the points listed above two units to the right and three units up. This gives the 12 points:

$(7, 3)$	$(-3, 3)$	$(2, 8)$	$(2, -2)$
$(5, 7)$	$(-1, 7)$	$(5, -1)$	$(-1, -1)$
$(6, 6)$	$(-2, 6)$	$(6, 0)$	$(-2, 0)$

39. Determine the domain and range of the function

$$f : \{r, s, t, u\} \to \{A, B, C, D, E\}$$

defined by $f(r) = A$, $f(s) = B$, $f(t) = B$, $f(u) = E$.

SOLUTION The domain is the set $D = \{r, s, t, u\}$; the range is the set $R = \{A, B, E\}$.

40. Give an example of a function whose domain D has three elements and whose range R has two elements. Does a function exist whose domain D has two elements and whose range R has three elements?

SOLUTION Define f by $f : \{a, b, c\} \to \{1, 2\}$ where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$.

There is no function whose domain has two elements and range has three elements. If that happened, one of the domain elements would get assigned to more than one element of the range, which would contradict the definition of a function.

In Exercises 41–48, find the domain and range of the function.

41. $f(x) = -x$

SOLUTION D : all reals; R : all reals

42. $g(t) = t^4$

SOLUTION D : all reals; R : $\{y : y \geq 0\}$

43. $f(x) = x^3$

SOLUTION D : all reals; R : all reals

44. $g(t) = \sqrt{2-t}$

SOLUTION $D : \{t : t \leq 2\}$; $R : \{y : y \geq 0\}$

45. $f(x) = |x|$

SOLUTION D : all reals; $R : \{y : y \geq 0\}$

46. $h(s) = \dfrac{1}{s}$

SOLUTION $D : \{s : s \neq 0\}$; $R : \{y : y \neq 0\}$

47. $f(x) = \dfrac{1}{x^2}$

SOLUTION $D : \{x : x \neq 0\}$; $R : \{y : y > 0\}$

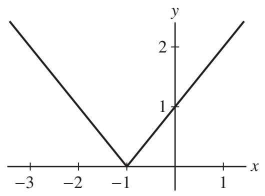
48. $g(t) = \cos \dfrac{1}{t}$

SOLUTION $D : \{t : t \neq 0\}$; $R : \{y : -1 \leq y \leq 1\}$

In Exercises 49–52, determine where $f(x)$ is increasing.

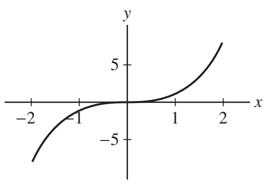
49. $f(x) = |x+1|$

SOLUTION A graph of the function $y = |x+1|$ is shown below. From the graph, we see that the function is increasing on the interval $(-1, \infty)$.



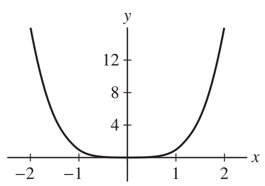
50. $f(x) = x^3$

SOLUTION A graph of the function $y = x^3$ is shown below. From the graph, we see that the function is increasing for all real numbers.



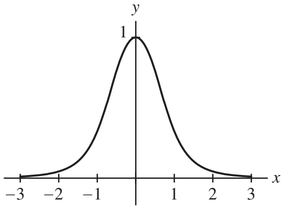
51. $f(x) = x^4$

SOLUTION A graph of the function $y = x^4$ is shown below. From the graph, we see that the function is increasing on the interval $(0, \infty)$.



52. $f(x) = \dfrac{1}{x^4 + x^2 + 1}$

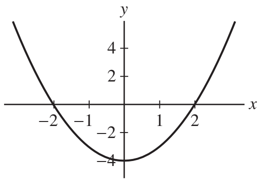
SOLUTION A graph of the function $y = \dfrac{1}{x^4 + x^2 + 1}$ is shown below. From the graph, we see that the function is increasing on the interval $(-\infty, 0)$.



In Exercises 53–58, find the zeros of $f(x)$ and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.

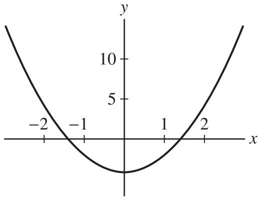
53. $f(x) = x^2 - 4$

SOLUTION Zeros: ± 2
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function). So, y-axis symmetry.



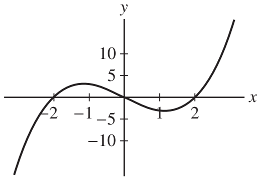
54. $f(x) = 2x^2 - 4$

SOLUTION Zeros: $\pm\sqrt{2}$
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function). So, y-axis symmetry.



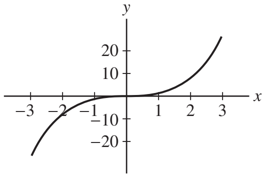
55. $f(x) = x^3 - 4x$

SOLUTION Zeros: $0, \pm 2$; Symmetry: $f(-x) = -f(x)$ (odd function). So origin symmetry.



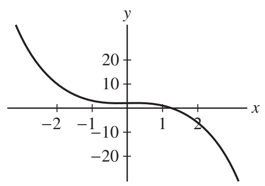
56. $f(x) = x^3$

SOLUTION Zeros: 0; Increasing for all x; Symmetry: $f(-x) = -f(x)$ (odd function). So origin symmetry.



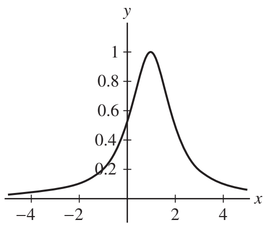
57. $f(x) = 2 - x^3$

SOLUTION This is an x-axis reflection of x^3 translated up 2 units. There is one zero at $x = \sqrt[3]{2}$.



58. $f(x) = \dfrac{1}{(x-1)^2 + 1}$

SOLUTION This is the graph of $\dfrac{1}{x^2+1}$ translated to the right 1 unit. The function has no zeros.



59. Which of the curves in Figure 26 is the graph of a function?

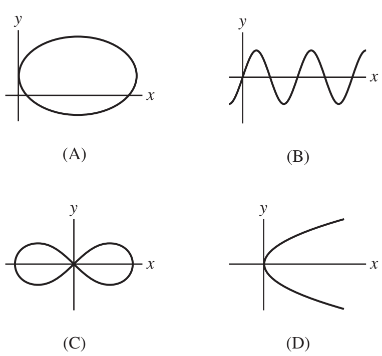


FIGURE 26

SOLUTION (B) is the graph of a function. (A), (C), and (D) all fail the vertical line test.

60. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^5$

(b) $g(t) = t^3 - t^2$

(c) $F(t) = \dfrac{1}{t^4 + t^2}$

SOLUTION

(a) $f(-x) = (-x)^5 = -x^5 = -f(x)$, so this function is odd.

(b) $g(-t) = (-t)^3 - (-t)^2 = -t^3 - t^2$ which is equal to neither $g(t)$ nor $-g(t)$, so this function is neither odd nor even.

(c) This function is even because

$$F(-t) = \frac{1}{(-t)^4 + (-t)^2} = \frac{1}{t^4 + t^2} = F(t).$$

61. Determine whether the function is even, odd, or neither.

(a) $f(t) = \dfrac{1}{t^4+t+1} - \dfrac{1}{t^4-t+1}$

(b) $g(t) = 2^t - 2^{-t}$

(c) $G(\theta) = \sin\theta + \cos\theta$

(d) $H(\theta) = \sin(\theta^2)$

SOLUTION

(a) This function is odd because

$$\begin{aligned} f(-t) &= \frac{1}{(-t)^4+(-t)+1} - \frac{1}{(-t)^4-(-t)+1} \\ &= \frac{1}{t^4-t+1} - \frac{1}{t^4+t+1} = -f(t). \end{aligned}$$

(b) $g(-t) = 2^{-t} - 2^{-(-t)} = 2^{-t} - 2^t = -g(t)$, so this function is odd.

(c) $G(-\theta) = \sin(-\theta) + \cos(-\theta) = -\sin\theta + \cos\theta$ which is equal to neither $G(\theta)$ nor $-G(\theta)$, so this function is neither odd nor even.

(d) $H(-\theta) = \sin((-\theta)^2) = \sin(\theta^2) = H(\theta)$, so this function is even.

62. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

SOLUTION Let $g(x) = 2x^4 + 12x^2 + 4$ and $h(x) = -5x^3 - 3x$, so that $f(x) = g(x) + h(x)$. Observe

$$g(-x) = 2(-x)^4 + 12(-x)^2 + 4 = 2x^4 + 12x^2 + 4 = g(x),$$

while

$$h(-x) = -5(-x)^3 - 3(-x) = 5x^3 + 3x = -h(x).$$

Thus, $g(x)$ is an even function, and $h(x)$ is an odd function.

63. Show that $f(x) = \ln\left(\dfrac{1-x}{1+x}\right)$ is an odd function.

SOLUTION

$$\begin{aligned} f(-x) &= \ln\left(\frac{1-(-x)}{1+(-x)}\right) \\ &= \ln\left(\frac{1+x}{1-x}\right) = -\ln\left(\frac{1-x}{1+x}\right) = -f(x), \end{aligned}$$

so this is an odd function.

64. State whether the function is increasing, decreasing, or neither.

(a) Surface area of a sphere as a function of its radius

(b) Temperature at a point on the equator as a function of time

(c) Price of an airline ticket as a function of the price of oil

(d) Pressure of the gas in a piston as a function of volume

SOLUTION

(a) Increasing **(b)** Neither **(c)** Increasing **(d)** Decreasing

In Exercises 65–70, let $f(x)$ be the function shown in Figure 27.

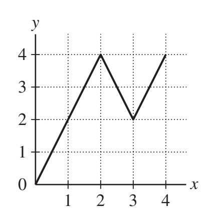


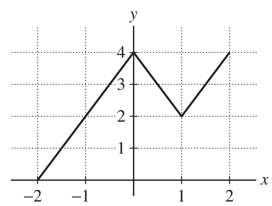
FIGURE 27

65. Find the domain and range of $f(x)$?

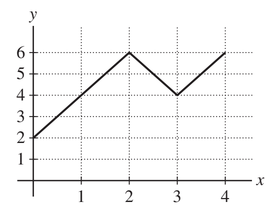
SOLUTION $D : [0, 4]$; $R : [0, 4]$

66. Sketch the graphs of $f(x+2)$ and $f(x)+2$.

SOLUTION The graph of $y = f(x+2)$ is obtained by shifting the graph of $y = f(x)$ two units to the left (see the graph below on the left). The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ two units up (see the graph below on the right).



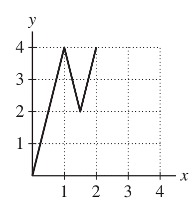
$f(x+2)$



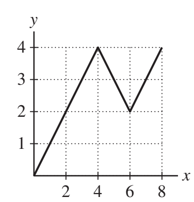
$f(x)+2$

67. Sketch the graphs of $f(2x)$, $f\left(\frac{1}{2}x\right)$, and $2f(x)$.

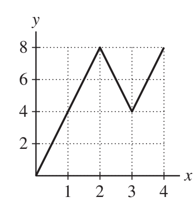
SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below in the middle). The graph of $y = 2f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below on the right).



$f(2x)$



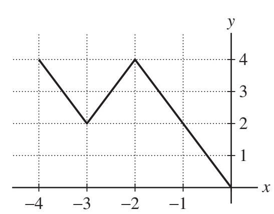
$f(x/2)$



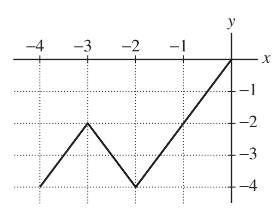
$2f(x)$

68. Sketch the graphs of $f(-x)$ and $-f(-x)$.

SOLUTION The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis (see the graph below on the left). The graph of $y = -f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across both the x- and y-axes, or equivalently, about the origin (see the graph below on the right).



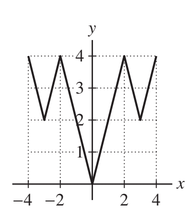
$f(-x)$



$-f(-x)$

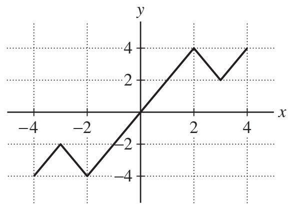
69. Extend the graph of $f(x)$ to $[-4, 4]$ so that it is an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph below).



70. Extend the graph of $f(x)$ to $[-4, 4]$ so that it is an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph below).



71. Suppose that $f(x)$ has domain $[4, 8]$ and range $[2, 6]$. Find the domain and range of:

(a) $f(x) + 3$ **(b)** $f(x + 3)$
(c) $f(3x)$ **(d)** $3f(x)$

SOLUTION

(a) $f(x) + 3$ is obtained by shifting $f(x)$ upward three units. Therefore, the domain remains $[4, 8]$, while the range becomes $[5, 9]$.

(b) $f(x + 3)$ is obtained by shifting $f(x)$ left three units. Therefore, the domain becomes $[1, 5]$, while the range remains $[2, 6]$.

(c) $f(3x)$ is obtained by compressing $f(x)$ horizontally by a factor of three. Therefore, the domain becomes $[\frac{4}{3}, \frac{8}{3}]$, while the range remains $[2, 6]$.

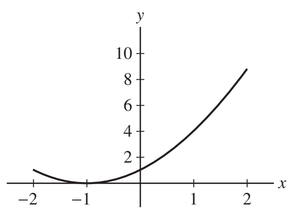
(d) $3f(x)$ is obtained by stretching $f(x)$ vertically by a factor of three. Therefore, the domain remains $[4, 8]$, while the range becomes $[6, 18]$.

72. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:

(a) $f(x + 1)$ **(b)** $f(x) + 1$
(c) $f(5x)$ **(d)** $5f(x)$

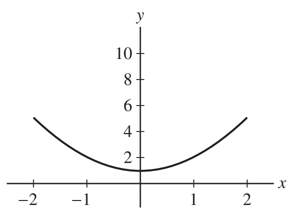
SOLUTION

(a) The graph of $y = f(x + 1)$ is obtained by shifting the graph of $y = f(x)$ one unit to the left.



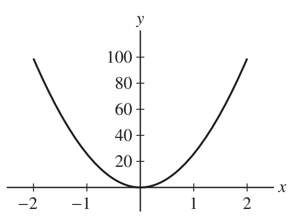
$f(x + 1)$

(b) The graph of $y = f(x) + 1$ is obtained by shifting the graph of $y = f(x)$ one unit up.



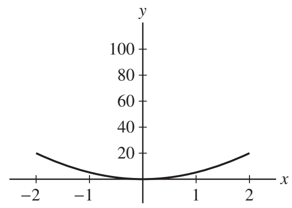
$f(x) + 1$

(c) The graph of $y = f(5x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 5.



$f(5x)$

(d) The graph of $y = 5f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 5.



$5f(x)$

73. Suppose that the graph of $f(x) = \sin x$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) $\boxed{\text{GU}}$ Verify your answers by plotting your equations.

SOLUTION

(a) Let $f(x) = \sin x$. After compressing the graph of f horizontally by a factor of 2, we obtain the function $g(x) = f(2x) = \sin 2x$. Shifting the graph 5 units to the right then yields

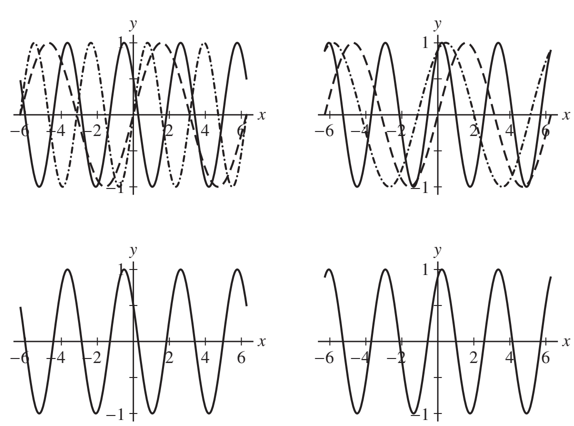
$$h(x) = g(x - 5) = \sin 2(x - 5) = \sin(2x - 10).$$

(b) Let $f(x) = \sin x$. After shifting the graph 5 units to the right, we obtain the function $g(x) = f(x - 5) = \sin(x - 5)$. Compressing the graph horizontally by a factor of 2 then yields

$$h(x) = g(2x) = \sin(2x - 5).$$

(c) The figure below at the top left shows the graphs of $y = \sin x$ (the dashed curve), the sine graph compressed horizontally by a factor of 2 (the dash, double dot curve) and then shifted right 5 units (the solid curve). Compare this last graph with the graph of $y = \sin(2x - 10)$ shown at the bottom left.

The figure below at the top right shows the graphs of $y = \sin x$ (the dashed curve), the sine graph shifted to the right 5 units (the dash, double dot curve) and then compressed horizontally by a factor of 2 (the solid curve). Compare this last graph with the graph of $y = \sin(2x - 5)$ shown at the bottom right.



74. Figure 28 shows the graph of $f(x) = |x| + 1$. Match the functions (a)–(e) with their graphs (i)–(v).

(a) $f(x - 1)$

(b) $-f(x)$

(c) $-f(x) + 2$

(d) $f(x - 1) - 2$

(e) $f(x + 1)$

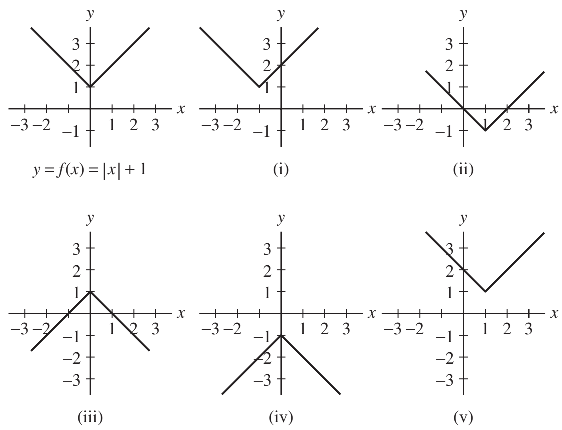


FIGURE 28

SOLUTION

(a) Shift graph to the right one unit: (v)

(b) Reflect graph across x-axis: (iv)

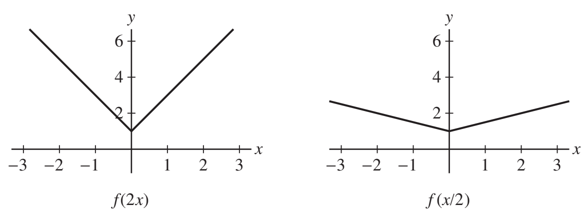
(c) Reflect graph across x-axis and then shift up two units: (iii)

(d) Shift graph to the right one unit and down two units: (ii)

(e) Shift graph to the left one unit: (i)

75. Sketch the graph of $f(2x)$ and $f\left(\frac{1}{2}x\right)$, where $f(x) = |x| + 1$ (Figure 28).

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the right).



76. Find the function $f(x)$ whose graph is obtained by shifting the parabola $y = x^2$ three units to the right and four units down, as in Figure 29.

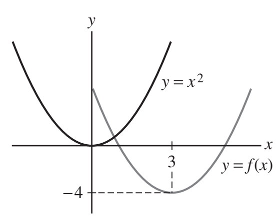
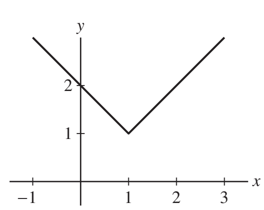


FIGURE 29

SOLUTION The new function is $f(x) = (x - 3)^2 - 4$

77. Define $f(x)$ to be the larger of x and $2 - x$. Sketch the graph of $f(x)$. What are its domain and range? Express $f(x)$ in terms of the absolute value function.

SOLUTION



The graph of $y = f(x)$ is shown above. Clearly, the domain of f is the set of all real numbers while the range is $\{y \mid y \geq 1\}$. Notice the graph has the standard V-shape associated with the absolute value function, but the base of the V has been translated to the point $(1, 1)$. Thus, $f(x) = |x - 1| + 1$.

78. For each curve in Figure 30, state whether it is symmetric with respect to the y-axis, the origin, both, or neither.

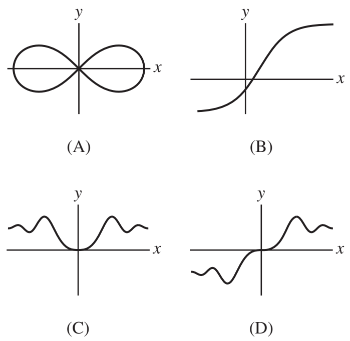


FIGURE 30

SOLUTION

(A) Both

(B) Neither

(C) y-axis

(D) Origin

79. Show that the sum of two even functions is even and the sum of two odd functions is odd.

SOLUTION Even: $(f+g)(-x) = f(-x) + g(-x) \overset{\text{even}}{=} f(x) + g(x) = (f+g)(x)$

Odd: $(f+g)(-x) = f(-x) + g(-x) \overset{\text{odd}}{=} -f(x) + -g(x) = -(f+g)(x)$

80. Suppose that $f(x)$ and $g(x)$ are both odd. Which of the following functions are even? Which are odd?

(a) $f(x)g(x)$

(b) $f(x)^3$

(c) $f(x) - g(x)$

(d) $\dfrac{f(x)}{g(x)}$

SOLUTION

(a) $f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) \Rightarrow$ Even

(b) $f(-x)^3 = [-f(x)]^3 = -f(x)^3 \Rightarrow$ Odd

(c) $f(-x) - g(-x) = -f(x) + g(x) = -(f(x) - g(x)) \Rightarrow$ Odd

(d) $\dfrac{f(-x)}{g(-x)} = \dfrac{-f(x)}{-g(x)} = \dfrac{f(x)}{g(x)} \Rightarrow$ Even

81. Prove that the only function whose graph is symmetric with respect to both the y-axis and the origin is the function $f(x) = 0$.

SOLUTION Suppose f is symmetric with respect to the y-axis. Then $f(-x) = f(x)$. If f is also symmetric with respect to the origin, then $f(-x) = -f(x)$. Thus $f(x) = -f(x)$ or $2f(x) = 0$. Finally, $f(x) = 0$.

Further Insights and Challenges

82. Prove the triangle inequality by adding the two inequalities

$$-|a| \leq a \leq |a|, \qquad -|b| \leq b \leq |b|$$

SOLUTION Adding the indicated inequalities gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

and this is equivalent to $|a + b| \leq |a| + |b|$.

83. Show that a fraction $r = a/b$ in lowest terms has a *finite* decimal expansion if and only if

$$b = 2^n 5^m \quad \text{for some } n, m \geq 0.$$

Hint: Observe that r has a finite decimal expansion when $10^N r$ is an integer for some $N \geq 0$ (and hence b divides 10^N).

SOLUTION Suppose r has a finite decimal expansion. Then there exists an integer $N \geq 0$ such that $10^N r$ is an integer, call it k. Thus, $r = k/10^N$. Because the only prime factors of 10 are 2 and 5, it follows that when r is written in lowest terms, its denominator must be of the form $2^n 5^m$ for some integers $n, m \geq 0$.

Conversely, suppose $r = a/b$ in lowest with $b = 2^n 5^m$ for some integers $n, m \geq 0$. Then $r = \dfrac{a}{b} = \dfrac{a}{2^n 5^m}$ or $2^n 5^m r = a$. If $m \geq n$, then $2^m 5^m r = a 2^{m-n}$ or $r = \dfrac{a 2^{m-n}}{10^m}$ and thus r has a finite decimal expansion (less than or equal to m terms, to be precise). On the other hand, if $n > m$, then $2^n 5^n r = a 5^{n-m}$ or $r = \dfrac{a 5^{n-m}}{10^n}$ and once again r has a finite decimal expansion.

84. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that

$$\frac{p}{10^s - 1} = 0.\overline{p_1 \dots p_s}$$

Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that

$$r = \frac{2}{11} = \frac{18}{10^2 - 1}$$

SOLUTION Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$, and let $\overline{p} = .\overline{p_1 \dots p_s}$. Then

$$10^s \overline{p} - \overline{p} = p_1 \dots p_s.\overline{p_1 \dots p_s} - .\overline{p_1 \dots p_s} = p_1 \dots p_s = p.$$

Thus,

$$\frac{p}{10^s - 1} = \overline{p} = .\overline{p_1 \dots p_s}.$$

Consider the rational number $r = 2/11$. Because

$$r = \frac{2}{11} = \frac{18}{99} = \frac{18}{10^2 - 1},$$

it follows that the decimal expansion of r is $0.\overline{18}$.

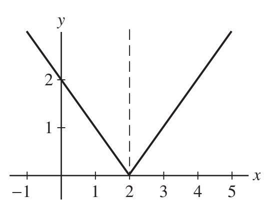
85. A function $f(x)$ is symmetric with respect to the vertical line $x = a$ if $f(a - x) = f(a + x)$.

(a) Draw the graph of a function that is symmetric with respect to $x = 2$.

(b) Show that if $f(x)$ is symmetric with respect to $x = a$, then $g(x) = f(x + a)$ is even.

SOLUTION

(a) There are many possibilities, one of which is



$y = |x - 2|$

(b) Let $g(x) = f(x + a)$. Then

$$\begin{aligned} g(-x) = f(-x + a) &= f(a - x) \\ &= f(a + x) \qquad \text{symmetry with respect to } x = a \\ &= g(x) \end{aligned}$$

Thus, $g(x)$ is even.

86. Formulate a condition for $f(x)$ to be symmetric with respect to the point $(a, 0)$ on the x-axis.

SOLUTION In order for $f(x)$ to be symmetrical with respect to the point $(a, 0)$, the value of f at a distance x units to the right of a must be opposite the value of f at a distance x units to the left of a. In other words, $f(x)$ is symmetrical with respect to $(a, 0)$ if $f(a + x) = -f(a - x)$.

1.2 Linear and Quadratic Functions

Preliminary Questions

1. What is the slope of the line $y = -4x - 9$?

SOLUTION The slope of the line $y = -4x - 9$ is -4, given by the coefficient of x.

2. Are the lines $y = 2x + 1$ and $y = -2x - 4$ perpendicular?

SOLUTION The slopes of perpendicular lines are negative reciprocals of one another. Because the slope of $y = 2x + 1$ is 2 and the slope of $y = -2x - 4$ is -2, these two lines are *not* perpendicular.

3. When is the line $ax + by = c$ parallel to the y-axis? To the x-axis?

SOLUTION The line $ax + by = c$ will be parallel to the y-axis when $b = 0$ and parallel to the x-axis when $a = 0$.

4. Suppose $y = 3x + 2$. What is Δy if x increases by 3?

SOLUTION Because $y = 3x + 2$ is a linear function with slope 3, increasing x by 3 will lead to $\Delta y = 3(3) = 9$.

5. What is the minimum of $f(x) = (x + 3)^2 - 4$?

SOLUTION Because $(x + 3)^2 \geq 0$, it follows that $(x + 3)^2 - 4 \geq -4$. Thus, the minimum value of $(x + 3)^2 - 4$ is -4.

6. What is the result of completing the square for $f(x) = x^2 + 1$?

SOLUTION Because there is no x term in $x^2 + 1$, completing the square on this expression leads to $(x - 0)^2 + 1$.

Exercises

In Exercises 1–4, find the slope, the y-intercept, and the x-intercept of the line with the given equation.

1. $y = 3x + 12$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term: that is, $m = 3$ and the y-intercept is 12. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 3x + 12$ or $x = -4$.

2. $y = 4 - x$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term: that is, $m = -1$ and the y-intercept is 4. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 4 - x$ or $x = 4$.

3. $4x + 9y = 3$

SOLUTION To determine the slope and y-intercept, we first solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{4}{9}x + \frac{1}{3}$. From here, we see that the slope is $m = -\frac{4}{9}$ and the y-intercept is $\frac{1}{3}$. To determine the x-intercept, substitute $y = 0$ and solve for x: $4x = 3$ or $x = \frac{3}{4}$.

4. $y - 3 = \frac{1}{2}(x - 6)$

SOLUTION The equation is in point-slope form, so we see that $m = \frac{1}{2}$. Substituting $x = 0$ yields $y - 3 = -3$ or $y = 0$. Thus, the x- and y-intercepts are both 0.

In Exercises 5–8, find the slope of the line.

5. $y = 3x + 2$

SOLUTION $m = 3$

6. $y = 3(x - 9) + 2$

SOLUTION $m = 3$

7. $3x + 4y = 12$

SOLUTION First solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x + 3$. The slope of the line is therefore $m = -\frac{3}{4}$.

8. $3x + 4y = -8$

SOLUTION First solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x - 2$. The slope of the line is therefore $m = -\frac{3}{4}$.

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, y-intercept 8

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = 3x + 8$.

10. Slope -2, y-intercept 3

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = -2x + 3$.

11. Slope 3, passes through $(7, 9)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 9 = 3(x - 7)$ or $y = 3x - 12$.

12. Slope -5, passes through $(0, 0)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 0 = -5(x - 0)$ or $y = -5x$.

13. Horizontal, passes through $(0, -2)$

SOLUTION A horizontal line has a slope of 0. Using the point-slope form for the equation of a line, we have $y - (-2) = 0(x - 0)$ or $y = -2$.

14. Passes through $(-1, 4)$ and $(2, 7)$

SOLUTION The slope of the line that passes through $(-1, 4)$ and $(2, 7)$ is

$$m = \frac{7 - 4}{2 - (-1)} = 1.$$

Using the point-slope form for the equation of a line, we have $y - 7 = 1(x - 2)$ or $y = x + 5$.

15. Parallel to $y = 3x - 4$, passes through $(1, 1)$

SOLUTION Because the equation $y = 3x - 4$ is in slope-intercept form, we can readily identify that it has a slope of 3. Parallel lines have the same slope, so the slope of the requested line is also 3. Using the point-slope form for the equation of a line, we have $y - 1 = 3(x - 1)$ or $y = 3x - 2$.

16. Passes through $(1, 4)$ and $(12, -3)$

SOLUTION The slope of the line that passes through $(1, 4)$ and $(12, -3)$ is

$$m = \frac{-3 - 4}{12 - 1} = \frac{-7}{11}.$$

Using the point-slope form for the equation of a line, we have $y - 4 = -\frac{7}{11}(x - 1)$ or $y = -\frac{7}{11}x + \frac{51}{11}$.

17. Perpendicular to $3x + 5y = 9$, passes through $(2, 3)$

SOLUTION We start by solving the equation $3x + 5y = 9$ for y to obtain the slope-intercept form for the equation of a line. This yields

$$y = -\frac{3}{5}x + \frac{9}{5},$$

from which we identify the slope as $-\frac{3}{5}$. Perpendicular lines have slopes that are negative reciprocals of one another, so the slope of the desired line is $m_{\perp} = \frac{5}{3}$. Using the point-slope form for the equation of a line, we have $y - 3 = \frac{5}{3}(x - 2)$ or $y = \frac{5}{3}x - \frac{1}{3}$.

18. Vertical, passes through $(-4, 9)$

SOLUTION A vertical line has the equation $x = c$ for some constant c. Because the line needs to pass through the point $(-4, 9)$, we must have $c = -4$. The equation of the desired line is then $x = -4$.

19. Horizontal, passes through $(8, 4)$

SOLUTION A horizontal line has slope 0. Using the point slope form for the equation of a line, we have $y - 4 = 0(x - 8)$ or $y = 4$.

20. Slope 3, x-intercept 6

SOLUTION If the x-intercept is 6, then the line passes through the point $(6, 0)$. Using the point-slope form for the equation of a line, we have $y - 0 = 3(x - 6)$ or $y = 3x - 18$.

21. Find the equation of the perpendicular bisector of the segment joining (1, 2) and (5, 4) (Figure 11). *Hint:* The midpoint Q of the segment joining (a, b) and (c, d) is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

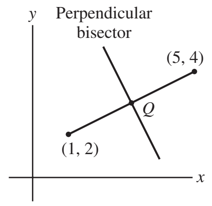


FIGURE 11

SOLUTION The slope of the segment joining (1, 2) and (5, 4) is

$$m = \frac{4-2}{5-1} = \frac{1}{2}$$

and the midpoint of the segment (Figure 11) is

$$\text{midpoint} = \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$$

The perpendicular bisector has slope $-1/m = -2$ and passes through (3, 3), so its equation is: $y - 3 = -2(x - 3)$ or $y = -2x + 9$.

22. Intercept-Intercept Form Show that if $a, b \neq 0$, then the line with x-intercept $x = a$ and y-intercept $y = b$ has equation (Figure 12)

$$\frac{x}{a} + \frac{y}{b} = 1$$

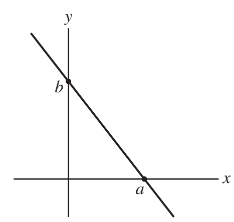


FIGURE 12

SOLUTION The line passes through the points $(a, 0)$ and $(0, b)$. Thus $m = -\frac{b}{a}$. Using the point-slope form for the equation of a line yields $y - 0 = -\frac{b}{a}(x - a) \Rightarrow y = -\frac{b}{a}x + b \Rightarrow \frac{b}{a}x + y = b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

23. Find an equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

SOLUTION From Exercise 22, $\frac{x}{4} + \frac{y}{3} = 1$ or $3x + 4y = 12$.

24. Find y such that $(3, y)$ lies on the line of slope $m = 2$ through (1, 4).

SOLUTION In order for the point $(3, y)$ to lie on the line through (1, 4) of slope 2, the slope of the segment connecting (1, 4) and $(3, y)$ must have slope 2. Therefore,

$$m = \frac{y-4}{3-1} = \frac{y-4}{2} = 2 \Rightarrow y - 4 = 4 \Rightarrow y = 8.$$

25. Determine whether there exists a constant c such that the line $x + cy = 1$:

(a) Has slope 4
(b) Passes through (3, 1)
(c) Is horizontal
(d) Is vertical

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = -\frac{x}{c} + \frac{1}{c}$. To have slope 4 requires $-\frac{1}{c} = 4$ or $c = -\frac{1}{4}$.

(b) Substituting $x = 3$ and $y = 1$ into the equation of the line gives $3 + c = 1$ or $c = -2$.

(c) From (a), we know the slope of the line is $-\frac{1}{c}$. There is no value for c that will make this slope equal to 0.

(d) With $c = 0$, the equation becomes $x = 1$. This is the equation of a vertical line.

26. Assume that the number N of concert tickets that can be sold at a price of P dollars per ticket is a linear function $N(P)$ for $10 \le P \le 40$. Determine $N(P)$ (called the demand function) if $N(10) = 500$ and $N(40) = 0$. What is the decrease ΔN in the number of tickets sold if the price is increased by $\Delta P = 5$ dollars?

SOLUTION We first determine the slope of the line:

$$m = \frac{500 - 0}{10 - 40} = \frac{500}{-30} = -\frac{50}{3}.$$

Knowing that $N(40) = 0$, it follows that

$$N(P) = -\frac{50}{3}(P - 40) = -\frac{50}{3}P + \frac{2000}{3}.$$

Because the slope of the demand function is $-\frac{50}{3}$, a 5 dollar increase in price will lead to a decrease in the number of tickets sold of $\frac{50}{3}(5) = \frac{250}{3} = 83\frac{1}{3}$, or about 83 tickets.

27. Materials expand when heated. Consider a metal rod of length L_0 at temperature T_0. If the temperature is changed by an amount ΔT, then the rod's length changes by $\Delta L = \alpha L_0 \Delta T$, where α is the thermal expansion coefficient. For steel, $\alpha = 1.24 \times 10^{-5}\,{}^\circ\mathrm{C}^{-1}$.

(a) A steel rod has length $L_0 = 40$ cm at $T_0 = 40^\circ$C. Find its length at $T = 90^\circ$C.

(b) Find its length at $T = 50^\circ$C if its length at $T_0 = 100^\circ$C is 65 cm.

(c) Express length L as a function of T if $L_0 = 65$ cm at $T_0 = 100^\circ$C.

SOLUTION

(a) With $T = 90^\circ$C and $T_0 = 40^\circ$C, $\Delta T = 50^\circ$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(40)(50) = .0248 \qquad \text{and} \qquad L = L_0 + \Delta L = 40.0248 \text{ cm}.$$

(b) With $T = 50^\circ$C and $T_0 = 100^\circ$C, $\Delta T = -50^\circ$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(65)(-50) = -.0403 \qquad \text{and} \qquad L = L_0 + \Delta L = 64.9597 \text{ cm}.$$

(c) $L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) = 65(1 + \alpha(T - 100))$

28. Do the points (0.5, 1), (1, 1.2), (2, 2) lie on a line?

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.2 - 1}{1 - 0.5} = \frac{0.2}{0.5} = 0.4,$$

while the second pair of data points yields a slope of

$$\frac{2 - 1.2}{2 - 1} = \frac{0.8}{1} = 0.8.$$

Because the slopes are not equal, the three points do not lie on a line.

29. Find b such that $(2, -1)$, $(3, 2)$, and $(b, 5)$ lie on a line.

SOLUTION The slope of the line determined by the points $(2, -1)$ and $(3, 2)$ is

$$\frac{2 - (-1)}{3 - 2} = 3.$$

To lie on the same line, the slope between $(3, 2)$ and $(b, 5)$ must also be 3. Thus, we require

$$\frac{5 - 2}{b - 3} = \frac{3}{b - 3} = 3,$$

or $b = 4$.

30. Find an expression for the velocity v as a linear function of t that matches the following data.

t (s)	0	2	4	6
v (m/s)	39.2	58.6	78	97.4

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{58.6 - 39.2}{2 - 0} = 9.7,$$

while the second pair of data points yields a slope of

$$\frac{78 - 58.6}{4 - 2} = 9.7,$$

and the last pair of data points yields a slope of

$$\frac{97.4 - 78}{6 - 4} = 9.7$$

Thus, the data suggests a linear function with slope 9.7. Finally,

$$v - 39.2 = 9.7(t - 0) \Rightarrow v = 9.7t + 39.2$$

31. The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.1 - 0.9}{30 - 20} = 0.02,$$

while the second pair of data points yields a slope of

$$\frac{1.27 - 1.1}{40 - 30} = 0.017,$$

and the last pair of data points yields a slope of

$$\frac{1.42 - 1.27}{50 - 40} = 0.015$$

Because the three slopes are not equal, T does not appear to be a linear function of L.

32. Show that $f(x)$ is linear of slope m if and only if

$$f(x+h) - f(x) = mh \quad \text{(for all } x \text{ and } h\text{)}$$

SOLUTION First, suppose $f(x)$ is linear. Then the slope between $(x, f(x))$ and $(x+h, f(x+h))$ is

$$m = \frac{f(x+h) - f(x)}{h} \Rightarrow mh = f(x+h) - f(x).$$

Conversely, suppose $f(x+h) - f(x) = mh$ for all x and for all h. Then

$$m = \frac{f(x+h) - f(x)}{h} = \frac{f(x+h) - f(x)}{x+h-x},$$

which is the slope between $(x, f(x))$ and $(x+h, f(x+h))$. Since this is true for all x and h, f must be linear (it has constant slope).

33. Find the roots of the quadratic polynomials:

(a) $4x^2 - 3x - 1$

(b) $x^2 - 2x - 1$

SOLUTION

(a) $x = \dfrac{3 \pm \sqrt{9 - 4(4)(-1)}}{2(4)} = \dfrac{3 \pm \sqrt{25}}{8} = 1 \text{ or } -\dfrac{1}{4}$

(b) $x = \dfrac{2 \pm \sqrt{4 - (4)(1)(-1)}}{2} = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

In Exercises 34–41, complete the square and find the minimum or maximum value of the quadratic function.

34. $y = x^2 + 2x + 5$

SOLUTION $y = x^2 + 2x + 1 - 1 + 5 = (x + 1)^2 + 4$; therefore, the minimum value of the quadratic polynomial is 4, and this occurs at $x = -1$.

35. $y = x^2 - 6x + 9$

SOLUTION $y = (x - 3)^2$; therefore, the minimum value of the quadratic polynomial is 0, and this occurs at $x = 3$.

36. $y = -9x^2 + x$

SOLUTION $y = -9(x^2 - x/9) = -9(x^2 - \frac{x}{9} + \frac{1}{324}) + \frac{9}{324} = -9(x - \frac{1}{18})^2 + \frac{1}{36}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{36}$, and this occurs at $x = \frac{1}{18}$.

37. $y = x^2 + 6x + 2$

SOLUTION $y = x^2 + 6x + 9 - 9 + 2 = (x + 3)^2 - 7$; therefore, the minimum value of the quadratic polynomial is -7, and this occurs at $x = -3$.

38. $y = 2x^2 - 4x - 7$

SOLUTION $y = 2(x^2 - 2x + 1 - 1) - 7 = 2(x^2 - 2x + 1) - 7 - 2 = 2(x - 1)^2 - 9$; therefore, the minimum value of the quadratic polynomial is -9, and this occurs at $x = 1$.

39. $y = -4x^2 + 3x + 8$

SOLUTION $y = -4x^2 + 3x + 8 = -4(x^2 - \frac{3}{4}x + \frac{9}{64}) + 8 + \frac{9}{16} = -4(x - \frac{3}{8})^2 + \frac{137}{16}$; therefore, the maximum value of the quadratic polynomial is $\frac{137}{16}$, and this occurs at $x = \frac{3}{8}$.

40. $y = 3x^2 + 12x - 5$

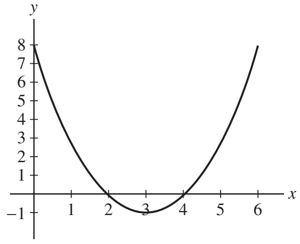
SOLUTION $y = 3(x^2 + 4x + 4) - 5 - 12 = 3(x + 2)^2 - 17$; therefore, the minimum value of the quadratic polynomial is -17, and this occurs at $x = -2$.

41. $y = 4x - 12x^2$

SOLUTION $y = -12(x^2 - \frac{x}{3}) = -12(x^2 - \frac{x}{3} + \frac{1}{36}) + \frac{1}{3} = -12(x - \frac{1}{6})^2 + \frac{1}{3}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{3}$, and this occurs at $x = \frac{1}{6}$.

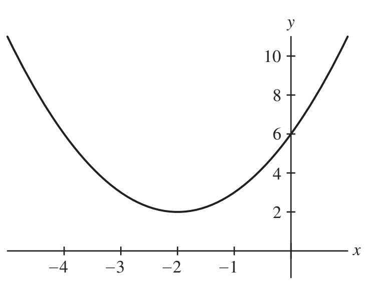
42. Sketch the graph of $y = x^2 - 6x + 8$ by plotting the roots and the minimum point.

SOLUTION $y = x^2 - 6x + 9 - 9 + 8 = (x - 3)^2 - 1$ so the vertex is located at $(3, -1)$ and the roots are $x = 2$ and $x = 4$. This is the graph of x^2 moved right 3 units and down 1 unit.



43. Sketch the graph of $y = x^2 + 4x + 6$ by plotting the minimum point, the y-intercept, and one other point.

SOLUTION $y = x^2 + 4x + 4 - 4 + 6 = (x + 2)^2 + 2$ so the minimum occurs at $(-2, 2)$. If $x = 0$, then $y = 6$ and if $x = -4$, $y = 6$. This is the graph of x^2 moved left 2 units and up 2 units.



44. If the alleles A and B of the cystic fibrosis gene occur in a population with frequencies p and $1 - p$ (where p is a fraction between 0 and 1), then the frequency of heterozygous carriers (carriers with both alleles) is $2p(1 - p)$. Which value of p gives the largest frequency of heterozygous carriers?

SOLUTION Let

$$f = 2p - 2p^2 = -2\left(p^2 - p + \frac{1}{4}\right) + \frac{1}{2} = -2\left(p - \frac{1}{2}\right)^2 + \frac{1}{2}.$$

Then $p = \frac{1}{2}$ yields a maximum.

45. For which values of c does $f(x) = x^2 + cx + 1$ have a double root? No real roots?

SOLUTION A double root occurs when $c^2 - 4(1)(1) = 0$ or $c^2 = 4$. Thus, $c = \pm 2$.

There are no real roots when $c^2 - 4(1)(1) < 0$ or $c^2 < 4$. Thus, $-2 < c < 2$.

46. Let $f(x)$ be a quadratic function and c a constant. Which of the following statements is correct? Explain graphically.

(a) There is a unique value of c such that $y = f(x) - c$ has a double root.

(b) There is a unique value of c such that $y = f(x - c)$ has a double root.

SOLUTION First note that because $f(x)$ is a quadratic function, its graph is a parabola.

(a) This is true. Because $f(x) - c$ is a vertical translation of the graph of $f(x)$, there is one and only one value of c that will move the vertex of the parabola to the x-axis.

(b) This is false. Observe that $f(x - c)$ is a horizontal translation of the graph of $f(x)$. If $f(x)$ has a double root, then $f(x - c)$ will have a double root for any value of c; on the other hand, if $f(x)$ does not have a double root, then there is no value of c for which $f(x - c)$ will have a double root.

47. Prove that $x + \frac{1}{x} \geq 2$ for all $x > 0$. *Hint:* Consider $(x^{1/2} - x^{-1/2})^2$.

SOLUTION Let $x > 0$. Then

$$\left(x^{1/2} - x^{-1/2}\right)^2 = x - 2 + \frac{1}{x}.$$

Because $(x^{1/2} - x^{-1/2})^2 \geq 0$, it follows that

$$x - 2 + \frac{1}{x} \geq 0 \qquad \text{or} \qquad x + \frac{1}{x} \geq 2.$$

48. Let $a, b > 0$. Show that the *geometric mean* $\sqrt{ab}$ is not larger than the *arithmetic mean* $(a + b)/2$. *Hint:* Use a variation of the hint given in Exercise 47.

SOLUTION Let $a, b > 0$ and note

$$0 \leq \left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b.$$

Therefore,

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

49. If objects of weights x and w_1 are suspended from the balance in Figure 13(A), the cross-beam is horizontal if $bx = aw_1$. If the lengths a and b are known, we may use this equation to determine an unknown weight x by selecting w_1 such that the cross-beam is horizontal. If a and b are not known precisely, we might proceed as follows. First balance x by w_1 on the left as in (A). Then switch places and balance x by w_2 on the right as in (B). The average $\bar{x} = \frac{1}{2}(w_1 + w_2)$ gives an estimate for x. Show that $\bar{x}$ is greater than or equal to the true weight x.

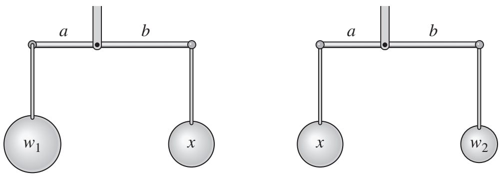


FIGURE 13

SOLUTION First note $bx = aw_1$ and $ax = bw_2$. Thus,

$$\bar{x} = \frac{1}{2}(w_1 + w_2)$$

$$\begin{aligned} &= \frac{1}{2}\left(\frac{bx}{a} + \frac{ax}{b}\right) \\ &= \frac{x}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\ &\geq \frac{x}{2}(2) \qquad \text{by Exercise 47} \\ &= x \end{aligned}$$

50. Find numbers x and y with sum 10 and product 24. *Hint:* Find a quadratic polynomial satisfied by x.

SOLUTION Let x and y be numbers whose sum is 10 and product is 24. Then $x + y = 10$ and $xy = 24$. From the second equation, $y = \frac{24}{x}$. Substituting this expression for y in the first equation gives $x + \frac{24}{x} = 10$ or $x^2 - 10x + 24 = (x-4)(x-6) = 0$, whence $x = 4$ or $x = 6$. If $x = 4$, then $y = \frac{24}{4} = 6$. On the other hand, if $x = 6$, then $y = \frac{24}{6} = 4$. Thus, the two numbers are 4 and 6.

51. Find a pair of numbers whose sum and product are both equal to 8.

SOLUTION Let x and y be numbers whose sum and product are both equal to 8. Then $x + y = 8$ and $xy = 8$. From the second equation, $y = \frac{8}{x}$. Substituting this expression for y in the first equation gives $x + \frac{8}{x} = 8$ or $x^2 - 8x + 8 = 0$. By the quadratic formula,

$$x = \frac{8 \pm \sqrt{64 - 32}}{2} = 4 \pm 2\sqrt{2}.$$

If $x = 4 + 2\sqrt{2}$, then

$$y = \frac{8}{4 + 2\sqrt{2}} = \frac{8}{4 + 2\sqrt{2}} \cdot \frac{4 - 2\sqrt{2}}{4 - 2\sqrt{2}} = 4 - 2\sqrt{2}.$$

On the other hand, if $x = 4 - 2\sqrt{2}$, then

$$y = \frac{8}{4 - 2\sqrt{2}} = \frac{8}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}} = 4 + 2\sqrt{2}.$$

Thus, the two numbers are $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$.

52. Show that the parabola $y = x^2$ consists of all points P such that $d_1 = d_2$, where d_1 is the distance from P to $\left(0, \frac{1}{4}\right)$ and d_2 is the distance from P to the line $y = -\frac{1}{4}$ (Figure 14).

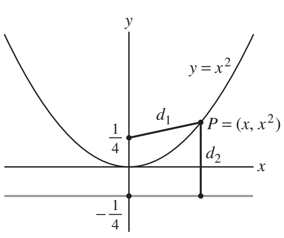


FIGURE 14

SOLUTION Let P be a point on the graph of the parabola $y = x^2$. Then P has coordinates (x, x^2) for some real number x. Now $d_2 = x^2 + \frac{1}{4}$ and

$$d_1 = \sqrt{(x-0)^2 + \left(x^2 - \frac{1}{4}\right)^2} = \sqrt{x^2 + x^4 - \frac{1}{2}x^2 + \frac{1}{16}} = \sqrt{\left(x^2 + \frac{1}{4}\right)^2} = x^2 + \frac{1}{4} = d_2.$$

Further Insights and Challenges

53. Show that if $f(x)$ and $g(x)$ are linear, then so is $f(x) + g(x)$. Is the same true of $f(x)g(x)$?

SOLUTION If $f(x) = mx + b$ and $g(x) = nx + d$, then

$$f(x) + g(x) = mx + b + nx + d = (m+n)x + (b+d),$$

which is linear. $f(x)g(x)$ is not generally linear. Take, for example, $f(x) = g(x) = x$. Then $f(x)g(x) = x^2$.

54. Show that if $f(x)$ and $g(x)$ are linear functions such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f(x) = g(x)$.

SOLUTION Suppose $f(x) = mx + b$ and $g(x) = nx + d$. Then $f(0) = b$ and $g(0) = d$, which implies $b = d$. Thus $f(x) = mx + b$ and $g(x) = nx + b$. Now, $f(1) = m + b$ and $g(1) = n + b$ so $m + b = n + b$ and $m = n$. Thus $f(x) = g(x)$.

55. Show that $\Delta y/\Delta x$ for the function $f(x) = x^2$ over the interval $[x_1, x_2]$ is not a constant, but depends on the interval. Determine the exact dependence of $\Delta y/\Delta x$ on x_1 and x_2.

SOLUTION For x^2, $\dfrac{\Delta y}{\Delta x} = \dfrac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$.

56. Use Eq. (2) to derive the quadratic formula for the roots of $ax^2 + bx + c = 0$.

SOLUTION Consider the equation $ax^2 + bx + c = 0$. First, complete the square to obtain

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{and} \qquad \left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Dropping the absolute values yields

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

57. Let $a, c \neq 0$. Show that the roots of

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad cx^2 + bx + a = 0$$

are reciprocals of each other.

SOLUTION Let r_1 and r_2 be the roots of $ax^2 + bx + c$ and r_3 and r_4 be the roots of $cx^2 + bx + a$. Without loss of generality, let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{1}{r_1} = \frac{2a}{-b + \sqrt{b^2 - 4ac}} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$
$$= \frac{2a(-b - \sqrt{b^2 - 4ac})}{b^2 - b^2 + 4ac} = \frac{-b - \sqrt{b^2 - 4ac}}{2c} = r_4.$$

Similarly, you can show $\dfrac{1}{r_2} = r_3$.

58. Show, by completing the square, that the parabola

$$y = ax^2 + bx + c$$

is congruent to $y = ax^2$ by a vertical and horizontal translation.

SOLUTION

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Thus, the first parabola is just the second translated horizontally by $-\frac{b}{2a}$ and vertically by $\frac{4ac-b^2}{4a}$.

59. Prove **Viète's Formulas**: The quadratic polynomial with α and β as roots is $x^2 + bx + c$, where $b = -\alpha - \beta$ and $c = \alpha\beta$.

SOLUTION If a quadratic polynomial has roots α and β, then the polynomial is

$$(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 + (-\alpha - \beta)x + \alpha\beta.$$

Thus, $b = -\alpha - \beta$ and $c = \alpha\beta$.

1.3 The Basic Classes of Functions

Preliminary Questions

1. Give an example of a rational function.

SOLUTION One example is $\dfrac{3x^2-2}{7x^3+x-1}$.

2. Is $|x|$ a polynomial function? What about $|x^2+1|$?

SOLUTION $|x|$ is not a polynomial; however, because $x^2+1>0$ for all x, it follows that $|x^2+1| = x^2+1$, which is a polynomial.

3. What is unusual about the domain of the composite function $f \circ g$ for the functions $f(x) = x^{1/2}$ and $g(x) = -1-|x|$?

SOLUTION Recall that $(f \circ g)(x) = f(g(x))$. Now, for any real number x, $g(x) = -1-|x| \le -1 < 0$. Because we cannot take the square root of a negative number, it follows that $f(g(x))$ is not defined for any real number. In other words, the domain of $f(g(x))$ is the empty set.

4. Is $f(x) = \left(\frac{1}{2}\right)^x$ increasing or decreasing?

SOLUTION The function $f(x) = (\frac{1}{2})^x$ is an exponential function with base $b = \frac{1}{2} < 1$. Therefore, f is a decreasing function.

5. Give an example of a transcendental function.

SOLUTION One possibility is $f(x) = e^x - \sin x$.

Exercises

In Exercises 1–12, determine the domain of the function.

1. $f(x) = x^{1/4}$

SOLUTION $x \ge 0$

2. $g(t) = t^{2/3}$

SOLUTION All reals

3. $f(x) = x^3 + 3x - 4$

SOLUTION All reals

4. $h(z) = z^3 + z^{-3}$

SOLUTION $z \ne 0$

5. $g(t) = \dfrac{1}{t+2}$

SOLUTION $t \ne -2$

6. $f(x) = \dfrac{1}{x^2+4}$

SOLUTION All reals

7. $G(u) = \dfrac{1}{u^2-4}$

SOLUTION $u \ne \pm 2$

8. $f(x) = \dfrac{\sqrt{x}}{x^2-9}$

SOLUTION $x \ge 0, x \ne 3$

9. $f(x) = x^{-4} + (x-1)^{-3}$

SOLUTION $x \ne 0, 1$

10. $F(s) = \sin\left(\dfrac{s}{s+1}\right)$

SOLUTION $s \ne -1$

11. $g(y) = 10^{\sqrt{y}+y^{-1}}$

SOLUTION $y > 0$

12. $f(x) = \dfrac{x + x^{-1}}{(x-3)(x+4)}$

SOLUTION $x \neq 0, 3, -4$

In Exercises 13–24, identify each of the following functions as polynomial, rational, algebraic, or transcendental.

13. $f(x) = 4x^3 + 9x^2 - 8$

SOLUTION Polynomial

14. $f(x) = x^{-4}$

SOLUTION Rational

15. $f(x) = \sqrt{x}$

SOLUTION Algebraic

16. $f(x) = \sqrt{1 - x^2}$

SOLUTION Algebraic

17. $f(x) = \dfrac{x^2}{x + \sin x}$

SOLUTION Transcendental

18. $f(x) = 2^x$

SOLUTION Transcendental

19. $f(x) = \dfrac{2x^3 + 3x}{9 - 7x^2}$

SOLUTION Rational

20. $f(x) = \dfrac{3x - 9x^{-1/2}}{9 - 7x^2}$

SOLUTION Algebraic

21. $f(x) = \sin(x^2)$

SOLUTION Transcendental

22. $f(x) = \dfrac{x}{\sqrt{x} + 1}$

SOLUTION Algebraic

23. $f(x) = x^2 + 3x^{-1}$

SOLUTION Rational

24. $f(x) = \sin(3^x)$

SOLUTION Transcendental

25. Is $f(x) = 2^{x^2}$ a transcendental function?

SOLUTION Yes.

26. Show that $f(x) = x^2 + 3x^{-1}$ and $g(x) = 3x^3 - 9x + x^{-2}$ are rational functions—that is, quotients of polynomials.

SOLUTION
$$f(x) = x^2 + 3x^{-1} = x^2 + \frac{3}{x} = \frac{x^3 + 3}{x}$$
$$g(x) = 3x^3 - 9x + x^{-2} = \frac{3x^5 - 9x^3 + 1}{x^2}$$

In Exercises 27–34, calculate the composite functions $f \circ g$ and $g \circ f$, and determine their domains.

27. $f(x) = \sqrt{x}, \quad g(x) = x + 1$

SOLUTION $f(g(x)) = \sqrt{x+1}$; D: $x \geq -1$, $\quad g(f(x)) = \sqrt{x} + 1$; D: $x \geq 0$

28. $f(x) = \frac{1}{x}, \quad g(x) = x^{-4}$

SOLUTION $f(g(x)) = x^4$; D: $x \neq 0$, $\quad g(f(x)) = x^4$: D: $x \neq 0$

29. $f(x) = 2^x, \quad g(x) = x^2$

SOLUTION $f(g(x)) = 2^{x^2}$; D: $\mathbf{R}$, $\quad g(f(x)) = (2^x)^2 = 2^{2x}$; D: $\mathbf{R}$

30. $f(x) = |x|, \quad g(\theta) = \sin\theta$

SOLUTION $f(g(\theta)) = |\sin\theta|$; D: $\mathbf{R}$, $\quad g(f(x)) = \sin|x|$; D: $\mathbf{R}$

31. $f(\theta) = \cos\theta, \quad g(x) = x^3 + x^2$

SOLUTION $f(g(x)) = \cos(x^3 + x^2)$; D: $\mathbf{R}$, $\quad g(f(\theta)) = \cos^3\theta + \cos^2\theta$; D: $\mathbf{R}$

32. $f(x) = \frac{1}{x^2+1}, \quad g(x) = x^{-2}$

SOLUTION $f(g(x)) = \frac{1}{(x^{-2})^2 + 1} = \frac{1}{x^{-4}+1}$; D: $x \neq 0$, $\quad g(f(x)) = \left(\frac{1}{x^2+1}\right)^{-2} = (x^2+1)^2$; D: $\mathbf{R}$

33. $f(t) = \frac{1}{\sqrt{t}}, \quad g(t) = -t^2$

SOLUTION $f(g(t)) = \frac{1}{\sqrt{-t^2}}$; D: Not valid for any t, $\quad g(f(t)) = -\left(\frac{1}{\sqrt{t}}\right)^2 = -\frac{1}{t}$; D: $t > 0$

34. $f(t) = \sqrt{t}, \quad g(t) = 1 - t^3$

SOLUTION $f(g(t)) = \sqrt{1-t^3}$; D: $t \leq 1$, $\quad g(f(t)) = 1 - t^{3/2}$; D: $t \geq 0$

35. The population (in millions) of a country as a function of time t (years) is $P(t) = 30.2^{0.1t}$. Show that the population doubles every 10 years. Show more generally that for any positive constants a and k, the function $g(t) = a2^{kt}$ doubles after $1/k$ years.

SOLUTION Let $P(t) = 30 \cdot 2^{0.1t}$. Then

$$P(t+10) = 30 \cdot 2^{0.1(t+10)} = 30 \cdot 2^{0.1t+1} = 2(30 \cdot 2^{0.1t}) = 2P(t).$$

Hence, the population doubles in size every 10 years. In the more general case, let $g(t) = a2^{kt}$. Then

$$g\left(t + \frac{1}{k}\right) = a2^{k(t+1/k)} = a2^{kt+1} = 2a2^{kt} = 2g(t).$$

Hence, the function g doubles after $1/k$ years.

36. Find all values of c such that $f(x) = \frac{x+1}{x^2+2cx+4}$ has domain $\mathbf{R}$.

SOLUTION The domain of f will consist of all real numbers provided the denominator has no real roots. The roots of $x^2 + 2cx + 4 = 0$ are

$$x = \frac{-2c \pm \sqrt{4c^2 - 16}}{2} = -c \pm \sqrt{c^2 - 4}.$$

There will be no real roots when $c^2 < 4$ or when $-2 < c < 2$.

Further Insights and Challenges

In Exercises 37–43, we define the first difference δf *of a function* $f(x)$ *by* $\delta f(x) = f(x+1) - f(x)$.

37. Show that if $f(x) = x^2$, then $\delta f(x) = 2x + 1$. Calculate δf for $f(x) = x$ and $f(x) = x^3$.

SOLUTION $f(x) = x^2$: $\delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$
$f(x) = x$: $\delta f(x) = x + 1 - x = 1$
$f(x) = x^3$: $\delta f(x) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$

38. Show that $\delta(10^x) = 9 \cdot 10^x$ and, more generally, that $\delta(b^x) = (b-1)b^x$.

SOLUTION $\delta(10^x) = 10^{x+1} - 10^x = 10 \cdot 10^x - 10^x = 10^x(10-1) = 9 \cdot 10^x$
$\delta(b^x) = b^{x+1} - b^x = b^x(b-1)$

39. Show that for any two functions f and g, $\delta(f+g) = \delta f + \delta g$ and $\delta(cf) = c\delta(f)$, where c is any constant.

SOLUTION $\delta(f+g) = (f(x+1)+g(x+1)) - (f(x) - g(x))$

$$= (f(x+1) - f(x)) + (g(x+1) - g(x)) = \delta f(x) + \delta g(x)$$

$$\delta(cf) = cf(x+1) - cf(x) = c(f(x+1) - f(x)) = c\delta f(x).$$

40. Suppose we can find a function $P(x)$ such that $\delta P = (x+1)^k$ and $P(0) = 0$. Prove that $P(1) = 1^k$, $P(2) = 1^k + 2^k$, and, more generally, for every whole number n,

$$P(n) = 1^k + 2^k + \cdots + n^k \qquad \boxed{1}$$

SOLUTION Suppose we have found a function $P(x)$ such that $\delta P(x) = (x+1)^k$ and $P(0) = 0$. Taking $x = 0$, we have $\delta P(0) = P(1) - P(0) = (0+1)^k = 1^k$. Therefore, $P(1) = P(0) + 1^k = 1^k$. Next, take $x = 1$. Then $\delta P(1) = P(2) - P(1) = (1+1)^k = 2^k$, and $P(2) = P(1) + 2^k = 1^k + 2^k$.

To prove the general result, we will proceed by induction. The basis step, proving that $P(1) = 1^k$ is given above, so we move on to the induction step. Assume that, for some integer j, $P(j) = 1^k + 2^k + \cdots + j^k$. Then $\delta P(j) = P(j+1) - P(j) = (j+1)^k$ and

$$P(j+1) = P(j) + (j+1)^k = 1^k + 2^k + \cdots + j^k + (j+1)^k.$$

Therefore, by mathematical induction, for every whole number n, $P(n) = 1^k + 2^k + \cdots + n^k$.

41. First show that

$$P(x) = \frac{x(x+1)}{2}$$

satisfies $\delta P = (x+1)$. Then apply Exercise 40 to conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION Let $P(x) = x(x+1)/2$. Then

$$\delta P(x) = P(x+1) - P(x) = \frac{(x+1)(x+2)}{2} - \frac{x(x+1)}{2} = \frac{(x+1)(x+2-x)}{2} = x+1.$$

Also, note that $P(0) = 0$. Thus, by Exercise 40, with $k = 1$, it follows that

$$P(n) = \frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n.$$

42. Calculate $\delta(x^3)$, $\delta(x^2)$, and $\delta(x)$. Then find a polynomial $P(x)$ of degree 3 such that $\delta P = (x+1)^2$ and $P(0) = 0$. Conclude that $P(n) = 1^2 + 2^2 + \cdots + n^2$.

SOLUTION From Exercise 37, we know

$$\delta x = 1, \qquad \delta x^2 = 2x + 1, \qquad \text{and} \qquad \delta x^3 = 3x^2 + 3x + 1.$$

Therefore,

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = x^2 + 2x + 1 = (x+1)^2.$$

Now, using the properties of the first difference from Exercise 39, it follows that

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = \delta\left(\frac{1}{3}x^3\right) + \delta\left(\frac{1}{2}x^2\right) + \delta\left(\frac{1}{6}x\right) = \delta\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right) = \delta\left(\frac{2x^3 + 3x^2 + x}{6}\right).$$

Finally, let

$$P(x) = \frac{2x^3 + 3x^2 + x}{6}.$$

Then $\delta P(x) = (x+1)^2$ and $P(0) = 0$, so by Exercise 40, with $k = 2$, it follows that

$$P(n) = \frac{2n^3 + 3n^2 + n}{6} = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

43. This exercise combined with Exercise 40 shows that for all whole numbers k, there exists a polynomial $P(x)$ satisfying Eq. (1). The solution requires the Binomial Theorem and proof by induction (see Appendix C).

(a) Show that $\delta(x^{k+1}) = (k+1)\,x^k + \cdots$, where the dots indicate terms involving smaller powers of x.

(b) Show by induction that there exists a polynomial of degree $k+1$ with leading coefficient $1/(k+1)$:

$$P(x) = \frac{1}{k+1}x^{k+1} + \cdots$$

such that $\delta P = (x+1)^k$ and $P(0) = 0$.

SOLUTION

(a) By the Binomial Theorem:

$$\begin{aligned}\delta(x^{n+1}) = (x+1)^{n+1} - x^{n+1} &= \left(x^{n+1} + \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1\right) - x^{n+1} \\ &= \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1\end{aligned}$$

Thus,

$$\delta(x^{n+1}) = (n+1)\,x^n + \cdots$$

where the dots indicate terms involving smaller powers of x.

(b) For $k = 0$, note that $P(x) = x$ satisfies $\delta P = (x+1)^0 = 1$ and $P(0) = 0$.

Now suppose the polynomial

$$P(x) = \frac{1}{k}x^k + p_{k-1}x^{k-1} + \cdots + p_1 x$$

which clearly satisfies $P(0) = 0$ also satisfies $\delta P = (x+1)^{k-1}$. We try to prove the existence of

$$Q(x) = \frac{1}{k+1}x^{k+1} + q_k x^k + \cdots + q_1 x$$

such that $\delta Q = (x+1)^k$. Observe that $Q(0) = 0$.

If $\delta Q = (x+1)^k$ and $\delta P = (x+1)^{k-1}$, then

$$\delta Q = (x+1)^k = (x+1)\delta P = x\delta P(x) + \delta P$$

By the linearity of δ (Exercise 39), we find $\delta Q - \delta P = x\delta P$ or $\delta(Q-P) = x\delta P$. By definition,

$$Q - P = \frac{1}{k+1}x^{k+1} + \left(q_k - \frac{1}{k}\right)x^k + \cdots + (q_1 - p_1)x,$$

so, by the linearity of δ,

$$\delta(Q-P) = \frac{1}{k+1}\delta(x^{k+1}) + \left(q_k - \frac{1}{k}\right)\delta(x^k) + \cdots + (q_1 - p_1) = x(x+1)^{k-1} \qquad \boxed{1}$$

By part (a),

$$\begin{aligned}\delta(x^{k+1}) &= (k+1)x^k + L_{k-1,k-1}x^{k-1} + \ldots + L_{k-1,1}x + 1 \\ \delta(x^k) &= kx^{k-1} + L_{k-2,k-2}x^{k-2} + \ldots + L_{k-2,1}x + 1 \\ &\vdots \\ \delta(x^2) &= 2x + 1\end{aligned}$$

where the $L_{i,j}$ are real numbers for each i, j.

To construct Q, we have to group like powers of x on both sides of Eq. (1). This yields the system of equations

$$\begin{aligned}\frac{1}{k+1}\left((k+1)x^k\right) &= x^k \\ \frac{1}{k+1}L_{k-1,k-1}x^{k-1} + \left(q_k - \frac{1}{k}\right)kx^{k-1} &= (k-1)x^{k-1} \\ &\vdots \\ \frac{1}{k+1} + \left(q_k - \frac{1}{k}\right) + (q_{k-1} - p_{k-1}) + \cdots + (q_1 - p_1) &= 0.\end{aligned}$$

The first equation is identically true, and the second equation can be solved immediately for q_k. Substituting the value of q_k into the third equation of the system, we can then solve for q_{k-1}. We continue this process until we substitute the values of $q_k, q_{k-1}, \dots q_2$ into the last equation, and then solve for q_1.

1.4 Trigonometric Functions

Preliminary Questions

1. How is it possible for two different rotations to define the same angle?

SOLUTION Working from the same initial radius, two rotations that differ by a whole number of full revolutions will have the same ending radius; consequently, the two rotations will define the same angle even though the measures of the rotations will be different.

2. Give two different positive rotations that define the angle $\pi/4$.

SOLUTION The angle $\pi/4$ is defined by any rotation of the form $\frac{\pi}{4} + 2\pi k$ where k is an integer. Thus, two different positive rotations that define the angle $\pi/4$ are

$$\frac{\pi}{4} + 2\pi(1) = \frac{9\pi}{4} \qquad \text{and} \qquad \frac{\pi}{4} + 2\pi(5) = \frac{41\pi}{4}.$$

3. Give a negative rotation that defines the angle $\pi/3$.

SOLUTION The angle $\pi/3$ is defined by any rotation of the form $\frac{\pi}{3} + 2\pi k$ where k is an integer. Thus, a negative rotation that defines the angle $\pi/3$ is

$$\frac{\pi}{3} + 2\pi(-1) = -\frac{5\pi}{3}.$$

4. The definition of $\cos\theta$ using right triangles applies when (choose the correct answer):

(a) $0 < \theta < \dfrac{\pi}{2}$ **(b)** $0 < \theta < \pi$ **(c)** $0 < \theta < 2\pi$

SOLUTION The correct response is **(a)**: $0 < \theta < \frac{\pi}{2}$.

5. What is the unit circle definition of $\sin\theta$?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\sin\theta$ is the y-coordinate of the point P.

6. How does the periodicity of $\sin\theta$ and $\cos\theta$ follow from the unit circle definition?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\cos\theta$ and $\sin\theta$ are the x- and y-coordinates, respectively, of the point P. The angle $\theta + 2\pi$ is obtained from the angle θ by making one full revolution around the circle. The angle $\theta + 2\pi$ will therefore have the radius $\overline{OP}$ as its terminal side. Thus

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{and} \qquad \sin(\theta + 2\pi) = \sin\theta.$$

In other words, $\sin\theta$ and $\cos\theta$ are periodic functions.

Exercises

1. Find the angle between 0 and 2π equivalent to $13\pi/4$.

SOLUTION Because $13\pi/4 > 2\pi$, we repeatedly subtract 2π until we arrive at a radian measure that is between 0 and 2π. After one subtraction, we have $13\pi/4 - 2\pi = 5\pi/4$. Because $0 < 5\pi/4 < 2\pi$, $5\pi/4$ is the angle measure between 0 and 2π that is equivalent to $13\pi/4$.

2. Describe $\theta = \pi/6$ by an angle of negative radian measure.

SOLUTION If we subtract 2π from $\pi/6$, we obtain $\theta = -11\pi/6$. Thus, the angle $\theta = \pi/6$ is equivalent to the angle $\theta = -11\pi/6$.

3. Convert from radians to degrees:

(a) 1 **(b)** $\dfrac{\pi}{3}$ **(c)** $\dfrac{5}{12}$ **(d)** $-\dfrac{3\pi}{4}$

SOLUTION

(a) $1\left(\dfrac{180^\circ}{\pi}\right) = \dfrac{180^\circ}{\pi} \approx 57.3^\circ$ **(b)** $\dfrac{\pi}{3}\left(\dfrac{180^\circ}{\pi}\right) = 60^\circ$

(c) $\dfrac{5}{12}\left(\dfrac{180^\circ}{\pi}\right) = \dfrac{75^\circ}{\pi} \approx 23.87^\circ$ **(d)** $-\dfrac{3\pi}{4}\left(\dfrac{180^\circ}{\pi}\right) = -135^\circ$

4. Convert from degrees to radians:

(a) 1° **(b)** 30° **(c)** 25° **(d)** 120°

SOLUTION

(a) $1^\circ\left(\dfrac{\pi}{180^\circ}\right)=\dfrac{\pi}{180}$ **(b)** $30^\circ\left(\dfrac{\pi}{180^\circ}\right)=\dfrac{\pi}{6}$ **(c)** $25^\circ\left(\dfrac{\pi}{180^\circ}\right)=\dfrac{5\pi}{36}$ **(d)** $120^\circ\left(\dfrac{\pi}{180^\circ}\right)=\dfrac{2\pi}{3}$

5. Find the lengths of the arcs subtended by the angles θ and ϕ radians in Figure 20.

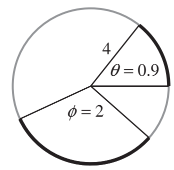


FIGURE 20 Circle of radius 4.

SOLUTION $s = r\theta = 4(.9) = 3.6$; $s = r\phi = 4(2) = 8$

6. Calculate the values of the six standard trigonometric functions for the angle θ in Figure 21.

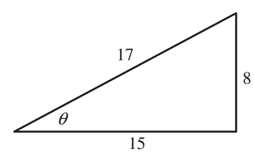


FIGURE 21

SOLUTION Using the definition of the six trigonometric functions in terms of the ratio of sides of a right triangle, we find $\sin\theta = 8/17$; $\cos\theta = 15/17$; $\tan\theta = 8/15$; $\csc\theta = 17/8$; $\sec\theta = 17/15$; $\cot\theta = 15/8$.

7. Fill in the remaining values of $(\cos\theta, \sin\theta)$ for the points in Figure 22.

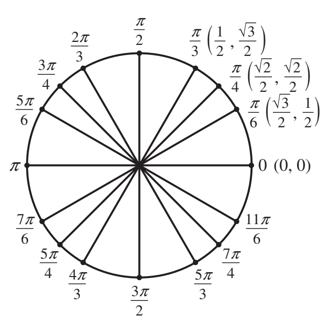


FIGURE 22

SOLUTION

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
$(\cos\theta, \sin\theta)$	$(0, 1)$	$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$	$(-1, 0)$	$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$

θ	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$(\cos\theta, \sin\theta)$	$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$	$(0, -1)$	$\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

8. Find the values of the six standard trigonometric functions at $\theta = 11\pi/6$.

SOLUTION From Figure 22, we see that

$$\sin\frac{11\pi}{6} = -\frac{1}{2} \qquad \text{and} \qquad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}.$$

Then,

$$\tan\frac{11\pi}{6} = \frac{\sin\frac{11\pi}{6}}{\cos\frac{11\pi}{6}} = -\frac{\sqrt{3}}{3};$$

$$\cot \frac{11\pi}{6} = \frac{\cos \frac{11\pi}{6}}{\sin \frac{11\pi}{6}} = -\sqrt{3};$$

$$\csc \frac{11\pi}{6} = \frac{1}{\sin \frac{11\pi}{6}} = -2;$$

$$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2\sqrt{3}}{3}.$$

In Exercises 9–14, use Figure 22 to find all angles between 0 *and* 2π *satisfying the given condition.*

9. $\cos\theta = \frac{1}{2}$

SOLUTION $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

10. $\tan\theta = 1$

SOLUTION $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

11. $\tan\theta = -1$

SOLUTION $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

12. $\csc\theta = 2$

SOLUTION $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

13. $\sin x = \frac{\sqrt{3}}{2}$

SOLUTION $x = \frac{\pi}{3}, \frac{2\pi}{3}$

14. $\sec t = 2$

SOLUTION $t = \frac{\pi}{3}, \frac{5\pi}{3}$

15. Fill in the following table of values:

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$							
$\sec\theta$							

SOLUTION

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$

16. Complete the following table of signs:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$0 < \theta < \frac{\pi}{2}$	+	+				
$\frac{\pi}{2} < \theta < \pi$						
$\pi < \theta < \frac{3\pi}{2}$						
$\frac{3\pi}{2} < \theta < 2\pi$						

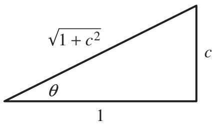
SOLUTION

θ	sin	cos	tan	cot	sec	csc
$0 < \theta < \dfrac{\pi}{2}$	$+$	$+$	$+$	$+$	$+$	$+$
$\dfrac{\pi}{2} < \theta < \pi$	$+$	$-$	$-$	$-$	$-$	$+$
$\pi < \theta < \dfrac{3\pi}{2}$	$-$	$-$	$+$	$+$	$-$	$-$
$\dfrac{3\pi}{2} < \theta < 2\pi$	$-$	$+$	$-$	$-$	$+$	$-$

17. Show that if $\tan\theta = c$ and $0 \le \theta < \pi/2$, then $\cos\theta = 1/\sqrt{1+c^2}$. *Hint:* Draw a right triangle whose opposite and adjacent sides have lengths c and 1.

SOLUTION Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\tan\theta$ is the ratio of the length of the side opposite the angle θ to the length of the adjacent side. With $c = \frac{c}{1}$, we label the length of the opposite side as c and the length of the adjacent side as 1 (see the diagram below). By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{1+c^2}$. Finally, we use the fact that $\cos\theta$ is the ratio of the length of the adjacent side to the length of the hypotenuse to obtain

$$\cos\theta = \frac{1}{\sqrt{1+c^2}}.$$



18. Suppose that $\cos\theta = \frac{1}{3}$.

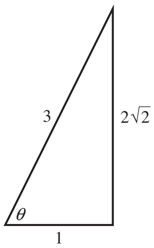
(a) Show that if $0 \le \theta < \pi/2$, then $\sin\theta = 2\sqrt{2}/3$ and $\tan\theta = 2\sqrt{2}$.

(b) Find $\sin\theta$ and $\tan\theta$ if $3\pi/2 \le \theta < 2\pi$.

SOLUTION

(a) Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\cos\theta$ is the ratio of the length of the side adjacent to the angle θ to the length of the hypotenuse, so we label the length of the adjacent side as 1 and the length of the hypotenuse as 3 (see the diagram below). By the Pythagorean theorem, the length of the side opposite the angle θ is $\sqrt{3^2 - 1^2} = 2\sqrt{2}$. Finally, we use the definitions of $\sin\theta$ as the ratio of the length of the opposite side to the length of the hypotenuse and of $\tan\theta$ as the ratio of the length of the opposite side to the length of the adjacent side to obtain

$$\sin\theta = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = \frac{2\sqrt{2}}{1} = 2\sqrt{2}.$$



(b) If $3\pi/2 \le \theta < 2\pi$, then θ is in the fourth quadrant and $\sin\theta$ and $\tan\theta$ are negative but have the same magnitude as found in part (a). Thus,

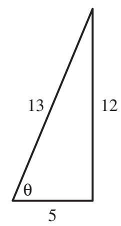
$$\sin\theta = -\frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = -2\sqrt{2}.$$

In Exercises 19–24, assume that $0 \leq \theta < \pi/2$.

19. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{5}{13}$.

SOLUTION Consider the triangle below. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\cos\theta = \frac{5}{13}$. The length of the side opposite the angle θ has been calculated using the Pythagorean theorem: $\sqrt{13^2 - 5^2} = 12$. From the triangle, we see that

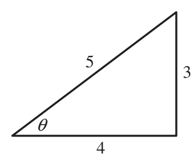
$$\sin\theta = \frac{12}{13} \quad \text{and} \quad \tan\theta = \frac{12}{5}.$$



20. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = \frac{3}{5}$.

SOLUTION Consider the triangle below. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = \frac{3}{5}$. The length of the side adjacent to the angle θ has been calculated using the Pythagorean theorem: $\sqrt{5^2 - 3^2} = 4$. From the triangle, we see that

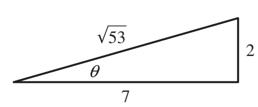
$$\cos\theta = \frac{4}{5} \quad \text{and} \quad \tan\theta = \frac{3}{4}.$$



21. Find $\sin\theta$, $\sec\theta$, and $\cot\theta$ if $\tan\theta = \frac{2}{7}$.

SOLUTION If $\tan\theta = \frac{2}{7}$, then $\cot\theta = \frac{7}{2}$. For the remaining trigonometric functions, consider the triangle below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \frac{2}{7}$. The length of the hypotenuse has been calculated using the Pythagorean theorem: $\sqrt{2^2 + 7^2} = \sqrt{53}$. From the triangle, we see that

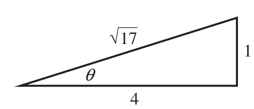
$$\sin\theta = \frac{2}{\sqrt{53}} = \frac{2\sqrt{53}}{53} \quad \text{and} \quad \sec\theta = \frac{\sqrt{53}}{7}.$$



22. Find $\sin\theta$, $\cos\theta$, and $\sec\theta$ if $\cot\theta = 4$.

SOLUTION Consider the triangle below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = 4 = \frac{4}{1}$. The length of the hypotenuse has been calculated using the Pythagorean theorem: $\sqrt{4^2 + 1^2} = \sqrt{17}$. From the triangle, we see that

$$\sin\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}, \quad \cos\theta = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \quad \text{and} \quad \sec\theta = \frac{\sqrt{17}}{4}.$$



23. Find $\cos 2\theta$ if $\sin\theta = \frac{1}{5}$.

SOLUTION Using the double angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we find that $\cos 2\theta = 1 - 2\sin^2\theta$. Thus, $\cos 2\theta = 1 - 2(1/25) = 23/25$.

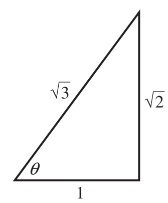
24. Find $\sin 2\theta$ and $\cos 2\theta$ if $\tan\theta = \sqrt{2}$.

SOLUTION By the double angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. We can determine $\sin\theta$ and $\cos\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \sqrt{2}$. The hypotenuse was calculated using the Pythagorean theorem: $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$. Thus,

$$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \qquad \text{and} \qquad \cos\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Finally,

$$\sin 2\theta = 2\frac{\sqrt{6}}{3}\cdot\frac{\sqrt{3}}{3} = \frac{2\sqrt{2}}{3}$$
$$\cos 2\theta = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}.$$



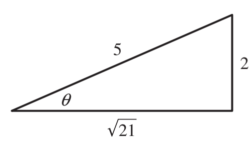
25. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = 0.4$ and $\pi/2 \le \theta < \pi$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\tan\theta$ using the triangle shown below. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = 0.4 = \frac{2}{5}$. The length of the side adjacent to the angle θ was calculated using the Pythagorean theorem: $\sqrt{5^2 - 2^2} = \sqrt{21}$. From the triangle, we see that

$$|\cos\theta| = \frac{\sqrt{21}}{5} \qquad \text{and} \qquad |\tan\theta| = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}.$$

Because $\pi/2 \le \theta < \pi$, both $\cos\theta$ and $\tan\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{21}}{5} \qquad \text{and} \qquad \tan\theta = -\frac{2\sqrt{21}}{21}.$$



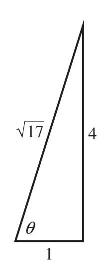
26. Find $\cos\theta$ and $\sin\theta$ if $\tan\theta = 4$ and $\pi \le \theta < 3\pi/2$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\sin\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = 4 = \frac{4}{1}$. The length of the hypotenuse was calculated using the Pythagorean theorem: $\sqrt{1^2 + 4^2} = \sqrt{17}$. From the triangle, we see that

$$|\cos\theta| = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17} \qquad \text{and} \qquad |\sin\theta| = \frac{4}{\sqrt{14}} = \frac{4\sqrt{17}}{17}.$$

Because $\pi \le \theta < 3\pi/2$, both $\cos\theta$ and $\sin\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{17}}{17} \qquad \text{and} \qquad \sin\theta = -\frac{4\sqrt{17}}{17}.$$



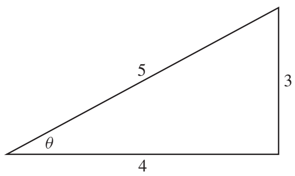
27. Find $\cos\theta$ if $\cot\theta = \frac{4}{3}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = \frac{4}{3}$. The length of the hypotenuse was calculated using the Pythagorean theorem: $\sqrt{3^2 + 4^2} = 5$. From the triangle, we see that

$$|\cos\theta| = \frac{4}{5}.$$

Because $\cot\theta = \frac{4}{3} > 0$ and $\sin\theta < 0$, the angle θ must be in the third quadrant; consequently, $\cos\theta$ will be negative and

$$\cos\theta = -\frac{4}{5}.$$



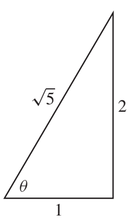
28. Find $\tan\theta$ if $\sec\theta = \sqrt{5}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\tan\theta$ using the triangle shown below. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\sec\theta = \sqrt{5}$. The length of the side opposite the angle θ was calculated using the Pythagorean theorem: $\sqrt{(\sqrt{5})^2 - 1^2} = 2$. From the triangle, we see that

$$|\tan\theta| = 2.$$

Because $\sec\theta = \sqrt{5} > 0$ and $\sin\theta < 0$, the angle θ must be in the fourth quadrant; consequently, $\tan\theta$ will be negative and

$$\tan\theta = -2.$$



29. Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the angles corresponding to the eight points in Figure 23(A) and (B).

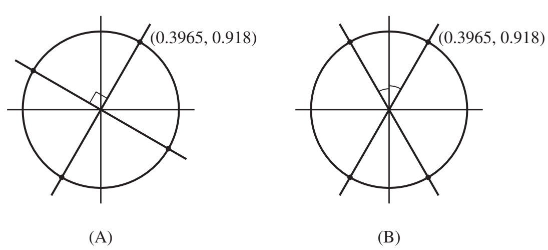


FIGURE 23

SOLUTION Let's start with the four points in Figure 23(A).

- The point in the first quadrant has coordinates $(0.3965, 0.918)$. Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153.$$

- The coordinates of the point in the second quadrant are $(-0.918, 0.3965)$. Therefore,

$$\sin\theta = 0.3965, \quad \cos\theta = -0.918, \quad \text{and} \quad \tan\theta = \frac{0.3965}{-0.918} = -0.4319.$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are $(-0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153.$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are $(0.918, -0.3965)$. Therefore,

$$\sin\theta = -0.3965, \quad \cos\theta = 0.918, \quad \text{and} \quad \tan\theta = \frac{-0.3965}{0.918} = -0.4319.$$

Now consider the four points in Figure 23(B).

- The point in the first quadrant has coordinates $(0.3965, 0.918)$. Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153.$$

- The point in the second quadrant is a reflection through the y-axis of the point in the first quadrant. Its coordinates are therefore $(-0.3965, 0.918)$ and

$$\sin\theta = 0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = -2.3153.$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are $(-0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153.$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are $(0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{0.3965} = -2.3153.$$

30. Refer to Figure 24(A). Express the functions $\sin\theta$, $\tan\theta$, and $\csc\theta$ in terms of c.

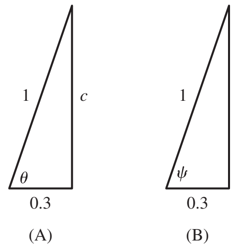


FIGURE 24

SOLUTION By the Pythagorean theorem, the length of the side adjacent to the angle θ in Figure 24(A) is $\sqrt{1-c^2}$. Consequently,

$$\sin\theta = \frac{c}{1} = c, \quad \cos\theta = \frac{\sqrt{1-c^2}}{1} = \sqrt{1-c^2}, \quad \text{and} \quad \tan\theta = \frac{c}{\sqrt{1-c^2}}.$$

31. Refer to Figure 24(B). Compute $\cos\psi$, $\sin\psi$, $\cot\psi$, and $\csc\psi$.

SOLUTION By the Pythagorean theorem, the length of the side opposite the angle ψ in Figure 24(B) is $\sqrt{1-0.3^2} = \sqrt{0.91}$. Consequently,

$$\cos\psi = \frac{0.3}{1} = 0.3, \quad \sin\psi = \frac{\sqrt{0.91}}{1} = \sqrt{0.91}, \quad \cot\psi = \frac{0.3}{\sqrt{0.91}} \qquad \text{and} \qquad \csc\psi = \frac{1}{\sqrt{0.91}}.$$

32. Express $\cos\left(\theta + \frac{\pi}{2}\right)$ and $\sin\left(\theta + \frac{\pi}{2}\right)$ in terms of $\cos\theta$ and $\sin\theta$. *Hint:* Find the relation between the coordinates (a, b) and (c, d) in Figure 25.

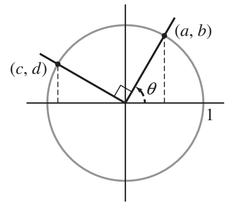


FIGURE 25

SOLUTION Note the triangle in the second quadrant in Figure 25 is congruent to the triangle in the first quadrant rotated 90° clockwise. Thus, $c = -b$ and $d = a$. But $a = \cos\theta$, $b = \sin\theta$, $c = \cos\left(\theta + \frac{\pi}{2}\right)$ and $d = \sin\left(\theta + \frac{\pi}{2}\right)$; therefore,

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \qquad \text{and} \qquad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta.$$

33. Use the addition formula to compute $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ exactly.

SOLUTION

$$\begin{aligned}\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}.\end{aligned}$$

34. Use the addition formula to compute $\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ exactly.

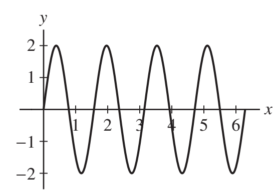
SOLUTION

$$\begin{aligned}\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

In Exercises 35–38, sketch the graph over $[0, 2\pi]$.

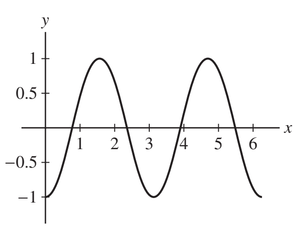
35. $2\sin 4\theta$

SOLUTION



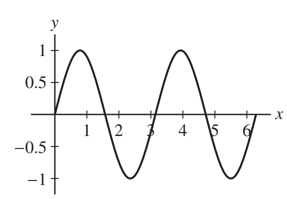
36. $\cos\left(2\left(\theta - \frac{\pi}{2}\right)\right)$

SOLUTION



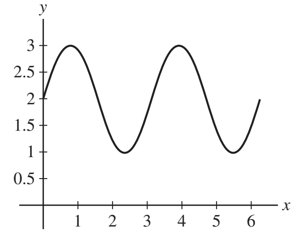
37. $\cos\left(2\theta - \frac{\pi}{2}\right)$

SOLUTION



38. $\sin\left(2\left(\theta - \frac{\pi}{2}\right) + \pi\right) + 2$

SOLUTION



39. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \sin x$ for $0 \le x < 2\pi$? *Hint:* The answer depends on c.

SOLUTION Recall that for any x, $-1 \le \sin x \le 1$. Thus, if $|c| > 1$, the horizontal line $y = c$ and the graph of $y = \sin x$ never intersect. If $c = +1$, then $y = c$ and $y = \sin x$ intersect at the peak of the sine curve; that is, they intersect at $x = \frac{\pi}{2}$. On the other hand, if $c = -1$, then $y = c$ and $y = \sin x$ intersect at the bottom of the sine curve; that is, they intersect at $x = \frac{3\pi}{2}$. Finally, if $|c| < 1$, the graphs of $y = c$ and $y = \sin x$ intersect twice.

40. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \tan x$ for $0 \le x < 2\pi$?

SOLUTION Recall that the graph of $y = \tan x$ consists of an infinite collection of "branches," each between two consecutive vertical asymptotes. Because each branch is increasing and has a range of all real numbers, the graph of the horizontal line $y = c$ will intersect each branch of the graph of $y = \tan x$ once, regardless of the value of c. The interval $0 \le x < 2\pi$ covers the equivalent of two branches of the tangent function, so over this interval there are two points of intersection for each value of c.

In Exercises 41–44, solve for $0 \le \theta < 2\pi$ *(see Example 4).*

41. $\sin 2\theta + \sin 3\theta = 0$

SOLUTION $\sin\alpha = -\sin\beta$ when $\alpha = -\beta + 2\pi k$ or $\alpha = \pi + \beta + 2\pi k$. Substituting $\alpha = 2\theta$ and $\beta = 3\theta$, we have either $2\theta = -3\theta + 2\pi k$ or $2\theta = \pi + 3\theta + 2\pi k$. Solving each of these equations for θ yields $\theta = \frac{2}{5}\pi k$ or $\theta = -\pi - 2\pi k$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{8\pi}{5}.$$

42. $\sin\theta = \sin 2\theta$

SOLUTION Using the double angle formula for the sine function, we rewrite the equation as $\sin\theta = 2\sin\theta\cos\theta$ or $\sin\theta(1 - 2\cos\theta) = 0$. Thus, either $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}.$$

43. $\cos 4\theta + \cos 2\theta = 0$

SOLUTION $\cos\alpha = -\cos\beta$ when $\alpha + \beta = \pi + 2\pi k$ or $\alpha = \beta + \pi + 2\pi k$. Substituting $\alpha = 4\theta$ and $\beta = 2\theta$, we have either $6\theta = \pi + 2\pi k$ or $4\theta = 2\theta + \pi + 2\pi k$. Solving each of these equations for θ yields $\theta = \frac{\pi}{6} + \frac{\pi}{3}k$ or $\theta = \frac{\pi}{2} + \pi k$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

44. $\sin\theta = \cos 2\theta$

SOLUTION Solving the double angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ for $\cos 2\theta$ yields $\cos 2\theta = 1 - 2\sin^2\theta$. We can therefore rewrite the original equation as $\sin\theta = 1 - 2\sin^2\theta$ or $2\sin^2\theta + \sin\theta - 1 = 0$. The left-hand side of this latter equation factors as $(2\sin\theta - 1)(\sin\theta + 1)$, so we have either $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$. The solutions on the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

In Exercises 45–54, derive the identity using the identities listed in this section.

45. $\cos 2\theta = 2\cos^2\theta - 1$

SOLUTION Starting from the double angle formula for cosine, $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we solve for $\cos 2\theta$. This gives $2\cos^2\theta = 1 + \cos 2\theta$ and then $\cos 2\theta = 2\cos^2\theta - 1$.

46. $\cos^2 \dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$

SOLUTION Substitute $x = \theta/2$ into the double angle formula for cosine, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to obtain $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos\theta}{2}$.

47. $\sin \dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{2}}$

SOLUTION Substitute $x = \theta/2$ into the double angle formula for sine, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ to obtain $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{2}$. Taking the square root of both sides yields $\sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{2}}$.

48. $\sin(\theta+\pi) = -\sin\theta$

SOLUTION From the addition formula for the sine function, we have

$$\sin(\theta+\pi) = \sin\theta\cos\pi + \cos\theta\sin\pi = -\sin\theta$$

49. $\cos(\theta+\pi) = -\cos\theta$

SOLUTION From the addition formula for the cosine function, we have

$$\cos(\theta+\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = \cos\theta(-1) = -\cos\theta$$

50. $\tan x = \cot\left(\dfrac{\pi}{2} - x\right)$

SOLUTION Using the Complementary Angle Identity,

$$\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)} = \frac{\sin x}{\cos x} = \tan x.$$

51. $\tan(\pi - \theta) = -\tan\theta$

SOLUTION Using Exercises 48 and 49,

$$\tan(\pi-\theta) = \frac{\sin(\pi-\theta)}{\cos(\pi-\theta)} = \frac{\sin(\pi+(-\theta))}{\cos(\pi+(-\theta))} = \frac{-\sin(-\theta)}{-\cos(-\theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta.$$

The second to last equality occurs because $\sin x$ is an odd function and $\cos x$ is an even function.

52. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

SOLUTION Using the definition of the tangent function and the double angle formulas for sine and cosine, we find

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}\cdot\frac{1/\cos^2 x}{1/\cos^2 x} = \frac{2\tan x}{1-\tan^2 x}.$$

53. $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$

SOLUTION Using the addition formula for the sine function, we find

$$\sin 2x = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x.$$

By Exercise 45, we know that $\cos 2x = 2\cos^2 x - 1$. Therefore,

$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{1+2\cos^2 x - 1} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin x}{\cos x} = \tan x.$$

54. $\sin^2 x\cos^2 x = \dfrac{1-\cos 4x}{8}$

SOLUTION Using the double angle formulas for sine and cosine, we find

$$\begin{aligned}\sin^2 x\cos^2 x &= \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x) = \frac{1}{4}(1-\cos^2 2x)\\ &= \frac{1}{4}\left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) = \frac{1}{8}(1-\cos 4x).\end{aligned}$$

55. Use Exercises 48 and 49 to show that $\tan\theta$ and $\cot\theta$ are periodic with period π.

SOLUTION By Exercises 48 and 49,

$$\tan(\theta+\pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta,$$

and

$$\cot(\theta+\pi) = \frac{\cos(\theta+\pi)}{\sin(\theta+\pi)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta.$$

Thus, both $\tan\theta$ and $\cot\theta$ are periodic with period π.

56. Use the identity of Exercise 45 to show that $\cos\frac{\pi}{8}$ is equal to $\sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}$.

SOLUTION Upon substituting $\theta = \frac{\pi}{8}$ into the identity

$$\cos 2\theta = 2\cos^2\theta - 1$$

we have

$$\frac{\sqrt{2}}{2} = \cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1.$$

Thus,

$$2\cos^2\frac{\pi}{8} = 1 + \frac{\sqrt{2}}{2} \quad \text{or} \quad \cos^2\frac{\pi}{8} = \frac{1}{2} + \frac{\sqrt{2}}{4}.$$

Taking the square root of both sides of this last expression and recognizing that $\cos\frac{\pi}{8} > 0$ because $0 < \frac{\pi}{8} < \frac{\pi}{2}$, it follows that

$$\cos\frac{\pi}{8} = \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}.$$

57. Use the Law of Cosines to find the distance from P to Q in Figure 26.

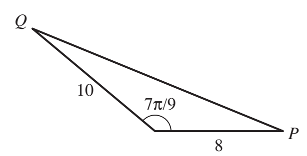


FIGURE 26

SOLUTION By the Law of Cosines, the distance from P to Q is

$$\sqrt{10^2 + 8^2 - 2(10)(8)\cos\frac{7\pi}{9}} = 16.928.$$

Further Insights and Challenges

58. Use Figure 27 to derive the Law of Cosines from the Pythagorean Theorem.

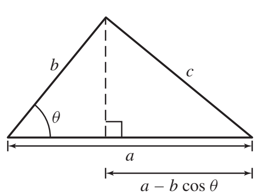


FIGURE 27

SOLUTION First note that the length of the altitude in Figure 27 is $b\sin\theta$. Applying the Pythagorean Theorem to the right triangle on the right in the figure, it then follows that

$$\begin{aligned} c^2 &= (a - b\cos\theta)^2 + b^2\sin^2\theta \\ &= a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

59. Use the addition formula to prove

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

SOLUTION

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= \cos\theta(2\cos^2\theta - 1 - 2\sin^2\theta) = \cos\theta(2\cos^2\theta - 1 - 2(1-\cos^2\theta)) \\ &= \cos\theta(2\cos^2\theta - 1 - 2 + 2\cos^2\theta) = 4\cos^3\theta - 3\cos\theta\end{aligned}$$

60. Use the addition formulas for sine and cosine to prove

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\cot(a-b) = \frac{\cot a\cot b + 1}{\cot b - \cot a}$$

SOLUTION

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a\cos b + \cos a\sin b}{\cos a\cos b - \sin a\sin b} = \frac{\frac{\sin a\cos b}{\cos a\cos b} + \frac{\cos a\sin b}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b} - \frac{\sin a\sin b}{\cos a\cos b}} = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\cot(a-b) = \frac{\cos(a-b)}{\sin(a-b)} = \frac{\cos a\cos b + \sin a\sin b}{\sin a\cos b - \cos a\sin b} = \frac{\frac{\cos a\cos b}{\sin a\sin b} + \frac{\sin a\sin b}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b} - \frac{\cos a\sin b}{\sin a\sin b}} = \frac{\cot a\cot b + 1}{\cot b - \cot a}$$

61. Let θ be the angle between the line $y = mx + b$ and the x-axis [Figure 28(A)]. Prove that $m = \tan\theta$.

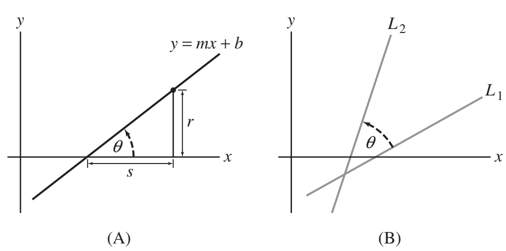


FIGURE 28

SOLUTION Using the distances labeled in Figure 28(A), we see that the slope of the line is given by the ratio r/s. The tangent of the angle θ is given by the same ratio. Therefore, $m = \tan\theta$.

62. Let L_1 and L_2 be the lines of slope m_1 and m_2 [Figure 28(B)]. Show that the angle θ between L_1 and L_2 satisfies $\cot\theta = \dfrac{m_2 m_1 + 1}{m_2 - m_1}$.

SOLUTION Measured from the positive x-axis, let α and β satisfy $\tan\alpha = m_1$ and $\tan\beta = m_2$. Without loss of generality, let $\beta \ge \alpha$. Then the angle between the two lines will be $\theta = \beta - \alpha$. Then from Exercise 60,

$$\cot\theta = \cot(\beta - \alpha) = \frac{\cot\beta\cot\alpha + 1}{\cot\alpha - \cot\beta} = \frac{(\frac{1}{m_1})(\frac{1}{m_2}) + 1}{\frac{1}{m_1} - \frac{1}{m_2}} = \frac{1 + m_1 m_2}{m_2 - m_1}$$

63. Perpendicular Lines Use Exercise 62 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

SOLUTION If lines are perpendicular, then the angle between them is $\theta = \pi/2 \Rightarrow$

$$\cot(\pi/2) = \frac{1 + m_1 m_2}{m_1 - m_2}$$

$$0 = \frac{1 + m_1 m_2}{m_1 - m_2}$$

$$\Rightarrow m_1 m_2 = -1 \Rightarrow m_1 = -\frac{1}{m_2}$$

64. Apply the double-angle formula to prove:

(a) $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

(b) $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

Guess the values of $\cos\frac{\pi}{32}$ and of $\cos\frac{\pi}{2^n}$ for all n.

SOLUTION

(a) $\cos\frac{\pi}{8} = \cos\frac{\pi/4}{2} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}.$

(b) $\cos\frac{\pi}{16} = \sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}.$

(c) Observe that $8 = 2^3$ and $\cos\frac{\pi}{8}$ involves two nested square roots of 2; further, $16 = 2^4$ and $\cos\frac{\pi}{16}$ involves three nested square roots of 2. Since $32 = 2^5$, it seems plausible that

$$\cos\frac{\pi}{32} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},$$

and that $\cos\frac{\pi}{2^n}$ involves $n-1$ nested square roots of 2. Note that the general case can be proven by induction.

1.5 Inverse Functions

Preliminary Questions

1. Which of the following satisfy $f^{-1}(x) = f(x)$?

(a) $f(x) = x$

(b) $f(x) = 1 - x$

(c) $f(x) = 1$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = |x|$

(f) $f(x) = x^{-1}$

SOLUTION The functions **(a)** $f(x) = x$, **(b)** $f(x) = 1 - x$ and **(f)** $f(x) = x^{-1}$ satisfy $f^{-1}(x) = f(x)$.

2. The graph of a function looks like the track of a roller coaster. Is the function one-to-one?

SOLUTION Because the graph looks like the track of a roller coaster, there will be several locations at which the graph has the same height. The graph will therefore fail the horizontal line test, meaning that the function is *not* one-to-one.

3. The function f maps teenagers in the United States to their last names. Explain why the inverse function f^{-1} does not exist.

SOLUTION Many different teenagers will have the same last name, so this function will not be one-to-one. Consequently, the function does not have an inverse.

4. The following fragment of a train schedule for the New Jersey Transit System defines a function f from towns to times. Is f one-to-one? What is $f^{-1}(6{:}27)$?

Trenton	6:21
Hamilton Township	6:27
Princeton Junction	6:34
New Brunswick	6:38

SOLUTION This function is one-to-one, and $f^{-1}(6{:}27) =$ Hamilton Township.

5. A homework problem asks for a sketch of the graph of the *inverse* of $f(x) = x + \cos x$. Frank, after trying but failing to find a formula for $f^{-1}(x)$, says it's impossible to graph the inverse. Bianca hands in an accurate sketch without solving for f^{-1}. How did Bianca complete the problem?

SOLUTION The graph of the inverse function is the reflection of the graph of $y = f(x)$ through the line $y = x$.

6. Which of the following quantities is undefined?

(a) $\sin^{-1}\left(-\frac{1}{2}\right)$

(b) $\cos^{-1}(2)$

(c) $\csc^{-1}\left(\frac{1}{2}\right)$

(d) $\csc^{-1}(2)$

SOLUTION **(b)** and **(c)** are undefined. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ and $\csc^{-1}(2) = \frac{\pi}{6}$.

7. Give an example of an angle θ such that $\cos^{-1}(\cos\theta) \neq \theta$. Does this contradict the definition of inverse function?

SOLUTION Any angle $\theta < 0$ or $\theta > \pi$ will work. No, this does not contradict the definition of inverse function.

Exercises

1. Show that $f(x) = 7x - 4$ is invertible and find its inverse.

SOLUTION Solving $y = 7x - 4$ for x yields $x = \frac{y+4}{7}$. Thus, $f^{-1}(x) = \frac{x+4}{7}$.

2. Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.

SOLUTION f is not one-to-one because $f(-1) = f(1) = 3$. However, if the domain is restricted to $x \geq 0$ or $x \leq 0$, then f is one-to-one.

3. What is the largest interval containing zero on which $f(x) = \sin x$ is one-to-one?

SOLUTION Looking at the graph of $\sin x$, the function is one-to-one on the interval $[-\pi/2, \pi/2]$.

4. Show that $f(x) = \frac{x-2}{x+3}$ is invertible and find its inverse.

(a) What is the domain of $f(x)$? The range of $f^{-1}(x)$?
(b) What is the domain of $f^{-1}(x)$? The range of $f(x)$?

SOLUTION We solve $y = f(x)$ for x as follows:

$$\begin{aligned} y &= \frac{x-2}{x+3} \\ yx + 3y &= x - 2 \\ yx - x &= -3y - 2 \\ x &= \frac{-3y-2}{y-1} = \frac{3y+2}{1-y}. \end{aligned}$$

Therefore,

$$f^{-1}(x) = \frac{3x+2}{1-x}.$$

(a) Domain of $f(x) = \{x | x \neq -3\} =$ Range of $f^{-1}(x)$.
(b) Domain of $f^{-1}(x) = \{x | x \neq 1\} =$ Range of $f(x)$.

5. Verify that $f(x) = x^3 + 3$ and $g(x) = (x-3)^{1/3}$ are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

SOLUTION

- $f(g(x)) = \left((x-3)^{1/3}\right)^3 + 3 = x - 3 + 3 = x.$
- $g(f(x)) = \left(x^3 + 3 - 3\right)^{1/3} = \left(x^3\right)^{1/3} = x.$

6. Repeat Exercise 5 for $f(t) = \frac{t+1}{t-1}$ and $g(t) = \frac{t+1}{t-1}$.

SOLUTION

$$f(g(t)) = \frac{\frac{t+1}{t-1} + 1}{\frac{t+1}{t-1} - 1} = \frac{t+1+t-1}{t+1-(t-1)} = t.$$

The calculations for $g(f(t))$ are identical.

7. The escape velocity from a planet of radius R is $v(R) = \sqrt{\frac{2GM}{R}}$, where G is the universal gravitational constant and M is the mass. Find the inverse of $v(R)$ expressing R in terms of v.

SOLUTION To find the inverse, we solve

$$y = \sqrt{\frac{2GM}{R}}$$

for R. This yields

$$R = \frac{2GM}{y^2}.$$

Therefore,

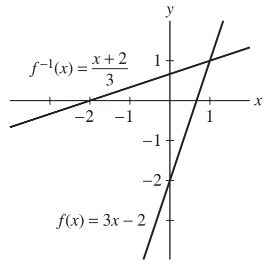
$$v^{-1}(R) = \frac{2GM}{R^2}.$$

In Exercises 8–15, find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain. Sketch the graphs of f and f^{-1}.

8. $f(x) = 3x - 2$

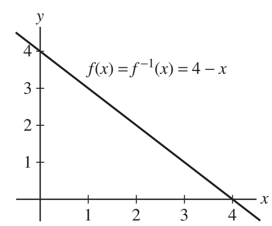
SOLUTION The linear function $f(x) = 3x - 2$ is one-to-one for all real numbers. Solving $y = 3x - 2$ for x gives $x = (y + 2)/3$. Thus,

$$f^{-1}(x) = \frac{x + 2}{3}.$$



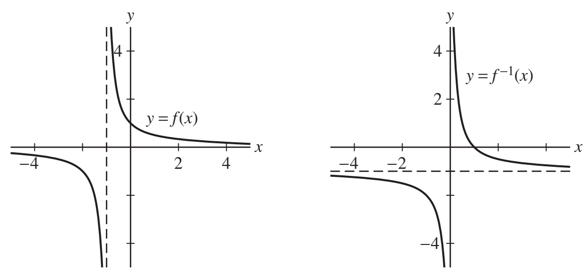
9. $f(x) = 4 - x$

SOLUTION The linear function $f(x) = 4 - x$ is one-to-one for all real numbers. Solving $y = x - 4$ for x gives $x = 4 - y$. Thus, $f^{-1}(x) = 4 - x$.



10. $f(x) = \dfrac{1}{x + 1}$

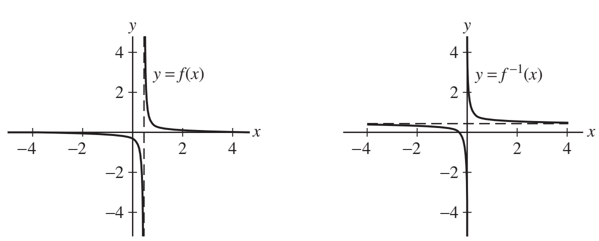
SOLUTION The graph of $f(x) = 1/(x + 1)$ given below shows that f passes the horizontal line test, and is therefore one-to-one, on its entire domain $\{x : x \neq -1\}$. Solving $y = \dfrac{1}{x+1}$ for x gives $x = \dfrac{1}{y} - 1$. Thus, $f^{-1}(x) = \dfrac{1}{x} - 1$.



11. $f(x) = \dfrac{1}{7x - 3}$

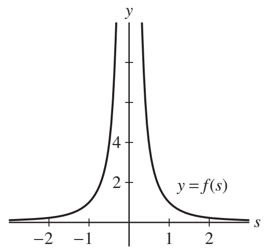
SOLUTION The graph of $f(x) = 1/(7x - 3)$ given below shows that f passes the horizontal line test, and is therefore one-to-one, on its entire domain $\{x : x \neq \frac{3}{7}\}$. Solving $y = 1/(7x - 3)$ for x gives

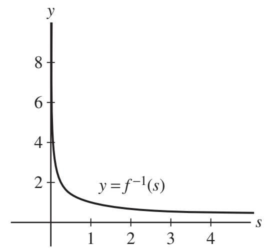
$$x = \frac{1}{7y} + \frac{3}{7}; \quad \text{thus,} \quad f^{-1}(x) = \frac{1}{7x} + \frac{3}{7}.$$



12. $f(s) = \dfrac{1}{s^2}$

SOLUTION To make $f(s) = s^{-2}$ one-to-one, we must restrict the domain to either $\{s : s > 0\}$ or $\{s : s < 0\}$. If we choose the domain $\{s : s > 0\}$, then solving $y = \dfrac{1}{s^2}$ for s yields $s = \dfrac{1}{\sqrt{y}}$. Hence, $f^{-1}(s) = \dfrac{1}{\sqrt{s}}$. Had we chosen the domain $\{s : s < 0\}$, the inverse would have been $f^{-1}(s) = -\dfrac{1}{\sqrt{s}}$.





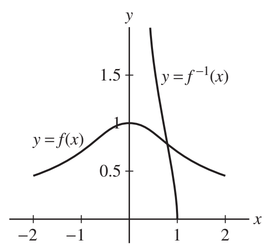
13. $f(x) = \dfrac{1}{\sqrt{x^2+1}}$

SOLUTION To make the function $f(x) = \dfrac{1}{\sqrt{x^2+1}}$ one-to-one, we must restrict the domain to either $\{x : x \geq 0\}$ or $\{x : x \leq 0\}$. If we choose the domain $\{x : x \geq 0\}$, then solving $y = \dfrac{1}{\sqrt{x^2+1}}$ for x yields

$$x = \frac{\sqrt{1-y^2}}{y}; \quad \text{hence,} \quad f^{-1}(x) = \frac{\sqrt{1-x^2}}{x}.$$

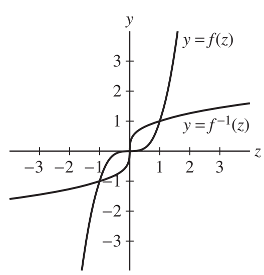
Had we chosen the domain $\{x : x \leq 0\}$, the inverse would have been

$$f^{-1}(x) = -\frac{\sqrt{1-x^2}}{x}.$$



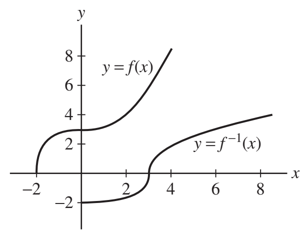
14. $f(z) = z^3$

SOLUTION The function $f(z) = z^3$ is one-to-one over its entire domain (see the graph below). Solving $y = z^3$ for z yields $y^{1/3} = z$. Thus, $f^{-1}(z) = z^{1/3}$.



15. $f(x) = \sqrt{x^3+9}$

SOLUTION The graph of $f(x) = \sqrt{x^3+9}$ given below shows that f passes the horizontal line test, and therefore is one-to-one, on its entire domain $\{x : x \geq -9^{1/3}\}$. Solving $y = \sqrt{x^3+9}$ for x yields $x = (y^2-9)^{1/3}$. Thus, $f^{-1}(x) = (x^2-9)^{1/3}$.



16. For each function shown in Figure 19, sketch the graph of the inverse (restrict the function's domain if necessary).

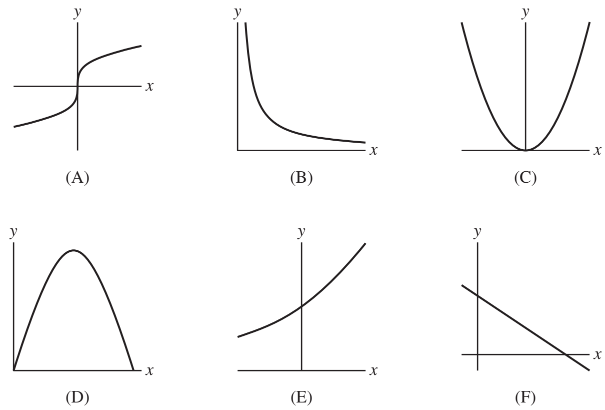
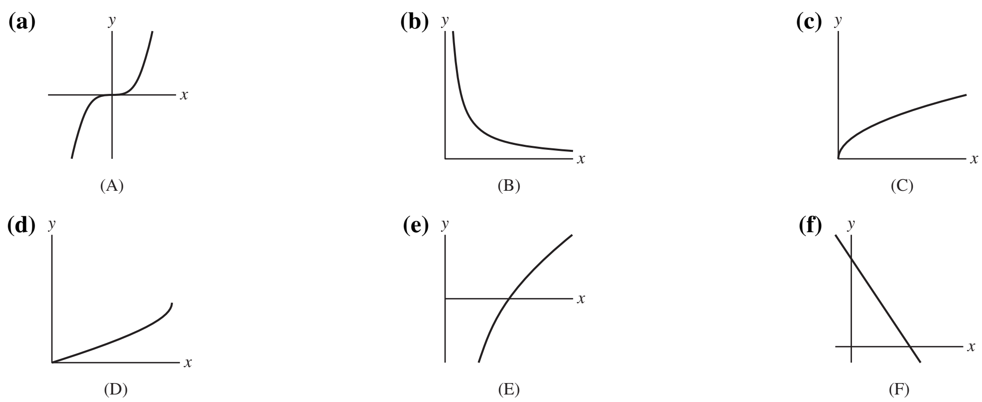


FIGURE 19

SOLUTION Here, we apply the rule that the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$. For (C) and (D), we must restrict the domain of f to make f one-to-one.



17. Which of the graphs in Figure 20 is the graph of a function satisfying $f^{-1} = f$?

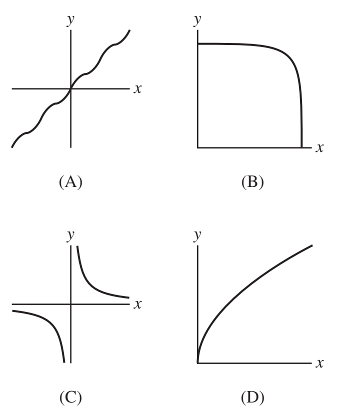


FIGURE 20

SOLUTION Figures (B) and (C) would not change when reflected around the line $y = x$. Therefore, these two satisfy $f^{-1} = f$.

18. Let n be a nonzero integer. Find a domain on which $f(x) = (1 - x^n)^{1/n}$ coincides with its inverse. *Hint:* The answer depends on whether n is even or odd.

SOLUTION First note

$$f(f(x)) = \left(1 - \left((1 - x^n)^{1/n}\right)^n\right)^{1/n} = \left(1 - (1 - x^n)\right)^{1/n} = (x^n)^{1/n} = x,$$

so $f(x)$ coincides with its inverse. For the domain and range of f, let's first consider the case when $n > 0$. If n is even, then $f(x)$ is defined only when $1 - x^n \geq 0$. Hence, the domain is $-1 \leq x \leq 1$. The range is $0 \leq y \leq 1$. If n is odd, then $f(x)$ is defined for all real numbers, and the range is also all real numbers. Now, suppose $n < 0$. Then $-n > 0$, and

$$f(x) = \left(1 - \frac{1}{x^{-n}}\right)^{-1/-n} = \left(\frac{x^{-n}}{x^{-n} - 1}\right)^{1/-n}.$$

If n is even, then $f(x)$ is defined only when $x^{-n} - 1 > 0$. Hence, the domain is $|x| > 1$. The range is $y > 1$. If n is odd, then $f(x)$ is defined for all real numbers except $x = 1$. The range is all real numbers except $y = 1$.

19. Let $f(x) = x^7 + x + 1$.

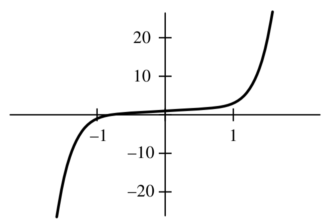
(a) Show that f^{-1} exists (but do not attempt to find it). *Hint:* Show that f is increasing.

(b) What is the domain of f^{-1}?

(c) Find $f^{-1}(3)$.

SOLUTION

(a) The graph of $f(x) = x^7 + x + 1$ is shown below. From this graph, we see that $f(x)$ is a strictly increasing function; by Example 3, it is therefore one-to-one. Because f is one-to-one, by Theorem 3, f^{-1} exists.

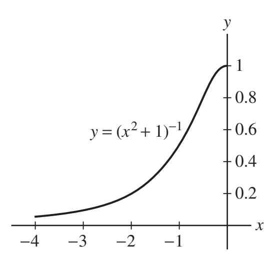


(b) The domain of $f^{-1}(x)$ is the range of $f(x) : (-\infty, \infty)$.

(c) Note that $f(1) = 1^7 + 1 + 1 = 3$; therefore, $f^{-1}(3) = 1$.

20. Show that $f(x) = (x^2 + 1)^{-1}$ is one-to-one on $(-\infty, 0]$, and find a formula for f^{-1} for this domain of f.

SOLUTION



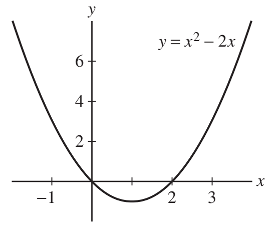
Notice that the graph of $f(x) = (x^2 + 1)^{-1}$ over the interval $(-\infty, 0]$ (shown above) passes the horizontal line test. Thus, $f(x)$ is one-to-one on $(-\infty, 0]$. To find a formula for f^{-1}, we solve $y = (x^2 + 1)^{-1}$ for x, which yields $x = \pm\sqrt{\frac{1}{y} - 1}$. Because the domain of f was restricted to $x \leq 0$, we choose the negative sign in front of the radical. Therefore, $f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$.

21. Let $f(x) = x^2 - 2x$. Determine a domain on which f^{-1} exists, and find a formula for f^{-1} for this domain of f.

SOLUTION From the graph of $y = x^2 - 2x$ shown below, we see that if the domain of f is restricted to either $x \leq 1$ or $x \geq 1$, then f is one-to-one and f^{-1} exists. To find a formula for f^{-1}, we solve $y = x^2 - 2x$ for x as follows:

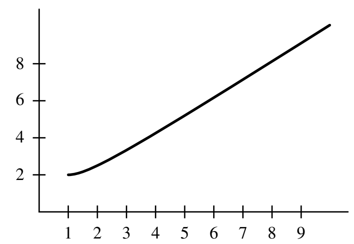
$$\begin{aligned} y + 1 &= x^2 - 2x + 1 = (x - 1)^2 \\ x - 1 &= \pm\sqrt{y + 1} \\ x &= 1 \pm \sqrt{y + 1} \end{aligned}$$

If the domain of f is restricted to $x \leq 1$, then we choose the negative sign in front of the radical and $f^{-1}(x) = 1 - \sqrt{x + 1}$. If the domain of f is restricted to $x \geq 1$, we choose the positive sign in front of the radical and $f^{-1}(x) = 1 + \sqrt{x + 1}$.



22. Show that $f(x) = x + x^{-1}$ is one-to-one on $[1, \infty)$, and find the corresponding inverse f^{-1}. What is the domain of f^{-1}?

SOLUTION The graph of $f(x) = x + x^{-1}$ on $[1, \infty)$ is shown below. From this graph, we see that for $x > 1$ the function is increasing, which implies that the function is one-to-one. Also, note that since f is increasing for $x > 1$, $f(x) \geq f(1) = 2$ for $x > 1$.



To find a formula for f^{-1}, let $y = x + x^{-1}$. Then $xy = x^2 + 1$ or $x^2 - xy + 1 = 0$. Using the quadratic formula, we find

$$x = \frac{y \pm \sqrt{y^2 - 4}}{2}.$$

To have $x \geq 1$ for $y \geq 2$, we must choose the positive sign in front of the radical. Thus,

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

for $x \geq 2$.

In Exercises 23–28, evaluate without using a calculator.

23. $\cos^{-1} 1$

SOLUTION $\cos^{-1} 1 = 0$.

24. $\sin^{-1} \dfrac{1}{2}$

SOLUTION $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$.

25. $\cot^{-1} 1$

SOLUTION $\cot^{-1} 1 = \frac{\pi}{4}$.

26. $\sec^{-1} \dfrac{2}{\sqrt{3}}$

SOLUTION $\sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$.

27. $\tan^{-1} \sqrt{3}$

SOLUTION $\tan^{-1} \sqrt{3} = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$.

28. $\sin^{-1}(-1)$

SOLUTION $\sin^{-1}(-1) = -\frac{\pi}{2}$.

In Exercises 29–38, compute without using a calculator.

29. $\sin^{-1}\left(\sin \dfrac{\pi}{3}\right)$

SOLUTION $\sin^{-1}(\sin \frac{\pi}{3}) = \frac{\pi}{3}$.

30. $\sin^{-1}\left(\sin\dfrac{4\pi}{3}\right)$

SOLUTION $\sin^{-1}(\sin\frac{4\pi}{3}) = \sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$. The answer is not $\frac{4\pi}{3}$ because $\frac{4\pi}{3}$ is not in the range of the inverse sine function.

31. $\cos^{-1}\left(\cos\dfrac{3\pi}{2}\right)$

SOLUTION $\cos^{-1}(\cos\frac{3\pi}{2}) = \cos^{-1}(0) = \frac{\pi}{2}$. The answer is not $\frac{3\pi}{2}$ because $\frac{3\pi}{2}$ is not in the range of the inverse cosine function.

32. $\sin^{-1}\left(\sin\left(-\dfrac{5\pi}{6}\right)\right)$

SOLUTION $\sin^{-1}(\sin(-\frac{5\pi}{6})) = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$. The answer is not $-\frac{5\pi}{6}$ because $-\frac{5\pi}{6}$ is not in the range of the inverse sine function.

33. $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

SOLUTION $\tan^{-1}(\tan\frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}$. The answer is not $\frac{3\pi}{4}$ because $\frac{3\pi}{4}$ is not in the range of the inverse tangent function.

34. $\tan^{-1}(\tan\pi)$

SOLUTION $\tan^{-1}(\tan\pi) = \tan^{-1}(0) = 0$. The answer is not π because π is not in the range of the inverse tangent function.

35. $\sec^{-1}(\sec 3\pi)$

SOLUTION $\sec^{-1}(\sec 3\pi) = \sec^{-1}(-1) = \pi$. The answer is not 3π because 3π is not in the range of the inverse secant function.

36. $\sec^{-1}\left(\sec\dfrac{3\pi}{2}\right)$

SOLUTION No inverse since $\sec\frac{3\pi}{2} = \frac{1}{\cos\frac{3\pi}{2}} = \frac{1}{0} \longrightarrow \infty$.

37. $\csc^{-1}\big(\csc(-\pi)\big)$

SOLUTION No inverse since $\csc(-\pi) = \frac{1}{\sin(-\pi)} = \frac{1}{0} \longrightarrow \infty$.

38. $\cot^{-1}\left(\cot\left(-\dfrac{\pi}{4}\right)\right)$

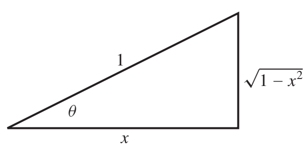
SOLUTION $\cot^{-1}\big(\cot(-\frac{\pi}{4})\big) = \cot^{-1}(-1) = \frac{3\pi}{4}$. The answer is not $-\frac{\pi}{4}$ because $-\frac{\pi}{4}$ is not in the range of the inverse cotangent function.

In Exercises 39–42, simplify by referring to the appropriate triangle or trigonometric identity.

39. $\tan(\cos^{-1} x)$

SOLUTION Let $\theta = \cos^{-1} x$. Then $\cos\theta = x$ and we generate the triangle shown below. From the triangle,

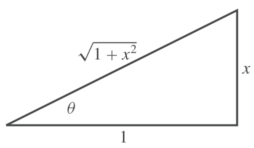
$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}.$$



40. $\cos(\tan^{-1} x)$

SOLUTION Let $\theta = \tan^{-1} x$. Then $\tan\theta = x$ and we generate the triangle shown below. From the triangle,

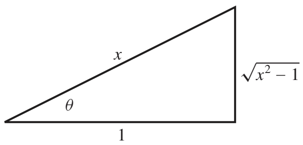
$$\cos(\tan^{-1} x) = \cos\theta = \frac{1}{\sqrt{x^2+1}}.$$



41. $\cot(\sec^{-1} x)$

SOLUTION Let $\theta = \sec^{-1} x$. Then $\sec\theta = x$ and we generate the triangle shown below. From the triangle,

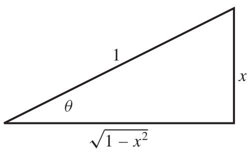
$$\cot(\sec^{-1} x) = \cot\theta = \frac{1}{\sqrt{x^2 - 1}}.$$



42. $\cot(\sin^{-1} x)$

SOLUTION Let $\theta = \sin^{-1} x$. Then $\sin\theta = x$ and we generate the triangle shown below. From the triangle,

$$\cot(\sin^{-1} x) = \cot\theta = \frac{\sqrt{1 - x^2}}{x}.$$

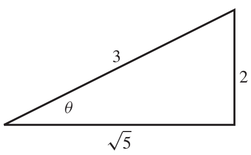


In Exercises 43–50, refer to the appropriate triangle or trigonometric identity to compute the given value.

43. $\cos\left(\sin^{-1} \frac{2}{3}\right)$

SOLUTION Let $\theta = \sin^{-1} \frac{2}{3}$. Then $\sin\theta = \frac{2}{3}$ and we generate the triangle shown below. From the triangle,

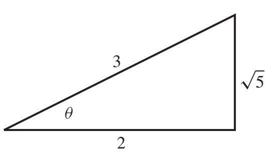
$$\cos\left(\sin^{-1} \frac{2}{3}\right) = \cos\theta = \frac{\sqrt{5}}{3}.$$



44. $\tan\left(\cos^{-1} \frac{2}{3}\right)$

SOLUTION Let $\theta = \cos^{-1} \frac{2}{3}$. Then $\cos\theta = \frac{2}{3}$ and we generate the triangle shown below. From the triangle,

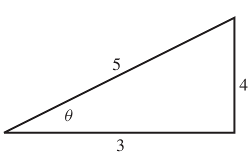
$$\tan\left(\cos^{-1} \frac{2}{3}\right) = \tan\theta = \frac{\sqrt{5}}{2}.$$



45. $\tan\left(\sin^{-1} 0.8\right)$

SOLUTION Let $\theta = \sin^{-1} 0.8$. Then $\sin\theta = 0.8 = \frac{4}{5}$ and we generate the triangle shown below. From the triangle,

$$\tan(\sin^{-1} 0.8) = \tan\theta = \frac{4}{3}.$$



46. $\cos\left(\cot^{-1} 1\right)$

SOLUTION $\cot^{-1} 1 = \frac{\pi}{4}$. Hence, $\cos(\cot^{-1} 1) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

47. $\cot\left(\csc^{-1} 2\right)$

SOLUTION $\csc^{-1} 2 = \frac{\pi}{6}$. Hence, $\cot(\csc^{-1} 2) = \cot\frac{\pi}{6} = \sqrt{3}$.

48. $\tan\left(\sec^{-1}(-2)\right)$

SOLUTION $\sec^{-1}(-2) = \frac{2\pi}{3}$. Hence $\tan(\sec^{-1}(-2)) = \tan\frac{2\pi}{3} = -\sqrt{3}$.

49. $\cot\left(\tan^{-1} 20\right)$

SOLUTION Let $\theta = \tan^{-1} 20$. Then $\tan\theta = 20$, so $\cot(\tan^{-1} 20) = \cot\theta = \frac{1}{\tan\theta} = \frac{1}{20}$.

50. $\sin\left(\csc^{-1} 20\right)$

SOLUTION Let $\theta = \csc^{-1} 20$. Then $\csc\theta = 20$, so $\sin(\csc^{-1} 20) = \sin\theta = \frac{1}{\csc\theta} = \frac{1}{20}$.

Further Insights and Challenges

51. Show that if $f(x)$ is odd and $f^{-1}(x)$ exists, then $f^{-1}(x)$ is odd. Show, on the other hand, that an even function does not have an inverse.

SOLUTION Suppose $f(x)$ is odd and $f^{-1}(x)$ exists. Because $f(x)$ is odd, $f(-x) = -f(x)$. Let $y = f^{-1}(x)$, then $f(y) = x$. Since $f(x)$ is odd, $f(-y) = -f(y) = -x$. Thus $f^{-1}(-x) = -y = -f^{-1}(x)$. Hence, f^{-1} is odd.

On the other hand, if $f(x)$ is even, then $f(-x) = f(x)$. Hence, f is not one-to-one and f^{-1} does not exist.

52. A cylindrical tank of radius R and length L lying horizontally as in Figure 21 is filled with oil to height h. Show that the volume $V(h)$ of oil in the tank as a function of height h is

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR - h^2}\right)$$

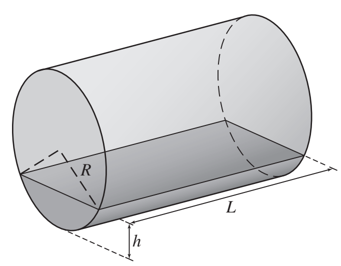


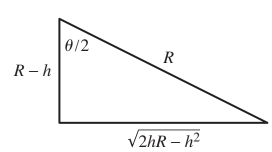
FIGURE 21 Oil in the tank has level h.

SOLUTION From Figure 21, we see that the volume of oil in the tank, $V(h)$, is equal to L times $A(h)$, the area of that portion of the circular cross section occupied by the oil. Now,

$$A(h) = \text{area of sector} - \text{area of triangle} = \frac{R^2\theta}{2} - \frac{R^2\sin\theta}{2},$$

where θ is the central angle of the sector. Referring to the diagram below,

$$\cos\frac{\theta}{2} = \frac{R-h}{R} \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{\sqrt{2hR - h^2}}{R}.$$



Thus,

$$\theta = 2\cos^{-1}\left(1 - \frac{h}{R}\right),$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{(R-h)\sqrt{2hR - h^2}}{R^2},$$

and

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR - h^2}\right).$$

1.6 Exponential and Logarithmic Functions

Preliminary Questions

1. Which of the following equations is incorrect?

(a) $3^2 \cdot 3^5 = 3^7$

(b) $(\sqrt{5})^{4/3} = 5^{2/3}$

(c) $3^2 \cdot 2^3 = 1$

(d) $(2^{-2})^{-2} = 16$

SOLUTION

(a) This equation is correct: $3^2 \cdot 3^5 = 3^{2+5} = 3^7$.

(b) This equation is correct: $(\sqrt{5})^{4/3} = (5^{1/2})^{4/3} = 5^{(1/2)\cdot(4/3)} = 5^{2/3}$.

(c) This equation is incorrect: $3^2 \cdot 2^3 = 9 \cdot 8 = 72 \neq 1$.

(d) this equation is correct: $(2^{-2})^{-2} = 2^{(-2)\cdot(-2)} = 2^4 = 16$.

2. Compute $\log_{b^2}(b^4)$.

SOLUTION Because $b^4 = (b^2)^2$, $\log_{b^2}(b^4) = 2$.

3. When is $\ln x$ negative?

SOLUTION $\ln x$ is negative for $0 < x < 1$.

4. What is $\ln(-3)$? Explain.

SOLUTION $\ln(-3)$ is not defined.

5. Explain the phrase "The logarithm converts multiplication into addition."

SOLUTION This phrase is a verbal description of the general property of logarithms that states

$$\log(ab) = \log a + \log b.$$

6. What are the domain and range of $\ln x$?

SOLUTION The domain of $\ln x$ is $x > 0$ and the range is all real numbers.

7. Which hyperbolic functions take on only positive values?

SOLUTION $\cosh x$ and $\operatorname{sech} x$ take on only positive values.

8. Which hyperbolic functions are increasing on their domains?

SOLUTION $\sinh x$ and $\tanh x$ are increasing on their domains.

9. Describe three properties of hyperbolic functions that have trigonometric analogs.

SOLUTION Hyperbolic functions have the following analogs with trigonometric functions: parity, identities and derivative formulas.

Exercises

1. Rewrite as a whole number (without using a calculator):

(a) 7^0

(b) $10^2(2^{-2} + 5^{-2})$

(c) $\dfrac{(4^3)^5}{(4^5)^3}$

(d) $27^{4/3}$

(e) $8^{-1/3} \cdot 8^{5/3}$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}$

SOLUTION

(a) $7^0 = 1$.
(b) $10^2(2^{-2} + 5^{-2}) = 100(1/4 + 1/25) = 25 + 4 = 29$.
(c) $(4^3)^5/(4^5)^3 = 4^{15}/4^{15} = 1$.
(d) $(27)^{4/3} = (27^{1/3})^4 = 3^4 = 81$.
(e) $8^{-1/3} \cdot 8^{5/3} = (8^{1/3})^5/8^{1/3} = 2^5/2 = 2^4 = 16$.
(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2} = 3 \cdot 2^{1/2} - 3 \cdot 2^2 \cdot 2^{-3/2} = 0$.

In Exercises 2–10, solve for the unknown variable.

2. $9^{2x} = 9^8$

SOLUTION If $9^{2x} = 9^8$, then $2x = 8$, and $x = 4$.

3. $e^{2x} = e^{x+1}$

SOLUTION If $e^{2x} = e^{x+1}$ then $2x = x + 1$, and $x = 1$.

4. $e^{t^2} = e^{4t-3}$

SOLUTION If $e^{t^2} = e^{4t-3}$, then $t^2 = 4t - 3$ or $t^2 - 4t + 3 = (t - 3)(t - 1) = 0$. Thus, $t = 1$ or $t = 3$.

5. $3^x = \left(\frac{1}{3}\right)^{x+1}$

SOLUTION Rewrite $(\frac{1}{3})^{x+1}$ as $(3^{-1})^{x+1} = 3^{-x-1}$. Then $3^x = 3^{-x-1}$, which requires $x = -x - 1$. Thus, $x = -1/2$.

6. $(\sqrt{5})^x = 125$

SOLUTION Rewrite $(\sqrt{5})^x$ as $(5^{1/2})^x = 5^{x/2}$ and 125 as 5^3. Then $5^{x/2} = 5^3$, so $x/2 = 3$ and $x = 6$.

7. $4^{-x} = 2^{x+1}$

SOLUTION Rewrite 4^{-x} as $(2^2)^{-x} = 2^{-2x}$. Then $2^{-2x} = 2^{x+1}$, which requires $-2x = x + 1$. Solving for x gives $x = -1/3$.

8. $b^4 = 10^{12}$

SOLUTION $b^4 = 10^{12}$ is equivalent to $b^4 = (10^3)^4$ so $b = 10^3$. Alternately, raise both sides of the equation to the one-fourth power. This gives $b = (10^{12})^{1/4} = 10^3$.

9. $k^{3/2} = 27$

SOLUTION Raise both sides of the equation to the two-thirds power. This gives $k = (27)^{2/3} = (27^{1/3})^2 = 3^2 = 9$.

10. $\left(b^2\right)^{x+1} = b^{-6}$

SOLUTION Rewrite $(b^2)^{x+1}$ as $b^{2(x+1)}$. Then $2(x + 1) = -6$, and $x = -4$.

In Exercises 11–26, calculate without using a calculator.

11. $\log_3 27$

SOLUTION $\log_3 27 = \log_3 3^3 = 3 \log_3 3 = 3$.

12. $\log_5 \frac{1}{25}$

SOLUTION $\log_5 \dfrac{1}{25} = \log_5 5^{-2} = -2 \log_5 5 = -2$.

13. $\ln 1$

SOLUTION $\ln 1 = 0$.

14. $\log_5(5^4)$

SOLUTION $\log_5(5^4) = 4 \log_5 5 = 4$.

15. $\log_2(2^{5/3})$

SOLUTION $\log_2 2^{5/3} = \dfrac{5}{3} \log_2 2 = \dfrac{5}{3}$.

16. $\log_2(8^{5/3})$

SOLUTION $\log_2(8^{5/3}) = \dfrac{5}{3} \log_2 2^3 = 5 \log_2 2 = 5$.

17. $\log_{64} 4$

SOLUTION $\log_{64} 4 = \log_{64} 64^{1/3} = \frac{1}{3}\log_{64} 64 = \frac{1}{3}$.

18. $\log_7(49^2)$

SOLUTION $\log_7 49^2 = 2\log_7 7^2 = 2 \cdot 2 \cdot \log_7 7 = 4$.

19. $\log_8 2 + \log_4 2$

SOLUTION $\log_8 2 + \log_4 2 = \log_8 8^{1/3} + \log_4 4^{1/2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

20. $\log_{25} 30 + \log_{25} \frac{5}{6}$

SOLUTION $\log_{25} 30 + \log_{25} \frac{5}{6} = \log_{25}\left(30 \cdot \frac{5}{6}\right) = \log_{25} 25 = 1$.

21. $\log_4 48 - \log_4 12$

SOLUTION $\log_4 48 - \log_4 12 = \log_4 \frac{48}{12} = \log_4 4 = 1$.

22. $\ln(\sqrt{e} \cdot e^{7/5})$

SOLUTION $\ln(\sqrt{e} \cdot e^{7/5}) = \ln(e^{1/2} \cdot e^{7/5}) = \ln(e^{1/2+7/5}) = \ln(e^{19/10}) = \frac{19}{10}$.

23. $\ln(e^3) + \ln(e^4)$

SOLUTION $\ln(e^3) + \ln(e^4) = 3 + 4 = 7$.

24. $\log_2 \frac{4}{3} + \log_2 24$

SOLUTION $\log_2 \frac{4}{3} + \log_2 24 = \log_2\left(\frac{4}{3} \cdot 24\right) = \log_2 32 = \log_2 2^5 = 5\log_2 2 = 5$.

25. $7^{\log_7(29)}$

SOLUTION $7^{\log_7(29)} = 29$.

26. $8^{3\log_8(2)}$

SOLUTION $8^{3\log_8(2)} = 8^{\log_8(2^3)} = 8^{\log_8(8)} = 8^1 = 8$.

27. Write as the natural log of a single expression:

(a) $2\ln 5 + 3\ln 4$

(b) $5\ln(x^{1/2}) + \ln(9x)$

SOLUTION

(a) $2\ln 5 + 3\ln 4 = \ln 5^2 + \ln 4^3 = \ln 25 + \ln 64 = \ln(25 \cdot 64) = \ln 1600$.

(b) $5\ln x^{1/2} + \ln 9x = \ln x^{5/2} + \ln 9x = \ln(x^{5/2} \cdot 9x) = \ln(9x^{7/2})$.

28. Solve for x: $\ln(x^2 + 1) - 3\ln x = \ln(2)$.

SOLUTION Combining terms on the left-hand side gives

$$\ln(x^2 + 1) - 3\ln x = \ln(x^2 + 1) - \ln x^3 = \ln \frac{x^2 + 1}{x^3}.$$

Therefore, $\frac{x^2+1}{x^3} = 2$ or $2x^3 - x^2 - 1 = 0$; $x = 1$ is the only real root to this equation. Substituting $x = 1$ into the original equation, we find

$$\ln 2 - 3\ln 1 = \ln 2 - 0 = \ln 2$$

as needed. Hence, $x = 1$ is the only solution.

In Exercises 29–34, solve for the unknown.

29. $7e^{5t} = 100$

SOLUTION Divide the equation by 7 and then take the natural logarithm of both sides. This gives

$$5t = \ln\left(\frac{100}{7}\right) \quad \text{or} \quad t = \frac{1}{5}\ln\left(\frac{100}{7}\right).$$

30. $6e^{-4t} = 2$

SOLUTION Divide the equation by 6 and then take the natural logarithm of both sides. This gives

$$-4t = \ln\left(\frac{1}{3}\right) \quad \text{or} \quad t = \frac{\ln 3}{4}.$$

31. $2^{x^2-2x} = 8$

SOLUTION Since $8 = 2^3$, we have $x^2 - 2x - 3 = 0$ or $(x-3)(x+1) = 0$. Thus, $x = -1$ or $x = 3$.

32. $e^{2t+1} = 9e^{1-t}$

SOLUTION Taking the natural logarithm of both sides of the equation gives

$$2t + 1 = \ln\left(9e^{1-t}\right) = \ln 9 + \ln e^{1-t} = \ln 9 + (1-t).$$

Thus, $3t = \ln 9$ or $t = \frac{1}{3}\ln 9$.

33. $\ln(x^4) - \ln(x^2) = 2$

SOLUTION $\ln(x^4) - \ln(x^2) = \ln\left(\dfrac{x^4}{x^2}\right) = \ln(x^2) = 2\ln x$. Thus, $2\ln x = 2$ or $\ln x = 1$. Hence, $x = e$.

34. $\log_3 y + 3\log_3(y^2) = 14$

SOLUTION $14 = \log_3 y + 3\log_3(y^2) = \log_3 y + \log_3 y^6 = \log_3 y^7$. Thus, $y^7 = 3^{14}$ or $y = 3^2 = 9$.

35. Use a calculator to compute $\sinh x$ and $\cosh x$ for $x = -3, 0, 5$.

SOLUTION

x	-3	0	5
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\dfrac{e^{-3} - e^3}{2} = -10.0179$	$\dfrac{e^0 - e^0}{2} = 0$	$\dfrac{e^5 - e^{-5}}{2} = 74.203$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\dfrac{e^{-3} + e^3}{2} = 10.0677$	$\dfrac{e^0 + e^0}{2} = 1$	$\dfrac{e^5 + e^{-5}}{2} = 74.210$

36. Compute $\sinh(\ln 5)$ and $\tanh(3\ln 5)$ without using a calculator.

SOLUTION

$$\sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - 1/5}{2} = \frac{24/5}{2} = 12/5;$$

$$\tanh(3\ln 5) = \frac{\sinh(3\ln 5)}{\cosh(3\ln 5)} = \frac{\frac{e^{3\ln 5} - e^{-3\ln 5}}{2}}{\frac{e^{3\ln 5} + e^{-3\ln 5}}{2}} = \frac{5^3 - 1/5^3}{5^3 + 1/5^3} = \frac{5^6 - 1}{5^6 + 1}.$$

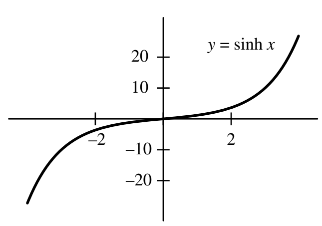
37. Show, by producing a counterexample, that $\ln(ab)$ is not equal to $(\ln a)(\ln b)$.

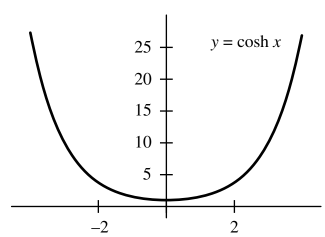
SOLUTION Let $a = e^2$ and $b = e^3$. Then $ab = e^5$ and $\ln(ab) = \ln(e^5) = 5$; however,

$$(\ln a)(\ln b) = (\ln e^2)(\ln e^3) = 2(3) = 6.$$

38. For which values of x are $y = \sinh x$ and $y = \cosh x$ increasing and decreasing?

SOLUTION The graph of $y = \sinh x$ is shown below on the left. From this graph, we see that $\sinh x$ is increasing for all x. On the other hand, from the graph of $y = \cosh x$ shown below on the right, we see that $\cosh x$ is decreasing for $x < 0$ and is increasing for $x > 0$.





39. Show that $y = \tanh x$ is an odd function.

SOLUTION $\tanh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \dfrac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = -\dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = -\tanh x.$

40. The population of a city (in millions) at time t (years) is $P(t) = 2.4e^{0.06t}$, where $t = 0$ is the year 2000. When will the population double from its size at $t = 0$?

SOLUTION Population doubles when $4.8 = 2.4e^{0.06t}$. Thus, $0.06t = \ln 2$ or $t = \dfrac{\ln 2}{0.06} \approx 11.55$ years.

41. The **Gutenberg–Richter Law** states that the number N of earthquakes per year worldwide of Richter magnitude at least M satisfies an approximate relation $\log_{10} N = a - M$ for some constant a. Find a, assuming that there is one earthquake of magnitude $M \geq 8$ per year. How many earthquakes of magnitude $M \geq 5$ occur per year?

SOLUTION Substituting $N = 1$ and $M = 8$ into the Gutenberg–Richter law and solving for a yields

$$a = 8 + \log_{10} 1 = 8.$$

The number N of earthquakes of Richter magnitude $M \geq 5$ then satisfies

$$\log_{10} N = 8 - 5 = 3.$$

Finally, $N = 10^3 = 1000$ earthquakes.

42. The energy E (in joules) radiated as seismic waves from an earthquake of Richter magnitude M is given by the formula $\log_{10} E = 4.8 + 1.5M$.

(a) Express E as a function of M.

(b) Show that when M increases by 1, the energy increases by a factor of approximately 31.6.

SOLUTION

(a) Solving $\log_{10} E = 4.8 + 1.5M$ for E yields

$$E = 10^{4.8+1.5M}.$$

(b) Using the formula from part (a), we find

$$\frac{E(M+1)}{E(M)} = \frac{10^{4.8+1.5(M+1)}}{10^{4.8+1.5M}} = \frac{10^{6.3+1.5M}}{10^{4.8+1.5M}} = 10^{1.5} \approx 31.6228.$$

43. Refer to the graphs to explain why the equation $\sinh x = t$ has a unique solution for every t and why $\cosh x = t$ has two solutions for every $t > 1$.

SOLUTION From its graph we see that $\sinh x$ is a one-to-one function with $\lim_{x\to-\infty} \sinh x = -\infty$ and $\lim_{x\to\infty} \sinh x = \infty$. Thus, for every real number t, the equation $\sinh x = t$ has a unique solution. On the other hand, from its graph, we see that $\cosh x$ is not one-to-one. Rather, it is an even function with a minimum value of $\cosh 0 = 1$. Thus, for every $t > 1$, the equation $\cosh x = t$ has two solutions: one positive, the other negative.

44. Compute $\cosh x$ and $\tanh x$, assuming that $\sinh x = 0.8$.

SOLUTION Using the identity $\cosh^2 x - \sinh^2 x = 1$, it follows that $\cosh^2 x - (\frac{4}{5})^2 = 1$, so that $\cosh^2 x = \frac{41}{25}$ and

$$\cosh x = \frac{\sqrt{41}}{5}.$$

Then, by definition,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{4}{5}}{\frac{\sqrt{41}}{5}} = \frac{4}{\sqrt{41}}.$$

45. Prove the addition formula for $\cosh x$.

SOLUTION

$$\begin{aligned}
\cosh(x+y) &= \frac{e^{x+y} + e^{-(x+y)}}{2} = \frac{2e^{x+y} + 2e^{-(x+y)}}{4} \\
&= \frac{e^{x+y} + e^{-x+y} + e^{x-y} + e^{-(x+y)}}{4} + \frac{e^{x+y} - e^{-x+y} - e^{x-y} + e^{-(x+y)}}{4} \\
&= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \cosh x \cosh y + \sinh x \sinh y.
\end{aligned}$$

46. Use the addition formulas to prove

$$\sinh(2x) = 2\cosh x \sinh x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

SOLUTION $\sinh(2x) = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2\cosh x \sinh x$ and $\cosh(2x) = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$.

47. An (imaginary) train moves along a track at velocity v. Bionica walks down the aisle of the train with velocity u in the direction of the train's motion. Compute the velocity w of Bionica relative to the ground using the laws of both Galileo and Einstein in the following cases.

(a) $v = 500$ m/s and $u = 10$ m/s. Is your calculator accurate enough to detect the difference between the two laws?

(b) $v = 10^7$ m/s and $u = 10^6$ m/s.

SOLUTION Recall that the speed of light is $c \approx 3 \times 10^8$ m/s.

(a) By Galileo's law, $w = 500 + 10 = 510$ m/s. Using Einstein's law and a calculator,

$$\tanh^{-1}\frac{w}{c} = \tanh^{-1}\frac{500}{c} + \tanh^{-1}\frac{10}{c} = 1.7 \times 10^{-6};$$

so $w = c \cdot \tanh(1.7 \times 10^{-6}) \approx 510$ m/s. No, the calculator was not accurate enough to detect the difference between the two laws.

(b) By Galileo's law, $u + v = 10^7 + 10^6 = 1.1 \times 10^7$ m/s. By Einstein's law,

$$\tanh^{-1}\frac{w}{c} = \tanh^{-1}\frac{10^7}{3 \times 10^8} + \tanh^{-1}\frac{10^6}{3 \times 10^8} \approx 0.036679,$$

so $w \approx c \cdot \tanh(0.036679) \approx 1.09988 \times 10^7$ m/s.

Further Insights and Challenges

48. Show that $\log_a b \log_b a = 1$.

SOLUTION $\log_a b = \dfrac{\ln b}{\ln a}$ and $\log_b a = \dfrac{\ln a}{\ln b}$. Thus $\log_a b \cdot \log_b a = \dfrac{\ln b}{\ln a} \cdot \dfrac{\ln a}{\ln b} = 1$.

49. Verify the formula $\log_b x = \dfrac{\log_a x}{\log_a b}$ for $a, b > 0$.

SOLUTION Let $y = \log_b x$. Then $x = b^y$ and $\log_a x = \log_a b^y = y \log_a b$. Thus, $y = \dfrac{\log_a x}{\log_a b}$.

50. **(a)** Use the addition formulas for $\sinh x$ and $\cosh x$ to prove

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

(b) Use (a) to show that Einstein's Law of Velocity Addition [Eq. (3)] is equivalent to

$$w = \frac{u + v}{1 + \frac{uv}{c^2}}$$

SOLUTION

(a)

$$\tanh(u + v) = \frac{\sinh(u + v)}{\cosh(u + v)} = \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v}$$

$$= \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v} \cdot \frac{1/(\cosh u \cosh v)}{1/(\cosh u \cosh v)} = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

(b) Einstein's law states: $\tanh^{-1}(w/c) = \tanh^{-1}(u/c) + \tanh^{-1}(v/c)$. Thus

$$\frac{w}{c} = \tanh\left(\tanh^{-1}(u/c) + \tanh^{-1}(v/c)\right) = \frac{\tanh(\tanh^{-1}(v/c)) + \tanh(\tanh^{-1}(u/c))}{1 + \tanh(\tanh^{-1}(v/c))\tanh(\tanh^{-1}(u/c))}$$

$$= \frac{\frac{v}{c} + \frac{u}{c}}{1 + \frac{v}{c}\frac{u}{c}} = \frac{(1/c)(u + v)}{1 + \frac{uv}{c^2}}.$$

Hence,

$$w = \frac{u + v}{1 + \frac{uv}{c^2}}.$$

51. Prove that every function $f(x)$ can be written as a sum $f(x) = f_+(x) + f_-(x)$ of an even function $f_+(x)$ and an odd function $f_-(x)$. Express $f(x) = 5e^x + 8e^{-x}$ in terms of $\cosh x$ and $\sinh x$.

SOLUTION Let $f_+(x) = \frac{f(x)+f(-x)}{2}$ and $f_-(x) = \frac{f(x)-f(-x)}{2}$. Then $f_+ + f_- = \frac{2f(x)}{2} = f(x)$. Moreover,

$$f_+(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_+(x),$$

so $f_+(x)$ is an even function, while

$$\begin{aligned} f_-(-x) &= \frac{f(-x) - f(-(-x))}{2} \\ &= \frac{f(-x) - f(x)}{2} = -\frac{(f(x) - f(-x))}{2} = -f_-(x), \end{aligned}$$

so $f_-(x)$ is an odd function.

For $f(x) = 5e^x + 8e^{-x}$, we have

$$f_+(x) = \frac{5e^x + 8e^{-x} + 5e^{-x} + 8e^x}{2} = 8\cosh x + 5\cosh x = 13\cosh x$$

and

$$f_-(x) = \frac{5e^x + 8e^{-x} - 5e^{-x} - 8e^x}{2} = 5\sinh x - 8\sinh x = -3\sinh x.$$

Therefore, $f(x) = f_+(x) + f_-(x) = 13\cosh x - 3\sinh x$.

1.7 Technology: Calculators and Computers

Preliminary Questions

1. Is there a definite way of choosing the optimal viewing rectangle, or is it best to experiment until you find a viewing rectangle appropriate to the problem at hand?

SOLUTION It is best to experiment with the window size until one is found that is appropriate for the problem at hand.

2. Describe the calculator screen produced when the function $y = 3 + x^2$ is plotted with viewing rectangle:

(a) $[-1, 1] \times [0, 2]$ **(b)** $[0, 1] \times [0, 4]$

SOLUTION

(a) Using the viewing rectangle $[-1, 1]$ by $[0, 2]$, the screen will display nothing as the minimum value of $y = 3 + x^2$ is $y = 3$.

(b) Using the viewing rectangle $[0, 1]$ by $[0, 4]$, the screen will display the portion of the parabola between the points $(0, 3)$ and $(1, 4)$.

3. According to the evidence in Example 4, it appears that $f(n) = (1 + 1/n)^n$ never takes on a value greater than 3 for $n > 0$. Does this evidence *prove* that $f(n) \le 3$ for $n > 0$?

SOLUTION No, this evidence does not constitute a proof that $f(n) \le 3$ for $n \ge 0$.

4. How can a graphing calculator be used to find the minimum value of a function?

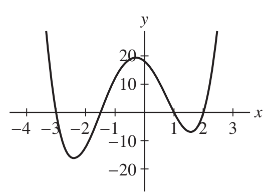
SOLUTION Experiment with the viewing window to zoom in on the lowest point on the graph of the function. The y-coordinate of the lowest point on the graph is the minimum value of the function.

Exercises

The exercises in this section should be done using a graphing calculator or computer algebra system.

1. Plot $f(x) = 2x^4 + 3x^3 - 14x^2 - 9x + 18$ in the appropriate viewing rectangles and determine its roots.

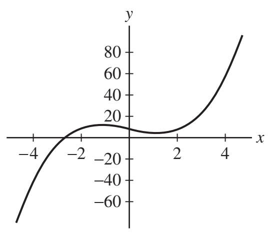
SOLUTION Using a viewing rectangle of $[-4, 3]$ by $[-20, 20]$, we obtain the plot below.



Now, the roots of $f(x)$ are the x-intercepts of the graph of $y = f(x)$. From the plot, we can identify the x-intercepts as -3, -1.5, 1, and 2. The roots of $f(x)$ are therefore $x = -3$, $x = -1.5$, $x = 1$, and $x = 2$.

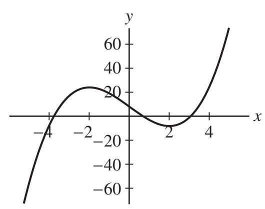
2. How many solutions does $x^3 - 4x + 8 = 0$ have?

SOLUTION Solutions to the equation $x^3 - 4x + 8 = 0$ are the x-intercepts of the graph of $y = x^3 - 4x + 8$. From the figure below, we see that the graph has one x-intercept (between $x = -4$ and $x = -2$), so the equation has one solution.



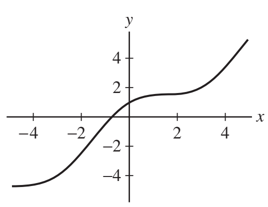
3. How many *positive* solutions does $x^3 - 12x + 8 = 0$ have?

SOLUTION The graph of $y = x^3 - 12x + 8$ shown below has two x-intercepts to the right of the origin; therefore the equation $x^3 - 12x + 8 = 0$ has two positive solutions.



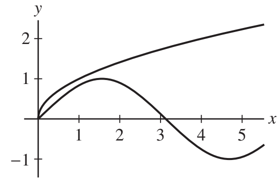
4. Does $\cos x + x = 0$ have a solution? A positive solution?

SOLUTION The graph of $y = \cos x + x$ shown below has one x-intercept; therefore, the equation $\cos x + x = 0$ has one solution. The lone x-intercept is to the left of the origin, so the equation has no positive solutions.



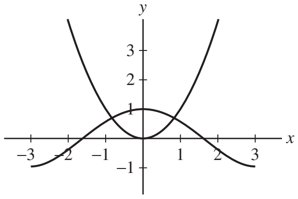
5. Find all the solutions of $\sin x = \sqrt{x}$ for $x > 0$.

SOLUTION Solutions to the equation $\sin x = \sqrt{x}$ correspond to points of intersection between the graphs of $y = \sin x$ and $y = \sqrt{x}$. The two graphs are shown below; the only point of intersection is at $x = 0$. Therefore, there are no solutions of $\sin x = \sqrt{x}$ for $x > 0$.



6. How many solutions does $\cos x = x^2$ have?

SOLUTION Solutions to the equation $\cos x = x^2$ correspond to points of intersection between the graphs of $y = \cos x$ and $y = x^2$. The two graphs are shown below; there are two points of intersection. Thus, the equation $\cos x = x^2$ has two solutions.

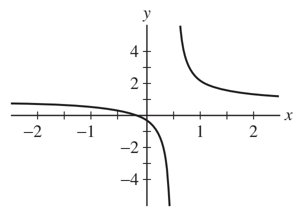


7. Let $f(x) = (x - 100)^2 + 1000$. What will the display show if you graph $f(x)$ in the viewing rectangle $[-10, 10]$ by $[-10, 10]$? Find an appropriate viewing rectangle.

SOLUTION Because $(x - 100)^2 \geq 0$ for all x, it follows that $f(x) = (x - 100)^2 + 1000 \geq 1000$ for all x. Thus, using a viewing rectangle of $[-10, 10]$ by $[-10, 10]$ will display nothing. The minimum value of the function occurs when $x = 100$, so an appropriate viewing rectangle would be $[50, 150]$ by $[1000, 2000]$.

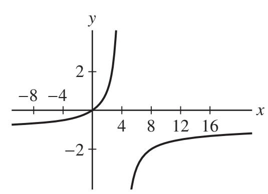
8. Plot $f(x) = \dfrac{8x + 1}{8x - 4}$ in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?

SOLUTION From the graph of $y = \dfrac{8x + 1}{8x - 4}$ shown below, we see that the vertical asymptote is $x = \frac{1}{2}$ and the horizontal asymptote is $y = 1$.



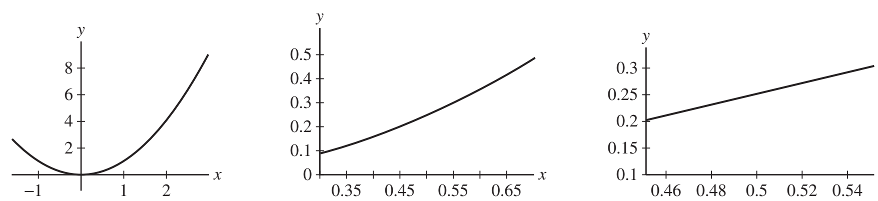
9. Plot the graph of $f(x) = x/(4 - x)$ in a viewing rectangle that clearly displays the vertical and horizontal asymptotes.

SOLUTION From the graph of $y = \dfrac{x}{4 - x}$ shown below, we see that the vertical asymptote is $x = 4$ and the horizontal asymptote is $y = -1$.



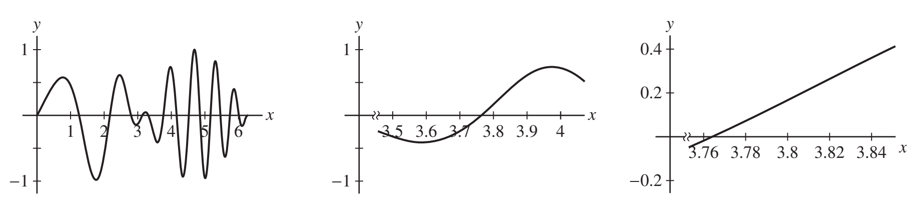
10. Illustrate local linearity for $f(x) = x^2$ by zooming in on the graph at $x = 0.5$ (see Example 6).

SOLUTION The following three graphs display $f(x) = x^2$ over the intervals $[-1, 3]$, $[0.3, 0.7]$ and $[0.45, 0.55]$. The final graph looks like a straight line.



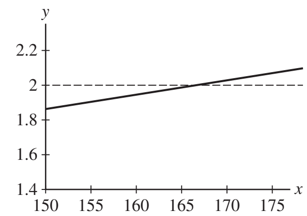
11. Plot $f(x) = \cos(x^2)\sin x$ for $0 \leq x \leq 2\pi$. Then illustrate local linearity at $x = 3.8$ by choosing appropriate viewing rectangles.

SOLUTION The following three graphs display $f(x) = \cos(x^2)\sin x$ over the intervals $[0, 2\pi]$, $[3.5, 4.1]$ and $[3.75, 3.85]$. The final graph looks like a straight line.



12. If P_0 dollars are deposited in a bank account paying 5% interest compounded monthly, then the account has value $P_0\left(1 + \frac{0.05}{12}\right)^N$ after N months. Find, to the nearest integer N, the number of months after which the account value doubles.

SOLUTION $P(N) = P_0(1 + \frac{0.05}{12})^N$. This doubles when $P(N) = 2P_0$, or when $2 = (1 + \frac{0.05}{12})^N$. The graphs of $y = 2$ and $y = (1 + \frac{0.05}{12})^N$ are shown below; they appear to intersect at $N = 167$. Thus, it will take approximately 167 months for money earning $r = 5\%$ interest compounded monthly to double in value.

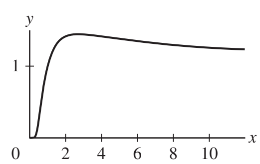


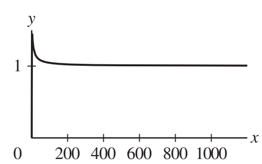
In Exercises 13–18, investigate the behavior of the function as n or x grows large by making a table of function values and plotting a graph (see Example 4). Describe the behavior in words.

13. $f(n) = n^{1/n}$

SOLUTION The table and graphs below suggest that as n gets large, $n^{1/n}$ approaches 1.

n	$n^{1/n}$
10	1.258925412
10^2	1.047128548
10^3	1.006931669
10^4	1.000921458
10^5	1.000115136
10^6	1.000013816

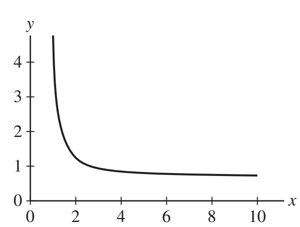


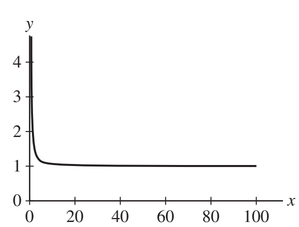


14. $f(n) = \dfrac{4n+1}{6n-5}$

SOLUTION The table and graphs below suggest that as n gets large, $\dfrac{4n+1}{6n-5}$ approaches $\dfrac{2}{3}$.

n	$\dfrac{4n+1}{6n-5}$
10	0.7454545455
10^2	0.6739495798
10^3	0.6673894912
10^4	0.6667388949
10^5	0.6666738889
10^6	0.6666673889

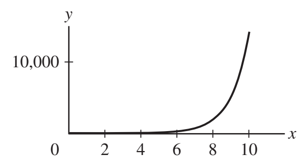


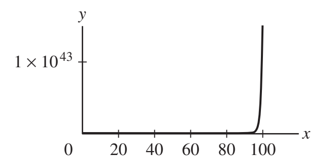


15. $f(n) = \left(1 + \dfrac{1}{n}\right)^{n^2}$

SOLUTION The table and graphs below suggest that as n gets large, $f(n)$ tends toward ∞.

n	$\left(1 + \dfrac{1}{n}\right)^{n^2}$
10	13780.61234
10^2	$1.635828711 \times 10^{43}$
10^3	$1.195306603 \times 10^{434}$
10^4	$5.341783312 \times 10^{4342}$
10^5	$1.702333054 \times 10^{43429}$
10^6	$1.839738749 \times 10^{434294}$

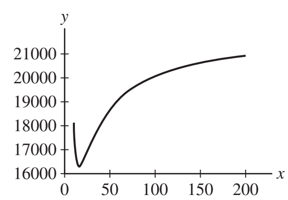


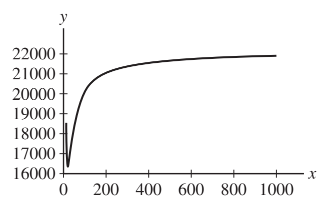


16. $f(x) = \left(\dfrac{x+6}{x-4}\right)^{x}$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ roughly tends toward 22026.

x	$\left(\dfrac{x+6}{x-4}\right)^{x}$
10	18183.91210
10^2	20112.36934
10^3	21809.33633
10^4	22004.43568
10^5	22024.26311
10^6	22025.36451
10^7	22026.35566

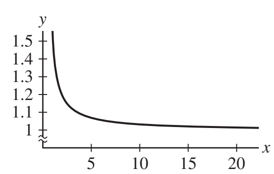


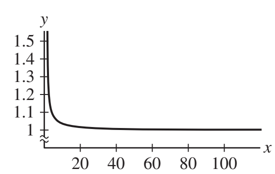


17. $f(x) = \left(x \tan \dfrac{1}{x}\right)^{x}$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ approaches 1.

x	$\left(x \tan \dfrac{1}{x}\right)^{x}$
10	1.033975759
10^2	1.003338973
10^3	1.000333389
10^4	1.000033334
10^5	1.000003333
10^6	1.000000333

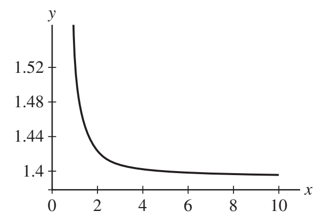


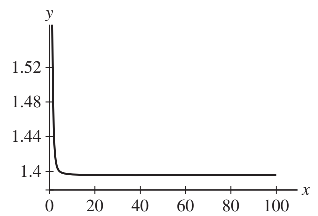


18. $f(x) = \left(x \tan \frac{1}{x}\right)^{x^2}$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ approaches 1.39561.

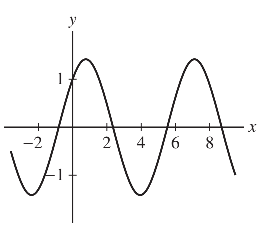
x	$\left(x \tan \frac{1}{x}\right)^{x^2}$
10	1.396701388
10^2	1.395623280
10^3	1.395612534
10^4	1.395612426
10^5	1.395612425
10^6	1.395612425



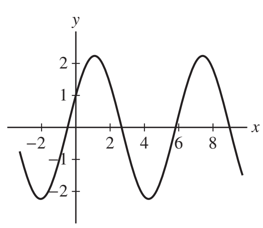


19. The graph of $f(\theta) = A\cos\theta + B\sin\theta$ is a sinusoidal wave for any constants A and B. Confirm this for $(A, B) = (1, 1)$, $(1, 2)$, and $(3, 4)$ by plotting $f(\theta)$.

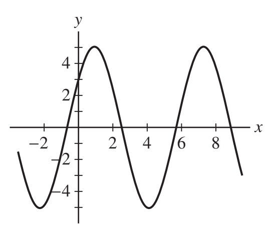
SOLUTION The graphs of $f(\theta) = \cos\theta + \sin\theta$, $f(\theta) = \cos\theta + 2\sin\theta$ and $f(\theta) = 3\cos\theta + 4\sin\theta$ are shown below.



$(A, B) = (1, 1)$



$(A, B) = (1, 2)$



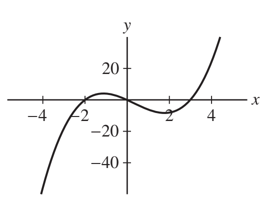
$(A, B) = (3, 4)$

20. Find the maximum value of $f(\theta)$ for the graphs produced in Exercise 19. Can you guess the formula for the maximum value in terms of A and B?

SOLUTION For $A = 1$ and $B = 1$, max $\approx 1.4 \approx \sqrt{2}$
For $A = 1$ and $B = 2$, max $\approx 2.25 \approx \sqrt{5}$
For $A = 3$ and $B = 4$, max $\approx 5 = \sqrt{3^2 + 4^2}$
Max $= \sqrt{A^2 + B^2}$

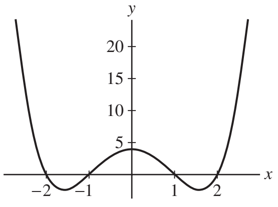
21. Find the intervals on which $f(x) = x(x + 2)(x - 3)$ is positive by plotting a graph.

SOLUTION The function $f(x) = x(x + 2)(x - 3)$ is positive when the graph of $y = x(x + 2)(x - 3)$ lies above the x-axis. The graph of $y = x(x + 2)(x - 3)$ is shown below. Clearly, the graph lies above the x-axis and the function is positive for $x \in (-2, 0) \cup (3, \infty)$.



22. Find the set of solutions to the inequality $(x^2-4)(x^2-1) < 0$ by plotting a graph.

SOLUTION To solve the inequality $(x^2-4)(x^2-1) < 0$, we can plot the graph of $y = (x^2-4)(x^2-1)$ and identify when the graph lies below the x-axis. The graph of $y = (x^2-4)(x^2-1)$ is shown below. The solution set for the inequality $(x^2-4)(x^2-1) < 0$ is clearly $x \in (-2,-1) \cup (1,2)$.



Further Insights and Challenges

23. CAS Let $f_1(x) = x$ and define a sequence of functions by $f_{n+1}(x) = \frac{1}{2}(f_n(x) + x/f_n(x))$. For example, $f_2(x) = \frac{1}{2}(x+1)$. Use a computer algebra system to compute $f_n(x)$ for $n = 3, 4, 5$ and plot $f_n(x)$ together with $\sqrt{x}$ for $x \geq 0$. What do you notice?

SOLUTION With $f_1(x) = x$ and $f_2(x) = \frac{1}{2}(x+1)$, we calculate

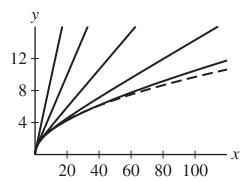
$$f_3(x) = \frac{1}{2}\left(\frac{1}{2}(x+1) + \frac{x}{\frac{1}{2}(x+1)}\right) = \frac{x^2+6x+1}{4(x+1)}$$

$$f_4(x) = \frac{1}{2}\left(\frac{x^2+6x+1}{4(x+1)} + \frac{x}{\frac{x^2+6x+1}{4(x+1)}}\right) = \frac{x^4+28x^3+70x^2+28x+1}{8(1+x)(1+6x+x^2)}$$

and

$$f_5(x) = \frac{1+120x+1820x^2+8008x^3+12870x^4+8008x^5+1820x^6+120x^7+x^8}{16(1+x)(1+6x+x^2)(1+28x+70x^2+28x^3+x^4)}.$$

A plot of $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, $f_5(x)$ and $\sqrt{x}$ is shown below, with the graph of $\sqrt{x}$ shown as a dashed curve. It seems as if the f_n are asymptotic to $\sqrt{x}$.



24. Set $P_0(x) = 1$ and $P_1(x) = x$. The **Chebyshev polynomials** (useful in approximation theory) are defined inductively by the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$.

(a) Show that $P_2(x) = 2x^2 - 1$.

(b) Compute $P_n(x)$ for $3 \leq n \leq 6$ using a computer algebra system or by hand, and plot $P_n(x)$ over $[-1,1]$.

(c) Check that your plots confirm two interesting properties: (a) $P_n(x)$ has n real roots in $[-1,1]$ and (b) for $x \in [-1,1]$, $P_n(x)$ lies between -1 and 1.

SOLUTION

(a) With $P_0(x) = 1$ and $P_1(x) = x$, we calculate

$$P_2(x) = 2x(P_1(x)) - P_0(x) = 2x(x) - 1 = 2x^2 - 1.$$

(b) Using the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$ with $n = 2, 3, 4$ and 5, we find

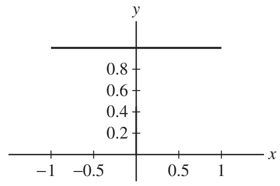
$$P_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$

$$P_4(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1$$

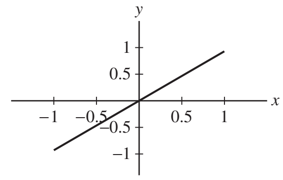
$$P_5(x) = 16x^5 - 20x^3 + 5x$$

$$P_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

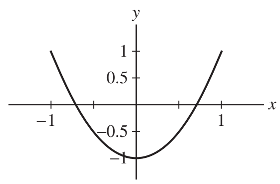
The graphs of the functions $P_n(x)$ for $0 \leq n \leq 6$ are shown below.



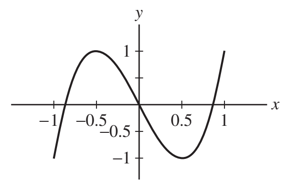
$P_0(x)$



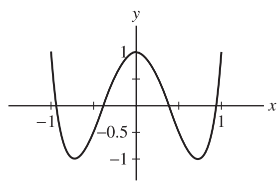
$P_1(x)$



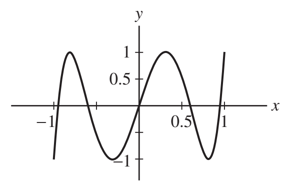
$P_2(x)$



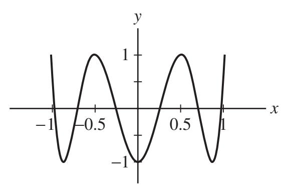
$P_3(x)$



$P_4(x)$



$P_5(x)$



$P_6(x)$

(c) From the graphs shown above, it is clear that for each n, the polynomial $P_n(x)$ has precisely n roots on the interval $[-1, 1]$ and that $-1 \le P_n(x) \le 1$ for $x \in [-1, 1]$.

CHAPTER REVIEW EXERCISES

1. Express $(4, 10)$ as a set $\{x : |x - a| < c\}$ for suitable a and c.

SOLUTION The center of the interval $(4, 10)$ is $\frac{4+10}{2} = 7$ and the radius is $\frac{10-4}{2} = 3$. Therefore, the interval $(4, 10)$ is equivalent to the set $\{x : |x - 7| < 3\}$.

2. Express as an interval:

(a) $\{x : |x - 5| < 4\}$ **(b)** $\{x : |5x + 3| \le 2\}$

SOLUTION

(a) Upon dropping the absolute value, the inequality $|x - 5| < 4$ becomes $-4 < x - 5 < 4$ or $1 < x < 9$. The set $\{x : |x - 5| < 4\}$ can therefore be expressed as the interval $(1, 9)$.

(b) Upon dropping the absolute value, the inequality $|5x + 3| \le 2$ becomes $-2 \le 5x + 3 \le 2$ or $-1 \le x \le -\frac{1}{5}$. The set $\{x : |5x + 3| \le 2\}$ can therefore be expressed as the interval $[-1, -\frac{1}{5}]$.

3. Express $\{x : 2 \le |x - 1| \le 6\}$ as a union of two intervals.

SOLUTION The set $\{x : 2 \le |x - 1| \le 6\}$ consists of those numbers that are at least 2 but at most 6 units from 1. The numbers larger than 1 that satisfy these conditions are $3 \le x \le 7$, while the numbers smaller than 1 that satisfy these conditions are $-5 \le x \le -1$. Therefore $\{x : 2 \le |x - 1| \le 6\} = [-5, -1] \cup [3, 7]$.

4. Give an example of numbers x, y such that $|x| + |y| = x - y$.

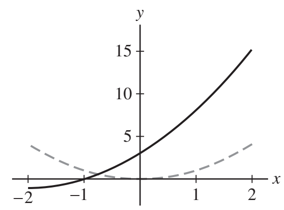
SOLUTION Let $x = 3$ and $y = -1$. Then $|x| + |y| = 3 + 1 = 4$ and $x - y = 3 - (-1) = 4$.

5. Describe the pairs of numbers x, y such that $|x + y| = x - y$.

SOLUTION First consider the case when $x + y \ge 0$. Then $|x + y| = x + y$ and we obtain the equation $x + y = x - y$. The solution of this equation is $y = 0$. Thus, the pairs $(x, 0)$ with $x \ge 0$ satisfy $|x + y| = x - y$. Next, consider the case when $x + y < 0$. Then $|x + y| = -(x + y) = -x - y$ and we obtain the equation $-x - y = x - y$. The solution of this equation is $x = 0$. Thus, the pairs $(0, y)$ with $y < 0$ also satisfy $|x + y| = x - y$.

6. Sketch the graph of $y = f(x+2) - 1$, where $f(x) = x^2$ for $-2 \le x \le 2$.

SOLUTION The graph of $y = f(x+2) - 1$ is obtained by shifting the graph of $y = f(x)$ two units to the left and one unit down. In the figure below, the graph of $y = f(x)$ is shown as the dashed curve, and the graph of $y = f(x+2) - 1$ is shown as the solid curve.



In Exercises 7–10, let $f(x)$ be the function shown in Figure 1.

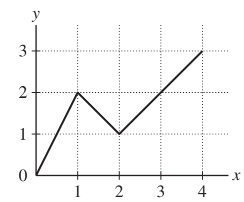
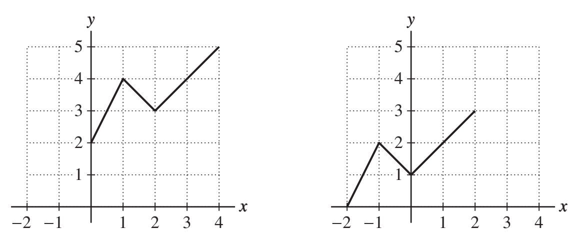


FIGURE 1

7. Sketch the graphs of $y = f(x) + 2$ and $y = f(x+2)$.

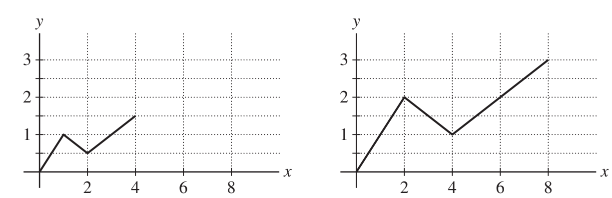
SOLUTION The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ up 2 units (see the graph below at the left). The graph of $y = f(x+2)$ is obtained by shifting the graph of $y = f(x)$ to the left 2 units (see the graph below at the right).



$f(x) + 2$ $\qquad$ $f(x + 2)$

8. Sketch the graphs of $y = \frac{1}{2} f(x)$ and $y = f\left(\frac{1}{2}x\right)$.

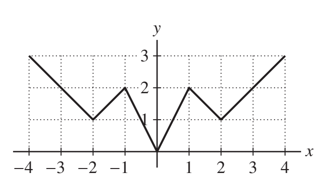
SOLUTION The graph of $y = \frac{1}{2} f(x)$ is obtained by compressing the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below at the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally be a factor of 2 (see the graph below at the right).



$f(x)/2$ $\qquad$ $f(x/2)$

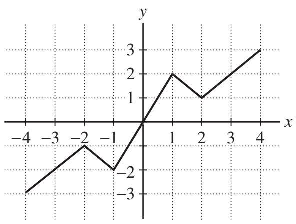
9. Continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph below).



10. Continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph below).



In Exercises 11–14, find the domain and range of the function.

11. $f(x) = \sqrt{x+1}$

SOLUTION The domain of the function $f(x) = \sqrt{x+1}$ is $\{x : x \geq -1\}$ and the range is $\{y : y \geq 0\}$.

12. $f(x) = \dfrac{4}{x^4+1}$

SOLUTION The domain of the function $f(x) = \dfrac{4}{x^4+1}$ is the set of all real numbers and the range is $\{y : 0 < y \leq 4\}$.

13. $f(x) = \dfrac{2}{3-x}$

SOLUTION The domain of the function $f(x) = \dfrac{2}{3-x}$ is $\{x : x \neq 3\}$ and the range is $\{y : y \neq 0\}$.

14. $f(x) = \sqrt{x^2 - x + 5}$

SOLUTION Because

$$x^2 - x + 5 = \left(x^2 - x + \frac{1}{4}\right) + 5 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{19}{4},$$

$x^2 - x + 5 \geq \frac{19}{4}$ for all x. It follows that the domain of the function $f(x) = \sqrt{x^2 - x + 5}$ is all real numbers and the range is $\{y : y \geq \sqrt{19}/2\}$.

15. Determine whether the function is increasing, decreasing, or neither:

(a) $f(x) = 3^{-x}$

(b) $f(x) = \dfrac{1}{x^2+1}$

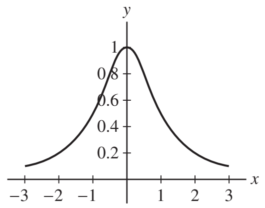
(c) $g(t) = t^2 + t$

(d) $g(t) = t^3 + t$

SOLUTION

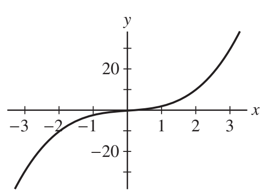
(a) The function $f(x) = 3^{-x}$ can be rewritten as $f(x) = (\frac{1}{3})^x$. This is an exponential function with a base less than 1; therefore, this is a decreasing function.

(b) From the graph of $y = 1/(x^2+1)$ shown below, we see that this function is neither increasing nor decreasing for all x (though it is increasing for $x < 0$ and decreasing for $x > 0$).



(c) The graph of $y = t^2 + t$ is an upward opening parabola; therefore, this function is neither increasing nor decreasing for all t. By completing the square we find $y = (t + \frac{1}{2})^2 - \frac{1}{4}$. The vertex of this parabola is then at $t = -\frac{1}{2}$, so the function is decreasing for $t < -\frac{1}{2}$ and increasing for $t > -\frac{1}{2}$.

(d) From the graph of $y = t^3 + t$ shown below, we see that this is an increasing function.



16. Determine whether the function is even, odd, or neither:

(a) $f(x) = x^4 - 3x^2$

(b) $g(x) = \sin(x+1)$

(c) $f(x) = 2^{-x^2}$

SOLUTION

(a) $f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2 = f(x)$, so this function is even.

(b) $g(-x) = \sin(-x+1)$, which is neither equal to $g(x)$ nor to $-g(x)$, so this function is neither even nor odd.

(c) $f(-x) = 2^{-(-x)^2} = 2^{-x^2} = f(x)$, so this function is even.

In Exercises 17–22, find the equation of the line.

17. Line passing through $(-1, 4)$ and $(2, 6)$

SOLUTION The slope of the line passing through $(-1, 4)$ and $(2, 6)$ is

$$m = \frac{6-4}{2-(-1)} = \frac{2}{3}.$$

The equation of the line passing through $(-1, 4)$ and $(2, 6)$ is therefore $y - 4 = \frac{2}{3}(x+1)$ or $2x - 3y = -14$.

18. Line passing through $(-1, 4)$ and $(-1, 6)$

SOLUTION The line passing through $(-1, 4)$ and $(-1, 6)$ is vertical with an x-coordinate of -1. Therefore, the equation of the line is $x = -1$.

19. Line of slope 6 through $(9, 1)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope 6 and passing through $(9, 1)$ is $y - 1 = 6(x - 9)$ or $6x - y = 53$.

20. Line of slope $-\frac{3}{2}$ through $(4, -12)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope $-\frac{3}{2}$ and passing through $(4, -12)$ is $y + 12 = -\frac{3}{2}(x - 4)$ or $3x + 2y = -12$.

21. Line through $(2, 3)$ parallel to $y = 4 - x$

SOLUTION The equation $y = 4 - x$ is in slope-intercept form; it follows that the slope of this line is -1. Any line parallel to $y = 4 - x$ will have the same slope, so we are looking for the equation of the line of slope -1 and passing through $(2, 3)$. The equation of this line is $y - 3 = -(x - 2)$ or $x + y = 5$.

22. Horizontal line through $(-3, 5)$

SOLUTION A horizontal line has a slope of 0; the equation of the specified line is therefore $y - 5 = 0(x + 3)$ or $y = 5$.

23. Does the following table of market data suggest a linear relationship between price and number of homes sold during a one-year period? Explain.

Price (thousands of \$)	180	195	220	240
No. of homes sold	127	118	103	91

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{118-127}{195-180} = -\frac{9}{15} = -\frac{3}{5},$$

while the second pair of data points yields a slope of

$$\frac{103-118}{220-195} = -\frac{15}{25} = -\frac{3}{5}$$

and the last pair of data points yields a slope of

$$\frac{91-103}{240-220} = -\frac{12}{20} = -\frac{3}{5}.$$

Because all three slopes are equal, the data does suggest a linear relationship between price and the number of homes sold.

24. Does the following table of revenue data for a computer manufacturer suggest a linear relation between revenue and time? Explain.

Year	2001	2005	2007	2010
Revenue (billions of \$)	13	18	15	11

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{18-13}{2005-2001}=\frac{5}{4},$$

while the second pair of data points yields a slope of

$$\frac{15-18}{2007-2005}=-\frac{3}{2}$$

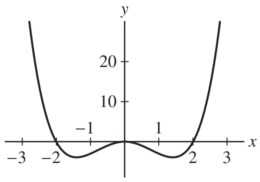
and the last pair of data points yields a slope of

$$\frac{11-15}{2010-2007}=-\frac{4}{3}.$$

Because the three slopes are not equal, the data does not suggest a linear relationship between revenue and time.

25. Find the roots of $f(x)=x^4-4x^2$ and sketch its graph. On which intervals is $f(x)$ decreasing?

SOLUTION The roots of $f(x)=x^4-4x^2$ are obtained by solving the equation $x^4-4x^2=x^2(x-2)(x+2)=0$, which yields $x=-2$, $x=0$ and $x=2$. The graph of $y=f(x)$ is shown below. From this graph we see that $f(x)$ is decreasing for x less than approximately -1.4 and for x between 0 and approximately 1.4.



26. Let $h(z)=2z^2+12z+3$. Complete the square and find the minimum value of $h(z)$.

SOLUTION Let $h(z)=2z^2+12z+3$. Completing the square yields

$$h(z)=2(z^2+6z)+3=2(z^2+6z+9)+3-18=2(z+3)^2-15.$$

Because $(z+3)^2\geq 0$ for all z, it follows that $h(z)=2(z+3)^2-15\geq -15$ for all z. Thus, the minimum value of $h(z)$ is -15.

27. Let $f(x)$ be the square of the distance from the point $(2,1)$ to a point $(x,3x+2)$ on the line $y=3x+2$. Show that $f(x)$ is a quadratic function, and find its minimum value by completing the square.

SOLUTION Let $f(x)$ denote the square of the distance from the point $(2,1)$ to a point $(x,3x+2)$ on the line $y=3x+2$. Then

$$f(x)=(x-2)^2+(3x+2-1)^2=x^2-4x+4+9x^2+6x+1=10x^2+2x+5,$$

which is a quadratic function. Completing the square, we find

$$f(x)=10\left(x^2+\frac{1}{5}x+\frac{1}{100}\right)+5-\frac{1}{10}=10\left(x+\frac{1}{10}\right)^2+\frac{49}{10}.$$

Because $(x+\frac{1}{10})^2\geq 0$ for all x, it follows that $f(x)\geq\frac{49}{10}$ for all x. Hence, the minimum value of $f(x)$ is $\frac{49}{10}$.

28. Prove that $x^2+3x+3\geq 0$ for all x.

SOLUTION Observe that

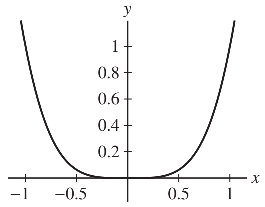
$$x^2+3x+3=\left(x^2+3x+\frac{9}{4}\right)+3-\frac{9}{4}=\left(x+\frac{3}{2}\right)^2+\frac{3}{4}.$$

Thus, $x^2+3x+3\geq\frac{3}{4}>0$ for all x.

In Exercises 29–34, sketch the graph by hand.

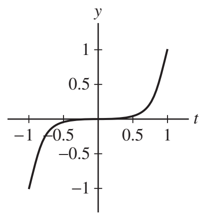
29. $y = t^4$

SOLUTION



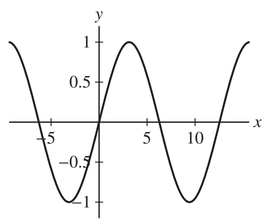
30. $y = t^5$

SOLUTION



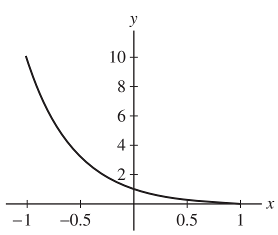
31. $y = \sin \dfrac{\theta}{2}$

SOLUTION



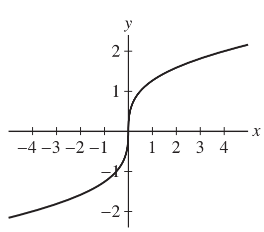
32. $y = 10^{-x}$

SOLUTION



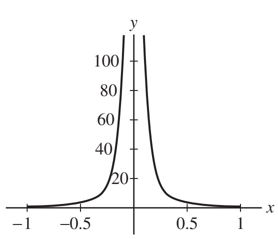
33. $y = x^{1/3}$

SOLUTION



34. $y = \dfrac{1}{x^2}$

SOLUTION



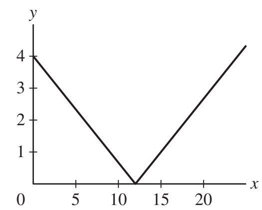
35. Show that the graph of $y = f\left(\frac{1}{3}x - b\right)$ is obtained by shifting the graph of $y = f\left(\frac{1}{3}x\right)$ to the right $3b$ units. Use this observation to sketch the graph of $y = \left|\frac{1}{3}x - 4\right|$.

SOLUTION Let $g(x) = f(\frac{1}{3}x)$. Then

$$g(x - 3b) = f\left(\frac{1}{3}(x - 3b)\right) = f\left(\frac{1}{3}x - b\right).$$

Thus, the graph of $y = f(\frac{1}{3}x - b)$ is obtained by shifting the graph of $y = f(\frac{1}{3}x)$ to the right $3b$ units.

The graph of $y = |\frac{1}{3}x - 4|$ is the graph of $y = |\frac{1}{3}x|$ shifted right 12 units (see the graph below).



36. Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Compute the composite functions $h(g(x))$ and $g(h(x))$, and find their domains.

SOLUTION Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Then

$$h(g(x)) = h(x^{-1}) = \cos x^{-1}.$$

The domain of this function is $x \neq 0$. On the other hand,

$$g(h(x)) = g(\cos x) = \frac{1}{\cos x} = \sec x.$$

The domain of this function is

$$x \neq \frac{(2n+1)\pi}{2} \text{ for any integer } n.$$

37. Find functions f and g such that the function

$$f(g(t)) = (12t + 9)^4$$

SOLUTION One possible choice is $f(t) = t^4$ and $g(t) = 12t + 9$. Then

$$f(g(t)) = f(12t + 9) = (12t + 9)^4$$

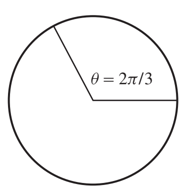
as desired.

38. Sketch the points on the unit circle corresponding to the following three angles, and find the values of the six standard trigonometric functions at each angle:

(a) $\frac{2\pi}{3}$ **(b)** $\frac{7\pi}{4}$ **(c)** $\frac{7\pi}{6}$

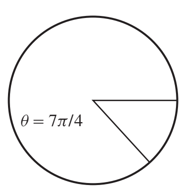
SOLUTION

(a)



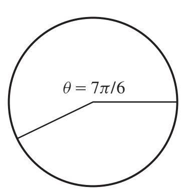
$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}$$
$$\tan\frac{2\pi}{3} = -\sqrt{3} \qquad \cot\frac{2\pi}{3} = -\frac{\sqrt{3}}{3}$$
$$\sec\frac{2\pi}{3} = -2 \qquad \csc\frac{2\pi}{3} = \frac{2\sqrt{3}}{3}$$

(b)



$$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$$
$$\tan\frac{7\pi}{4} = -1 \qquad \cot\frac{7\pi}{4} = -1$$
$$\sec\frac{7\pi}{4} = \sqrt{2} \qquad \csc\frac{7\pi}{4} = -\sqrt{2}$$

(c)



$$\sin\frac{7\pi}{6} = -\frac{1}{2} \qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3} \qquad \cot\frac{7\pi}{6} = \sqrt{3}$$
$$\sec\frac{7\pi}{6} = -\frac{2\sqrt{3}}{2} \qquad \csc\frac{7\pi}{6} = -2$$

39. What is the period of the function $g(\theta) = \sin 2\theta + \sin \frac{\theta}{2}$?

SOLUTION The function $\sin 2\theta$ has a period of π, and the function $\sin(\theta/2)$ has a period of 4π. Because 4π is a multiple of π, the period of the function $g(\theta) = \sin 2\theta + \sin \theta/2$ is 4π.

40. Assume that $\sin\theta = \frac{4}{5}$, where $\pi/2 < \theta < \pi$. Find:

(a) $\tan\theta$ **(b)** $\sin 2\theta$ **(c)** $\csc\dfrac{\theta}{2}$

SOLUTION If $\sin\theta = 4/5$, then by the fundamental trigonometric identity,

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}.$$

Because $\pi/2 < \theta < \pi$, it follows that $\cos\theta$ must be negative. Hence, $\cos\theta = -3/5$.

(a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{4/5}{-3/5} = -\dfrac{4}{3}$.

(b) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\dfrac{4}{5}\cdot -\dfrac{3}{5} = -\dfrac{24}{25}$.

(c) We first note that

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-3/5)}{2}} = 2\frac{\sqrt{5}}{5}.$$

Thus,

$$\csc\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{2}.$$

41. Give an example of values a, b such that

(a) $\cos(a+b) \neq \cos a + \cos b$ **(b)** $\cos\dfrac{a}{2} \neq \dfrac{\cos a}{2}$

SOLUTION

(a) Take $a = b = \pi/2$. Then $\cos(a+b) = \cos\pi = -1$ but

$$\cos a + \cos b = \cos\frac{\pi}{2} + \cos\frac{\pi}{2} = 0 + 0 = 0.$$

(b) Take $a = \pi$. Then

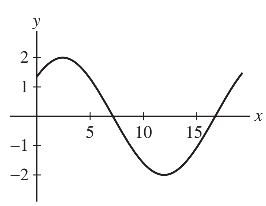
$$\cos\left(\frac{a}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

but

$$\frac{\cos a}{2} = \frac{\cos\pi}{2} = \frac{-1}{2} = -\frac{1}{2}.$$

42. Let $f(x) = \cos x$. Sketch the graph of $y = 2f\left(\frac{1}{3}x - \frac{\pi}{4}\right)$ for $0 \le x \le 6\pi$.

SOLUTION



43. Solve $\sin 2x + \cos x = 0$ for $0 \le x < 2\pi$.

SOLUTION Using the double angle formula for the sine function, we rewrite the equation as $2\sin x\cos x + \cos x = \cos x(2\sin x + 1) = 0$. Thus, either $\cos x = 0$ or $\sin x = -1/2$. From here we see that the solutions are $x = \pi/2$, $x = 7\pi/6$, $x = 3\pi/2$ and $x = 11\pi/6$.

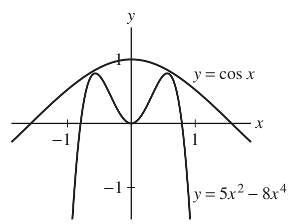
44. How does $h(n) = n^2/2^n$ behave for large whole-number values of n? Does $h(n)$ tend to infinity?

SOLUTION The table below suggests that for large whole number values of n, $h(n) = \dfrac{n^2}{2^n}$ tends toward 0.

n	$h(n) = n^2/2^n$
10	0.09765625000
10^2	$7.888609052 \times 10^{-27}$
10^3	$9.332636185 \times 10^{-296}$
10^4	$5.012372749 \times 10^{-3003}$
10^5	$1.000998904 \times 10^{-30093}$
10^6	$1.010034059 \times 10^{-301018}$

45. GU Use a graphing calculator to determine whether the equation $\cos x = 5x^2 - 8x^4$ has any solutions.

SOLUTION The graphs of $y = \cos x$ and $y = 5x^2 - 8x^4$ are shown below. Because the graphs do not intersect, there are no solutions to the equation $\cos x = 5x^2 - 8x^4$.

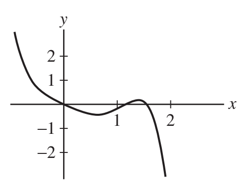


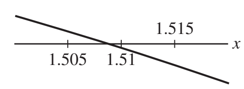
46. GU Using a graphing calculator, find the number of real roots and estimate the largest root to two decimal places:

(a) $f(x) = 1.8x^4 - x^5 - x$
(b) $g(x) = 1.7x^4 - x^5 - x$

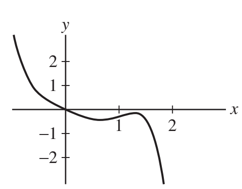
SOLUTION

(a) The graph of $y = 1.8x^4 - x^5 - x$ is shown below at the left. Because the graph has three x-intercepts, the function $f(x) = 1.8x^4 - x^5 - x$ has three real roots. From the graph shown below at the right, we see that the largest root of $f(x) = 1.8x^4 - x^5 - x$ is approximately $x = 1.51$.





(b) The graph of $y = 1.7x^4 - x^5 - x$ is shown below. Because the graph has only one x-intercept, the function $f(x) = 1.7x^4 - x^5 - x$ has only one real root. From the graph, we see that the largest root of $f(x) = 1.7x^4 - x^5 - x$ is approximately $x = 0$.



47. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.

(a) $2^a 3^b$
(b) $\dfrac{2^a}{3^b}$
(c) $(2^a)^b$
(d) $2^{a-b} 3^{b-a}$

(i) 2^{ab}
(ii) 6^{a+b}
(iii) $\left(\frac{2}{3}\right)^{a-b}$

SOLUTION

(a) No match.
(b) No match.
(c) **(i)**: $(2^a)^b = 2^{ab}$.
(d) **(iii)**: $2^{a-b}3^{b-a} = 2^{a-b}\left(\dfrac{1}{3}\right)^{a-b} = \left(\dfrac{2}{3}\right)^{a-b}$.

48. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.

(a) $\ln\left(\frac{a}{b}\right)$ **(b)** $\frac{\ln a}{\ln b}$

(c) $e^{\ln a - \ln b}$ **(d)** $(\ln a)(\ln b)$

(i) $\ln a + \ln b$ **(ii)** $\ln a - \ln b$ **(iii)** $\frac{a}{b}$

SOLUTION

(a) **(ii)**: $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

(b) No match.

(c) **(iii)**: $e^{\ln a - \ln b} = e^{\ln a}\frac{1}{e^{\ln b}} = \frac{a}{b}$.

(d) No match.

49. Find the inverse of $f(x) = \sqrt{x^3 - 8}$ and determine its domain and range.

SOLUTION To find the inverse of $f(x) = \sqrt{x^3 - 8}$, we solve $y = \sqrt{x^3 - 8}$ for x as follows:

$$y^2 = x^3 - 8$$
$$x^3 = y^2 + 8$$
$$x = \sqrt[3]{y^2 + 8}.$$

Therefore, $f^{-1}(x) = \sqrt[3]{x^2 + 8}$. The domain of f^{-1} is the range of f, namely $\{x : x \geq 0\}$; the range of f^{-1} is the domain of f, namely $\{y : y \geq 2\}$.

50. Find the inverse of $f(x) = \frac{x-2}{x-1}$ and determine its domain and range.

SOLUTION To find the inverse of $f(x) = \frac{x-2}{x-1}$, we solve $y = \frac{x-2}{x-1}$ for x as follows:

$$x - 2 = y(x-1) = yx - y$$
$$x - yx = 2 - y$$
$$x = \frac{2-y}{1-y}.$$

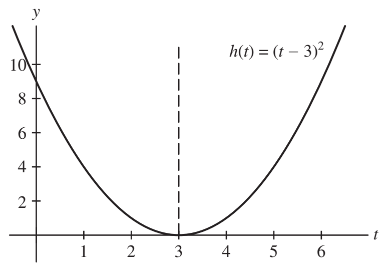
Therefore,

$$f^{-1}(x) = \frac{2-x}{1-x} = \frac{x-2}{x-1}.$$

The domain of f^{-1} is the range of f, namely $\{x : x \neq 1\}$; the range of f^{-1} is the domain of f, namely $\{y : y \neq 1\}$.

51. Find a domain on which $h(t) = (t-3)^2$ is one-to-one and determine the inverse on this domain.

SOLUTION From the graph of $h(t) = (t-3)^2$ shown below, we see that h is one-to-one on each of the intervals $t \geq 3$ and $t \leq 3$.



We find the inverse of $h(t) = (t-3)^2$ on the domain $\{t : t \leq 3\}$ by solving $y = (t-3)^2$ for t. First, we find

$$\sqrt{y} = \sqrt{(t-3)^2} = |t-3|.$$

Having restricted the domain to $\{t : t \leq 3\}$, $|t-3| = -(t-3) = 3-t$. Thus,

$$\sqrt{y} = 3 - t$$
$$t = 3 - \sqrt{y}.$$

The inverse function is $h^{-1}(t) = 3 - \sqrt{t}$. For $t \geq 3$, $h^{-1}(t) = 3 + \sqrt{t}$.

52. Show that $g(x) = \dfrac{x}{x-1}$ is equal to its inverse on the domain $\{x : x \neq 1\}$.

SOLUTION To show that $g(x) = \frac{x}{x-1}$ is equal to its inverse, we need to show that for $x \neq 1$,

$$g\left(g(x)\right) = x.$$

First, we notice that for $x \neq 1$, $g(x) \neq 1$. Therefore,

$$g\left(g(x)\right) = g\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x-(x-1)} = \frac{x}{1} = x.$$

53. Suppose that $g(x)$ is the inverse of $f(x)$. Match the functions (a)–(d) with their inverses (i)–(iv).

(a) $f(x) + 1$ **(b)** $f(x+1)$ **(c)** $4f(x)$ **(d)** $f(4x)$

(i) $g(x)/4$ **(ii)** $g(x/4)$ **(iii)** $g(x-1)$ **(iv)** $g(x) - 1$

SOLUTION

(a) **(iii)**: $f(x) + 1$ and $g(x-1)$ are inverse functions:

$$f(g(x-1)) + 1 = (x-1) + 1 = x;$$
$$g(f(x) + 1 - 1) = g(f(x)) = x.$$

(b) **(iv)**: $f(x+1)$ and $g(x) - 1$ are inverse functions:

$$f(g(x) - 1 + 1) = f(g(x)) = x;$$
$$g(f(x+1)) - 1 = (x+1) - 1 = x.$$

(c) **(ii)**: $4f(x)$ and $g(x/4)$ are inverse functions:

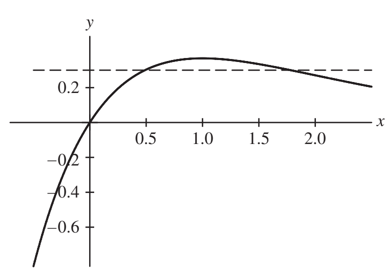
$$4f(g(x/4)) = 4(x/4) = x;$$
$$g(4f(x)/4) = g(f(x)) = x.$$

(d) **(i)**: $f(4x)$ and $g(x)/4$ are inverse functions:

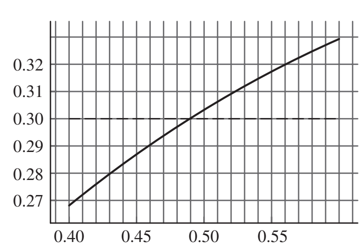
$$f(4 \cdot g(x)/4) = f(g(x)) = x;$$
$$\frac{1}{4}g(f(4x)) = \frac{1}{4}(4x) = x.$$

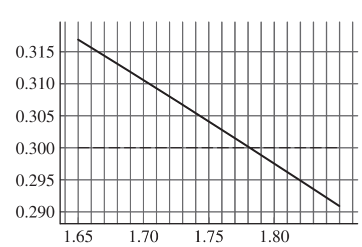
54. $\boxed{\text{GU}}$ Plot $f(x) = xe^{-x}$ and use the zoom feature to find two solutions of $f(x) = 0.3$.

SOLUTION The graph of $f(x) = xe^{-x}$ is shown below. Based on this graph, we should zoom in near $x = 0.5$ and near $x = 1.75$ to find solutions of $f(x) = 0.3$.



From the figure below at the left, we see that one solution of $f(x) = 0.3$ is approximately $x = 0.49$; from the figure below at the right, we see that a second solution of $f(x) = 0.3$ is approximately $x = 1.78$.





2 LIMITS

2.1 Limits, Rates of Change, and Tangent Lines

Preliminary Questions

1. Average velocity is equal to the slope of a secant line through two points on a graph. Which graph?

SOLUTION Average velocity is the slope of a secant line through two points on the graph of position as a function of time.

2. Can instantaneous velocity be defined as a ratio? If not, how is instantaneous velocity computed?

SOLUTION Instantaneous velocity cannot be defined as a ratio. It is defined as the limit of average velocity as time elapsed shrinks to zero.

3. What is the graphical interpretation of instantaneous velocity at a moment $t = t_0$?

SOLUTION Instantaneous velocity at time $t = t_0$ is the slope of the line tangent to the graph of position as a function of time at $t = t_0$.

4. What is the graphical interpretation of the following statement? The average rate of change approaches the instantaneous rate of change as the interval $[x_0, x_1]$ shrinks to x_0.

SOLUTION The slope of the secant line over the interval $[x_0, x_1]$ approaches the slope of the tangent line at $x = x_0$.

5. The rate of change of atmospheric temperature with respect to altitude is equal to the slope of the tangent line to a graph. Which graph? What are possible units for this rate?

SOLUTION The rate of change of atmospheric temperature with respect to altitude is the slope of the line tangent to the graph of atmospheric temperature as a function of altitude. Possible units for this rate of change are °F/ft or °C/m.

Exercises

1. A ball dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds.

(a) How far does the ball travel during the time interval [2, 2.5]?

(b) Compute the average velocity over [2, 2.5].

(c) Compute the average velocity for the time intervals in the table and estimate the ball's instantaneous velocity at $t = 2$.

Interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
Average velocity				

SOLUTION

(a) During the time interval [2, 2.5], the ball travels $\Delta s = s(2.5) - s(2) = 4.9(2.5)^2 - 4.9(2)^2 = 11.025$ m.

(b) The average velocity over [2, 2.5] is

$$\frac{\Delta s}{\Delta t} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{11.025}{0.5} = 22.05 \text{ m/s}.$$

(c)

time interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
average velocity	19.649	19.6245	19.6049	19.600049

The instantaneous velocity at $t = 2$ is 19.6 m/s.

2. A wrench released from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds. Estimate the instantaneous velocity at $t = 3$.

SOLUTION To find the instantaneous velocity, we compute the average velocities:

time interval	[3, 3.01]	[3, 3.005]	[3, 3.001]	[3, 3.00001]
average velocity	29.449	29.4245	29.4049	29.400049

The instantaneous velocity is approximately 29.4 m/s.

3. Let $v = 20\sqrt{T}$ as in Example 2. Estimate the instantaneous rate of change of v with respect to T when $T = 300\,\mathrm{K}$.

SOLUTION

T interval	[300, 300.01]	[300, 300.005]
average rate of change	0.577345	0.577348
T interval	[300, 300.001]	[300, 300.00001]
average rate of change	0.57735	0.57735

The instantaneous rate of change is approximately $0.57735\ \mathrm{m/(s \cdot K)}$.

4. Compute $\Delta y/\Delta x$ for the interval [2, 5], where $y = 4x - 9$. What is the instantaneous rate of change of y with respect to x at $x = 2$?

SOLUTION $\Delta y/\Delta x = ((4(5) - 9) - (4(2) - 9))/(5 - 2) = 4$. Because the graph of $y = 4x - 9$ is a line with slope 4, the average rate of change of y calculated over any interval will be equal to 4; hence, the instantaneous rate of change at any x will also be equal to 4.

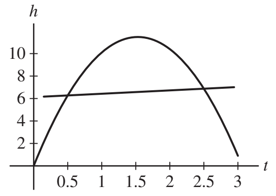
In Exercises 5 and 6, a stone is tossed vertically into the air from ground level with an initial velocity of 15 m/s. *Its height at time t is* $h(t) = 15t - 4.9t^2$ m.

5. Compute the stone's average velocity over the time interval [0.5, 2.5] and indicate the corresponding secant line on a sketch of the graph of $h(t)$.

SOLUTION The average velocity is equal to

$$\frac{h(2.5) - h(0.5)}{2} = 0.3.$$

The secant line is plotted with $h(t)$ below.



6. Compute the stone's average velocity over the time intervals [1, 1.01], [1, 1.001], [1, 1.0001] and [0.99, 1], [0.999, 1], [0.9999, 1], and then estimate the instantaneous velocity at $t = 1$.

SOLUTION With $h(t) = 15t - 4.9t^2$, the average velocity over the time interval $[t_1, t_2]$ is given by

$$\frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(t_1)}{t_2 - t_1}.$$

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average velocity	5.151	5.1951	5.1995	5.249	5.2049	5.2005

The instantaneous velocity at $t = 1$ second is 5.2 m/s.

7. With an initial deposit of \$100, the balance in a bank account after t years is $f(t) = 100(1.08)^t$ dollars.

(a) What are the units of the rate of change of $f(t)$?

(b) Find the average rate of change over [0, 0.5] and [0, 1].

(c) Estimate the instantaneous rate of change at $t = 0.5$ by computing the average rate of change over intervals to the left and right of $t = 0.5$.

SOLUTION

(a) The units of the rate of change of $f(t)$ are dollars/year or \$/yr.

(b) The average rate of change of $f(t) = 100(1.08)^t$ over the time interval $[t_1, t_2]$ is given by

$$\frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

time interval	[0, .5]	[0, 1]
average rate of change	7.8461	8

(c)

time interval	[0.5, 0.51]	[0.5, 0.501]	[0.5, 0.5001]
average rate of change	8.0011	7.9983	7.9981
time interval	[0.49, 0.5]	[0.499, 0.5]	[0.4999, 0.5]
average rate of change	7.9949	7.9977	7.998

The rate of change at $t = 0.5$ is approximately \$8/yr.

8. The position of a particle at time t is $s(t) = t^3 + t$. Compute the average velocity over the time interval [1, 4] and estimate the instantaneous velocity at $t = 1$.

SOLUTION The average velocity over the time interval [1, 4] is

$$\frac{s(4) - s(1)}{4 - 1} = \frac{68 - 2}{3} = 22.$$

To estimate the instantaneous velocity at $t = 1$, we examine the following table.

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average rate of change	4.0301	4.0030	4.0003	3.9701	3.9970	3.9997

The rate of change at $t = 1$ is approximately 4.

9. Figure 8 shows the estimated number N of Internet users in Chile, based on data from the United Nations Statistics Division.

(a) Estimate the rate of change of N at $t = 2003.5$.

(b) Does the rate of change increase or decrease as t increases? Explain graphically.

(c) Let R be the average rate of change over [2001, 2005]. Compute R.

(d) Is the rate of change at $t = 2002$ greater than or less than the average rate R? Explain graphically.

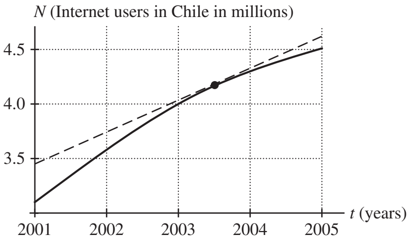


FIGURE 8

SOLUTION

(a) The tangent line shown in Figure 8 appears to pass through the points (2002, 3.75) and (2005, 4.6). Thus, the rate of change of N at $t = 2003.5$ is approximately

$$\frac{4.6 - 3.75}{2005 - 2002} = 0.283$$

million Internet users per year.

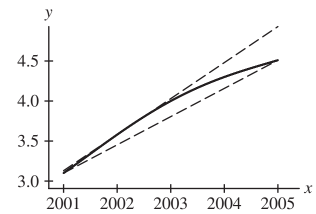
(b) As t increases, we move from left to right along the graph in Figure 8. Moreover, as we move from left to right along the graph, the slope of the tangent line decreases. Thus, the rate of change decreases as t increases.

(c) The graph of $N(t)$ appear to pass through the points (2001, 3.1) and (2005, 4.5). Thus, the average rate of change over [2001, 2005] is approximately

$$R = \frac{4.5 - 3.1}{2005 - 2001} = 0.35$$

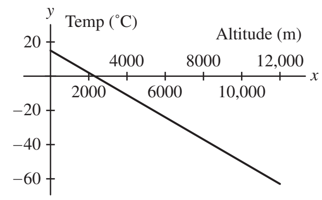
million Internet users per year.

(d) For the figure below, we see that the slope of the tangent line at $t = 2002$ is larger than the slope of the secant line through the endpoints of the graph of $N(t)$. Thus, the rate of change at $t = 2002$ is greater than the average rate of change R.



10. The **atmospheric temperature** T (in °C) at altitude h meters above a certain point on earth is $T = 15 - 0.0065h$ for $h \le 12{,}000$ m. What are the average and instantaneous rates of change of T with respect to h? Why are they the same? Sketch the graph of T for $h \le 12{,}000$.

SOLUTION The average and instantaneous rates of change of T with respect to h are both -0.0065°C/m. The rates of change are the same because T is a linear function of h with slope -0.0065.



In Exercises 11–18, estimate the instantaneous rate of change at the point indicated.

11. $P(x) = 3x^2 - 5; \quad x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
average rate of change	12.03	12.003	12.0003	11.97	11.997	11.9997

The rate of change at $x = 2$ is approximately 12.

12. $f(t) = 12t - 7; \quad t = -4$

SOLUTION

t interval	[−4, −3.99]	[−4, −3.999]	[−4, −3.9999]
average rate of change	12	12	12
t interval	[−4.01, −4]	[−4.001, −4]	[−4.0001, −4]
average rate of change	12	12	12

The rate of change at $t = -4$ is 12, as the graph of $y = f(t)$ is a line with slope 12.

13. $y(x) = \dfrac{1}{x+2}; \quad x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
average rate of change	−0.0623	−0.0625	−0.0625	−0.0627	−0.0625	−0.0625

The rate of change at $x = 2$ is approximately -0.06.

14. $y(t) = \sqrt{3t+1}; \quad t = 1$

SOLUTION

t interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average rate of change	0.7486	0.7499	0.7500	0.7514	0.7501	0.7500

The rate of change at $t = 1$ is approximately 0.75.

15. $f(x) = e^x; \quad x = 0$

SOLUTION

x interval	$[-0.01, 0]$	$[-0.001, 0]$	$[-0.0001, 0]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
average rate of change	0.9950	0.9995	0.99995	1.0050	1.0005	1.00005

The rate of change at $x = 0$ is approximately 1.00.

16. $f(x) = e^x; \quad x = e$

SOLUTION

x interval	$[e - 0.01, e]$	$[e - 0.001, e]$	$[e - 0.0001, e]$	$[e, e + 0.01]$	$[e, e + 0.001]$	$[e, e + 0.0001]$
average rate of change	15.0787	15.1467	15.1535	15.2303	15.1618	15.1550

The rate of change at $x = e$ is approximately 15.15.

17. $f(x) = \ln x; \quad x = 3$

SOLUTION

x interval	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
average rate of change	0.33389	0.33339	0.33334	0.33278	0.33328	0.33333

The rate of change at $x = 3$ is approximately 0.333.

18. $f(x) = \tan^{-1} x; \quad x = \frac{\pi}{4}$

SOLUTION

x interval	$\left[\frac{\pi}{4} - 0.01, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4} - 0.001, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4} - 0.0001, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.01\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.001\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.0001\right]$
average rate of change	0.6215	0.6188	0.6185	0.6155	0.6182	0.6185

The rate of change at $x = \frac{\pi}{4}$ is approximately 0.619.

19. The height (in centimeters) at time t (in seconds) of a small mass oscillating at the end of a spring is $h(t) = 8\cos(12\pi t)$.

(a) Calculate the mass's average velocity over the time intervals $[0, 0.1]$ and $[3, 3.5]$.

(b) Estimate its instantaneous velocity at $t = 3$.

SOLUTION

(a) The average velocity over the time interval $[t_1, t_2]$ is given by $\dfrac{\Delta h}{\Delta t} = \dfrac{h(t_2) - h(t_1)}{t_2 - t_1}$.

time interval	$[0, 0.1]$	$[3, 3.5]$
average velocity	-144.721 cm/s	0 cm/s

(b)

time interval	$[3, 3.0001]$	$[3, 3.00001]$	$[3, 3.000001]$	$[2.9999, 3]$	$[2.99999, 3]$	$[2.999999, 3]$
average velocity	-0.5685	-0.05685	-0.005685	0.5685	0.05685	0.005685

The instantaneous velocity at $t = 3$ seconds is approximately 0 cm/s.

20. The number $P(t)$ of *E. coli* cells at time t (hours) in a petri dish is plotted in Figure 9.

(a) Calculate the average rate of change of $P(t)$ over the time interval $[1, 3]$ and draw the corresponding secant line.

(b) Estimate the slope m of the line in Figure 9. What does m represent?

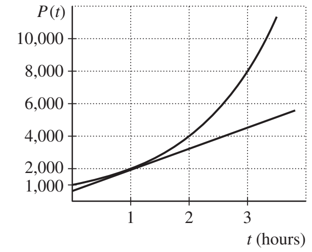


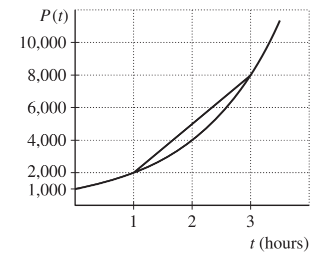
FIGURE 9 Number of *E. coli* cells at time t.

SOLUTION

(a) Looking at the graph, we can estimate $P(1) = 2000$ and $P(3) = 8000$. Assuming these values of $P(t)$, the average rate of change is

$$\frac{P(3) - P(1)}{3 - 1} = \frac{6000}{2} = 3000 \text{ cells/hour.}$$

The secant line is here:



(b) The line in Figure 9 goes through two points with approximate coordinates (1, 2000) and (2.5, 4000). This line has approximate slope

$$m = \frac{4000 - 2000}{2.5 - 1} = \frac{4000}{3} \text{ cells/hour.}$$

m is close to the slope of the line tangent to the graph of $P(t)$ at $t = 1$, and so m represents the instantaneous rate of change of $P(t)$ at $t = 1$ hour.

21. Assume that the period T (in seconds) of a pendulum (the time required for a complete back-and-forth cycle) is $T = \frac{3}{2}\sqrt{L}$, where L is the pendulum's length (in meters).

(a) What are the units for the rate of change of T with respect to L? Explain what this rate measures.

(b) Which quantities are represented by the slopes of lines A and B in Figure 10?

(c) Estimate the instantaneous rate of change of T with respect to L when $L = 3$ m.

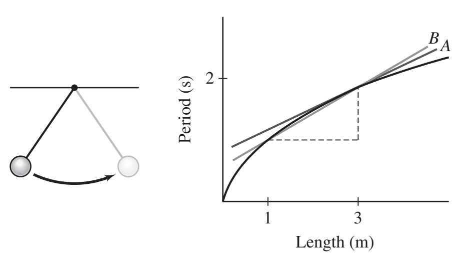


FIGURE 10 The period T is the time required for a pendulum to swing back and forth.

SOLUTION

(a) The units for the rate of change of T with respect to L are seconds per meter. This rate measures the sensitivity of the period of the pendulum to a change in the length of the pendulum.

(b) The slope of the line B represents the average rate of change in T from $L = 1$ m to $L = 3$ m. The slope of the line A represents the instantaneous rate of change of T at $L = 3$ m.

(c)

time interval	[3, 3.01]	[3, 3.001]	[3, 3.0001]	[2.99, 3]	[2.999, 3]	[2.9999, 3]
average velocity	0.4327	0.4330	0.4330	0.4334	0.4330	0.4330

The instantaneous rate of change at $L = 1$ m is approximately 0.4330 s/m.

22. The graphs in Figure 11 represent the positions of moving particles as functions of time.

(a) Do the instantaneous velocities at times t_1, t_2, t_3 in (A) form an increasing or a decreasing sequence?

(b) Is the particle speeding up or slowing down in (A)?

(c) Is the particle speeding up or slowing down in (B)?

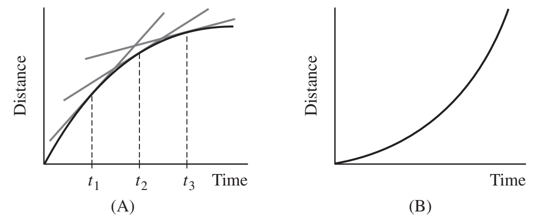


FIGURE 11

SOLUTION

(a) As the value of the independent variable increases, we note that the slope of the tangent lines decreases. Since Figure 11(A) displays position as a function of time, the slope of each tangent line is equal to the velocity of the particle; consequently, the velocities at t_1, t_2, t_3 form a decreasing sequence.

(b) Based on the solution to part (a), the velocity of the particle is decreasing; hence, the particle is slowing down.

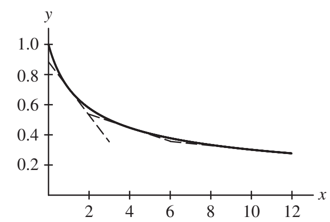
(c) If we were to draw several lines tangent to the graph in Figure 11(B), we would find that the slopes would be increasing. Accordingly, the velocity of the particle associated with Figure 11(B) is increasing, and the particle is speeding up.

23. GU An advertising campaign boosted sales of Crunchy Crust frozen pizza to a peak level of S_0 dollars per month. A marketing study showed that after t months, monthly sales declined to

$$S(t) = S_0 g(t), \quad \text{where } g(t) = \frac{1}{\sqrt{1+t}}.$$

Do sales decline more slowly or more rapidly as time increases? Answer by referring to a sketch the graph of $g(t)$ together with several tangent lines.

SOLUTION We notice from the figure below that, as time increases, the slopes of the tangent lines to the graph of $g(t)$ become less negative. Thus, sales decline more slowly as time increases.



24. The fraction of a city's population infected by a flu virus is plotted as a function of time (in weeks) in Figure 12.

(a) Which quantities are represented by the slopes of lines A and B? Estimate these slopes.

(b) Is the flu spreading more rapidly at $t = 1$, 2, or 3?

(c) Is the flu spreading more rapidly at $t = 4$, 5, or 6?

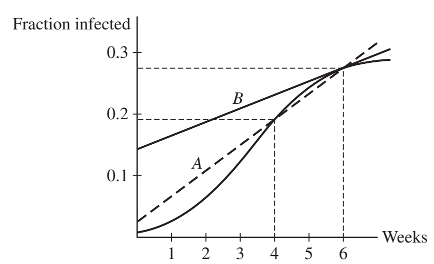


FIGURE 12

SOLUTION

(a) The slope of line A is the average rate of change over the interval $[4, 6]$, whereas the slope of the line B is the instantaneous rate of change at $t = 6$. Thus, the slope of the line $A \approx (0.28 - 0.19)/2 = 0.045$/week, whereas the slope of the line $B \approx (0.28 - 0.15)/6 = 0.0217$/week.

(b) Among times $t = 1, 2, 3$, the flu is spreading most rapidly at $t = 3$ since the slope is greatest at that instant; hence, the rate of change is greatest at that instant.

(c) Among times $t = 4, 5, 6$, the flu is spreading most rapidly at $t = 4$ since the slope is greatest at that instant; hence, the rate of change is greatest at that instant.

25. The graphs in Figure 13 represent the positions s of moving particles as functions of time t. Match each graph with a description:

(a) Speeding up

(b) Speeding up and then slowing down

(c) Slowing down

(d) Slowing down and then speeding up

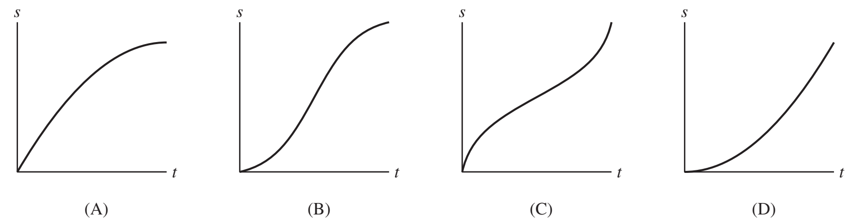


FIGURE 13

SOLUTION When a particle is speeding up over a time interval, its graph is bent upward over that interval. When a particle is slowing down, its graph is bent downward over that interval. Accordingly,

- In graph (A), the particle is (c) slowing down.
- In graph (B), the particle is (b) speeding up and then slowing down.
- In graph (C), the particle is (d) slowing down and then speeding up.
- In graph (D), the particle is (a) speeding up.

26. An epidemiologist finds that the percentage $N(t)$ of susceptible children who were infected on day t during the first three weeks of a measles outbreak is given, to a reasonable approximation, by the formula (Figure 14)

$$N(t) = \frac{100t^2}{t^3 + 5t^2 - 100t + 380}$$

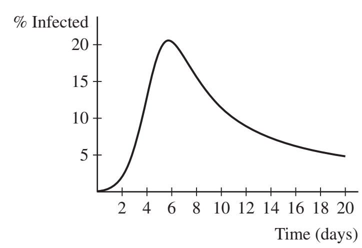


FIGURE 14 Graph of $N(t)$.

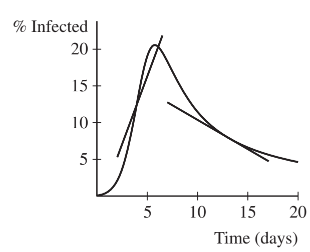
(a) Draw the secant line whose slope is the average rate of change in infected children over the intervals [4, 6] and [12, 14]. Then compute these average rates (in units of percent per day).

(b) Is the rate of decline greater at $t = 8$ or $t = 16$?

(c) Estimate the rate of change of $N(t)$ on day 12.

SOLUTION

(a)



The average rate of change of $N(t)$ over the interval between day 4 and day 6 is given by

$$\frac{\Delta N}{\Delta t} = \frac{N(6) - N(4)}{6 - 4} = 3.776\%/\text{day}.$$

Similarly, we calculate the average rate of change of $N(t)$ over the interval between day 12 and day 14 as

$$\frac{\Delta N}{\Delta t} = \frac{N(14) - N(12)}{14 - 12} = -0.7983\%/\text{day}.$$

(b) The slope of the tangent line at $t = 8$ would be more negative than the slope of the tangent line at $t = 16$. Thus, the rate of decline is greater at $t = 8$ than at $t = 16$.

(c)

time interval	[12, 12.5]	[12, 12.2]	[12, 12.01]	[12, 12.001]
average rate of change	-0.9288	-0.9598	-0.9805	-0.9815

time interval	[11.5, 12]	[11.8, 12]	[11.99, 12]	[11.999, 12]
average rate of change	-1.0402	-1.0043	-0.9827	-0.9817

The instantaneous rate of change of $N(t)$ on day 12 is $-0.9816\%/\text{day}$.

27. The fungus *Fusarium exosporium* infects a field of flax plants through the roots and causes the plants to wilt. Eventually, the entire field is infected. The percentage $f(t)$ of infected plants as a function of time t (in days) since planting is shown in Figure 15.

(a) What are the units of the rate of change of $f(t)$ with respect to t? What does this rate measure?

(b) Use the graph to rank (from smallest to largest) the average infection rates over the intervals [0, 12], [20, 32], and [40, 52].

(c) Use the following table to compute the average rates of infection over the intervals [30, 40], [40, 50], [30, 50].

Days	0	10	20	30	40	50	60
Percent infected	0	18	56	82	91	96	98

(d) Draw the tangent line at $t = 40$ and estimate its slope.

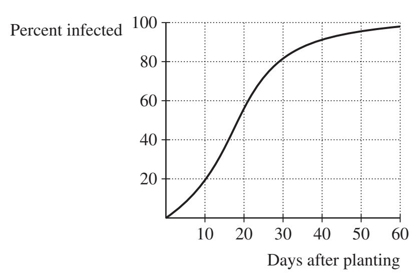


FIGURE 15

SOLUTION

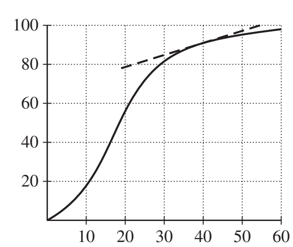
(a) The units of the rate of change of $f(t)$ with respect to t are percent /day or %/d. This rate measures how quickly the population of flax plants is becoming infected.

(b) From smallest to largest, the average rates of infection are those over the intervals [40, 52], [0, 12], [20, 32]. This is because the slopes of the secant lines over these intervals are arranged from smallest to largest.

(c) The average rates of infection over the intervals [30, 40], [40, 50], [30, 50] are 0.9, 0.5, 0.7 %/d, respectively.

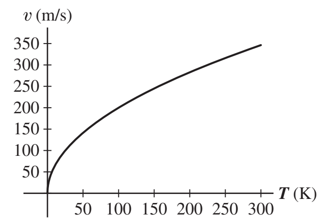
(d) The tangent line sketched in the graph below appears to pass through the points (20, 80) and (40, 91). The estimate of the instantaneous rate of infection at $t = 40$ days is therefore

$$\frac{91 - 80}{40 - 20} = \frac{11}{20} = 0.55\%/\text{d}.$$



28. Let $v = 20\sqrt{T}$ as in Example 2. Is the rate of change of v with respect to T greater at low temperatures or high temperatures? Explain in terms of the graph.

SOLUTION



As the graph progresses to the right, the graph bends progressively downward, meaning that the slope of the tangent lines becomes smaller. This means that the rate of change of v with respect to T is lower at high temperatures.

29. If an object in linear motion (but with changing velocity) covers Δs meters in Δt seconds, then its average velocity is $v_0 = \Delta s/\Delta t$ m/s. Show that it would cover the same distance if it traveled at constant velocity v_0 over the same time interval. This justifies our calling $\Delta s/\Delta t$ the *average velocity*.

SOLUTION At constant velocity, the distance traveled is equal to velocity times time, so an object moving at constant velocity v_0 for Δt seconds travels $v_0 \delta t$ meters. Since $v_0 = \Delta s/\Delta t$, we find

$$\text{distance traveled} = v_0 \delta t = \left(\frac{\Delta s}{\Delta t}\right) \Delta t = \Delta s$$

So the object covers the same distance Δs by traveling at constant velocity v_0.

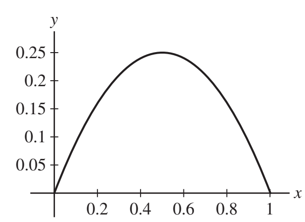
30. Sketch the graph of $f(x) = x(1-x)$ over [0, 1]. Refer to the graph and, without making any computations, find:

(a) The average rate of change over [0, 1]

(b) The (instantaneous) rate of change at $x = \frac{1}{2}$

(c) The values of x at which the rate of change is positive

SOLUTION



(a) $f(0) = f(1)$, so there is no change between $x = 0$ and $x = 1$. The average rate of change is zero.

(b) The tangent line to the graph of $f(x)$ is horizontal at $x = \frac{1}{2}$; the instantaneous rate of change is zero at this point.

(c) The rate of change is positive at all points where the graph is rising, because the slope of the tangent line is positive at these points. This is so for all x between $x = 0$ and $x = 0.5$.

31. Which graph in Figure 16 has the following property: For all x, the average rate of change over $[0, x]$ is greater than the instantaneous rate of change at x. Explain.

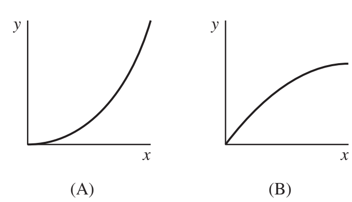


FIGURE 16

SOLUTION

(a) The average rate of change over $[0, x]$ is greater than the instantaneous rate of change at x: (B).

(b) The average rate of change over $[0, x]$ is less than the instantaneous rate of change at x: (A)

The graph in (B) bends downward, so the slope of the secant line through $(0, 0)$ and $(x, f(x))$ is larger than the slope of the tangent line at $(x, f(x))$. On the other hand, the graph in (A) bends upward, so the slope of the tangent line at $(x, f(x))$ is larger than the slope of the secant line through $(0, 0)$ and $(x, f(x))$.

Further Insights and Challenges

32. The height of a projectile fired in the air vertically with initial velocity 25 m/s is

$$h(t) = 25t - 4.9t^2 \text{ m}.$$

(a) Compute $h(1)$. Show that $h(t) - h(1)$ can be factored with $(t - 1)$ as a factor.

(b) Using part (a), show that the average velocity over the interval $[1, t]$ is $20.1 - 4.9t$.

(c) Use this formula to find the average velocity over several intervals $[1, t]$ with t close to 1. Then estimate the instantaneous velocity at time $t = 1$.

SOLUTION

(a) With $h(t) = 25t - 4.9t^2$, we have $h(1) = 20.1$ m, so

$$h(t) - h(1) = -4.9t^2 + 25t - 20.1.$$

Factoring the quadratic, we obtain

$$h(t) - h(1) = (t - 1)(-4.9t + 20.1).$$

(b) The average velocity over the interval $[1, t]$ is

$$\frac{h(t) - h(1)}{t - 1} = \frac{(t - 1)(-4.9t + 20.1)}{t - 1} = 20.1 - 4.9t.$$

(c)

t	1.01	1.001	1.0001	1.00001
average velocity over $[1, t]$	15.151	15.1951	15.19951	15.199951

The instantaneous velocity is approximately 15.2 m/s. Plugging $t = 1$ second into the formula in (b) yields $20.1 - 4.9(1) = 15.2$ m/s exactly.

33. Let $Q(t) = t^2$. As in the previous exercise, find a formula for the average rate of change of Q over the interval $[1, t]$ and use it to estimate the instantaneous rate of change at $t = 1$. Repeat for the interval $[2, t]$ and estimate the rate of change at $t = 2$.

SOLUTION The average rate of change is

$$\frac{Q(t) - Q(1)}{t - 1} = \frac{t^2 - 1}{t - 1}.$$

Applying the difference of squares formula gives that the average rate of change is $((t + 1)(t - 1))/(t - 1) = (t + 1)$ for $t \neq 1$. As t gets closer to 1, this gets closer to $1 + 1 = 2$. The instantaneous rate of change is 2.

For $t_0 = 2$, the average rate of change is

$$\frac{Q(t) - Q(2)}{t - 2} = \frac{t^2 - 4}{t - 2},$$

which simplifies to $t + 2$ for $t \neq 2$. As t approaches 2, the average rate of change approaches 4. The instantaneous rate of change is therefore 4.

34. Show that the average rate of change of $f(x) = x^3$ over $[1, x]$ is equal to

$$x^2 + x + 1.$$

Use this to estimate the instantaneous rate of change of $f(x)$ at $x = 1$.

SOLUTION The average rate of change is

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 1}{x - 1}.$$

Factoring the numerator as the difference of cubes means the average rate of change is

$$\frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$$

(for all $x \neq 1$). The closer x gets to 1, the closer the average rate of change gets to $1^2 + 1 + 1 = 3$. The instantaneous rate of change is 3.

35. Find a formula for the average rate of change of $f(x) = x^3$ over $[2, x]$ and use it to estimate the instantaneous rate of change at $x = 2$.

SOLUTION The average rate of change is

$$\frac{f(x) - f(2)}{x - 2} = \frac{x^3 - 8}{x - 2}.$$

Applying the difference of cubes formula to the numerator, we find that the average rate of change is

$$\frac{(x^2 + 2x + 4)(x - 2)}{x - 2} = x^2 + 2x + 4$$

for $x \neq 2$. The closer x gets to 2, the closer the average rate of change gets to $2^2 + 2(2) + 4 = 12$.

36. Let $T = \frac{3}{2}\sqrt{L}$ as in Exercise 21. The numbers in the second column of Table 1 are increasing, and those in the last column are decreasing. Explain why in terms of the graph of T as a function of L. Also, explain graphically why the instantaneous rate of change at $L = 3$ lies between 0.4329 and 0.4331.

TABLE 1 **Average Rates of Change of T with Respect to L**

Interval	Average rate of change	Interval	Average rate of change
[3, 3.2]	0.42603	[2.8, 3]	0.44048
[3, 3.1]	0.42946	[2.9, 3]	0.43668
[3, 3.001]	0.43298	[2.999, 3]	0.43305
[3, 3.0005]	0.43299	[2.9995, 3]	0.43303

SOLUTION Since the average rate of change is increasing on the intervals $[3, L]$ as L get close to 3, we know that the slopes of the secant lines between points on the graph over these intervals are increasing. The more rows we add with smaller intervals, the greater the average rate of change. This means that the instantaneous rate of change is probably greater than all of the numbers in this column.

Likewise, since the average rate of change is *decreasing* on the intervals $[L, 3]$ as L gets closer to 3, we know that the slopes of the secant lines between points over these intervals are decreasing. This means that the instantaneous rate of change is probably less than all the numbers in this column.

The tangent slope is somewhere between the greatest value in the first column and the least value in the second column. Hence, it is between 0.43299 and 0.43303. The first column underestimates the instantaneous rate of change by secant slopes; this estimate improves as L decreases toward $L = 3$. The second column overestimates the instantaneous rate of change by secant slopes; this estimate improves as L increases toward $L = 3$.

2.2 Limits: A Numerical and Graphical Approach

Preliminary Questions

1. What is the limit of $f(x) = 1$ as $x \to \pi$?

SOLUTION $\lim_{x\to\pi} 1 = 1$.

2. What is the limit of $g(t) = t$ as $t \to \pi$?

SOLUTION $\lim_{t\to\pi} t = \pi$.

3. Is $\lim\limits_{x\to 10} 20$ equal to 10 or 20?

SOLUTION $\lim_{x\to 10} 20 = 20$.

4. Can $f(x)$ approach a limit as $x \to c$ if $f(c)$ is undefined? If so, give an example.

SOLUTION Yes. The limit of a function f as $x \to c$ does not depend on what happens *at* $x = c$, only on the behavior of f as $x \to c$. As an example, consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The function is clearly not defined at $x = 1$ but

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

5. What does the following table suggest about $\lim_{x\to 1-} f(x)$ and $\lim_{x\to 1+} f(x)$?

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	7	25	4317	3.0126	3.0047	3.00011

SOLUTION The values in the table suggest that $\lim_{x\to 1-} f(x) = \infty$ and $\lim_{x\to 1+} f(x) = 3$.

6. Can you tell whether $\lim_{x\to 5} f(x)$ exists from a plot of $f(x)$ for $x > 5$? Explain.

SOLUTION No. By examining values of $f(x)$ for x close to but greater than 5, we can determine whether the one-sided limit $\lim_{x\to 5+} f(x)$ exists. To determine whether $\lim_{x\to 5} f(x)$ exists, we must examine value of $f(x)$ on both sides of $x = 5$.

7. If you know in advance that $\lim_{x\to 5} f(x)$ exists, can you determine its value from a plot of $f(x)$ for all $x > 5$?

SOLUTION Yes. If $\lim_{x\to 5} f(x)$ exists, then both one-sided limits must exist and be equal.

Exercises

In Exercises 1–4, fill in the tables and guess the value of the limit.

1. $\lim_{x\to 1} f(x)$, where $f(x) = \dfrac{x^3-1}{x^2-1}$.

x	$f(x)$	x	$f(x)$
1.002		0.998	
1.001		0.999	
1.0005		0.9995	
1.00001		0.99999	

SOLUTION

x	0.998	0.999	0.9995	0.99999	1.00001	1.0005	1.001	1.002
$f(x)$	1.498501	1.499250	1.499625	1.499993	1.500008	1.500375	1.500750	1.501500

The limit as $x \to 1$ is $\frac{3}{2}$.

2. $\lim_{t\to 0} h(t)$, where $h(t) = \dfrac{\cos t - 1}{t^2}$. Note that $h(t)$ is even; that is, $h(t) = h(-t)$.

t	± 0.002	± 0.0001	± 0.00005	± 0.00001
$h(t)$				

SOLUTION

t	± 0.002	± 0.0001
$h(t)$	-0.499999833333	-0.499999999583
t	± 0.00005	± 0.00001
$h(t)$	-0.499999999896	-0.500000000000

The limit as $t \to 0$ is $-\frac{1}{2}$.

3. $\lim_{y \to 2} f(y)$, where $f(y) = \dfrac{y^2 - y - 2}{y^2 + y - 6}$.

y	$f(y)$	y	$f(y)$
2.002		1.998	
2.001		1.999	
2.0001		1.9999	

SOLUTION

y	1.998	1.999	1.9999	2.0001	2.001	2.02
$f(y)$	0.59984	0.59992	0.599992	0.600008	0.60008	0.601594

The limit as $y \to 2$ is $\frac{3}{5}$.

4. $\lim_{x \to 0+} f(x)$, where $f(x) = x \ln x$.

x	1	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$							

SOLUTION

x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$	0	-0.34657	-0.23026	-0.14979	-0.04605	-0.02649	-0.00691

The limit as $x \to 0+$ is 0.

5. Determine $\lim_{x \to 0.5} f(x)$ for $f(x)$ as in Figure 9.

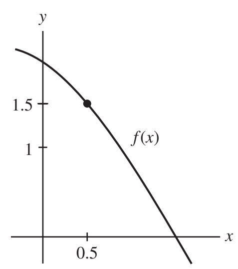


FIGURE 9

SOLUTION The graph suggests that $f(x) \to 1.5$ as $x \to 0.5$.

6. Determine $\lim_{x \to 0.5} g(x)$ for $g(x)$ as in Figure 10.

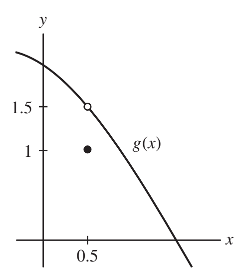


FIGURE 10

SOLUTION The graph suggests that $g(x) \to 1.5$ as $x \to 0.5$. The value $g(0.5)$, which happens to be 1, does not affect the limit.

In Exercises 7 and 8, evaluate the limit.

7. $\lim_{x \to 21} x$

SOLUTION As $x \to 21$, $f(x) = x \to 21$. You can see this, for example, on the graph of $f(x) = x$.

8. $\lim_{x \to 4.2} \sqrt{3}$

SOLUTION The graph of $f(x) = \sqrt{3}$ is a horizontal line. $f(x) = \sqrt{3}$ for all values of x, so the limit is also equal to $\sqrt{3}$.

In Exercises 9–16, verify each limit using the limit definition. For example, in Exercise 9, show that $|3x - 12|$ can be made as small as desired by taking x close to 4.

9. $\lim_{x \to 4} 3x = 12$

SOLUTION $|3x - 12| = 3|x - 4|$. $|3x - 12|$ can be made arbitrarily small by making x close enough to 4, thus making $|x - 4|$ small.

10. $\lim_{x \to 5} 3 = 3$

SOLUTION $|f(x) - 3| = |3 - 3| = 0$ for all values of x so $f(x) - 3$ is already smaller than any positive number as $x \to 5$.

11. $\lim_{x \to 3} (5x + 2) = 17$

SOLUTION $|(5x + 2) - 17| = |5x - 15| = 5|x - 3|$. Therefore, if you make $|x - 3|$ small enough, you can make $|(5x + 2) - 17|$ as small as desired.

12. $\lim_{x \to 2} (7x - 4) = 10$

SOLUTION As $x \to 2$, note that $|(7x - 4) - 10| = |7x - 14| = 7\,|x - 2|$. If you make $|x - 2|$ small enough, you can make $|(7x - 4) - 10|$ as small as desired.

13. $\lim_{x \to 0} x^2 = 0$

SOLUTION As $x \to 0$, we have $|x^2 - 0| = |x + 0||x - 0|$. To simplify things, suppose that $|x| < 1$, so that $|x + 0||x - 0| = |x||x| < |x|$. By making $|x|$ sufficiently small, so that $|x + 0||x - 0| = x^2$ is even smaller, you can make $|x^2 - 0|$ as small as desired.

14. $\lim_{x \to 0} (3x^2 - 9) = -9$

SOLUTION $|3x^2 - 9 - (-9)| = |3x^2| = 3|x^2|$. If you make $|x| < 1$, $|x^2| < |x|$, so that making $|x - 0|$ small enough can make $|3x^2 - 9 - (-9)|$ as small as desired.

15. $\lim_{x \to 0} (4x^2 + 2x + 5) = 5$

SOLUTION As $x \to 0$, we have $|4x^2 + 2x + 5 - 5| = |4x^2 + 2x| = |x||4x + 2|$. If $|x| < 1$, $|4x + 2|$ can be no bigger than 6, so $|x||4x + 2| < 6|x|$. Therefore, by making $|x - 0| = |x|$ sufficiently small, you can make $|4x^2 + 2x + 5 - 5| = |x||4x + 2|$ as small as desired.

16. $\lim_{x \to 0} (x^3 + 12) = 12$

SOLUTION $|(x^3 + 12) - 12| = |x^3|$. If we make $|x| < 1$, then $|x^3| < |x|$. Therefore, by making $|x - 0| = |x|$ sufficiently small, we can make $|(x^3 + 12) - 12|$ as small as desired.

In Exercises 17–36, estimate the limit numerically or state that the limit does not exist. If infinite, state whether the one-sided limits are ∞ or $-\infty$.

17. $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

SOLUTION

x	0.9995	0.99999	1.00001	1.0005
$f(x)$	0.500063	0.500001	0.49999	0.499938

The limit as $x \to 1$ is $\frac{1}{2}$.

18. $\lim_{x \to -4} \dfrac{2x^2 - 32}{x + 4}$

SOLUTION

x	-4.001	-4.0001	-3.9999	-3.999
$f(x)$	-16.002	-16.0002	-15.9998	-15.998

The limit as $x \to -4$ is -16.

19. $\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2}$

SOLUTION

x	1.999	1.99999	2.00001	2.001
$f(x)$	1.666889	1.666669	1.666664	1.666445

The limit as $x \to 2$ is $\frac{5}{3}$.

20. $\lim_{x\to 3} \frac{x^3-2x^2-9}{x^2-2x-3}$

SOLUTION

x	2.99	2.995	3.005	3.01
$f(x)$	3.741880	3.745939	3.754064	3.758130

The limit as $x \to 3$ is 3.75.

21. $\lim_{x\to 0} \frac{\sin 2x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	1.999867	1.999967	1.999967	1.999867

The limit as $x \to 0$ is 2.

22. $\lim_{x\to 0} \frac{\sin 5x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	4.997917	4.999479	4.999479	4.997917

The limit as $x \to 0$ is 5.

23. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$

SOLUTION

x	-0.05	-0.001	0.001	0.05
$f(x)$	0.0249948	0.0005	-0.0005	-0.0249948

The limit as $x \to 0$ is 0.

24. $\lim_{x\to 0} \frac{\sin x}{x^2}$

SOLUTION

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	-99.9983	-999.9998	-10000.0	10000.0	999.9998	99.9983

The limit does not exist. As $x \to 0-$, $f(x) \to -\infty$; similarly, as $x \to 0+$, $f(x) \to \infty$.

25. $\lim_{x\to 4} \frac{1}{(x-4)^3}$

SOLUTION

x	3.99	3.999	3.9999	4.0001	4.001	4.01
$f(x)$	-10^6	-10^9	-10^{12}	10^{12}	10^9	10^6

The limit does not exist. As $x \to 4-$, $f(x) \to -\infty$; similarly, as $x \to 4+$, $f(x) \to \infty$.

26. $\lim_{x \to 1-} \frac{3-x}{x-1}$

SOLUTION

x	0.99	0.999	0.9999	0.99999
$f(x)$	-201	-2001	-20001	-200001

As $x \to 1-$, $f(x) \to -\infty$.

27. $\lim_{x \to 3+} \frac{x-4}{x^2-9}$

SOLUTION

x	3.01	3.001	3.0001	3.00001
$f(x)$	-16.473	-166.473	-1666.473	-16666.473

As $x \to 3+$, $f(x) \to -\infty$.

28. $\lim_{h \to 0} \frac{3^h - 1}{h}$

SOLUTION

h	-0.05	-0.001	-0.0001	0.0001	0.001	0.05
$f(h)$	1.06898	1.09801	1.09855	1.09867	1.09922	1.12935

The limit as $x \to 0$ is approximately 1.099. (The exact answer is $\ln 3$.)

29. $\lim_{h \to 0} \sin h \cos \frac{1}{h}$

SOLUTION

h	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(h)$	-0.008623	-0.000562	0.000095	-0.000095	0.000562	0.008623

The limit as $x \to 0$ is 0.

30. $\lim_{h \to 0} \cos \frac{1}{h}$

SOLUTION

h	± 0.1	± 0.01	± 0.001	± 0.0001
$f(h)$	-0.839072	0.862319	0.562379	-0.952155

The limit does not exist since $\cos(1/h)$ oscillates infinitely often as $h \to 0$.

31. $\lim_{x \to 0} |x|^x$

SOLUTION

x	-0.05	-0.001	-0.00001	0.00001	0.001	0.05
$f(x)$	1.161586	1.006932	1.000115	0.999885	0.993116	0.860892

The limit as $x \to 0$ is 1.

32. $\lim_{x \to 1+} \frac{\sec^{-1} x}{\sqrt{x-1}}$

SOLUTION

x	1.05	1.01	1.005	1.001
$f(x)$	1.3857	1.4084	1.4113	1.4136

The limit as $x \to 1+$ is approximately 1.414. (The exact answer is $\sqrt{2}$.)

33. $\lim_{t\to e} \dfrac{t-e}{\ln t - 1}$

SOLUTION

r	$e-0.01$	$e-0.001$	$e-0.0001$	$e+0.0001$	$e+0.001$	$e+0.01$
$f(t)$	2.713279	2.717782	2.718232	2.718332	2.718782	2.723279

The limit as $t \to 0$ is approximately 2.718. (The exact answer is e.)

34. $\lim_{r\to 0} (1+r)^{1/r}$

SOLUTION

r	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(r)$	2.731999	2.719642	2.718418	2.718146	2.716924	2.704814

The limit as $r \to 0$ is approximately 2.718. (The exact answer is e.)

35. $\lim_{x\to 1-} \dfrac{\tan^{-1} x}{\cos^{-1} x}$

SOLUTION

x	0.999	0.9999	0.99999	0.999999	0.9999999
$f(x)$	17.549	55.532	175.619	555.360	1756.204

The limit as $x \to 1-$ does not exist.

36. $\lim_{x\to 0} \dfrac{\tan^{-1} x - x}{\sin^{-1} x - x}$

SOLUTION

x	-0.01	-0.001	0.001	0.01
$f(x)$	-1.999791	-2.000066	-2.000066	-1.999791

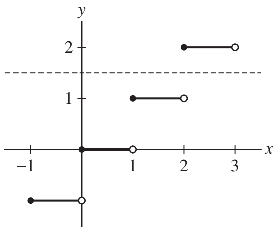
The limit as $x \to 0$ is approximately -2.00. (The exact answer is -2.)

37. The **greatest integer function** is defined by $[x] = n$, where n is the unique integer such that $n \le x < n+1$. Sketch the graph of $y = [x]$. Calculate, for c an integer:

(a) $\lim_{x\to c-} [x]$

(b) $\lim_{x\to c+} [x]$

SOLUTION Here is a graph of the greatest integer function:



(a) From the graph, we see that, for c an integer,

$$\lim_{x\to c-} [x] = c - 1.$$

(b) From the graph, we see that, for c an integer,

$$\lim_{x\to c+} [x] = c.$$

38. Determine the one-sided limits at $c = 1, 2,$ and 4 of the function $g(x)$ shown in Figure 11, and state whether the limit exists at these points.

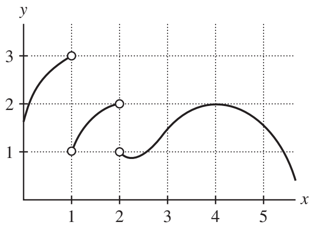


FIGURE 11

SOLUTION

- At $c = 1$, the left-hand limit is $\lim_{x\to 1-} g(x) = 3$, whereas the right-hand limit is $\lim_{x\to 1+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 1$.
- At $c = 2$, the left-hand limit is $\lim_{x\to 2-} g(x) = 2$, whereas the right-hand limit is $\lim_{x\to 2+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 2$.
- At $c = 4$, the left-hand limit is $\lim_{x\to 4-} g(x) = 2$, whereas the right-hand limit is $\lim_{x\to 4+} g(x) = 2$. Accordingly, the two-sided limit exists at $c = 4$ and equals 2.

In Exercises 39–46, determine the one-sided limits numerically or graphically. If infinite, state whether the one-sided limits are ∞ or $-\infty$, and describe the corresponding vertical asymptote. In Exercise 46, $[x]$ is the greatest integer function defined in Exercise 37.

39. $\lim_{x\to 0\pm} \dfrac{\sin x}{|x|}$

SOLUTION

x	-0.2	-0.02	0.02	0.2
$f(x)$	-0.993347	-0.999933	0.999933	0.993347

The left-hand limit is $\lim_{x\to 0-} f(x) = -1$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = 1$.

40. $\lim_{x\to 0\pm} |x|^{1/x}$

SOLUTION

x	-0.2	-0.1	0.15	0.2
$f(x)$	3125.0	10^{10}	0.000003	0.000320

The left-hand limit is $\lim_{x\to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = 0$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = |x|^{1/x}$.

41. $\lim_{x\to 0\pm} \dfrac{x - \sin|x|}{x^3}$

SOLUTION

x	-0.1	-0.01	0.01	0.1
$f(x)$	199.853	19999.8	0.166666	0.166583

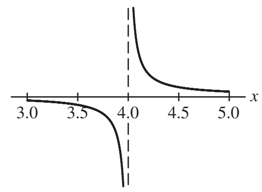
The left-hand limit is $\lim_{x\to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = \dfrac{1}{6}$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = \frac{x-\sin|x|}{x^3}$.

42. $\lim_{x\to 4\pm} \dfrac{x+1}{x-4}$

SOLUTION The graph of $y = \frac{x+1}{x-4}$ for x near 4 is shown below. From this graph, we see that

$$\lim_{x\to 4-} \frac{x+1}{x-4} = -\infty \quad \text{while} \quad \lim_{x\to 4+} \frac{x+1}{x-4} = \infty.$$

Thus, the line $x = 4$ is a vertical asymptote for the graph of $y = \frac{x+1}{x-4}$.

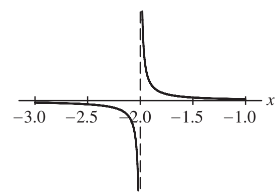


43. $\lim\limits_{x\to -2\pm} \dfrac{4x^2+7}{x^3+8}$

SOLUTION The graph of $y = \frac{4x^2+7}{x^3+8}$ for x near -2 is shown below. From this graph, we see that

$$\lim_{x\to -2-} \frac{4x^2+7}{x^3+8} = -\infty \quad \text{while} \quad \lim_{x\to -2+} \frac{4x^2+7}{x^3+8} = \infty.$$

Thus, the line $x = -2$ is a vertical asymptote for the graph of $y = \frac{4x^2+7}{x^3+8}$.

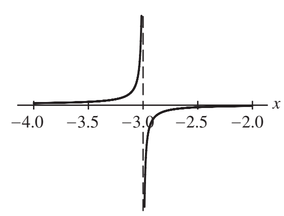


44. $\lim\limits_{x\to -3\pm} \dfrac{x^2}{x^2-9}$

SOLUTION The graph of $y = \frac{x^2}{x^2-9}$ for x near -3 is shown below. From this graph, we see that

$$\lim_{x\to -3-} \frac{x^2}{x^2-9} = \infty \quad \text{while} \quad \lim_{x\to -3+} \frac{x^2}{x^2-9} = -\infty.$$

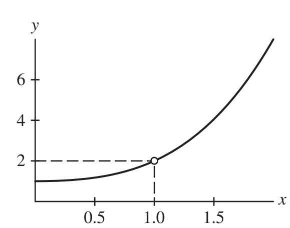
Thus, the line $x = -3$ is a vertical asymptote for the graph of $y = \frac{x^2}{x^2-9}$.



45. $\lim\limits_{x\to 1\pm} \dfrac{x^5+x-2}{x^2+x-2}$

SOLUTION The graph of $y = \frac{x^5+x-2}{x^2+x-2}$ for x near 1 is shown below. From this graph, we see that

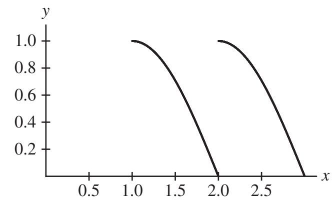
$$\lim_{x\to 1\pm} \frac{x^5+x-2}{x^2+x-2} = 2.$$



46. $\lim\limits_{x\to 2\pm} \cos\left(\dfrac{\pi}{2}(x - [x])\right)$

SOLUTION The graph of $y = \cos\left(\frac{\pi}{2}(x - [x])\right)$ for x near 2 is shown below. From this graph, we see that

$$\lim_{x\to 2-} \cos\left(\frac{\pi}{2}(x - [x])\right) = 0 \quad \text{while} \quad \lim_{x\to 2+} \cos\left(\frac{\pi}{2}(x - [x])\right) = 1.$$



47. Determine the one-sided limits at $c = 2, 4$ of the function $f(x)$ in Figure 12. What are the vertical asymptotes of $f(x)$?

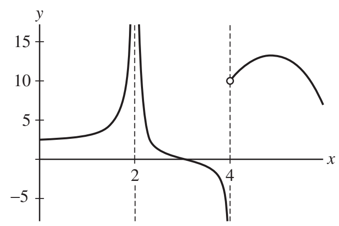


FIGURE 12

SOLUTION

- For $c = 2$, we have $\lim_{x \to 2-} f(x) = \infty$ and $\lim_{x \to 2+} f(x) = \infty$.
- For $c = 4$, we have $\lim_{x \to 4-} f(x) = -\infty$ and $\lim_{x \to 4+} f(x) = 10$.

The vertical asymptotes are the vertical lines $x = 2$ and $x = 4$.

48. Determine the infinite one- and two-sided limits in Figure 13.

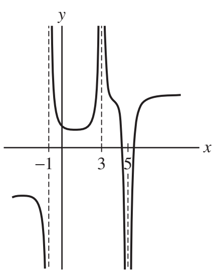


FIGURE 13

SOLUTION

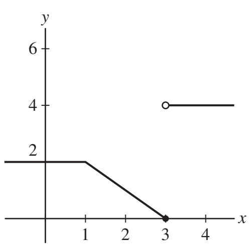
- $\lim_{x \to -1-} f(x) = -\infty$
- $\lim_{x \to -1+} f(x) = \infty$
- $\lim_{x \to 3} f(x) = \infty$
- $\lim_{x \to 5} f(x) = -\infty$

The vertical asymptotes are the vertical lines $x = 1$, $x = 3$, and $x = 5$.

In Exercises 49–52, sketch the graph of a function with the given limits.

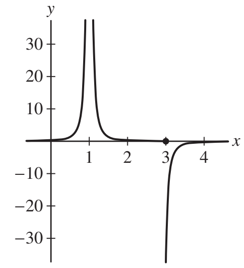
49. $\lim_{x \to 1} f(x) = 2, \quad \lim_{x \to 3-} f(x) = 0, \quad \lim_{x \to 3+} f(x) = 4$

SOLUTION



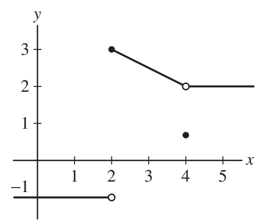
50. $\lim_{x\to 1} f(x) = \infty, \quad \lim_{x\to 3-} f(x) = 0, \quad \lim_{x\to 3+} f(x) = -\infty$

SOLUTION



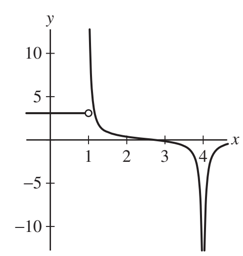
51. $\lim_{x\to 2+} f(x) = f(2) = 3, \quad \lim_{x\to 2-} f(x) = -1, \; \lim_{x\to 4} f(x) = 2 \neq f(4)$

SOLUTION



52. $\lim_{x\to 1+} f(x) = \infty, \quad \lim_{x\to 1-} f(x) = 3, \quad \lim_{x\to 4} f(x) = -\infty$

SOLUTION



53. Determine the one-sided limits of the function $f(x)$ in Figure 14, at the points $c = 1, 3, 5, 6$.

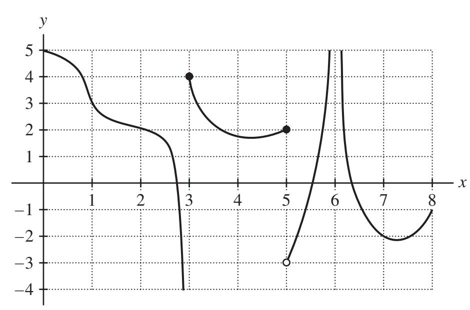


FIGURE 14 Graph of $f(x)$

SOLUTION

- $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 3$
- $\lim_{x\to 3-} f(x) = -\infty$
- $\lim_{x\to 3+} f(x) = 4$
- $\lim_{x\to 5-} f(x) = 2$
- $\lim_{x\to 5+} f(x) = -3$
- $\lim_{x\to 6-} f(x) = \lim_{x\to 6+} f(x) = \infty$

54. Does either of the two oscillating functions in Figure 15 appear to approach a limit as $x \to 0$?

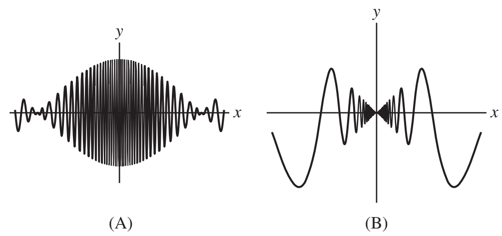


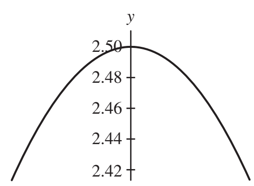
FIGURE 15

SOLUTION (A) does not appear to approach a limit as $x \to 0$; the values of the function oscillate wildly as $x \to 0$. The values of the function graphed in (B) seem to settle to 0 as $x \to 0$, so the limit seems to exist.

GU *In Exercises 55–60, plot the function and use the graph to estimate the value of the limit.*

55. $\displaystyle\lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 2\theta}$

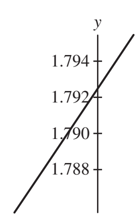
SOLUTION



From the graph of $y = \dfrac{\sin 5\theta}{\sin 2\theta}$ shown above, we see that the limit as $\theta \to 0$ is $\frac{5}{2}$.

56. $\displaystyle\lim_{x \to 0} \frac{12^x - 1}{4^x - 1}$

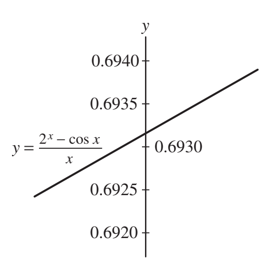
SOLUTION



From the graph of $y = \dfrac{12^x - 1}{4^x - 1}$ shown above, we see that the limit as $x \to 0$ is approximately 1.7925. (The exact answer is $\ln 12/\ln 4$.)

57. $\displaystyle\lim_{x \to 0} \frac{2^x - \cos x}{x}$

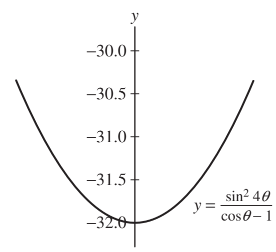
SOLUTION



The limit as $x \to 0$ is approximately 0.693. (The exact answer is $\ln 2$.)

58. $\lim_{\theta \to 0} \frac{\sin^2 4\theta}{\cos\theta - 1}$

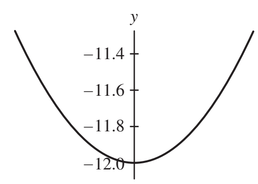
SOLUTION



The limit as $\theta \to 0$ is -32.

59. $\lim_{\theta \to 0} \frac{\cos 7\theta - \cos 5\theta}{\theta^2}$

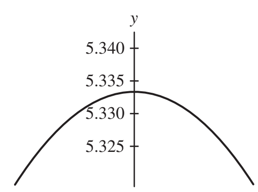
SOLUTION



From the graph of $y = \frac{\cos 7\theta - \cos 5\theta}{\theta^2}$ shown above, we see that the limit as $\theta \to 0$ is -12.

60. $\lim_{\theta \to 0} \frac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$

SOLUTION



From the graph of $y = \frac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$ shown above, we see that the limit as $\theta \to 0$ is approximately 5.333. (The exact answer is $\frac{16}{3}$.)

61. Let n be a positive integer. For which n are the two infinite one-sided limits $\lim_{x \to 0\pm} 1/x^n$ equal?

SOLUTION First, suppose that n is even. Then $x^n \ge 0$ for all x, and $\frac{1}{x^n} > 0$ for all $x \neq 0$. Hence,

$$\lim_{x \to 0-} \frac{1}{x^n} = \lim_{x \to 0+} \frac{1}{x^n} = \infty.$$

Next, suppose that n is odd. Then $\frac{1}{x^n} > 0$ for all $x > 0$ but $\frac{1}{x^n} < 0$ for all $x < 0$. Thus,

$$\lim_{x \to 0-} \frac{1}{x^n} = -\infty \quad \text{but} \quad \lim_{x \to 0+} \frac{1}{x^n} = \infty.$$

Finally, the two infinite one-sided limits are equal whenever n is even.

62. Let $L(n) = \lim_{x \to 1} \left(\frac{n}{1 - x^n} - \frac{1}{1 - x} \right)$ for n a positive integer. Investigate $L(n)$ numerically for several values of n, and then guess the value of of $L(n)$ in general.

SOLUTION

- We first notice that for $n = 1$,

$$\frac{1}{1 - x} - \frac{1}{1 - x} = 0,$$

so $L(1) = 0$.

- Next, let's try $n = 3$. From the table below, it appears that $L(3) = 1$.

x	0.99	0.999	1.001	1.01
$f(x)$	1.006700	1.000667	0.999334	0.993367

- For $n = 6$, we find

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$f(x)$	2.529312	2.502919	2.500392	2.499375	2.497082	2.470980

Thus, $L(6) = 2.5 = \frac{5}{2}$

From these values, we conjecture that $L(n) = \frac{n-1}{2}$.

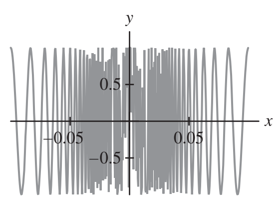
63. GU In some cases, numerical investigations can be misleading. Plot $f(x) = \cos \frac{\pi}{x}$.

(a) Does $\lim_{x\to 0} f(x)$ exist?

(b) Show, by evaluating $f(x)$ at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$, that you might be able to trick your friends into believing that the limit exists and is equal to $L = 1$.

(c) Which sequence of evaluations might trick them into believing that the limit is $L = -1$.

SOLUTION Here is the graph of $f(x)$.



(a) From the graph of $f(x)$, which shows that the value of $f(x)$ oscillates more and more rapidly as $x \to 0$, it follows that $\lim_{x\to 0} f(x)$ does not exist.

(b) Notice that

$$f\left(\frac{1}{2}\right) = \cos\frac{\pi}{1/2} = \cos 2\pi = 1;$$

$$f\left(\frac{1}{4}\right) = \cos\frac{\pi}{1/4} = \cos 4\pi = 1;$$

$$f\left(\frac{1}{6}\right) = \cos\frac{\pi}{1/6} = \cos 6\pi = 1;$$

and, in general, $f(\frac{1}{2n}) = 1$ for all integers n.

(c) At $x = 1, \frac{1}{3}, \frac{1}{5}, \ldots$, the value of $f(x)$ is always -1.

Further Insights and Challenges

64. Light waves of frequency λ passing through a slit of width a produce a **Fraunhofer diffraction pattern** of light and dark fringes (Figure 16). The intensity as a function of the angle θ is

$$I(\theta) = I_m\left(\frac{\sin(R\sin\theta)}{R\sin\theta}\right)^2$$

where $R = \pi a/\lambda$ and I_m is a constant. Show that the intensity function is not defined at $\theta = 0$. Then choose any two values for R and check numerically that $I(\theta)$ approaches I_m as $\theta \to 0$.

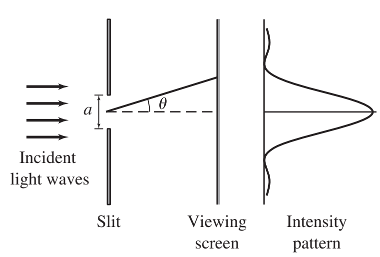


FIGURE 16 Fraunhofer diffraction pattern.

SOLUTION If you plug in $\theta = 0$, you get a division by zero in the expression

$$\frac{\sin\left(R\sin\theta\right)}{R\sin\theta};$$

thus, $I(0)$ is undefined. If $R = 2$, a table of values as $\theta \to 0$ follows:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.998667\ I_m$	$0.9999667\ I_m$	$0.9999667\ I_m$	$0.9998667\ I_m$

The limit as $\theta \to 0$ is $1 \cdot I_m = I_m$.

If $R = 3$, the table becomes:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.999700\ I_m$	$0.999925\ I_m$	$0.999925\ I_m$	$0.999700\ I_m$

Again, the limit as $\theta \to 0$ is $1I_m = I_m$.

65. Investigate $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta}$ numerically for several values of n. Then guess the value in general.

SOLUTION

- For $n = 3$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	2.955202	2.999550	2.999996	2.999996	2.999550	2.955202

  The limit as $\theta \to 0$ is 3.

- For $n = -5$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	-4.794255	-4.997917	-4.999979	-4.999979	-4.997917	-4.794255

  The limit as $\theta \to 0$ is -5.

- We surmise that, in general, $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n$.

66. Show numerically that $\lim_{x\to 0} \frac{b^x - 1}{x}$ for $b = 3, 5$ appears to equal $\ln 3, \ln 5$, where $\ln x$ is the natural logarithm. Then make a conjecture (guess) for the value in general and test your conjecture for two additional values of b.

SOLUTION

-

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{5^x - 1}{x}$	1.486601	1.596556	1.608144	1.610734	1.622459	1.746189

  We have $\ln 5 \approx 1.6094$.

•

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\dfrac{3^x - 1}{x}$	1.040415	1.092600	1.098009	1.099216	1.104669	1.161232

We have $\ln 3 \approx 1.0986$.

• We conjecture that $\lim_{x \to 0} \dfrac{b^x - 1}{x} = \ln b$ for any positive number b. Here are two additional test cases.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\dfrac{\left(\frac{1}{2}\right)^x - 1}{x}$	−0.717735	−0.695555	−0.693387	−0.692907	−0.690750	−0.669670

We have $\ln \frac{1}{2} \approx -0.69315$.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\dfrac{7^x - 1}{x}$	1.768287	1.927100	1.944018	1.947805	1.964966	2.148140

We have $\ln 7 \approx 1.9459$.

67. Investigate $\lim_{x \to 1} \dfrac{x^n - 1}{x^m - 1}$ for (m, n) equal to (2, 1), (1, 2), (2, 3), and (3, 2). Then guess the value of the limit in general and check your guess for two additional pairs.

SOLUTION

•

x	0.99	0.9999	1.0001	1.01
$\dfrac{x - 1}{x^2 - 1}$	0.502513	0.500025	0.499975	0.497512

The limit as $x \to 1$ is $\frac{1}{2}$.

x	0.99	0.9999	1.0001	1.01
$\dfrac{x^2 - 1}{x - 1}$	1.99	1.9999	2.0001	2.01

The limit as $x \to 1$ is 2.

x	0.99	0.9999	1.0001	1.01
$\dfrac{x^2 - 1}{x^3 - 1}$	0.670011	0.666700	0.666633	0.663344

The limit as $x \to 1$ is $\frac{2}{3}$.

x	0.99	0.9999	1.0001	1.01
$\dfrac{x^3 - 1}{x^2 - 1}$	1.492513	1.499925	1.500075	1.507512

The limit as $x \to 1$ is $\frac{3}{2}$.

- For general m and n, we have $\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$.
-

x	0.99	0.9999	1.0001	1.01
$\frac{x-1}{x^3-1}$	0.336689	0.333367	0.333300	0.330022

The limit as $x \to 1$ is $\frac{1}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x-1}$	2.9701	2.9997	3.0003	3.0301

The limit as $x \to 1$ is 3.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^7-1}$	0.437200	0.428657	0.428486	0.420058

The limit as $x \to 1$ is $\frac{3}{7} \approx 0.428571$.

68. Find by numerical experimentation the positive integers k such that $\lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^k}$ exists.

SOLUTION

- For $k = 1$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x} = 0$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-0.01	-0.0001	0.0001	0.01

- For $k = 2$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^2} = 1$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	0.999967	1.000000	1.000000	0.999967

- For $k = 3$, the limit does not exist.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-10^2	-10^4	10^4	10^2

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to \infty$.

- For $k = 4$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^4} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^4	10^8	10^8	10^4

- For $k = 5$, the limit does not exist.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-10^6	-10^{12}	10^{12}	10^6

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to \infty$.

- For $k = 6$, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(\sin^2 x)}{x^6} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^8	10^{16}	10^{16}	10^8

- SUMMARY
  - For $k = 1$, the limit is 0.
  - For $k = 2$, the limit is 1.
  - For odd $k > 2$, the limit does not exist.
  - For even $k > 2$, the limit is ∞.

69. GU Plot the graph of $f(x) = \frac{2^x - 8}{x - 3}$.

(a) Zoom in on the graph to estimate $L = \lim_{x \to 3} f(x)$.

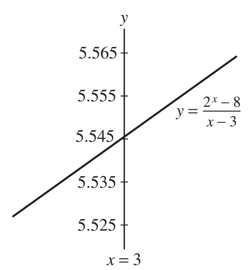
(b) Explain why

$$f(2.99999) \le L \le f(3.00001)$$

Use this to determine L to three decimal places.

SOLUTION

(a)



(b) It is clear that the graph of f rises as we move to the right. Mathematically, we may express this observation as: whenever $u < v$, $f(u) < f(v)$. Because

$$2.99999 < 3 = \lim_{x \to 3} f(x) < 3.00001,$$

it follows that

$$f(2.99999) < L = \lim_{x \to 3} f(x) < f(3.00001).$$

With $f(2.99999) \approx 5.54516$ and $f(3.00001) \approx 5.545195$, the above inequality becomes $5.54516 < L < 5.545195$; hence, to three decimal places, $L = 5.545$.

70. GU The function $f(x) = \frac{2^{1/x} - 2^{-1/x}}{2^{1/x} + 2^{-1/x}}$ is defined for $x \neq 0$.

(a) Investigate $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ numerically.

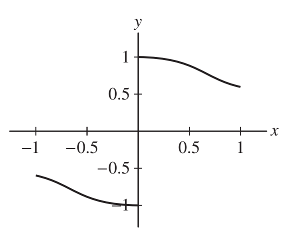
(b) Plot the graph of f and describe its behavior near $x = 0$.

SOLUTION

(a)

x	-0.3	-0.2	-0.1	0.1	0.2	0.3
$f(x)$	-0.980506	-0.998049	-0.999998	0.999998	0.998049	0.980506

(b) As $x \to 0-$, $f(x) \to -1$, whereas as $x \to 0+$, $f(x) \to 1$.



2.3 Basic Limit Laws

Preliminary Questions

1. State the Sum Law and Quotient Law.

SOLUTION Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. The Sum Law states that

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

Provided $\lim_{x\to c} g(x) \neq 0$, the Quotient Law states that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

2. Which of the following is a verbal version of the Product Law (assuming the limits exist)?

(a) The product of two functions has a limit.

(b) The limit of the product is the product of the limits.

(c) The product of a limit is a product of functions.

(d) A limit produces a product of functions.

SOLUTION The verbal version of the Product Law is **(b)**: The limit of the product is the product of the limits.

3. Which statement is correct? The Quotient Law does not hold if:

(a) The limit of the denominator is zero.

(b) The limit of the numerator is zero.

SOLUTION Statements **(a)** is correct. The Quotient Law does not hold if the limit of the denominator is zero.

Exercises

In Exercises 1–24, evaluate the limit using the Basic Limit Laws and the limits $\lim_{x\to c} x^{p/q} = c^{p/q}$ *and* $\lim_{x\to c} k = k$.

1. $\lim_{x\to 9} x$

SOLUTION $\lim_{x\to 9} x = 9.$

2. $\lim_{x\to -3} 14$

SOLUTION $\lim_{x\to -3} 14 = 14.$

3. $\lim_{x\to \frac{1}{2}} x^4$

SOLUTION $\lim_{x\to \frac{1}{2}} x^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$

4. $\lim_{z\to 27} z^{2/3}$

SOLUTION $\lim_{z\to 27} z^{2/3} = 27^{2/3} = 9.$

5. $\lim_{t\to 2} t^{-1}$

SOLUTION $\lim_{t\to 2} t^{-1} = 2^{-1} = \frac{1}{2}.$

6. $\lim_{x\to 5} x^{-2}$

SOLUTION $\lim_{x\to 5} x^{-2} = 5^{-2} = \frac{1}{25}.$

7. $\lim_{x\to 0.2} (3x+4)$

SOLUTION Using the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 0.2}(3x+4) &= \lim_{x\to 0.2} 3x + \lim_{x\to 0.2} 4\\ &= 3\lim_{x\to 0.2} x + \lim_{x\to 0.2} 4 = 3(0.2)+4 = 4.6.\end{aligned}$$

8. $\lim\limits_{x\to\frac{1}{3}}(3x^3+2x^2)$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to\frac{1}{3}}(3x^3+2x^2) &= \lim_{x\to\frac{1}{3}} 3x^3 + \lim_{x\to\frac{1}{3}} 2x^2\\ &= 3\lim_{x\to\frac{1}{3}} x^3 + 2\lim_{x\to\frac{1}{3}} x^2\\ &= 3\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 = \frac{1}{3}.\end{aligned}$$

9. $\lim\limits_{x\to-1}(3x^4-2x^3+4x)$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to-1}(3x^4-2x^3+4x) &= \lim_{x\to-1} 3x^4 - \lim_{x\to-1} 2x^3 + \lim_{x\to-1} 4x\\ &= 3\lim_{x\to-1} x^4 - 2\lim_{x\to-1} x^3 + 4\lim_{x\to-1} x\\ &= 3(-1)^4 - 2(-1)^3 + 4(-1) = 3+2-4 = 1.\end{aligned}$$

10. $\lim\limits_{x\to 8}(3x^{2/3}-16x^{-1})$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to 8}(3x^{2/3}-16x^{-1}) &= \lim_{x\to 8} 3x^{2/3} - \lim_{x\to 8} 16x^{-1}\\ &= 3\lim_{x\to 8} x^{2/3} - 16\lim_{x\to 8} x^{-1}\\ &= 3(8)^{2/3} - 16(8)^{-1} = 3(4) - 2 = 10.\end{aligned}$$

11. $\lim\limits_{x\to 2}(x+1)(3x^2-9)$

SOLUTION Using the Product Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 2}(x+1)\left(3x^2-9\right) &= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} 3x^2 - \lim_{x\to 2} 9\right)\\ &= (2+1)\left(3\lim_{x\to 2} x^2 - 9\right)\\ &= 3(3(2)^2-9) = 9.\end{aligned}$$

12. $\lim\limits_{x\to\frac{1}{2}}(4x+1)(6x-1)$

SOLUTION Using the Product Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 1/2}(4x+1)(6x-1) &= \left(\lim_{x\to 1/2}(4x+1)\right)\left(\lim_{x\to 1/2}(6x-1)\right)\\ &= \left(\lim_{x\to 1/2} 4x + \lim_{x\to 1/2} 1\right)\left(\lim_{x\to 1/2} 6x - \lim_{x\to 1/2} 1\right)\\ &= \left(4\lim_{x\to 1/2} x + \lim_{x\to 1/2} 1\right)\left(6\lim_{x\to 1/2} x - \lim_{x\to 1/2} 1\right)\\ &= \left(4\cdot\frac{1}{2}+1\right)\left(6\cdot\frac{1}{2}-1\right) = 3(2) = 6.\end{aligned}$$

13. $\lim\limits_{t\to 4}\dfrac{3t-14}{t+1}$

SOLUTION Using the Quotient Law, the Sum Law and the Constant Multiple Law:

$$\lim_{t\to 4}\frac{3t-14}{t+1} = \frac{\lim_{t\to 4}(3t-14)}{\lim_{t\to 4}(t+1)} = \frac{3\lim_{t\to 4} t - \lim_{t\to 4} 14}{\lim_{t\to 4} t + \lim_{t\to 4} 1} = \frac{3\cdot 4 - 14}{4+1} = -\frac{2}{5}.$$

14. $\lim_{z\to 9} \frac{\sqrt{z}}{z-2}$

SOLUTION Using the Quotient Law, the Powers Law and the Sum Law:

$$\lim_{z\to 9} \frac{\sqrt{z}}{z-2} = \frac{\lim_{z\to 9}\sqrt{z}}{\lim_{z\to 9}(z-2)} = \frac{\lim_{z\to 9}\sqrt{z}}{\lim_{z\to 9} z - \lim_{z\to 9} 2} = \frac{3}{7}.$$

15. $\lim_{y\to \frac{1}{4}} (16y+1)(2y^{1/2}+1)$

SOLUTION Using the Product Law, the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{y\to \frac{1}{4}} (16y+1)(2y^{1/2}+1) &= \left(\lim_{y\to \frac{1}{4}} (16y+1)\right)\left(\lim_{y\to \frac{1}{4}} (2y^{1/2}+1)\right)\\ &= \left(16 \lim_{y\to \frac{1}{4}} y + \lim_{y\to \frac{1}{4}} 1\right)\left(2 \lim_{y\to \frac{1}{4}} y^{1/2} + \lim_{y\to \frac{1}{4}} 1\right)\\ &= \left(16\left(\frac{1}{4}\right)+1\right)\left(2\left(\frac{1}{2}\right)+1\right) = 10.\end{aligned}$$

16. $\lim_{x\to 2} x(x+1)(x+2)$

SOLUTION Using the Product Law and Sum Law:

$$\begin{aligned}\lim_{x\to 2} x(x+1)(x+2) &= \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} (x+1)\right)\left(\lim_{x\to 2} (x+2)\right)\\ &= 2\left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} x + \lim_{x\to 2} 2\right)\\ &= 2(2+1)(2+2) = 24\end{aligned}$$

17. $\lim_{y\to 4} \frac{1}{\sqrt{6y+1}}$

SOLUTION Using the Quotient Law, the Powers Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{y\to 4} \frac{1}{\sqrt{6y+1}} &= \frac{1}{\lim_{y\to 4}\sqrt{6y+1}} = \frac{1}{\sqrt{6\lim_{y\to 4} y + 1}}\\ &= \frac{1}{\sqrt{6(4)+1}} = \frac{1}{5}.\end{aligned}$$

18. $\lim_{w\to 7} \frac{\sqrt{w+2}+1}{\sqrt{w-3}-1}$

SOLUTION Using the Quotient Law, the Sum Law and the Powers Law:

$$\begin{aligned}\lim_{w\to 7} \frac{\sqrt{w+2}+1}{\sqrt{w-3}-1} &= \frac{\lim_{w\to 7}(\sqrt{w+2}+1)}{\lim_{w\to 7}(\sqrt{w-3}-1)}\\ &= \frac{\sqrt{\lim_{w\to 7}(w+2)}+1}{\sqrt{\lim_{w\to 7}(w-3)}-1}\\ &= \frac{\sqrt{9}+1}{\sqrt{4}-1} = 4.\end{aligned}$$

19. $\lim_{x\to -1} \frac{x}{x^3+4x}$

SOLUTION Using the Quotient Law, the Sum Law, the Powers Law and the Constant Multiple Law:

$$\lim_{x\to -1} \frac{x}{x^3+4x} = \frac{\lim_{x\to -1} x}{\lim_{x\to -1} x^3 + 4 \lim_{x\to -1} x} = \frac{-1}{(-1)^3+4(-1)} = \frac{1}{5}.$$

20. $\lim_{t\to -1} \dfrac{t^2+1}{(t^3+2)(t^4+1)}$

SOLUTION Using the Quotient Law, the Product Law, the Sum Law and the Powers Law:

$$\lim_{x\to -1} \frac{t^2+1}{(t^3+2)(t^4+1)} = \frac{\lim_{x\to -1} t^2 + \lim_{x\to -1} 1}{\left(\lim_{x\to -1} t^3 + \lim_{x\to -1} 2\right)\left(\lim_{x\to -1} t^4 + \lim_{x\to -1} 1\right)}$$
$$= \frac{(-1)^2+1}{((-1)^3+2)((-1)^4+1)} = \frac{2}{(1)(2)} = 1.$$

21. $\lim_{t\to 25} \dfrac{3\sqrt{t} - \frac{1}{5}t}{(t-20)^2}$

SOLUTION Using the Quotient Law, the Sum Law, the Constant Multiple Law and the Powers Law:

$$\lim_{t\to 25} \frac{3\sqrt{t} - \frac{1}{5}t}{(t-20)^2} = \frac{3\sqrt{\lim_{t\to 25} t} - \frac{1}{5}\lim_{t\to 25} t}{\left(\lim_{t\to 25} t - 20\right)^2} = \frac{3(5) - \frac{1}{5}(25)}{5^2} = \frac{2}{5}.$$

22. $\lim_{y\to \frac{1}{3}} (18y^2-4)^4$

SOLUTION Using the Powers Law, the Sum Law and the Constant Multiple Law:

$$\lim_{y\to \frac{1}{3}} (18y^2-4)^4 = \left(18 \lim_{y\to \frac{1}{3}} y^2 - 4\right)^4 = (2-4)^4 = 16.$$

23. $\lim_{t\to \frac{3}{2}} (4t^2+8t-5)^{3/2}$

SOLUTION Using the Powers Law, the Sum Law and the Constant Multiple Law:

$$\lim_{t\to \frac{3}{2}} (4t^2+8t-5)^{3/2} = \left(4 \lim_{t\to \frac{3}{2}} t^2 + 8 \lim_{t\to \frac{3}{2}} t - 5\right)^{3/2} = (9+12-5)^{3/2} = 64.$$

24. $\lim_{t\to 7} \dfrac{(t+2)^{1/2}}{(t+1)^{2/3}}$

SOLUTION Using the Quotient Law, the Powers Law and the Sum Law:

$$\lim_{t\to 7} \frac{(t+2)^{1/2}}{(t+1)^{2/3}} = \frac{\left(\lim_{t\to 7} t + 2\right)^{1/2}}{\left(\lim_{t\to 7} t + 1\right)^{2/3}} = \frac{9^{1/2}}{8^{2/3}} = \frac{3}{4}.$$

25. Use the Quotient Law to prove that if $\lim_{x\to c} f(x)$ exists and is nonzero, then

$$\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{\lim_{x\to c} f(x)}$$

SOLUTION Since $\lim_{x\to c} f(x)$ is nonzero, we can apply the Quotient Law:

$$\lim_{x\to c} \left(\frac{1}{f(x)}\right) = \frac{\left(\lim_{x\to c} 1\right)}{\left(\lim_{x\to c} f(x)\right)} = \frac{1}{\lim_{x\to c} f(x)}.$$

26. Assuming that $\lim_{x\to 6} f(x) = 4$, compute:

(a) $\lim_{x\to 6} f(x)^2$ **(b)** $\lim_{x\to 6} \dfrac{1}{f(x)}$ **(c)** $\lim_{x\to 6} x\sqrt{f(x)}$

SOLUTION

(a) Using the Powers Law:

$$\lim_{x\to 6} f(x)^2 = \left(\lim_{x\to 6} f(x)\right)^2 = 4^2 = 16.$$

(b) Since $\lim_{x\to 6} f(x) \neq 0$, we may apply the Quotient Law:

$$\lim_{x\to 6} \frac{1}{f(x)} = \frac{1}{\lim_{x\to 6} f(x)} = \frac{1}{4}.$$

(c) Using the Product Law and Powers Law:

$$\lim_{x\to 6} x\sqrt{f(x)} = \left(\lim_{x\to 6} x\right)\left(\lim_{x\to 6} f(x)\right)^{1/2} = 6(4)^{1/2} = 12.$$

In Exercises 27–30, evaluate the limit assuming that $\lim_{x\to -4} f(x) = 3$ *and* $\lim_{x\to -4} g(x) = 1$.

27. $\lim_{x\to -4} f(x)g(x)$

SOLUTION $\lim_{x\to -4} f(x)g(x) = \lim_{x\to -4} f(x) \lim_{x\to -4} g(x) = 3 \cdot 1 = 3.$

28. $\lim_{x\to -4} (2f(x) + 3g(x))$

SOLUTION

$$\begin{aligned}\lim_{x\to -4} (2f(x) + 3g(x)) &= 2 \lim_{x\to -4} f(x) + 3 \lim_{x\to -4} g(x) \\ &= 2 \cdot 3 + 3 \cdot 1 = 6 + 3 = 9.\end{aligned}$$

29. $\lim_{x\to -4} \frac{g(x)}{x^2}$

SOLUTION Since $\lim_{x\to -4} x^2 \neq 0$, we may apply the Quotient Law, then applying the Powers Law:

$$\lim_{x\to -4} \frac{g(x)}{x^2} = \frac{\lim_{x\to -4} g(x)}{\lim_{x\to -4} x^2} = \frac{1}{\left(\lim_{x\to -4} x\right)^2} = \frac{1}{16}.$$

30. $\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9}$

SOLUTION

$$\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9} = \frac{\lim_{x\to -4} f(x) + \lim_{x\to -4} 1}{3\lim_{x\to -4} g(x) - \lim_{x\to -4} 9} = \frac{3+1}{3\cdot 1 - 9} = \frac{4}{-6} = -\frac{2}{3}.$$

31. Can the Quotient Law be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$? Explain.

SOLUTION The limit Quotient Law *cannot* be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ since $\lim_{x\to 0} x = 0$. This violates a condition of the Quotient Law. Accordingly, the rule *cannot* be employed.

32. Show that the Product Law cannot be used to evaluate the limit $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \tan x$.

SOLUTION The limit Product Law *cannot* be applied to evaluate $\lim_{x\to \pi/2} (x - \pi/2) \tan x$ since $\lim_{x\to \pi/2} \tan x$ does not exist (for example, as $x \to \pi/2-$, $\tan x \to \infty$). This violates a hypothesis of the Product Law. Accordingly, the rule *cannot* be employed.

33. Give an example where $\lim_{x\to 0} (f(x) + g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

SOLUTION Let $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} 0 = 0$ However, $\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/x$ and $\lim_{x\to 0} g(x) = \lim_{x\to 0} -1/x$ do not exist.

Further Insights and Challenges

34. Show that if both $\lim_{x\to c} f(x)\,g(x)$ and $\lim_{x\to c} g(x)$ exist and $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} f(x)$ exists. *Hint:* Write $f(x) = \dfrac{f(x)\,g(x)}{g(x)}$.

SOLUTION Given that $\lim_{x\to c} f(x)g(x) = L$ and $\lim_{x\to c} g(x) = M \neq 0$ both exist, observe that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)g(x)}{g(x)} = \frac{\lim_{x\to c} f(x)g(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$$

also exists.

35. Suppose that $\lim_{t\to 3} tg(t) = 12$. Show that $\lim_{t\to 3} g(t)$ exists and equals 4.

SOLUTION We are given that $\lim_{t\to 3} tg(t) = 12$. Since $\lim_{t\to 3} t = 3 \neq 0$, we may apply the Quotient Law:

$$\lim_{t\to 3} g(t) = \lim_{t\to 3} \frac{tg(t)}{t} = \frac{\lim_{t\to 3} tg(t)}{\lim_{t\to 3} t} = \frac{12}{3} = 4.$$

36. Prove that if $\lim_{t\to 3} \frac{h(t)}{t} = 5$, then $\lim_{t\to 3} h(t) = 15$.

SOLUTION Given that $\lim_{t\to 3} \dfrac{h(t)}{t} = 5$, observe that $\lim_{t\to 3} t = 3$. Now use the Product Law:

$$\lim_{t\to 3} h(t) = \lim_{t\to 3} t\,\frac{h(t)}{t} = \left(\lim_{t\to 3} t\right)\left(\lim_{t\to 3} \frac{h(t)}{t}\right) = 3 \cdot 5 = 15.$$

37. Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true? Why?

(a) $f(0) = 0$

(b) $\lim_{x\to 0} f(x) = 0$

SOLUTION

(a) Given that $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, it is not necessarily true that $f(0) = 0$. A counterexample is provided by $f(x) = \begin{cases} x, & x \neq 0 \\ 5, & x = 0 \end{cases}$.

(b) Given that $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, it is necessarily true that $\lim_{x\to 0} f(x) = 0$. For note that $\lim_{x\to 0} x = 0$, whence

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\,\frac{f(x)}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{f(x)}{x}\right) = 0 \cdot 1 = 0.$$

38. Prove that if $\lim_{x\to c} f(x) = L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

SOLUTION Suppose that $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ exists. Then

$$L = \lim_{x\to c} f(x) = \lim_{x\to c} g(x) \cdot \frac{f(x)}{g(x)} = \lim_{x\to c} g(x) \cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0 \cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0.$$

But, we were given that $L \neq 0$, so we have arrived at a contradiction. Thus, $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ does not exist.

39. Suppose that $\lim_{h\to 0} g(h) = L$.

(a) Explain why $\lim_{h\to 0} g(ah) = L$ for any constant $a \neq 0$.

(b) If we assume instead that $\lim_{h\to 1} g(h) = L$, is it still necessarily true that $\lim_{h\to 1} g(ah) = L$?

(c) Illustrate (a) and (b) with the function $f(x) = x^2$.

SOLUTION

(a) As $h \to 0$, $ah \to 0$ as well; hence, if we make the change of variable $w = ah$, then

$$\lim_{h\to 0} g(ah) = \lim_{w\to 0} g(w) = L.$$

(b) No. As $h \to 1$, $ah \to a$, so we should not expect $\lim_{h\to 1} g(ah) = \lim_{h\to 1} g(h)$.

(c) Let $g(x) = x^2$. Then

$$\lim_{h\to 0} g(h) = 0 \quad \text{and} \quad \lim_{h\to 0} g(ah) = \lim_{h\to 0} (ah)^2 = 0.$$

On the other hand,

$$\lim_{h\to 1} g(h) = 1 \quad \text{while} \quad \lim_{h\to 1} g(ah) = \lim_{h\to 1} (ah)^2 = a^2,$$

which is equal to the previous limit if and only if $a = \pm 1$.

40. Assume that $L(a) = \lim_{x\to 0} \frac{a^x - 1}{x}$ exists for all $a > 0$. Assume also that $\lim_{x\to 0} a^x = 1$.

(a) Prove that $L(ab) = L(a) + L(b)$ for $a, b > 0$. *Hint:* $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$. This shows that $L(a)$ "behaves" like a logarithm. We will see that $L(a) = \ln a$ in Section 3.10.

(b) Verify numerically that $L(12) = L(3) + L(4)$.

SOLUTION

(a) Let $a, b > 0$. Then

$$\begin{aligned} L(ab) = \lim_{x\to 0} \frac{(ab)^x - 1}{x} &= \lim_{x\to 0} \frac{a^x(b^x - 1) + (a^x - 1)}{x} \\ &= \lim_{x\to 0} a^x \cdot \lim_{x\to 0} \frac{b^x - 1}{x} + \lim_{x\to 0} \frac{a^x - 1}{x} \\ &= 1 \cdot L(b) + L(a) = L(a) + L(b). \end{aligned}$$

(b) From the table below, we estimate that, to three decimal places, $L(3) = 1.099$, $L(4) = 1.386$ and $L(12) = 2.485$. Thus,

$$L(12) = 2.485 = 1.099 + 1.386 = L(3) + L(4).$$

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$(3^x - 1)/x$	1.092600	1.098009	1.098552	1.098673	1.099216	1.104669
$(4^x - 1)/x$	1.376730	1.385334	1.386198	1.386390	1.387256	1.395948
$(12^x - 1)/x$	2.454287	2.481822	2.484600	2.485215	2.488000	2.516038

2.4 Limits and Continuity

Preliminary Questions

1. Which property of $f(x) = x^3$ allows us to conclude that $\lim_{x\to 2} x^3 = 8$?

SOLUTION We can conclude that $\lim_{x\to 2} x^3 = 8$ because the function x^3 is continuous at $x = 2$.

2. What can be said about $f(3)$ if f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$?

SOLUTION If f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$, then $f(3) = \frac{1}{2}$.

3. Suppose that $f(x) < 0$ if x is positive and $f(x) > 1$ if x is negative. Can f be continuous at $x = 0$?

SOLUTION Since $f(x) < 0$ when x is positive and $f(x) > 1$ when x is negative, it follows that

$$\lim_{x\to 0+} f(x) \le 0 \quad \text{and} \quad \lim_{x\to 0-} f(x) \ge 1.$$

Thus, $\lim_{x\to 0} f(x)$ does not exist, so f cannot be continuous at $x = 0$.

4. Is it possible to determine $f(7)$ if $f(x) = 3$ for all $x < 7$ and f is right-continuous at $x = 7$? What if f is left-continuous?

SOLUTION No. To determine $f(7)$, we need to combine either knowledge of the values of $f(x)$ for $x < 7$ with *left*-continuity or knowledge of the values of $f(x)$ for $x > 7$ with right-continuity.

5. Are the following true or false? If false, state a correct version.

(a) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal.

(b) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$.

(c) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist, then f has a removable discontinuity at $x = a$.

(d) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

(e) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)/g(x)$ is continuous at $x = a$.

SOLUTION

(a) False. The correct statement is "$f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$."

(b) True.

(c) False. The correct statement is "If the left- and right-hand limits of $f(x)$ as $x \to a$ are equal but not equal to $f(a)$, then f has a removable discontinuity at $x = a$."

(d) True.

(e) False. The correct statement is "If $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$."

Exercises

1. Referring to Figure 14, state whether $f(x)$ is left- or right-continuous (or neither) at each point of discontinuity. Does $f(x)$ have any removable discontinuities?

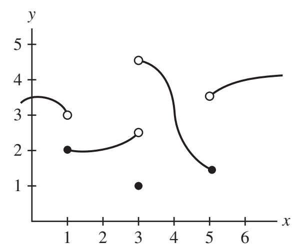


FIGURE 14 Graph of $y = f(x)$

SOLUTION

- The function f is discontinuous at $x = 1$; it is right-continuous there.
- The function f is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function f is discontinuous at $x = 5$; it is left-continuous there.

However, these discontinuities are not removable.

Exercises 2–4 refer to the function $g(x)$ in Figure 15.

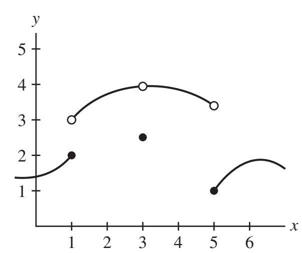


FIGURE 15 Graph of $y = g(x)$

2. State whether $g(x)$ is left- or right-continuous (or neither) at each of its points of discontinuity.

SOLUTION

- The function g is discontinuous at $x = 1$; it is left-continuous there.
- The function g is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function g is discontinuous at $x = 5$; it is right-continuous there.

3. At which point c does $g(x)$ have a removable discontinuity? How should $g(c)$ be redefined to make g continuous at $x = c$?

SOLUTION Because $\lim_{x\to 3} g(x)$ exists, the function g has a removable discontinuity at $x = 3$. Assigning $g(3) = 4$ makes g continuous at $x = 3$.

4. Find the point c_1 at which $g(x)$ has a jump discontinuity but is left-continuous. How should $g(c_1)$ be redefined to make g right-continuous at $x = c_1$?

SOLUTION The function g has a jump discontinuity at $x = 1$, but is left-continuous there. Assigning $g(1) = 3$ makes g right-continuous at $x = 1$ (but no longer left-continuous).

5. In Figure 16, determine the one-sided limits at the points of discontinuity. Which discontinuity is removable and how should f be redefined to make it continuous at this point?

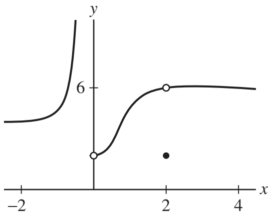


FIGURE 16

SOLUTION The function f is discontinuous at $x = 0$, at which $\lim_{x\to 0-} f(x) = \infty$ and $\lim_{x\to 0+} f(x) = 2$. The function f is also discontinuous at $x = 2$, at which $\lim_{x\to 2-} f(x) = 6$ and $\lim_{x\to 2+} f(x) = 6$. Because the two one-sided limits exist and are equal at $x = 2$, the discontinuity at $x = 2$ is removable. Assigning $f(2) = 6$ makes f continuous at $x = 2$.

6. Suppose that $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

(a) What is $f(3)$ if f is left-continuous at $x = 3$?

(b) What is $f(3)$ if f is right-continuous at $x = 3$?

SOLUTION $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

- If f is left-continuous at $x = 3$, then $f(3) = \lim_{x\to 3-} f(x) = 2$.
- If f is right-continuous at $x = 3$, then $f(3) = \lim_{x\to 0+} f(x) = -4$.

In Exercises 7–16, use the Laws of Continuity and Theorems 2 and 3 to show that the function is continuous.

7. $f(x) = x + \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x + \sin x$ by Continuity Law (i).

8. $f(x) = x \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x \sin x$ by Continuity Law (iii).

9. $f(x) = 3x + 4 \sin x$

SOLUTION Since x and $\sin x$ are continuous, so are $3x$ and $4 \sin x$ by Continuity Law (ii). Thus $3x + 4 \sin x$ is continuous by Continuity Law (i).

10. $f(x) = 3x^3 + 8x^2 - 20x$

SOLUTION

- Since x is continuous, so are x^3 and x^2 by repeated applications of Continuity Law (iii).
- Hence $3x^3$, $8x^2$, and $-20x$ are continuous by Continuity Law (ii).
- Finally, $3x^3 + 8x^2 - 20x$ is continuous by Continuity Law (i).

11. $f(x) = \dfrac{1}{x^2 + 1}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Recall that constant functions, such as 1, are continuous. Thus $x^2 + 1$ is continuous.
- Finally, $\dfrac{1}{x^2 + 1}$ is continuous by Continuity Law (iv) because $x^2 + 1$ is never 0.

12. $f(x) = \dfrac{x^2 - \cos x}{3 + \cos x}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Since $\cos x$ is continuous, so is $-\cos x$ by Continuity Law (ii).

- Accordingly, $x^2 - \cos x$ is continuous by Continuity Law (i).
- Since 3 (a constant function) and $\cos x$ are continuous, so is $3 + \cos x$ by Continuity Law (i).
- Finally, $\dfrac{x^2 - \cos x}{3 + \cos x}$ is continuous by Continuity Law (iv) because $3 + \cos x$ is never 0.

13. $f(x) = \cos(x^2)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\cos x$ and x^2, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

14. $f(x) = \tan^{-1}(4^x)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\tan^{-1} x$ and 4^x, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

15. $f(x) = e^x \cos 3x$

SOLUTION e^x and $\cos 3x$ are continuous, so $e^x \cos 3x$ is continuous by Continuity Law (iii).

16. $f(x) = \ln(x^4 + 1)$

SOLUTION

- Since x is continuous, so is x^4 by repeated application of Continuity Law (iii).
- Since 1 (a constant function) and x^4 are continuous, so is $x^4 + 1$ by Continuity Law (i).
- Finally, because $x^4 + 1 > 0$ for all x and $\ln x$ is continuous for $x > 0$, the composite function $\ln(x^4 + 1)$ is continuous.

In Exercises 17–34, determine the points of discontinuity. State the type of discontinuity (removable, jump, infinite, or none of these) and whether the function is left- or right-continuous.

17. $f(x) = \dfrac{1}{x}$

SOLUTION The function $1/x$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

18. $f(x) = |x|$

SOLUTION The function $f(x) = |x|$ is continuous everywhere.

19. $f(x) = \dfrac{x - 2}{|x - 1|}$

SOLUTION The function $\dfrac{x - 2}{|x - 1|}$ is discontinuous at $x = 1$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 1$.

20. $f(x) = [x]$

SOLUTION This function has a jump discontinuity at $x = n$ for every integer n. It is continuous at all other values of x. For every integer n,

$$\lim_{x \to n+} [x] = n$$

since $[x] = n$ for all x between n and $n + 1$. This shows that $[x]$ is *right-continuous* at $x = n$. On the other hand,

$$\lim_{x \to n-} [x] = n - 1$$

since $[x] = n - 1$ for all x between $n - 1$ and n. Thus $[x]$ is not left-continuous.

21. $f(x) = \left[\dfrac{1}{2}x\right]$

SOLUTION The function $\left[\frac{1}{2}x\right]$ is discontinuous at even integers, at which there are jump discontinuities. Because

$$\lim_{x \to 2n+} \left[\frac{1}{2}x\right] = n$$

but

$$\lim_{x \to 2n-} \left[\frac{1}{2}x\right] = n - 1,$$

it follows that this function is right-continuous at the even integers but not left-continuous.

22. $g(t) = \dfrac{1}{t^2 - 1}$

SOLUTION The function $f(t) = \dfrac{1}{t^2-1} = \dfrac{1}{(t-1)(t+1)}$ is discontinuous at $t = -1$ and $t = 1$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

23. $f(x) = \dfrac{x+1}{4x-2}$

SOLUTION The function $f(x) = \dfrac{x+1}{4x-2}$ is discontinuous at $x = \frac{1}{2}$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = \frac{1}{2}$.

24. $h(z) = \dfrac{1-2z}{z^2 - z - 6}$

SOLUTION The function $f(z) = \dfrac{1-2z}{z^2-z-6} = \dfrac{1-2z}{(z+2)(z-3)}$ is discontinuous at $z = -2$ and $z = 3$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

25. $f(x) = 3x^{2/3} - 9x^3$

SOLUTION The function $f(x) = 3x^{2/3} - 9x^3$ is defined and continuous for all x.

26. $g(t) = 3t^{-2/3} - 9t^3$

SOLUTION The function $g(t) = 3t^{-2/3} - 9t^3$ is discontinuous at $t = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $t = 0$.

27. $f(x) = \begin{cases} \dfrac{x-2}{|x-2|} & x \neq 2 \\ -1 & x = 2 \end{cases}$

SOLUTION For $x > 2$, $f(x) = \dfrac{x-2}{(x-2)} = 1$. For $x < 2$, $f(x) = \dfrac{(x-2)}{(2-x)} = -1$. The function has a jump discontinuity at $x = 2$. Because

$$\lim_{x \to 2-} f(x) = -1 = f(2)$$

but

$$\lim_{x \to 2+} f(x) = 1 \neq f(2),$$

it follows that this function is left-continuous at $x = 2$ but not right-continuous.

28. $f(x) = \begin{cases} \cos \dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION The function $\cos\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$, at which there is an oscillatory discontinuity. Because neither

$$\lim_{x \to 0-} f(x) \quad \text{nor} \quad \lim_{x \to 0+} f(x)$$

exist, the function is neither left- nor right-continuous at $x = 0$.

29. $g(t) = \tan 2t$

SOLUTION The function $g(t) = \tan 2t = \dfrac{\sin 2t}{\cos 2t}$ is discontinuous whenever $\cos 2t = 0$; i.e., whenever

$$2t = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad t = \frac{(2n+1)\pi}{4},$$

where n is an integer. At every such value of t there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

30. $f(x) = \csc(x^2)$

SOLUTION The function $f(x) = \csc(x^2) = \dfrac{1}{\sin(x^2)}$ is discontinuous whenever $\sin(x^2) = 0$; i.e., whenever $x^2 = n\pi$ or $x = \pm\sqrt{n\pi}$, where n is a positive integer. At every such value of x there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

31. $f(x) = \tan(\sin x)$

SOLUTION The function $f(x) = \tan(\sin x)$ is continuous everywhere. Reason: $\sin x$ is continuous everywhere and $\tan u$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$—and in particular on $-1 \le u = \sin x \le 1$. Continuity of $\tan(\sin x)$ follows by the continuity of composite functions.

32. $f(x) = \cos(\pi[x])$

SOLUTION The function $f(x) = \cos(\pi[x])$ has a jump discontinuity at $x = n$ for every integer n. The function is right-continuous but not left-continuous at each of these points of discontinuity.

33. $f(x) = \dfrac{1}{e^x - e^{-x}}$

SOLUTION The function $f(x) = \dfrac{1}{e^x - e^{-x}}$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

34. $f(x) = \ln|x - 4|$

SOLUTION The function $f(x) = \ln|x - 4|$ is discontinuous at $x = 4$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 4$.

In Exercises 35–48, determine the domain of the function and prove that it is continuous on its domain using the Laws of Continuity and the facts quoted in this section.

35. $f(x) = 2\sin x + 3\cos x$

SOLUTION The domain of $2\sin x + 3\cos x$ is all real numbers. Both $\sin x$ and $\cos x$ are continuous on this domain, so $2\sin x + 3\cos x$ is continuous by Continuity Laws (i) and (ii).

36. $f(x) = \sqrt{x^2 + 9}$

SOLUTION The domain of $\sqrt{x^2 + 9}$ is all real numbers, as $x^2 + 9 > 0$ for all x. Since $\sqrt{x}$ and the polynomial $x^2 + 9$ are both continuous, so is the composite function $\sqrt{x^2 + 9}$.

37. $f(x) = \sqrt{x}\sin x$

SOLUTION This function is defined as long as $x \ge 0$. Since $\sqrt{x}$ and $\sin x$ are continuous, so is $\sqrt{x}\sin x$ by Continuity Law (iii).

38. $f(x) = \dfrac{x^2}{x + x^{1/4}}$

SOLUTION This function is defined as long as $x \ge 0$ and $x + x^{1/4} \ne 0$, and so the domain is all $x > 0$. Since x is continuous, so are x^2 and $x + x^{1/4}$ by Continuity Laws (iii) and (i); hence, by Continuity Law (iv), so is $\dfrac{x^2}{x + x^{1/4}}$.

39. $f(x) = x^{2/3}2^x$

SOLUTION The domain of $x^{2/3}2^x$ is all real numbers as the denominator of the rational exponent is odd. Both $x^{2/3}$ and 2^x are continuous on this domain, so $x^{2/3}2^x$ is continuous by Continuity Law (iii).

40. $f(x) = x^{1/3} + x^{3/4}$

SOLUTION The domain of $x^{1/3} + x^{3/4}$ is $x \ge 0$. On this domain, both $x^{1/3}$ and $x^{3/4}$ are continuous, so $x^{1/3} + x^{3/4}$ is continuous by Continuity Law (i).

41. $f(x) = x^{-4/3}$

SOLUTION This function is defined for all $x \ne 0$. Because the function $x^{4/3}$ is continuous and not equal to zero for $x \ne 0$, it follows that

$$x^{-4/3} = \frac{1}{x^{4/3}}$$

is continuous for $x \ne 0$ by Continuity Law (iv).

42. $f(x) = \ln(9 - x^2)$

SOLUTION The domain of $\ln(9 - x^2)$ is all x such that $9 - x^2 > 0$, or $|x| < 3$. The polynomial $9 - x^2$ is continuous for all real numbers and $\ln x$ is continuous for $x > 0$; therefore, the composite function $\ln(9 - x^2)$ is continuous for $|x| < 3$.

43. $f(x) = \tan^2 x$

SOLUTION The domain of $\tan^2 x$ is all $x \ne \pm(2n - 1)\pi/2$ where n is a positive integer. Because $\tan x$ is continuous on this domain, it follows from Continuity Law (iii) that $\tan^2 x$ is also continuous on this domain.

44. $f(x) = \cos(2^x)$

SOLUTION The domain of $\cos(2^x)$ is all real numbers. Because the functions $\cos x$ and 2^x are continuous on this domain, so is the composite function $\cos(2^x)$.

45. $f(x) = (x^4 + 1)^{3/2}$

SOLUTION The domain of $(x^4 + 1)^{3/2}$ is all real numbers as $x^4 + 1 > 0$ for all x. Because $x^{3/2}$ and the polynomial $x^4 + 1$ are both continuous, so is the composite function $(x^4 + 1)^{3/2}$.

46. $f(x) = e^{-x^2}$

SOLUTION The domain of e^{-x^2} is all real numbers. Because e^x and the polynomial $-x^2$ are both continuous for all real numbers, so is the composite function e^{-x^2}.

47. $f(x) = \dfrac{\cos(x^2)}{x^2 - 1}$

SOLUTION The domain for this function is all $x \neq \pm 1$. Because the functions $\cos x$ and x^2 are continuous on this domain, so is the composite function $\cos(x^2)$. Finally, because the polynomial $x^2 - 1$ is continuous and not equal to zero for $x \neq \pm 1$, the function $\dfrac{\cos(x^2)}{x^2 - 1}$ is continuous by Continuity Law (iv).

48. $f(x) = 9^{\tan x}$

SOLUTION The domain of $9^{\tan x}$ is all $x \neq \pm(2n - 1)\pi/2$ where n is a positive integer. Because $\tan x$ and 9^x are continuous on this domain, it follows that the composite function $9^{\tan x}$ is also continuous on this domain.

49. Show that the function

$$f(x) = \begin{cases} x^2 + 3 & \text{for } x < 1 \\ 10 - x & \text{for } 1 \le x \le 2 \\ 6x - x^2 & \text{for } x > 2 \end{cases}$$

is continuous for $x \neq 1, 2$. Then compute the right- and left-hand limits at $x = 1, 2$, and determine whether $f(x)$ is left-continuous, right-continuous, or continuous at these points (Figure 17).

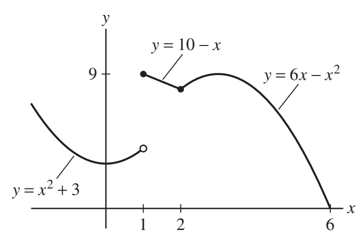


FIGURE 17

SOLUTION Let's start with $x \neq 1, 2$.

- Because x is continuous, so is x^2 by Continuity Law (iii). The constant function 3 is also continuous, so $x^2 + 3$ is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $x < 1$.
- Because x and the constant function 10 are continuous, the function $10 - x$ is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $1 < x < 2$.
- Because x is continuous, x^2 is continuous by Continuity Law (iii) and $6x$ is continuous by Continuity Law (ii). Therefore, $6x - x^2$ is continuous by Continuity Law (i), so $f(x)$ is continuous for $x > 2$.

At $x = 1$, $f(x)$ has a jump discontinuity because the one-sided limits exist but are not equal:

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x^2 + 3) = 4, \qquad \lim_{x \to 1+} f(x) = \lim_{x \to 1+} (10 - x) = 9.$$

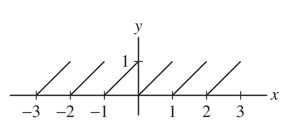
Furthermore, the right-hand limit equals the function value $f(1) = 9$, so $f(x)$ is right-continuous at $x = 1$. At $x = 2$,

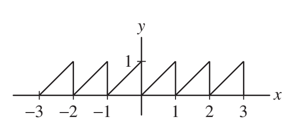
$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (10 - x) = 8, \qquad \lim_{x \to 2+} f(x) = \lim_{x \to 2+} (6x - x^2) = 8.$$

The left- and right-hand limits exist and are equal to $f(2)$, so $f(x)$ is continuous at $x = 2$.

50. Sawtooth Function Draw the graph of $f(x) = x - [x]$. At which points is f discontinuous? Is it left- or right-continuous at those points?

SOLUTION Two views of the sawtooth function $f(x) = x - [x]$ appear below. The first is the actual graph. In the second, the jumps are "connected" so as to better illustrate its "sawtooth" nature. The function is right-continuous at integer values of x.

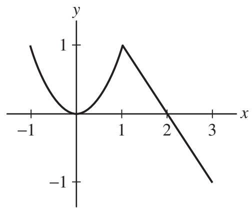




In Exercises 51–54, sketch the graph of $f(x)$. At each point of discontinuity, state whether f is left- or right-continuous.

51. $f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

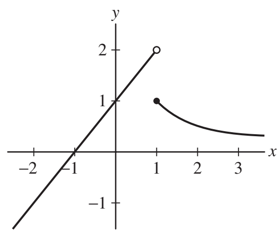
SOLUTION



The function f is continuous everywhere.

52. $f(x) = \begin{cases} x + 1 & \text{for } x < 1 \\ \dfrac{1}{x} & \text{for } x \ge 1 \end{cases}$

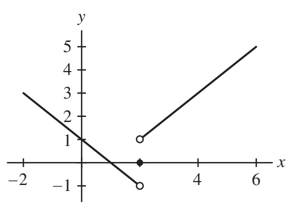
SOLUTION



The function f is right-continuous at $x = 1$.

53. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{|x - 2|} & x \ne 2 \\ 0 & x = 2 \end{cases}$

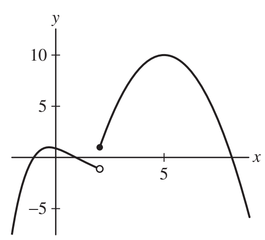
SOLUTION



The function f is neither left- nor right-continuous at $x = 2$.

54. $f(x) = \begin{cases} x^3 + 1 & \text{for } -\infty < x \le 0 \\ -x + 1 & \text{for } 0 < x < 2 \\ -x^2 + 10x - 15 & \text{for } x \ge 2 \end{cases}$

SOLUTION



The function f is right-continuous at $x = 2$.

55. Show that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & x \neq 4 \\ 10 & x = 4 \end{cases}$$

has a removable discontinuity at $x = 4$.

SOLUTION To show that $f(x)$ has a removable discontinuity at $x = 4$, we must establish that

$$\lim_{x \to 4} f(x)$$

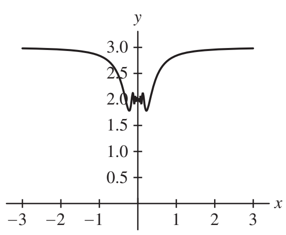
exists but does not equal $f(4)$. Now,

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8 \neq 10 = f(4);$$

thus, $f(x)$ has a removable discontinuity at $x = 4$. To remove the discontinuity, we must redefine $f(4) = 8$.

56. GU Define $f(x) = x \sin \frac{1}{x} + 2$ for $x \neq 0$. Plot $f(x)$. How should $f(0)$ be defined so that f is continuous at $x = 0$?

SOLUTION



From the graph, it appears that $f(0)$ should be defined equal to 2 to make f continuous at $x = 0$.

In Exercises 57–59, find the value of the constant (a, b, or c) that makes the function continuous.

57. $f(x) = \begin{cases} x^2 - c & \text{for } x < 5 \\ 4x + 2c & \text{for } x \geq 5 \end{cases}$

SOLUTION As $x \to 5-$, we have $x^2 - c \to 25 - c = L$. As $x \to 5+$, we have $4x + 2c \to 20 + 2c = R$. Match the limits: $L = R$ or $25 - c = 20 + 2c$ implies $c = \frac{5}{3}$.

58. $f(x) = \begin{cases} 2x + 9x^{-1} & \text{for } x \leq 3 \\ -4x + c & \text{for } x > 3 \end{cases}$

SOLUTION As $x \to 3-$, we have $2x + 9x^{-1} \to 9 = L$. As $x \to 3+$, we have $-4x + c \to c - 12 = R$. Match the limits: $L = R$ or $9 = c - 12$ implies $c = 21$.

59. $f(x) = \begin{cases} x^{-1} & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x \leq \frac{1}{2} \\ x^{-1} & \text{for } x > \frac{1}{2} \end{cases}$

SOLUTION As $x \to -1-$, $x^{-1} \to -1$ while as $x \to -1+$, $ax + b \to b - a$. For f to be continuous at $x = -1$, we must therefore have $b - a = -1$. Now, as $x \to \frac{1}{2}-$, $ax + b \to \frac{1}{2}a + b$ while as $x \to \frac{1}{2}+$, $x^{-1} \to 2$. For f to be continuous at $x = \frac{1}{2}$, we must therefore have $\frac{1}{2}a + b = 2$. Solving these two equations for a and b yields $a = 2$ and $b = 1$.

60. Define

$$g(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ cx & \text{for } -1 \leq x \leq 2 \\ x + 2 & \text{for } x > 2 \end{cases}$$

Find a value of c such that $g(x)$ is

(a) left-continuous

(b) right-continuous

In each case, sketch the graph of $g(x)$.

SOLUTION

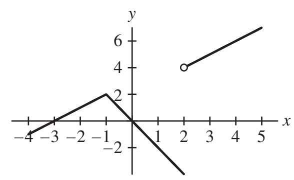
(a) In order for $g(x)$ to be left-continuous, we need

$$\lim_{x\to -1-} g(x) = \lim_{x\to -1-} (x+3) = 2$$

to be equal to

$$\lim_{x\to -1+} g(x) = \lim_{x\to -1+} cx = -c.$$

Therefore, we must have $c = -2$. The graph of $g(x)$ with $c = -2$ is shown below.



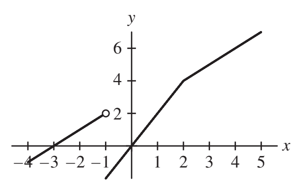
(b) In order for $g(x)$ to be right-continuous, we need

$$\lim_{x\to 2-} g(x) = \lim_{x\to 2-} cx = 2c$$

to be equal to

$$\lim_{x\to 2+} g(x) = \lim_{x\to 2+} (x+2) = 4.$$

Therefore, we must have $c = 2$. The graph of $g(x)$ with $c = 2$ is shown below.



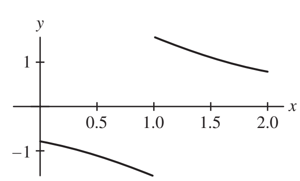
61. Define $g(t) = \tan^{-1}\left(\dfrac{1}{t-1}\right)$ for $t \neq 1$. Answer the following questions, using a plot if necessary.

(a) Can $g(1)$ be defined so that $g(t)$ is continuous at $t = 1$?

(b) How should $g(1)$ be defined so that $g(t)$ is left-continuous at $t = 1$?

SOLUTION

(a) From the graph of $g(t)$ shown below, we see that g has a jump discontinuity at $t = 1$; therefore, $g(a)$ cannot be defined so that g is continuous at $t = 1$.



(b) To make g left-continuous at $t = 1$, we should define

$$g(1) = \lim_{t\to 1-} \tan^{-1}\left(\tfrac{1}{t-1}\right) = -\frac{\pi}{2}.$$

62. Each of the following statements is *false*. For each statement, sketch the graph of a function that provides a counterexample.

(a) If $\lim_{x\to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

(b) If $f(x)$ has a jump discontinuity at $x = a$, then $f(a)$ is equal to either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a+} f(x)$.

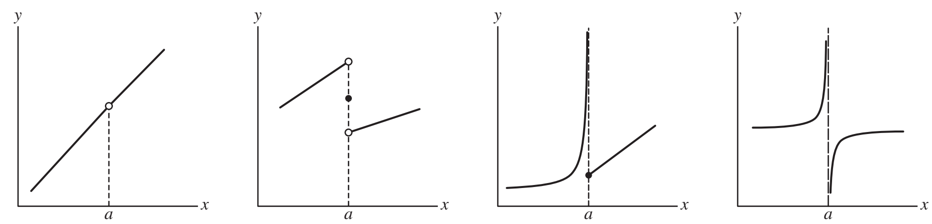
SOLUTION Refer to the four figures shown below.

(a) The figure at the top left shows a function for which $\lim_{x\to a} f(x)$ exists, but the function is not continuous at $x = a$ because the function is not defined at $x = a$.

(b) The figure at the top right shows a function that has a jump discontinuity at $x = a$ but $f(a)$ is not equal to either $\lim_{x \to a-} f(x)$ or $\lim_{x \to a-} f(x)$.

(c) This statement can be false either when the two one-sided limits exist and are equal or when one or both of the one-sided limits does not exist. The figure at the top left shows a function that has a discontinuity at $x = a$ with both one-sided limits being equal; the figure at the bottom left shows a function that has a discontinuity at $x = a$ with a one-sided limit that does not exist.

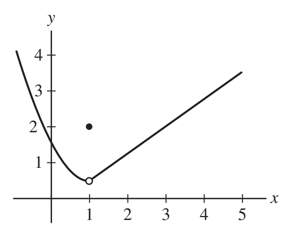
(d) The figure at the bottom left shows a function for which $\lim_{x \to a} f(x)$ does not exist and one of the one-sided limits also does not exist; the figure at the bottom right shows a function for which $\lim_{x \to a} f(x)$ does not exist and neither of the one-sided limits exists.



In Exercises 63–66, draw the graph of a function on [0, 5] *with the given properties.*

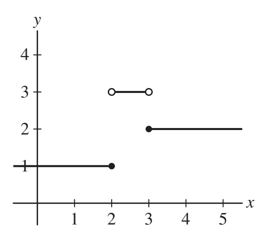
63. $f(x)$ is not continuous at $x = 1$, but $\lim_{x \to 1+} f(x)$ and $\lim_{x \to 1-} f(x)$ exist and are equal.

SOLUTION



64. $f(x)$ is left-continuous but not continuous at $x = 2$ and right-continuous but not continuous at $x = 3$.

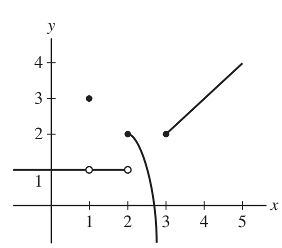
SOLUTION



65. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and

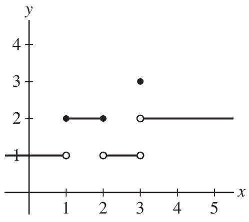
$$\lim_{x \to 3-} f(x) = -\infty, \qquad \lim_{x \to 3+} f(x) = 2$$

SOLUTION



66. $f(x)$ is right- but not left-continuous at $x = 1$, left- but not right-continuous at $x = 2$, and neither left- nor right-continuous at $x = 3$.

SOLUTION



In Exercises 67–80, evaluate using substitution.

67. $\lim_{x\to -1}(2x^3 - 4)$

SOLUTION $\lim_{x\to -1}(2x^3 - 4) = 2(-1)^3 - 4 = -6.$

68. $\lim_{x\to 2}(5x - 12x^{-2})$

SOLUTION $\lim_{x\to 2}(5x - 12x^{-2}) = 5(2) - 12(2^{-2}) = 10 - 12(\frac{1}{4}) = 7.$

69. $\lim_{x\to 3} \dfrac{x+2}{x^2+2x}$

SOLUTION $\lim_{x\to 3} \dfrac{x+2}{x^2+2x} = \dfrac{3+2}{3^2+2\cdot 3} = \dfrac{5}{15} = \dfrac{1}{3}$

70. $\lim_{x\to \pi} \sin\left(\dfrac{x}{2} - \pi\right)$

SOLUTION $\lim_{x\to \pi} \sin(\frac{x}{2} - \pi) = \sin(-\frac{\pi}{2}) = -1.$

71. $\lim_{x\to \frac{\pi}{4}} \tan(3x)$

SOLUTION $\lim_{x\to \frac{\pi}{4}} \tan(3x) = \tan(3\cdot\frac{\pi}{4}) = \tan(\frac{3\pi}{4}) = -1$

72. $\lim_{x\to \pi} \dfrac{1}{\cos x}$

SOLUTION $\lim_{x\to \pi} \dfrac{1}{\cos x} = \dfrac{1}{\cos \pi} = \dfrac{1}{-1} = -1.$

73. $\lim_{x\to 4} x^{-5/2}$

SOLUTION $\lim_{x\to 4} x^{-5/2} = 4^{-5/2} = \dfrac{1}{32}.$

74. $\lim_{x\to 2} \sqrt{x^3 + 4x}$

SOLUTION $\lim_{x\to 2} \sqrt{x^3 + 4x} = \sqrt{2^3 + 4(2)} = 4.$

75. $\lim_{x\to -1}(1 - 8x^3)^{3/2}$

SOLUTION $\lim_{x\to -1}(1 - 8x^3)^{3/2} = (1 - 8(-1)^3)^{3/2} = 27.$

76. $\lim_{x\to 2} \left(\dfrac{7x+2}{4-x}\right)^{2/3}$

SOLUTION $\lim_{x\to 2} \left(\dfrac{7x+2}{4-x}\right)^{2/3} = \left(\dfrac{7(2)+2}{4-2}\right)^{2/3} = 4.$

77. $\lim_{x\to 3} 10^{x^2-2x}$

SOLUTION $\lim_{x\to 3} 10^{x^2-2x} = 10^{3^2-2(3)} = 1000.$

78. $\lim_{x \to -\frac{\pi}{2}} 3^{\sin x}$

SOLUTION $\lim_{x \to -\frac{\pi}{2}} 3^{-\sin x} = 3^{-\sin(\pi/2)} = \frac{1}{3}$.

79. $\lim_{x \to 4} \sin^{-1}\left(\frac{x}{4}\right)$

SOLUTION $\lim_{x \to 4} \sin^{-1}\left(\frac{x}{4}\right) = \sin^{-1}\left(\lim_{x \to 4} \frac{x}{4}\right) = \sin^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{2}$

80. $\lim_{x \to 0} \tan^{-1}(e^x)$

SOLUTION $\lim_{x \to 0} \tan^{-1}(e^x) = \tan^{-1}\left(\lim_{x \to 0} e^x\right) = \tan^{-1}(e^0) = \tan^{-1} 1 = \frac{\pi}{4}$

81. Suppose that $f(x)$ and $g(x)$ are discontinuous at $x = c$. Does it follow that $f(x) + g(x)$ is discontinuous at $x = c$? If not, give a counterexample. Does this contradict Theorem 1 (i)?

SOLUTION Even if $f(x)$ and $g(x)$ are discontinuous at $x = c$, it is *not* necessarily true that $f(x) + g(x)$ is discontinuous at $x = c$. For example, suppose $f(x) = -x^{-1}$ and $g(x) = x^{-1}$. Both $f(x)$ and $g(x)$ are discontinuous at $x = 0$; however, the function $f(x) + g(x) = 0$, which is continuous everywhere, including $x = 0$. This does not contradict Theorem 1 (i), which deals only with continuous functions.

82. Prove that $f(x) = |x|$ is continuous for all x. *Hint:* To prove continuity at $x = 0$, consider the one-sided limits.

SOLUTION Let $c < 0$. Then

$$\lim_{x \to c} |x| = \lim_{x \to c} -x = -c = |c|.$$

Next, let $c > 0$. Then

$$\lim_{x \to c} |x| = \lim_{x \to c} x = c = |c|.$$

Finally,

$$\lim_{x \to 0-} |x| = \lim_{x \to 0-} -x = 0,$$

$$\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0$$

and we recall that $|0| = 0$. Thus, $|x|$ is continuous for all x.

83. Use the result of Exercise 82 to prove that if $g(x)$ is continuous, then $f(x) = |g(x)|$ is also continuous.

SOLUTION Recall that the composition of two continuous functions is continuous. Now, $f(x) = |g(x)|$ is a composition of the continuous functions $g(x)$ and $|x|$, so is also continuous.

84. Which of the following quantities would be represented by continuous functions of time and which would have one or more discontinuities?

(a) Velocity of an airplane during a flight

(b) Temperature in a room under ordinary conditions

(c) Value of a bank account with interest paid yearly

(d) The salary of a teacher

(e) The population of the world

SOLUTION

(a) The velocity of an airplane during a flight from Boston to Chicago is a continuous function of time.

(b) The temperature of a room under ordinary conditions is a continuous function of time.

(c) The value of a bank account with interest paid yearly is *not* a continuous function of time. It has discontinuities when deposits or withdrawals are made and when interest is paid.

(d) The salary of a teacher is *not* a continuous function of time. It has discontinuities whenever the teacher gets a raise (or whenever his or her salary is lowered).

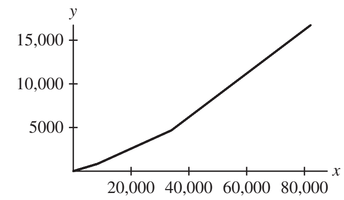
(e) The population of the world is *not* a continuous function of time since it changes by a discrete amount with each birth or death. Since it takes on such large numbers (many billions), it is often treated as a continuous function for the purposes of mathematical modeling.

85. In 2009, the federal income tax $T(x)$ on income of x dollars (up to \$82,250) was determined by the formula

$$T(x) = \begin{cases} 0.10x & \text{for } 0 \le x < 8350 \\ 0.15x - 417.50 & \text{for } 8350 \le x < 33{,}950 \\ 0.25x - 3812.50 & \text{for } 33{,}950 \le x < 82{,}250 \end{cases}$$

Sketch the graph of $T(x)$. Does $T(x)$ have any discontinuities? Explain why, if $T(x)$ had a jump discontinuity, it might be advantageous in some situations to earn *less* money.

SOLUTION $T(x)$, the amount of federal income tax owed on an income of x dollars in 2009, might be a discontinuous function depending upon how the tax tables are constructed (as determined by that year's regulations). Here is a graph of $T(x)$ for that particular year.



If $T(x)$ had a jump discontinuity (say at $x = c$), it might be advantageous to earn slightly less income than c (say $c - \epsilon$) and be taxed at a lower rate than to earn c or more and be taxed at a higher rate. Your net earnings may actually be more in the former case than in the latter one.

Further Insights and Challenges

86. If $f(x)$ has a removable discontinuity at $x = c$, then it is possible to redefine $f(c)$ so that $f(x)$ is continuous at $x = c$. Can this be done in more than one way?

SOLUTION In order for $f(x)$ to have a removable discontinuity at $x = c$, $\lim_{x\to c} f(x) = L$ must exist. To remove the discontinuity, we define $f(c) = L$. Then f is continuous at $x = c$ since $\lim_{x\to c} f(x) = L = f(c)$. Now *assume* that we may define $f(c) = M \neq L$ and still have f continuous at $x = c$. Then $\lim_{x\to c} f(x) = f(c) = M$. Therefore $M = L$, a contradiction. Roughly speaking, there's only one way to fill in the hole in the graph of f!

87. Give an example of functions $f(x)$ and $g(x)$ such that $f(g(x))$ is continuous but $g(x)$ has at least one discontinuity.

SOLUTION Answers may vary. The simplest examples are the functions $f(g(x))$ where $f(x) = C$ is a constant function, and $g(x)$ is defined for all x. In these cases, $f(g(x)) = C$. For example, if $f(x) = 3$ and $g(x) = [x]$, g is discontinuous at all integer values $x = n$, but $f(g(x)) = 3$ is continuous.

88. Continuous at Only One Point Show that the following function is continuous only at $x = 0$:

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational} \end{cases}$$

SOLUTION Let $f(x) = x$ for x rational and $f(x) = -x$ for x irrational.

- Now $f(0) = 0$ since 0 is rational. Moreover, as $x \to 0$, we have $|f(x) - f(0)| = |f(x) - 0| = |x| \to 0$. Thus $\lim_{x\to 0} f(x) = f(0)$ and f is continuous at $x = 0$.
- Let $c \neq 0$ be any nonzero rational number. Let $\{x_1, x_2, \dots\}$ be a sequence of irrational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |-x_n - c| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all rational numbers.
- Let $c \neq 0$ be any nonzero irrational number. Let $\{x_1, x_2, \dots\}$ be a sequence of rational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |x_n - (-c)| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all irrational numbers.
- CONCLUSION: f is continuous at $x = 0$ and is discontinuous at all points $x \neq 0$.

89. Show that $f(x)$ is a discontinuous function for all x where $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ -1 & \text{for } x \text{ irrational} \end{cases}$$

Show that $f(x)^2$ is continuous for all x.

SOLUTION $\lim_{x\to c} f(x)$ does not exist for any c. If c is irrational, then there is always a rational number r arbitrarily close to c so that $|f(c) - f(r)| = 2$. If, on the other hand, c is rational, there is always an *irrational* number z arbitrarily close to c so that $|f(c) - f(z)| = 2$.

On the other hand, $f(x)^2$ is a constant function that always has value 1, which is obviously continuous.

2.5 Evaluating Limits Algebraically

Preliminary Questions

1. Which of the following is indeterminate at $x = 1$?

$$\frac{x^2+1}{x-1}, \qquad \frac{x^2-1}{x+2}, \qquad \frac{x^2-1}{\sqrt{x+3}-2}, \qquad \frac{x^2+1}{\sqrt{x+3}-2}$$

SOLUTION At $x = 1$, $\frac{x^2-1}{\sqrt{x+3}-2}$ is of the form $\frac{0}{0}$; hence, this function is indeterminate. None of the remaining functions is indeterminate at $x = 1$: $\frac{x^2+1}{x-1}$ and $\frac{x^2+1}{\sqrt{x+3}-2}$ are undefined because the denominator is zero but the numerator is not, while $\frac{x^2-1}{x+2}$ is equal to 0.

2. Give counterexamples to show that these statements are false:

(a) If $f(c)$ is indeterminate, then the right- and left-hand limits as $x \to c$ are not equal.

(b) If $\lim_{x\to c} f(x)$ exists, then $f(c)$ is not indeterminate.

(c) If $f(x)$ is undefined at $x = c$, then $f(x)$ has an indeterminate form at $x = c$.

SOLUTION

(a) Let $f(x) = \frac{x^2-1}{x-1}$. At $x = 1$, f is indeterminate of the form $\frac{0}{0}$ but

$$\lim_{x\to 1-} \frac{x^2-1}{x-1} = \lim_{x\to 1-} (x+1) = 2 = \lim_{x\to 1+} (x+1) = \lim_{x\to 1+} \frac{x^2-1}{x-1}.$$

(b) Again, let $f(x) = \frac{x^2-1}{x-1}$. Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$$

but $f(1)$ is indeterminate of the form $\frac{0}{0}$.

(c) Let $f(x) = \frac{1}{x}$. Then f is undefined at $x = 0$ but does not have an indeterminate form at $x = 0$.

3. The method for evaluating limits discussed in this section is sometimes called "simplify and plug in." Explain how it actually relies on the property of continuity.

SOLUTION If f is continuous at $x = c$, then, by definition, $\lim_{x\to c} f(x) = f(c)$; in other words, the limit of a continuous function at $x = c$ is the value of the function at $x = c$. The "simplify and plug-in" strategy is based on simplifying a function which is indeterminate to a continuous function. Once the simplification has been made, the limit of the remaining continuous function is obtained by evaluation.

Exercises

In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.

1. $\lim_{x\to 6} \frac{x^2-36}{x-6}$

SOLUTION When we substitute $x = 6$ into $\frac{x^2-36}{x-6}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x\to 6} \frac{x^2-36}{x-6} = \lim_{x\to 6} \frac{(x-6)(x+6)}{x-6} = \lim_{x\to 6} (x+6) = 12.$$

2. $\lim_{h\to 3} \frac{9-h^2}{h-3}$

SOLUTION When we substitute $h = 3$ into $\frac{9-h^2}{h-3}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the denominator and simplifying, we find

$$\lim_{h\to 3} \frac{9-h^2}{h-3} = \lim_{h\to 3} \frac{(3-h)(3+h)}{h-3} = \lim_{h\to 3} -(3+h) = -6.$$

3. $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$

SOLUTION When we substitute $x = -1$ into $\frac{x^2+2x+1}{x+1}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x\to -1} \frac{x^2+2x+1}{x+1} = \lim_{x\to -1} \frac{(x+1)^2}{x+1} = \lim_{x\to -1} (x+1) = 0.$$

4. $\lim_{t\to 9} \frac{2t-18}{5t-45}$

SOLUTION When we substitute $t = 9$ into $\frac{2t-18}{5t-45}$, we obtain the indeterminate form $\frac{0}{0}$. Upon dividing out the common factor of $t - 9$ from both the numerator and denominator, we find

$$\lim_{t\to 9} \frac{2t-18}{5t-45} = \lim_{t\to 9} \frac{2(t-9)}{5(t-9)} = \lim_{t\to 9} \frac{2}{5} = \frac{2}{5}.$$

In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

5. $\lim_{x\to 7} \frac{x-7}{x^2-49}$

SOLUTION $\lim_{x\to 7} \frac{x-7}{x^2-49} = \lim_{x\to 7} \frac{x-7}{(x-7)(x+7)} = \lim_{x\to 7} \frac{1}{x+7} = \frac{1}{14}.$

6. $\lim_{x\to 8} \frac{x^2-64}{x-9}$

SOLUTION $\lim_{x\to 8} \frac{x^2-64}{x-9} = \frac{0}{-1} = 0$

7. $\lim_{x\to -2} \frac{x^2+3x+2}{x+2}$

SOLUTION $\lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+1)(x+2)}{x+2} = \lim_{x\to -2} (x+1) = -1.$

8. $\lim_{x\to 8} \frac{x^3-64x}{x-8}$

SOLUTION $\lim_{x\to 8} \frac{x^3-64x}{x-8} = \lim_{x\to 8} \frac{x(x-8)(x+8)}{x-8} = \lim_{x\to 8} x(x+8) = 8(16) = 128.$

9. $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25}$

SOLUTION $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25} = \lim_{x\to 5} \frac{(x-5)(2x+1)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{2x+1}{x+5} = \frac{11}{10}.$

10. $\displaystyle\lim_{h\to 0}\frac{(1+h)^3-1}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h}=\lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h}=\lim_{h\to 0}\frac{3h+3h^2+h^3}{h}$$
$$=\lim_{h\to 0}(3+3h+h^2)=3+3(0)+0^2=3.$$

11. $\displaystyle\lim_{x\to -\frac{1}{2}}\frac{2x+1}{2x^2+3x+1}$

SOLUTION $\displaystyle\lim_{x\to -\frac{1}{2}}\frac{2x+1}{2x^2+3x+1}=\lim_{x\to -\frac{1}{2}}\frac{2x+1}{(2x+1)(x+1)}=\lim_{x\to -\frac{1}{2}}\frac{1}{x+1}=2.$

12. $\displaystyle\lim_{x\to 3}\frac{x^2-x}{x^2-9}$

SOLUTION As $x\to 3$, the numerator $x^2-x\to 6$ while the denominator $x^2-9\to 0$; thus, this limit does not exist. Checking the one-sided limits, we find

$$\lim_{x\to 3-}\frac{x^2-x}{x^2-9}=\lim_{x\to 3-}\frac{x(x-1)}{(x-3)(x+3)}=-\infty$$

while

$$\lim_{x\to 3+}\frac{x^2-x}{x^2-9}=\lim_{x\to 3+}\frac{x(x-1)}{(x-3)(x+3)}=\infty.$$

13. $\displaystyle\lim_{x\to 2}\frac{3x^2-4x-4}{2x^2-8}$

SOLUTION $\displaystyle\lim_{x\to 2}\frac{3x^2-4x-4}{2x^2-8}=\lim_{x\to 2}\frac{(3x+2)(x-2)}{2(x-2)(x+2)}=\lim_{x\to 2}\frac{3x+2}{2(x+2)}=\frac{8}{8}=1.$

14. $\displaystyle\lim_{h\to 0}\frac{(3+h)^3-27}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{(3+h)^3-27}{h}=\lim_{h\to 0}\frac{27+27h+9h^2+h^3-27}{h}=\lim_{h\to 0}\frac{27h+9h^2+h^3}{h}$$
$$=\lim_{h\to 0}(27+9h+h^2)=27+9(0)+0^2=27.$$

15. $\displaystyle\lim_{t\to 0}\frac{4^{2t}-1}{4^t-1}$

SOLUTION $\displaystyle\lim_{t\ to 0}\frac{4^{2t}-1}{4^t-1}=\lim_{t\ to 0}\frac{(4^t-1)(4^t+1)}{4^t-1}=\lim_{t\to 0}(4^t+1)=2.$

16. $\displaystyle\lim_{h\to 4}\frac{(h+2)^2-9h}{h-4}$

SOLUTION $\displaystyle\lim_{h\to 4}\frac{(h+2)^2-9h}{h-4}=\lim_{h\to 4}\frac{h^2-5h+4}{h-4}=\lim_{h\to 4}\frac{(h-1)(h-4)}{h-4}=\lim_{h\to 4}(h-1)=3.$

17. $\displaystyle\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}$

SOLUTION $\displaystyle\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}=\lim_{x\to 16}\frac{\sqrt{x}-4}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)}=\lim_{x\to 16}\frac{1}{\sqrt{x}+4}=\frac{1}{8}.$

18. $\displaystyle\lim_{t\to -2}\frac{2t+4}{12-3t^2}$

SOLUTION $\displaystyle\lim_{t\to -2}\frac{2t+4}{12-3t^2}=\lim_{t\to -2}\frac{2(t+2)}{-3(t-2)(t+2)}=\lim_{t\to -2}\frac{2}{-3(t-2)}=\frac{1}{6}.$

19. $\displaystyle\lim_{y\to 3}\frac{y^2+y-12}{y^3-10y+3}$

SOLUTION $\displaystyle\lim_{y\to 3}\frac{y^2+y-12}{y^3-10y+3}=\lim_{y\to 3}\frac{(y-3)(y+4)}{(y-3)(y^2+3y-1)}=\lim_{y\to 3}\frac{(y+4)}{(y^2+3y-1)}=\frac{7}{17}.$

20. $\displaystyle\lim_{h\to 0}\frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}=\lim_{h\to 0}\frac{\frac{4-(h+2)^2}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{\frac{4-(h^2+4h+4)}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{\frac{-h^2-4h}{4(h+2)^2}}{h}$$
$$=\lim_{h\to 0}\frac{h\frac{-h-4}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{-h-4}{4(h+2)^2}=\frac{-4}{16}=-\frac{1}{4}.$$

21. $\displaystyle\lim_{h\to 0}\frac{\sqrt{2+h}-2}{h}$

SOLUTION $\displaystyle\lim_{h\to 0}\frac{\sqrt{h+2}-2}{h}$ does not exist.

- As $h\to 0+$, we have $\displaystyle\frac{\sqrt{h+2}-2}{h}=\frac{\left(\sqrt{h+2}-2\right)\left(\sqrt{h+2}+2\right)}{h(\sqrt{h+2}+2)}=\frac{h-2}{h(\sqrt{h+2}+2)}\to-\infty.$
- As $h\to 0-$, we have $\displaystyle\frac{\sqrt{h+2}-2}{h}=\frac{\left(\sqrt{h+2}-2\right)\left(\sqrt{h+2}+2\right)}{h(\sqrt{h+2}+2)}=\frac{h-2}{h(\sqrt{h+2}+2)}\to\infty.$

22. $\displaystyle\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}$

SOLUTION

$$\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}=\lim_{x\to 8}\frac{(\sqrt{x-4}-2)(\sqrt{x-4}+2)}{(x-8)(\sqrt{x-4}+2)}=\lim_{x\to 8}\frac{x-4-4}{(x-8)(\sqrt{x-4}+2)}$$
$$=\lim_{x\to 8}\frac{1}{\sqrt{x-4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}.$$

23. $\displaystyle\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

SOLUTION

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{(\sqrt{x}-\sqrt{8-x})(\sqrt{x}+\sqrt{8-x})}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)}$$
$$=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2x-8}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)}$$
$$=\lim_{x\to 4}\frac{(\sqrt{x}+\sqrt{8-x})}{2}=\frac{\sqrt{4}+\sqrt{4}}{2}=2.$$

24. $\displaystyle\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}$

SOLUTION

$$\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}=\lim_{x\to 4}\left(\frac{\sqrt{5-x}-1}{2-\sqrt{x}}\cdot\frac{\sqrt{5-x}+1}{\sqrt{5-x}+1}\right)=\lim_{x\to 4}\frac{4-x}{(2-\sqrt{x})(\sqrt{5-x}+1)}$$
$$=\lim_{x\to 4}\frac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})(\sqrt{5-x}+1)}=\lim_{x\to 4}\frac{2+\sqrt{x}}{\sqrt{5-x}+1}=2.$$

25. $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)$

SOLUTION $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)=\lim_{x\to 4}\frac{\sqrt{x}+2-4}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}=\lim_{x\to 4}\frac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}=\frac{1}{4}.$

26. $\lim_{x\to 0+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)$

SOLUTION

$$\lim_{x\to 0+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)=\lim_{x\to 0+}\frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}}=\lim_{x\to 0+}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}$$
$$=\lim_{x\to 0+}\frac{x}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0+}\frac{\sqrt{x}}{\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=0.$$

27. $\lim_{x\to 0}\frac{\cot x}{\csc x}$

SOLUTION $\lim_{x\to 0}\frac{\cot x}{\csc x}=\lim_{x\to 0}\frac{\cos x}{\sin x}\cdot\sin x=\cos 0=1.$

28. $\lim_{\theta\to\frac{\pi}{2}}\frac{\cot\theta}{\csc\theta}$

SOLUTION $\lim_{\theta\to\frac{\pi}{2}}\frac{\cot\theta}{\csc\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta}\cdot\sin\theta=\cos\frac{\pi}{2}=0.$

29. $\lim_{t\to 2}\frac{2^{2t}+2^t-20}{2^t-4}$

SOLUTION $\lim_{t\to 2}\frac{2^{2t}+2^t-20}{2^t-4}=\lim_{t\to 2}\frac{(2^t+5)(2^t-4)}{2^t-4}=\lim_{t\to 2}(2^t+5)=9.$

30. $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$

SOLUTION $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)=\lim_{x\to 1}\frac{(1+x)-2}{(1-x)(1+x)}=\lim_{x\to 1}\frac{-1}{1+x}=-\frac{1}{2}.$

31. $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}$

SOLUTION $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}\cdot\frac{\cos x}{\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{\sin x-\cos x}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$

32. $\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)$

SOLUTION

$$\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin^2\theta}{\cos\theta\,(1+\sin\theta)}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{1+\sin\theta}=\frac{0}{2}=0.$$

33. $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)$

SOLUTION $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)=\lim_{\theta\to\frac{\pi}{4}}\frac{(\tan\theta+1)-2}{(\tan\theta+1)(\tan\theta-1)}=\lim_{\theta\to\frac{\pi}{4}}\frac{1}{\tan\theta+1}=\frac{1}{2}.$

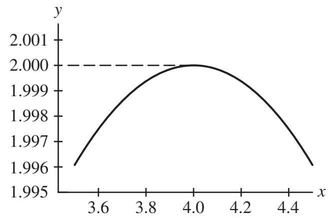
34. $\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}$

SOLUTION

$$\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\frac{(2\cos x-1)\,(\cos x+2)}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\cos x+2=\cos\frac{\pi}{3}+2=\frac{5}{2}.$$

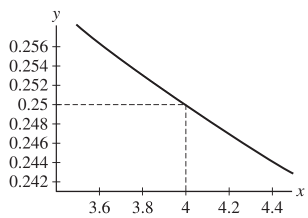
35. GU Use a plot of $f(x) = \dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$ to estimate $\lim_{x\to 4} f(x)$ to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

SOLUTION Let $f(x) = \frac{x-4}{\sqrt{x}-\sqrt{8-x}}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4} f(x) \approx 2.00$; to two decimal places, this matches the value of 2 obtained in Exercise 23.



36. GU Use a plot of $f(x) = \dfrac{1}{\sqrt{x}-2} - \dfrac{4}{x-4}$ to estimate $\lim_{x\to 4} f(x)$ numerically. Compare with the answer obtained algebraically in Exercise 25.

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}-2} - \frac{4}{x-4}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4} f(x) \approx 0.25$; to two decimal places, this matches the value of $\frac{1}{4}$ obtained in Exercise 25.



In Exercises 37–42, evaluate using the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

37. $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$

SOLUTION $\lim_{x\to 2} \dfrac{x^3-8}{x-2} = \lim_{x\to 2} \dfrac{(x-2)\left(x^2+2x+4\right)}{x-2} = \lim_{x\to 2} \left(x^2+2x+4\right) = 12.$

38. $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9}$

SOLUTION $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9} = \lim_{x\to 3} \dfrac{(x-3)\left(x^2+3x+9\right)}{(x-3)(x+3)} = \lim_{x\to 3} \dfrac{\left(x^2+3x+9\right)}{x+3} = \dfrac{27}{6} = \dfrac{9}{2}.$

39. $\lim_{x\to 1} \dfrac{x^2-5x+4}{x^3-1}$

SOLUTION $\lim_{x\to 1} \dfrac{x^2-5x+4}{x^3-1} = \lim_{x\to 1} \dfrac{(x-1)(x-4)}{(x-1)\left(x^2+x+1\right)} = \lim_{x\to 1} \dfrac{x-4}{x^2+x+1} = -1.$

40. $\lim_{x\to -2} \dfrac{x^3+8}{x^2+6x+8}$

SOLUTION $\lim_{x\to -2} \dfrac{x^3+8}{x^2+6x+8} = \lim_{x\to -2} \dfrac{(x+2)(x^2-2x+4)}{(x+2)(x+4)} = \lim_{x\to -2} \dfrac{(x^2-2x+4)}{x+4} = \dfrac{12}{2} = 6.$

41. $\lim_{x\to 1} \dfrac{x^4-1}{x^3-1}$

SOLUTION

$$\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x+1)(x^2+1)}{(x^2+x+1)} = \frac{4}{3}.$$

42. $\lim_{x\to 27} \frac{x-27}{x^{1/3}-3}$

SOLUTION $\lim_{x\to 27} \frac{x-27}{x^{1/3}-3} = \lim_{x\to 27} \frac{(x^{1/3}-3)(x^{2/3}+3x^{1/3}+9)}{x^{1/3}-3} = \lim_{x\to 27}(x^{2/3}+3x^{1/3}+9) = 27$

43. Evaluate $\lim_{h\to 0} \frac{\sqrt[4]{1+h}-1}{h}$. *Hint:* Set $x = \sqrt[4]{1+h}$ and rewrite as a limit as $x \to 1$.

SOLUTION Let $x = \sqrt[4]{1+h}$. Then $h = x^4 - 1 = (x-1)(x+1)(x^2+1)$, $x \to 1$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sqrt[4]{1+h}-1}{h} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)(x^2+1)} = \lim_{x\to 1} \frac{1}{(x+1)(x^2+1)} = \frac{1}{4}.$$

44. Evaluate $\lim_{h\to 0} \frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$. *Hint:* Set $x = \sqrt[6]{1+h}$ and rewrite as a limit as $x \to 1$.

SOLUTION Let $x = \sqrt[6]{1+h}$. Then $\sqrt[3]{1+h} - 1 = x^2 - 1 = (x-1)(x+1)$, $\sqrt{1+h} - 1 = x^3 - 1 = (x-1)(x^2 + x + 1)$, $x \to 1$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}.$$

In Exercises 45–54, evaluate in terms of the constant a.

45. $\lim_{x\to 0}(2a+x)$

SOLUTION $\lim_{x\to 0}(2a+x) = 2a$.

46. $\lim_{h\to -2}(4ah+7a)$

SOLUTION $\lim_{h\to -2}(4ah+7a) = -a$.

47. $\lim_{t\to -1}(4t-2at+3a)$

SOLUTION $\lim_{t\to -1}(4t-2at+3a) = -4+5a$.

48. $\lim_{h\to 0} \frac{(3a+h)^2-9a^2}{h}$

SOLUTION $\lim_{h\to 0} \frac{(3a+h)^2-9a^2}{h} = \lim_{h\to 0} \frac{6ah+h^2}{h} = \lim_{h\to 0}(6a+h) = 6a$.

49. $\lim_{h\to 0} \frac{2(a+h)^2-2a^2}{h}$

SOLUTION $\lim_{h\to 0} \frac{2(a+h)^2-2a^2}{h} = \lim_{h\to 0} \frac{4ha+2h^2}{h} = \lim_{h\to 0}(4a+2h) = 4a$.

50. $\lim_{x\to a} \frac{(x+a)^2-4x^2}{x-a}$

SOLUTION

$$\lim_{x\to a} \frac{(x+a)^2-4x^2}{x-a} = \lim_{x\to a} \frac{(x^2+2ax+a^2)-4x^2}{x-a} = \lim_{x\to a} \frac{-3x^2+2ax+a^2}{x-a}$$
$$= \lim_{x\to a} \frac{(a-x)(a+3x)}{x-a} = \lim_{x\to a}(-(a+3x)) = -4a.$$

51. $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$

SOLUTION $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$.

52. $\displaystyle\lim_{h\to 0}\frac{\sqrt{a+2h}-\sqrt{a}}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{\sqrt{a+2h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{a+2h}-\sqrt{a}\right)\left(\sqrt{a+2h}+\sqrt{a}\right)}{h\left(\sqrt{a+2h}+\sqrt{a}\right)}$$
$$=\lim_{h\to 0}\frac{2h}{h\left(\sqrt{a+2h}+\sqrt{a}\right)}=\lim_{h\to 0}\frac{2}{\sqrt{a+2h}+\sqrt{a}}=\frac{1}{\sqrt{a}}.$$

53. $\displaystyle\lim_{x\to 0}\frac{(x+a)^3-a^3}{x}$

SOLUTION $\displaystyle\lim_{x\to 0}\frac{(x+a)^3-a^3}{x}=\lim_{x\to 0}\frac{x^3+3x^2a+3xa^2+a^3-a^3}{x}=\lim_{x\to 0}(x^2+3xa+3a^2)=3a^2.$

54. $\displaystyle\lim_{h\to a}\frac{\frac{1}{h}-\frac{1}{a}}{h-a}$

SOLUTION $\displaystyle\lim_{h\to a}\frac{\frac{1}{h}-\frac{1}{a}}{h-a}=\lim_{h\to a}\frac{\frac{a-h}{ah}}{h-a}=\lim_{h\to a}\frac{a-h}{ah}\frac{1}{h-a}=\lim_{h\to a}\frac{-1}{ah}=-\frac{1}{a^2}$

Further Insights and Challenges

In Exercises 55–58, find all values of c such that the limit exists.

55. $\displaystyle\lim_{x\to c}\frac{x^2-5x-6}{x-c}$

SOLUTION $\displaystyle\lim_{x\to c}\frac{x^2-5x-6}{x-c}$ will exist provided that $x-c$ is a factor of the numerator. (Otherwise there will be an infinite discontinuity at $x=c$.) Since $x^2-5x-6=(x+1)(x-6)$, this occurs for $c=-1$ and $c=6$.

56. $\displaystyle\lim_{x\to 1}\frac{x^2+3x+c}{x-1}$

SOLUTION $\displaystyle\lim_{x\to 1}\frac{x^2+3x+c}{x-1}$ exists as long as $(x-1)$ is a factor of x^2+3x+c. If $x^2+3x+c=(x-1)(x+q)$, then $q-1=3$ and $-q=c$. Hence $q=4$ and $c=-4$.

57. $\displaystyle\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{c}{x^3-1}\right)$

SOLUTION Simplifying, we find

$$\frac{1}{x-1}-\frac{c}{x^3-1}=\frac{x^2+x+1-c}{(x-1)(x^2+x+1)}.$$

In order for the limit to exist as $x\to 1$, the numerator must evaluate to 0 at $x=1$. Thus, we must have $3-c=0$, which implies $c=3$.

58. $\displaystyle\lim_{x\to 0}\frac{1+cx^2-\sqrt{1+x^2}}{x^4}$

SOLUTION Rationalizing the numerator, we find

$$\frac{1+cx^2-\sqrt{1+x^2}}{x^4}=\frac{(1+cx^2-\sqrt{1+x^2})(1+cx^2+\sqrt{1+x^2})}{x^4(1+cx^2+\sqrt{1+x^2})}=\frac{(1+cx^2)^2-(1+x^2)}{x^4(1+cx^2+\sqrt{1+x^2})}$$
$$=\frac{(2c-1)x^2+c^2x^4}{x^4(1+cx^2+\sqrt{1+x^2})}.$$

In order for the limit to exist as $x\to 0$, the coefficient of x^2 in the numerator must be zero. Thus, we need $2c-1=0$, which implies $c=\frac{1}{2}$.

59. For which sign $\pm$ does the following limit exist?

$$\lim_{x\to 0}\left(\frac{1}{x}\pm\frac{1}{x(x-1)}\right)$$

SOLUTION

- The limit $\lim_{x\to 0}\left(\frac{1}{x}+\frac{1}{x(x-1)}\right) = \lim_{x\to 0}\frac{(x-1)+1}{x(x-1)} = \lim_{x\to 0}\frac{1}{x-1} = -1.$
- The limit $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x(x-1)}\right)$ does not exist.
  - As $x \to 0+$, we have $\frac{1}{x}-\frac{1}{x(x-1)} = \frac{(x-1)-1}{x(x-1)} = \frac{x-2}{x(x-1)} \to \infty.$
  - As $x \to 0-$, we have $\frac{1}{x}-\frac{1}{x(x-1)} = \frac{(x-1)-1}{x(x-1)} = \frac{x-2}{x(x-1)} \to -\infty.$

2.6 Trigonometric Limits

Preliminary Questions

1. Assume that $-x^4 \le f(x) \le x^2$. What is $\lim_{x\to 0} f(x)$? Is there enough information to evaluate $\lim_{x\to \frac{1}{2}} f(x)$? Explain.

SOLUTION Since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^2 = 0$, the squeeze theorem guarantees that $\lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to \frac{1}{2}} -x^4 = -\frac{1}{16} \neq \frac{1}{4} = \lim_{x\to \frac{1}{2}} x^2$, we do not have enough information to determine $\lim_{x\to \frac{1}{2}} f(x)$.

2. State the Squeeze Theorem carefully.

SOLUTION Assume that for $x \neq c$ (in some open interval containing c),

$$l(x) \le f(x) \le u(x)$$

and that $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L.$$

3. If you want to evaluate $\lim_{h\to 0}\frac{\sin 5h}{3h}$, it is a good idea to rewrite the limit in terms of the variable (choose one):

(a) $\theta = 5h$ **(b)** $\theta = 3h$ **(c)** $\theta = \frac{5h}{3}$

SOLUTION To match the given limit to the pattern of

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta},$$

it is best to substitute for the argument of the sine function; thus, rewrite the limit in terms of **(a):** $\theta = 5h$**.**

Exercises

1. State precisely the hypothesis and conclusions of the Squeeze Theorem for the situation in Figure 6.

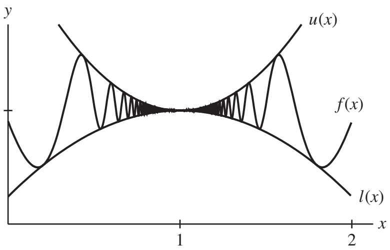


FIGURE 6

SOLUTION For all $x \neq 1$ on the open interval $(0, 2)$ containing $x = 1$, $\ell(x) \le f(x) \le u(x)$. Moreover,

$$\lim_{x\to 1}\ell(x) = \lim_{x\to 1} u(x) = 2.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 1} f(x) = 2.$$

2. In Figure 7, is $f(x)$ squeezed by $u(x)$ and $l(x)$ at $x = 3$? At $x = 2$?

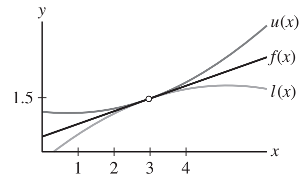


FIGURE 7

SOLUTION Because there is an open interval containing $x = 3$ on which $l(x) \le f(x) \le u(x)$ and $\lim_{x\to 3} l(x) = \lim_{x\to 3} u(x)$, $f(x)$ is *squeezed* by $u(x)$ and $l(x)$ at $x = 3$. Because there is an open interval containing $x = 2$ on which $l(x) \le f(x) \le u(x)$ but $\lim_{x\to 2} l(x) \ne \lim_{x\to 2} u(x)$, $f(x)$ is *trapped* by $u(x)$ and $l(x)$ at $x = 2$ but not *squeezed*.

3. What does the Squeeze Theorem say about $\lim_{x\to 7} f(x)$ if $\lim_{x\to 7} l(x) = \lim_{x\to 7} u(x) = 6$ and $f(x)$, $u(x)$, and $l(x)$ are related as in Figure 8? The inequality $f(x) \le u(x)$ is not satisfied for all x. Does this affect the validity of your conclusion?

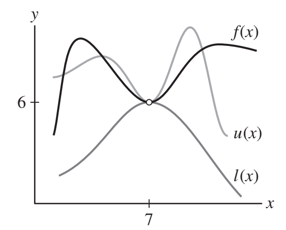


FIGURE 8

SOLUTION The Squeeze Theorem does not require that the inequalities $l(x) \le f(x) \le u(x)$ hold for all x, only that the inequalities hold on some open interval containing $x = c$. In Figure 8, it is clear that $l(x) \le f(x) \le u(x)$ on some open interval containing $x = 7$. Because $\lim_{x\to 7} u(x) = \lim_{x\to 7} l(x) = 6$, the Squeeze Theorem guarantees that $\lim_{x\to 7} f(x) = 6$.

4. Determine $\lim_{x\to 0} f(x)$ assuming that $\cos x \le f(x) \le 1$.

SOLUTION Because $\lim_{x\to 0} \cos x = \lim_{x\to 0} 1 = 1$, it follows that $\lim_{x\to 0} f(x) = 1$ by the Squeeze Theorem.

5. State whether the inequality provides sufficient information to determine $\lim_{x\to 1} f(x)$, and if so, find the limit.

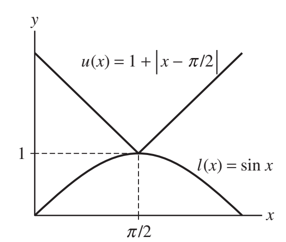
(a) $4x - 5 \le f(x) \le x^2$
(b) $2x - 1 \le f(x) \le x^2$
(c) $4x - x^2 \le f(x) \le x^2 + 2$

SOLUTION

(a) Because $\lim_{x\to 1} (4x - 5) = -1 \ne 1 = \lim_{x\to 1} x^2$, the given inequality does *not* provide sufficient information to determine $\lim_{x\to 1} f(x)$.
(b) Because $\lim_{x\to 1} (2x - 1) = 1 = \lim_{x\to 1} x^2$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 1$.
(c) Because $\lim_{x\to 1} (4x - x^2) = 3 = \lim_{x\to 1} (x^2 + 2)$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 3$.

6. $\boxed{\text{GU}}$ Plot the graphs of $u(x) = 1 + \left|x - \frac{\pi}{2}\right|$ and $l(x) = \sin x$ on the same set of axes. What can you say about $\lim_{x\to \frac{\pi}{2}} f(x)$ if $f(x)$ is squeezed by $l(x)$ and $u(x)$ at $x = \frac{\pi}{2}$?

SOLUTION



$\lim_{x\to \pi/2} u(x) = 1$ and $\lim_{x\to \pi/2} l(x) = 1$, so any function $f(x)$ satisfying $l(x) \le f(x) \le u(x)$ for all x near $\pi/2$ will satisfy $\lim_{x\to \pi/2} f(x) = 1$.

In Exercises 7–16, evaluate using the Squeeze Theorem.

7. $\lim_{x\to 0} x^2 \cos \frac{1}{x}$

SOLUTION Multiplying the inequality $-1 \le \cos \frac{1}{x} \le 1$, which holds for all $x \ne 0$, by x^2 yields $-x^2 \le x^2 \cos \frac{1}{x} \le x^2$. Because

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 0} x^2 \cos \frac{1}{x} = 0.$$

8. $\lim_{x\to 0} x \sin \frac{1}{x^2}$

SOLUTION Multiplying the inequality $\left|\sin \frac{1}{x^2}\right| \le 1$, which holds for $x \ne 0$, by $|x|$ yields $\left|x \sin \frac{1}{x^2}\right| \le |x|$ or $-|x| \le x \sin \frac{1}{x^2} \le |x|$. Because

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 0} x \sin \frac{1}{x^2} = 0.$$

9. $\lim_{x\to 1} (x-1) \sin \frac{\pi}{x-1}$

SOLUTION Multiplying the inequality $\left|\sin \frac{\pi}{x-1}\right| \le 1$, which holds for $x \ne 1$, by $|x-1|$ yields $\left|(x-1)\sin \frac{\pi}{x-1}\right| \le |x-1|$ or $-|x-1| \le (x-1)\sin \frac{\pi}{x-1} \le |x-1|$. Because

$$\lim_{x\to 1} -|x-1| = \lim_{x\to 1} |x-1| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 1} (x-1) \sin \frac{\pi}{x-1} = 0.$$

10. $\lim_{x\to 3} (x^2-9) \frac{x-3}{|x-3|}$

SOLUTION For $x \ne 3$, $\frac{x-3}{|x-3|} = \pm 1$; thus

$$-|x^2-9| \le (x^2-9)\frac{x-3}{|x-3|} \le |x^2-9|.$$

Because

$$\lim_{x\to 3} -|x^2-9| = \lim_{x\to 3} |x^2-9| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 3} (x^2-9)\frac{x-3}{|x-3|} = 0.$$

11. $\lim_{t\to 0} (2^t-1) \cos \frac{1}{t}$

SOLUTION Multiplying the inequality $\left|\cos \frac{1}{t}\right| \le 1$, which holds for $t \ne 0$, by $|2^t-1|$ yields $\left|(2^t-1)\cos \frac{1}{t}\right| \le |2^t-1|$ or $-|2^t-1| \le (2^t-1)\cos \frac{1}{t} \le |2^t-1|$. Because

$$\lim_{t\to 0} -|2^t-1| = \lim_{t\to 0} |2^t-1| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{t\to 0} (2^t-1)\cos \frac{1}{t} = 0.$$

12. $\lim_{x\to 0+} \sqrt{x}\, e^{\cos(\pi/x)}$

SOLUTION Since $-1 \le \cos\frac{\pi}{x} \le 1$ and e^x is an increasing function, it follows that

$$\frac{1}{e} \le e^{\cos(\pi/x)} \le e \quad \text{and} \quad \frac{1}{e}\sqrt{x} \le \sqrt{x}e^{\cos(\pi/x)} \le e\sqrt{x}.$$

Because

$$\lim_{x\to 0+} \frac{1}{e}\sqrt{x} = \lim_{x\to 0+} e\sqrt{x} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0+} \sqrt{x}e^{\cos(\pi/x)} = 0.$$

13. $\lim_{t\to 2} (t^2-4)\cos\dfrac{1}{t-2}$

SOLUTION Multiplying the inequality $\left|\cos\frac{1}{t-2}\right| \le 1$, which holds for $t \ne 2$, by $|t^2-4|$ yields $\left|(t^2-4)\cos\frac{1}{t-2}\right| \le |t^2-4|$ or $-|t^2-4| \le (t^2-4)\cos\frac{1}{t-2} \le |t^2-4|$. Because

$$\lim_{t\to 2} -|t^2-4| = \lim_{t\to 2} |t^2-4| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{t\to 2} (t^2-4)\cos\frac{1}{t-2} = 0.$$

14. $\lim_{x\to 0} \tan x \cos\left(\sin\dfrac{1}{x}\right)$

SOLUTION Multiplying the inequality $\left|\cos\left(\sin\frac{1}{x}\right)\right| \le 1$, which holds for $x \ne 0$, by $|\tan x|$ yields $\left|\tan x\cos\left(\sin\frac{1}{x}\right)\right| \le |\tan x|$ or $-|\tan x| \le \tan x \cos\left(\sin\frac{1}{x}\right) \le |\tan x|$. Because

$$\lim_{x\to 0} -|\tan x| = \lim_{x\to 0} |\tan x| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 0} \tan x \cos\left(\sin\frac{1}{x}\right) = 0.$$

15. $\lim_{\theta\to\frac{\pi}{2}} \cos\theta\cos(\tan\theta)$

SOLUTION Multiplying the inequality $|\cos(\tan\theta)| \le 1$, which holds for all θ near $\frac{\pi}{2}$ but not equal to $\frac{\pi}{2}$, by $|\cos\theta|$ yields $|\cos\theta\cos(\tan\theta)| \le |\cos\theta|$ or $-|\cos\theta| \le \cos\theta\cos(\tan\theta) \le |\cos\theta|$. Because

$$\lim_{\theta\to\frac{\pi}{2}} -|\cos\theta| = \lim_{\theta\to\frac{\pi}{2}} |\cos\theta| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{\theta\to\frac{\pi}{2}} \cos\theta\cos(\tan\theta) = 0.$$

16. $\lim_{t\to 0+} \sin t \tan^{-1}(\ln t)$

SOLUTION Multiplying the inequality $|\tan^{-1}(\ln t)| \le \frac{\pi}{2}$, which holds for all $t > 0$, by $|\sin t|$ yields $|\sin t\tan^{-1}(\ln t)| \le \frac{\pi}{2}|\sin t|$ or $-\frac{\pi}{2}|\sin t| \le \sin t\tan^{-1}(\ln t) \le \frac{\pi}{2}|\sin t|$. Because

$$\lim_{t\to 0+} -|\sin t| = \lim_{t\to 0+} |\sin t| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{t\to 0+} \sin t\tan^{-1}(\ln t) = 0.$$

In Exercises 17–26, evaluate using Theorem 2 as necessary.

17. $\lim_{x\to 0} \frac{\tan x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \frac{1}{\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1.$

18. $\lim_{x\to 0} \frac{\sin x \sec x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\sin x \sec x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \sec x = 1 \cdot 1 = 1.$

19. $\lim_{t\to 0} \frac{\sqrt{t^3+9}\sin t}{t}$

SOLUTION $\lim_{t\to 0} \frac{\sqrt{t^3+9}\sin t}{t} = \lim_{t\to 0} \sqrt{t^3+9} \cdot \lim_{t\to 0} \frac{\sin t}{t} = \sqrt{9} \cdot 1 = 3.$

20. $\lim_{t\to 0} \frac{\sin^2 t}{t}$

SOLUTION $\lim_{t\to 0} \frac{\sin^2 t}{t} = \lim_{t\to 0} \frac{\sin t}{t} \sin t = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \sin t = 1 \cdot 0 = 0.$

21. $\lim_{x\to 0} \frac{x^2}{\sin^2 x}$

SOLUTION $\lim_{x\to 0} \frac{x^2}{\sin^2 x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x} \frac{\sin x}{x}} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} \cdot \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} \cdot \frac{1}{1} = 1.$

22. $\lim_{t\to \frac{\pi}{2}} \frac{1-\cos t}{t}$

SOLUTION The function $\frac{1-\cos t}{t}$ is continuous at $\frac{\pi}{2}$; evaluate using substitution:

$$\lim_{t\to \frac{\pi}{2}} \frac{1-\cos t}{t} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

23. $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}$

SOLUTION $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta\cos\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta} = 0 \cdot 1 = 0.$

24. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 0 \cdot 1 = 0.$$

25. $\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t}$

SOLUTION $\frac{\sin t}{t}$ is continuous at $t = \frac{\pi}{4}$. Hence, by substitution

$$\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}.$$

26. $\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t}$

SOLUTION By factoring and applying the Product Law:

$$\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t} = \lim_{t\to 0} \cos t \cdot \lim_{t\to 0} \frac{1-\cos t}{t} = 1(0) = 0.$$

27. Let $L = \lim_{x\to 0} \dfrac{\sin 14x}{x}$.

(a) Show, by letting $\theta = 14x$, that $L = \lim_{\theta\to 0} 14\dfrac{\sin\theta}{\theta}$.

(b) Compute L.

SOLUTION

(a) Let $\theta = 14x$. Then $x = \frac{\theta}{14}$ and $\theta \to 0$ as $x \to 0$, so

$$L = \lim_{x\to 0} \frac{\sin 14x}{x} = \lim_{\theta\to 0} \frac{\sin\theta}{(\theta/14)} = \lim_{\theta\to 0} 14\frac{\sin\theta}{\theta}.$$

(b) Based on part (a),

$$L = 14 \lim_{\theta\to 0} \cdot \frac{\sin\theta}{\theta} = 14.$$

28. Evaluate $\lim_{h\to 0} \dfrac{\sin 9h}{\sin 7h}$. *Hint:* $\dfrac{\sin 9h}{\sin 7h} = \left(\dfrac{9}{7}\right)\left(\dfrac{\sin 9h}{9h}\right)\left(\dfrac{7h}{\sin 7h}\right)$.

SOLUTION

$$\lim_{h\to 0} \frac{\sin 9h}{\sin 7h} = \lim_{h\to 0} \frac{9}{7} \frac{(\sin 9h)\;/\;(9h)}{(\sin 7h)\;/\;(7h)} = \frac{9}{7} \frac{\lim_{h\to 0}(\sin 9h)/(9h)}{\lim_{h\to 0}(\sin 7h)/(7h)} = \frac{9}{7} \cdot \frac{1}{1} = \frac{9}{7}.$$

In Exercises 29–48, evaluate the limit.

29. $\lim_{h\to 0} \dfrac{\sin 9h}{h}$

SOLUTION $\lim_{h\to 0} \dfrac{\sin 9h}{h} = \lim_{h\to 0} 9\,\dfrac{\sin 9h}{9h} = 9.$

30. $\lim_{h\to 0} \dfrac{\sin 4h}{4h}$

SOLUTION Let $x = 4h$. Then $x \to 0$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sin 4h}{4h} = \lim x \to 0 \frac{\sin x}{x} = 1.$$

31. $\lim_{h\to 0} \dfrac{\sin h}{5h}$

SOLUTION $\lim_{h\to 0} \dfrac{\sin h}{5h} = \lim_{h\to 0} \dfrac{1}{5}\dfrac{\sin h}{h} = \dfrac{1}{5}.$

32. $\lim_{x\to \frac{\pi}{6}} \dfrac{x}{\sin 3x}$

SOLUTION $\lim_{x\to \frac{\pi}{6}} \dfrac{x}{\sin 3x} = \dfrac{\pi/6}{\sin(\pi/2)} = \dfrac{\pi}{6}.$

33. $\lim_{\theta\to 0} \dfrac{\sin 7\theta}{\sin 3\theta}$

SOLUTION We have

$$\frac{\sin 7\theta}{\sin 3\theta} = \frac{7}{3}\left(\frac{\sin 7\theta}{7\theta}\right)\left(\frac{3\theta}{\sin 3\theta}\right)$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin 7\theta}{3\theta} = \frac{7}{3}\left(\lim_{\theta\to 0} \frac{\sin 7\theta}{7\theta}\right)\left(\lim_{\theta\to 0} \frac{3\theta}{\sin 3\theta}\right) = \frac{7}{3}(1)(1) = \frac{7}{3}$$

34. $\lim_{x\to 0} \dfrac{\tan 4x}{9x}$

SOLUTION $\lim_{x\to 0} \dfrac{\tan 4x}{9x} = \lim_{x\to 0} \dfrac{1}{9} \cdot \dfrac{\sin 4x}{4x} \cdot \dfrac{4}{\cos 4x} = \dfrac{4}{9}.$

35. $\lim_{x\to 0} x \csc 25x$

SOLUTION Let $h = 25x$. Then

$$\lim_{x\to 0} x \csc 25x = \lim_{h\to 0} \frac{h}{25} \csc h = \frac{1}{25} \lim_{h\to 0} \frac{h}{\sin h} = \frac{1}{25}.$$

36. $\lim_{t\to 0} \frac{\tan 4t}{t \sec t}$

SOLUTION $\lim_{t\to 0} \frac{\tan 4t}{t \sec t} = \lim_{t\to 0} \frac{4 \sin 4t}{4t \cos(4t) \sec(t)} = \lim_{t\to 0} \frac{4 \cos t}{\cos 4t} \cdot \frac{\sin 4t}{4t} = 4.$

37. $\lim_{h\to 0} \frac{\sin 2h \ \sin 3h}{h^2}$

SOLUTION

$$\begin{aligned}\lim_{h\to 0} \frac{\sin 2h \ \sin 3h}{h^2} &= \lim_{h\to 0} \frac{\sin 2h \ \sin 3h}{h \cdot h} = \lim_{h\to 0} \frac{\sin 2h}{h} \frac{\sin 3h}{h} \\ &= \lim_{h\to 0} 2\frac{\sin 2h}{2h} 3\frac{\sin 3h}{3h} = \lim_{h\to 0} 2\frac{\sin 2h}{2h} \lim_{h\to 0} 3\frac{\sin 3h}{3h} = 2 \cdot 3 = 6.\end{aligned}$$

38. $\lim_{z\to 0} \frac{\sin(z/3)}{\sin z}$

SOLUTION $\lim_{z\to 0} \frac{\sin(z/3)}{\sin z} \cdot \frac{z/3}{z/3} = \lim_{z\to 0} \frac{1}{3} \cdot \frac{z}{\sin z} \cdot \frac{\sin(z/3)}{z/3} = \frac{1}{3}.$

39. $\lim_{\theta\to 0} \frac{\sin(-3\theta)}{\sin(4\theta)}$

SOLUTION $\lim_{\theta\to 0} \frac{\sin(-3\theta)}{\sin(4\theta)} = \lim_{\theta\to 0} \frac{-\sin(3\theta)}{3\theta} \cdot \frac{3}{4} \cdot \frac{4\theta}{\sin(4\theta)} = -\frac{3}{4}.$

40. $\lim_{x\to 0} \frac{\tan 4x}{\tan 9x}$

SOLUTION $\lim_{x\to 0} \frac{\tan 4x}{\tan 9x} = \lim_{x\to 0} \frac{\cos 9x}{\cos 4x} \cdot \frac{\sin 4x}{4x} \cdot \frac{4}{9} \cdot \frac{9x}{\sin 9x} = \frac{4}{9}.$

41. $\lim_{t\to 0} \frac{\csc 8t}{\csc 4t}$

SOLUTION $\lim_{t\to 0} \frac{\csc 8t}{\csc 4t} = \lim_{t\to 0} \frac{\sin 4t}{\sin 8t} \cdot \frac{8t}{4t} \cdot \frac{1}{2} = \frac{1}{2}.$

42. $\lim_{x\to 0} \frac{\sin 5x \sin 2x}{\sin 3x \sin 5x}$

SOLUTION $\lim_{x\to 0} \frac{\sin 5x \sin 2x}{\sin 3x \sin 5x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{3} \cdot \frac{3x}{\sin 3x} = \frac{2}{3}.$

43. $\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x}$

SOLUTION $\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x} = \lim_{x\to 0} \left(3 \frac{\sin 3x}{3x} \cdot \frac{2}{5} \frac{(\sin 2x) \,/\, (2x)}{(\sin 5x) \,/\, (5x)}\right) = \frac{6}{5}.$

44. $\lim_{h\to 0} \frac{1 - \cos 2h}{h}$

SOLUTION $\lim_{h\to 0} \frac{1 - \cos 2h}{h} = \lim_{h\to 0} 2\frac{1 - \cos 2h}{2h} = 2 \lim_{h\to 0} \frac{1 - \cos 2h}{2h} = 2 \cdot 0 = 0.$

45. $\lim_{h\to 0} \frac{\sin(2h)(1 - \cos h)}{h^2}$

SOLUTION $\lim_{h\to 0} \frac{\sin(2h)(1 - \cos h)}{h^2} = \lim_{h\to 0} \frac{\sin(2h)}{h} \lim_{h\to 0} \frac{1 - \cos h}{h} = 1 \cdot 0 = 0.$

46. $\lim_{t\to 0} \dfrac{1-\cos 2t}{\sin^2 3t}$

SOLUTION Using the identity $\cos 2t = 1 - 2\sin^2 t$, we find

$$\frac{1-\cos 2t}{\sin^2 3t} = \frac{2\sin^2 t}{\sin^2 3t} = \frac{2}{9}\left(\frac{\sin t}{t}\right)^2\left(\frac{3t}{\sin 3t}\right)^2.$$

Thus,

$$\lim_{t\to 0}\frac{1-\cos 2t}{\sin^2 3t} = \lim_{t\to 0}\frac{2}{9}\left(\frac{\sin t}{t}\right)^2\left(\frac{3t}{\sin 3t}\right)^2 = \frac{2}{9}.$$

47. $\lim_{\theta\to 0} \dfrac{\cos 2\theta - \cos\theta}{\theta}$

SOLUTION

$$\lim_{\theta\to 0}\frac{\cos 2\theta-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{(\cos 2\theta - 1)+(1-\cos\theta)}{\theta} = \lim_{\theta\to 0}\frac{\cos 2\theta - 1}{\theta} + \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$

$$= -2\lim_{\theta\to 0}\frac{1-\cos 2\theta}{2\theta} + \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = -2\cdot 0 + 0 = 0.$$

48. $\lim_{h\to\frac{\pi}{2}} \dfrac{1-\cos 3h}{h}$

SOLUTION The function is continuous at $\frac{\pi}{2}$, so we may use substitution:

$$\lim_{h\to\frac{\pi}{2}}\frac{1-\cos 3h}{h} = \frac{1-\cos 3\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

49. Calculate $\lim_{x\to 0-}\dfrac{\sin x}{|x|}$.

SOLUTION

$$\lim_{x\to 0-}\frac{\sin x}{|x|} = \lim_{x\to 0-}\frac{\sin x}{-x} = -1$$

50. Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to evaluate the limit $\lim_{\theta\to 0}\dfrac{\sin 3\theta - 3\sin\theta}{\theta^3}$.

SOLUTION Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we find

$$\frac{\sin 3\theta - 3\sin\theta}{\theta^3} = -4\left(\frac{\sin\theta}{\theta}\right)^3.$$

Therefore,

$$\lim_{\theta\to 0}\frac{\sin 3\theta - 3\sin\theta}{\theta^3} = -4\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3 = -4(1)^3 = -4.$$

51. Prove the following result stated in Theorem 2:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

7

Hint: $\dfrac{1-\cos\theta}{\theta} = \dfrac{1}{1+\cos\theta}\cdot\dfrac{1-\cos^2\theta}{\theta}$.

SOLUTION

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\frac{1-\cos^2\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\frac{\sin^2\theta}{\theta}$$

$$= \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\sin\theta\,\frac{\sin\theta}{\theta}$$

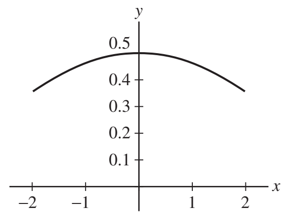
$$= \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\sin\theta\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{1}{2}\cdot 0\cdot 1 = 0.$$

52. GU Investigate $\lim_{h\to 0} \dfrac{1-\cos h}{h^2}$ numerically (and graphically if you have a graphing utility). Then prove that the limit is equal to $\frac{1}{2}$. *Hint:* See the hint for Exercise 51.

SOLUTION

h	-0.1	-0.01	0.01	0.1
$\dfrac{1-\cos h}{h^2}$	0.499583	0.499996	0.499996	0.499583

  The limit is $\frac{1}{2}$.

- 

- $\lim_{h\to 0} \dfrac{1-\cos h}{h^2} = \lim_{h\to 0} \dfrac{1-\cos^2 h}{h^2(1+\cos h)} = \lim_{h\to 0} \left(\dfrac{\sin h}{h}\right)^2 \dfrac{1}{1+\cos h} = \dfrac{1}{2}.$

In Exercises 53–55, evaluate using the result of Exercise 52.

53. $\lim_{h\to 0} \dfrac{\cos 3h - 1}{h^2}$

SOLUTION We make the substitution $\theta = 3h$. Then $h = \theta/3$, and

$$\lim_{h\to 0} \frac{\cos 3h - 1}{h^2} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{(\theta/3)^2} = -9 \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = -\frac{9}{2}.$$

54. $\lim_{h\to 0} \dfrac{\cos 3h - 1}{\cos 2h - 1}$

SOLUTION Write

$$\frac{\cos 3h - 1}{\cos 2h - 1} = \frac{1-\cos 3h}{(3h)^2} \cdot \frac{(2h)^2}{1-\cos 2h} \cdot \frac{9h^2}{4h^2}.$$

Then

$$\lim_{h\to 0} \frac{\cos 3h - 1}{\cos 2h - 1} = \frac{9}{4} \lim_{h\to 0} \frac{1-\cos 3h}{(3h)^2} \cdot \lim_{h\to 0} \frac{(2h)^2}{1-\cos 2h} = \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{1}{1/2} = \frac{9}{4}.$$

55. $\lim_{t\to 0} \dfrac{\sqrt{1-\cos t}}{t}$

SOLUTION $\lim_{t\to 0+} \dfrac{\sqrt{1-\cos t}}{t} = \sqrt{\lim_{t\to 0+} \dfrac{1-\cos t}{t^2}} = \sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$; on the other hand, $\lim_{t\to 0-} \dfrac{\sqrt{1-\cos t}}{t} = -\sqrt{\lim_{t\to 0-} \dfrac{1-\cos t}{t^2}} = -\sqrt{\dfrac{1}{2}} = -\dfrac{\sqrt{2}}{2}$.

56. Use the Squeeze Theorem to prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

SOLUTION Suppose $\lim_{x\to c} |f(x)| = 0$. Then

$$\lim_{x\to c} -|f(x)| = -\lim_{x\to c} |f(x)| = 0.$$

Now, for all x, the inequalities

$$-|f(x)| \le f(x) \le |f(x)|$$

hold. Because $\lim_{x\to c} |f(x)| = 0$ and $\lim_{x\to c} -|f(x)| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to c} f(x) = 0$.

Further Insights and Challenges

57. Use the result of Exercise 52 to prove that for $m \neq 0$,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = -\frac{m^2}{2}$$

SOLUTION Substitute $u = mx$ into $\dfrac{\cos mx - 1}{x^2}$. We obtain $x = \frac{u}{m}$. As $x \to 0$, $u \to 0$; therefore,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = \lim_{u\to 0} \frac{\cos u - 1}{(u/m)^2} = \lim_{u\to 0} m^2 \frac{\cos u - 1}{u^2} = m^2\left(-\frac{1}{2}\right) = -\frac{m^2}{2}.$$

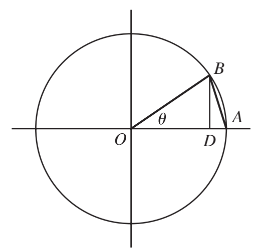
58. Using a diagram of the unit circle and the Pythagorean Theorem, show that

$$\sin^2\theta \le (1-\cos\theta)^2 + \sin^2\theta \le \theta^2$$

Conclude that $\sin^2\theta \le 2(1-\cos\theta) \le \theta^2$ and use this to give an alternative proof of Eq. (7) in Exercise 51. Then give an alternative proof of the result in Exercise 52.

SOLUTION

- Consider the unit circle shown below. The triangle BDA is a right triangle. It has base $1-\cos\theta$, altitude $\sin\theta$, and hypotenuse h. Observe that the hypotenuse h is less than the arc length $AB = \text{radius}\cdot\text{angle} = 1\cdot\theta = \theta$. Apply the Pythagorean Theorem to obtain $(1-\cos\theta)^2 + \sin^2\theta = h^2 \le \theta^2$. The inequality $\sin^2\theta \le (1-\cos\theta)^2 + \sin^2\theta$ follows from the fact that $(1-\cos\theta)^2 \ge 0$.



- Note that

$$(1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta = 2(1-\cos\theta).$$

  Therefore,

$$\sin^2\theta \le 2(1-\cos\theta) \le \theta^2.$$

- Divide the previous inequality by 2θ to obtain

$$\frac{\sin^2\theta}{2\theta} \le \frac{1-\cos\theta}{\theta} \le \frac{\theta}{2}.$$

  Because

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{2\theta} = \frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\sin\theta = \frac{1}{2}(1)(0) = 0,$$

  and $\displaystyle\lim_{h\to 0}\frac{\theta}{2} = 0$, it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$$

- Divide the inequality

$$\sin^2\theta \le 2(1-\cos\theta) \le \theta^2$$

  by $2\theta^2$ to obtain

$$\frac{\sin^2\theta}{2\theta^2} \le \frac{1-\cos\theta}{\theta^2} \le \frac{1}{2}.$$

Because

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{2\theta^2} = \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2 = \frac{1}{2}(1^2) = \frac{1}{2},$$

and $\lim_{h\to 0} \frac{1}{2} = \frac{1}{2}$, it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

59. **(a)** Investigate $\lim_{x\to c} \frac{\sin x - \sin c}{x - c}$ numerically for the five values $c = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.
(b) Can you guess the answer for general c?
(c) Check that your answer to (b) works for two other values of c.

SOLUTION

(a)

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.999983	0.99999983	0.99999983	0.999983

Here $c = 0$ and $\cos c = 1$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.868511	0.866275	0.865775	0.863511

Here $c = \frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx 0.866025$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.504322	0.500433	0.499567	0.495662

Here $c = \frac{\pi}{3}$ and $\cos c = \frac{1}{2}$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.710631	0.707460	0.706753	0.703559

Here $c = \frac{\pi}{4}$ and $\cos c = \frac{\sqrt{2}}{2} \approx 0.707107$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.005000	0.000500	−0.000500	−0.005000

Here $c = \frac{\pi}{2}$ and $\cos c = 0$.

(b) $\lim_{x\to c} \frac{\sin x - \sin c}{x - c} = \cos c$.

(c)

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	−0.411593	−0.415692	−0.416601	−0.420686

Here $c = 2$ and $\cos c = \cos 2 \approx -0.416147$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.863511	0.865775	0.866275	0.868511

Here $c = -\frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx 0.866025$.

2.7 Limits at Infinity

Preliminary Questions

1. Assume that

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad \lim_{x\to L} g(x) = \infty$$

Which of the following statements are correct?

(a) $x = L$ is a vertical asymptote of $g(x)$.

(b) $y = L$ is a horizontal asymptote of $g(x)$.

(c) $x = L$ is a vertical asymptote of $f(x)$.

(d) $y = L$ is a horizontal asymptote of $f(x)$.

SOLUTION

(a) Because $\lim_{x\to L} g(x) = \infty$, $x = L$ is a vertical asymptote of $g(x)$. This statement is correct.

(b) This statement is not correct.

(c) This statement is not correct.

(d) Because $\lim_{x\to\infty} f(x) = L$, $y = L$ is a horizontal asymptote of $f(x)$. This statement is correct.

2. What are the following limits?

(a) $\lim_{x\to\infty} x^3$ **(b)** $\lim_{x\to-\infty} x^3$ **(c)** $\lim_{x\to-\infty} x^4$

SOLUTION

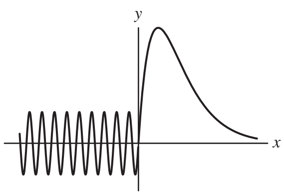
(a) $\lim_{x\to\infty} x^3 = \infty$

(b) $\lim_{x\to-\infty} x^3 = -\infty$

(c) $\lim_{x\to-\infty} x^4 = \infty$

3. Sketch the graph of a function that approaches a limit as $x \to \infty$ but does not approach a limit (either finite or infinite) as $x \to -\infty$.

SOLUTION



4. What is the sign of a if $f(x) = ax^3 + x + 1$ satisfies $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION Because $\lim_{x\to-\infty} x^3 = -\infty$, a must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

5. What is the sign of the leading coefficient a_7 if $f(x)$ is a polynomial of degree 7 such that $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION The behavior of $f(x)$ as $x \to -\infty$ is controlled by the leading term; that is, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} a_7x^7$. Because $x^7 \to -\infty$ as $x \to -\infty$, a_7 must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

6. Explain why $\lim_{x\to\infty} \sin \frac{1}{x}$ exists but $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist. What is $\lim_{x\to\infty} \sin \frac{1}{x}$?

SOLUTION As $x \to \infty$, $\frac{1}{x} \to 0$, so

$$\lim_{x\to\infty} \sin \frac{1}{x} = \sin 0 = 0.$$

On the other hand, $\frac{1}{x} \to \pm\infty$ as $x \to 0$, and as $\frac{1}{x} \to \pm\infty$, $\sin \frac{1}{x}$ oscillates infinitely often. Thus

$$\lim_{x\to 0} \sin \frac{1}{x}$$

does not exist.

Exercises

1. What are the horizontal asymptotes of the function in Figure 6?

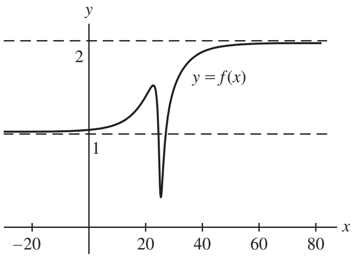


FIGURE 6

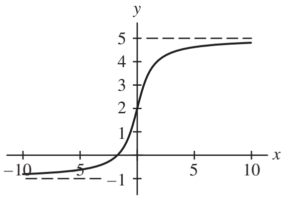
SOLUTION Because

$$\lim_{x\to-\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 2,$$

the function $f(x)$ has horizontal asymptotes of $y = 1$ and $y = 2$.

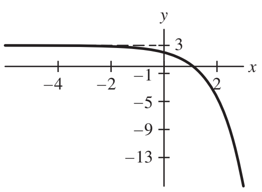
2. Sketch the graph of a function $f(x)$ that has both $y = -1$ and $y = 5$ as horizontal asymptotes.

SOLUTION



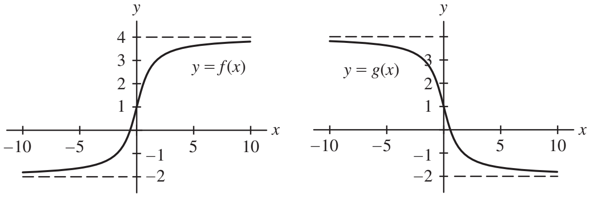
3. Sketch the graph of a function $f(x)$ with a single horizontal asymptote $y = 3$.

SOLUTION



4. Sketch the graphs of two functions $f(x)$ and $g(x)$ that have both $y = -2$ and $y = 4$ as horizontal asymptotes but $\lim_{x\to\infty} f(x) \neq \lim_{x\to\infty} g(x)$.

SOLUTION



5. GU Investigate the asymptotic behavior of $f(x) = \dfrac{x^3}{x^3 + x}$ numerically and graphically:

(a) Make a table of values of $f(x)$ for $x = \pm 50, \pm 100, \pm 500, \pm 1000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

SOLUTION

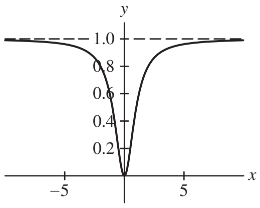
(a) From the table below, it appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3+x} = 1.$$

x	± 50	± 100	± 500	± 1000
$f(x)$	0.999600	0.999900	0.999996	0.999999

(b) From the graph below, it also appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3+x} = 1.$$



(c) The horizontal asymptote of $f(x)$ is $y = 1$.

6. GU Investigate $\lim_{x\to\pm\infty} \frac{12x+1}{\sqrt{4x^2+9}}$ numerically and graphically:

(a) Make a table of values of $f(x) = \frac{12x+1}{\sqrt{4x^2+9}}$ for $x = \pm 100, \pm 500, \pm 1000, \pm 10{,}000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

SOLUTION

(a) From the tables below, it appears that

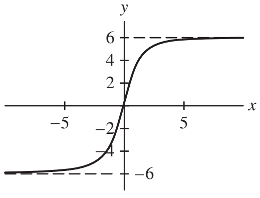
$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6.$$

x	-100	-500	-1000	-10000
$f(x)$	-5.994326	-5.998973	-5.999493	-5.999950

x	100	500	1000	10000
$f(x)$	6.004325	6.000973	6.000493	6.000050

(b) From the graph below, it also appears that

$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6.$$



(c) The horizontal asymptotes of $f(x)$ are $y = -6$ and $y = 6$.

In Exercises 7–16, evaluate the limit.

7. $\lim_{x\to\infty} \frac{x}{x+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{x}{x+9} = \lim_{x\to\infty} \frac{x^{-1}(x)}{x^{-1}(x+9)} = \lim_{x\to\infty} \frac{1}{1+\frac{9}{x}} = \frac{1}{1+0} = 1.$$

8. $\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9} = \lim_{x\to\infty} \frac{x^{-2}(3x^2+20x)}{x^{-2}(4x^2+9)} = \lim_{x\to\infty} \frac{3+\frac{20}{x}}{4+\frac{9}{x^2}} = \frac{3+0}{4+0} = \frac{3}{4}.$$

9. $\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29} = \lim_{x\to\infty} \frac{x^{-4}(3x^2+20x)}{x^{-4}(2x^4+3x^3-29)} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+\frac{20}{x^3}}{2+\frac{3}{x}-\frac{29}{x^4}} = \frac{0}{2} = 0.$$

10. $\lim_{x\to\infty} \frac{4}{x+5}$

SOLUTION

$$\lim_{x\to\infty} \frac{4}{x+5} = \lim_{x\to\infty} \frac{x^{-1}(4)}{x^{-1}(x+5)} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{5}{x}} = \frac{0}{1} = 0.$$

11. $\lim_{x\to\infty} \frac{7x-9}{4x+3}$

SOLUTION

$$\lim_{x\to\infty} \frac{7x-9}{4x+3} = \lim_{x\to\infty} \frac{x^{-1}(7x-9)}{x^{-1}(4x+3)} = \lim_{x\to\infty} \frac{7-\frac{9}{x}}{4+\frac{3}{x}} = \frac{7}{4}.$$

12. $\lim_{x\to\infty} \frac{9x^2-2}{6-29x}$

SOLUTION

$$\lim_{x\to\infty} \frac{9x^2-2}{6-29x} = \lim_{x\to\infty} \frac{x^{-1}(9x^2-2)}{x^{-1}(6-29x)} = \lim_{x\to\infty} \frac{9x-\frac{2}{x}}{\frac{6}{x}-29} = \frac{\infty}{-29} = -\infty.$$

13. $\lim_{x\to-\infty} \frac{7x^2-9}{4x+3}$

SOLUTION

$$\lim_{x\to-\infty} \frac{7x^2-9}{4x+3} = \lim_{x\to-\infty} \frac{x^{-1}(7x^2-9)}{x^{-1}(4x+3)} = \lim_{x\to-\infty} \frac{7x-\frac{9}{x}}{4+\frac{3}{x}} = -\infty.$$

14. $\lim_{x\to-\infty} \frac{5x-9}{4x^3+2x+7}$

SOLUTION

$$\lim_{x\to-\infty} \frac{5x-9}{4x^3+2x+7} = \lim_{x\to-\infty} \frac{x^{-3}(5x-9)}{x^{-3}(4x^3+2x+7)} = \lim_{x\to-\infty} \frac{\frac{5}{x^2}-\frac{9}{x^3}}{4+\frac{2}{x^2}+\frac{7}{x^3}} = \frac{0}{4} = 0.$$

15. $\lim_{x\to-\infty} \frac{3x^3 - 10}{x + 4}$

SOLUTION

$$\lim_{x\to-\infty} \frac{3x^3 - 10}{x + 4} = \lim_{x\to-\infty} \frac{x^{-1}(3x^3 - 10)}{x^{-1}(x + 4)} = \lim_{x\to-\infty} \frac{3x^2 - \frac{10}{x}}{1 + \frac{4}{x}} = \frac{\infty}{1} = \infty.$$

16. $\lim_{x\to-\infty} \frac{2x^5 + 3x^4 - 31x}{8x^4 - 31x^2 + 12}$

SOLUTION

$$\lim_{x\to-\infty} \frac{2x^5 + 3x^4 - 31x}{8x^4 - 31x^2 + 12} = \lim_{x\to-\infty} \frac{x^{-4}(2x^5 + 3x^4 - 31x)}{x^{-4}(8x^4 - 31x^2 + 12)} = \lim_{x\to-\infty} \frac{2x + 3 - \frac{31}{x^3}}{8 - \frac{31}{x^2} + \frac{12}{x^4}} = \frac{-\infty}{8} = -\infty.$$

In Exercises 17–22, find the horizontal asymptotes.

17. $f(x) = \frac{2x^2 - 3x}{8x^2 + 8}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{2x^2 - 3x}{8x^2 + 8} = \lim_{x\to\infty} \frac{2 - \frac{3}{x}}{8 + \frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}.$$

Similarly,

$$\lim_{x\to-\infty} \frac{2x^2 - 3x}{8x^2 + 8} = \lim_{x\to-\infty} \frac{2 - \frac{3}{x}}{8 + \frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}.$$

Thus, the horizontal asymptote of $f(x)$ is $y = \frac{1}{4}$.

18. $f(x) = \frac{8x^3 - x^2}{7 + 11x - 4x^4}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to-\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0.$$

Thus, the horizontal asymptote of $f(x)$ is $y = 0$.

19. $f(x) = \dfrac{\sqrt{36x^2+7}}{9x+4}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^2+7}}{9x+4} = \lim_{x\to\infty} \frac{\sqrt{36+\frac{7}{x^2}}}{9+\frac{4}{x}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

On the other hand, for $x < 0$, $x^{-1} = -|x^{-1}| = -\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{\sqrt{36x^2+7}}{9x+4} = \lim_{x\to-\infty} \frac{-\sqrt{36+\frac{7}{x^2}}}{9+\frac{4}{x}} = \frac{-\sqrt{36}}{9} = -\frac{2}{3}.$$

Thus, the horizontal asymptotes of $f(x)$ are $y = \frac{2}{3}$ and $y = -\frac{2}{3}$.

20. $f(x) = \dfrac{\sqrt{36x^4+7}}{9x^2+4}$

SOLUTION For all $x \neq 0$, $x^{-2} = |x^{-2}| = \sqrt{x^{-4}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} = \lim_{x\to\infty} \frac{\sqrt{36+\frac{7}{x^4}}}{9+\frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

Similarly,

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} = \lim_{x\to-\infty} \frac{\sqrt{36+\frac{7}{x^4}}}{9+\frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

Thus, the horizontal asymptote of $f(x)$ is $y = \frac{2}{3}$.

21. $f(t) = \dfrac{e^t}{1+e^{-t}}$

SOLUTION With

$$\lim_{t\to\infty} \frac{e^t}{1+e^{-t}} = \frac{\infty}{1} = \infty$$

and

$$\lim_{t\to-\infty} \frac{e^t}{1+e^{-t}} = 0,$$

the function $f(t)$ has one horizontal asymptote, $y = 0$.

22. $f(t) = \dfrac{t^{1/3}}{(64t^2+9)^{1/6}}$

SOLUTION For $t > 0$, $t^{-1/3} = |t^{-1/3}| = (t^{-2})^{1/6}$, so

$$\lim_{t\to\infty} \frac{t^{1/3}}{(64t^2+9)^{1/6}} = \lim_{t\to\infty} \frac{1}{(64+\frac{9}{t^2})^{1/6}} = \frac{1}{2}.$$

On the other hand, for $t < 0$, $t^{-1/3} = -|t^{-1/3}| = -(t^{-2})^{1/6}$, so

$$\lim_{t\to-\infty} \frac{t^{1/3}}{(64t^2+9)^{1/6}} = \lim_{t\to-\infty} \frac{1}{-(64+\frac{9}{t^2})^{1/6}} = -\frac{1}{2}.$$

Thus, the horizontal asymptotes for $f(t)$ are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

In Exercises 23–30, evaluate the limit.

23. $\lim_{x\to\infty} \frac{\sqrt{9x^4+3x+2}}{4x^3+1}$

SOLUTION For $x > 0$, $x^{-3} = |x^{-3}| = \sqrt{x^{-6}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{9x^4+3x+2}}{4x^3+1} = \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^2}+\frac{3}{x^5}+\frac{2}{x^6}}}{4+\frac{1}{x^3}} = 0.$$

24. $\lim_{x\to\infty} \frac{\sqrt{x^3+20x}}{10x-2}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{x^3+20x}}{10x-2} = \lim_{x\to\infty} \frac{\sqrt{x+\frac{20}{x}}}{10-\frac{2}{x}} = \frac{\infty}{10} = \infty.$$

25. $\lim_{x\to-\infty} \frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}}$

SOLUTION For $x < 0$, $x^{-2} = |x^{-2}| = \sqrt{x^{-4}}$, so

$$\lim_{x\to-\infty} \frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}} = \lim_{x\to-\infty} \frac{8+\frac{7}{x^{5/3}}}{\sqrt{16+\frac{6}{x^4}}} = \frac{8}{\sqrt{16}} = 2.$$

26. $\lim_{x\to-\infty} \frac{4x-3}{\sqrt{25x^2+4x}}$

SOLUTION For $x < 0$, $x^{-1} = -|x^{-1}| = -\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{4x-3}{\sqrt{25x^2+4x}} = \lim_{x\to-\infty} \frac{4-\frac{3}{x}}{-\sqrt{25+\frac{4}{x}}} = \frac{4}{-\sqrt{25}} = -\frac{4}{5}.$$

27. $\lim_{t\to\infty} \frac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2}$

SOLUTION $\lim_{t\to\infty} \frac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2} = \lim_{t\to\infty} \frac{1+\frac{1}{t}}{(4+\frac{1}{t^{2/3}})^2} = \frac{1}{16}.$

28. $\lim_{t\to\infty} \frac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}}$

SOLUTION $\lim_{t\to\infty} \frac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}} = \lim_{t\to\infty} \frac{1-\frac{9}{t}}{(8+\frac{2}{t^4})^{1/3}} = \frac{1}{2}.$

29. $\lim_{x\to-\infty} \frac{|x|+x}{x+1}$

SOLUTION For $x < 0$, $|x| = -x$. Therefore, for all $x < 0$,

$$\frac{|x|+x}{x+1} = \frac{-x+x}{x+1} = 0;$$

consequently,

$$\lim_{x\to-\infty} \frac{|x|+x}{x+1} = 0.$$

30. $\displaystyle\lim_{t\to-\infty} \frac{4+6e^{2t}}{5-9e^{3t}}$

SOLUTION Because

$$\lim_{t\to-\infty} e^{2t} = \lim_{t\to-\infty} e^{3t} = 0,$$

it follows that

$$\lim_{t\to-\infty} \frac{4+6e^{2t}}{5-9e^{3t}} = \frac{4+0}{5-0} = \frac{4}{5}.$$

31. Determine $\displaystyle\lim_{x\to\infty} \tan^{-1} x$. Explain geometrically.

SOLUTION As an angle θ increases from 0 to $\frac{\pi}{2}$, its tangent $x = \tan\theta$ approaches ∞. Therefore,

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}.$$

Geometrically, this means that the graph of $y = \tan^{-1} x$ has a horizontal asymptote at $y = \frac{\pi}{2}$.

32. Show that $\displaystyle\lim_{x\to\infty} (\sqrt{x^2+1} - x) = 0$. *Hint:* Observe that

$$\sqrt{x^2+1} - x = \frac{1}{\sqrt{x^2+1}+x}$$

SOLUTION Rationalizing the "numerator," we find

$$\begin{aligned}\sqrt{x^2+1} - x &= (\sqrt{x^2+1} - x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\ &= \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}.\end{aligned}$$

Thus,

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x) = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0.$$

33. According to the **Michaelis–Menten equation** (Figure 7), when an enzyme is combined with a substrate of concentration s (in millimolars), the reaction rate (in micromolars/min) is

$$R(s) = \frac{As}{K+s} \qquad (A,\ K \text{ constants})$$

(a) Show, by computing $\displaystyle\lim_{s\to\infty} R(s)$, that A is the limiting reaction rate as the concentration s approaches ∞.

(b) Show that the reaction rate $R(s)$ attains one-half of the limiting value A when $s = K$.

(c) For a certain reaction, $K = 1.25$ mM and $A = 0.1$. For which concentration s is $R(s)$ equal to 75% of its limiting value?

Leonor Michaelis 1875–1949 Maud Menten 1879–1960

FIGURE 7 Canadian-born biochemist Maud Menten is best known for her fundamental work on enzyme kinetics with German scientist Leonor Michaelis. She was also an accomplished painter, clarinetist, mountain climber, and master of numerous languages.

SOLUTION

(a) $\lim_{s\to\infty} R(s) = \lim_{s\to\infty} \frac{As}{K+s} = \lim_{s\to\infty} \frac{A}{1+\frac{K}{s}} = A.$

(b) Observe that

$$R(K) = \frac{AK}{K+K} = \frac{AK}{2K} = \frac{A}{2},$$

have of the limiting value.

(c) By part (a), the limiting value is 0.1, so we need to determine the value of s that satisfies

$$R(s) = \frac{0.1s}{1.25+s} = 0.075.$$

Solving this equation for s yields

$$s = \frac{(1.25)(0.075)}{0.025} = 3.75 \text{ mM}.$$

34. Suppose that the average temperature of the earth is $T(t) = 283 + 3(1 - e^{-0.03t})$ kelvins, where t is the number of years since 2000.

(a) Calculate the long-term average $L = \lim_{t\to\infty} T(t)$.

(b) At what time is $T(t)$ within one-half a degree of its limiting value?

SOLUTION

(a) $L = \lim_{t\to\infty} T(t) = \lim_{t\to\infty}(283 + 3(1 - e^{-0.03t})) = 286$ kelvins.

(b) We need to solve the equation

$$T(t) = 283 + 3(1 - e^{-0.03t}) = 285.5.$$

This yields

$$t = \frac{1}{0.03}\ln 6 \approx 59.73.$$

The average temperature of the earth will be within one-half a degree of its limiting value in roughly 2060.

In Exercises 35–42, calculate the limit.

35. $\lim_{x\to\infty}\left(\sqrt{4x^4+9x} - 2x^2\right)$

SOLUTION Write

$$\begin{aligned}\sqrt{4x^4+9x} - 2x^2 &= \left(\sqrt{4x^4+9x} - 2x^2\right)\frac{\sqrt{4x^4+9x}+2x^2}{\sqrt{4x^4+9x}+2x^2}\\ &= \frac{(4x^4+9x)-4x^4}{\sqrt{4x^4+9x}+2x^2} = \frac{9x}{\sqrt{4x^4+9x}+2x^2}.\end{aligned}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{4x^4+9x} - 2x^2) = \lim_{x\to\infty}\frac{9x}{\sqrt{4x^4+9x}+2x^2} = 0.$$

36. $\lim_{x\to\infty}(\sqrt{9x^3+x} - x^{3/2})$

SOLUTION Write

$$\begin{aligned}\sqrt{9x^3+x} - x^{3/2} &= \left(\sqrt{9x^3+x} - x^{3/2}\right)\frac{\sqrt{9x^3+x}+x^{3/2}}{\sqrt{9x^3+x}+x^{3/2}}\\ &= \frac{(9x^3+x)-x^3}{\sqrt{9x^3+x}+x^{3/2}} = \frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}}.\end{aligned}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{9x^3+x} - x^{3/2}) = \lim_{t\to\infty}\frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}} = \infty.$$

37. $\lim_{x\to\infty}\left(2\sqrt{x}-\sqrt{x+2}\right)$

SOLUTION Write

$$\begin{aligned}2\sqrt{x}-\sqrt{x+2} &= \left(2\sqrt{x}-\sqrt{x+2}\right)\frac{2\sqrt{x}+\sqrt{x+2}}{2\sqrt{x}+\sqrt{x+2}}\\ &= \frac{4x-(x+2)}{2\sqrt{x}+\sqrt{x+2}}=\frac{3x-2}{2\sqrt{x}+\sqrt{x+2}}.\end{aligned}$$

Thus,

$$\lim_{x\to\infty}(2\sqrt{x}-\sqrt{x+2})=\lim_{x\to\infty}\frac{3x-2}{2\sqrt{x}+\sqrt{x+2}}=\infty.$$

38. $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+2}\right)$

SOLUTION $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+2}\right)=\lim_{x\to\infty}\frac{2}{x(x+2)}=0.$

39. $\lim_{x\to\infty}(\ln(3x+1)-\ln(2x+1))$

SOLUTION Because

$$\ln(3x+1)-\ln(2x+1)=\ln\frac{3x+1}{2x+1}$$

and

$$\lim_{x\to\infty}\frac{3x+1}{2x+1}=\frac{3}{2},$$

it follows that

$$\lim_{x\to\infty}(\ln(3x+1)-\ln(2x+1))=\ln\frac{3}{2}.$$

40. $\lim_{x\to\infty}\left(\ln(\sqrt{5x^2+2})-\ln x\right)$

SOLUTION Because

$$\ln(\sqrt{5x^2+2})-\ln x=\ln\frac{\sqrt{5x^2+2}}{x}$$

and

$$\lim_{x\to\infty}\frac{\sqrt{5x^2+2}}{x}=\lim_{x\to\infty}\frac{\sqrt{5+\frac{2}{x^2}}}{1}=\sqrt{5},$$

it follows that

$$\lim_{x\to\infty}\left(\ln(\sqrt{5x^2+2})-\ln x\right)=\ln\sqrt{5}=\frac{1}{2}\ln 5.$$

41. $\lim_{x\to\infty}\tan^{-1}\left(\frac{x^2+9}{9-x}\right)$

SOLUTION Because

$$\lim_{x\to\infty}\frac{x^2+9}{9-x}=\lim_{x\to\infty}\frac{x+\frac{9}{x}}{\frac{9}{x}-1}=\frac{\infty}{-1}=-\infty,$$

it follows that

$$\lim_{x\to\infty}\tan^{-1}\left(\frac{x^2+9}{9-x}\right)=-\frac{\pi}{2}.$$

42. $\lim_{x \to \infty} \tan^{-1}\left(\frac{1+x}{1-x}\right)$

SOLUTION Because

$$\lim_{x \to \infty} \frac{1+x}{1-x} = -1,$$

it follows that

$$\lim_{x \to \infty} \tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

43. Let $P(n)$ be the perimeter of an n-gon inscribed in a unit circle (Figure 8).

(a) Explain, intuitively, why $P(n)$ approaches 2π as $n \to \infty$.

(b) Show that $P(n) = 2n \sin\left(\frac{\pi}{n}\right)$.

(c) Combine (a) and (b) to conclude that $\lim_{n \to \infty} \frac{n}{\pi} \sin\left(\frac{\pi}{n}\right) = 1$.

(d) Use this to give another argument that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

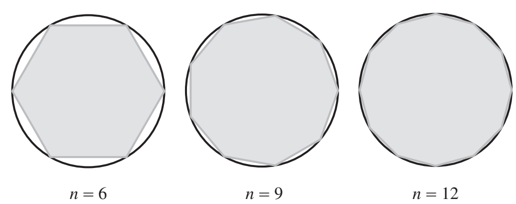


FIGURE 8

SOLUTION

(a) As $n \to \infty$, the n-gon approaches a circle of radius 1. Therefore, the perimeter of the n-gon approaches the circumference of the unit circle as $n \to \infty$. That is, $P(n) \to 2\pi$ as $n \to \infty$.

(b) Each side of the n-gon is the third side of an isosceles triangle with equal length sides of length 1 and angle $\theta = \frac{2\pi}{n}$ between the equal length sides. The length of each side of the n-gon is therefore

$$\sqrt{1^2 + 1^2 - 2\cos\frac{2\pi}{n}} = \sqrt{2(1 - \cos\frac{2\pi}{n})} = \sqrt{4\sin^2\frac{\pi}{n}} = 2\sin\frac{\pi}{n}.$$

Finally,

$$P(n) = 2n \sin\frac{\pi}{n}.$$

(c) Combining parts (a) and (b),

$$\lim_{n \to \infty} P(n) = \lim_{n \to \infty} 2n \sin\frac{\pi}{n} = 2\pi.$$

Dividing both sides of this last expression by 2π yields

$$\lim_{n \to \infty} \frac{n}{\pi} \sin\frac{\pi}{n} = 1.$$

(d) Let $\theta = \frac{\pi}{n}$. Then $\theta \to 0$ as $n \to \infty$,

$$\frac{n}{\pi} \sin\frac{\pi}{n} = \frac{1}{\theta}\sin\theta = \frac{\sin\theta}{\theta},$$

and

$$\lim_{n \to \infty} \frac{n}{\pi} \sin\frac{\pi}{n} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

44. Physicists have observed that Einstein's theory of **special relativity** reduces to Newtonian mechanics in the limit as $c \to \infty$, where c is the speed of light. This is illustrated by a stone tossed up vertically from ground level so that it returns to earth one second later. Using Newton's Laws, we find that the stone's maximum height is $h = g/8$ meters ($g = 9.8$ m/s^2). According to special relativity, the stone's mass depends on its velocity divided by c, and the maximum height is

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g$$

Prove that $\lim_{c \to \infty} h(c) = g/8$.

SOLUTION Write

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g = \left(c\sqrt{c^2/g^2 + 1/4} - c^2/g\right)\frac{c\sqrt{c^2/g^2 + 1/4} + c^2/g}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}$$
$$= \frac{c^2(c^2/g^2 + 1/4) - c^4/g^2}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}.$$

Thus,

$$\lim_{c\to\infty} h(c) = \lim_{c\to\infty} \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{2c^2/g} = \frac{g}{8}.$$

Further Insights and Challenges

45. Every limit as $x \to \infty$ can be rewritten as a one-sided limit as $t \to 0+$, where $t = x^{-1}$. Setting $g(t) = f(t^{-1})$, we have

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0+} g(t)$$

Show that $\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2}$, and evaluate using the Quotient Law.

SOLUTION Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3x^2 - x}{2x^2 + 5} = \frac{3t^{-2} - t^{-1}}{2t^{-2} + 5} = \frac{3 - t}{2 + 5t^2}.$$

Thus,

$$\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2} = \frac{3}{2}.$$

46. Rewrite the following as one-sided limits as in Exercise 45 and evaluate.

(a) $\lim_{x\to\infty} \frac{3 - 12x^3}{4x^3 + 3x + 1}$

(b) $\lim_{x\to\infty} e^{1/x}$

(c) $\lim_{x\to\infty} x \sin\frac{1}{x}$

(d) $\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right)$

SOLUTION

(a) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3 - 12x^3}{4x^3 + 3x + 1} = \frac{3 - 12t^{-3}}{4t^{-3} + 3t^{-1} + 1} = \frac{3t^3 - 12}{4 + 3t^2 + t^3}.$$

Thus,

$$\lim_{x\to\infty} \frac{3 - 12x^3}{4x^3 + 3x + 1} = \lim_{t\to 0+} \frac{3t^3 - 12}{4 + 3t^2 + t^3} = \frac{-12}{4} = -3.$$

(b) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and $e^{1/x} = e^t$. Thus,

$$\lim_{x\to\infty} e^{1/x} = \lim_{t\to 0+} e^t = e^0 = 1.$$

(c) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$x \sin\frac{1}{x} = \frac{1}{t}\sin t = \frac{\sin t}{t}.$$

Thus,

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0+} \frac{\sin t}{t} = 1.$$

(d) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{x+1}{x-1} = \frac{t^{-1}+1}{t^{-1}-1} = \frac{1+t}{1-t}.$$

Thus,

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right) = \lim_{t\to 0+} \ln\left(\frac{1+t}{1-t}\right) = \ln 1 = 0.$$

47. Let $G(b) = \lim_{x\to\infty}(1+b^x)^{1/x}$ for $b \geq 0$. Investigate $G(b)$ numerically and graphically for $b = 0.2, 0.8, 2, 3, 5$ (and additional values if necessary). Then make a conjecture for the value of $G(b)$ as a function of b. Draw a graph of $y = G(b)$. Does $G(b)$ appear to be continuous? We will evaluate $G(b)$ using L'Hôpital's Rule in Section 4.5 (see Exercise 69 in Section 4.5).

SOLUTION

- $b = 0.2$:

x	5	10	50	100
$f(x)$	1.000064	1.000000	1.000000	1.000000

It appears that $G(0.2) = 1$.

- $b = 0.8$:

x	5	10	50	100
$f(x)$	1.058324	1.010251	1.000000	1.000000

It appears that $G(0.8) = 1$.

- $b = 2$:

x	5	10	50	100
$f(x)$	2.012347	2.000195	2.000000	2.000000

It appears that $G(2) = 2$.

- $b = 3$:

x	5	10	50	100
$f(x)$	3.002465	3.000005	3.000000	3.000000

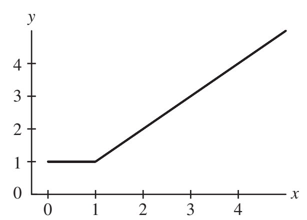
It appears that $G(3) = 3$.

- $b = 5$:

x	5	10	50	100
$f(x)$	5.000320	5.000000	5.000000	5.000000

It appears that $G(5) = 5$.

Based on these observations we conjecture that $G(b) = 1$ if $0 \leq b \leq 1$ and $G(b) = b$ for $b > 1$. The graph of $y = G(b)$ is shown below; the graph does appear to be continuous.



2.8 Intermediate Value Theorem

Preliminary Questions

1. Prove that $f(x) = x^2$ takes on the value 0.5 in the interval [0, 1].

SOLUTION Observe that $f(x) = x^2$ is continuous on [0, 1] with $f(0) = 0$ and $f(1) = 1$. Because $f(0) < 0.5 < f(1)$, the Intermediate Value Theorem guarantees there is a $c \in [0, 1]$ such that $f(c) = 0.5$.

2. The temperature in Vancouver was 8°C at 6 AM and rose to 20°C at noon. Which assumption about temperature allows us to conclude that the temperature was 15°C at some moment of time between 6 AM and noon?

SOLUTION We must assume that temperature is a continuous function of time.

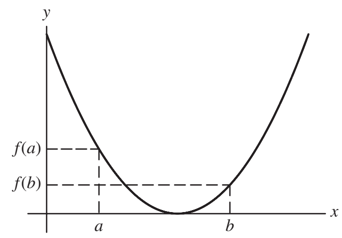
3. What is the graphical interpretation of the IVT?

SOLUTION If f is continuous on $[a, b]$, then the horizontal line $y = k$ for every k between $f(a)$ and $f(b)$ intersects the graph of $y = f(x)$ at least once.

4. Show that the following statement is false by drawing a graph that provides a counterexample:

If $f(x)$ is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

SOLUTION



5. Assume that $f(t)$ is continuous on [1, 5] and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.

(a) $f(c) = 3$ has a solution with $c \in [1, 5]$.
(b) $f(c) = 75$ has a solution with $c \in [1, 5]$.
(c) $f(c) = 50$ has no solution with $c \in [1, 5]$.
(d) $f(c) = 30$ has exactly one solution with $c \in [1, 5]$.

SOLUTION

(a) This statement is sometimes true.
(b) This statement is always true.
(c) This statement is never true.
(d) This statement is sometimes true.

Exercises

1. Use the IVT to show that $f(x) = x^3 + x$ takes on the value 9 for some x in [1, 2].

SOLUTION Observe that $f(1) = 2$ and $f(2) = 10$. Since f is a polynomial, it is continuous everywhere; in particular on [1, 2]. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = 9$.

2. Show that $g(t) = \dfrac{t}{t+1}$ takes on the value 0.499 for some t in [0, 1].

SOLUTION $g(0) = 0$ and $g(1) = \frac{1}{2}$. Since $g(t)$ is continuous for all $x \neq -1$, and since $0 < 0.4999 < \frac{1}{2}$, the IVT states that $g(t) = 0.4999$ for some t between 0 and 1.

3. Show that $g(t) = t^2 \tan t$ takes on the value $\frac{1}{2}$ for some t in $\left[0, \frac{\pi}{4}\right]$.

SOLUTION $g(0) = 0$ and $g(\frac{\pi}{4}) = \frac{\pi^2}{16}$. $g(t)$ is continuous for all t between 0 and $\frac{\pi}{4}$, and $0 < \frac{1}{2} < \frac{\pi^2}{16}$; therefore, by the IVT, there is a $c \in [0, \frac{\pi}{4}]$ such that $g(c) = \frac{1}{2}$.

4. Show that $f(x) = \dfrac{x^2}{x^7+1}$ takes on the value 0.4.

SOLUTION $f(0) = 0 < 0.4$. $f(1) = \frac{1}{2} > 0.4$. $f(x)$ is continuous at all points x where $x \neq -1$, therefore $f(x) = 0.4$ for some x between 0 and 1.

5. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. *Hint:* Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.

SOLUTION Let $f(x) = x - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(1) = 1 - \cos 1 \approx 0.46$. Therefore, by the IVT there is a $c \in [0, 1]$ such that $f(c) = c - \cos c = 0$. Thus $c = \cos c$ and hence the equation $\cos x = x$ has a solution c in $[0, 1]$.

6. Use the IVT to find an interval of length $\frac{1}{2}$ containing a root of $f(x) = x^3 + 2x + 1$.

SOLUTION Let $f(x) = x^3 + 2x + 1$. Observe that $f(-1) = -2$ and $f(0) = 1$. Since f is continuous, we may conclude by the IVT that f has a root in $[-1, 0]$. Now, $f(-\frac{1}{2}) = -\frac{1}{8}$ so $f(-\frac{1}{2})$ and $f(0)$ are of opposite sign. Therefore, the IVT guarantees that f has a root on $[-\frac{1}{2}, 0]$.

In Exercises 7–16, prove using the IVT.

7. $\sqrt{c} + \sqrt{c+2} = 3$ has a solution.

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{x+2} - 3$. Note that f is continuous on $\left[\frac{1}{4}, 2\right]$ with $f(\frac{1}{4}) = \sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - 3 = -1$ and $f(2) = \sqrt{2} - 1 \approx 0.41$. Therefore, by the IVT there is a $c \in \left[\frac{1}{4}, 2\right]$ such that $f(c) = \sqrt{c} + \sqrt{c+2} - 3 = 0$. Thus $\sqrt{c} + \sqrt{c+2} = 3$ and hence the equation $\sqrt{x} + \sqrt{x+2} = 3$ has a solution c in $\left[\frac{1}{4}, 2\right]$.

8. For all integers n, $\sin nx = \cos x$ for some $x \in [0, \pi]$.

SOLUTION For each integer n, let $f(x) = \sin nx - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(\pi) = 1$. Therefore, by the IVT there is a $c \in [0, \pi]$ such that $f(c) = \sin nc - \cos c = 0$. Thus $\sin nc = \cos c$ and hence the equation $\sin nx = \cos x$ has a solution c in the interval $[0, \pi]$.

9. $\sqrt{2}$ exists. *Hint:* Consider $f(x) = x^2$.

SOLUTION Let $f(x) = x^2$. Observe that f is continuous with $f(1) = 1$ and $f(2) = 4$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = c^2 = 2$. This proves the existence of $\sqrt{2}$, a number whose square is 2.

10. A positive number c has an nth root for all positive integers n.

SOLUTION If $c = 1$, then $\sqrt[n]{c} = 1$. Now, suppose $c \neq 1$. Let $f(x) = x^n - c$, and let $b = \max\{1, c\}$. Then, if $c > 1$, $b^n = c^n > c$, and if $c < 1$, $b^n = 1 > c$. So $b^n > c$. Now observe that $f(0) = -c < 0$ and $f(b) = b^n - c > 0$. Since f is continuous on $[0, b]$, by the intermediate value theorem, there is some $d \in [0, b]$ such that $f(d) = 0$. We can refer to d as $\sqrt[n]{c}$.

11. For all positive integers k, $\cos x = x^k$ has a solution.

SOLUTION For each positive integer k, let $f(x) = x^k - \cos x$. Observe that f is continuous on $\left[0, \frac{\pi}{2}\right]$ with $f(0) = -1$ and $f(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^k > 0$. Therefore, by the IVT there is a $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = c^k - \cos(c) = 0$. Thus $\cos c = c^k$ and hence the equation $\cos x = x^k$ has a solution c in the interval $\left[0, \frac{\pi}{2}\right]$.

12. $2^x = bx$ has a solution if $b > 2$.

SOLUTION Let $f(x) = 2^x - bx$. Observe that f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = 2 - b < 0$. Therefore, by the IVT, there is a $c \in [0, 1]$ such that $f(c) = 2^c - bc = 0$.

13. $2^x + 3^x = 4^x$ has a solution.

SOLUTION Let $f(x) = 2^x + 3^x - 4^x$. Observe that f is continuous on $[0, 2]$ with $f(0) = 1 > 0$ and $f(2) = -3 < 0$. Therefore, by the IVT, there is a $c \in (0, 2)$ such that $f(c) = 2^c + 3^c - 4^c = 0$.

14. $\cos x = \cos^{-1} x$ has a solution in $(0, 1)$.

SOLUTION Let $f(x) = \cos x - \cos^{-1} x$. Observe that f is continuous on $[0, 1]$ with $f(0) = 1 - \frac{\pi}{2} < 0$ and $f(1) = \cos 1 - 0 \approx 0.54 > 0$. Therefore, by the IVT, there is a $c \in (0, 1)$ such that $f(c) = \cos c - \cos^{-1} c = 0$.

15. $e^x + \ln x = 0$ has a solution.

SOLUTION Let $f(x) = e^x + \ln x$. Observe that f is continuous on $[e^{-2}, 1]$ with $f(e^{-2}) = e^{e^{-2}} - 2 < 0$ and $f(1) = e > 0$. Therefore, by the IVT, there is a $c \in (e^{-2}, 1) \subset (0, 1)$ such that $f(c) = e^c + \ln c = 0$.

16. $\tan^{-1} x = \cos^{-1} x$ has a solution.

SOLUTION Let $f(x) = \tan^{-1} x - \cos^{-1} x$. Observe that f is continuous on $[0, 1]$ with $f(0) = \tan^{-1} 0 - \cos^{-1} 0 = -\frac{\pi}{2} < 0$ and $f(1) = \tan^{-1} 1 - \cos^{-1} 1 = \frac{\pi}{4} > 0$. Therefore, by the IVT, there is a $c \in (0, 1)$ such that $f(c) = \tan^{-1} c - \cos^{-1} c = 0$.

17. Carry out three steps of the Bisection Method for $f(x) = 2^x - x^3$ as follows:

(a) Show that $f(x)$ has a zero in $[1, 1.5]$.

(b) Show that $f(x)$ has a zero in $[1.25, 1.5]$.

(c) Determine whether $[1.25, 1.375]$ or $[1.375, 1.5]$ contains a zero.

SOLUTION Note that $f(x)$ is continuous for all x.

(a) $f(1) = 1$, $f(1.5) = 2^{1.5} - (1.5)^3 < 3 - 3.375 < 0$. Hence, $f(x) = 0$ for some x between 1 and 1.5.

(b) $f(1.25) \approx 0.4253 > 0$ and $f(1.5) < 0$. Hence, $f(x) = 0$ for some x between 1.25 and 1.5.

(c) $f(1.375) \approx -0.0059$. Hence, $f(x) = 0$ for some x between 1.25 and 1.375.

18. Figure 4 shows that $f(x) = x^3 - 8x - 1$ has a root in the interval $[2.75, 3]$. Apply the Bisection Method twice to find an interval of length $\frac{1}{16}$ containing this root.

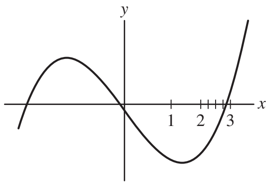


FIGURE 4 Graph of $y = x^3 - 8x - 1$.

SOLUTION Let $f(x) = x^3 - 8x - 1$. Observe that f is continuous with $f(2.75) = -2.203125$ and $f(3) = 2$. Therefore, by the IVT there is a $c \in [2.75, 3]$ such that $f(c) = 0$. The midpoint of the interval $[2.75, 3]$ is 2.875 and $f(2.875) = -0.236$. Hence, $f(x) = 0$ for some x between 2.875 and 3. The midpoint of the interval $[2.875, 3]$ is 2.9375 and $f(2.9375) = 0.84$. Thus, $f(x) = 0$ for some x between 2.875 and 2.9375.

19. Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$.

SOLUTION Let $f(x) = x^7 + 3x - 10$. Observe that f is continuous with $f(1) = -6$ and $f(2) = 124$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = 0$. $f(1.5) \approx 11.59 > 0$, so $f(c) = 0$ for some $c \in [1, 1.5]$. $f(1.25) \approx -1.48 < 0$, and so $f(c) = 0$ for some $c \in [1.25, 1.5]$. This means that $[1.25, 1.5]$ is an interval of length 0.25 containing a root of $f(x)$.

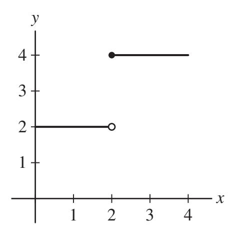
20. Show that $\tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8 = 0$ has a root in $[0.5, 0.6]$. Apply the Bisection Method twice to find an interval of length 0.025 containing this root.

SOLUTION Let $f(x) = \tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8$. Since $f(0.5) = -0.937387 < 0$ and $f(0.6) = 0.206186 > 0$, we conclude that $f(x) = 0$ has a root in $[0.5, 0.6]$. Since $f(0.55) = -0.35393 < 0$ and $f(0.6) > 0$, we can conclude that $f(x) = 0$ has a root in $[0.55, 0.6]$. Since $f(0.575) = -0.0707752 < 0$, we can conclude that f has a root on $[0.575, 0.6]$.

In Exercises 21–24, draw the graph of a function $f(x)$ *on* $[0, 4]$ *with the given property.*

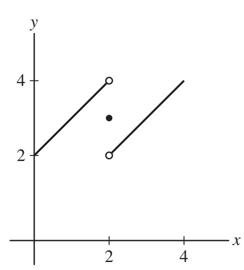
21. Jump discontinuity at $x = 2$ and does not satisfy the conclusion of the IVT.

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



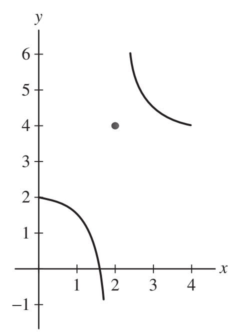
22. Jump discontinuity at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$.

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that for every value M between $f(0) = 2$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



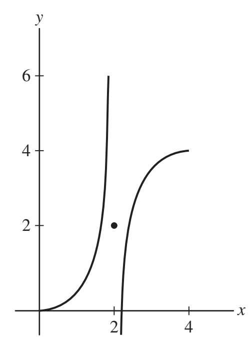
23. Infinite one-sided limits at $x = 2$ and does not satisfy the conclusion of the IVT.

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



24. Infinite one-sided limits at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$.

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that for every value M between $f(0) = 0$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



25. Can Corollary 2 be applied to $f(x) = x^{-1}$ on $[-1, 1]$? Does $f(x)$ have any roots?

SOLUTION No, because $f(x) = x^{-1}$ is not continuous on $[-1, 1]$. Even though $f(-1) = -1 < 0$ and $f(1) = 1 > 0$, the function has no roots between $x = -1$ and $x = 1$. In fact, this function has no roots at all.

Further Insights and Challenges

26. Take any map and draw a circle on it anywhere (Figure 5). Prove that at any moment in time there exists a pair of diametrically opposite points A and B on that circle corresponding to locations where the temperatures at that moment are equal. *Hint:* Let θ be an angular coordinate along the circle and let $f(\theta)$ be the difference in temperatures at the locations corresponding to θ and $\theta + \pi$.

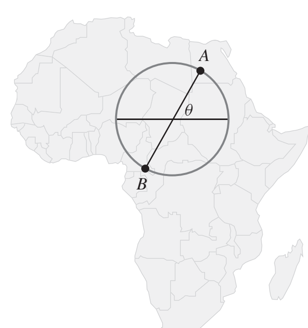


FIGURE 5 $f(\theta)$ is the difference between the temperatures at A and B.

SOLUTION Say the circle has (fixed but arbitrary) radius r and use polar coordinates with the pole at the center of the circle. For $0 \le \theta \le 2\pi$, let $T(\theta)$ be the temperature at the point $(r\cos\theta, r\sin\theta)$. We assume this temperature varies continuously. For $0 \le \theta \le \pi$, define f as the difference $f(\theta) = T(\theta) - T(\theta + \pi)$. Then f is continuous on $[0, \pi]$. There are three cases.

- If $f(0) = T(0) - T(\pi) = 0$, then $T(0) = T(\pi)$ and we have found a pair of diametrically opposite points on the circle at which the temperatures are equal.

- If $f(0) = T(0) - T(\pi) > 0$, then

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) < 0.$$

[Note that the angles 0 and 2π correspond to the same point, $(x, y) = (r, 0)$.] Since f is continuous on $[0, \pi]$, we have by the IVT that $f(c) = T(c) - T(c + \pi) = 0$ for some $c \in [0, \pi]$. Accordingly, $T(c) = T(c + \pi)$ and we have again found a pair of diametrically opposite points on the circle at which the temperatures are equal.

- If $f(0) = T(0) - T(\pi) < 0$, then

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) > 0.$$

Since f is continuous on $[0, \pi]$, we have by the IVT that $f(d) = T(d) - T(d + \pi) = 0$ for some $d \in [0, \pi]$. Accordingly, $T(d) = T(d + \pi)$ and once more we have found a pair of diametrically opposite points on the circle at which the temperatures are equal.

CONCLUSION: There is always a pair of diametrically opposite points on the circle at which the temperatures are equal.

27. Show that if $f(x)$ is continuous and $0 \le f(x) \le 1$ for $0 \le x \le 1$, then $f(c) = c$ for some c in $[0, 1]$ (Figure 6).

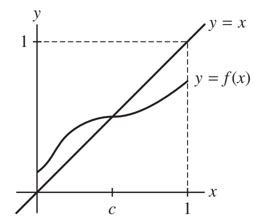


FIGURE 6 A function satisfying $0 \le f(x) \le 1$ for $0 \le x \le 1$.

SOLUTION If $f(0) = 0$, the proof is done with $c = 0$. We may assume that $f(0) > 0$. Let $g(x) = f(x) - x$. $g(0) = f(0) - 0 = f(0) > 0$. Since $f(x)$ is continuous, the Rule of Differences dictates that $g(x)$ is continuous. We need to prove that $g(c) = 0$ for some $c \in [0, 1]$. Since $f(1) \le 1$, $g(1) = f(1) - 1 \le 0$. If $g(1) = 0$, the proof is done with $c = 1$, so let's assume that $g(1) < 0$.

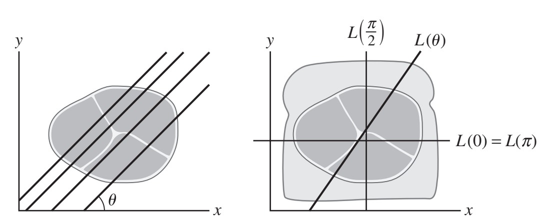
We now have a continuous function $g(x)$ on the interval $[0, 1]$ such that $g(0) > 0$ and $g(1) < 0$. From the IVT, there must be some $c \in [0, 1]$ so that $g(c) = 0$, so $f(c) - c = 0$ and so $f(c) = c$.

This is a simple case of a very general, useful, and beautiful theorem called the **Brouwer fixed point theorem**.

28. Use the IVT to show that if $f(x)$ is continuous and one-to-one on an interval $[a, b]$, then $f(x)$ is either an increasing or a decreasing function.

SOLUTION Let $f(x)$ be a continuous, one-to-one function on the interval $[a, b]$. Suppose for sake of contradiction that $f(x)$ is neither increasing nor decreasing on $[a, b]$. Now, $f(x)$ cannot be constant for that would contradict the condition that $f(x)$ is one-to-one. It follows that somewhere on $[a, b]$, $f(x)$ must transition from increasing to decreasing or from decreasing to increasing. To be specific, suppose $f(x)$ is increasing for $x_1 < x < x_2$ and decreasing for $x_2 < x < x_3$. Let k be any number between $\max\{f(x_1), f(x_3)\}$ and $f(x_2)$. Because $f(x)$ is continuous, the IVT guarantees there exists a $c_1 \in (x_1, x_2)$ such that $f(c_1) = k$; moreover, there exists a $c_2 \in (x_2, x_3)$ such that $f(c_2) = k$. However, this contradicts the condition that $f(x)$ is one-to-one. A similar analysis for the case when $f(x)$ is decreasing for $x_1 < x < x_2$ and increasing for $x_2 < x < x_3$ again leads to a contradiction. Therefore, $f(x)$ must either be increasing or decreasing on $[a, b]$.

29. **Ham Sandwich Theorem** Figure 7(A) shows a slice of ham. Prove that for any angle θ $(0 \le \theta \le \pi)$, it is possible to cut the slice in half with a cut of incline θ. *Hint:* The lines of inclination θ are given by the equations $y = (\tan\theta)x + b$, where b varies from $-\infty$ to ∞. Each such line divides the slice into two pieces (one of which may be empty). Let $A(b)$ be the amount of ham to the left of the line minus the amount to the right, and let A be the total area of the ham. Show that $A(b) = -A$ if b is sufficiently large and $A(b) = A$ if b is sufficiently negative. Then use the IVT. This works if $\theta \ne 0$ or $\frac{\pi}{2}$. If $\theta = 0$, define $A(b)$ as the amount of ham above the line $y = b$ minus the amount below. How can you modify the argument to work when $\theta = \frac{\pi}{2}$ (in which case $\tan\theta = \infty$)?



(A) Cutting a slice of ham at an angle θ.

(B) A slice of ham on top of a slice of bread.

FIGURE 7

SOLUTION Let θ be such that $\theta \neq \frac{\pi}{2}$. For any b, consider the line $L(\theta)$ drawn at angle θ to the x axis starting at $(0, b)$. This line has formula $y = (\tan\theta)x + b$. Let $A(b)$ be the amount of ham above the line minus that below the line.

Let $A > 0$ be the area of the ham. We have to accept the following (reasonable) assumptions:

- For low enough $b = b_0$, the line $L(\theta)$ lies entirely below the ham, so that $A(b_0) = A - 0 = A$.
- For high enough b_1, the line $L(\theta)$ lies entirely above the ham, so that $A(b_1) = 0 - A = -A$.
- $A(b)$ is continuous as a function of b.

Under these assumptions, we see $A(b)$ is a continuous function satisfying $A(b_0) > 0$ and $A(b_1) < 0$ for some $b_0 < b_1$. By the IVT, $A(b) = 0$ for some $b \in [b_0, b_1]$.

Suppose that $\theta = \frac{\pi}{2}$. Let the line $L(c)$ be the vertical line through $(c, 0)$ $(x = c)$. Let $A(c)$ be the area of ham to the left of $L(c)$ minus that to the right of $L(c)$. Since $L(0)$ lies entirely to the left of the ham, $A(0) = 0 - A = -A$. For some $c = c_1$ sufficiently large, $L(c)$ lies entirely to the right of the ham, so that $A(c_1) = A - 0 = A$. Hence $A(c)$ is a continuous function of c such that $A(0) < 0$ and $A(c_1) > 0$. By the IVT, there is some $c \in [0, c_1]$ such that $A(c) = 0$.

30. Figure 7(B) shows a slice of ham on a piece of bread. Prove that it is possible to slice this open-faced sandwich so that each part has equal amounts of ham and bread. *Hint:* By Exercise 29, for all $0 \le \theta \le \pi$ there is a line $L(\theta)$ of incline θ (which we assume is unique) that divides the ham into two equal pieces. Let $B(\theta)$ denote the amount of bread to the left of (or above) $L(\theta)$ minus the amount to the right (or below). Notice that $L(\pi)$ and $L(0)$ are the same line, but $B(\pi) = -B(0)$ since left and right get interchanged as the angle moves from 0 to π. Assume that $B(\theta)$ is continuous and apply the IVT. (By a further extension of this argument, one can prove the full "Ham Sandwich Theorem," which states that if you allow the knife to cut at a slant, then it is possible to cut a sandwich consisting of a slice of ham and two slices of bread so that all three layers are divided in half.)

SOLUTION For each angle θ, $0 \le \theta < \pi$, let $L(\theta)$ be the line at angle θ to the x-axis that slices the ham exactly in half, as shown in Figure 7. Let $L(0) = L(\pi)$ be the horizontal line cutting the ham in half, also as shown. For θ and $L(\theta)$ thus defined, let $B(\theta) =$ the amount of bread to the left of $L(\theta)$ minus that to the right of $L(\theta)$.

To understand this argument, one must understand what we mean by "to the left" or "to the right". Here, we mean to the left or right of the line as viewed in the direction θ. Imagine you are walking along the line in direction θ (directly right if $\theta = 0$, directly left if $\theta = \pi$, etc).

We will further accept the fact that B is continuous as a function of θ, which seems intuitively obvious. We need to prove that $B(c) = 0$ for some angle c.

Since $L(0)$ and $L(\pi)$ are drawn in opposite direction, $B(0) = -B(\pi)$. If $B(0) > 0$, we apply the IVT on $[0, \pi]$ with $B(0) > 0$, $B(\pi) < 0$, and B continuous on $[0, \pi]$; by IVT, $B(c) = 0$ for some $c \in [0, \pi]$. On the other hand, if $B(0) < 0$, then we apply the IVT with $B(0) < 0$ and $B(\pi) > 0$. If $B(0) = 0$, we are also done; $L(0)$ is the appropriate line.

2.9 The Formal Definition of a Limit

Preliminary Questions

1. Given that $\lim_{x\to 0} \cos x = 1$, which of the following statements is true?

(a) If $|\cos x - 1|$ is very small, then x is close to 0.

(b) There is an $\epsilon > 0$ such that $|x| < 10^{-5}$ if $0 < |\cos x - 1| < \epsilon$.

(c) There is a $\delta > 0$ such that $|\cos x - 1| < 10^{-5}$ if $0 < |x| < \delta$.

(d) There is a $\delta > 0$ such that $|\cos x| < 10^{-5}$ if $0 < |x - 1| < \delta$.

SOLUTION The true statement is **(c)**: There is a $\delta > 0$ such that $|\cos x - 1| < 10^{-5}$ if $0 < |x| < \delta$.

2. Suppose it is known that for a given ϵ and δ, $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < \delta$. Which of the following statements must also be true?

(a) $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < 2\delta$

(b) $|f(x) - 2| < 2\epsilon$ if $0 < |x - 3| < \delta$

(c) $|f(x) - 2| < \dfrac{\epsilon}{2}$ if $0 < |x - 3| < \dfrac{\delta}{2}$

(d) $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < \dfrac{\delta}{2}$

SOLUTION Statements **(b)** and **(d)** are true.

Exercises

1. Based on the information conveyed in Figure 5(A), find values of L, ϵ, and $\delta > 0$ such that the following statement holds: $|f(x) - L| < \epsilon$ if $|x| < \delta$.

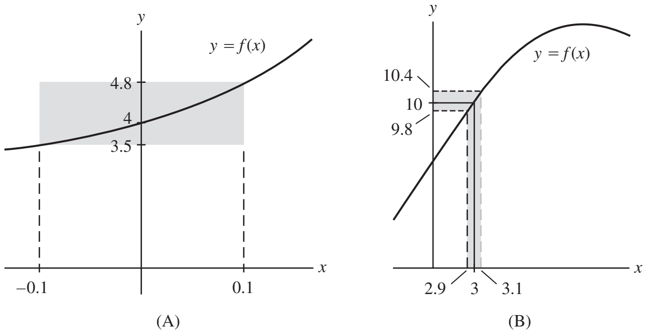


FIGURE 5

SOLUTION We see $-0.1 < x < 0.1$ forces $3.5 < f(x) < 4.8$. Rewritten, this means that $|x - 0| < 0.1$ implies that $|f(x) - 4| < 0.8$. Replacing numbers where appropriate in the definition of the limit $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$, we get $L = 4$, $\epsilon = 0.8$, $c = 0$, and $\delta = 0.1$.

2. Based on the information conveyed in Figure 5(B), find values of c, L, ϵ, and $\delta > 0$ such that the following statement holds: $|f(x) - L| < \epsilon$ if $|x - c| < \delta$.

SOLUTION From the shaded region in the graph, we can see that $9.8 < f(x) < 10.4$ whenever $2.9 < x < 3.1$. Rewriting these double inequalities as absolute value inequalities, we get $|f(x) - 10| < 0.4$ whenever $|x - 3| < 0.1$. Replacing numbers where appropriate in the definition of the limit $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$, we get $L = 10$, $\epsilon = 0.4$, $c = 3$, and $\delta = 0.1$.

3. Consider $\lim_{x\to 4} f(x)$, where $f(x) = 8x + 3$.

(a) Show that $|f(x) - 35| = 8|x - 4|$.

(b) Show that for any $\epsilon > 0$, $|f(x) - 35| < \epsilon$ if $|x - 4| < \delta$, where $\delta = \frac{\epsilon}{8}$. Explain how this proves rigorously that $\lim_{x\to 4} f(x) = 35$.

SOLUTION

(a) $|f(x) - 35| = |8x + 3 - 35| = |8x - 32| = |8(x - 4)| = 8\,|x - 4|$. (Remember that the last step is justified because $8 > 0$).

(b) Let $\epsilon > 0$. Let $\delta = \epsilon/8$ and suppose $|x - 4| < \delta$. By part **(a)**, $|f(x) - 35| = 8|x - 4| < 8\delta$. Substituting $\delta = \epsilon/8$, we see $|f(x) - 35| < 8\epsilon/8 = \epsilon$. We see that, for any $\epsilon > 0$, we found an appropriate δ so that $|x - 4| < \delta$ implies $|f(x) - 35| < \epsilon$. Hence $\lim_{x\to 4} f(x) = 35$.

4. Consider $\lim_{x\to 2} f(x)$, where $f(x) = 4x - 1$.

(a) Show that $|f(x) - 7| < 4\delta$ if $|x - 2| < \delta$.

(b) Find a δ such that

$$|f(x) - 7| < 0.01 \qquad \text{if} \qquad |x - 2| < \delta$$

(c) Prove rigorously that $\lim_{x\to 2} f(x) = 7$.

SOLUTION

(a) If $0 < |x - 2| < \delta$, then $|(4x - 1) - 7| = 4|x - 2| < 4\delta$.

(b) If $0 < |x - 2| < \delta = 0.0025$, then $|(4x - 1) - 7| = 4|x - 2| < 4\delta = 0.01$.

(c) Let $\epsilon > 0$ be given. Then whenever $0 < |x - 2| < \delta = \epsilon/4$, we have $|(4x - 1) - 7| = 4|x - 2| < 4\delta = \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to 2}(4x - 1) = 7$.

5. Consider $\lim_{x\to 2} x^2 = 4$ (refer to Example 2).

(a) Show that $|x^2 - 4| < 0.05$ if $0 < |x - 2| < 0.01$.

(b) Show that $|x^2 - 4| < 0.0009$ if $0 < |x - 2| < 0.0002$.

(c) Find a value of δ such that $|x^2 - 4|$ is less than 10^{-4} if $0 < |x - 2| < \delta$.

SOLUTION

(a) If $0 < |x - 2| < \delta = 0.01$, then $|x| < 3$ and $\left|x^2 - 4\right| = |x - 2||x + 2| \le |x - 2|\,(|x| + 2) < 5|x - 2| < 0.05$.

(b) If $0 < |x-2| < \delta = 0.0002$, then $|x| < 2.0002$ and

$$\left|x^2-4\right| = |x-2||x+2| \le |x-2|\,(|x|+2) < 4.0002|x-2| < 0.00080004 < 0.0009.$$

(c) Note that $\left|x^2-4\right| = |(x+2)(x-2)| \le |x+2|\,|x-2|$. Since $|x-2|$ can get arbitrarily small, we can require $|x-2| < 1$ so that $1 < x < 3$. This ensures that $|x+2|$ is at most 5. Now we know that $\left|x^2-4\right| \le 5|x-2|$. Let $\delta = 10^{-5}$. Then, if $|x-2| < \delta$, we get $\left|x^2-4\right| \le 5|x-2| < 5 \times 10^{-5} < 10^{-4}$ as desired.

6. With regard to the limit $\lim_{x\to 5} x^2 = 25$,

(a) Show that $|x^2-25| < 11|x-5|$ if $4 < x < 6$. *Hint:* Write $|x^2-25| = |x+5| \cdot |x-5|$.
(b) Find a δ such that $|x^2-25| < 10^{-3}$ if $|x-5| < \delta$.
(c) Give a rigorous proof of the limit by showing that $|x^2-25| < \epsilon$ if $0 < |x-5| < \delta$, where δ is the smaller of $\frac{\epsilon}{11}$ and 1.

SOLUTION

(a) If $4 < x < 6$, then $|x-5| < \delta = 1$ and $\left|x^2-25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < 11|x-5|$.

(b) If $0 < |x-5| < \delta = \frac{0.001}{11}$, then $x < 6$ and $\left|x^2-25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < 11|x-5| < 0.001$.

(c) Let $0 < |x-5| < \delta = \min\left\{1, \frac{\epsilon}{11}\right\}$. Since $\delta < 1$, $|x-5| < \delta < 1$ implies $4 < x < 6$. Specifically, $x < 6$ and

$$\left|x^2-25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < |x-5|(6+5) = 11|x-5|.$$

Since δ is also less than $\epsilon/11$, we can conclude $11|x-5| < 11(\epsilon/11) = \epsilon$, thus completing the rigorous proof that $|x^2-25| < \epsilon$ if $|x-5| < \delta$.

7. Refer to Example 3 to find a value of $\delta > 0$ such that

$$\left|\frac{1}{x} - \frac{1}{3}\right| < 10^{-4} \qquad \text{if} \qquad 0 < |x-3| < \delta$$

SOLUTION The Example shows that for any $\epsilon > 0$ we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| \le \epsilon \quad \text{if } |x-3| < \delta$$

where δ is the smaller of the numbers 6ϵ and 1. In our case, we may take $\delta = 6 \times 10^{-4}$.

8. Use Figure 6 to find a value of $\delta > 0$ such that the following statement holds: $\left|1/x^2 - \frac{1}{4}\right| < \epsilon$ if $|x-2| < \delta$ for $\epsilon = 0.03$. Then find a value of δ that works for $\epsilon = 0.01$.

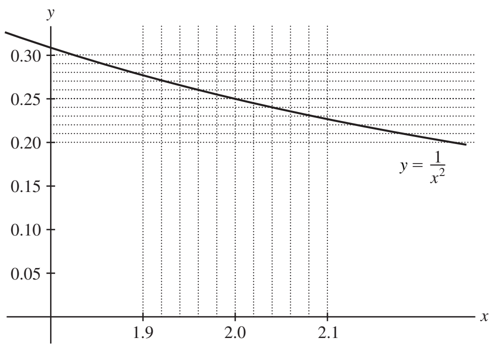


FIGURE 6

SOLUTION From Figure 6, we see that $0.22 < \frac{1}{x^2} < 0.28$ for $1.9 < x < 2.1$. Rewriting these expressions using absolute values yields

$$\left|\frac{1}{x^2} - \frac{1}{4}\right| < 0.03$$

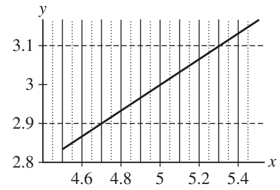
for $|x-2| < 0.1$. Thus, for $\epsilon = 0.03$, we may take $\delta = 0.1$. Additionally, we see that $0.24 < \frac{1}{x^2} < 0.26$ for $1.96 < x < 2.04$. Rewriting these expressions using absolute values yields

$$\left|\frac{1}{x^2} - \frac{1}{4}\right| < 0.01$$

for $|x-2| < 0.04$. Thus, for $\epsilon = 0.01$, we may take $\delta = 0.04$.

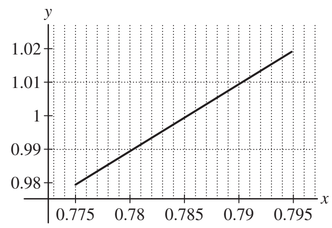
9. GU Plot $f(x) = \sqrt{2x-1}$ together with the horizontal lines $y = 2.9$ and $y = 3.1$. Use this plot to find a value of $\delta > 0$ such that $|\sqrt{2x-1} - 3| < 0.1$ if $|x - 5| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.25$ will guarantee that $|\sqrt{2x-1} - 3| < 0.1$ whenever $|x - 5| \le \delta$.



10. GU Plot $f(x) = \tan x$ together with the horizontal lines $y = 0.99$ and $y = 1.01$. Use this plot to find a value of $\delta > 0$ such that $|\tan x - 1| < 0.01$ if $\left|x - \frac{\pi}{4}\right| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.005$ will guarantee that $|\tan x - 1| < 0.01$ whenever $|x - \frac{\pi}{4}| \le \delta$.

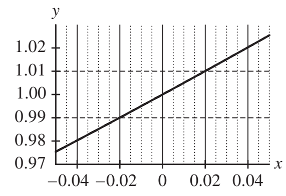


11. GU The number e has the following property: $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. Use a plot of $f(x) = \frac{e^x - 1}{x}$ to find a value of $\delta > 0$ such that $|f(x) - 1| < 0.01$ if $|x - 1| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.02$ will guarantee that

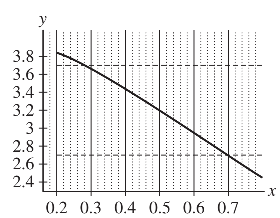
$$\left|\frac{e^x - 1}{x} - 1\right| < 0.01$$

whenever $|x| < \delta$.

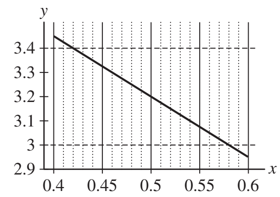


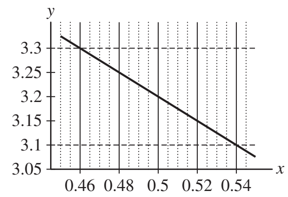
12. GU Let $f(x) = \frac{4}{x^2+1}$ and $\epsilon = 0.5$. Using a plot of $f(x)$, find a value of $\delta > 0$ such that $\left|f(x) - \frac{16}{5}\right| < \epsilon$ for $\left|x - \frac{1}{2}\right| < \delta$. Repeat for $\epsilon = 0.2$ and 0.1.

SOLUTION From the plot below, we see that $\delta = 0.18$ will guarantee that $|f(x) - \frac{16}{5}| < 0.5$ whenever $|x - \frac{1}{2}| < \delta$.



When $\epsilon = 0.2$, we see that $\delta = 0.075$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $|x - \frac{1}{2}| < \delta$ (examine the plot below at the left); when $\epsilon = 0.1$, $\delta = 0.035$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $|x - \frac{1}{2}| < \delta$ (examine the plot below at the right).





13. Consider $\lim_{x\to 2} \dfrac{1}{x}$.

(a) Show that if $|x-2| < 1$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x-2|$$

(b) Let δ be the smaller of 1 and 2ϵ. Prove:

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon \qquad \text{if} \qquad 0 < |x-2| < \delta$$

(c) Find a $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < 0.01$ if $|x-2| < \delta$.

(d) Prove rigorously that $\lim_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

SOLUTION

(a) Since $|x-2| < 1$, it follows that $1 < x < 3$, in particular that $x > 1$. Because $x > 1$, then $\dfrac{1}{x} < 1$ and

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2x} < \frac{1}{2}|x-2|.$$

(b) Let $\delta = \min\{1, 2\epsilon\}$ and suppose that $|x-2| < \delta$. Then by part (a) we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x-2| < \frac{1}{2}\delta < \frac{1}{2}\cdot 2\epsilon = \epsilon.$$

(c) Choose $\delta = 0.02$. Then $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| < \dfrac{1}{2}\delta = 0.01$ by part (b).

(d) Let $\epsilon > 0$ be given. Then whenever $0 < |x-2| < \delta = \min\{1, 2\epsilon\}$, we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}\delta \le \epsilon.$$

Since ϵ was arbitrary, we conclude that $\lim_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

14. Consider $\lim_{x\to 1} \sqrt{x+3}$.

(a) Show that $|\sqrt{x+3} - 2| < \frac{1}{2}|x-1|$ if $|x-1| < 4$. *Hint:* Multiply the inequality by $|\sqrt{x+3} + 2|$ and observe that $|\sqrt{x+3} + 2| > 2$.

(b) Find $\delta > 0$ such that $|\sqrt{x+3} - 2| < 10^{-4}$ for $|x-1| < \delta$.

(c) Prove rigorously that the limit is equal to 2.

SOLUTION

(a) $|x-1| < 4$ implies that $-3 < x < 5$. Since $x > -3$, then $\sqrt{x+3}$ is defined (and positive), whence

$$\left|\sqrt{x+3} - 2\right| = \left|\frac{\left(\sqrt{x+3} - 2\right)}{1}\,\frac{\left(\sqrt{x+3} + 2\right)}{\left(\sqrt{x+3} + 2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{|x-1|}{2}.$$

(b) Choose $\delta = 0.0002$. Then provided $0 < |x-1| < \delta$, we have $x > -3$ and therefore

$$\left|\sqrt{x+3} - 2\right| < \frac{|x-1|}{2} < \frac{\delta}{2} = 0.0001$$

by part (a).

(c) Let $\epsilon > 0$ be given. Then whenever $0 < |x-1| < \delta = \min\{2\epsilon, 4\}$, we have $x > -3$ and thus

$$\left|\sqrt{x+3}-2\right| = \left|\frac{\left(\sqrt{x+3}-2\right)}{1}\,\frac{\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}+2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{2\epsilon}{2} = \epsilon.$$

Since ϵ was arbitrary, we conclude that $\lim_{x\to 1}\sqrt{x+3} = 2$.

15. Let $f(x) = \sin x$. Using a calculator, we find:

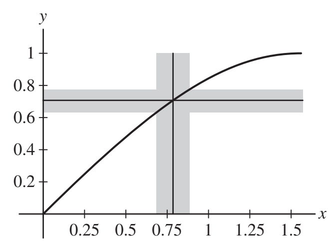
$$f\left(\frac{\pi}{4}-0.1\right) \approx 0.633, \quad f\left(\frac{\pi}{4}\right) \approx 0.707, \quad f\left(\frac{\pi}{4}+0.1\right) \approx 0.774$$

Use these values and the fact that $f(x)$ is increasing on $\left[0, \frac{\pi}{2}\right]$ to justify the statement

$$\left|f(x) - f\left(\frac{\pi}{4}\right)\right| < 0.08 \qquad \text{if} \qquad \left|x - \frac{\pi}{4}\right| < 0.1$$

Then draw a figure like Figure 3 to illustrate this statement.

SOLUTION Since $f(x)$ is increasing on the interval, the three $f(x)$ values tell us that $0.633 \le f(x) \le 0.774$ for all x between $\frac{\pi}{4} - 0.1$ and $\frac{\pi}{4} + 0.1$. We may subtract $f(\frac{\pi}{4})$ from the inequality for $f(x)$. This show that, for $\frac{\pi}{4} - 0.1 < x < \frac{\pi}{4} + 0.1$, $0.633 - f(\frac{\pi}{4}) \le f(x) - f(\frac{\pi}{4}) \le 0.774 - f(\frac{\pi}{4})$. This means that, if $|x - \frac{\pi}{4}| < 0.1$, then $0.633 - 0.707 \le f(x) - f(\frac{\pi}{4}) \le 0.774 - 0.707$, so $-0.074 \le f(x) - f(\frac{\pi}{4}) \le 0.067$. Then $-0.08 < f(x) - f(\frac{\pi}{4}) < 0.08$ follows from this, so $|x - \frac{\pi}{4}| < 0.1$ implies $|f(x) - f(\frac{\pi}{4})| < 0.08$. The figure below illustrates this.



16. Adapt the argument in Example 1 to prove rigorously that $\lim_{x\to c}(ax+b) = ac+b$, where a, b, c are arbitrary.

SOLUTION $|f(x) - (ac+b)| = |(ax+b) - (ac+b)| = |a(x-c)| = |a|\,|x-c|$. This says the gap is $|a|$ times as large as $|x-c|$. Let $\epsilon > 0$. Let $\delta = \epsilon/|a|$. If $|x-c| < \delta$, we get $|f(x) - (ac+b)| = |a|\,|x-c| < |a|\epsilon/|a| = \epsilon$, which is what we had to prove.

17. Adapt the argument in Example 2 to prove rigorously that $\lim_{x\to c} x^2 = c^2$ for all c.

SOLUTION To relate the gap to $|x-c|$, we take

$$\left|x^2 - c^2\right| = |(x+c)(x-c)| = |x+c|\,|x-c|.$$

We choose δ in two steps. First, since we are requiring $|x-c|$ to be small, we require $\delta < |c|$, so that x lies between 0 and $2c$. This means that $|x+c| < 3|c|$, so $|x-c||x+c| < 3|c|\delta$. Next, we require that $\delta < \dfrac{\epsilon}{3|c|}$, so

$$|x-c||x+c| < \frac{\epsilon}{3|c|}3|c| = \epsilon,$$

and we are done.

Therefore, given $\epsilon > 0$, we let

$$\delta = \min\left\{|c|, \frac{\epsilon}{3|c|}\right\}.$$

Then, for $|x-c| < \delta$, we have

$$|x^2 - c^2| = |x-c|\,|x+c| < 3|c|\delta < 3|c|\frac{\epsilon}{3|c|} = \epsilon.$$

18. Adapt the argument in Example 3 to prove rigorously that $\lim_{x\to c} x^{-1} = \frac{1}{c}$ for all $c \neq 0$.

SOLUTION Suppose that $c \neq 0$. To relate the gap to $|x-c|$, we find:

$$\left|x^{-1} - \frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \frac{|x-c|}{|cx|}$$

Since $|x-c|$ is required to be small, we may assume from the outset that $|x-c| < |c|/2$, so that x is between $|c|/2$ and $3|c|/2$. This forces $|cx| > |c|/2$, from which

$$\frac{|x-c|}{|cx|} < \frac{2}{|c|}|x-c|.$$

If $\delta < \epsilon(\frac{|c|}{2})$,

$$\left|x^{-1} - \frac{1}{c}\right| < \frac{2}{|c|}|x-c| < \frac{2}{|c|}\frac{|c|}{2}\epsilon = \epsilon.$$

Therefore, given $\epsilon > 0$ we let

$$\delta = \min\left(\frac{|c|}{2}, \epsilon\left(\frac{|c|}{2}\right)\right).$$

We have shown that $|x^{-1} - \frac{1}{c}| < \epsilon$ if $0 < |x-c| < \delta$.

In Exercises 19–24, use the formal definition of the limit to prove the statement rigorously.

19. $\lim_{x\to 4} \sqrt{x} = 2$

SOLUTION Let $\epsilon > 0$ be given. We bound $|\sqrt{x} - 2|$ by multiplying $\dfrac{\sqrt{x}+2}{\sqrt{x}+2}$.

$$|\sqrt{x}-2| = \left|\sqrt{x}-2\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right|.$$

We can assume $\delta < 1$, so that $|x-4| < 1$, and hence $\sqrt{x}+2 > \sqrt{3}+2 > 3$. This gives us

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3}.$$

Let $\delta = \min(1, 3\epsilon)$. If $|x-4| < \delta$,

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3} < \delta\frac{1}{3} < 3\epsilon\frac{1}{3} = \epsilon,$$

thus proving the limit rigorously.

20. $\lim_{x\to 1}(3x^2 + x) = 4$

SOLUTION Let $\epsilon > 0$ be given. We bound $\left|(3x^2+x)-4\right|$ using quadratic factoring.

$$\left|(3x^2+x)-4\right| = \left|3x^2+x-4\right| = |(3x+4)(x-1)| = |x-1||3x+4|.$$

Let $\delta = \min(1, \frac{\epsilon}{10})$. Since $\delta < 1$, we get $|3x+4| < 10$, so that

$$\left|(3x^2+x)-4\right| = |x-1||3x+4| < 10|x-1|.$$

Since $\delta < \frac{\epsilon}{10}$, we get

$$\left|(3x^2+x)-4\right| < 10|x-1| < 10\frac{\epsilon}{10} = \epsilon.$$

21. $\lim_{x\to 1} x^3 = 1$

SOLUTION Let $\epsilon > 0$ be given. We bound $\left|x^3 - 1\right|$ by factoring the difference of cubes:

$$\left|x^3-1\right| = \left|(x^2+x+1)(x-1)\right| = |x-1|\left|x^2+x+1\right|.$$

Let $\delta = \min(1, \frac{\epsilon}{7})$, and assume $|x-1| < \delta$. Since $\delta < 1$, $0 < x < 2$. Since $x^2 + x + 1$ increases as x increases for $x > 0$, $x^2 + x + 1 < 7$ for $0 < x < 2$, and so

$$\left|x^3 - 1\right| = |x-1|\left|x^2 + x + 1\right| < 7|x-1| < 7\frac{\epsilon}{7} = \epsilon$$

and the limit is rigorously proven.

22. $\lim_{x\to 0}(x^2 + x^3) = 0$

SOLUTION Let $\epsilon > 0$ be given. Now,

$$|(x^2 + x^3) - 0| = |x|\,|x|\,|x+1|.$$

Let $\delta = \min(1, \frac{1}{2}\epsilon)$, and suppose $|x| < \delta$. Since $\delta < 1$, $|x| < 1$, so $-1 < x < 1$. This means $|1 + x| < 2$, so that $|x|\,|x+1| < 2$. Thus,

$$\left|(x^2 + x^3) - 0\right| = |x|\,|x|\,|x+1| < 2|x| < 2 \cdot \frac{1}{2}\epsilon = \epsilon.$$

and the limit is rigorously proven.

23. $\lim_{x\to 2} x^{-2} = \frac{1}{4}$

SOLUTION Let $\epsilon > 0$ be given. First, we bound $x^{-2} - \frac{1}{4}$:

$$\left|x^{-2} - \frac{1}{4}\right| = \left|\frac{4 - x^2}{4x^2}\right| = |2-x|\left|\frac{2+x}{4x^2}\right|.$$

Let $\delta = \min(1, \frac{4}{5}\epsilon)$, and suppose $|x-2| < \delta$. Since $\delta < 1$, $|x-2| < 1$, so $1 < x < 3$. This means that $4x^2 > 4$ and $|2+x| < 5$, so that $\dfrac{2+x}{4x^2} < \frac{5}{4}$. We get:

$$\left|x^{-2} - \frac{1}{4}\right| = |2-x|\left|\frac{2+x}{4x^2}\right| < \frac{5}{4}|x-2| < \frac{5}{4}\cdot\frac{4}{5}\epsilon = \epsilon.$$

and the limit is rigorously proven.

24. $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

SOLUTION Let $\epsilon > 0$ be given. Let $\delta = \epsilon$, and assume $|x - 0| = |x| < \delta$. We bound $x \sin\frac{1}{x}$.

$$\left|x\sin\frac{1}{x} - 0\right| = |x|\left|\sin\frac{1}{x}\right| < |x| < \delta = \epsilon.$$

25. Let $f(x) = \dfrac{x}{|x|}$. Prove rigorously that $\lim_{x\to 0} f(x)$ does not exist. *Hint:* Show that for any L, there always exists some x such that $|x| < \delta$ but $|f(x) - L| \ge \frac{1}{2}$, no matter how small δ is taken.

SOLUTION Let L be any real number. Let $\delta > 0$ be any small positive number. Let $x = \frac{\delta}{2}$, which satisfies $|x| < \delta$, and $f(x) = 1$. We consider two cases:

- $(|f(x) - L| \ge \frac{1}{2})$: we are done.
- $(|f(x) - L| < \frac{1}{2})$: This means $\frac{1}{2} < L < \frac{3}{2}$. In this case, let $x = -\frac{\delta}{2}$. $f(x) = -1$, and so $\frac{3}{2} < L - f(x)$.

In either case, there exists an x such that $|x| < \frac{\delta}{2}$, but $|f(x) - L| \ge \frac{1}{2}$.

26. Prove rigorously that $\lim_{x\to 0}|x| = 0$.

SOLUTION Let $\epsilon > 0$ be given and take $\delta = \epsilon$. Then, whenever $|x| < \delta$,

$$||x| - 0| = |x| < \delta = \epsilon,$$

thus proving the limit rigorously.

27. Let $f(x) = \min(x, x^2)$, where $\min(a, b)$ is the minimum of a and b. Prove rigorously that $\lim_{x\to 1} f(x) = 1$.

SOLUTION Let $\epsilon > 0$ and let $\delta = \min(1, \frac{\epsilon}{2})$. Then, whenever $|x - 1| < \delta$, it follows that $0 < x < 2$. If $1 < x < 2$, then $\min(x, x^2) = x$ and

$$|f(x) - 1| = |x - 1| < \delta < \frac{\epsilon}{2} < \epsilon.$$

On the other hand, if $0 < x < 1$, then $\min(x, x^2) = x^2$, $|x + 1| < 2$ and

$$|f(x) - 1| = |x^2 - 1| = |x - 1|\,|x + 1| < 2\delta < \epsilon.$$

Thus, whenever $|x - 1| < \delta$, $|f(x) - 1| < \epsilon$.

28. Prove rigorously that $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

SOLUTION Let $\delta > 0$ be a given small positive number, and let L be any real number. We will prove that $\left|\sin \frac{1}{x} - L\right| \geq \frac{1}{2}$ for some x such that $|x| < \delta$.

Let $N > 0$ be a positive integer large enough so that $\frac{2}{(4N+1)\pi} < \delta$. Let

$$x_1 = \frac{2}{(4N+1)\pi},$$

$$x_2 = \frac{2}{(4N+3)\pi}.$$

$$x_2 < x_1 < \delta.$$

$$\sin\frac{1}{x_1} = \sin\frac{(4N+1)\pi}{2} = 1 \quad \text{and} \quad \sin\frac{1}{x_2} = \sin\frac{(4N+3)\pi}{2} = -1.$$

If $|\sin \frac{1}{x_1} - L| \geq \frac{1}{2}$, we are done. Therefore, let's assume that $|\sin \frac{1}{x_1} - L| < \frac{1}{2}$. $-\frac{1}{2} < \sin\frac{1}{x_1} - L < \frac{1}{2}$, so $L - \frac{1}{2} < \sin\frac{1}{x_1} = 1 < L + \frac{1}{2}$. This means $L > \frac{1}{2}$, so that $|\sin\frac{1}{x_2} - L| = |-1 - L| > \frac{3}{2}$. In either case, there is an x such that $|x| < \delta$ but $|\sin\frac{1}{x} - L| \geq \frac{1}{2}$, so no limit L can exist.

29. First, use the identity

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

to verify the relation

$$\sin(a+h) - \sin a = h\frac{\sin(h/2)}{h/2}\cos\left(a + \frac{h}{2}\right) \qquad \boxed{6}$$

Then use the inequality $\left|\frac{\sin x}{x}\right| \leq 1$ for $x \neq 0$ to show that $|\sin(a+h) - \sin a| < |h|$ for all a. Finally, prove rigorously that $\lim_{x\to a} \sin x = \sin a$.

SOLUTION We first write

$$\sin(a+h) - \sin a = \sin(a+h) + \sin(-a).$$

Applying the identity with $x = a + h$, $y = -a$, yields:

$$\begin{aligned}\sin(a+h) - \sin a &= \sin(a+h) + \sin(-a) = 2\sin\left(\frac{a+h-a}{2}\right)\cos\left(\frac{2a+h}{2}\right)\\ &= 2\sin\left(\frac{h}{2}\right)\cos\left(a+\frac{h}{2}\right) = 2\left(\frac{h}{h}\right)\sin\left(\frac{h}{2}\right)\cos\left(a+\frac{h}{2}\right) = h\frac{\sin(h/2)}{h/2}\cos\left(a+\frac{h}{2}\right).\end{aligned}$$

Therefore,

$$|\sin(a+h) - \sin a| = |h|\left|\frac{\sin(h/2)}{h/2}\right|\left|\cos\left(a+\frac{h}{2}\right)\right|.$$

Using the fact that $\left|\frac{\sin\theta}{\theta}\right| < 1$ and that $|\cos\theta| \leq 1$, and making the substitution $h = x - a$, we see that this last relation is equivalent to

$$|\sin x - \sin a| < |x - a|.$$

Now, to prove the desired limit, let $\epsilon > 0$, and take $\delta = \epsilon$. If $|x - a| < \delta$, then

$$|\sin x - \sin a| < |x - a| < \delta = \epsilon,$$

Therefore, a δ was found for arbitrary ϵ, and the proof is complete.

Further Insights and Challenges

30. Uniqueness of the Limit Prove that a function converges to at most one limiting value. In other words, use the limit definition to prove that if $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

SOLUTION Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L_1$, there exists δ_1 such that if $|x - c| < \delta_1$ then $|f(x) - L_1| < \epsilon$. Similarly, since $\lim_{x\to c} f(x) = L_2$, there exists δ_2 such that if $|x - c| < \delta_2$ then $|f(x) - L_2| < \epsilon$. Now let $|x - c| < \min(\delta_1, \delta_2)$ and observe that

$$\begin{aligned}|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |f(x) - L_2| \\ &= |f(x) - L_1| + |f(x) - L_2| < 2\epsilon.\end{aligned}$$

So, $|L_1 - L_2| < 2\epsilon$ for any $\epsilon > 0$. We have $|L_1 - L_2| = \lim_{\epsilon\to 0} |L_1 - L_2| < \lim_{\epsilon\to 0} 2\epsilon = 0$. Therefore, $|L_1 - L_2| = 0$ and, hence, $L_1 = L_2$.

In Exercises 31–33, prove the statement using the formal limit definition.

31. The Constant Multiple Law [Theorem 1, part (ii) in Section 2.3, p. 77]

SOLUTION Suppose that $\lim_{x\to c} f(x) = L$. We wish to prove that $\lim_{x\to c} af(x) = aL$.

Let $\epsilon > 0$ be given. $\epsilon/|a|$ is also a positive number. Since $\lim_{x\to c} f(x) = L$, we know there is a $\delta > 0$ such that $|x - c| < \delta$ forces $|f(x) - L| < \epsilon/|a|$. Suppose $|x - c| < \delta$. $|af(x) - aL| = |a||f(x) - aL| < |a|(\epsilon/|a|) = \epsilon$, so the rule is proven.

32. The Squeeze Theorem. (Theorem 1 in Section 2.6, p. 96)

SOLUTION *Proof of the Squeeze Theorem.* Suppose that (i) the inequalities $h(x) \le f(x) \le g(x)$ hold for all x near (but not equal to) a and (ii) $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$. Let $\epsilon > 0$ be given.

- By (i), there exists a $\delta_1 > 0$ such that $h(x) \le f(x) \le g(x)$ whenever $0 < |x - a| < \delta_1$.
- By (ii), there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that $|h(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_2$ and $|g(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_3$.
- Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |x - a| < \delta$ we have $L - \epsilon < h(x) \le f(x) \le g(x) < L + \epsilon$; i.e., $|f(x) - L| < \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to a} f(x) = L$.

33. The Product Law [Theorem 1, part (iii) in Section 2.3, p. 77]. *Hint:* Use the identity

$$f(x)g(x) - LM = (f(x) - L)\,g(x) + L(g(x) - M)$$

SOLUTION Before we can prove the Product Law, we need to establish one preliminary result. We are given that $\lim_{x\to c} g(x) = M$. Consequently, if we set $\epsilon = 1$, then the definition of a limit guarantees the existence of a $\delta_1 > 0$ such that whenever $0 < |x - c| < \delta_1$, $|g(x) - M| < 1$. Applying the inequality $|g(x)| - |M| \le |g(x) - M|$, it follows that $|g(x)| < 1 + |M|$. In other words, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$.

We can now prove the Product Law. Let $\epsilon > 0$. As proven above, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$. Furthermore, by the definition of a limit, $\lim_{x\to c} g(x) = M$ implies there exists a $\delta_2 > 0$ such that $|g(x) - M| < \frac{\epsilon}{2(1+|L|)}$ whenever $0 < |x - c| < \delta_2$. We have included the "1+" in the denominator to avoid division by zero in case $L = 0$. The reason for including the factor of 2 in the denominator will become clear shortly. Finally, because $\lim_{x\to c} f(x) = L$, there exists a $\delta_3 > 0$ such that $|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$ whenever $0 < |x - c| < \delta_3$. Now, let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then, for all x satisfying $0 < |x - c| < \delta$, we have

$$\begin{aligned}|f(x)g(x) - LM| &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\epsilon}{2(1+|M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1+|L|)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence,

$$\lim_{x\to c} f(x)g(x) = LM = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x).$$

34. Let $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Prove that $\lim_{x\to c} f(x)$ does not exist for any c.

SOLUTION Let c be any number, and let $\delta > 0$ be an arbitrary small number. We will prove that there is an x such that $|x - c| < \delta$, but $|f(x) - f(c)| > \frac{1}{2}$. c must be either irrational or rational. If c is rational, then $f(c) = 1$. Since the irrational numbers are dense, there is at least one irrational number z such that $|z - c| < \delta$. $|f(z) - f(c)| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$. On the other hand, if c is irrational, then there is a *rational* number q such that $|q - c| < \delta$. $|f(q) - f(c)| = |1 - 0| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$.

35. Here is a function with strange continuity properties:

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is the rational number } p/q \text{ in lowest terms} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Show that $f(x)$ is discontinuous at c if c is rational. *Hint:* There exist irrational numbers arbitrarily close to c.

(b) Show that $f(x)$ is continuous at c if c is irrational. *Hint:* Let I be the interval $\{x : |x - c| < 1\}$. Show that for any $Q > 0$, I contains at most finitely many fractions p/q with $q < Q$. Conclude that there is a δ such that all fractions in $\{x : |x - c| < \delta\}$ have a denominator larger than Q.

SOLUTION

(a) Let c be any rational number and suppose that, in lowest terms, $c = p/q$, where p and q are integers. To prove the discontinuity of f at c, we must show there is an $\epsilon > 0$ such that for any $\delta > 0$ there is an x for which $|x - c| < \delta$, but that $|f(x) - f(c)| > \epsilon$. Let $\epsilon = \frac{1}{2q}$ and $\delta > 0$. Since there is at least one irrational number between any two distinct real numbers, there is some irrational x between c and $c + \delta$. Hence, $|x - c| < \delta$, but $|f(x) - f(c)| = |0 - \frac{1}{q}| = \frac{1}{q} > \frac{1}{2q} = \epsilon$.

(b) Let c be irrational, let $\epsilon > 0$ be given, and let $N > 0$ be a prime integer sufficiently large so that $\frac{1}{N} < \epsilon$. Let $\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}$ be all rational numbers $\frac{p}{q}$ in lowest terms such that $|\frac{p}{q} - c| < 1$ and $q < N$. Since N is finite, this is a finite list; hence, one number $\frac{p_i}{q_i}$ in the list must be closest to c. Let $\delta = \frac{1}{2}|\frac{p_i}{q_i} - c|$. By construction, $|\frac{p_i}{q_i} - c| > \delta$ for all $i = 1 \dots m$. Therefore, for any rational number $\frac{p}{q}$ such that $|\frac{p}{q} - c| < \delta$, $q > N$, so $\frac{1}{q} < \frac{1}{N} < \epsilon$.

Therefore, for any *rational* number x such that $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. $|f(x) - f(c)| = 0$ for any irrational number x, so $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$ for any number x.

CHAPTER REVIEW EXERCISES

1. The position of a particle at time t (s) is $s(t) = \sqrt{t^2 + 1}$ m. Compute its average velocity over [2, 5] and estimate its instantaneous velocity at $t = 2$.

SOLUTION Let $s(t) = \sqrt{t^2 + 1}$. The average velocity over [2, 5] is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{\sqrt{26} - \sqrt{5}}{3} \approx 0.954 \text{ m/s}.$$

From the data in the table below, we estimate that the instantaneous velocity at $t = 2$ is approximately 0.894 m/s.

interval	[1.9, 2]	[1.99, 2]	[1.999, 2]	[2, 2.001]	[2, 2.01]	[2, 2.1]
average ROC	0.889769	0.893978	0.894382	0.894472	0.894873	0.898727

2. The "wellhead" price p of natural gas in the United States (in dollars per 1000 ft^3) on the first day of each month in 2008 is listed in the table below.

J	F	M	A	M	J
6.99	7.55	8.29	8.94	9.81	10.82
J	A	S	O	N	D
10.62	8.32	7.27	6.36	5.97	5.87

Compute the average rate of change of p (in dollars per 1000 ft^3 per month) over the quarterly periods January–March, April–June, and July–September.

SOLUTION To determine the average rate of change in price over the first quarter, divide the difference between the April and January prices by the three-month duration of the quarter. This yields

$$\frac{8.94 - 6.99}{3} = 0.65 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

In a similar manner, we calculate the average rates of change for the second and third quarters of the year to be

$$\frac{10.62 - 8.94}{3} = 0.56 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

and

$$\frac{6.36 - 10.62}{3} = -1.42 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

3. For a whole number n, let $P(n)$ be the number of *partitions* of n, that is, the number of ways of writing n as a sum of one or more whole numbers. For example, $P(4) = 5$ since the number 4 can be partitioned in five different ways: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. Treating $P(n)$ as a continuous function, use Figure 1 to estimate the rate of change of $P(n)$ at $n = 12$.

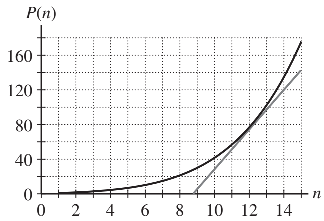


FIGURE 1 Graph of $P(n)$.

SOLUTION The tangent line drawn in the figure appears to pass through the points (15, 140) and (10.5, 40). We therefore estimate that the rate of change of $P(n)$ at $n = 12$ is

$$\frac{140 - 40}{15 - 10.5} = \frac{100}{4.5} = \frac{200}{9}.$$

4. The average velocity v (m/s) of an oxygen molecule in the air at temperature T (°C) is $v = 25.7\sqrt{273.15 + T}$. What is the average speed at $T = 25°$ (room temperature)? Estimate the rate of change of average velocity with respect to temperature at $T = 25°$. What are the units of this rate?

SOLUTION Let $v(T) = 25.7\sqrt{273.15 + T}$. The average velocity at $T = 25°$C is

$$v(25) = 25.7\sqrt{273.15 + 25} \approx 443.76 \text{ m/s.}$$

From the data in the table below, we estimate that the rate of change of velocity with respect to temperature when $T = 25°$C is 0.7442 m/s^2.

interval	[24.9, 25]	[24.99, 25]	[24.999, 25]	[25, 25.001]	[25, 25.01]	[25, 25.1]
average ROC	0.744256	0.744199	0.744193	0.744195	0.744187	0.744131

In Exercises 5–10, estimate the limit numerically to two decimal places or state that the limit does not exist.

5. $\lim_{x\to 0} \frac{1 - \cos^3(x)}{x^2}$

SOLUTION Let $f(x) = \frac{1-\cos^3 x}{x^2}$. The data in the table below suggests that

$$\lim_{x\to 0} \frac{1 - \cos^3 x}{x^2} \approx 1.50.$$

In constructing the table, we take advantage of the fact that f is an even function.

x	± 0.001	± 0.01	± 0.1
$f(x)$	1.500000	1.499912	1.491275

(The exact value is $\frac{3}{2}$.)

6. $\lim_{x \to 1} x^{1/(x-1)}$

SOLUTION Let $f(x) = x^{1/(x-1)}$. The data in the table below suggests that

$$\lim_{x \to 1} x^{1/(x-1)} \approx 2.72.$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.867972	2.731999	2.719642	2.716924	2.704814	2.593742

(The exact value is e.)

7. $\lim_{x \to 2} \frac{x^x - 4}{x^2 - 4}$

SOLUTION Let $f(x) = \frac{x^x - 4}{x^2 - 4}$. The data in the table below suggests that

$$\lim_{x \to 2} \frac{x^x - 4}{x^2 - 4} \approx 1.69.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	1.575461	1.680633	1.691888	1.694408	1.705836	1.828386

(The exact value is $1 + \ln 2$.)

8. $\lim_{x \to 2} \frac{x - 2}{\ln(3x - 5)}$

SOLUTION Let $f(x) = \frac{x-2}{\ln(3x-5)}$. The data in the table below suggests that

$$\lim_{x \to 2} \frac{x - 2}{\ln(3x - 5)} \approx 0.33.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.280367	0.328308	0.332833	0.333833	0.338309	0.381149

(The exact value is $1/3$.)

9. $\lim_{x \to 1} \left(\frac{7}{1 - x^7} - \frac{3}{1 - x^3} \right)$

SOLUTION Let $f(x) = \left(\frac{7}{1-x^7} - \frac{3}{1-x^3} \right)$. The data in the table below suggests that

$$\lim_{x \to 1} \left(\frac{7}{1 - x^7} - \frac{3}{1 - x^3} \right) \approx 2.00.$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.347483	2.033498	2.003335	1.996668	1.966835	1.685059

(The exact value is 2.)

10. $\lim_{x \to 2} \frac{3^x - 9}{5^x - 25}$

SOLUTION Let $f(x) = \frac{3^x - 9}{5^x - 25}$. The data in the table below suggests that

$$\lim_{x \to 2} \frac{3^x - 9}{5^x - 25} \approx 0.246.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.251950	0.246365	0.245801	0.245675	0.245110	0.239403

(The exact value is $\frac{9}{25} \frac{\ln 3}{\ln 5}$.)

In Exercises 11–50, evaluate the limit if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

11. $\lim_{x\to 4}(3+x^{1/2})$

SOLUTION $\lim_{x\to 4}(3+x^{1/2}) = 3+\sqrt{4} = 5.$

12. $\lim_{x\to 1}\frac{5-x^2}{4x+7}$

SOLUTION $\lim_{x\to 1}\frac{5-x^2}{4x+7} = \frac{5-1^2}{4(1)+7} = \frac{4}{11}.$

13. $\lim_{x\to -2}\frac{4}{x^3}$

SOLUTION $\lim_{x\to -2}\frac{4}{x^3} = \frac{4}{(-2)^3} = -\frac{1}{2}.$

14. $\lim_{x\to -1}\frac{3x^2+4x+1}{x+1}$

SOLUTION $\lim_{x\to -1}\frac{3x^2+4x+1}{x+1} = \lim_{x\to -1}\frac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1}(3x+1) = 3(-1)+1 = -2.$

15. $\lim_{t\to 9}\frac{\sqrt{t}-3}{t-9}$

SOLUTION $\lim_{t\to 9}\frac{\sqrt{t}-3}{t-9} = \lim_{t\to 9}\frac{\sqrt{t}-3}{(\sqrt{t}-3)(\sqrt{t}+3)} = \lim_{t\to 9}\frac{1}{\sqrt{t}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$

16. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

SOLUTION

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} &= \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3}\frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}\\ &= \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{4}.\end{aligned}$$

17. $\lim_{x\to 1}\frac{x^3-x}{x-1}$

SOLUTION $\lim_{x\to 1}\frac{x^3-x}{x-1} = \lim_{x\to 1}\frac{x(x-1)(x+1)}{x-1} = \lim_{x\to 1}x(x+1) = 1(1+1) = 2.$

18. $\lim_{h\to 0}\frac{2(a+h)^2-2a^2}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{2(a+h)^2-2a^2}{h} = \lim_{h\to 0}\frac{2a^2+4ah+2h^2-2a^2}{h} = \lim_{h\to 0}\frac{h(4a+2h)}{h} = \lim_{h\to 0}(4a+2h) = 4a+2(0) = 4a.$$

19. $\lim_{t\to 9}\frac{t-6}{\sqrt{t}-3}$

SOLUTION Because the one-sided limits

$$\lim_{t\to 9-}\frac{t-6}{\sqrt{t}-3} = -\infty \qquad \text{and} \qquad \lim_{t\to 9+}\frac{t-6}{\sqrt{t}-3} = \infty,$$

are not equal, the two-sided limit

$$\lim_{t\to 9}\frac{t-6}{\sqrt{t}-3} \qquad \text{does not exist.}$$

20. $\lim_{s \to 0} \frac{1 - \sqrt{s^2 + 1}}{s^2}$

SOLUTION

$$\lim_{s \to 0} \frac{1 - \sqrt{s^2+1}}{s^2} = \lim_{s \to 0} \frac{1 - \sqrt{s^2+1}}{s^2} \cdot \frac{1 + \sqrt{s^2+1}}{1 + \sqrt{s^2+1}} = \lim_{s \to 0} \frac{1 - (s^2+1)}{s^2(1 + \sqrt{s^2+1})}$$
$$= \lim_{s \to 0} \frac{-1}{1 + \sqrt{s^2+1}} = \frac{-1}{1 + \sqrt{0^2+1}} = -\frac{1}{2}.$$

21. $\lim_{x \to -1+} \frac{1}{x+1}$

SOLUTION For $x > -1$, $x + 1 > 0$. Therefore,

$$\lim_{x \to -1+} \frac{1}{x+1} = \infty.$$

22. $\lim_{y \to \frac{1}{3}} \frac{3y^2 + 5y - 2}{6y^2 - 5y + 1}$

SOLUTION

$$\lim_{y \to \frac{1}{3}} \frac{3y^2 + 5y - 2}{6y^2 - 5y + 1} = \lim_{y \to \frac{1}{3}} \frac{(3y-1)(y+2)}{(3y-1)(2y-1)} = \lim_{y \to \frac{1}{3}} \frac{y+2}{2y-1} = -7.$$

23. $\lim_{x \to 1} \frac{x^3 - 2x}{x - 1}$

SOLUTION Because the one-sided limits

$$\lim_{x \to 1-} \frac{x^3 - 2x}{x - 1} = \infty \qquad \text{and} \qquad \lim_{x \to 1+} \frac{x^3 - 2x}{x - 1} = -\infty,$$

are not equal, the two-sided limit

$$\lim_{x \to 1} \frac{x^3 - 2x}{x - 1} \qquad \text{does not exist.}$$

24. $\lim_{a \to b} \frac{a^2 - 3ab + 2b^2}{a - b}$

SOLUTION $\lim_{a \to b} \frac{a^2 - 3ab + 2b^2}{a - b} = \lim_{a \to b} \frac{(a-b)(a-2b)}{a-b} = \lim_{a \to b} (a - 2b) = b - 2b = -b.$

25. $\lim_{x \to 0} \frac{e^{3x} - e^x}{e^x - 1}$

SOLUTION

$$\lim_{x \to 0} \frac{e^{3x} - e^x}{e^x - 1} = \lim_{x \to 0} \frac{e^x(e^x - 1)(e^x + 1)}{e^x - 1} = \lim_{x \to 0} e^x(e^x + 1) = 1 \cdot 2 = 2.$$

26. $\lim_{\theta \to 0} \frac{\sin 5\theta}{\theta}$

SOLUTION

$$\lim_{\theta \to 0} \frac{\sin 5\theta}{\theta} = 5 \lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta} = 5(1) = 5.$$

27. $\lim_{x \to 1.5} \frac{[x]}{x}$

SOLUTION $\lim_{x \to 1.5} \frac{[x]}{x} = \frac{[1.5]}{1.5} = \frac{1}{1.5} = \frac{2}{3}.$

28. $\lim_{\theta \to \frac{\pi}{4}} \sec\theta$

SOLUTION

$$\lim_{\theta \to \frac{\pi}{4}} \sec\theta = \sec\frac{\pi}{4} = \sqrt{2}.$$

29. $\lim_{z\to -3} \frac{z+3}{z^2+4z+3}$

SOLUTION

$$\lim_{z\to -3} \frac{z+3}{z^2+4z+3} = \lim_{z\to -3} \frac{z+3}{(z+3)(z+1)} = \lim_{z\to -3} \frac{1}{z+1} = -\frac{1}{2}.$$

30. $\lim_{x\to 1} \frac{x^3-ax^2+ax-1}{x-1}$

SOLUTION Using

$$x^3-ax^2+ax-1 = (x-1)(x^2+x+1) - ax(x-1) = (x-1)(x^2+x-ax+1)$$

we find

$$\lim_{x\to 1} \frac{x^3-ax^2+ax-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x-ax+1)}{x-1} = \lim_{x\to 1}(x^2+x-ax+1)$$
$$= 1^2+1-a(1)+1 = 3-a.$$

31. $\lim_{x\to b} \frac{x^3-b^3}{x-b}$

SOLUTION $\lim_{x\to b} \frac{x^3-b^3}{x-b} = \lim_{x\to b} \frac{(x-b)(x^2+xb+b^2)}{x-b} = \lim_{x\to b}(x^2+xb+b^2) = b^2+b(b)+b^2 = 3b^2.$

32. $\lim_{x\to 0} \frac{\sin 4x}{\sin 3x}$

SOLUTION

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 3x} = \frac{4}{3} \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{3x}{\sin 3x} = \frac{4}{3} \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \lim_{x\to 0} \frac{3x}{\sin 3x} = \frac{4}{3}(1)(1) = \frac{4}{3}.$$

33. $\lim_{x\to 0} \left(\frac{1}{3x} - \frac{1}{x(x+3)}\right)$

SOLUTION $\lim_{x\to 0} \left(\frac{1}{3x} - \frac{1}{x(x+3)}\right) = \lim_{x\to 0} \frac{(x+3)-3}{3x(x+3)} = \lim_{x\to 0} \frac{1}{3(x+3)} = \frac{1}{3(0+3)} = \frac{1}{9}.$

34. $\lim_{\theta\to \frac{1}{4}} 3^{\tan(\pi\theta)}$

SOLUTION

$$\lim_{\theta\to \frac{1}{4}} 3^{\tan(\pi\theta)} = 3^{\tan(\pi/4)} = 3^1 = 3.$$

35. $\lim_{x\to 0-} \frac{[x]}{x}$

SOLUTION For x sufficiently close to zero but negative, $[x] = -1$. Therefore,

$$\lim_{x\to 0-} \frac{[x]}{x} = \lim_{x\to 0-} \frac{-1}{x} = \infty.$$

36. $\lim_{x\to 0+} \frac{[x]}{x}$

SOLUTION For x sufficiently close to zero but positive, $[x] = 0$. Therefore,

$$\lim_{x\to 0+} \frac{[x]}{x} = \lim_{x\to 0+} \frac{0}{x} = 0.$$

37. $\lim_{\theta\to \frac{\pi}{2}} \theta \sec\theta$

SOLUTION Because the one-sided limits

$$\lim_{\theta\to \frac{\pi}{2}-} \theta\sec\theta = \infty \qquad \text{and} \qquad \lim_{\theta\to \frac{\pi}{2}+} \theta\sec\theta = -\infty$$

are not equal, the two-sided limit

$$\lim_{\theta \to \frac{\pi}{2}} \theta \sec \theta \qquad \text{does not exist.}$$

38. $\lim_{y \to 2} \ln\left(\sin \frac{\pi}{y}\right)$

SOLUTION

$$\lim_{y \to 2} \ln\left(\sin \frac{\pi}{y}\right) = \ln\left(\sin \frac{\pi}{2}\right) = \ln 1 = 0.$$

39. $\lim_{\theta \to 0} \frac{\cos \theta - 2}{\theta}$

SOLUTION Because the one-sided limits

$$\lim_{\theta \to 0-} \frac{\cos \theta - 2}{\theta} = \infty \qquad \text{and} \qquad \lim_{\theta \to 0+} \frac{\cos \theta - 2}{\theta} = -\infty$$

are not equal, the two-sided limit

$$\lim_{\theta \to 0} \frac{\cos \theta - 2}{\theta} \qquad \text{does not exist.}$$

40. $\lim_{x \to 4.3} \frac{1}{x - [x]}$

SOLUTION $\lim_{x \to 4.3} \frac{1}{x - [x]} = \frac{1}{4.3 - [4.3]} = \frac{1}{0.3} = \frac{10}{3}.$

41. $\lim_{x \to 2-} \frac{x - 3}{x - 2}$

SOLUTION For x close to 2 but less than 2, $x - 3 < 0$ and $x - 2 < 0$. Therefore,

$$\lim_{x \to 2-} \frac{x - 3}{x - 2} = \infty.$$

42. $\lim_{t \to 0} \frac{\sin^2 t}{t^3}$

SOLUTION Note that

$$\frac{\sin^2 t}{t^3} = \frac{\sin t}{t} \cdot \frac{\sin t}{t} \cdot \frac{1}{t}.$$

As $t \to 0$, each factor of $\frac{\sin t}{t}$ approaches 1; however, the factor $\frac{1}{t}$ tends to $-\infty$ as $t \to 0-$ and tends to ∞ as $t \to 0+$. Consequently,

$$\lim_{t \to 0-} \frac{\sin^2 t}{t^3} = -\infty, \quad \lim_{t \to 0+} \frac{\sin^2 t}{t^3} = \infty$$

and

$$\lim_{t \to 0} \frac{\sin^2 t}{t^3} \qquad \text{does not exist.}$$

43. $\lim_{x \to 1+} \left(\frac{1}{\sqrt{x - 1}} - \frac{1}{\sqrt{x^2 - 1}}\right)$

SOLUTION $\lim_{x \to 1+} \left(\frac{1}{\sqrt{x - 1}} - \frac{1}{\sqrt{x^2 - 1}}\right) = \lim_{x \to 1+} \frac{\sqrt{x + 1} - 1}{\sqrt{x^2 - 1}} = \infty.$

44. $\lim_{t \to e} \sqrt{t}(\ln t - 1)$

SOLUTION

$$\lim_{t \to e} \sqrt{t}(\ln t - 1) = \lim_{t \to e} \sqrt{t} \cdot \lim_{t \to e} (\ln t - 1) = \sqrt{e}(\ln e - 1) = 0.$$

45. $\lim_{x\to\frac{\pi}{2}} \tan x$

SOLUTION Because the one-sided limits

$$\lim_{x\to\frac{\pi}{2}-} \tan x = \infty \qquad \text{and} \qquad \lim_{x\to\frac{\pi}{2}+} \tan x = -\infty$$

are not equal, the two-sided limit

$$\lim_{x\to\frac{\pi}{2}} \tan x \qquad \text{does not exist.}$$

46. $\lim_{t\to 0} \cos \frac{1}{t}$

SOLUTION As $t \to 0$, $\frac{1}{t}$ grows without bound and $\cos(\frac{1}{t})$ oscillates faster and faster. Consequently,

$$\lim_{t\to 0} \cos\left(\frac{1}{t}\right) \qquad \text{does not exist.}$$

The same is true for both one-sided limits.

47. $\lim_{t\to 0+} \sqrt{t}\cos \frac{1}{t}$

SOLUTION For $t > 0$,

$$-1 \le \cos\left(\frac{1}{t}\right) \le 1,$$

so

$$-\sqrt{t} \le \sqrt{t}\cos\left(\frac{1}{t}\right) \le \sqrt{t}.$$

Because

$$\lim_{t\to 0+} -\sqrt{t} = \lim_{t\to 0+} \sqrt{t} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{t\to 0+} \sqrt{t}\cos\left(\frac{1}{t}\right) = 0.$$

48. $\lim_{x\to 5+} \frac{x^2-24}{x^2-25}$

SOLUTION For x close to 5 but larger than 5, $x^2 - 24 > 0$ and $x^2 - 25 > 0$. Therefore,

$$\lim_{x\to 5+} \frac{x^2-24}{x^2-25} = \infty.$$

49. $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

SOLUTION

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{-\sin^2 x}{\sin x(\cos x + 1)} = -\lim_{x\to 0} \frac{\sin x}{\cos x + 1} = -\frac{0}{1+1} = 0.$$

50. $\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta} \cdot \frac{\sec\theta + 1}{\sec\theta + 1} = \lim_{\theta\to 0} \frac{\tan^2\theta}{\sin^2\theta(\sec\theta + 1)}$$

$$= \lim_{\theta\to 0} \frac{\sec^2\theta}{\sec\theta + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

51. Find the left- and right-hand limits of the function $f(x)$ in Figure 2 at $x = 0, 2, 4$. State whether $f(x)$ is left- or right-continuous (or both) at these points.

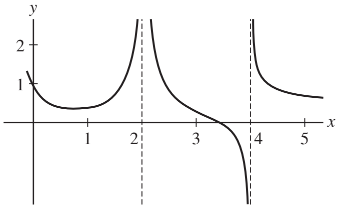


FIGURE 2

SOLUTION According to the graph of $f(x)$,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = 1$$
$$\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = \infty$$
$$\lim_{x\to 4-} f(x) = -\infty$$
$$\lim_{x\to 4+} f(x) = \infty.$$

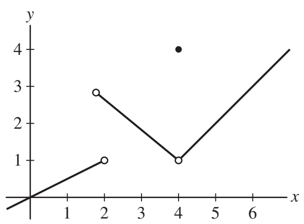
The function is both left- and right-continuous at $x = 0$ and neither left- nor right-continuous at $x = 2$ and $x = 4$.

52. Sketch the graph of a function $f(x)$ such that

(a) $\lim_{x\to 2-} f(x) = 1$, $\quad \lim_{x\to 2+} f(x) = 3$

(b) $\lim_{x\to 4} f(x)$ exists but does not equal $f(4)$.

SOLUTION

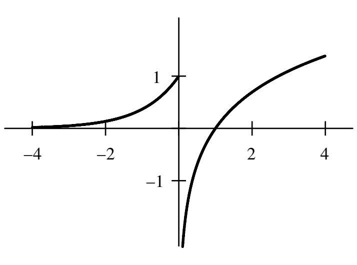


53. Graph $h(x)$ and describe the discontinuity:

$$h(x) = \begin{cases} e^x & \text{for } x \le 0 \\ \ln x & \text{for } x > 0 \end{cases}$$

Is $h(x)$ left- or right-continuous?

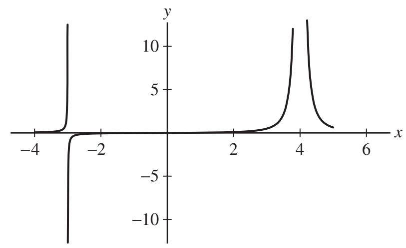
SOLUTION The graph of $h(x)$ is shown below. At $x = 0$, the function has an infinite discontinuity but is left-continuous.



54. Sketch the graph of a function $g(x)$ such that

$$\lim_{x\to -3-} g(x) = \infty, \qquad \lim_{x\to -3+} g(x) = -\infty, \qquad \lim_{x\to 4} g(x) = \infty$$

SOLUTION



55. Find the points of discontinuity of

$$g(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{for } |x| < 1 \\ |x-1| & \text{for } |x| \ge 1 \end{cases}$$

Determine the type of discontinuity and whether $g(x)$ is left- or right-continuous.

SOLUTION First note that $\cos\left(\frac{\pi x}{2}\right)$ is continuous for $-1 < x < 1$ and that $|x-1|$ is continuous for $x \le -1$ and for $x \ge 1$. Thus, the only points at which $g(x)$ might be discontinuous are $x = \pm 1$. At $x = 1$, we have

$$\lim_{x\to 1-} g(x) = \lim_{x\to 1-} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to 1+} g(x) = \lim_{x\to 1+} |x-1| = |1-1| = 0,$$

so $g(x)$ is continuous at $x = 1$. On the other hand, at $x = -1$,

$$\lim_{x\to -1+} g(x) = \lim_{x\to -1+} \cos\left(\frac{\pi x}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to -1-} g(x) = \lim_{x\to -1-} |x-1| = |-1-1| = 2,$$

so $g(x)$ has a jump discontinuity at $x = -1$. Since $g(-1) = 2$, $g(x)$ is left-continuous at $x = -1$.

56. Show that $f(x) = xe^{\sin x}$ is continuous on its domain.

SOLUTION Because e^x and $\sin x$ are continuous for all real numbers, their composition, $e^{\sin x}$ is continuous for all real numbers. Moreover, x is continuous for all real numbers, so the product $xe^{\sin x}$ is continuous for all real numbers. Thus, $f(x) = xe^{\sin x}$ is continuous for all real numbers.

57. Find a constant b such that $h(x)$ is continuous at $x = 2$, where

$$h(x) = \begin{cases} x+1 & \text{for } |x| < 2 \\ b - x^2 & \text{for } |x| \ge 2 \end{cases}$$

With this choice of b, find all points of discontinuity.

SOLUTION To make $h(x)$ continuous at $x = 2$, we must have the two one-sided limits as x approaches 2 be equal. With

$$\lim_{x\to 2-} h(x) = \lim_{x\to 2-} (x+1) = 2+1 = 3$$

and

$$\lim_{x\to 2+} h(x) = \lim_{x\to 2+} (b - x^2) = b - 4,$$

it follows that we must choose $b = 7$. Because $x+1$ is continuous for $-2 < x < 2$ and $7 - x^2$ is continuous for $x \le -2$ and for $x \ge 2$, the only possible point of discontinuity is $x = -2$. At $x = -2$,

$$\lim_{x\to -2+} h(x) = \lim_{x\to -2+} (x+1) = -2+1 = -1$$

and

$$\lim_{x\to -2-} h(x) = \lim_{x\to -2-} (7 - x^2) = 7 - (-2)^2 = 3,$$

so $h(x)$ has a jump discontinuity at $x = -2$.

In Exercises 58–63, find the horizontal asymptotes of the function by computing the limits at infinity.

58. $f(x) = \dfrac{9x^2 - 4}{2x^2 - x}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{9x^2 - 4}{2x^2 - x} = \lim_{x\to\infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2}$$

and

$$\lim_{x\to-\infty} \frac{9x^2 - 4}{2x^2 - x} = \lim_{x\to-\infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2},$$

it follows that the graph of $y = \dfrac{9x^2 - 4}{2x^2 - x}$ has a horizontal asymptote of $\frac{9}{2}$.

59. $f(x) = \dfrac{x^2 - 3x^4}{x - 1}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{x^2 - 3x^4}{x - 1} = \lim_{x\to\infty} \frac{1/x^2 - 3}{1/x^3 - 1/x^4} = -\infty$$

and

$$\lim_{x\to-\infty} \frac{x^2 - 3x^4}{x - 1} = \lim_{x\to-\infty} \frac{1/x^2 - 3}{1/x^3 - 1/x^4} = \infty,$$

it follows that the graph of $y = \dfrac{x^2 - 3x^4}{x - 1}$ does not have any horizontal asymptotes.

60. $f(u) = \dfrac{8u - 3}{\sqrt{16u^2 + 6}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{8u - 3}{\sqrt{16u^2 + 6}} = \lim_{u\to\infty} \frac{8 - 3/u}{\sqrt{16 + 6/u^2}} = \frac{8}{\sqrt{16}} = 2$$

and

$$\lim_{u\to-\infty} \frac{8u - 3}{\sqrt{16u^2 + 6}} = \lim_{u\to-\infty} \frac{8 - 3/u}{-\sqrt{16 + 6/u^2}} = \frac{8}{-\sqrt{16}} = -2,$$

it follows that the graph of $y = \dfrac{8u - 3}{\sqrt{16u^2 + 6}}$ has horizontal asymptotes of $y = \pm 2$.

61. $f(u) = \dfrac{2u^2 - 1}{\sqrt{6 + u^4}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{2u^2 - 1}{\sqrt{6 + u^4}} = \lim_{u\to\infty} \frac{2 - 1/u^2}{\sqrt{6/u^4 + 1}} = \frac{2}{\sqrt{1}} = 2$$

and

$$\lim_{u\to-\infty} \frac{2u^2 - 1}{\sqrt{6 + u^4}} = \lim_{u\to-\infty} \frac{2 - 1/u^2}{\sqrt{6/u^4 + 1}} = \frac{2}{\sqrt{1}} = 2,$$

it follows that the graph of $y = \dfrac{2u^2 - 1}{\sqrt{6 + u^4}}$ has a horizontal asymptote of $y = 2$.

62. $f(x) = \dfrac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} = \lim_{x\to\infty} \frac{3x^{-2/15} + 9x^{-13/35}}{7 - x^{-17/15}} = 0$$

and

$$\lim_{x\to-\infty} \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} = \lim_{x\to-\infty} \frac{3x^{-2/15} + 9x^{-13/35}}{7 - x^{-17/15}} = 0,$$

it follows that the graph of $y = \dfrac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}}$ has a horizontal asymptote of $y = 0$.

63. $f(t) = \dfrac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}}$

SOLUTION Because

$$\lim_{t\to\infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t\to\infty} \frac{1 - t^{-2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1$$

and

$$\lim_{t\to-\infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t\to-\infty} \frac{1 - t^{-2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1,$$

it follows that the graph of $y = \dfrac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}}$ has a horizontal asymptote of $y = 1$.

64. Calculate (a)–(d), assuming that

$$\lim_{x\to 3} f(x) = 6, \qquad \lim_{x\to 3} g(x) = 4$$

(a) $\lim_{x\to 3} (f(x) - 2g(x))$

(b) $\lim_{x\to 3} x^2 f(x)$

(c) $\lim_{x\to 3} \dfrac{f(x)}{g(x) + x}$

(d) $\lim_{x\to 3} (2g(x)^3 - g(x)^{3/2})$

SOLUTION

(a) $\lim_{x\to 3} (f(x) - 2g(x)) = \lim_{x\to 3} f(x) - 2\lim_{x\to 3} g(x) = 6 - 2(4) = -2.$

(b) $\lim_{x\to 3} x^2 f(x) = \lim_{x\to 3} x^2 \cdot \lim_{x\to 3} f(x) = 3^2 \cdot 6 = 54.$

(c) $\lim_{x\to 3} \dfrac{f(x)}{g(x) + x} = \dfrac{\lim_{x\to 3} f(x)}{\lim_{x\to 3}(g(x) + x)} = \dfrac{6}{\lim_{x\to 3} g(x) + \lim_{x\to 3} x} = \dfrac{6}{4 + 3} = \dfrac{6}{7}.$

(d) $\lim_{x\to 3} (2g(x)^3 - g(x)^{3/2}) = 2\left(\lim_{x\to 3} g(x)\right)^3 - \left(\lim_{x\to 3} g(x)\right)^{3/2} = 2(4)^3 - 4^{3/2} = 120.$

65. Assume that the following limits exist:

$$A = \lim_{x\to a} f(x), \qquad B = \lim_{x\to a} g(x), \qquad L = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Prove that if $L = 1$, then $A = B$. *Hint:* You cannot use the Quotient Law if $B = 0$, so apply the Product Law to L and B instead.

SOLUTION Suppose the limits A, B, and L all exist and $L = 1$. Then

$$B = B \cdot 1 = B \cdot L = \lim_{x\to a} g(x) \cdot \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} g(x) \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) = A.$$

66. $\boxed{\text{GU}}$ Define $g(t) = (1 + 2^{1/t})^{-1}$ for $t \neq 0$. How should $g(0)$ be defined to make $g(t)$ left-continuous at $t = 0$?

SOLUTION Because

$$\lim_{t\to 0-} (1 + 2^{1/t})^{-1} = \left[\lim_{t\to 0-} (1 + 2^{1/t})\right]^{-1} = 1^{-1} = 1,$$

we should define $g(0) = 1$ to make $g(t)$ left-continuous at $t = 0$.

67. In the notation of Exercise 65, give an example where L exists but neither A nor B exists.

SOLUTION Suppose

$$f(x) = \frac{1}{(x-a)^3} \qquad \text{and} \qquad g(x) = \frac{1}{(x-a)^5}.$$

Then, neither A nor B exists, but

$$L = \lim_{x\to a} \frac{(x-a)^{-3}}{(x-a)^{-5}} = \lim_{x\to a} (x-a)^2 = 0.$$

68. True or false?

(a) If $\lim_{x\to 3} f(x)$ exists, then $\lim_{x\to 3} f(x) = f(3)$.

(b) If $\lim_{x\to 0} \frac{f(x)}{x} = 1$, then $f(0) = 0$.

(c) If $\lim_{x\to -7} f(x) = 8$, then $\lim_{x\to -7} \frac{1}{f(x)} = \frac{1}{8}$.

(d) If $\lim_{x\to 5+} f(x) = 4$ and $\lim_{x\to 5-} f(x) = 8$, then $\lim_{x\to 5} f(x) = 6$.

(e) If $\lim_{x\to 0} \frac{f(x)}{x} = 1$, then $\lim_{x\to 0} f(x) = 0$.

(f) If $\lim_{x\to 5} f(x) = 2$, then $\lim_{x\to 5} f(x)^3 = 8$.

SOLUTION

(a) False. The limit $\lim_{x\to 3} f(x)$ may exist and need not equal $f(3)$. The limit is equal to $f(3)$ if $f(x)$ is continuous at $x = 3$.

(b) False. The value of the limit $\lim_{x\to 0} \frac{f(x)}{x} = 1$ does not depend on the value $f(0)$, so $f(0)$ can have any value.

(c) True, by the Limit Laws.

(d) False. If the two one-sided limits are not equal, then the two-sided limit does not exist.

(e) True. Apply the Product Law to the functions $\frac{f(x)}{x}$ and x.

(f) True, by the Limit Laws.

69. Let $f(x) = x\left[\frac{1}{x}\right]$, where $[x]$ is the greatest integer function. Show that for $x \neq 0$,

$$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \leq \frac{1}{x}$$

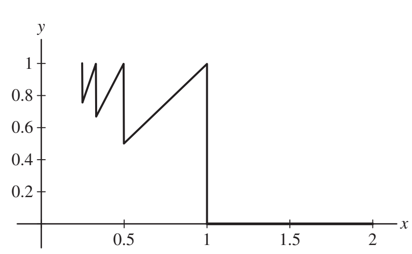
Then use the Squeeze Theorem to prove that

$$\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1$$

Hint: Treat the one-sided limits separately.

SOLUTION

(a) The graph of $f(x) = x\left[\frac{1}{x}\right]$ over $[\frac{1}{4}, 2]$ is shown below.



(b) Let y be any real number. From the definition of the greatest integer function, it follows that $y - 1 < [y] \leq y$, with equality holding if and only if y is an integer. If $x \neq 0$, then $\frac{1}{x}$ is a real number, so

$$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \leq \frac{1}{x}.$$

Upon multiplying this inequality through by x, we find

$$1 - x < x\left[\frac{1}{x}\right] \le 1.$$

Because

$$\lim_{x\to 0}(1-x) = \lim_{x\to 0} 1 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1.$$

70. Let r_1 and r_2 be the roots of $f(x) = ax^2 - 2x + 20$. Observe that $f(x)$ "approaches" the linear function $L(x) = -2x + 20$ as $a \to 0$. Because $r = 10$ is the unique root of $L(x)$, we might expect one of the roots of $f(x)$ to approach 10 as $a \to 0$ (Figure 3). Prove that the roots can be labeled so that $\lim_{a\to 0} r_1 = 10$ and $\lim_{a\to 0} r_2 = \infty$.

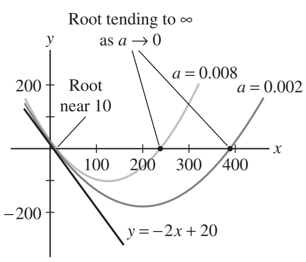


FIGURE 3 Graphs of $f(x) = ax^2 - 2x + 20$.

SOLUTION Using the quadratic formula, we find that the roots of the quadratic polynomial $f(x) = ax^2 - 2x + 20$ are

$$\frac{2 \pm \sqrt{4 - 80a}}{2a} = \frac{1 \pm \sqrt{1 - 20a}}{a} = \frac{20}{1 \pm \sqrt{1 - 20a}}.$$

Now let

$$r_1 = \frac{20}{1 + \sqrt{1 - 20a}} \qquad \text{and} \qquad r_2 = \frac{20}{1 - \sqrt{1 - 20a}}.$$

It is straightforward to calculate that

$$\lim_{a\to 0} r_1 = \lim_{a\to 0} \frac{20}{1 + \sqrt{1 - 20a}} = \frac{20}{2} = 10$$

and that

$$\lim_{a\to 0} r_2 = \lim_{a\to 0} \frac{20}{1 - \sqrt{1 - 20a}} = \infty$$

as desired.

71. Use the IVT to prove that the curves $y = x^2$ and $y = \cos x$ intersect.

SOLUTION Let $f(x) = x^2 - \cos x$. Note that any root of $f(x)$ corresponds to a point of intersection between the curves $y = x^2$ and $y = \cos x$. Now, $f(x)$ is continuous over the interval $[0, \frac{\pi}{2}]$, $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$; consequently, the curves $y = x^2$ and $y = \cos x$ intersect.

72. Use the IVT to prove that $f(x) = x^3 - \dfrac{x^2 + 2}{\cos x + 2}$ has a root in the interval $[0, 2]$.

SOLUTION Let $f(x) = x^3 - \frac{x^2+2}{\cos x+2}$. Because $\cos x + 2$ is never zero, $f(x)$ is continuous for all real numbers. Because

$$f(0) = -\frac{2}{3} < 0 \qquad \text{and} \qquad f(2) = 8 - \frac{6}{\cos 2 + 2} \approx 4.21 > 0,$$

the Intermediate Value Theorem guarantees there exists a $c \in (0, 2)$ such that $f(c) = 0$.

73. Use the IVT to show that $e^{-x^2} = x$ has a solution on $(0, 1)$.

SOLUTION Let $f(x) = e^{-x^2} - x$. Observe that f is continuous on $[0, 1]$ with $f(0) = e^0 - 0 = 1 > 0$ and $f(1) = e^{-1} - 1 < 0$. Therefore, the IVT guarantees there exists a $c \in (0, 1)$ such that $f(c) = e^{-c^2} - c = 0$.

74. Use the Bisection Method to locate a solution of $x^2 - 7 = 0$ to two decimal places.

SOLUTION Let $f(x) = x^2 - 7$. By trial and error, we find that $f(2.6) = -0.24 < 0$ and $f(2.7) = 0.29 > 0$. Because $f(x)$ is continuous on $[2.6, 2.7]$, it follows from the Intermediate Value Theorem that $f(x)$ has a root on $(2.6, 2.7)$. We approximate the root by the midpoint of the interval: $x = 2.65$. Now, $f(2.65) = 0.0225 > 0$. Because $f(2.6)$ and $f(2.65)$ are of opposite sign, the root must lie on $(2.6, 2.65)$. The midpoint of this interval is $x = 2.625$ and $f(2.625) < 0$; hence, the root must be on the interval $(2.625, 2.65)$. Continuing in this fashion, we construct the following sequence of intervals and midpoints.

interval	midpoint
(2.625, 2.65)	2.6375
(2.6375, 2.65)	2.64375
(2.64375, 2.65)	2.646875
(2.64375, 2.646875)	2.6453125
(2.6453125, 2.646875)	2.64609375

At this point, we note that, to two decimal places, one root of $x^2 - 7 = 0$ is 2.65.

75. Give an example of a (discontinuous) function that does not satisfy the conclusion of the IVT on $[-1, 1]$. Then show that the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies the conclusion of the IVT on every interval $[-a, a]$, even though f is discontinuous at $x = 0$.

SOLUTION Let $g(x) = [x]$. This function is discontinuous on $[-1, 1]$ with $g(-1) = -1$ and $g(1) = 1$. For all $c \neq 0$, there is no x such that $g(x) = c$; thus, $g(x)$ does not satisfy the conclusion of the Intermediate Value Theorem on $[-1, 1]$.

Now, let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

and let $a > 0$. On the interval

$$x \in \left[\frac{a}{2 + 2\pi a}, \frac{a}{2}\right] \subset [-a, a],$$

$\frac{1}{x}$ runs from $\frac{2}{a}$ to $\frac{2}{a} + 2\pi$, so the sine function covers one full period and clearly takes on every value from $-\sin a$ through $\sin a$.

76. Let $f(x) = \dfrac{1}{x+2}$.

(a) Show that $\left|f(x) - \frac{1}{4}\right| < \dfrac{|x-2|}{12}$ if $|x - 2| < 1$. *Hint:* Observe that $|4(x + 2)| > 12$ if $|x - 2| < 1$.

(b) Find $\delta > 0$ such that $\left|f(x) - \frac{1}{4}\right| < 0.01$ for $|x - 2| < \delta$.

(c) Prove rigorously that $\lim_{x \to 2} f(x) = \frac{1}{4}$.

SOLUTION

(a) Let $f(x) = \frac{1}{x+2}$. Then

$$\left|f(x) - \frac{1}{4}\right| = \left|\frac{1}{x+2} - \frac{1}{4}\right| = \left|\frac{4 - (x+2)}{4(x+2)}\right| = \frac{|x-2|}{|4(x+2)|}.$$

If $|x - 2| < 1$, then $1 < x < 3$, so $3 < x + 2 < 5$ and $12 < 4(x + 2) < 20$. Hence,

$$\frac{1}{|4(x+2)|} < \frac{1}{12} \qquad \text{and} \qquad \left|f(x) - \frac{1}{4}\right| < \frac{|x-2|}{12}.$$

(b) If $|x-2|<\delta$, then by part (a),

$$\left|f(x)-\frac{1}{4}\right|<\frac{\delta}{12}.$$

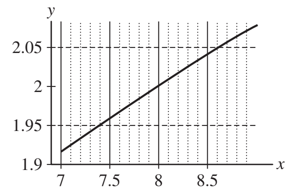
Choosing $\delta = 0.12$ will then guarantee that $|f(x)-\frac{1}{4}|<0.01$.

(c) Let $\epsilon > 0$ and take $\delta = \min\{1, 12\epsilon\}$. Then, whenever $|x-2|<\delta$,

$$\left|f(x)-\frac{1}{4}\right| = \left|\frac{1}{x+2}-\frac{1}{4}\right| = \frac{|2-x|}{4|x+2|} \le \frac{|x-2|}{12} < \frac{\delta}{12} = \epsilon.$$

77. GU Plot the function $f(x) = x^{1/3}$. Use the zoom feature to find a $\delta > 0$ such that $|x^{1/3}-2| < 0.05$ for $|x-8|<\delta$.

SOLUTION The graphs of $y = f(x) = x^{1/3}$ and the horizontal lines $y = 1.95$ and $y = 2.05$ are shown below. From this plot, we see that $\delta = 0.55$ guarantees that $|x^{1/3}-2| < 0.05$ whenever $|x-8|<\delta$.

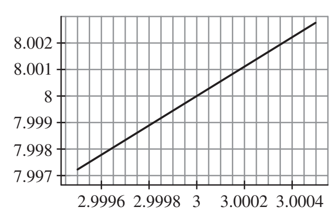


78. Use the fact that $f(x) = 2^x$ is increasing to find a value of δ such that $|2^x - 8| < 0.001$ if $|x-2|<\delta$. *Hint:* Find c_1 and c_2 such that $7.999 < f(c_1) < f(c_2) < 8.001$.

SOLUTION From the graph below, we see that

$$7.999 < f(2.99985) < f(3.00015) < 8.001.$$

Thus, with $\delta = 0.00015$, it follows that $|2^x - 8| < 0.001$ if $|x-3|<\delta$.



79. Prove rigorously that $\lim_{x\to -1}(4+8x) = -4$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \epsilon/8$. Then, whenever $|x-(-1)| = |x+1| < \delta$,

$$|f(x)-(-4)| = |4+8x+4| = 8|x+1| < 8\delta = \epsilon.$$

80. Prove rigorously that $\lim_{x\to 3}(x^2-x) = 6$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \min\{1, \epsilon/6\}$. Because $\delta \le 1$, $|x-3|<\delta$ guarantees $|x+2|<6$. Therefore, whenever $|x-3|<\delta$,

$$|f(x)-6| = |x^2-x-6| = |x-3|\,|x+2| < 6|x-3| < 6\delta \le \epsilon.$$

3 DIFFERENTIATION

3.1 Definition of the Derivative

Preliminary Questions

1. Which of the lines in Figure 10 are tangent to the curve?

FIGURE 10

SOLUTION Lines B and D are tangent to the curve.

2. What are the two ways of writing the difference quotient?

SOLUTION The difference quotient may be written either as

$$\frac{f(x) - f(a)}{x - a}$$

or as

$$\frac{f(a+h) - f(a)}{h}.$$

3. Find a and h such that $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between $(3, f(3))$ and $(5, f(5))$.

SOLUTION With $a = 3$ and $h = 2$, $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between the points $(3, f(3))$ and $(5, f(5))$ on the graph of $f(x)$.

4. Which derivative is approximated by $\frac{\tan\left(\frac{\pi}{4} + 0.0001\right) - 1}{0.0001}$?

SOLUTION $\frac{\tan(\frac{\pi}{4} + 0.0001) - 1}{0.0001}$ is a good approximation to the derivative of the function $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

5. What do the following quantities represent in terms of the graph of $f(x) = \sin x$?

(a) $\sin 1.3 - \sin 0.9$ **(b)** $\frac{\sin 1.3 - \sin 0.9}{0.4}$ **(c)** $f'(0.9)$

SOLUTION Consider the graph of $y = \sin x$.

(a) The quantity $\sin 1.3 - \sin 0.9$ represents the difference in height between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$.

(b) The quantity $\frac{\sin 1.3 - \sin 0.9}{0.4}$ represents the slope of the secant line between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$ on the graph.

(c) The quantity $f'(0.9)$ represents the slope of the tangent line to the graph at $x = 0.9$.

Exercises

1. Let $f(x) = 5x^2$. Show that $f(3 + h) = 5h^2 + 30h + 45$. Then show that

$$\frac{f(3+h) - f(3)}{h} = 5h + 30$$

and compute $f'(3)$ by taking the limit as $h \to 0$.

SOLUTION With $f(x) = 5x^2$, it follows that

$$f(3+h) = 5(3+h)^2 = 5(9+6h+h^2) = 45 + 30h + 5h^2.$$

Using this result, we find

$$\frac{f(3+h) - f(3)}{h} = \frac{45 + 30h + 5h^2 - 5 \cdot 9}{h} = \frac{45 + 30h + 5h^2 - 45}{h} = \frac{30h + 5h^2}{h} = 30 + 5h.$$

As $h \to 0$, $30 + 5h \to 30$, so $f'(3) = 30$.

2. Let $f(x) = 2x^2 - 3x - 5$. Show that the secant line through $(2, f(2))$ and $(2+h, f(2+h))$ has slope $2h + 5$. Then use this formula to compute the slope of:

(a) The secant line through $(2, f(2))$ and $(3, f(3))$

(b) The tangent line at $x = 2$ (by taking a limit)

SOLUTION The formula for the slope of the secant line is

$$\frac{f(2+h) - f(2)}{2+h-2} = \frac{[2(2+h)^2 - 3(2+h) - 5] - (8 - 6 - 5)}{h} = \frac{2h^2 + 5h}{h} = 2h + 5$$

(a) To find the slope of the secant line through $(2, f(2))$ and $(3, f(3))$, we take $h = 1$, so the slope is $2(1) + 5 = 7$.

(b) As $h \to 0$, the slope of the secant line approaches $2(0) + 5 = 5$. Hence, the slope of the tangent line at $x = 2$ is 5.

In Exercises 3–6, compute $f'(a)$ in two ways, using Eq. (1) and Eq. (2).

3. $f(x) = x^2 + 9x, \quad a = 0$

SOLUTION Let $f(x) = x^2 + 9x$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 + 9(0+h) - 0}{h} = \lim_{h\to 0} \frac{9h + h^2}{h} = \lim_{h\to 0}(9+h) = 9.$$

Alternately,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 + 9x - 0}{x} = \lim_{x\to 0}(x+9) = 9.$$

4. $f(x) = x^2 + 9x, \quad a = 2$

SOLUTION Let $f(x) = x^2 + 9x$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 + 9(2+h) - 22}{h} = \lim_{h\to 0} \frac{13h + h^2}{h} = \lim_{h\to 0}(13+h) = 13.$$

Alternately,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^2 + 9x - (2^2 + 9(2))}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+11)}{x - 2} = \lim_{x\to 2}(x+11) = 13.$$

5. $f(x) = 3x^2 + 4x + 2, \quad a = -1$

SOLUTION Let $f(x) = 3x^2 + 4x + 2$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{3(-1+h)^2 + 4(-1+h) + 2 - 1}{h}$$
$$= \lim_{h\to 0} \frac{3h^2 - 2h}{h} = \lim_{h\to 0} (3h - 2) = -2.$$

Alternately,

$$f'(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{3x^2 + 4x + 2 - 1}{x + 1}$$
$$= \lim_{x\to -1} \frac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1} (3x + 1) = -2.$$

6. $f(x) = x^3, \quad a = 2$

SOLUTION Let $f(x) = x^3$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$
$$= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0} (12 + 6h + h^2) = 12.$$

Alternately,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^3 - 8}{x - 2}$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x\to 2} (x^2 + 2x + 4) = 12.$$

In Exercises 7–10, refer to Figure 11.

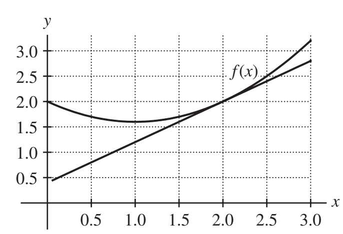


FIGURE 11

7. Find the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$. Is it larger or smaller than $f'(2)$? Explain.

SOLUTION From the graph, it appears that $f(2.5) = 2.5$ and $f(2) = 2$. Thus, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{2.5 - 2}{2.5 - 2} = 1.$$

From the graph, it is also clear that the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ has a larger slope than the tangent line at $x = 2$. In other words, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is larger than $f'(2)$.

8. Estimate $\dfrac{f(2+h) - f(2)}{h}$ for $h = -0.5$. What does this quantity represent? Is it larger or smaller than $f'(2)$? Explain.

SOLUTION With $h = -0.5$, $2 + h = 1.5$. Moreover, from the graph it appears that $f(1.5) = 1.7$ and $f(2) = 2$. Thus,

$$\frac{f(2+h) - f(2)}{h} = \frac{1.7 - 2}{-0.5} = 0.6.$$

This quantity represents the slope of the secant line through the points $(2, f(2))$ and $(1.5, f(1.5))$. It is clear from the graph that the secant line through the points $(2, f(2))$ and $(1.5, f(1.5))$ has a smaller slope than the tangent line at $x = 2$. In other words, $\dfrac{f(2+h) - f(2)}{h}$ for $h = -0.5$ is smaller than $f'(2)$.

9. Estimate $f'(1)$ and $f'(2)$.

SOLUTION From the graph, it appears that the tangent line at $x = 1$ would be horizontal. Thus, $f'(1) \approx 0$. The tangent line at $x = 2$ appears to pass through the points $(0.5, 0.8)$ and $(2, 2)$. Thus

$$f'(2) \approx \frac{2 - 0.8}{2 - 0.5} = 0.8.$$

10. Find a value of h for which $\dfrac{f(2+h) - f(2)}{h} = 0$.

SOLUTION In order for

$$\frac{f(2+h) - f(2)}{h}$$

to be equal to zero, we must have $f(2+h) = f(2)$. Now, $f(2) = 2$, and the only other point on the graph with a y-coordinate of 2 is $f(0) = 2$. Thus, $2 + h = 0$, or $h = -2$.

In Exercises 11–14, refer to Figure 12.

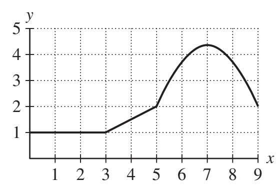


FIGURE 12 Graph of $f(x)$.

11. Determine $f'(a)$ for $a = 1, 2, 4, 7$.

SOLUTION Remember that the value of the derivative of f at $x = a$ can be interpreted as the slope of the line tangent to the graph of $y = f(x)$ at $x = a$. From Figure 12, we see that the graph of $y = f(x)$ is a horizontal line (that is, a line with zero slope) on the interval $0 \le x \le 3$. Accordingly, $f'(1) = f'(2) = 0$. On the interval $3 \le x \le 5$, the graph of $y = f(x)$ is a line of slope $\frac{1}{2}$; thus, $f'(4) = \frac{1}{2}$. Finally, the line tangent to the graph of $y = f(x)$ at $x = 7$ is horizontal, so $f'(7) = 0$.

12. For which values of x is $f'(x) < 0$?

SOLUTION If $f'(x) < 0$, then the slope of the tangent line at x is negative. Graphically, this would mean that the value of the function was decreasing for increasing x. From the graph, it follows that $f'(x) < 0$ for $7 < x < 9$.

13. Which is larger, $f'(5.5)$ or $f'(6.5)$?

SOLUTION The line tangent to the graph of $y = f(x)$ at $x = 5.5$ has a larger slope than the line tangent to the graph of $y = f(x)$ at $x = 6.5$. Therefore, $f'(5.5)$ is larger than $f'(6.5)$.

14. Show that $f'(3)$ does not exist.

SOLUTION Because

$$\lim_{h \to 0-} \frac{f(3+h) - f(3)}{h} = 0 \qquad \text{but} \qquad \lim_{h \to 0+} \frac{f(3+h) - f(3)}{h} = \frac{1}{2},$$

it follows that

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

does not exist.

In Exercises 15–18, use the limit definition to calculate the derivative of the linear function.

15. $f(x) = 7x - 9$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{7(a+h) - 9 - (7a - 9)}{h} = \lim_{h\to 0} 7 = 7.$$

16. $f(x) = 12$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{12 - 12}{h} = \lim_{h\to 0} 0 = 0.$$

17. $g(t) = 8 - 3t$

SOLUTION

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{8 - 3(a+h) - (8 - 3a)}{h} = \lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} (-3) = -3.$$

18. $k(z) = 14z + 12$

SOLUTION

$$\lim_{h\to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h\to 0} \frac{14(a+h) + 12 - (14a + 12)}{h} = \lim_{h\to 0} \frac{14h}{h} = \lim_{h\to 0} 14 = 14.$$

19. Find an equation of the tangent line at $x = 3$, assuming that $f(3) = 5$ and $f'(3) = 2$?

SOLUTION By definition, the equation of the tangent line to the graph of $f(x)$ at $x = 3$ is $y = f(3) + f'(3)(x - 3) = 5 + 2(x - 3) = 2x - 1$.

20. Find $f(3)$ and $f'(3)$, assuming that the tangent line to $y = f(x)$ at $a = 3$ has equation $y = 5x + 2$.

SOLUTION The slope of the tangent line to $y = f(x)$ at $a = 3$ is $f'(3)$ by definition, therefore $f'(3) = 5$. Also by definition, the tangent line to $y = f(x)$ at $a = 3$ goes through $(3, f(3))$, so $f(3) = 17$.

21. Describe the tangent line at an arbitrary point on the "curve" $y = 2x + 8$.

SOLUTION Since $y = 2x + 8$ represents a straight line, the tangent line at any point is the line itself, $y = 2x + 8$.

22. Suppose that $f(2 + h) - f(2) = 3h^2 + 5h$. Calculate:

(a) The slope of the secant line through $(2, f(2))$ and $(6, f(6))$

(b) $f'(2)$

SOLUTION Let f be a function such that $f(2 + h) - f(2) = 3h^2 + 5h$.

(a) We take $h = 4$ to compute the slope of the secant line through $(2, f(2))$ and $(6, f(6))$:

$$\frac{f(4+2) - f(2)}{(4+2) - 2} = \frac{3(4)^2 + 5(4)}{4} = 17$$

(b) $f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3h^2 + 5h}{h} = \lim_{h\to 0} (3h + 5) = 5.$

23. Let $f(x) = \frac{1}{x}$. Does $f(-2 + h)$ equal $\frac{1}{-2 + h}$ or $\frac{1}{-2} + \frac{1}{h}$? Compute the difference quotient at $a = -2$ with $h = 0.5$.

SOLUTION Let $f(x) = \frac{1}{x}$. Then

$$f(-2 + h) = \frac{1}{-2 + h}.$$

With $a = -2$ and $h = 0.5$, the difference quotient is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(-1.5) - f(-2)}{0.5} = \frac{\frac{1}{-1.5} - \frac{1}{-2}}{0.5} = -\frac{1}{3}.$$

24. Let $f(x) = \sqrt{x}$. Does $f(5+h)$ equal $\sqrt{5+h}$ or $\sqrt{5} + \sqrt{h}$? Compute the difference quotient at $a = 5$ with $h = 1$.

SOLUTION Let $f(x) = \sqrt{x}$. Then $f(5+h) = \sqrt{5+h}$. With $a = 5$ and $h = 1$, the difference quotient is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(5+1) - f(5)}{1} = \frac{\sqrt{6} - \sqrt{5}}{1} = \sqrt{6} - \sqrt{5}.$$

25. Let $f(x) = 1/\sqrt{x}$. Compute $f'(5)$ by showing that

$$\frac{f(5+h) - f(5)}{h} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})}$$

SOLUTION Let $f(x) = 1/\sqrt{x}$. Then

$$\begin{aligned}\frac{f(5+h) - f(5)}{h} &= \frac{\frac{1}{\sqrt{5+h}} - \frac{1}{\sqrt{5}}}{h} = \frac{\sqrt{5} - \sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}} \\ &= \frac{\sqrt{5} - \sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}}\left(\frac{\sqrt{5} + \sqrt{5+h}}{\sqrt{5} + \sqrt{5+h}}\right) \\ &= \frac{5 - (5+h)}{h\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})}.\end{aligned}$$

Thus,

$$\begin{aligned}f'(5) &= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})} \\ &= -\frac{1}{\sqrt{5}\,\sqrt{5}(\sqrt{5} + \sqrt{5})} = -\frac{1}{10\sqrt{5}}.\end{aligned}$$

26. Find an equation of the tangent line to the graph of $f(x) = 1/\sqrt{x}$ at $x = 9$.

SOLUTION Let $f(x) = 1/\sqrt{x}$. Then

$$\begin{aligned}\frac{f(9+h) - f(9)}{h} &= \frac{\frac{1}{\sqrt{9+h}} - \frac{1}{3}}{h} = \frac{3 - \sqrt{9+h}}{3h\sqrt{9+h}} \\ &= \frac{3 - \sqrt{9+h}}{3h\sqrt{9+h}}\left(\frac{3 + \sqrt{9+h}}{3 + \sqrt{9+h}}\right) \\ &= \frac{9 - (9+h)}{3h\sqrt{9+h}(\sqrt{9+h} + 3)} = -\frac{1}{3\sqrt{9+h}(\sqrt{9+h} + 3)}.\end{aligned}$$

Thus,

$$\begin{aligned}f'(9) &= \lim_{h \to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h \to 0} -\frac{1}{3\sqrt{9+h}(\sqrt{9+h} + 3)} \\ &= -\frac{1}{9(3+3)} = -\frac{1}{54}.\end{aligned}$$

Because $f(9) = \frac{1}{3}$, it follows that an equation of the tangent line to the graph of $f(x) = 1/\sqrt{x}$ at $x = 9$ is

$$y = f'(9)(x - 9) + f(9) = -\frac{1}{54}(x - 9) + \frac{1}{3}.$$

In Exercises 27–44, use the limit definition to compute $f'(a)$ and find an equation of the tangent line.

27. $f(x) = 2x^2 + 10x, \quad a = 3$

SOLUTION Let $f(x) = 2x^2 + 10x$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{2(3+h)^2 + 10(3+h) - 48}{h}$$
$$= \lim_{h\to 0} \frac{18 + 12h + 2h^2 + 30 + 10h - 48}{h} = \lim_{h\to 0} (22 + 2h) = 22.$$

At $a = 3$, the tangent line is

$$y = f'(3)(x-3) + f(3) = 22(x-3) + 48 = 22x - 18.$$

28. $f(x) = 4 - x^2, \quad a = -1$

SOLUTION Let $f(x) = 4 - x^2$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{4 - (-1+h)^2 - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 - (1 - 2h + h^2) - 3}{h}$$
$$= \lim_{h\to 0} (2 - h) = 2.$$

At $a = -1$, the tangent line is

$$y = f'(-1)(x+1) + f(-1) = 2(x+1) + 3 = 2x + 5.$$

29. $f(t) = t - 2t^2, \quad a = 3$

SOLUTION Let $f(t) = t - 2t^2$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h) - 2(3+h)^2 - (-15)}{h}$$
$$= \lim_{h\to 0} \frac{3 + h - 18 - 12h - 2h^2 + 15}{h}$$
$$= \lim_{h\to 0} (-11 - 2h) = -11.$$

At $a = 3$, the tangent line is

$$y = f'(3)(t-3) + f(3) = -11(t-3) - 15 = -11t + 18.$$

30. $f(x) = 8x^3, \quad a = 1$

SOLUTION Let $f(x) = 8x^3$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{8(1+h)^3 - 8}{h}$$
$$= \lim_{h\to 0} \frac{8 + 24h + 24h^2 + 8h^3 - 8}{h}$$
$$= \lim_{h\to 0} (24 + 24h + 8h^2) = 24.$$

At $a = 1$, the tangent line is

$$y = f'(1)(x-1) + f(1) = 24(x-1) + 8 = 24x - 16.$$

31. $f(x) = x^3 + x, \quad a = 0$

SOLUTION Let $f(x) = x^3 + x$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^3 + h - 0}{h}$$
$$= \lim_{h\to 0} (h^2 + 1) = 1.$$

At $a = 0$, the tangent line is

$$y = f'(0)(x - 0) + f(0) = x.$$

32. $f(t) = 2t^3 + 4t, \quad a = 4$

SOLUTION Let $f(t) = 2t^3 + 4t$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{2(4+h)^3 + 4(4+h) - 144}{h}$$
$$= \lim_{h\to 0} \frac{128 + 96h + 24h^2 + 2h^3 + 16 + 4h - 144}{h}$$
$$= \lim_{h\to 0} (100 + 24h + 2h^2) = 100.$$

At $a = 4$, the tangent line is

$$y = f'(4)(t - 4) + f(4) = 100(t - 4) + 144 = 100t - 256.$$

33. $f(x) = x^{-1}, \quad a = 8$

SOLUTION Let $f(x) = x^{-1}$. Then

$$f'(8) = \lim_{h\to 0} \frac{f(8+h) - f(8)}{h} = \lim_{h\to 0} \frac{\frac{1}{8+h} - \left(\frac{1}{8}\right)}{h} = \lim_{h\to 0} \frac{\frac{8-8-h}{8(8+h)}}{h} = \lim_{h\to 0} \frac{-h}{(64+8h)h} = -\frac{1}{64}$$

The tangent at $a = 8$ is

$$y = f'(8)(x - 8) + f(8) = -\frac{1}{64}(x - 8) + \frac{1}{8} = -\frac{1}{64}x + \frac{1}{4}.$$

34. $f(x) = x + x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x + x^{-1}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{4 + h + \frac{1}{4+h} - 4 - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{h + \frac{4-4-h}{4(4+h)}}{h} = \lim_{h\to 0} \left(1 - \frac{1}{16+4h}\right) = \frac{15}{16}$$

The tangent at $a = 4$ is

$$y = f'(4)(x - 4) + f(4) = \frac{15}{16}(x - 4) + \frac{17}{4} = \frac{15}{16}x + \frac{1}{2}.$$

35. $f(x) = \dfrac{1}{x+3}, \quad a = -2$

SOLUTION Let $f(x) = \frac{1}{x+3}$. Then

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{-2+h+3} - 1}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-h}{h(1+h)} = \lim_{h\to 0} \frac{-1}{1+h} = -1.$$

The tangent line at $a = -2$ is

$$y = f'(-2)(x + 2) + f(-2) = -1(x + 2) + 1 = -x - 1.$$

36. $f(t) = \dfrac{2}{1-t}, \quad a = -1$

SOLUTION Let $f(t) = \frac{2}{1-t}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{2}{1-(-1+h)} - 1}{h} = \lim_{h\to 0} \frac{2-(2-h)}{h(2-h)} = \lim_{h\to 0} \frac{1}{2-h} = \frac{1}{2}.$$

At $a = -1$, the tangent line is

$$y = f'(-1)(x+1) + f(-1) = \frac{1}{2}(x+1) + 1 = \frac{1}{2}x + \frac{3}{2}.$$

37. $f(x) = \sqrt{x+4}, \quad a = 1$

SOLUTION Let $f(x) = \sqrt{x+4}$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} \cdot \frac{\sqrt{h+5} + \sqrt{5}}{\sqrt{h+5} + \sqrt{5}}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{h+5} + \sqrt{5})} = \lim_{h\to 0} \frac{1}{\sqrt{h+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}}.$$

The tangent line at $a = 1$ is

$$y = f'(1)(x-1) + f(1) = \frac{1}{2\sqrt{5}}(x-1) + \sqrt{5} = \frac{1}{2\sqrt{5}}x + \frac{9}{2\sqrt{5}}.$$

38. $f(t) = \sqrt{3t+5}, \quad a = -1$

SOLUTION Let $f(t) = \sqrt{3t+5}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\sqrt{3h+2} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\sqrt{3h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{3h+2} + \sqrt{2}}{\sqrt{3h+2} + \sqrt{2}}$$
$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{3h+2} + \sqrt{2})} = \lim_{h\to 0} \frac{3}{\sqrt{3h+2} + \sqrt{2}} = \frac{3}{2\sqrt{2}}.$$

The tangent line at $a = -1$ is

$$y = f'(-1)(t+1) + f(-1) = \frac{3}{2\sqrt{2}}(t+1) + \sqrt{2} = \frac{3}{2\sqrt{2}}t + \frac{7}{2\sqrt{2}}.$$

39. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{x}}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{4\sqrt{4+h}+2(4+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-1}{4\sqrt{4+h} + 2(4+h)} = -\frac{1}{16}.$$

At $a = 4$ the tangent line is

$$y = f'(4)(x-4) + f(4) = -\frac{1}{16}(x-4) + \frac{1}{2} = -\frac{1}{16}x + \frac{3}{4}.$$

40. $f(x) = \dfrac{1}{\sqrt{2x+1}}, \quad a = 4$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{2x+1}}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2h+9}} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-\sqrt{2h+9}}{3\sqrt{2h+9}} \cdot \frac{3+\sqrt{2h+9}}{3+\sqrt{2h+9}}}{h} = \lim_{h\to 0} \frac{\frac{9-2h-9}{9\sqrt{2h+9}+3(2h+9)}}{h}$$
$$= \lim_{h\to 0} \frac{-2}{9\sqrt{2h+9} + 3(2h+9)} = -\frac{1}{27}.$$

At $a = 4$ the tangent line is

$$y = f'(4)(x-4) + f(4) = -\frac{1}{27}(x-4) + \frac{1}{3} = -\frac{1}{27}x + \frac{13}{27}.$$

41. $f(t) = \sqrt{t^2+1}, \quad a = 3$

SOLUTION Let $f(t) = \sqrt{t^2+1}$. Then

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h} \cdot \frac{\sqrt{10+6h+h^2} + \sqrt{10}}{\sqrt{10+6h+h^2} + \sqrt{10}} \\ &= \lim_{h\to 0} \frac{6h+h^2}{h(\sqrt{10+6h+h^2} + \sqrt{10})} = \lim_{h\to 0} \frac{6+h}{\sqrt{10+6h+h^2} + \sqrt{10}} = \frac{3}{\sqrt{10}}. \end{aligned}$$

The tangent line at $a = 3$ is

$$y = f'(3)(t-3) + f(3) = \frac{3}{\sqrt{10}}(t-3) + \sqrt{10} = \frac{3}{\sqrt{10}}t + \frac{1}{\sqrt{10}}.$$

42. $f(x) = x^{-2}, \quad a = -1$

SOLUTION Let $f(x) = \frac{1}{x^2}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(-1+h)^2} - 1}{h} = \lim_{h\to 0} \frac{\frac{h(2-h)}{(-1+h)^2}}{h} = \lim_{h\to 0} \frac{2-h}{(-1+h)^2} = 2.$$

The tangent line at $a = -1$ is

$$y = f'(-1)(x+1) + f(-1) = 2(x+1) + 1 = 2x + 3.$$

43. $f(x) = \dfrac{1}{x^2+1}, \quad a = 0$

SOLUTION Let $f(x) = \dfrac{1}{x^2+1}$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{(0+h)^2+1} - 1}{h} = \lim_{h\to 0} \frac{\frac{-h^2}{h^2+1}}{h} = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at $a = 0$ is

$$y = f(0) + f'(0)(x-0) = 1 + 0(x-1) = 1.$$

44. $f(t) = t^{-3}, \quad a = 1$

SOLUTION Let $f(t) = \dfrac{1}{t^3}$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(h)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^3} - 1}{h} = \lim_{h\to 0} \frac{\frac{-h\left(3+3h+h^2\right)}{(1+h)^3}}{h} = \lim_{h\to 0} \frac{-(3+3h+h^2)}{(1+h)^3} = -3.$$

The tangent line at $a = 1$ is

$$y = f'(1)(t-1) + f(1) = -3(t-1) + 1 = -3t + 4.$$

45. Figure 13 displays data collected by the biologist Julian Huxley (1887–1975) on the average antler weight W of male red deer as a function of age t. Estimate the derivative at $t = 4$. For which values of t is the slope of the tangent line equal to zero? For which values is it negative?

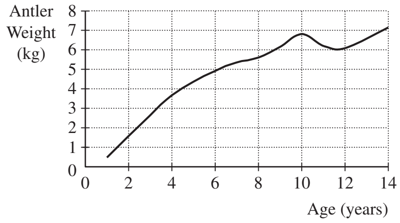
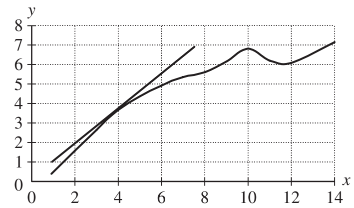


FIGURE 13

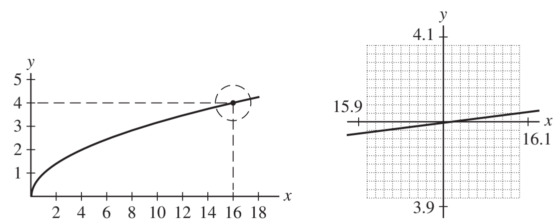
SOLUTION Let $W(t)$ denote the antler weight as a function of age. The "tangent line" sketched in the figure below passes through the points $(1, 1)$ and $(6, 5.5)$. Therefore

$$W'(4) \approx \frac{5.5 - 1}{6 - 1} = 0.9 \text{ kg/year}.$$

If the slope of the tangent is zero, the tangent line is horizontal. This appears to happen at roughly $t = 10$ and at $t = 11.6$. The slope of the tangent line is negative when the height of the graph decreases as we move to the right. For the graph in Figure 13, this occurs for $10 < t < 11.6$.



46. Figure 14(A) shows the graph of $f(x) = \sqrt{x}$. The close-up in Figure 14(B) shows that the graph is nearly a straight line near $x = 16$. Estimate the slope of this line and take it as an estimate for $f'(16)$. Then compute $f'(16)$ and compare with your estimate.



(A) Graph of $y = \sqrt{x}$ (B) Zoom view near (16, 4)

FIGURE 14

SOLUTION From the close-up in Figure 14(B), the line appears to pass through the points $(15.92, 3.99)$ and $(16.08, 4.01)$. Thus,

$$f'(16) \approx \frac{4.01 - 3.99}{16.08 - 15.92} = \frac{0.02}{0.16} = 0.125.$$

With $f(x) = \sqrt{x}$,

$$f'(16) = \lim_{h \to 0} \frac{\sqrt{16 + h} - 4}{h} \cdot \frac{\sqrt{16 + h} + 4}{\sqrt{16 + h} + 4} = \lim_{h \to 0} \frac{16 + h - 16}{h(\sqrt{16 + h} + 4)} = \lim_{h \to 0} \frac{1}{\sqrt{16 + h} + 4} = \frac{1}{8} = 0.125,$$

which is consistent with the approximation obtained from the close-up graph.

47. $\boxed{\text{GU}}$ Let $f(x) = \dfrac{4}{1 + 2^x}$.

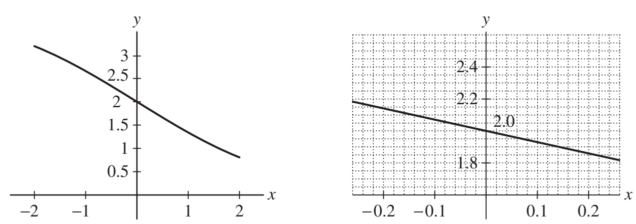
(a) Plot $f(x)$ over $[-2, 2]$. Then zoom in near $x = 0$ until the graph appears straight, and estimate the slope $f'(0)$.

(b) Use (a) to find an approximate equation to the tangent line at $x = 0$. Plot this line and $f(x)$ on the same set of axes.

SOLUTION

(a) The figure below at the left shows the graph of $f(x) = \frac{4}{1+2^x}$ over $[-2, 2]$. The figure below at the right is a close-up near $x = 0$. From the close-up, we see that the graph is nearly straight and passes through the points $(-0.22, 2.15)$ and $(0.22, 1.85)$. We therefore estimate

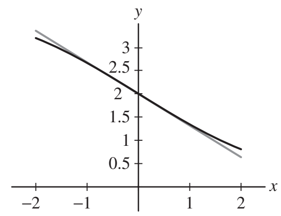
$$f'(0) \approx \frac{1.85 - 2.15}{0.22 - (-0.22)} = \frac{-0.3}{0.44} = -0.68$$



(b) Using the estimate for $f'(0)$ obtained in part (a), the approximate equation of the tangent line is

$$y = f'(0)(x-0) + f(0) = -0.68x + 2.$$

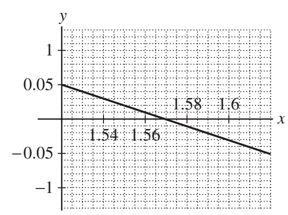
The figure below shows the graph of $f(x)$ and the approximate tangent line.



48. GU Let $f(x) = \cot x$. Estimate $f'\left(\frac{\pi}{2}\right)$ graphically by zooming in on a plot of $f(x)$ near $x = \frac{\pi}{2}$.

SOLUTION The figure below shows a close-up of the graph of $f(x) = \cot x$ near $x = \frac{\pi}{2} \approx 1.5708$. From the close-up, we see that the graph is nearly straight and passes through the points $(1.53, 0.04)$ and $(1.61, -0.04)$. We therefore estimate

$$f'\left(\frac{\pi}{2}\right) \approx \frac{-0.04 - 0.04}{1.61 - 1.53} = \frac{-0.08}{0.08} = -1$$



49. Determine the intervals along the x-axis on which the derivative in Figure 15 is positive.

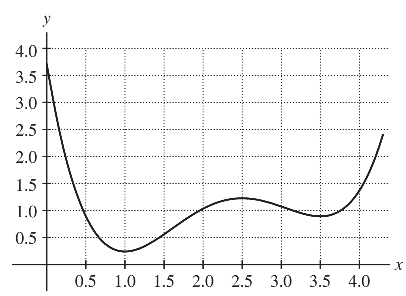
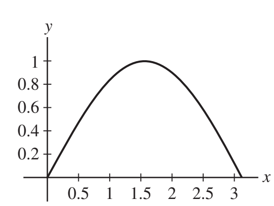


FIGURE 15

SOLUTION The derivative (that is, the slope of the tangent line) is positive when the height of the graph increases as we move to the right. From Figure 15, this appears to be true for $1 < x < 2.5$ and for $x > 3.5$.

50. Sketch the graph of $f(x) = \sin x$ on $[0, \pi]$ and guess the value of $f'\left(\frac{\pi}{2}\right)$. Then calculate the difference quotient at $x = \frac{\pi}{2}$ for two small positive and negative values of h. Are these calculations consistent with your guess?

SOLUTION Here is the graph of $y = \sin x$ on $[0, \pi]$.



At $x = \frac{\pi}{2}$, we're at the peak of the sine graph. The tangent line appears to be horizontal, so the slope is 0; hence, $f'(\frac{\pi}{2})$ appears to be 0.

h	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\dfrac{\sin(\frac{\pi}{2}+h)-1}{h}$	0.005	0.0005	0.00005	-0.00005	-0.0005	-0.005

These numerical calculations are consistent with our guess.

In Exercises 51–56, each limit represents a derivative $f'(a)$. Find $f(x)$ and a.

51. $\lim\limits_{h\to 0} \dfrac{(5+h)^3 - 125}{h}$

SOLUTION The difference quotient $\dfrac{(5+h)^3-125}{h}$ has the form $\dfrac{f(a+h)-f(a)}{h}$ where $f(x) = x^3$ and $a = 5$.

52. $\lim\limits_{x\to 5} \dfrac{x^3 - 125}{x - 5}$

SOLUTION The difference quotient $\dfrac{x^3-125}{x-5}$ has the form $\dfrac{f(x)-f(a)}{x-a}$ where $f(x) = x^3$ and $a = 5$.

53. $\lim\limits_{h\to 0} \dfrac{\sin\left(\frac{\pi}{6}+h\right) - 0.5}{h}$

SOLUTION The difference quotient $\dfrac{\sin(\frac{\pi}{6}+h)-.5}{h}$ has the form $\dfrac{f(a+h)-f(a)}{h}$ where $f(x) = \sin x$ and $a = \frac{\pi}{6}$.

54. $\lim\limits_{x\to \frac{1}{4}} \dfrac{x^{-1} - 4}{x - \frac{1}{4}}$

SOLUTION The difference quotient $\dfrac{\frac{1}{x}-4}{x-\frac{1}{4}}$ has the form $\dfrac{f(x)-f(a)}{x-a}$ where $f(x) = \frac{1}{x}$ and $a = \frac{1}{4}$.

55. $\lim\limits_{h\to 0} \dfrac{5^{2+h} - 25}{h}$

SOLUTION The difference quotient $\dfrac{5^{(2+h)}-25}{h}$ has the form $\dfrac{f(a+h)-f(a)}{h}$ where $f(x) = 5^x$ and $a = 2$.

56. $\lim\limits_{h\to 0} \dfrac{5^h - 1}{h}$

SOLUTION The difference quotient $\dfrac{5^h-1}{h}$ has the form $\dfrac{f(a+h)-f(a)}{h}$ where $f(x) = 5^x$ and $a = 0$.

57. Apply the method of Example 6 to $f(x) = \sin x$ to determine $f'\left(\frac{\pi}{4}\right)$ accurately to four decimal places.

SOLUTION We know that

$$f'(\pi/4) = \lim_{h\to 0} \frac{f(\pi/4+h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin(\pi/4+h) - \sqrt{2}/2}{h}.$$

Creating a table of values of h close to zero:

h	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001
$\dfrac{\sin(\frac{\pi}{4}+h) - (\sqrt{2}/2)}{h}$	0.7074602	0.7071421	0.7071103	0.7071033	0.7070714	0.7067531

Accurate up to four decimal places, $f'(\frac{\pi}{4}) \approx 0.7071$.

58. Apply the method of Example 6 to $f(x) = \cos x$ to determine $f'\left(\frac{\pi}{5}\right)$ accurately to four decimal places. Use a graph of $f(x)$ to explain how the method works in this case.

SOLUTION We know that

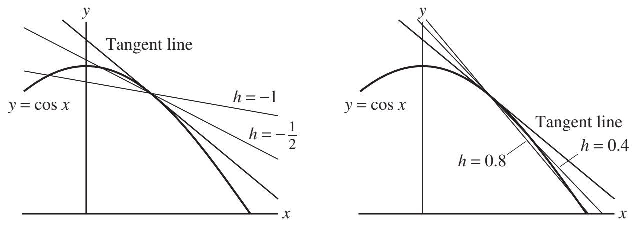
$$f'\left(\frac{\pi}{5}\right) = \lim_{h\to 0} \frac{f(\pi/5+h) - f(\pi/5)}{h} = \lim_{h\to 0} \frac{\cos(\frac{\pi}{5}+h) - \cos(\frac{\pi}{5})}{h}.$$

We make a chart using values of h close to zero:

h	-0.001	-0.0001	-0.00001
$\dfrac{\cos(\frac{\pi}{5}+h) - \cos(\frac{\pi}{5})}{h}$	-0.587381	-0.587745	-0.587781
h	0.001	0.0001	0.00001
$\dfrac{\cos(\frac{\pi}{5}+h) - \cos(\frac{\pi}{5})}{h}$	-0.588190	-0.587826	-0.587789

$f'(\frac{\pi}{5}) \approx -0.5878$.

The figures shown below illustrate why this procedure works. From the figure on the left, we see that for $h < 0$, the slope of the secant line is greater (less negative) than the slope of the tangent line. On the other hand, from the figure on the right, we see that for $h > 0$, the slope of the secant line is less (more negative) than the slope of the tangent line. Thus, the slope of the tangent line must fall between the slope of a secant line with $h > 0$ and the slope of a secant line with $h < 0$.



59. For each graph in Figure 16, determine whether $f'(1)$ is larger or smaller than the slope of the secant line between $x = 1$ and $x = 1 + h$ for $h > 0$. Explain.

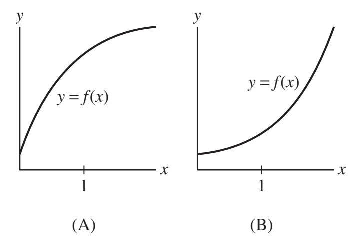


FIGURE 16

SOLUTION

- On curve (A), $f'(1)$ is larger than

$$\frac{f(1+h) - f(1)}{h};$$

  the curve is bending downwards, so that the secant line to the right is at a lower angle than the tangent line. We say such a curve is **concave down**, and that its derivative is *decreasing*.
- On curve (B), $f'(1)$ is smaller than

$$\frac{f(1+h) - f(1)}{h};$$

  the curve is bending upwards, so that the secant line to the right is at a steeper angle than the tangent line. We say such a curve is **concave up**, and that its derivative is *increasing*.

60. Refer to the graph of $f(x) = 2^x$ in Figure 17.

(a) Explain graphically why, for $h > 0$,

$$\frac{f(-h) - f(0)}{-h} \le f'(0) \le \frac{f(h) - f(0)}{h}$$

(b) Use (a) to show that $0.69314 \le f'(0) \le 0.69315$.

(c) Similarly, compute $f'(x)$ to four decimal places for $x = 1, 2, 3, 4$.

(d) Now compute the ratios $f'(x)/f'(0)$ for $x = 1, 2, 3, 4$. Can you guess an approximate formula for $f'(x)$?

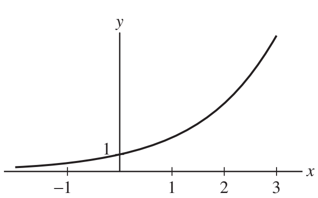


FIGURE 17 Graph of $f(x) = 2^x$.

SOLUTION

(a) In the graph, the inequality

$$f'(0) \le \frac{f(h) - f(0)}{h}$$

holds for positive values of h, since the difference quotient

$$\frac{f(h) - f(0)}{h}$$

is an increasing function of h. (The slopes of the secant lines between $(0, f(0))$ and a nearby point increase as the nearby point moves from left to right.) Hence the slopes of the secant lines between $(0, f(0))$ and a nearby point to the right, $(h, f(h))$ (where h is positive) exceed $f'(0)$. Similarly, for $h > 0$, $-h$ is negative and 0 lies to the right of $-h$. Consequently, the slope of the secant line between $(0, f(0))$ and a nearby point to the left, $(-h, f(-h))$ is less than $f'(0)$. Therefore, the inequality

$$f'(0) \ge \frac{f(-h) - f(0)}{-h}$$

holds for $h > 0$.

(b) For $h = 0.00001$, we have

$$\frac{f(h) - f(0)}{h} = \frac{2^h - 1}{h} \approx 0.69315,$$

and

$$\frac{f(-h) - f(0)}{-h} \approx 0.69314.$$

In light of (a), $0.69314 \le f'(0) \le 0.69315$.

(c) We'll use the same values of $h = \pm 0.00001$ and compute difference quotients at $x = 1, 2, 3, 4$.

- Since $1.386290 \le f'(1) \le 1.386299$, we conclude that $f'(1) \approx 1.3863$ to four decimal places.
- Since $2.772579 \le f'(2) \le 2.772598$, we conclude that $f'(2) \approx 2.7726$ to four decimal places.
- Since $5.545158 \le f'(3) \le 5.545197$, we conclude that $f'(3) \approx 5.5452$ to four decimal places.
- With $h = \pm 0.000001$, $11.090351 \le f'(4) \le 11.090359$, so we conclude that $f'(4) \approx 11.0904$ to four decimal places.

(d)

x	1	2	3	4
$f'(x)/f'(0)$	2	4	8	16

Looking at this table, we guess that $f'(x)/f'(0) = 2^x$. In other words, $f'(x) = 2^x f'(0)$.

61. $\boxed{\text{GU}}$ Sketch the graph of $f(x) = x^{5/2}$ on $[0, 6]$.

(a) Use the sketch to justify the inequalities for $h > 0$:

$$\frac{f(4) - f(4-h)}{h} \le f'(4) \le \frac{f(4+h) - f(4)}{h}$$

(b) Use (a) to compute $f'(4)$ to four decimal places.

(c) Use a graphing utility to plot $f(x)$ and the tangent line at $x = 4$, using your estimate for $f'(4)$.

SOLUTION

(a) The slope of the secant line between points $(4, f(4))$ and $(4+h, f(4+h))$ is

$$\frac{f(4+h) - f(4)}{h}.$$

$x^{5/2}$ is a smooth curve increasing at a faster rate as $x \to \infty$. Therefore, if $h > 0$, then the slope of the secant line is greater than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$. Likewise, if $h < 0$, the slope of the secant line is less than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$.

(b) We know that

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{(4+h)^{5/2} - 32}{h}.$$

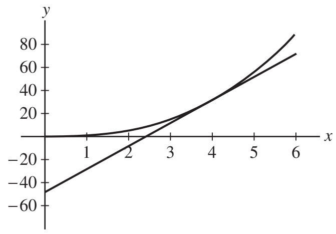
Creating a table with values of h close to zero:

h	-0.0001	-0.00001	0.00001	0.0001
$\dfrac{(4+h)^{5/2}-32}{h}$	19.999625	19.99999	20.0000	20.0000375

Thus, $f'(4) \approx 20.0000$.

(c) Using the estimate for $f'(4)$ obtained in part (b), the equation of the line tangent to $f(x) = x^{5/2}$ at $x = 4$ is

$$y = f'(4)(x-4) + f(4) = 20(x-4) + 32 = 20x - 48.$$

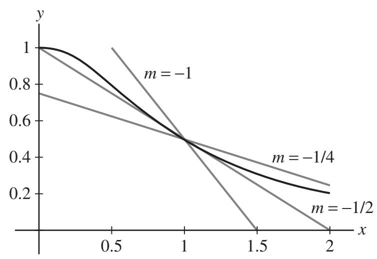


62. GU Verify that $P = \left(1, \frac{1}{2}\right)$ lies on the graphs of both $f(x) = 1/(1+x^2)$ and $L(x) = \frac{1}{2} + m(x-1)$ for every slope m. Plot $f(x)$ and $L(x)$ on the same axes for several values of m until you find a value of m for which $y = L(x)$ appears tangent to the graph of $f(x)$. What is your estimate for $f'(1)$?

SOLUTION Let $f(x) = \dfrac{1}{1+x^2}$ and $L(x) = \frac{1}{2} + m(x-1)$. Because

$$f(1) = \frac{1}{1+1^2} = \frac{1}{2} \qquad \text{and} \qquad L(1) = \frac{1}{2} + m(1-1) = \frac{1}{2},$$

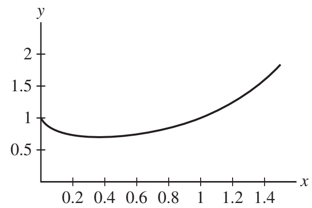
it follows that $P = (1, \frac{1}{2})$ lies on the graphs of both functions. A plot of $f(x)$ and $L(x)$ on the same axes for several values of m is shown below. The graph of $L(x)$ with $m = -\frac{1}{2}$ appears to be tangent to the graph of $f(x)$ at $x = 1$. We therefore estimate $f'(1) = -\frac{1}{2}$.



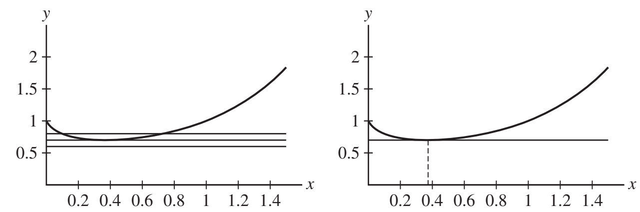
63. GU Use a plot of $f(x) = x^x$ to estimate the value c such that $f'(c) = 0$. Find c to sufficient accuracy so that

$$\left|\frac{f(c+h) - f(c)}{h}\right| \le 0.006 \qquad \text{for} \quad h = \pm 0.001$$

SOLUTION Here is a graph of $f(x) = x^x$ over the interval $[0, 1.5]$.



The graph shows one location with a horizontal tangent line. The figure below at the left shows the graph of $f(x)$ together with the horizontal lines $y = 0.6$, $y = 0.7$ and $y = 0.8$. The line $y = 0.7$ is very close to being tangent to the graph of $f(x)$. The figure below at the right refines this estimate by graphing $f(x)$ and $y = 0.69$ on the same set of axes. The point of tangency has an x-coordinate of roughly 0.37, so $c \approx 0.37$.



We note that

$$\left| \frac{f(0.37 + 0.001) - f(0.37)}{0.001} \right| \approx 0.00491 < 0.006$$

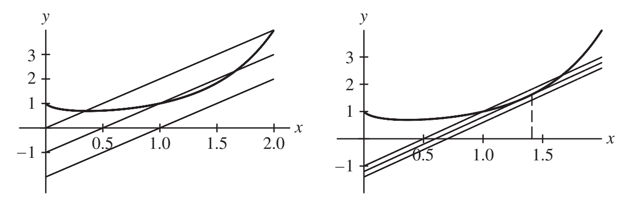
and

$$\left| \frac{f(0.37 - 0.001) - f(0.37)}{0.001} \right| \approx 0.00304 < 0.006,$$

so we have determined c to the desired accuracy.

64. GU Plot $f(x) = x^x$ and $y = 2x + a$ on the same set of axes for several values of a until the line becomes tangent to the graph. Then estimate the value c such that $f'(c) = 2$.

SOLUTION The figure below on the left shows the graphs of the function $f(x) = x^x$ together with the lines $y = 2x$, $y = 2x - 1$, and $y = 2x - 2$; the figure on the right shows the graphs of $f(x) = x^x$ together with the lines $y = 2x - 1$, $y = 2x - 1.2$, and $y = 2x - 1.4$. The graph of $y = 2x - 1.2$ appears to be tangent to the graph of $f(x)$ at $x \approx 1.4$. We therefore estimate that $f'(1.4) = 2$.



In Exercises 65–71, estimate derivatives using the ***symmetric difference quotient*** *(SDQ), defined as the average of the difference quotients at h and $-h$:*

$$\frac{1}{2}\left(\frac{f(a+h) - f(a)}{h} + \frac{f(a-h) - f(a)}{-h} \right) = \frac{f(a+h) - f(a-h)}{2h} \qquad \boxed{4}$$

The SDQ usually gives a better approximation to the derivative than the difference quotient.

65. The vapor pressure of water at temperature T (in kelvins) is the atmospheric pressure P at which no net evaporation takes place. Use the following table to estimate $P'(T)$ for $T = 303, 313, 323, 333, 343$ by computing the SDQ given by Eq. (4) with $h = 10$.

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

SOLUTION Using equation (4),

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};$$

$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};$$

$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};$$

$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};$$

$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}$$

66. Use the SDQ with $h = 1$ year to estimate $P'(T)$ in the years 2000, 2002, 2004, 2006, where $P(T)$ is the U.S. ethanol production (Figure 18). Express your answer in the correct units.

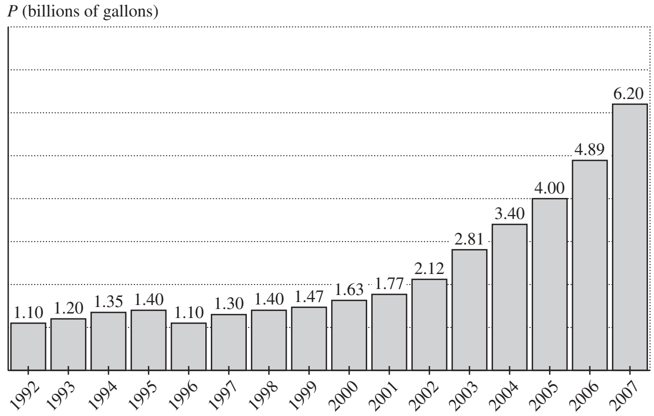


FIGURE 18 U.S. Ethanol Production

SOLUTION Using equation (4),

$$P'(2000) \approx \frac{P(2001) - P(1999)}{2} = \frac{1.77 - 1.47}{2} = 0.15 \text{ billions of gallons/yr;}$$

$$P'(2002) \approx \frac{P(2003) - P(2001)}{2} = \frac{2.81 - 1.77}{2} = 0.52 \text{ billions of gallons/yr;}$$

$$P'(2004) \approx \frac{P(2005) - P(2003)}{2} = \frac{4 - 2.81}{2} = 0.595 \text{ billions of gallons/yr;}$$

$$P'(2006) \approx \frac{P(2007) - P(2005)}{2} = \frac{6.2 - 4}{2} = 1.1 \text{ billions of gallons/yr}$$

In Exercises 67 and 68, traffic speed S along a certain road (in km/h) varies as a function of traffic density q (number of cars per km of road). Use the following data to answer the questions:

q (density)	60	70	80	90	100
S (speed)	72.5	67.5	63.5	60	56

67. Estimate $S'(80)$.

SOLUTION Let $S(q)$ be the function determining S given q. Using equation (4) with $h = 10$,

$$S'(80) \approx \frac{S(90) - S(70)}{20} = \frac{60 - 67.5}{20} = -0.375;$$

with $h = 20$,

$$S'(80) \approx \frac{S(100) - S(60)}{40} = \frac{56 - 72.5}{40} = -0.4125;$$

The mean of these two symmetric difference quotients is -0.39375 kph·km/car.

68. Explain why $V = qS$, called *traffic volume*, is equal to the number of cars passing a point per hour. Use the data to estimate $V'(80)$.

SOLUTION The traffic speed S has units of km/hour, and the traffic density has units of cars/km. Therefore, the traffic volume $V = Sq$ has units of cars/hour. A table giving the values of V follows.

q	60	70	80	90	100
V	4350	4725	5080	5400	5600

To estimate dV/dq, we take the mean of the symmetric difference quotients. With $h = 10$,

$$V'(80) \approx \frac{V(90) - V(70)}{20} = \frac{5400 - 4725}{20} = 33.75;$$

with $h = 20$,

$$V'(80) \approx \frac{V(100) - V(60)}{40} = \frac{5600 - 4350}{40} = 31.25;$$

The mean of the symmetric difference quotients is 32.5. Hence $dV/dq \approx 32.5$ cars per hour when $q = 80$.

Exercises 69–71: The current (in amperes) at time t (in seconds) flowing in the circuit in Figure 19 is given by Kirchhoff's Law:

$$i(t) = Cv'(t) + R^{-1}v(t)$$

where $v(t)$ is the voltage (in volts), C the capacitance (in farads), and R the resistance (in ohms, Ω).

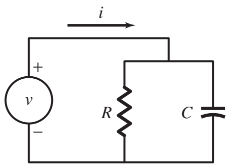


FIGURE 19

69. Calculate the current at $t = 3$ if

$$v(t) = 0.5t + 4 \text{ V}$$

where $C = 0.01$ F and $R = 100\ \Omega$.

SOLUTION Since $v(t)$ is a line with slope 0.5, $v'(t) = 0.5$ volts/s for all t. From the formula, $i(3) = Cv'(3) + (1/R)v(3) = 0.01(0.5) + (1/100)(5.5) = 0.005 + 0.055 = 0.06$ amperes.

70. Use the following data to estimate $v'(10)$ (by an SDQ). Then estimate $i(10)$, assuming $C = 0.03$ and $R = 1000$.

t	9.8	9.9	10	10.1	10.2
$v(t)$	256.52	257.32	258.11	258.9	259.69

SOLUTION Taking $h = 0.1$, we find

$$v'(10) \approx \frac{v(10.1) - v(9.9)}{0.2} = \frac{258.9 - 257.32}{0.2} = 7.9 \text{ volts/s.}$$

Thus,

$$i(10) = 0.03(7.9) + \frac{1}{1000}(258.11) = 0.49511 \text{ amperes.}$$

71. Assume that $R = 200\ \Omega$ but C is unknown. Use the following data to estimate $v'(4)$ (by an SDQ) and deduce an approximate value for the capacitance C.

t	3.8	3.9	4	4.1	4.2
$v(t)$	388.8	404.2	420	436.2	452.8
$i(t)$	32.34	33.22	34.1	34.98	35.86

SOLUTION Solving $i(4) = Cv'(4) + (1/R)v(4)$ for C yields

$$C = \frac{i(4) - (1/R)v(4)}{v'(4)} = \frac{34.1 - \frac{420}{200}}{v'(4)}.$$

To compute C, we first approximate $v'(4)$. Taking $h = 0.1$, we find

$$v'(4) \approx \frac{v(4.1) - v(3.9)}{0.2} = \frac{436.2 - 404.2}{0.2} = 160.$$

Plugging this in to the equation above yields

$$C \approx \frac{34.1 - 2.1}{160} = 0.2 \text{ farads.}$$

Further Insights and Challenges

72. The SDQ usually approximates the derivative much more closely than does the ordinary difference quotient. Let $f(x) = 2^x$ and $a = 0$. Compute the SDQ with $h = 0.001$ and the ordinary difference quotients with $h = \pm 0.001$. Compare with the actual value, which is $f'(0) = \ln 2$.

SOLUTION Let $f(x) = 2^x$ and $a = 0$.

- The ordinary difference quotient for $h = -0.001$ is 0.69290701 and for $h = 0.001$ is 0.69338746.
- The symmetric difference quotient for $h = 0.001$ is 0.69314724.
- Clearly the symmetric difference quotient gives a better estimate of the derivative $f'(0) = \ln 2 \approx 0.69314718$.

73. Explain how the symmetric difference quotient defined by Eq. (4) can be interpreted as the slope of a secant line.

SOLUTION The symmetric difference quotient

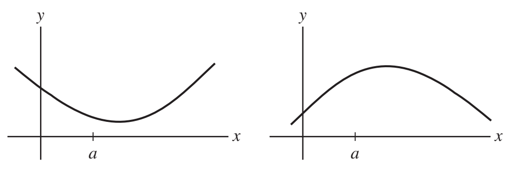
$$\frac{f(a+h) - f(a-h)}{2h}$$

is the slope of the secant line connecting the points $(a - h, f(a - h))$ and $(a + h, f(a + h))$ on the graph of f; the difference in the function values is divided by the difference in the x-values.

74. Which of the two functions in Figure 20 satisfies the inequality

$$\frac{f(a+h) - f(a-h)}{2h} \le \frac{f(a+h) - f(a)}{h}$$

for $h > 0$? Explain in terms of secant lines.



(A) (B)

FIGURE 20

SOLUTION Figure (A) satisfies the inequality

$$\frac{f(a+h) - f(a-h)}{2h} \le \frac{f(a+h) - f(a)}{h}$$

since in this graph the symmetric difference quotient has a larger negative slope than the ordinary right difference quotient. [In figure (B), the symmetric difference quotient has a larger positive slope than the ordinary right difference quotient and therefore does *not* satisfy the stated inequality.]

75. Show that if $f(x)$ is a quadratic polynomial, then the SDQ at $x = a$ (for any $h \neq 0$) is *equal* to $f'(a)$. Explain the graphical meaning of this result.

SOLUTION Let $f(x) = px^2 + qx + r$ be a quadratic polynomial. We compute the SDQ at $x = a$.

$$\begin{aligned}\frac{f(a+h) - f(a-h)}{2h} &= \frac{p(a+h)^2 + q(a+h) + r - (p(a-h)^2 + q(a-h) + r)}{2h} \\ &= \frac{pa^2 + 2pah + ph^2 + qa + qh + r - pa^2 + 2pah - ph^2 - qa + qh - r}{2h} \\ &= \frac{4pah + 2qh}{2h} = \frac{2h(2pa + q)}{2h} = 2pa + q\end{aligned}$$

Since this doesn't depend on h, the limit, which is equal to $f'(a)$, is also $2pa + q$. Graphically, this result tells us that the secant line to a parabola passing through points chosen symmetrically about $x = a$ is always parallel to the tangent line at $x = a$.

76. Let $f(x) = x^{-2}$. Compute $f'(1)$ by taking the limit of the SDQs (with $a = 1$) as $h \to 0$.

SOLUTION Let $f(x) = x^{-2}$. With $a = 1$, the symmetric difference quotient is

$$\frac{f(1+h) - f(1-h)}{2h} = \frac{\frac{1}{(1+h)^2} - \frac{1}{(1-h)^2}}{2h} = \frac{(1-h)^2 - (1+h)^2}{2h(1-h)^2(1+h)^2} = \frac{-4h}{2h(1-h)^2(1+h)^2} = -\frac{2}{(1-h)^2(1+h)^2}.$$

Therefore,

$$f'(1) = \lim_{h \to 0} -\frac{2}{(1-h)^2(1+h)^2} = -2.$$

3.2 The Derivative as a Function

Preliminary Questions

1. What is the slope of the tangent line through the point $(2, f(2))$ if $f'(x) = x^3$?

SOLUTION The slope of the tangent line through the point $(2, f(2))$ is given by $f'(2)$. Since $f'(x) = x^3$, it follows that $f'(2) = 2^3 = 8$.

2. Evaluate $(f - g)'(1)$ and $(3f + 2g)'(1)$ assuming that $f'(1) = 3$ and $g'(1) = 5$.

SOLUTION $(f - g)'(1) = f'(1) - g'(1) = 3 - 5 = -2$ and $(3f + 2g)'(1) = 3f'(1) + 2g'(1) = 3(3) + 2(5) = 19$.

3. To which of the following does the Power Rule apply?

(a) $f(x) = x^2$ **(b)** $f(x) = 2^e$
(c) $f(x) = x^e$ **(d)** $f(x) = e^x$
(e) $f(x) = x^x$ **(f)** $f(x) = x^{-4/5}$

SOLUTION

(a) Yes. x^2 is a power function, so the Power Rule can be applied.
(b) Yes. 2^e is a constant function, so the Power Rule can be applied.
(c) Yes. x^e is a power function, so the Power Rule can be applied.
(d) No. e^x is an exponential function (the base is constant while the exponent is a variable), so the Power Rule does not apply.
(e) No. x^x is not a power function because both the base and the exponent are variable, so the Power Rule does not apply.
(f) Yes. $x^{-4/5}$ is a power function, so the Power Rule can be applied.

4. Choose (a) or (b). The derivative does not exist if the tangent line is: (a) horizontal (b) vertical.

SOLUTION The derivative does not exist when: (b) the tangent line is vertical. At a horizontal tangent, the derivative is zero.

5. Which property distinguishes $f(x) = e^x$ from all other exponential functions $g(x) = b^x$?

SOLUTION The line tangent to $f(x) = e^x$ at $x = 0$ has slope equal to 1.

Exercises

In Exercises 1–6, compute $f'(x)$ using the limit definition.

1. $f(x) = 3x - 7$

SOLUTION Let $f(x) = 3x - 7$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - 7 - (3x - 7)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3.$$

2. $f(x) = x^2 + 3x$

SOLUTION Let $f(x) = x^2 + 3x$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h\to 0} (2x + h + 3) = 2x + 3.$$

3. $f(x) = x^3$

SOLUTION Let $f(x) = x^3$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

4. $f(x) = 1 - x^{-1}$

SOLUTION Let $f(x) = 1 - x^{-1}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1 - \frac{1}{x+h} - \left(1 - \frac{1}{x}\right)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)-x}{x(x+h)}}{h} = \lim_{h\to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}.$$

5. $f(x) = x - \sqrt{x}$

SOLUTION Let $f(x) = x - \sqrt{x}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - \sqrt{x+h} - (x - \sqrt{x})}{h} = 1 - \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= 1 - \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = 1 - \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = 1 - \frac{1}{2\sqrt{x}}.$$

6. $f(x) = x^{-1/2}$

SOLUTION Let $f(x) = x^{-1/2}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}$$

Multiplying the numerator and denominator of the expression by $\sqrt{x} + \sqrt{x+h}$, we obtain:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x}(2\sqrt{x})} = \frac{-1}{2x\sqrt{x}}.$$

In Exercises 7–14, use the Power Rule to compute the derivative.

7. $\left.\frac{d}{dx} x^4\right|_{x=-2}$

SOLUTION $\frac{d}{dx}\left(x^4\right) = 4x^3$ so $\left.\frac{d}{dx} x^4\right|_{x=-2} = 4(-2)^3 = -32.$

8. $\left.\frac{d}{dt} t^{-3}\right|_{t=4}$

SOLUTION $\frac{d}{dt}\left(t^{-3}\right) = -3t^{-4}$ so $\left.\frac{d}{dt} t^{-3}\right|_{t=4} = -3(4)^{-4} = -\frac{3}{256}.$

9. $\left.\frac{d}{dt} t^{2/3}\right|_{t=8}$

SOLUTION $\frac{d}{dt}\left(t^{2/3}\right) = \frac{2}{3}t^{-1/3}$ so $\left.\frac{d}{dt} t^{2/3}\right|_{t=8} = \frac{2}{3}(8)^{-1/3} = \frac{1}{3}.$

10. $\left.\frac{d}{dt} t^{-2/5}\right|_{t=1}$

SOLUTION $\frac{d}{dt}\left(t^{-2/5}\right) = -\frac{2}{5}t^{-7/5}$ so $\left.\frac{d}{dt} t^{-2/5}\right|_{t=1} = -\frac{2}{5}(1)^{-7/5} = -\frac{2}{5}.$

11. $\frac{d}{dx} x^{0.35}$

SOLUTION $\frac{d}{dx}\left(x^{0.35}\right) = 0.35(x^{0.35-1}) = 0.35x^{-0.65}.$

12. $\dfrac{d}{dx}\,x^{14/3}$

SOLUTION $\dfrac{d}{dx}\left(x^{14/3}\right) = \dfrac{14}{3}\left(x^{(14/3)-1}\right) = \dfrac{14}{3}x^{11/3}.$

13. $\dfrac{d}{dt}\,t^{\sqrt{17}}$

SOLUTION $\dfrac{d}{dt}\left(t^{\sqrt{17}}\right) = \sqrt{17}t^{\sqrt{17}-1}$

14. $\dfrac{d}{dt}\,t^{-\pi^2}$

SOLUTION $\dfrac{d}{dt}(t^{-\pi^2}) = -\pi^2 t^{-\pi^2-1}$

In Exercises 15–18, compute $f'(x)$ and find an equation of the tangent line to the graph at $x = a$.

15. $f(x) = x^4, \quad a = 2$

SOLUTION Let $f(x) = x^4$. Then, by the Power Rule, $f'(x) = 4x^3$. The equation of the tangent line to the graph of $f(x)$ at $x = 2$ is

$$y = f'(2)(x-2) + f(2) = 32(x-2) + 16 = 32x - 48.$$

16. $f(x) = x^{-2}, \quad a = 5$

SOLUTION Let $f(x) = x^{-2}$. Using the Power Rule, $f'(x) = -2x^{-3}$. The equation of the tangent line to the graph of $f(x)$ at $x = 5$ is

$$y = f'(5)(x-5) + f(5) = -\frac{2}{125}(x-5) + \frac{1}{25} = -\frac{2}{125}x + \frac{3}{25}.$$

17. $f(x) = 5x - 32\sqrt{x}, \quad a = 4$

SOLUTION Let $f(x) = 5x - 32x^{1/2}$. Then $f'(x) = 5 - 16x^{-1/2}$. In particular, $f'(4) = -3$. The tangent line at $x = 4$ is

$$y = f'(4)(x-4) + f(4) = -3(x-4) - 44 = -3x - 32.$$

18. $f(x) = \sqrt[3]{x}, \quad a = 8$

SOLUTION Let $f(x) = \sqrt[3]{x} = x^{1/3}$. Then $f'(x) = \frac{1}{3}(x^{1/3-1}) = \frac{1}{3}x^{-2/3}$. In particular, $f'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$. $f(8) = 2$, so the tangent line at $x = 8$ is

$$y = f'(8)(x-8) + f(8) = \frac{1}{12}(x-8) + 2 = \frac{1}{12}x + \frac{4}{3}.$$

19. Calculate:

(a) $\dfrac{d}{dx}12e^x$ **(b)** $\dfrac{d}{dt}(25t - 8e^t)$ **(c)** $\dfrac{d}{dt}e^{t-3}$

Hint for (c): Write e^{t-3} as $e^{-3}e^t$.

SOLUTION

(a) $\dfrac{d}{dx}12e^x = 12\dfrac{d}{dx}e^x = 12e^x.$

(b) $\dfrac{d}{dt}(25t - 8e^t) = 25\dfrac{d}{dt}t - 8\dfrac{d}{dt}e^t = 25 - 8e^t.$

(c) $\dfrac{d}{dt}e^{t-3} = e^{-3}\dfrac{d}{dt}e^t = e^{-3}\cdot e^t = e^{t-3}.$

20. Find an equation of the tangent line to $y = 24e^x$ at $x = 2$.

SOLUTION Let $f(x) = 24e^x$. Then $f(2) = 24e^2$, $f'(x) = 24e^x$, and $f'(2) = 24e^2$. The equation of the tangent line is

$$y = f'(2)(x-2) + f(2) = 24e^2(x-2) + 24e^2.$$

In Exercises 21–32, calculate the derivative.

21. $f(x) = 2x^3 - 3x^2 + 5$

SOLUTION $\dfrac{d}{dx}\left(2x^3 - 3x^2 + 5\right) = 6x^2 - 6x.$

22. $f(x) = 2x^3 - 3x^2 + 2x$

SOLUTION $\frac{d}{dx}\left(2x^3 - 3x^2 + 2x\right) = 6x^2 - 6x + 2.$

23. $f(x) = 4x^{5/3} - 3x^{-2} - 12$

SOLUTION $\frac{d}{dx}\left(4x^{5/3} - 3x^{-2} - 12\right) = \frac{20}{3}x^{2/3} + 6x^{-3}.$

24. $f(x) = x^{5/4} + 4x^{-3/2} + 11x$

SOLUTION $\frac{d}{dx}\left(x^{5/4} + 4x^{-3/2} + 11x\right) = \frac{5}{4}x^{1/4} - 6x^{-5/2} + 11.$

25. $g(z) = 7z^{-5/14} + z^{-5} + 9$

SOLUTION $\frac{d}{dz}\left(7z^{-5/14} + z^{-5} + 9\right) = -\frac{5}{2}z^{-19/14} - 5z^{-6}.$

26. $h(t) = 6\sqrt{t} + \frac{1}{\sqrt{t}}$

SOLUTION $\frac{d}{dt}\left(6t^{1/2} + t^{-1/2}\right) = 3t^{-1/2} - \frac{1}{2}t^{-3/2}.$

27. $f(s) = \sqrt[4]{s} + \sqrt[3]{s}$

SOLUTION $f(s) = \sqrt[4]{s} + \sqrt[3]{s} = s^{1/4} + s^{1/3}$. In this form, we can apply the Sum and Power Rules.

$$\frac{d}{ds}\left(s^{1/4} + s^{1/3}\right) = \frac{1}{4}(s^{(1/4)-1}) + \frac{1}{3}(s^{(1/3)-1}) = \frac{1}{4}s^{-3/4} + \frac{1}{3}s^{-2/3}.$$

28. $W(y) = 6y^4 + 7y^{2/3}$

SOLUTION $\frac{d}{dy}\left(6y^4 + 7y^{2/3}\right) = 24y^3 + \frac{14}{3}y^{-1/3}.$

29. $g(x) = e^2$

SOLUTION Because e^2 is a constant, $\frac{d}{dx}e^2 = 0.$

30. $f(x) = 3e^x - x^3$

SOLUTION $\frac{d}{dx}\left(3e^x - x^3\right) = 3e^x - 3x^2.$

31. $h(t) = 5e^{t-3}$

SOLUTION $\frac{d}{dt}5e^{t-3} = 5e^{-3}\frac{d}{dt}e^t = 5e^{-3}e^t = 5e^{t-3}.$

32. $f(x) = 9 - 12x^{1/3} + 8e^x$

SOLUTION $\frac{d}{dx}\left(9 - 12x^{1/3} + 8e^x\right) = -4x^{-2/3} + 8e^x.$

In Exercises 33–36, calculate the derivative by expanding or simplifying the function.

33. $P(s) = (4s - 3)^2$

SOLUTION $P(s) = (4s - 3)^2 = 16s^2 - 24s + 9$. Thus,

$$\frac{dP}{ds} = 32s - 24.$$

34. $Q(r) = (1 - 2r)(3r + 5)$

SOLUTION $Q(r) = (1 - 2r)(3r + 5) = -6r^2 - 7r + 5$. Thus,

$$\frac{dQ}{dr} = -12r - 7.$$

35. $g(x) = \dfrac{x^2 + 4x^{1/2}}{x^2}$

SOLUTION $g(x) = \dfrac{x^2 + 4x^{1/2}}{x^2} = 1 + 4x^{-3/2}$. Thus,

$$\frac{dg}{dx} = -6x^{-5/2}.$$

36. $s(t) = \dfrac{1 - 2t}{t^{1/2}}$

SOLUTION $s(t) = \dfrac{1 - 2t}{t^{1/2}} = t^{-1/2} - 2t^{1/2}$. Thus,

$$\frac{ds}{dt} = -\frac{1}{2}t^{-3/2} - t^{-1/2}.$$

In Exercises 37–42, calculate the derivative indicated.

37. $\left.\dfrac{dT}{dC}\right|_{C=8}$, $\quad T = 3C^{2/3}$

SOLUTION With $T(C) = 3C^{2/3}$, we have $\dfrac{dT}{dC} = 2C^{-1/3}$. Therefore,

$$\left.\frac{dT}{dC}\right|_{C=8} = 2(8)^{-1/3} = 1.$$

38. $\left.\dfrac{dP}{dV}\right|_{V=-2}$, $\quad P = \dfrac{7}{V}$

SOLUTION With $P = 7V^{-1}$, we have $\dfrac{dP}{dV} = -7V^{-2}$. Therefore,

$$\left.\frac{dP}{dV}\right|_{V=-2} = -7(-2)^{-2} = -\frac{7}{4}.$$

39. $\left.\dfrac{ds}{dz}\right|_{z=2}$, $\quad s = 4z - 16z^2$

SOLUTION With $s = 4z - 16z^2$, we have $\dfrac{ds}{dz} = 4 - 32z$. Therefore,

$$\left.\frac{ds}{dz}\right|_{z=2} = 4 - 32(2) = -60.$$

40. $\left.\dfrac{dR}{dW}\right|_{W=1}$, $\quad R = W^{\pi}$

SOLUTION Let $R(W) = W^{\pi}$. Then $dR/dW = \pi W^{\pi-1}$. Therefore,

$$\left.\frac{dR}{dW}\right|_{W=1} = \pi(1)^{\pi-1} = \pi.$$

41. $\left.\dfrac{dr}{dt}\right|_{t=4}$, $\quad r = t - e^t$

SOLUTION With $r = t - e^t$, we have $\dfrac{dr}{dt} = 1 - e^t$. Therefore,

$$\left.\frac{dr}{dt}\right|_{t=4} = 1 - e^4.$$

42. $\left.\dfrac{dp}{dh}\right|_{h=4}, \quad p = 7e^{h-2}$

SOLUTION With $p = 7e^{h-2}$, we have $\dfrac{dp}{dh} = 7e^{h-2}$. Therefore,

$$\left.\frac{dp}{dh}\right|_{h=4} = 7e^{4-2} = 7e^2.$$

43. Match the functions in graphs (A)–(D) with their derivatives (I)–(III) in Figure 13. Note that two of the functions have the same derivative. Explain why.

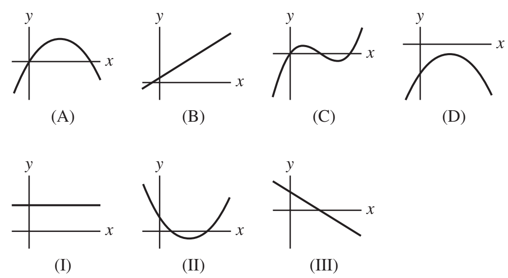


FIGURE 13

SOLUTION

- Consider the graph in (A). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).
- Consider the graph in (B). This is a linear function, so its slope is constant. Thus the derivative is constant, which matches the graph in (I).
- Consider the graph in (C). Moving from left to right, the slope of the tangent line transitions from positive to negative then back to positive. The derivative should therefore be negative in the middle and positive to either side. This matches the graph in (II).
- Consider the graph in (D). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).

Note that the functions whose graphs are shown in (A) and (D) have the same derivative. This happens because the graph in (D) is just a vertical translation of the graph in (A), which means the two functions differ by a constant. The derivative of a constant is zero, so the two functions end up with the same derivative.

44. Of the two functions f and g in Figure 14, which is the derivative of the other? Justify your answer.

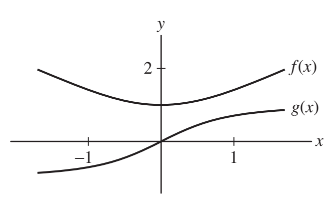


FIGURE 14

SOLUTION $g(x)$ is the derivative of $f(x)$. For $f(x)$ the slope is negative for negative values of x until $x = 0$, where there is a horizontal tangent, and then the slope is positive for positive values of x. Notice that $g(x)$ is negative for negative values of x, goes through the origin at $x = 0$, and then is positive for positive values of x.

45. Assign the labels $f(x)$, $g(x)$, and $h(x)$ to the graphs in Figure 15 in such a way that $f'(x) = g(x)$ and $g'(x) = h(x)$.

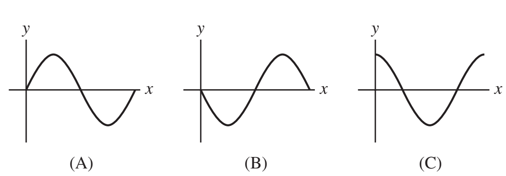


FIGURE 15

SOLUTION Consider the graph in (A). Moving from left to right, the slope of the tangent line is positive over the first quarter of the graph, negative in the middle half and positive again over the final quarter. The derivative of this function must therefore be negative in the middle and positive on either side. This matches the graph in (C).

Now focus on the graph in (C). The slope of the tangent line is negative over the left half and positive on the right half. The derivative of this function therefore needs to be negative on the left and positive on the right. This description matches the graph in (B).

We should therefore label the graph in (A) as $f(x)$, the graph in (B) as $h(x)$, and the graph in (C) as $g(x)$. Then $f'(x) = g(x)$ and $g'(x) = h(x)$.

46. According to the *peak oil theory*, first proposed in 1956 by geophysicist M. Hubbert, the total amount of crude oil $Q(t)$ produced worldwide up to time t has a graph like that in Figure 16.

(a) Sketch the derivative $Q'(t)$ for $1900 \le t \le 2150$. What does $Q'(t)$ represent?

(b) In which year (approximately) does $Q'(t)$ take on its maximum value?

(c) What is $L = \lim_{t\to\infty} Q(t)$? And what is its interpretation?

(d) What is the value of $\lim_{t\to\infty} Q'(t)$?

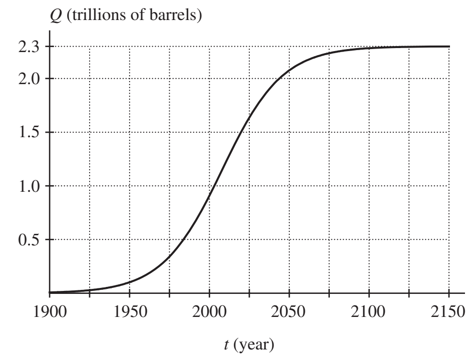
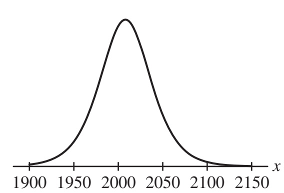


FIGURE 16 Total oil production up to time t

SOLUTION

(a) One possible derivative sketch is shown below. Because the graph of $Q(t)$ is roughly horizontal around $t = 1900$, the graph of $Q'(t)$ begins near zero. Until roughly $t = 2000$, the graph of $Q(t)$ increases more and more rapidly, so the graph of $Q'(t)$ increases. Thereafter, the graph of $Q(t)$ increases more and more gradually, so the graph of $Q'(t)$ decreases. Around $t = 2150$, the graph of $Q(t)$ is again roughly horizontal, so the graph of $Q'(t)$ returns to zero. Note that $Q'(t)$ represents the rate of change in total worldwide oil production; that is, the number of barrels produced per year.



(b) The graph of $Q(t)$ appears to be increasing most rapidly around the year 2000, so $Q'(t)$ takes on its maximum value around the year 2000.

(c) From Figure 16

$$L = \lim_{t\to\infty} Q(t) = 2.3$$

trillion barrels of oil. This value represents the total number of barrels of oil that can be produced by the planet.

(d) Because the graph of $Q(t)$ appears to approach a horizontal line as $t \to \infty$, it appears that

$$\lim_{t\to\infty} Q'(t) = 0.$$

47.  Use the table of values of $f(x)$ to determine which of (A) or (B) in Figure 17 is the graph of $f'(x)$. Explain.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	10	55	98	139	177	210	237	257	268

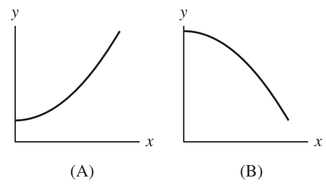


FIGURE 17 Which is the graph of $f'(x)$?

SOLUTION The increment between successive x values in the table is a constant 0.5 but the increment between successive $f(x)$ values decreases from 45 to 43 to 41 to 38 and so on. Thus the difference quotients decrease with increasing x, suggesting that $f'(x)$ decreases as a function of x. Because the graph in (B) depicts a decreasing function, (B) might be the graph of the derivative of $f(x)$.

48. Let R be a variable and r a constant. Compute the derivatives:

(a) $\frac{d}{dR}R$ **(b)** $\frac{d}{dR}r$ **(c)** $\frac{d}{dR}r^2R^3$

SOLUTION

(a) $\frac{d}{dR}R = 1$, since R is a linear function of R with slope 1.

(b) $\frac{d}{dR}r = 0$, since r is a constant.

(c) We apply the Linearity and Power Rules:

$$\frac{d}{dR}r^2R^3 = r^2\frac{d}{dR}R^3 = r^2\big(3(R^2)\big) = 3r^2R^2.$$

49. Compute the derivatives, where c is a constant.

(a) $\frac{d}{dt}ct^3$ **(b)** $\frac{d}{dy}(9c^2y^3 - 24c)$ **(c)** $\frac{d}{dz}(5z + 4cz^2)$

SOLUTION

(a) $\frac{d}{dt}ct^3 = 3ct^2$.

(b) $\frac{d}{dz}(5z + 4cz^2) = 5 + 8cz$.

(c) $\frac{d}{dy}(9c^2y^3 - 24c) = 27c^2y^2$.

50. Find the points on the graph of $f(x) = 12x - x^3$ where the tangent line is horizontal.

SOLUTION Let $f(x) = 12x - x^3$. Solve $f'(x) = 12 - 2x^2 = 0$ to obtain $x = \pm\sqrt{6}$. Thus, the graph of $f(x) = 12x - x^3$ has a horizontal tangent line at two points: $(\sqrt{6}, 6\sqrt{6})$ and $(-\sqrt{6}, -6\sqrt{6})$.

51. Find the points on the graph of $y = x^2 + 3x - 7$ at which the slope of the tangent line is equal to 4.

SOLUTION Let $y = x^2 + 3x - 7$. Solving $dy/dx = 2x + 3 = 4$ yields $x = \frac{1}{2}$.

52. 3.2.52 Find the values of x where $y = x^3$ and $y = x^2 + 5x$ have parallel tangent lines.

SOLUTION Let $f(x) = x^3$ and $g(x) = x^2 + 5x$. The graphs have parallel tangent lines when $f'(x) = g'(x)$. Hence, we solve $f'(x) = 3x^2 = 2x + 5 = g'(x)$ to obtain $x = \frac{5}{3}$ and $x = -1$.

53. Determine a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 0$ and $p'(1) = 4$.

SOLUTION Let $p(x) = x^2 + ax + b$ satisfy $p(1) = 0$ and $p'(1) = 4$. Now, $p'(x) = 2x + a$. Therefore $0 = p(1) = 1 + a + b$ and $4 = p'(1) = 2 + a$; i.e., $a = 2$ and $b = -3$.

54. Find all values of x such that the tangent line to $y = 4x^2 + 11x + 2$ is steeper than the tangent line to $y = x^3$.

SOLUTION Let $f(x) = 4x^2 + 11x + 2$ and let $g(x) = x^3$. We need all x such that $f'(x) > g'(x)$.

$$\begin{aligned} f'(x) &> g'(x) \\ 8x + 11 &> 3x^2 \\ 0 &> 3x^2 - 8x - 11 \\ 0 &> (3x - 11)(x + 1). \end{aligned}$$

The product $(3x - 11)(x + 1) = 0$ when $x = -1$ and when $x = \frac{11}{3}$. We therefore examine the intervals $x < -1$, $-1 < x < \frac{11}{3}$ and $x > \frac{11}{3}$. For $x < -1$ and for $x > \frac{11}{3}$, $(3x - 11)(x + 1) > 0$, whereas for $-1 < x < \frac{11}{3}$, $(3x - 11)(x + 1) < 0$. The solution set is therefore $-1 < x < \frac{11}{3}$.

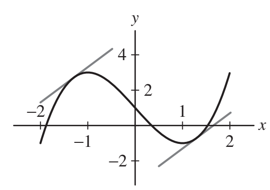
55. Let $f(x) = x^3 - 3x + 1$. Show that $f'(x) \geq -3$ for all x and that, for every $m > -3$, there are precisely two points where $f'(x) = m$. Indicate the position of these points and the corresponding tangent lines for one value of m in a sketch of the graph of $f(x)$.

SOLUTION Let $P = (a, b)$ be a point on the graph of $f(x) = x^3 - 3x + 1$.

- The derivative satisfies $f'(x) = 3x^2 - 3 \geq -3$ since $3x^2$ is nonnegative.
- Suppose the slope m of the tangent line is greater than -3. Then $f'(a) = 3a^2 - 3 = m$, whence

$$a^2 = \frac{m+3}{3} > 0 \quad \text{and thus} \quad a = \pm\sqrt{\frac{m+3}{3}}.$$

- The two parallel tangent lines with slope 2 are shown with the graph of $f(x)$ here.



56. Show that the tangent lines to $y = \frac{1}{3}x^3 - x^2$ at $x = a$ and at $x = b$ are parallel if $a = b$ or $a + b = 2$.

SOLUTION Let $P = (a, f(a))$ and $Q = (b, f(b))$ be points on the graph of $y = f(x) = \frac{1}{3}x^3 - x^2$. Equate the slopes of the tangent lines at the points P and Q: $a^2 - 2a = b^2 - 2b$. Thus $a^2 - 2a - b^2 + 2b = 0$. Now,

$$a^2 - 2a - b^2 + 2b = (a - b)(a + b) - 2(a - b) = (a - 2 + b)(a - b);$$

therefore, either $a = b$ (i.e., P and Q are the same point) or $a + b = 2$.

57. Compute the derivative of $f(x) = x^{3/2}$ using the limit definition. *Hint:* Show that

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}\left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right)$$

SOLUTION Once we have the difference of square roots, we multiply by the conjugate to solve the problem.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} &= \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h}\left(\frac{\sqrt{(x+h)^3} + \sqrt{x^3}}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right) \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}\left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right). \end{aligned}$$

The first factor of the expression in the last line is clearly the limit definition of the derivative of x^3, which is $3x^2$. The second factor can be evaluated, so

$$\frac{d}{dx}x^{3/2} = 3x^2 \frac{1}{2\sqrt{x^3}} = \frac{3}{2}x^{1/2}.$$

58. Use the limit definition of $m(b)$ to approximate $m(4)$. Then estimate the slope of the tangent line to $y = 4^x$ at $x = 0$ and $x = 2$.

SOLUTION Recall

$$m(4) = \lim_{h\to 0}\left(\frac{4^h - 1}{h}\right).$$

Using a table of values, we find

h	$\frac{4^h - 1}{h}$
0.01	1.39595
0.001	1.38726
0.0001	1.38639
0.00001	1.38630

Thus $m(4) \approx 1.386$. Knowing that $y'(x) = m(4)\cdot 4^x$, it follows that $y'(0) \approx 1.386$ and $y'(2) \approx 1.386 \cdot 16 = 22.176$.

59. Let $f(x) = xe^x$. Use the limit definition to compute $f'(0)$, and find the equation of the tangent line at $x = 0$.

SOLUTION Let $f(x) = xe^x$. Then $f(0) = 0$, and

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{he^h - 0}{h} = \lim_{h\to 0} e^h = 1.$$

The equation of the tangent line is

$$y = f'(0)(x-0) + f(0) = 1(x-0) + 0 = x.$$

60. The average speed (in meters per second) of a gas molecule is

$$v_{\text{avg}} = \sqrt{\frac{8RT}{\pi M}}$$

where T is the temperature (in kelvins), M is the molar mass (in kilograms per mole), and $R = 8.31$. Calculate dv_{avg}/dT at $T = 300$ K for oxygen, which has a molar mass of 0.032 kg/mol.

SOLUTION Using the form $v_{av} = (8RT/(\pi M))^{1/2} = \sqrt{8R/(\pi M)}T^{1/2}$, where M and R are constant, we use the Power Rule to compute the derivative dv_{av}/dT.

$$\frac{d}{dT}\sqrt{8R/(\pi M)}T^{1/2} = \sqrt{8R/(\pi M)}\frac{d}{dT}T^{1/2} = \sqrt{8R/(\pi M)}\frac{1}{2}(T^{(1/2)-1}).$$

In particular, if $T = 300^\circ$K,

$$\frac{d}{dT}v_{av} = \sqrt{8(8.31)/(\pi(0.032))}\frac{1}{2}(300)^{-1/2} = 0.74234 \text{ m/(s}\cdot\text{K)}.$$

61. Biologists have observed that the pulse rate P (in beats per minute) in animals is related to body mass (in kilograms) by the approximate formula $P = 200m^{-1/4}$. This is one of many *allometric scaling laws* prevalent in biology. Is $|dP/dm|$ an increasing or decreasing function of m? Find an equation of the tangent line at the points on the graph in Figure 18 that represent goat ($m = 33$) and man ($m = 68$).

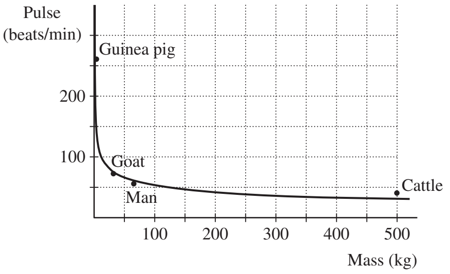


FIGURE 18

SOLUTION $dP/dm = -50m^{-5/4}$. For $m > 0$, $|dP/dm| = |50m^{-5/4}|$. $|dP/dm| \to 0$ as m gets larger; $|dP/dm|$ gets smaller as m gets bigger.

For each $m = c$, the equation of the tangent line to the graph of P at m is

$$y = P'(c)(m - c) + P(c).$$

For a goat ($m = 33$ kg), $P(33) = 83.445$ beats per minute (bpm) and

$$\frac{dP}{dm} = -50(33)^{-5/4} \approx -0.63216 \text{ bpm/kg}.$$

Hence, $y = -0.63216(m - 33) + 83.445$.

For a man ($m = 68$ kg), we have $P(68) = 69.647$ bpm and

$$\frac{dP}{dm} = -50(68)^{-5/4} \approx -0.25606 \text{ bpm/kg}.$$

Hence, the tangent line has formula $y = -0.25606(m - 68) + 69.647$.

62. Some studies suggest that kidney mass K in mammals (in kilograms) is related to body mass m (in kilograms) by the approximate formula $K = 0.007m^{0.85}$. Calculate dK/dm at $m = 68$. Then calculate the derivative with respect to m of the relative kidney-to-mass ratio K/m at $m = 68$.

SOLUTION

$$\frac{dK}{dm} = 0.007(0.85)m^{-0.15} = 0.00595m^{-0.15};$$

hence,

$$\left.\frac{dK}{dm}\right|_{m=68} = 0.00595(68)^{-0.15} = 0.00315966.$$

Because

$$\frac{K}{m} = 0.007\frac{m^{0.85}}{m} = 0.007m^{-0.15},$$

we find

$$\frac{d}{dm}\left(\frac{K}{m}\right) = 0.007\frac{d}{dm}m^{-0.15} = -0.00105m^{-1.15},$$

and

$$\left.\frac{d}{dm}\left(\frac{K}{m}\right)\right|_{m=68} = -8.19981 \times 10^{-6} \text{ kg}^{-1}.$$

63. The Clausius–Clapeyron Law relates the *vapor pressure* of water P (in atmospheres) to the temperature T (in kelvins):

$$\frac{dP}{dT} = k\frac{P}{T^2}$$

where k is a constant. Estimate dP/dT for $T = 303, 313, 323, 333, 343$ using the data and the approximation

$$\frac{dP}{dT} \approx \frac{P(T+10) - P(T-10)}{20}$$

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

Do your estimates seem to confirm the Clausius–Clapeyron Law? What is the approximate value of k?

SOLUTION Using the indicated approximation to the first derivative, we calculate

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};$$
$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};$$
$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};$$

$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};$$
$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}$$

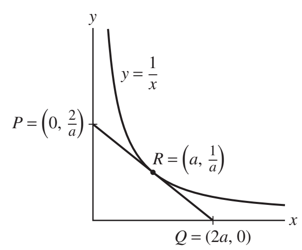
If the Clausius–Clapeyron law is valid, then $\frac{T^2}{P}\frac{dP}{dT}$ should remain constant as T varies. Using the data for vapor pressure and temperature and the approximate derivative values calculated above, we find

T (K)	303	313	323	333	343
$\frac{T^2}{P}\frac{dP}{dT}$	5047.59	5025.76	5009.54	4994.57	4981.45

These values are roughly constant, suggesting that the Clausius–Clapeyron law is valid, and that $k \approx 5000$.

64. Let L be the tangent line to the hyperbola $xy = 1$ at $x = a$, where $a > 0$. Show that the area of the triangle bounded by L and the coordinate axes does not depend on a.

SOLUTION Let $f(x) = x^{-1}$. The tangent line to f at $x = a$ is $y = f'(a)(x - a) + f(a) = -\frac{1}{a^2}(x - a) + \frac{1}{a}$. The y-intercept of this line (where $x = 0$) is $\frac{2}{a}$. Its x-intercept (where $y = 0$) is $2a$. Hence the area of the triangle bounded by the tangent line and the coordinate axes is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$, which is independent of a.



65. In the setting of Exercise 64, show that the point of tangency is the midpoint of the segment of L lying in the first quadrant.

SOLUTION In the previous exercise, we saw that the tangent line to the hyperbola $xy = 1$ or $y = \frac{1}{x}$ at $x = a$ has y-intercept $P = (0, \frac{2}{a})$ and x-intercept $Q = (2a, 0)$. The midpoint of the line segment connecting P and Q is thus

$$\left(\frac{0 + 2a}{2}, \frac{\frac{2}{a} + 0}{2}\right) = \left(a, \frac{1}{a}\right),$$

which is the point of tangency.

66. Match functions (A)–(C) with their derivatives (I)–(III) in Figure 19.

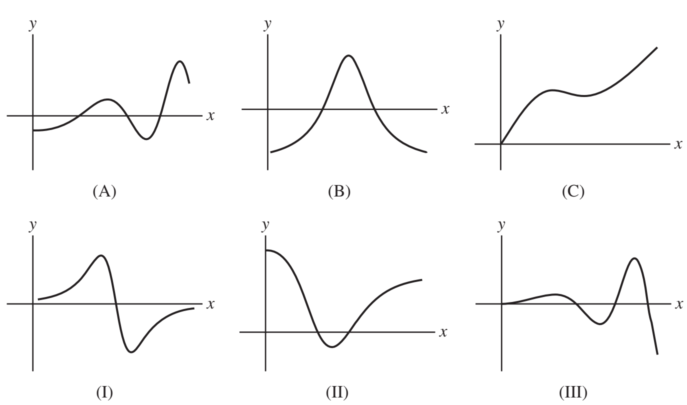


FIGURE 19

SOLUTION Note that the graph in (A) has three locations with a horizontal tangent line. The derivative must therefore cross the x-axis in three locations, which matches (III).

The graph in (B) has only one location with a horizontal tangent line, so its derivative should cross the x-axis only once. Thus, (I) is the graph corresponding to the derivative of (B).

Finally, the graph in (B) has two locations with a horizontal tangent line, so its derivative should cross the x-axis twice. Thus, (II) is the graph corresponding to the derivative of (C).

67. Make a rough sketch of the graph of the derivative of the function in Figure 20(A).

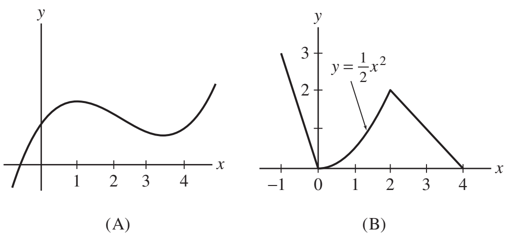
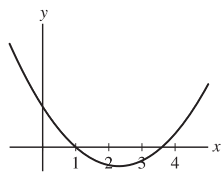


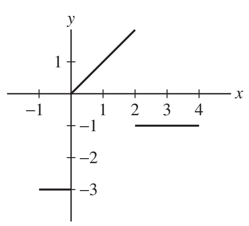
FIGURE 20

SOLUTION The graph has a tangent line with negative slope approximately on the interval (1, 3.6), and has a tangent line with a positive slope elsewhere. This implies that the derivative must be negative on the interval (1, 3.6) and positive elsewhere. The graph may therefore look like this:



68. Graph the derivative of the function in Figure 20(B), omitting points where the derivative is not defined.

SOLUTION On $(-1, 0)$, the graph is a line with slope -3, so the derivative is equal to -3. The derivative on $(0, 2)$ is x. Finally, on $(2, 4)$ the function is a line with slope -1, so the derivative is equal to -1. Combining this information leads to the graph:



69. Sketch the graph of $f(x) = x\,|x|$. Then show that $f'(0)$ exists.

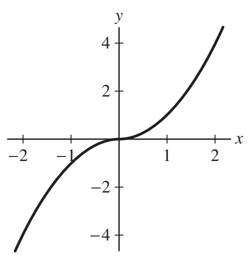
SOLUTION For $x < 0$, $f(x) = -x^2$, and $f'(x) = -2x$. For $x > 0$, $f(x) = x^2$, and $f'(x) = 2x$. At $x = 0$, we find

$$\lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0+} \frac{h^2}{h} = 0$$

and

$$\lim_{h\to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0-} \frac{-h^2}{h} = 0.$$

Because the two one-sided limits exist and are equal, it follows that $f'(0)$ exists and is equal to zero. Here is the graph of $f(x) = x|x|$.



70. Determine the values of x at which the function in Figure 21 is: (a) discontinuous, and (b) nondifferentiable.

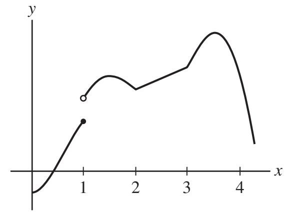


FIGURE 21

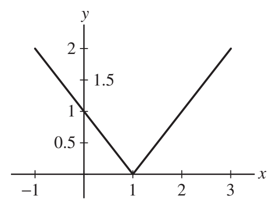
SOLUTION The function is discontinuous at those points where it is undefined or there is a break in the graph. On the interval $[0, 4]$, there is only one such point, at $x = 1$.

The function is nondifferentiable at those points where it is discontinuous or where it has a corner or cusp. In addition to the point $x = 1$ we already know about, the function is nondifferentiable at $x = 2$ and $x = 3$.

In Exercises 71–76, find the points c (if any) such that $f'(c)$ does not exist.

71. $f(x) = |x - 1|$

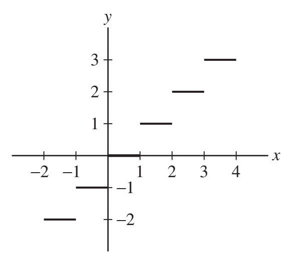
SOLUTION



Here is the graph of $f(x) = |x - 1|$. Its derivative does not exist at $x = 1$. At that value of x there is a sharp corner.

72. $f(x) = [x]$

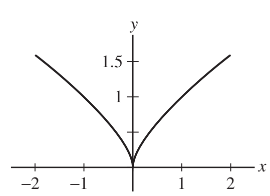
SOLUTION



Here is the graph of $f(x) = [x]$. This is the integer step function graph. Its derivative does not exist at all x values that are integers. At those values of x the graph is discontinuous.

73. $f(x) = x^{2/3}$

SOLUTION Here is the graph of $f(x) = x^{2/3}$. Its derivative does not exist at $x = 0$. At that value of x, there is a sharp corner or "cusp".

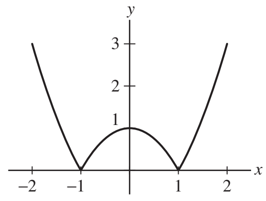


74. $f(x) = x^{3/2}$

SOLUTION The function is differentiable on its entire domain, $\{x : x \geq 0\}$. The formula is $\frac{d}{dx}x^{3/2} = \frac{3}{2}x^{1/2}$.

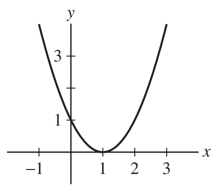
75. $f(x) = |x^2 - 1|$

SOLUTION Here is the graph of $f(x) = \left|x^2 - 1\right|$. Its derivative does not exist at $x = -1$ or at $x = 1$. At these values of x, the graph has sharp corners.



76. $f(x) = |x - 1|^2$

SOLUTION

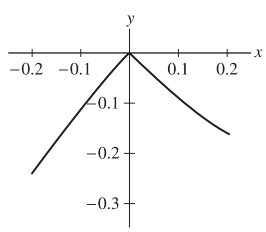


This is the graph of $f(x) = |x - 1|^2$. Its derivative exists everywhere.

GU *In Exercises 77–82, zoom in on a plot of $f(x)$ at the point $(a, f(a))$ and state whether or not $f(x)$ appears to be differentiable at $x = a$. If it is nondifferentiable, state whether the tangent line appears to be vertical or does not exist.*

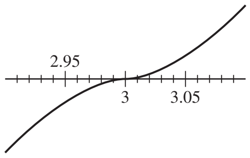
77. $f(x) = (x - 1)|x|, \quad a = 0$

SOLUTION The graph of $f(x) = (x - 1)|x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



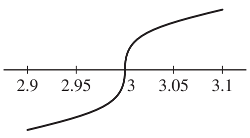
78. $f(x) = (x - 3)^{5/3}, \quad a = 3$

SOLUTION The graph of $f(x) = (x - 3)^{5/3}$ for x near 3 is shown below. From this graph, it appears that f is differentiable at $x = 3$, with a horizontal tangent line.



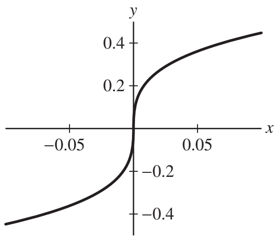
79. $f(x) = (x - 3)^{1/3}, \quad a = 3$

SOLUTION The graph of $f(x) = (x - 3)^{1/3}$ for x near 3 is shown below. From this graph, it appears that f is not differentiable at $x = 3$. Moreover, the tangent line appears to be vertical.



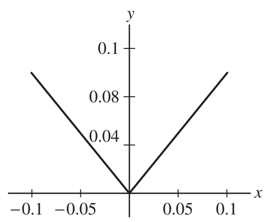
80. $f(x) = \sin(x^{1/3})$, $\quad a = 0$

SOLUTION The graph of $f(x) = \sin(x^{1/3})$ for x near 0 is shown below. From this graph, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line appears to be vertical.



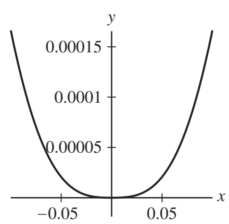
81. $f(x) = |\sin x|$, $\quad a = 0$

SOLUTION The graph of $f(x) = |\sin x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



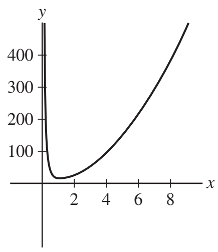
82. $f(x) = |x - \sin x|$, $\quad a = 0$

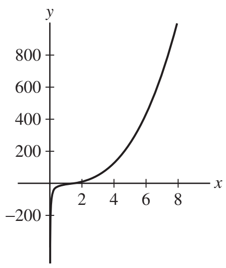
SOLUTION The graph of $f(x) = |x - \sin x|$ for x near 0 is shown below. From this graph, it appears that f is differentiable at $x = 0$, with a horizontal tangent line.



83. GU Plot the derivative $f'(x)$ of $f(x) = 2x^3 - 10x^{-1}$ for $x > 0$ (set the bounds of the viewing box appropriately) and observe that $f'(x) > 0$. What does the positivity of $f'(x)$ tell us about the graph of $f(x)$ itself? Plot $f(x)$ and confirm this conclusion.

SOLUTION Let $f(x) = 2x^3 - 10x^{-1}$. Then $f'(x) = 6x^2 + 10x^{-2}$. The graph of $f'(x)$ is shown in the figure below at the left and it is clear that $f'(x) > 0$ for all $x > 0$. The positivity of $f'(x)$ tells us that the graph of $f(x)$ is increasing for $x > 0$. This is confirmed in the figure below at the right, which shows the graph of $f(x)$.





84. Find the coordinates of the point P in Figure 22 at which the tangent line passes through $(5, 0)$.

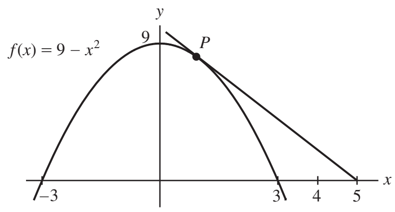


FIGURE 22 Graph of $f(x) = 9 - x^2$.

SOLUTION Let $f(x) = 9 - x^2$, and suppose P has coordinates $(a, 9 - a^2)$. Because $f'(x) = -2x$, the slope of the line tangent to the graph of $f(x)$ at P is $-2a$, and the equation of the tangent line is

$$y = f'(a)(x - a) + f(a) = -2a(x - a) + 9 - a^2 = -2ax + 9 + a^2.$$

In order for this line to pass through the point $(5, 0)$, we must have

$$0 = -10a + 9 + a^2 = (a - 9)(a - 1).$$

Thus, $a = 1$ or $a = 9$. We exclude $a = 9$ because from Figure 22 we are looking for an x-coordinate between 0 and 5. Thus, the point P has coordinates $(1, 8)$.

Exercises 85–88 refer to Figure 23. Length QR is called the subtangent *at P, and length RT is called the* subnormal*.*

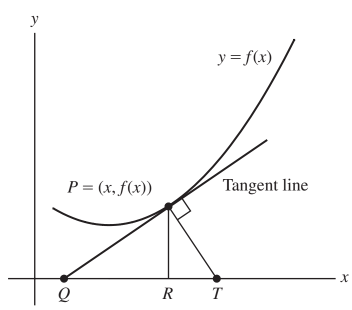


FIGURE 23

85. Calculate the subtangent of

$$f(x) = x^2 + 3x \quad \text{at } x = 2$$

SOLUTION Let $f(x) = x^2 + 3x$. Then $f'(x) = 2x + 3$, and the equation of the tangent line at $x = 2$ is

$$y = f'(2)(x - 2) + f(2) = 7(x - 2) + 10 = 7x - 4.$$

This line intersects the x-axis at $x = \frac{4}{7}$. Thus Q has coordinates $(\frac{4}{7}, 0)$, R has coordinates $(2, 0)$ and the subtangent is

$$2 - \frac{4}{7} = \frac{10}{7}.$$

86. Show that the subtangent of $f(x) = e^x$ is everywhere equal to 1.

SOLUTION Let $f(x) = e^x$. Then $f'(x) = e^x$, and the equation of the tangent line at $x = a$ is

$$y = f'(a)(x - a) + f(a) = e^a(x - a) + e^a.$$

This line intersects the x-axis at $x = a - 1$. Thus, Q has coordinates $(a - 1, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - (a - 1) = 1.$$

87. Prove in general that the subnormal at P is $|f'(x)f(x)|$.

SOLUTION The slope of the tangent line at P is $f'(x)$. The slope of the line normal to the graph at P is then $-1/f'(x)$, and the normal line intersects the x-axis at the point T with coordinates $(x + f(x)f'(x), 0)$. The point R has coordinates $(x, 0)$, so the subnormal is

$$|x + f(x)f'(x) - x| = |f(x)f'(x)|.$$

88. Show that $\overline{PQ}$ has length $|f(x)|\sqrt{1+f'(x)^{-2}}$.

SOLUTION The coordinates of the point P are $(x, f(x))$, the coordinates of the point R are $(x, 0)$ and the coordinates of the point Q are

$$\left(x - \frac{f(x)}{f'(x)}, 0\right).$$

Thus, $\overline{PR} = |f(x)|$, $\overline{QR} = \left|\frac{f(x)}{f'(x)}\right|$, and by the Pythagorean Theorem

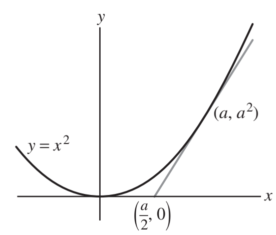
$$\overline{PQ} = \sqrt{\left(\frac{f(x)}{f'(x)}\right)^2 + (f(x))^2} = |f(x)|\sqrt{1+f'(x)^{-2}}.$$

89. Prove the following theorem of Apollonius of Perga (the Greek mathematician born in 262 BCE who gave the parabola, ellipse, and hyperbola their names): The subtangent of the parabola $y = x^2$ at $x = a$ is equal to $a/2$.

SOLUTION Let $f(x) = x^2$. The tangent line to f at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 2a(x-a) + a^2 = 2ax - a^2.$$

The x-intercept of this line (where $y = 0$) is $\frac{a}{2}$ as claimed.



90. Show that the subtangent to $y = x^3$ at $x = a$ is equal to $\frac{1}{3}a$.

SOLUTION Let $f(x) = x^3$. Then $f'(x) = 3x^2$, and the equation of the tangent line t $x = a$ is

$$y = f'(a)(x-a) + f(a) = 3a^2(x-a) + a^3 = 3a^2x - 2a^3.$$

This line intersects the x-axis at $x = 2a/3$. Thus, Q has coordinates $(2a/3, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - \frac{2}{3}a = \frac{1}{3}a.$$

91. Formulate and prove a generalization of Exercise 90 for $y = x^n$.

SOLUTION Let $f(x) = x^n$. Then $f'(x) = nx^{n-1}$, and the equation of the tangent line t $x = a$ is

$$y = f'(a)(x-a) + f(a) = na^{n-1}(x-a) + a^n = na^{n-1}x - (n-1)a^n.$$

This line intersects the x-axis at $x = (n-1)a/n$. Thus, Q has coordinates $((n-1)a/n, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - \frac{n-1}{n}a = \frac{1}{n}a.$$

Further Insights and Challenges

92. Two small arches have the shape of parabolas. The first is given by $f(x) = 1 - x^2$ for $-1 \le x \le 1$ and the second by $g(x) = 4 - (x-4)^2$ for $2 \le x \le 6$. A board is placed on top of these arches so it rests on both (Figure 24). What is the slope of the board? *Hint:* Find the tangent line to $y = f(x)$ that intersects $y = g(x)$ in exactly one point.

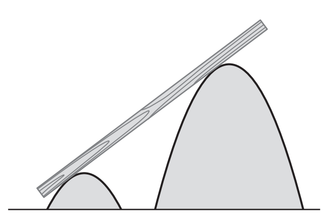


FIGURE 24

SOLUTION At the points where the board makes contact with the arches the slope of the board must be equal to the slope of the arches (and hence they are equal to each other). Suppose $(t, f(t))$ is the point where the board touches the left hand arch. The tangent line here (the line the board defines) is given by

$$y = f'(t)(x - t) + f(t).$$

This line must hit the other arch in exactly one point. In other words, if we plug in $y = g(x)$ to get

$$g(x) = f'(t)(x - t) + f(t)$$

there can only be one solution for x in terms of t. Computing f' and plugging in we get

$$4 - (x^2 - 8x + 16) = -2tx + 2t^2 + 1 - t^2$$

which simplifies to

$$x^2 - 2tx - 8x + t^2 + 13 = 0.$$

This is a quadratic equation $ax^2 + bx + c = 0$ with $a = 1$, $b = (-2t - 8)$ and $c = t^2 + 13$. By the quadratic formula we know there is a unique solution for x iff $b^2 - 4ac = 0$. In our case this means

$$(2t + 8)^2 = 4(t^2 + 13).$$

Solving this gives $t = -3/8$ and plugging into f' shows the slope of the board must be $3/4$.

93. A vase is formed by rotating $y = x^2$ around the y-axis. If we drop in a marble, it will either touch the bottom point of the vase or be suspended above the bottom by touching the sides (Figure 25). How small must the marble be to touch the bottom?

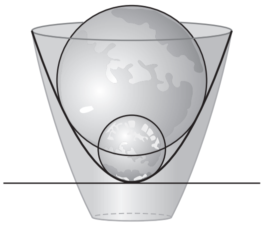


FIGURE 25

SOLUTION Suppose a circle is tangent to the parabola $y = x^2$ at the point (t, t^2). The slope of the parabola at this point is $2t$, so the slope of the radius of the circle at this point is $-\frac{1}{2t}$ (since it is perpendicular to the tangent line of the circle). Thus the center of the circle must be where the line given by $y = -\frac{1}{2t}(x - t) + t^2$ crosses the y-axis. We can find the y-coordinate by setting $x = 0$: we get $y = \frac{1}{2} + t^2$. Thus, the radius extends from $(0, \frac{1}{2} + t^2)$ to (t, t^2) and

$$r = \sqrt{\left(\frac{1}{2} + t^2 - t^2\right)^2 + t^2} = \sqrt{\frac{1}{4} + t^2}.$$

This radius is greater than $\frac{1}{2}$ whenever $t > 0$; so, if a marble has radius $> 1/2$ it sits on the edge of the vase, but if it has radius $\leq 1/2$ it rolls all the way to the bottom.

94. Let $f(x)$ be a differentiable function, and set $g(x) = f(x + c)$, where c is a constant. Use the limit definition to show that $g'(x) = f'(x + c)$. Explain this result graphically, recalling that the graph of $g(x)$ is obtained by shifting the graph of $f(x)$ c units to the left (if $c > 0$) or right (if $c < 0$).

SOLUTION

- Let $g(x) = f(x + c)$. Using the limit definition,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f((x+h)+c) - f(x+c)}{h}$$
$$= \lim_{h\to 0} \frac{f((x+c)+h) - f(x+c)}{h} = f'(x+c).$$

- The graph of $g(x)$ is obtained by shifting $f(x)$ to the left by c units. This implies that $g'(x)$ is equal to $f'(x)$ shifted to the left by c units, which happens to be $f'(x + c)$. Therefore, $g'(x) = f'(x + c)$.

95. Negative Exponents Let n be a whole number. Use the Power Rule for x^n to calculate the derivative of $f(x) = x^{-n}$ by showing that

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}$$

SOLUTION Let $f(x) = x^{-n}$ where n is a positive integer.

- The difference quotient for f is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^{-n} - x^{-n}}{h} = \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h} = \frac{\frac{x^n-(x+h)^n}{x^n(x+h)^n}}{h}$$
$$= \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}.$$

- Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0} \frac{-1}{x^n(x+h)^n} \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = -x^{-2n} \frac{d}{dx}\left(x^n\right).$$

- From above, we continue: $f'(x) = -x^{-2n}\frac{d}{dx}\left(x^n\right) = -x^{-2n} \cdot nx^{n-1} = -nx^{-n-1}$. Since n is a positive integer, $k = -n$ is a negative integer and we have $\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1} = kx^{k-1}$; i.e. $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ for negative integers k.

96. Verify the Power Rule for the exponent $1/n$, where n is a positive integer, using the following trick: Rewrite the difference quotient for $y = x^{1/n}$ at $x = b$ in terms of $u = (b+h)^{1/n}$ and $a = b^{1/n}$.

SOLUTION Substituting $x = (b+h)^{1/n}$ and $a = b^{1/n}$ into the left-hand side of equation (3) yields

$$\frac{x^n - a^n}{x - a} = \frac{(b+h) - b}{(b+h)^{1/n} - b^{1/n}} = \frac{h}{(b+h)^{1/n} - b^{1/n}}$$

whereas substituting these same expressions into the right-hand side of equation (3) produces

$$\frac{x^n - a^n}{x - a} = (b+h)^{\frac{n-1}{n}} + (b+h)^{\frac{n-2}{n}} b^{1/n} + (b+h)^{\frac{n-3}{n}} b^{2/n} + \cdots + b^{\frac{n-1}{n}};$$

hence,

$$\frac{(b+h)^{1/n} - b^{1/n}}{h} = \frac{1}{(b+h)^{\frac{n-1}{n}} + (b+h)^{\frac{n-2}{n}} b^{1/n} + (b+h)^{\frac{n-3}{n}} b^{2/n} + \cdots + b^{\frac{n-1}{n}}}.$$

If we take $f(x) = x^{1/n}$, then, using the previous expression,

$$f'(b) = \lim_{h\to 0} \frac{(b+h)^{1/n} - b^{1/n}}{h} = \frac{1}{nb^{\frac{n-1}{n}}} = \frac{1}{n} b^{\frac{1}{n}-1}.$$

Replacing b by x, we have $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$.

97. Infinitely Rapid Oscillations Define

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ but $f'(0)$ does not exist (see Figure 24).

SOLUTION Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. As $x \to 0$,

$$|f(x) - f(0)| = \left| x \sin\left(\frac{1}{x}\right) - 0\right| = |x| \left|\sin\left(\frac{1}{x}\right)\right| \to 0$$

since the values of the sine lie between -1 and 1. Hence, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = f(0)$ and thus f is continuous at $x = 0$.

As $x \to 0$, the difference quotient at $x = 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x \sin\left(\frac{1}{x}\right) - 0}{x - 0} = \sin\left(\frac{1}{x}\right)$$

does *not* converge to a limit since it oscillates infinitely through every value between -1 and 1. Accordingly, $f'(0)$ does not exist.

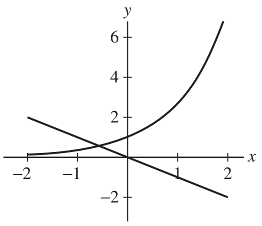
98. For which value of λ does the equation $e^x = \lambda x$ have a unique solution? For which values of λ does it have at least one solution? For intuition, plot $y = e^x$ and the line $y = \lambda x$.

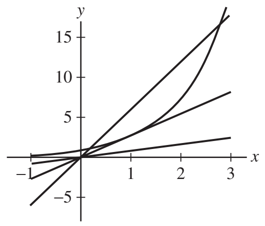
SOLUTION First, note that when $\lambda = 0$, the equation $e^x = 0 \cdot x = 0$ has no real solution. For $\lambda \neq 0$, we observe that solutions to the equation $e^x = \lambda x$ correspond to points of intersection between the graphs of $y = e^x$ and $y = \lambda x$. When $\lambda < 0$, the two graphs intersect at only one location (see the graph below at the left). On the other hand, when $\lambda > 0$, the graphs may have zero, one or two points of intersection (see the graph below at the right). Note that the graphs have one point of intersection when $y = \lambda x$ is the tangent line to $y = e^x$. Thus, not only do we require $e^x = \lambda x$, but also $e^x = \lambda$. It then follows that the point of intersection satisfies $\lambda = \lambda x$, so $x = 1$. This then gives $\lambda = e$.

Therefore the equation $e^x = \lambda x$:

(a) has at least one solution when $\lambda < 0$ and when $\lambda \geq e$;

(b) has a unique solution when $\lambda < 0$ and when $\lambda = e$.





3.3 Product and Quotient Rules

Preliminary Questions

1. Are the following statements true or false? If false, state the correct version.

(a) fg denotes the function whose value at x is $f(g(x))$.

(b) f/g denotes the function whose value at x is $f(x)/g(x)$.

(c) The derivative of the product is the product of the derivatives.

(d) $\left.\dfrac{d}{dx}(fg)\right|_{x=4} = f(4)g'(4) - g(4)f'(4)$

(e) $\left.\dfrac{d}{dx}(fg)\right|_{x=0} = f(0)g'(0) + g(0)f'(0)$

SOLUTION

(a) False. The notation fg denotes the function whose value at x is $f(x)g(x)$.

(b) True.

(c) False. The derivative of a product fg is $f'(x)g(x) + f(x)g'(x)$.

(d) False. $\left.\dfrac{d}{dx}(fg)\right|_{x=4} = f(4)g'(4) + g(4)f'(4)$.

(e) True.

2. Find $(f/g)'(1)$ if $f(1) = f'(1) = g(1) = 2$ and $g'(1) = 4$.

SOLUTION $\left.\dfrac{d}{dx}(f/g)\right|_{x=1} = [g(1)f'(1) - f(1)g'(1)]/g(1)^2 = [2(2) - 2(4)]/2^2 = -1$.

3. Find $g(1)$ if $f(1) = 0$, $f'(1) = 2$, and $(fg)'(1) = 10$.

SOLUTION $(fg)'(1) = f(1)g'(1) + f'(1)g(1)$, so $10 = 0 \cdot g'(1) + 2g(1)$ and $g(1) = 5$.

Exercises

In Exercises 1–6, use the Product Rule to calculate the derivative.

1. $f(x) = x^3(2x^2 + 1)$

SOLUTION Let $f(x) = x^3(2x^2 + 1)$. Then

$$f'(x) = x^3 \frac{d}{dx}(2x^2 + 1) + (2x^2 + 1)\frac{d}{dx}x^3 = x^3(4x) + (2x^2 + 1)(3x^2) = 10x^4 + 3x^2.$$

2. $f(x) = (3x - 5)(2x^2 - 3)$

SOLUTION Let $f(x) = (3x - 5)(2x^2 - 3)$. Then

$$f'(x) = (3x - 5)\frac{d}{dx}(2x^2 - 3) + (2x^2 - 3)\frac{d}{dx}(3x - 5) = (3x - 5)(4x) + (2x^2 - 3)(3) = 18x^2 - 20x - 9.$$

3. $f(x) = x^2 e^x$

SOLUTION Let $f(x) = x^2 e^x$. Then

$$f'(x) = x^2 \frac{d}{dx}e^x + e^x \frac{d}{dx}x^2 = x^2 e^x + e^x(2x) = e^x(x^2 + 2x).$$

4. $f(x) = (2x - 9)(4e^x + 1)$

SOLUTION Let $f(x) = (2x - 9)(4e^x + 1)$. Then

$$f'(x) = (2x - 9)\frac{d}{dx}(4e^x + 1) + (4e^x + 1)\frac{d}{dx}(2x - 9) = (2x - 9)(4e^x) + (4e^x + 1)(2) = 8xe^x - 28e^x + 2.$$

5. $\left.\dfrac{dh}{ds}\right|_{s=4}$, $\quad h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$

SOLUTION Let $h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$. Then

$$\begin{aligned}\frac{dh}{ds} &= (s^{-1/2} + 2s)\frac{d}{dx}(7 - s^{-1}) + (7 - s^{-1})\frac{d}{ds}\left(s^{-1/2} + 2s\right)\\ &= (s^{-1/2} + 2s)(s^{-2}) + (7 - s^{-1})\left(-\frac{1}{2}s^{-3/2} + 2\right) = -\frac{7}{2}s^{-3/2} + \frac{3}{2}s^{-5/2} + 14.\end{aligned}$$

Therefore,

$$\left.\frac{dh}{ds}\right|_{s=4} = -\frac{7}{2}(4)^{-3/2} + \frac{3}{2}(4)^{-5/2} + 14 = \frac{871}{64}.$$

6. $\left.\dfrac{dy}{dt}\right|_{t=2}$, $\quad y = (t - 8t^{-1})(e^t + t^2)$

SOLUTION Let $y(t) = (t - 8t^{-1})(e^t + t^2)$. Then

$$\begin{aligned}\frac{dy}{dt} &= (t - 8t^{-1})\frac{d}{dt}(e^t + t^2) + (e^t + t^2)\frac{d}{dt}(t - 8t^{-1})\\ &= (t - 8t^{-1})(e^t + 2t) + (e^t + t^2)(1 + 8t^{-2}).\end{aligned}$$

Therefore,

$$\left.\frac{dy}{dt}\right|_{t=2} = (2 - 4)(e^2 + 4) + (e^2 + 4)(1 + 2) = e^2 + 4.$$

In Exercises 7–12, use the Quotient Rule to calculate the derivative.

7. $f(x) = \dfrac{x}{x - 2}$

SOLUTION Let $f(x) = \frac{x}{x-2}$. Then

$$f'(x) = \frac{(x - 2)\frac{d}{dx}x - x\frac{d}{dx}(x - 2)}{(x - 2)^2} = \frac{(x - 2) - x}{(x - 2)^2} = \frac{-2}{(x - 2)^2}.$$

8. $f(x) = \dfrac{x+4}{x^2+x+1}$

SOLUTION Let $f(x) = \frac{x+4}{x^2+x+1}$. Then

$$f'(x) = \frac{(x^2+x+1)\frac{d}{dx}(x+4) - (x+4)\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1)-(x+4)(2x+1)}{(x^2+x+1)^2} = \frac{-x^2-8x-3}{(x^2+x+1)^2}.$$

9. $\left.\dfrac{dg}{dt}\right|_{t=-2}, \quad g(t) = \dfrac{t^2+1}{t^2-1}$

SOLUTION Let $g(t) = \dfrac{t^2+1}{t^2-1}$. Then

$$\frac{dg}{dt} = \frac{(t^2-1)\frac{d}{dt}(t^2+1) - (t^2+1)\frac{d}{dt}(t^2-1)}{(t^2-1)^2} = \frac{(t^2-1)(2t)-(t^2+1)(2t)}{(t^2-1)^2} = -\frac{4t}{(t^2-1)^2}.$$

Therefore,

$$\left.\frac{dg}{dt}\right|_{t=-2} = -\frac{4(-2)}{((-2)^2-1)^2} = \frac{8}{9}.$$

10. $\left.\dfrac{dw}{dz}\right|_{z=9}, \quad w = \dfrac{z^2}{\sqrt{z}+z}$

SOLUTION Let $w(z) = \dfrac{z^2}{\sqrt{z}+z}$. Then

$$\frac{dw}{dz} = \frac{(\sqrt{z}+z)\frac{d}{dz}z^2 - z^2\frac{d}{dz}(\sqrt{z}+z)}{(\sqrt{z}+z)^2} = \frac{2z(\sqrt{z}+z) - z^2((1/2)z^{-1/2}+1)}{(\sqrt{z}+z)^2} = \frac{(3/2)z^{3/2}+z^2}{(\sqrt{z}+z)^2}.$$

Therefore,

$$\left.\frac{dw}{dz}\right|_{z=9} = \frac{(3/2)(9)^{3/2}+9^2}{(\sqrt{9}+9)^2} = \frac{27}{32}.$$

11. $g(x) = \dfrac{1}{1+e^x}$

SOLUTION Let $g(x) = \dfrac{1}{1+e^x}$. Then

$$\frac{dg}{dx} = \frac{(1+e^x)\frac{d}{dx}1 - 1\frac{d}{dx}(1+e^x)}{(1+e^x)^2} = \frac{(1+e^x)(0)-e^x}{(1+e^x)^2} = -\frac{e^x}{(1+e^x)^2}.$$

12. $f(x) = \dfrac{e^x}{x^2+1}$

SOLUTION Let $f(x) = \dfrac{e^x}{x^2+1}$. Then

$$\frac{df}{dx} = \frac{(x^2+1)\frac{d}{dx}e^x - e^x\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}.$$

In Exercises 13–16, calculate the derivative in two ways. First use the Product or Quotient Rule; then rewrite the function algebraically and apply the Power Rule directly.

13. $f(t) = (2t+1)(t^2-2)$

SOLUTION Let $f(t) = (2t+1)(t^2-2)$. Then, using the Product Rule,

$$f'(t) = (2t+1)(2t) + (t^2-2)(2) = 6t^2+2t-4.$$

Multiplying out first, we find $f(t) = 2t^3+t^2-4t-2$. Therefore, $f'(t) = 6t^2+2t-4$.

14. $f(x) = x^2(3 + x^{-1})$

SOLUTION Let $f(x) = x^2(3 + x^{-1})$. Then, using the product rule, and then power and sum rules,

$$f'(x) = x^2(-x^{-2}) + (3 + x^{-1})(2x) = 6x + 1.$$

Multiplying out first, we find $f(x) = 3x^2 + x$. Then $f'(x) = 6x + 1$.

15. $h(t) = \dfrac{t^2 - 1}{t - 1}$

SOLUTION Let $h(t) = \frac{t^2-1}{t-1}$. Using the quotient rule,

$$f'(t) = \frac{(t-1)(2t) - (t^2-1)(1)}{(t-1)^2} = \frac{t^2 - 2t + 1}{(t-1)^2} = 1$$

for $t \neq 1$. Simplifying first, we find for $t \neq 1$,

$$h(t) = \frac{(t-1)(t+1)}{(t-1)} = t + 1.$$

Hence $h'(t) = 1$ for $t \neq 1$.

16. $g(x) = \dfrac{x^3 + 2x^2 + 3x^{-1}}{x}$

SOLUTION Let $g(x) = \frac{x^3+2x^2+3x^{-1}}{x}$. Using the quotient rule and the sum and power rules, and simplifying

$$g'(x) = \frac{x(3x^2 + 4x - 3x^{-2}) - (x^3 + 2x^2 + 3x^{-1})1}{x^2} = \frac{1}{x^2}\left(2x^3 + 2x^2 - 6x^{-1}\right) = 2x + 2 - 6x^{-3}.$$

Simplifying first yields $g(x) = x^2 + 2x + 3x^{-2}$, from which we calculate $g'(x) = 2x + 2 - 6x^{-3}$.

In Exercises 17–38, calculate the derivative.

17. $f(x) = (x^3 + 5)(x^3 + x + 1)$

SOLUTION Let $f(x) = (x^3 + 5)(x^3 + x + 1)$. Then

$$f'(x) = (x^3 + 5)(3x^2 + 1) + (x^3 + x + 1)(3x^2) = 6x^5 + 4x^3 + 18x^2 + 5.$$

18. $f(x) = (4e^x - x^2)(x^3 + 1)$

SOLUTION Let $f(x) = (4e^x - x^2)(x^3 + 1)$. Then

$$f'(x) = (4e^x - x^2)(3x^2) + (x^3 + 1)(4e^x - 2x) = e^x(4x^3 + 12x^2 + 4) - 5x^4 - 2x.$$

19. $\left.\dfrac{dy}{dx}\right|_{x=3}, \quad y = \dfrac{1}{x + 10}$

SOLUTION Let $y = \frac{1}{x+10}$. Using the quotient rule:

$$\frac{dy}{dx} = \frac{(x+10)(0) - 1(1)}{(x+10)^2} = -\frac{1}{(x+10)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{1}{(3+10)^2} = -\frac{1}{169}.$$

20. $\left.\dfrac{dz}{dx}\right|_{x=-2}, \quad z = \dfrac{x}{3x^2 + 1}$

SOLUTION Let $z = \frac{x}{3x^2+1}$. Using the quotient rule:

$$\frac{dz}{dx} = \frac{(3x^2+1)(1) - x(6x)}{(3x^2+1)^2} = \frac{1 - 3x^2}{(3x^2+1)^2}.$$

Therefore,

$$\left.\frac{dz}{dx}\right|_{x=-2} = \frac{1 - 3(-2)^2}{(3(-2)^2 + 1)^2} = -\frac{11}{169}.$$

21. $f(x) = (\sqrt{x}+1)(\sqrt{x}-1)$

SOLUTION Let $f(x) = (\sqrt{x}+1)(\sqrt{x}-1)$. Multiplying through first yields $f(x) = x - 1$ for $x \geq 0$. Therefore, $f'(x) = 1$ for $x \geq 0$. If we carry out the product rule on $f(x) = (x^{1/2}+1)(x^{1/2}-1)$, we get

$$f'(x) = (x^{1/2}+1)\left(\frac{1}{2}(x^{-1/2})\right) + (x^{1/2}-1)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2} + \frac{1}{2}x^{-1/2} + \frac{1}{2} - \frac{1}{2}x^{-1/2} = 1.$$

22. $f(x) = \dfrac{9x^{5/2}-2}{x}$

SOLUTION Let $f(x) = \frac{9x^{5/2}-2}{x} = 9x^{3/2} - 2x^{-1}$. Then $f'(x) = \frac{27}{2}x^{1/2} + 2x^{-2}$.

23. $\left.\dfrac{dy}{dx}\right|_{x=2}$, $\quad y = \dfrac{x^4-4}{x^2-5}$

SOLUTION Let $y = \dfrac{x^4-4}{x^2-5}$. Then

$$\frac{dy}{dx} = \frac{\left(x^2-5\right)\left(4x^3\right) - \left(x^4-4\right)(2x)}{\left(x^2-5\right)^2} = \frac{2x^5 - 20x^3 + 8x}{\left(x^2-5\right)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{2(2)^5 - 20(2)^3 + 8(2)}{(2^2-5)^2} = -80.$$

24. $f(x) = \dfrac{x^4+e^x}{x+1}$

SOLUTION Let $f(x) = \dfrac{x^4+e^x}{x+1}$. Then

$$\frac{df}{dx} = \frac{(x+1)(4x^3+e^x) - (x^4+e^x)(1)}{(x+1)^2} = \frac{(x+1)(4x^3+e^x) - x^4 - e^x}{(x+1)^2}.$$

25. $\left.\dfrac{dz}{dx}\right|_{x=1}$, $\quad z = \dfrac{1}{x^3+1}$

SOLUTION Let $z = \frac{1}{x^3+1}$. Using the quotient rule:

$$\frac{dz}{dx} = \frac{(x^3+1)(0) - 1(3x^2)}{(x^3+1)^2} = -\frac{3x^2}{(x^3+1)^2}.$$

Therefore,

$$\left.\frac{dz}{dx}\right|_{x=1} = -\frac{3(1)^2}{(1^3+1)^2} = -\frac{3}{4}.$$

26. $f(x) = \dfrac{3x^3 - x^2 + 2}{\sqrt{x}}$

SOLUTION Let

$$f(x) = \frac{3x^3 - x^2 + 2}{\sqrt{x}} = \frac{3x^3 - x^2 + 2}{x^{1/2}}.$$

Using the quotient rule, and then simplifying by taking out the greatest negative factor:

$$f'(x) = \frac{(x^{1/2})(9x^2 - 2x) - (3x^3 - x^2 + 2)(\frac{1}{2}x^{-1/2})}{x} = \frac{1}{x^{3/2}}\left((9x^3 - 2x^2) - \frac{1}{2}(3x^3 - x^2 + 2)\right)$$
$$= \frac{1}{x^{3/2}}\left(\frac{15}{2}x^3 - \frac{3}{2}x^2 - 1\right).$$

Alternately, since there is a single exponent of x in the denominator, we could also simplify $f(x)$ first, getting $f(x) = 3x^{5/2} - x^{3/2} + 2x^{-1/2}$. Then $f'(x) = \frac{15}{2}x^{3/2} - \frac{3}{2}x^{1/2} - x^{-3/2}$. The two answers are the same.

27. $h(t) = \dfrac{t}{(t+1)(t^2+1)}$

SOLUTION Let $h(t) = \dfrac{t}{(t+1)(t^2+1)} = \dfrac{t}{t^3+t^2+t+1}$. Then

$$h'(t) = \frac{\left(t^3+t^2+t+1\right)(1) - t\left(3t^2+2t+1\right)}{\left(t^3+t^2+t+1\right)^2} = \frac{-2t^3-t^2+1}{\left(t^3+t^2+t+1\right)^2}.$$

28. $f(x) = x^{3/2}\left(2x^4 - 3x + x^{-1/2}\right)$

SOLUTION Let $f(x) = x^{3/2}(2x^4 - 3x + x^{-1/2})$. We multiply through the $x^{3/2}$ to get $f(x) = 2x^{11/2} - 3x^{5/2} + x$. Then $f'(x) = 11x^{9/2} - \frac{15}{2}x^{3/2} + 1$.

29. $f(t) = 3^{1/2} \cdot 5^{1/2}$

SOLUTION Let $f(t) = \sqrt{3}\sqrt{5}$. Then $f'(t) = 0$, since $f(t)$ is a *constant* function!

30. $h(x) = \pi^2(x-1)$

SOLUTION Let $h(x) = \pi^2(x-1)$. Then $h'(x) = \pi^2$.

31. $f(x) = (x+3)(x-1)(x-5)$

SOLUTION Let $f(x) = (x+3)(x-1)(x-5)$. Using the Product Rule inside the Product Rule with a first factor of $(x+3)$ and a second factor of $(x-1)(x-5)$, we find

$$f'(x) = (x+3)\left((x-1)(1) + (x-5)(1)\right) + (x-1)(x-5)(1) = 3x^2 - 6x - 13.$$

Alternatively,

$$f(x) = (x+3)\left(x^2 - 6x + 5\right) = x^3 - 3x^2 - 13x + 15.$$

Therefore, $f'(x) = 3x^2 - 6x - 13$.

32. $f(x) = e^x(x^2+1)(x+4)$

SOLUTION Let $f(x) = e^x(x^2+1)(x+4)$. Using the Product Rule inside the Product Rule with a first factor of e^x and a second factor of $(x^2+1)(x+4)$, we find

$$f'(x) = e^x\left((x^2+1)(1) + (x+4)(2x)\right) + (x^2+1)(x+4)e^x = (x^3+7x^2+9x+5)e^x.$$

33. $f(x) = \dfrac{e^x}{x+1}$

SOLUTION Let $f(x) = \dfrac{e^x}{(e^x+1)(x+1)}$. Then

$$f'(x) = \frac{(e^x+1)(x+1)e^x - e^x\left((e^x+1)(1) + (x+1)e^x\right)}{(e^x+1)^2(x+1)^2} = \frac{e^x(x-e^x)}{(e^x+1)^2(x+1)^2}.$$

34. $g(x) = \dfrac{e^{x+1}+e^x}{e+1}$

SOLUTION Let

$$g(x) = \frac{e^{x+1}+e^x}{e+1} = \frac{e^x(e+1)}{e+1} = e^x.$$

Then $g'(x) = e^x$.

35. $g(z) = \left(\dfrac{z^2-4}{z-1}\right)\left(\dfrac{z^2-1}{z+2}\right)$ *Hint:* Simplify first.

SOLUTION Let

$$g(z) = \left(\frac{z^2-4}{z-1}\right)\left(\frac{z^2-1}{z+2}\right) = \left(\frac{(z+2)(z-2)}{z-1}\right)\left(\frac{(z+1)(z-1)}{z+2}\right) = (z-2)(z+1)$$

for $z \neq -2$ and $z \neq 1$. Then,

$$g'(z) = (z+1)(1) + (z-2)(1) = 2z - 1.$$

36. $\dfrac{d}{dx}\Big((ax+b)(abx^2+1)\Big)$ $\quad$ (a, b constants)

SOLUTION Let $f(x) = (ax+b)(abx^2+1)$. Then

$$f'(x) = (ax+b)(2abx) + (abx^2+1)(a) = 3a^2bx^2 + a + 2ab^2x.$$

37. $\dfrac{d}{dt}\left(\dfrac{xt-4}{t^2-x}\right)$ $\quad$ (x constant)

SOLUTION Let $f(t) = \frac{xt-4}{t^2-x}$. Using the quotient rule:

$$f'(t) = \frac{(t^2-x)(x) - (xt-4)(2t)}{(t^2-x)^2} = \frac{xt^2 - x^2 - 2xt^2 + 8t}{(t^2-x)^2} = \frac{-xt^2 + 8t - x^2}{(t^2-x)^2}.$$

38. $\dfrac{d}{dx}\left(\dfrac{ax+b}{cx+d}\right)$ $\quad$ (a, b, c, d constants)

SOLUTION Let $f(x) = \left(\dfrac{ax+b}{cx+d}\right)$. Using the quotient rule:

$$f'(x) = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{(ad-bc)}{(cx+d)^2}.$$

In Exercises 39–42, calculate the derivative using the values:

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
10	-2	5	-1

39. $(fg)'(4)$ and $(f/g)'(4)$.

SOLUTION Let $h = fg$ and $H = f/g$. Then $h' = fg' + gf'$ and $H' = \frac{gf' - fg'}{g^2}$. Finally,

$$h'(4) = f(4)g'(4) + g(4)f'(4) = (10)(-1) + (5)(-2) = -20,$$

and

$$H'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{(g(4))^2} = \frac{(5)(-2) - (10)(-1)}{(5)^2} = 0.$$

40. $F'(4)$, where $F(x) = x^2 f(x)$.

SOLUTION Let $F(x) = x^2 f(x)$. Then $F'(x) = x^2 f'(x) + 2xf(x)$, and

$$F'(4) = 16f'(4) + 8f(4) = (16)(-2) + (8)(10) = 48.$$

41. $G'(4)$, where $G(x) = g(x)^2$.

SOLUTION Let $G(x) = g(x)^2 = g(x)g(x)$. Then $G'(x) = g(x)g'(x) + g(x)g'(x) = 2g(x)g'(x)$, and

$$G'(4) = 2g(4)g'(4) = 2(5)(-1) = -10.$$

42. $H'(4)$, where $H(x) = \dfrac{x}{g(x)f(x)}$.

SOLUTION Let $H(x) = \dfrac{x}{g(x)f(x)}$. Then

$$H'(x) = \frac{g(x)f(x)\cdot 1 - x(g(x)f'(x) + f(x)g'(x))}{(g(x)f(x))^2},$$

and

$$H'(4) = \frac{(5)(10) - 4((5)(-2) + (10)(-1))}{((5)(10))^2} = \frac{13}{250}.$$

43. Calculate $F'(0)$, where

$$F(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$$

Hint: Do not calculate $F'(x)$. Instead, write $F(x) = f(x)/g(x)$ and express $F'(0)$ directly in terms of $f(0)$, $f'(0)$, $g(0)$, $g'(0)$.

SOLUTION Taking the hint, let

$$f(x) = x^9 + x^8 + 4x^5 - 7x$$

and let

$$g(x) = x^4 - 3x^2 + 2x + 1.$$

Then $F(x) = \frac{f(x)}{g(x)}$. Now,

$$f'(x) = 9x^8 + 8x^7 + 20x^4 - 7 \quad \text{and} \quad g'(x) = 4x^3 - 6x + 2.$$

Moreover, $f(0) = 0$, $f'(0) = -7$, $g(0) = 1$, and $g'(0) = 2$.

Using the quotient rule:

$$F'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2} = \frac{-7 - 0}{1} = -7.$$

44. Proceed as in Exercise 43 to calculate $F'(0)$, where

$$F(x) = \left(1 + x + x^{4/3} + x^{5/3}\right)\frac{3x^5 + 5x^4 + 5x + 1}{8x^9 - 7x^4 + 1}$$

SOLUTION Write $F(x) = f(x)(g(x)/h(x))$, where

$$f(x) = (1 + x + x^{4/3} + x^{5/3})$$
$$g(x) = 3x^5 + 5x^4 + 5x + 1$$

and

$$h(x) = 8x^9 - 7x^4 + 1.$$

Now, $f'(x) = 1 + \frac{4}{3}x^{\frac{1}{3}} + \frac{5}{3}x^{\frac{2}{3}}$, $g'(x) = 15x^4 + 20x^3 + 5$, and $h'(x) = 72x^8 - 28x^3$. Moreover, $f(0) = 1$, $f'(0) = 1$, $g(0) = 1$, $g'(0) = 5$, $h(0) = 1$, and $h'(0) = 0$. From the product and quotient rules,

$$F'(0) = f(0)\frac{h(0)g'(0) - g(0)h'(0)}{h(0)^2} + f'(0)(g(0)/h(0)) = 1\frac{1(5) - 1(0)}{1} + 1(1/1) = 6.$$

45. Use the Product Rule to calculate $\frac{d}{dx}e^{2x}$.

SOLUTION Note that $e^{2x} = e^x \cdot e^x$. Therefore

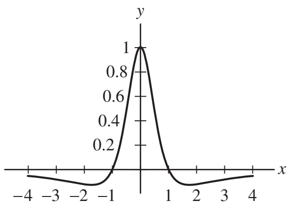
$$\frac{d}{dx}e^{2x} = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}.$$

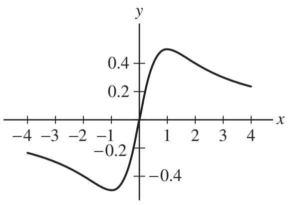
46. $\boxed{\text{GU}}$ Plot the derivative of $f(x) = x/(x^2 + 1)$ over $[-4, 4]$. Use the graph to determine the intervals on which $f'(x) > 0$ and $f'(x) < 0$. Then plot $f(x)$ and describe how the sign of $f'(x)$ is reflected in the graph of $f(x)$.

SOLUTION Let $f(x) = \dfrac{x}{x^2 + 1}$. Then

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The derivative is shown in the figure below at the left. From this plot we see that $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ for $|x| > 1$. The original function is plotted in the figure below at the right. Observe that the graph of $f(x)$ is increasing whenever $f'(x) > 0$ and that $f(x)$ is decreasing whenever $f'(x) < 0$.



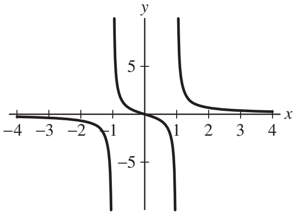


47. GU Plot $f(x) = x/(x^2 - 1)$ (in a suitably bounded viewing box). Use the plot to determine whether $f'(x)$ is positive or negative on its domain $\{x : x \neq \pm 1\}$. Then compute $f'(x)$ and confirm your conclusion algebraically.

SOLUTION Let $f(x) = \dfrac{x}{x^2 - 1}$. The graph of $f(x)$ is shown below. From this plot, we see that $f(x)$ is decreasing on its domain $\{x : x \neq \pm 1\}$. Consequently, $f'(x)$ must be negative. Using the quotient rule, we find

$$f'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2},$$

which is negative for all $x \neq \pm 1$.



48. Let $P = V^2 R/(R + r)^2$ as in Example 7. Calculate dP/dr, assuming that r is variable and R is constant.

SOLUTION Note that V is also constant. Let

$$f(r) = \frac{V^2 R}{(R + r)^2} = \frac{V^2 R}{R^2 + 2Rr + r^2}.$$

Using the quotient rule:

$$f'(r) = \frac{(R^2 + 2Rr + r^2)(0) - (V^2 R)(2R + 2r)}{(R + r)^4} = -\frac{2V^2 R(R + r)}{(R + r)^4} = -\frac{2V^2 R}{(R + r)^3}.$$

49. Find $a > 0$ such that the tangent line to the graph of

$$f(x) = x^2 e^{-x} \quad \text{at } x = a$$

passes through the origin (Figure 4).

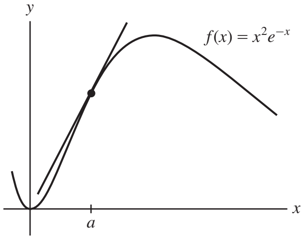


FIGURE 4

SOLUTION Let $f(x) = x^2 e^{-x}$. Then $f(a) = a^2 e^{-a}$,

$$f'(x) = -x^2 e^{-x} + 2xe^{-x} = e^{-x}(2x - x^2),$$

$f'(a) = (2a - a^2)e^{-a}$, and the equation of the tangent line to f at $x = a$ is

$$y = f'(a)(x - a) + f(a) = (2a - a^2)e^{-a}(x - a) + a^2 e^{-a}.$$

For this line to pass through the origin, we must have

$$0 = (2a - a^2)e^{-a}(-a) + a^2 e^{-a} = e^{-a}\left(a^2 - 2a^2 + a^3\right) = a^2 e^{-a}(a - 1).$$

Thus, $a = 0$ or $a = 1$. The only value $a > 0$ such that the tangent line to $f(x) = x^2 e^{-x}$ passes through the origin is therefore $a = 1$.

50. Current I (amperes), voltage V (volts), and resistance R (ohms) in a circuit are related by Ohm's Law, $I = V/R$.

(a) Calculate $\left.\frac{dI}{dR}\right|_{R=6}$ if V is constant with value $V = 24$.

(b) Calculate $\left.\frac{dV}{dR}\right|_{R=6}$ if I is constant with value $I = 4$.

SOLUTION

(a) According to Ohm's Law, $I = V/R = VR^{-1}$. Thus, using the power rule,

$$\frac{dI}{dR} = -VR^{-2}.$$

With $V = 24$ volts, it follows that

$$\left.\frac{dI}{dR}\right|_{R=6} = -24(6)^{-2} = -\frac{2}{3}\frac{\text{amps}}{\Omega}.$$

(b) Solving Ohm's Law for V yields $V = RI$. Thus

$$\frac{dV}{dR} = I \qquad \text{and} \qquad \left.\frac{dV}{dR}\right|_{I=4} = 4 \text{ amps}.$$

51. The revenue per month earned by the Couture clothing chain at time t is $R(t) = N(t)S(t)$, where $N(t)$ is the number of stores and $S(t)$ is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of 5 stores per month, and (B) to use advertising to increase average revenue per store at a rate of \$10,000 per month. Assume that $N(0) = 50$ and $S(0) = \$150{,}000$.

(a) Show that total revenue will increase at the rate

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t)$$

Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores on the one hand, and the average revenue per store on the other.

(b) Calculate $\left.\frac{dR}{dt}\right|_{t=0}$.

(c) If Couture can implement only one leg (A or B) of its expansion at $t = 0$, which choice will grow revenue most rapidly?

SOLUTION

(a) Given $R(t) = N(t)S(t)$, it follows that

$$\frac{dR}{dt} = N(t)S'(t) + S(t)N'(t).$$

We are told that $N'(t) = 5$ stores per month and $S'(t) = 10{,}000$ dollars per month. Therefore,

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t).$$

(b) Using part (a) and the given values of $N(0)$ and $S(0)$, we find

$$\left.\frac{dR}{dt}\right|_{t=0} = 5(150{,}000) + 10{,}000(50) = 1{,}250{,}000.$$

(c) From part (b), we see that of the two terms contributing to total revenue growth, the term $5S(0)$ is larger than the term $10{,}000N(0)$. Thus, if only one leg of the campaign can be implemented, it should be part A: increase the number of stores by 5 per month.

52. The **tip speed ratio** of a turbine (Figure 5) is the ratio $R = T/W$, where T is the speed of the tip of a blade and W is the speed of the wind. (Engineers have found empirically that a turbine with n blades extracts maximum power from the wind when $R = 2\pi/n$.) Calculate dR/dt (t in minutes) if $W = 35$ km/h and W decreases at a rate of 4 km/h per minute, and the tip speed has constant value $T = 150$ km/h.

FIGURE 5 Turbines on a wind farm

SOLUTION Let $R = T/W$. Then

$$\frac{dR}{dt} = \frac{WT' - TW'}{W^2}.$$

Using the values $T = 150$, $T' = 0$, $W = 35$ and $W' = -4$, we find

$$\frac{dR}{dt} = \frac{(35)(0) - 150(-4)}{35^2} = \frac{24}{49}.$$

53. The curve $y = 1/(x^2 + 1)$ is called the *witch of Agnesi* (Figure 6) after the Italian mathematician Maria Agnesi (1718–1799), who wrote one of the first books on calculus. This strange name is the result of a mistranslation of the Italian word *la versiera*, meaning "that which turns." Find equations of the tangent lines at $x = \pm 1$.

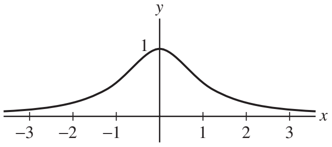


FIGURE 6 The witch of Agnesi.

SOLUTION Let $f(x) = \dfrac{1}{x^2 + 1}$. Then $f'(x) = \dfrac{(x^2 + 1)(0) - 1(2x)}{(x^2 + 1)^2} = -\dfrac{2x}{\left(x^2 + 1\right)^2}$.

- At $x = -1$, the tangent line is

$$y = f'(-1)(x + 1) + f(-1) = \frac{1}{2}(x + 1) + \frac{1}{2} = \frac{1}{2}x + 1.$$

- At $x = 1$, the tangent line is

$$y = f'(1)(x - 1) + f(1) = -\frac{1}{2}(x - 1) + \frac{1}{2} = -\frac{1}{2}x + 1.$$

54. Let $f(x) = g(x) = x$. Show that $(f/g)' \neq f'/g'$.

SOLUTION $(f/g) = (x/x) = 1$, so $(f/g)' = 0$. On the other hand, $(f'/g') = (x'/x') = (1/1) = 1$. We see that $0 \neq 1$.

55. Use the Product Rule to show that $(f^2)' = 2ff'$.

SOLUTION Let $g = f^2 = ff$. Then $g' = \left(f^2\right)' = (ff)' = ff' + ff' = 2ff'$.

56. Show that $(f^3)' = 3f^2 f'$.

SOLUTION Let $g = f^3 = fff$. Then

$$g' = \left(f^3\right)' = [f\,(ff)]' = f\left(ff' + ff'\right) + ff(f') = 3f^2 f'.$$

Further Insights and Challenges

57. Let f, g, h be differentiable functions. Show that $(fgh)'(x)$ is equal to

$$f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

Hint: Write fgh as $f(gh)$.

SOLUTION Let $p = fgh$. Then

$$p' = (fgh)' = f\left(gh' + hg'\right) + ghf' = f'gh + fg'h + fgh'.$$

58. Prove the Quotient Rule using the limit definition of the derivative.

SOLUTION Let $p = \dfrac{f}{g}$. Suppose that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Then

$$p'(a) = \lim_{h\to 0} \frac{p(a+h) - p(a)}{h} = \lim_{h\to 0} \frac{\dfrac{f(a+h)}{g(a+h)} - \dfrac{f(a)}{g(a)}}{h} = \lim_{h\to 0} \frac{\dfrac{f(a+h)g(a) - f(a)g(a+h)}{g(a+h)g(a)}}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(a+h)}{hg(a+h)g(a)}$$

$$= \lim_{h\to 0} \left(\frac{1}{g(a+h)g(a)} \left(g(a)\,\frac{f(a+h) - f(a)}{h} - f(a)\,\frac{g(a+h) - g(a)}{h} \right)\right)$$

$$= \left(\lim_{h\to 0} \frac{1}{g(a+h)g(a)}\right)\left(\left(g(a)\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right) - \left(f(a)\lim_{h\to 0} \frac{g(a+h) - g(a)}{h}\right)\right)$$

$$= \frac{1}{(g(a))^2}\left(g(a)f'(a) - f(a)g'(a)\right) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}$$

In other words, $p' = \left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$.

59. Derivative of the Reciprocal Use the limit definition to prove

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)} \qquad \boxed{7}$$

Hint: Show that the difference quotient for $1/f(x)$ is equal to

$$\frac{f(x) - f(x+h)}{hf(x)f(x+h)}$$

SOLUTION Let $g(x) = \frac{1}{f(x)}$. We then compute the derivative of $g(x)$ using the difference quotient:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x) - f(x+h)}{f(x)f(x+h)}\right)$$

$$= -\lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right)\left(\frac{1}{f(x)f(x+h)}\right).$$

We can apply the rule of products for limits. The first parenthetical expression is the difference quotient definition of $f'(x)$. The second can be evaluated at $h = 0$ to give $\frac{1}{(f(x))^2}$. Hence

$$g'(x) = \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}.$$

60. Prove the Quotient Rule using Eq. (7) and the Product Rule.

SOLUTION Let $h(x) = \frac{f(x)}{g(x)}$. We can write $h(x) = f(x)\frac{1}{g(x)}$. Applying Eq. (7),

$$h'(x) = f(x)\left(\left(\frac{1}{g(x)}\right)'\right) + f'(x)\left(\frac{1}{g(x)}\right) = -f(x)\left(\frac{g'(x)}{(g(x))^2}\right) + \frac{f'(x)}{g(x)} = \frac{-f(x)g'(x) + f'(x)g(x)}{(g(x))^2}.$$

61. Use the limit definition of the derivative to prove the following special case of the Product Rule:

$$\frac{d}{dx}(xf(x)) = xf'(x) + f(x)$$

SOLUTION First note that because $f(x)$ is differentiable, it is also continuous. It follows that

$$\lim_{h\to 0} f(x+h) = f(x).$$

Now we tackle the derivative:

$$\begin{aligned}\frac{d}{dx}(xf(x)) &= \lim_{h\to 0}\frac{(x+h)f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(x\frac{f(x+h)-f(x)}{h} + f(x+h)\right)\\ &= x\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x+h)\\ &= xf'(x) + f(x).\end{aligned}$$

62. Carry out Maria Agnesi's proof of the Quotient Rule from her book on calculus, published in 1748: Assume that f, g, and $h = f/g$ are differentiable. Compute the derivative of $hg = f$ using the Product Rule, and solve for h'.

SOLUTION Suppose that f, g, and h are differentiable functions with $h = f/g$.

- Then $hg = f$ and via the product rule $hg' + gh' = f'$.
- Solving for h' yields $h' = \dfrac{f' - hg'}{g} = \dfrac{f' - \frac{f}{g}g'}{g} = \dfrac{gf' - fg'}{g^2}$.

63. The Power Rule Revisited If you are familiar with *proof by induction*, use induction to prove the Power Rule for all whole numbers n. Show that the Power Rule holds for $n = 1$; then write x^n as $x \cdot x^{n-1}$ and use the Product Rule.

SOLUTION Let k be a positive integer. If $k = 1$, then $x^k = x$. Note that

$$\frac{d}{dx}\left(x^1\right) = \frac{d}{dx}(x) = 1 = 1x^0.$$

Hence the Power Rule holds for $k = 1$. Assume it holds for $k = n$ where $n \geq 2$. Then for $k = n+1$, we have

$$\begin{aligned}\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{n+1}\right) &= \frac{d}{dx}\left(x\cdot x^n\right) = x\frac{d}{dx}\left(x^n\right) + x^n\frac{d}{dx}(x)\\ &= x\cdot nx^{n-1} + x^n \cdot 1 = (n+1)x^n = kx^{k-1}\end{aligned}$$

Accordingly, the Power Rule holds for all positive integers by induction.

Exercises 64 and 65: A basic fact of algebra states that c is a root of a polynomial $f(x)$ if and only if $f(x) = (x-c)g(x)$ for some polynomial $g(x)$. We say that c is a ***multiple root*** *if $f(x) = (x-c)^2h(x)$, where $h(x)$ is a polynomial.*

64. Show that c is a multiple root of $f(x)$ if and only if c is a root of both $f(x)$ and $f'(x)$.

SOLUTION Assume first that $f(c) = f'(c) = 0$ and let us show that c is a multiple root of $f(x)$. We have $f(x) = (x-c)g(x)$ for some polynomial $g(x)$ and so $f'(x) = (x-c)g'(x) + g(x)$. However, $f'(c) = 0 + g(c) = 0$, so c is also a root of $g(x)$ and hence $g(x) = (x-c)h(x)$ for some polynomial $h(x)$. We conclude that $f(x) = (x-c)^2h(x)$, which shows that c is a multiple root of $f(x)$.

Conversely, assume that c is a multiple root. Then $f(c) = 0$ and $f(x) = (x-c)^2g(x)$ for some polynomial $g(x)$. Then $f'(x) = (x-c)^2g'(x) + 2g(x)(x-c)$. Therefore, $f'(c) = (c-c)^2g'(c) + 2g(c)(c-c) = 0$.

65. Use Exercise 64 to determine whether $c = -1$ is a multiple root:

(a) $x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2$

(b) $x^4 + x^3 - 5x^2 - 3x + 2$

SOLUTION

(a) To show that -1 is a multiple root of

$$f(x) = x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2,$$

it suffices to check that $f(-1) = f'(-1) = 0$. We have $f(-1) = -1 + 2 + 4 - 8 + 1 + 2 = 0$ and

$$\begin{aligned}f'(x) &= 5x^4 + 8x^3 - 12x^2 - 16x - 1\\ f'(-1) &= 5 - 8 - 12 + 16 - 1 = 0\end{aligned}$$

(b) Let $f(x) = x^4 + x^3 - 5x^2 - 3x + 2$. Then $f'(x) = 4x^3 + 3x^2 - 10x - 3$. Because

$$f(-1) = 1 - 1 - 5 + 3 + 2 = 0$$

but

$$f'(-1) = -4 + 3 + 10 - 3 = 6 \neq 0,$$

it follows that $x = -1$ is a root of f, but not a multiple root.

66. Figure 7 is the graph of a polynomial with roots at A, B, and C. Which of these is a multiple root? Explain your reasoning using Exercise 64.

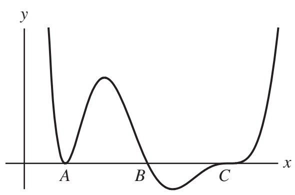


FIGURE 7

SOLUTION A on the figure is a multiple root. It is a multiple root because $f(x) = 0$ at A and because the tangent line to the graph at A is horizontal, so that $f'(x) = 0$ at A. For the same reasons, f also has a multiple root at C.

67. According to Eq. (6) in Section 3.2, $\frac{d}{dx}b^x = m(b)\,b^x$. Use the Product Rule to show that $m(ab) = m(a) + m(b)$.

SOLUTION

$$m(ab)(ab)^x = \frac{d}{dx}(ab)^x = \frac{d}{dx}\left(a^x b^x\right) = a^x \frac{d}{dx}b^x + b^x \frac{d}{dx}a^x = m(b)a^x b^x + m(a)a^x b^x = (m(a) + m(b))(ab)^x.$$

Thus, $m(ab) = m(a) + m(b)$.

3.4 Rates of Change

Preliminary Questions

1. Which units might be used for each rate of change?

(a) Pressure (in atmospheres) in a water tank with respect to depth

(b) The rate of a chemical reaction (change in concentration with respect to time with concentration in moles per liter)

SOLUTION

(a) The rate of change of pressure with respect to depth might be measured in atmospheres/meter.

(b) The reaction rate of a chemical reaction might be measured in moles/(liter·hour).

2. Two trains travel from New Orleans to Memphis in 4 hours. The first train travels at a constant velocity of 90 mph, but the velocity of the second train varies. What was the second train's average velocity during the trip?

SOLUTION Since both trains travel the same distance in the same amount of time, they have the same average velocity: 90 mph.

3. Estimate $f(26)$, assuming that $f(25) = 43$, $f'(25) = 0.75$.

SOLUTION $f(x) \approx f(25) + f'(25)(x - 25)$, so $f(26) \approx 43 + 0.75(26 - 25) = 43.75$.

4. The population $P(t)$ of Freedonia in 2009 was $P(2009) = 5$ million.

(a) What is the meaning of $P'(2009)$?

(b) Estimate $P(2010)$ if $P'(2009) = 0.2$.

SOLUTION

(a) Because $P(t)$ measures the population of Freedonia as a function of time, the derivative $P'(2009)$ measures the rate of change of the population of Freedonia in the year 2009.

(b) $P(2010) \approx P(2009) + P'(2010)$. Thus, if $P'(2009) = 0.2$, then $P(2009) \approx 5.2$ million.

Exercises

In Exercises 1–8, find the rate of change.

1. Area of a square with respect to its side s when $s = 5$.

SOLUTION Let the area be $A = f(s) = s^2$. Then the rate of change of A with respect to s is $d/ds(s^2) = 2s$. When $s = 5$, the area changes at a rate of 10 square units per unit increase. (Draw a 5×5 square on graph paper and trace the area added by increasing each side length by 1, excluding the corner, to see what this means.)

2. Volume of a cube with respect to its side s when $s = 5$.

SOLUTION Let the volume be $V = f(s) = s^3$. Then the rate of change of V with respect to s is $\frac{d}{ds}s^3 = 3s^2$. When $s = 5$, the volume changes at a rate of $3(5^2) = 75$ cubic units per unit increase.

3. Cube root $\sqrt[3]{x}$ with respect to x when $x = 1, 8, 27$.

SOLUTION Let $f(x) = \sqrt[3]{x}$. Writing $f(x) = x^{1/3}$, we see the rate of change of $f(x)$ with respect to x is given by $f'(x) = \frac{1}{3}x^{-2/3}$. The requested rates of change are given in the table that follows:

c	ROC of $f(x)$ with respect to x at $x = c$.
1	$f'(1) = \frac{1}{3}(1) = \frac{1}{3}$
8	$f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{3}(\frac{1}{4}) = \frac{1}{12}$
27	$f'(27) = \frac{1}{3}(27^{-2/3}) = \frac{1}{3}(\frac{1}{9}) = \frac{1}{27}$

4. The reciprocal $1/x$ with respect to x when $x = 1, 2, 3$.

SOLUTION Let $f(x) = x^{-1}$. The rate of change of $f(x)$ with respect to x is given by $f'(x) = -x^{-2}$. The requested rates of change are then -1 when $x = 1$, $-\frac{1}{4}$ when $x = 2$ and $-\frac{1}{9}$ when $x = 3$.

5. The diameter of a circle with respect to radius.

SOLUTION The relationship between the diameter d of a circle and its radius r is $d = 2r$. The rate of change of the diameter with respect to the radius is then $d' = 2$.

6. Surface area A of a sphere with respect to radius r ($A = 4\pi r^2$).

SOLUTION Because $A = 4\pi r^2$, the rate of change of the surface area of a sphere with respect to the radius is $A' = 8\pi r$.

7. Volume V of a cylinder with respect to radius if the height is equal to the radius.

SOLUTION The volume of the cylinder is $V = \pi r^2 h = \pi r^3$. Thus $dV/dr = 3\pi r^2$.

8. Speed of sound v (in m/s) with respect to air temperature T (in kelvins), where $v = 20\sqrt{T}$.

SOLUTION Because, $v = 20\sqrt{T} = 20T^{1/2}$, the rate of change of the speed of sound with respect to temperature is $v' = 10T^{-1/2} = \frac{10}{\sqrt{T}}$.

In Exercises 9–11, refer to Figure 10, the graph of distance $s(t)$ from the origin as a function of time for a car trip.

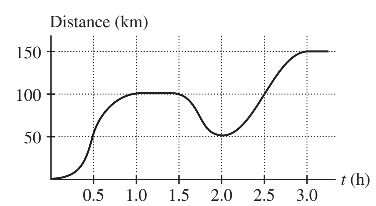


FIGURE 10 Distance from the origin versus time for a car trip.

9. Find the average velocity over each interval.

(a) [0, 0.5] **(b)** [0.5, 1] **(c)** [1, 1.5] **(d)** [1, 2]

SOLUTION

(a) The average velocity over the interval [0, 0.5] is

$$\frac{50 - 0}{0.5 - 0} = 100 \text{ km/hour}.$$

(b) The average velocity over the interval [0.5, 1] is

$$\frac{100-50}{1-0.5} = 100 \text{ km/hour.}$$

(c) The average velocity over the interval [1, 1.5] is

$$\frac{100-100}{1.5-1} = 0 \text{ km/hour.}$$

(d) The average velocity over the interval [1, 2] is

$$\frac{50-100}{2-1} = -50 \text{ km/hour.}$$

10. At what time is velocity at a maximum?

SOLUTION The velocity is maximum when the slope of the distance versus time curve is most positive. This appears to happen when $t = 0.5$ hours.

11. Match the descriptions (i)–(iii) with the intervals (a)–(c).

(i) Velocity increasing
(ii) Velocity decreasing
(iii) Velocity negative
(a) [0, 0.5]
(b) [2.5, 3]
(c) [1.5, 2]

SOLUTION

(a) **(i)** : The distance curve is increasing, and is also *bending* upward, so that distance is increasing at an increasing rate.

(b) **(ii)** : Over the interval [2.5, 3], the distance curve is flattening, showing that the car is slowing down; that is, the velocity is decreasing.

(c) **(iii)** : The distance curve is decreasing, so the tangent line has negative slope; this means the velocity is negative.

12. Use the data from Table 1 in Example 1 to calculate the average rate of change of Martian temperature T with respect to time t over the interval from 8:36 AM to 9:34 AM.

SOLUTION The time interval from 8:36 AM to 9:34 AM has length 58 minutes, and the change in temperature over this time interval is

$$\Delta T = -42 - (-47.7) = 5.7^\circ\text{C}.$$

The average rate of change is then

$$\frac{\Delta T}{\Delta t} = \frac{5.7}{58} \approx 0.0983^\circ\text{C/min} = 5.897^\circ\text{C/hr}.$$

13. Use Figure 3 from Example 1 to estimate the instantaneous rate of change of Martian temperature with respect to time (in degrees Celsius per hour) at $t = 4$ AM.

SOLUTION The segment of the temperature graph around $t = 4$ AM appears to be a straight line passing through roughly (1:36, -70) and (4:48, -75). The instantaneous rate of change of Martian temperature with respect to time at $t = 4$ AM is therefore approximately

$$\frac{dT}{dt} = \frac{-75-(-70)}{3.2} = -1.5625^\circ\text{C/hour}.$$

14. The temperature (in °C) of an object at time t (in minutes) is $T(t) = \frac{3}{8}t^2 - 15t + 180$ for $0 \le t \le 20$. At what rate is the object cooling at $t = 10$? (Give correct units.)

SOLUTION Given $T(t) = \frac{3}{8}t^2 - 15t + 180$, it follows that

$$T'(t) = \frac{3}{4}t - 15 \quad \text{and} \quad T'(10) = \frac{3}{4}(10) - 15 = -7.5^\circ\text{C/min}.$$

At $t = 10$, the object is cooling at the rate of 7.5°C/min.

15. The velocity (in cm/s) of blood molecules flowing through a capillary of radius 0.008 cm is $v = 6.4 \times 10^{-8} - 0.001r^2$, where r is the distance from the molecule to the center of the capillary. Find the rate of change of velocity with respect to r when $r = 0.004$ cm.

SOLUTION The rate of change of the velocity of the blood molecules is $v'(r) = -0.002r$. When $r = 0.004$ cm, this rate is -8×10^{-6} 1/s.

16. Figure 11 displays the voltage V across a capacitor as a function of time while the capacitor is being charged. Estimate the rate of change of voltage at $t = 20$ s. Indicate the values in your calculation and include proper units. Does voltage change more quickly or more slowly as time goes on? Explain in terms of tangent lines.

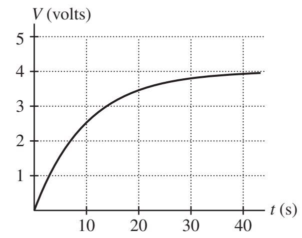
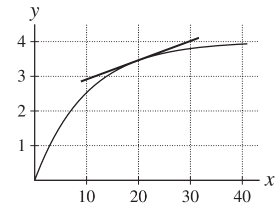


FIGURE 11

SOLUTION The tangent line sketched in the figure below appears to pass through the points (10, 3) and (30, 4). Thus, the rate of change of voltage at $t = 20$ seconds is approximately

$$\frac{4-3}{30-10} = 0.05 \text{ V/s}.$$

As we move to the right of the graph, the tangent lines to it grow shallower, indicating that the voltage changes more slowly as time goes on.



17. Use Figure 12 to estimate dT/dh at $h = 30$ and 70, where T is atmospheric temperature (in degrees Celsius) and h is altitude (in kilometers). Where is dT/dh equal to zero?

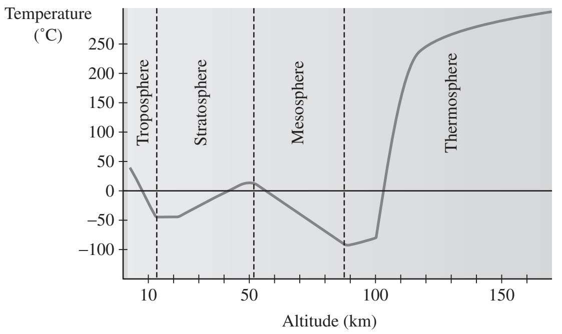


FIGURE 12 Atmospheric temperature versus altitude.

SOLUTION At $h = 30$ km, the graph of atmospheric temperature appears to be linear passing through the points $(23, -50)$ and (40, 0). The slope of this segment of the graph is then

$$\frac{0-(-50)}{40-23} = \frac{50}{17} = 2.94;$$

so

$$\left.\frac{dT}{dh}\right|_{h=30} \approx 2.94°\text{C/km}.$$

At $h = 70$ km, the graph of atmospheric temperature appears to be linear passing through the points (58, 0) and $(88, -100)$. The slope of this segment of the graph is then

$$\frac{-100-0}{88-58} = \frac{-100}{30} = -3.33;$$

so

$$\left.\frac{dT}{dh}\right|_{h=70} \approx -3.33°\text{C/km}.$$

$\frac{dT}{dh} = 0$ at those points where the tangent line on the graph is horizontal. This appears to happen over the interval [13, 23], and near the points $h = 50$ and $h = 90$.

18. The earth exerts a gravitational force of $F(r) = (2.99 \times 10^{16})/r^2$ newtons on an object with a mass of 75 kg located r meters from the center of the earth. Find the rate of change of force with respect to distance r at the surface of the earth.

SOLUTION The rate of change of force is $F'(r) = -5.98 \times 10^{16}/r^3$. Therefore,

$$F'(6.77 \times 10^6) = -5.98 \times 10^{16}/(6.77 \times 10^6)^3 = -1.93 \times 10^{-4} \text{ N/m}.$$

19. Calculate the rate of change of escape velocity $v_{\text{esc}} = (2.82 \times 10^7)r^{-1/2}$ m/s with respect to distance r from the center of the earth.

SOLUTION The rate that escape velocity changes is $v'_{\text{esc}}(r) = -1.41 \times 10^7 r^{-3/2}$.

20. The power delivered by a battery to an apparatus of resistance R (in ohms) is $P = 2.25R/(R + 0.5)^2$ watts. Find the rate of change of power with respect to resistance for $R = 3\ \Omega$ and $R = 5\ \Omega$.

SOLUTION

$$P'(R) = \frac{(R + 0.5)^2 2.25 - 2.25R(2R + 1)}{(R + 0.5)^4}.$$

Therefore, $P'(3) = -0.1312$ W/Ω and $P'(5) = -0.0609$ W/Ω.

21. The position of a particle moving in a straight line during a 5-s trip is $s(t) = t^2 - t + 10$ cm. Find a time t at which the instantaneous velocity is equal to the average velocity for the entire trip.

SOLUTION Let $s(t) = t^2 - t + 10$, $0 \le t \le 5$, with s in centimeters (cm) and t in seconds (s). The average velocity over the t-interval [0, 5] is

$$\frac{s(5) - s(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ cm/s}.$$

The (instantaneous) velocity is $v(t) = s'(t) = 2t - 1$. Solving $2t - 1 = 4$ yields $t = \frac{5}{2}$ s, the time at which the instantaneous velocity equals the calculated average velocity.

22. The height (in meters) of a helicopter at time t (in minutes) is $s(t) = 600t - 3t^3$ for $0 \le t \le 12$.

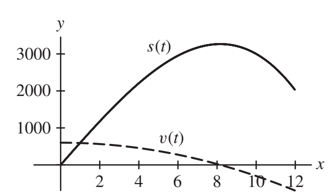
(a) Plot $s(t)$ and velocity $v(t)$.

(b) Find the velocity at $t = 8$ and $t = 10$.

(c) Find the maximum height of the helicopter.

SOLUTION

(a) With $s(t) = 600t - 3t^3$, it follows that $v(t) = 600 - 9t^2$. Plots of the position and the velocity are shown below.



(b) From part (a), we have $v(t) = 600 - 9t^2$. Thus, $v'(8) = 24$ meters/minute and $v'(10) = -300$ meters/minute.

(c) From the graph in part (a), we see that the helicopter achieves its maximum height when the velocity is zero. Solving $600 - 9t^2 = 0$ for t yields

$$t = \sqrt{\frac{600}{9}} = \frac{10}{3}\sqrt{6} \text{ minutes}.$$

The maximum height of the helicopter is then

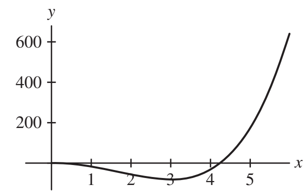
$$s\left(\frac{10}{3}\sqrt{6}\right) = \frac{4000}{3}\sqrt{6} \approx 3266 \text{ meters}.$$

23. A particle moving along a line has position $s(t) = t^4 - 18t^2$ m at time t seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (that is, it has zero velocity)?

SOLUTION The particle passes through the origin when $s(t) = t^4 - 18t^2 = t^2(t^2 - 18) = 0$. This happens when $t = 0$ seconds and when $t = 3\sqrt{2} \approx 4.24$ seconds. With $s(t) = t^4 - 18t^2$, it follows that $v(t) = s'(t) = 4t^3 - 36t = 4t(t^2 - 9)$. The particle is therefore instantaneously motionless when $t = 0$ seconds and when $t = 3$ seconds.

24. $\boxed{\text{GU}}$ Plot the position of the particle in Exercise 23. What is the farthest distance to the left of the origin attained by the particle?

SOLUTION The plot of the position of the particle in Exercise 23 is shown below. Positive values of position correspond to distance to the right of the origin and negative values correspond to distance to the left of the origin. The most negative value of $s(t)$ occurs at $t = 3$ and is equal to $s(3) = 3^4 - 18(3)^2 = -81$. Thus, the particle achieves a maximum distance to the left of the origin of 81 meters.



25. A bullet is fired in the air vertically from ground level with an initial velocity 200 m/s. Find the bullet's maximum velocity and maximum height.

SOLUTION We employ Galileo's formula, $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = 200t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). The velocity is $v(t) = 200 - 9.8t$. The maximum velocity of 200 m/s occurs at $t = 0$. This is the initial velocity. The bullet reaches its maximum height when $v(t) = 200 - 9.8t = 0$; i.e., when $t \approx 20.41$ s. At this point, the height is 2040.82 m.

26. Find the velocity of an object dropped from a height of 300 m at the moment it hits the ground.

SOLUTION We employ Galileo's formula, $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = 300 - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground its height is 0. Solve $s(t) = 300 - 4.9t^2 = 0$ to obtain $t \approx 7.8246$ s. (We discard the negative time, which took place before the ball was dropped.) The velocity at impact is $v(7.8246) = -9.8(7.8246) \approx -76.68$ m/s. This signifies that the ball is *falling* at 76.68 m/s.

27. A ball tossed in the air vertically from ground level returns to earth 4 s later. Find the initial velocity and maximum height of the ball.

SOLUTION Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = v_0 t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground after 4 seconds its height is 0. Solve $0 = s(4) = 4v_0 - 4.9(4)^2$ to obtain $v_0 = 19.6$ m/s. The ball reaches its maximum height when $s'(t) = 0$, that is, when $19.6 - 9.8t = 0$, or $t = 2$ s. At this time, $t = 2$ s,

$$s(2) = 0 + 19.6(2) - \frac{1}{2}(9.8)(4) = 19.6 \text{ m}.$$

28. Olivia is gazing out a window from the tenth floor of a building when a bucket (dropped by a window washer) passes by. She notes that it hits the ground 1.5 s later. Determine the floor from which the bucket was dropped if each floor is 5 m high and the window is in the middle of the tenth floor. Neglect air friction.

SOLUTION Suppose H is the unknown height from which the bucket fell starting at time $t = 0$. The height of the bucket at time t is $s(t) = H - 4.9t^2$. Let T be the time when the bucket hits the ground (thus $S(T) = 0$). Olivia saw the bucket at time $T - 1.5$. The window is located 9.5 floors or 47.5 m above ground. So we have the equations

$$s(T - 1.5) = H - 4.9(T - 1.5)^2 = 47.5 \quad \text{and} \quad s(T) = H - 4.9T^2 = 0$$

Subtracting the second equation from the first, we obtain $-4.9(-3T + 2.25) = 47.5$, so $T \approx 4$ s. The second equation gives us $H = 4.9T^2 = 4.9(4)^2 \approx 78.4$ m. Since there are 5 m in a floor, the bucket was dropped $78.4/5 \approx 15.7$ floors above the ground. The bucket was dropped from the top of the 15th floor.

29. Show that for an object falling according to Galileo's formula, the average velocity over any time interval $[t_1, t_2]$ is equal to the average of the instantaneous velocities at t_1 and t_2.

SOLUTION The simplest way to proceed is to compute both values and show that they are equal. The average velocity over $[t_1, t_2]$ is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{(s_0 + v_0 t_2 - \frac{1}{2}gt_2^2) - (s_0 + v_0 t_1 - \frac{1}{2}gt_1^2)}{t_2 - t_1} = \frac{v_0(t_2 - t_1) + \frac{g}{2}(t_2{}^2 - t_1{}^2)}{t_2 - t_1}$$

$$= \frac{v_0(t_2 - t_1)}{t_2 - t_1} - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1)$$

Whereas the average of the instantaneous velocities at the beginning and end of $[t_1, t_2]$ is

$$\frac{s'(t_1) + s'(t_2)}{2} = \frac{1}{2}\Big((v_0 - gt_1) + (v_0 - gt_2)\Big) = \frac{1}{2}(2v_0) - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1).$$

The two quantities are the same.

30. An object falls under the influence of gravity near the earth's surface. Which of the following statements is true? Explain.

(a) Distance traveled increases by equal amounts in equal time intervals.
(b) Velocity increases by equal amounts in equal time intervals.
(c) The derivative of velocity increases with time.

SOLUTION For an object falling under the influence of gravity, Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$.

(a) Since the height of the object varies quadratically with respect to time, it is *not* true that the object covers equal distance in equal time intervals.

(b) The velocity is $v(t) = s'(t) = v_0 - gt$. The velocity varies linearly with respect to time. Accordingly, the velocity decreases (becomes more negative) by equal amounts in equal time intervals. Moreover, its *speed* (the magnitude of velocity) increases by equal amounts in equal time intervals.

(c) Acceleration, the derivative of velocity with respect to time, is given by $a(t) = v'(t) = -g$. This is a *constant*; it does not change with time. Hence it is *not* true that acceleration (the derivative of velocity) increases with time.

31. By Faraday's Law, if a conducting wire of length ℓ meters moves at velocity v m/s perpendicular to a magnetic field of strength B (in teslas), a voltage of size $V = -B\ell v$ is induced in the wire. Assume that $B = 2$ and $\ell = 0.5$.

(a) Calculate dV/dv.
(b) Find the rate of change of V with respect to time t if $v = 4t + 9$.

SOLUTION

(a) Assuming that $B = 2$ and $l = 0.5$, $V = -2(.5)v = -v$. Therefore,

$$\frac{dV}{dv} = -1.$$

(b) If $v = 4t + 9$, then $V = -2(.5)(4t + 9) = -(4t + 9)$. Therefore, $\frac{dV}{dt} = -4$.

32. The voltage V, current I, and resistance R in a circuit are related by Ohm's Law: $V = IR$, where the units are volts, amperes, and ohms. Assume that voltage is constant with $V = 12$ volts. Calculate (specifying units):

(a) The average rate of change of I with respect to R for the interval from $R = 8$ to $R = 8.1$
(b) The rate of change of I with respect to R when $R = 8$
(c) The rate of change of R with respect to I when $I = 1.5$

SOLUTION Let $V = IR$ or $I = V/R = 12/R$ (since we are assuming $V = 12$ volts).

(a) The average rate of change is

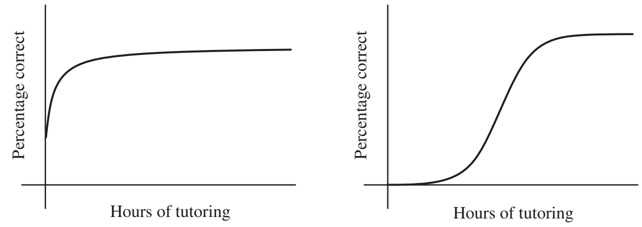
$$\frac{\Delta I}{\Delta R} = \frac{I(8.1) - I(8)}{8.1 - 8} = \frac{\frac{12}{8.1} - \frac{12}{8}}{0.1} \approx -0.185\ \text{A}/\Omega.$$

(b) $dI/dR = -12/R^2 = -12/8^2 = -0.1875\ \text{A}/\Omega$.
(c) With $R = 12/I$, we have $dR/dI = -12/I^2 = -12/1.5^2 \approx -5.33\ \Omega/\text{A}$.

33. Ethan finds that with h hours of tutoring, he is able to answer correctly $S(h)$ percent of the problems on a math exam. Which would you expect to be larger: $S'(3)$ or $S'(30)$? Explain.

SOLUTION One possible graph of $S(h)$ is shown in the figure below on the left. This graph indicates that in the early hours of working with the tutor, Ethan makes rapid progress in learning the material but eventually approaches either the limit of his ability to learn the material or the maximum possible score on the exam. In this scenario, $S'(3)$ would be larger than $S'(30)$.

An alternative graph of $S(h)$ is shown below on the right. Here, in the early hours of working with the tutor little progress is made (perhaps the tutor is assessing how much Ethan already knows, his learning style, his personality, etc.). This is followed by a period of rapid improvement and finally a leveling off as Ethan reaches his maximum score. In this scenario, $S'(3)$ and $S'(30)$ might be roughly equal.



34. Suppose $\theta(t)$ measures the angle between a clock's minute and hour hands. What is $\theta'(t)$ at 3 o'clock?

SOLUTION The minute hand makes one full revolution every 60 minutes, so the minute hand moves at a rate of

$$\frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/min.}$$

The hour hand makes one-twelfth of a revolution every 60 minutes, so the hour hand moves with a rate of

$$\frac{\pi}{360} \text{ rad/min.}$$

At 3 o'clock, the movement of the minute hand works to decrease the angle between the minute and hour hands while the movement of the hour hand works to increase the angle. Therefore, at 3 o'clock,

$$\theta'(t) = \frac{\pi}{360} - \frac{\pi}{30} = -\frac{11\pi}{360} \text{ rad/min.}$$

35. To determine drug dosages, doctors estimate a person's body surface area (BSA) (in meters squared) using the formula $\text{BSA} = \sqrt{hm}/60$, where h is the height in centimeters and m the mass in kilograms. Calculate the rate of change of BSA with respect to mass for a person of constant height $h = 180$. What is this rate at $m = 70$ and $m = 80$? Express your result in the correct units. Does BSA increase more rapidly with respect to mass at lower or higher body mass?

SOLUTION Assuming constant height $h = 180$ cm, let $f(m) = \sqrt{hm}/60 = \frac{\sqrt{5}}{10}m$ be the formula for body surface area in terms of weight. The rate of change of BSA with respect to mass is

$$f'(m) = \frac{\sqrt{5}}{10}\left(\frac{1}{2}m^{-1/2}\right) = \frac{\sqrt{5}}{20\sqrt{m}}.$$

If $m = 70$ kg, this is

$$f'(70) = \frac{\sqrt{5}}{20\sqrt{70}} = \frac{\sqrt{14}}{280} \approx 0.0133631\frac{\text{m}^2}{\text{kg}}.$$

If $m = 80$ kg,

$$f'(80) = \frac{\sqrt{5}}{20\sqrt{80}} = \frac{1}{20\sqrt{16}} = \frac{1}{80}\frac{\text{m}^2}{\text{kg}}.$$

Because the rate of change of BSA depends on $1/\sqrt{m}$, it is clear that BSA increases more rapidly at lower body mass.

36. The atmospheric CO_2 level $A(t)$ at Mauna Loa, Hawaii at time t (in parts per million by volume) is recorded by the Scripps Institution of Oceanography. The values for the months January–December 2007 were

382.45, 383.68, 384.23, 386.26, 386.39, 385.87,
384.39, 381.78, 380.73, 380.81, 382.33, 383.69

(a) Assuming that the measurements were made on the first of each month, estimate $A'(t)$ on the 15th of the months January–November.

(b) In which months did $A'(t)$ take on its largest and smallest values?

(c) In which month was the CO_2 level most nearly constant?

SOLUTION

(a) The rate of change in the atmospheric CO_2 level on the 15th of each month can be estimated using the monthly differences $A(n) - A(n-1)$ for $2 \le n \le 12$. The estimates we obtain are:

$$1.23, 0.55, 2.03, 0.13, -0.52, -1.48, -2.61, -1.05, 0.08, 1.52, 1.36$$

t	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov
$P'(t)$	1.23	0.55	2.03	0.13	−0.52	−1.48	−2.61	−1.05	0.08	1.52	1.36

(b) According to the table in part (a), the maximum rate of change occurs in March and the minimum rate is in July.

(c) According to the table in part (a), the CO_2 level is most nearly constant in September.

37. The tangent lines to the graph of $f(x) = x^2$ grow steeper as x increases. At what rate do the slopes of the tangent lines increase?

SOLUTION Let $f(x) = x^2$. The slopes s of the tangent lines are given by $s = f'(x) = 2x$. The rate at which these slopes are increasing is $ds/dx = 2$.

38. Figure 13 shows the height y of a mass oscillating at the end of a spring. through one cycle of the oscillation. Sketch the graph of velocity as a function of time.

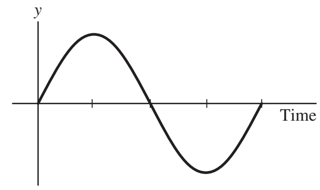
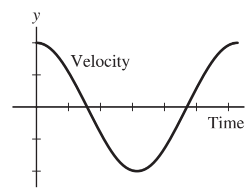


FIGURE 13

SOLUTION The position graph appears to break into four equal-sized components. Over the first quarter of the time interval, the position graph is rising but bending downward, eventually reaching a horizontal tangent. Thus, over the first quarter of the time interval, the velocity is positive but decreasing, eventually reaching 0. Continuing to examine the structure of the position graph produces the following graph of velocity:



In Exercises 39–46, use Eq. (3) to estimate the unit change.

39. Estimate $\sqrt{2} - \sqrt{1}$ and $\sqrt{101} - \sqrt{100}$. Compare your estimates with the actual values.

SOLUTION Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = \frac{1}{2}(x^{-1/2})$. We are using the derivative to estimate the average rate of change. That is,

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \approx f'(x),$$

so that

$$\sqrt{x+h} - \sqrt{x} \approx hf'(x).$$

Thus, $\sqrt{2} - \sqrt{1} \approx 1f'(1) = \frac{1}{2}(1) = \frac{1}{2}$. The actual value, to six decimal places, is 0.414214. Also, $\sqrt{101} - \sqrt{100} \approx 1f'(100) = \frac{1}{2}\left(\frac{1}{10}\right) = .05$. The actual value, to six decimal places, is 0.0498756.

40. Estimate $f(4) - f(3)$ if $f'(x) = 2^{-x}$. Then estimate $f(4)$, assuming that $f(3) = 12$.

SOLUTION Using the estimate that

$$\frac{f(x+h) - f(x)}{h} \approx f'(x),$$

so that $f(x+h) - f(x) \approx f'(x)h$ with $x = 3$, $h = 1$, we get

$$f(4) - f(3) \approx 2^{-3}(1) = \frac{1}{8}.$$

If $f(3) = 12$, then $f(4) \approx 12\frac{1}{8} = \frac{97}{8}$.

41. Let $F(s) = 1.1s + 0.05s^2$ be the stopping distance as in Example 3. Calculate $F(65)$ and estimate the increase in stopping distance if speed is increased from 65 to 66 mph. Compare your estimate with the actual increase.

SOLUTION Let $F(s) = 1.1s + .05s^2$ be as in Example 3. $F'(s) = 1.1 + 0.1s$.

- Then $F(65) = 282.75$ ft and $F'(65) = 7.6$ ft/mph.
- $F'(65) \approx F(66) - F(65)$ is approximately equal to the change in stopping distance per 1 mph increase in speed when traveling at 65 mph. Increasing speed from 65 to 66 therefore increases stopping distance by approximately 7.6 ft.
- The actual increase in stopping distance when speed increases from 65 mph to 66 mph is $F(66) - F(65) = 290.4 - 282.75 = 7.65$ feet, which differs by less than one percent from the estimate found using the derivative.

42. According to Kleiber's Law, the metabolic rate P (in kilocalories per day) and body mass m (in kilograms) of an animal are related by a *three-quarter-power law* $P = 73.3m^{3/4}$. Estimate the increase in metabolic rate when body mass increases from 60 to 61 kg.

SOLUTION Let $P(m) = 73.3m^{3/4}$ be the function relating body mass m to metabolic rate P. Then,

$$P'(m) = \frac{3}{4}(73.3)m^{-1/4} = 54.975m^{-1/4}$$

$$P(61) - P(60) \approx P'(60) = 54.975(60^{-1/4}) = 19.7527.$$

As body mass is increased from 60 to 61 kg, metabolic rate is increased by approximately 19.7527 kcal/day.

43. The dollar cost of producing x bagels is $C(x) = 300 + 0.25x - 0.5(x/1000)^3$. Determine the cost of producing 2000 bagels and estimate the cost of the 2001st bagel. Compare your estimate with the actual cost of the 2001st bagel.

SOLUTION Expanding the power of 3 yields

$$C(x) = 300 + 0.25x - 5 \times 10^{-10}x^3.$$

This allows us to get the derivative $C'(x) = 0.25 - 1.5 \times 10^{-9}x^2$. The cost of producing 2000 bagels is

$$C(2000) = 300 + 0.25(2000) - 0.5(2000/1000)^3 = 796$$

dollars. The cost of the 2001st bagel is, by definition, $C(2001) - C(2000)$. By the derivative estimate, $C(2001) - C(2000) \approx C'(2000)(1)$, so the cost of the 2001st bagel is approximately

$$C'(2000) = 0.25 - 1.5 \times 10^{-9}(2000^2) = \$0.244.$$

$C(2001) = 796.244$, so the *exact* cost of the 2001st bagel is indistinguishable from the estimated cost. The function is very nearly linear at this point.

44. Suppose the dollar cost of producing x video cameras is $C(x) = 500x - 0.003x^2 + 10^{-8}x^3$.

(a) Estimate the marginal cost at production level $x = 5000$ and compare it with the actual cost $C(5001) - C(5000)$.

(b) Compare the marginal cost at $x = 5000$ with the average cost per camera, defined as $C(x)/x$.

SOLUTION Let $C(x) = 500x - 0.003x^2 + 10^{-8}x^3$. Then

$$C'(x) = 500 - 0.006x + (3 \times 10^{-8})x^2.$$

(a) The cost difference is approximately $C'(5000) = 470.75$. The actual cost is $C(5001) - C(5000) = 470.747$, which is quite close to the marginal cost computed using the derivative.

(b) The average cost per camera is

$$\frac{C(5000)}{5000} = \frac{2426250}{5000} = 485.25,$$

which is slightly higher than the marginal cost.

45. Demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is $D(p) = 900/p$ barrels, where p is the dollar price per barrel. Find the demand when $p = \$40$. Estimate the decrease in demand if p rises to \$41 and the increase if p declines to \$39.

SOLUTION $D(p) = 900p^{-1}$, so $D'(p) = -900p^{-2}$. When the price is \$40 a barrel, the per capita demand is $D(40) = 22.5$ barrels per year. With an increase in price from \$40 to \$41 a barrel, the change in demand $D(41) - D(40)$ is approximately $D'(40) = -900(40^{-2}) = -0.5625$ barrels a year. With a decrease in price from \$40 to \$39 a barrel, the change in demand $D(39) - D(40)$ is approximately $-D'(40) = +0.5625$. An increase in oil prices of a dollar leads to a decrease in demand of 0.5625 barrels a year, and a decrease of a dollar leads to an *increase* in demand of 0.5625 barrels a year.

46. The reproduction rate f of the fruit fly *Drosophila melanogaster*, grown in bottles in a laboratory, decreases with the number p of flies in the bottle. A researcher has found the number of offspring per female per day to be approximately $f(p) = (34 - 0.612p)p^{-0.658}$.

(a) Calculate $f(15)$ and $f'(15)$.

(b) Estimate the decrease in daily offspring per female when p is increased from 15 to 16. Is this estimate larger or smaller than the actual value $f(16) - f(15)$?

(c) $\boxed{\text{GU}}$ Plot $f(p)$ for $5 \le p \le 25$ and verify that $f(p)$ is a decreasing function of p. Do you expect $f'(p)$ to be positive or negative? Plot $f'(p)$ and confirm your expectation.

SOLUTION Let

$$f(p) = (34 - 0.612p)p^{-0.658} = 34p^{-0.658} - 0.612p^{0.342}.$$

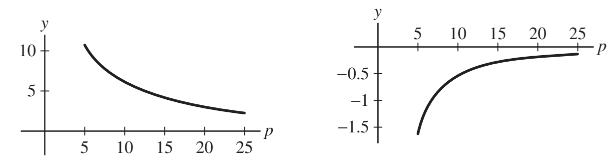
Then

$$f'(p) = -22.372p^{-1.658} - 0.209304p^{-0.658}.$$

(a) $f(15) = 34(15)^{-0.658} - 0.612(15)^{0.342} \approx 4.17767$ offspring per female per day and $f'(15) = -22.372(15)^{-1.658} - 0.209304(15)^{-0.658} \approx -0.28627$ offspring per female per day per fly.

(b) $f(16) - f(15) \approx f'(15) \approx -0.28627$. The decrease in daily offspring per female is estimated at 0.28627. $f(16) - f(15) = -0.272424$. The actual decrease in daily offspring per female is 0.272424. The actual decrease in daily offspring per female is less than the estimated decrease. This is because the graph of the function bends towards the x axis.

(c) The function $f(p)$ is plotted below at the left and is clearly a decreasing function of p; we therefore expect that $f'(p)$ will be negative. The plot of the derivative shown below at the right confirms our expectation.



47. According to Stevens' Law in psychology, the perceived magnitude of a stimulus is proportional (approximately) to a power of the actual intensity I of the stimulus. Experiments show that the *perceived brightness* B of a light satisfies $B = kI^{2/3}$, where I is the light intensity, whereas the *perceived heaviness* H of a weight W satisfies $H = kW^{3/2}$ (k is a constant that is different in the two cases). Compute dB/dI and dH/dW and state whether they are increasing or decreasing functions. Then explain the following statements:

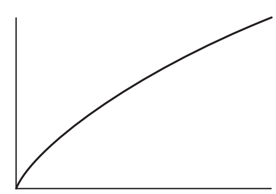
(a) A one-unit increase in light intensity is felt more strongly when I is small than when I is large.

(b) Adding another pound to a load W is felt more strongly when W is large than when W is small.

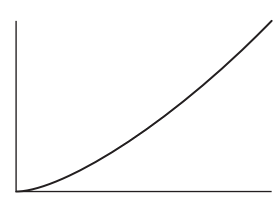
SOLUTION

(a) $dB/dI = \frac{2k}{3} I^{-1/3} = \frac{2k}{3I^{1/3}}$.

As I increases, dB/dI shrinks, so that the rate of change of perceived intensity decreases as the actual intensity increases. Increased light intensity has a *diminished return* in perceived intensity. A sketch of B against I is shown: See that the height of the graph increases more slowly as you move to the right.



(b) $dH/dW = \frac{3k}{2} W^{1/2}$. As W increases, dH/dW increases as well, so that the rate of change of perceived weight increases as weight increases. A sketch of H against W is shown: See that the graph becomes steeper as you move to the right.



48. Let $M(t)$ be the mass (in kilograms) of a plant as a function of time (in years). Recent studies by Niklas and Enquist have suggested that a remarkably wide range of plants (from algae and grass to palm trees) obey a *three-quarter-power growth law*—that is, $dM/dt = CM^{3/4}$ for some constant C.

(a) If a tree has a growth rate of 6 kg/yr when $M = 100$ kg, what is its growth rate when $M = 125$ kg?

(b) If $M = 0.5$ kg, how much more mass must the plant acquire to double its growth rate?

SOLUTION

(a) Suppose a tree has a growth rate dM/dt of 6 kg/yr when $M = 100$, then $6 = C(100^{3/4}) = 10C\sqrt{10}$, so that $C = \frac{3\sqrt{10}}{50}$. When $M = 125$,

$$\frac{dM}{dt} = C(125^{3/4}) = \frac{3\sqrt{10}}{50} 25(5^{1/4}) = 7.09306.$$

(b) The growth rate when $M = 0.5$ kg is $dM/dt = C(0.5^{3/4})$. To double the rate, we must find M so that $dM/dt = CM^{3/4} = 2C(0.5^{3/4})$. We solve for M.

$$CM^{3/4} = 2C(0.5^{3/4})$$

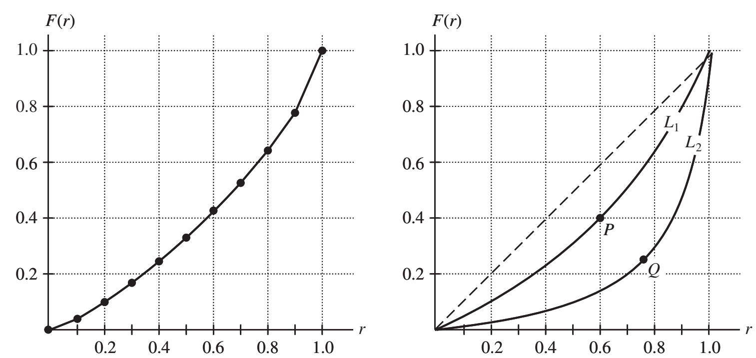
$$M^{3/4} = 2(0.5^{3/4})$$
$$M = (2(0.5^{3/4}))^{4/3} = 1.25992.$$

The plant must acquire the difference $1.25992 - 0.5 = 0.75992$ kg in order to double its growth rate.

Note that a doubling of growth rate requires *more* than a doubling of mass.

Further Insights and Challenges

Exercises 49–51: The ***Lorenz curve*** $y = F(r)$ *is used by economists to study income distribution in a given country (see Figure 14). By definition,* $F(r)$ *is the fraction of the total income that goes to the bottom* r*th part of the population, where* $0 \le r \le 1$*. For example, if* $F(0.4) = 0.245$*, then the bottom* 40% *of households receive* 24.5% *of the total income. Note that* $F(0) = 0$ *and* $F(1) = 1$.



(A) Lorenz curve for Sweden in 2004

(B) Two Lorenz curves: The tangent lines at P and Q have slope 1.

FIGURE 14

49. Our goal is to find an interpretation for $F'(r)$. The average income for a group of households is the total income going to the group divided by the number of households in the group. The national average income is $A = T/N$, where N is the total number of households and T is the total income earned by the entire population.

(a) Show that the average income among households in the bottom rth part is equal to $(F(r)/r)A$.

(b) Show more generally that the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\left(\frac{F(r + \Delta r) - F(r)}{\Delta r}\right) A$$

(c) Let $0 \le r \le 1$. A household belongs to the $100r$th percentile if its income is greater than or equal to the income of $100r$ % of all households. Pass to the limit as $\Delta r \to 0$ in (b) to derive the following interpretation: A household in the $100r$th percentile has income $F'(r)A$. In particular, a household in the $100r$th percentile receives more than the national average if $F'(r) > 1$ and less if $F'(r) < 1$.

(d) For the Lorenz curves L_1 and L_2 in Figure 14(B), what percentage of households have above-average income?

SOLUTION

(a) The total income among households in the bottom rth part is $F(r)T$ and there are rN households in this part of the population. Thus, the average income among households in the bottom rth part is equal to

$$\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r}A.$$

(b) Consider the interval $[r, r + \Delta r]$. The total income among households between the bottom rth part and the bottom $r + \Delta r$-th part is $F(r + \Delta r)T - F(r)T$. Moreover, the number of households covered by this interval is $(r + \Delta r)N - rN = \Delta r N$. Thus, the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\frac{F(r + \Delta r)T - F(r)T}{\Delta r N} = \frac{F(r + \Delta r) - F(r)}{\Delta r} \cdot \frac{T}{N} = \frac{F(r + \Delta r) - F(r)}{\Delta r}A.$$

(c) Take the result from part (b) and let $\Delta r \to 0$. Because

$$\lim_{\Delta r \to 0} \frac{F(r + \Delta r) - F(r)}{\Delta r} = F'(r),$$

we find that a household in the $100r$th percentile has income $F'(r)A$.

(d) The point P in Figure 14(B) has an r-coordinate of 0.6, while the point Q has an r-coordinate of roughly 0.75. Thus, on curve L_1, 40% of households have $F'(r) > 1$ and therefore have above-average income. On curve L_2, roughly 25% of households have above-average income.

50. The following table provides values of $F(r)$ for Sweden in 2004. Assume that the national average income was $A = 30{,}000$ euros.

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.01	0.245	0.423	0.642	1

(a) What was the average income in the lowest 40% of households?

(b) Show that the average income of the households belonging to the interval [0.4, 0.6] was 26,700 euros.

(c) Estimate $F'(0.5)$. Estimate the income of households in the 50th percentile? Was it greater or less than the national average?

SOLUTION

(a) The average income in the lowest 40% of households is $F'(0.4)A = 0.245(30{,}000) = 7350$ euros.

(b) The average income of the households belonging to the interval [0.4, 0.6] is

$$\frac{F(0.6) - F(0.4)}{0.2}A = \frac{0.423 - 0.245}{0.2}(30{,}000) = (0.89)(30{,}000) = 26700$$

euros.

(c) We estimate

$$F'(0.5) \approx \frac{F(0.6) - F(0.4)}{0.2} = \frac{0.423 - 0.245}{0.2} = 0.89.$$

The income of households in the 50th percentile is then $F'(0.5)A = 0.89(30{,}000) = 26{,}700$ euros, which is less than the national average.

51. Use Exercise 49 (c) to prove:

(a) $F'(r)$ is an increasing function of r.

(b) Income is distributed equally (all households have the same income) if and only if $F(r) = r$ for $0 \le r \le 1$.

SOLUTION

(a) Recall from Exercise 49 (c) that $F'(r)A$ is the income of a household in the $100r$-th percentile. Suppose $0 \le r_1 < r_2 \le 1$. Because $r_2 > r_1$, a household in the $100r_2$-th percentile must have income at least as large as a household in the $100r_1$-th percentile. Thus, $F'(r_2)A \ge F'(r_1)A$, or $F'(r_2) \ge F'(r_1)$. This implies $F'(r)$ is an increasing function of r.

(b) If $F(r) = r$ for $0 \le r \le 1$, then $F'(r) = 1$ and households in all percentiles have income equal to the national average; that is, income is distributed equally. Alternately, if income is distributed equally (all households have the same income), then $F'(r) = 1$ for $0 \le r \le 1$. Thus, F must be a linear function in r with slope 1. Moreover, the condition $F(0) = 0$ requires the F intercept of the line to be 0. Hence, $F(r) = 1 \cdot r + 0 = r$.

52. CAS Studies of Internet usage show that website popularity is described quite well by Zipf's Law, according to which the nth most popular website receives roughly the fraction $1/n$ of all visits. Suppose that on a particular day, the nth most popular site had approximately $V(n) = 10^6/n$ visitors (for $n \le 15{,}000$).

(a) Verify that the top 50 websites received nearly 45% of the visits. *Hint:* Let $T(N)$ denote the sum of $V(n)$ for $1 \le n \le N$. Use a computer algebra system to compute $T(45)$ and $T(15{,}000)$.

(b) Verify, by numerical experimentation, that when Eq. (3) is used to estimate $V(n+1) - V(n)$, the error in the estimate decreases as n grows larger. Find (again, by experimentation) an N such that the error is at most 10 for $n \ge N$.

(c) Using Eq. (3), show that for $n \ge 100$, the nth website received at most 100 more visitors than the $(n+1)$st website.

SOLUTION

(a) In Mathematica, using the command `Sum[10^6/n,{n,50}]` yields 4.49921×10^6 and the command `Sum[10^6/n,{n,15000}]` yields 1.01931×10^7. We see that the first 50 sites get around 4.4 million hits, nearly half the 10.19 million hits of the first 15000 sites.

(b) We use `V[n_] := 10^6/n`, and compute the error $V(n+1) - V(n) - V'(n)$ for various values of n. The table of values computed follows:

n	10	20	30	40	50
$(V(n+1) - V(n)) - V'(n)$	909.091	119.048	35.8423	15.2489	7.84314

The error decreases in every entry. Furthermore, for $n > 50$, the error appears to be less than 10.

(c) Since $V(n) = 10^6 n^{-1}$, $V'(n) = -10^6 n^{-2}$. The marginal derivative estimate Eq. (3) tells us that

$$V(n) - V(n+1) \approx -V'(n) = 10^6 n^{-2}.$$

If $n \ge 100$, $-V'(n) \le 10^6(100)^{-2} = 10^6(10^{-4}) = 100$. Therefore $V(n) - V(n+1) < 100$ for $n \ge 100$.

In Exercises 53 and 54, the average cost per unit at production level x is defined as $C_{\text{avg}}(x) = C(x)/x$, where $C(x)$ is the cost function. Average cost is a measure of the efficiency of the production process.

53. Show that $C_{\text{avg}}(x)$ is equal to the slope of the line through the origin and the point $(x, C(x))$ on the graph of $C(x)$. Using this interpretation, determine whether average cost or marginal cost is greater at points A, B, C, D in Figure 15.

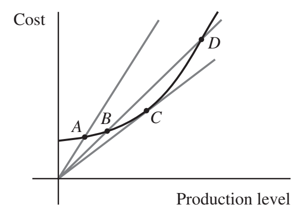


FIGURE 15 Graph of $C(x)$.

SOLUTION By definition, the slope of the line through the origin and $(x, C(x))$, that is, between $(0, 0)$ and $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{\text{av}}.$$

At point A, average cost is greater than marginal cost, as the line from the origin to A is steeper than the curve at this point (we see this because the line, tracing from the origin, crosses the curve from below). At point B, the average cost is still greater than the marginal cost. At the point C, the average cost and the marginal cost are nearly the same, since the tangent line and the line from the origin are nearly the same. The line from the origin to D crosses the cost curve from above, and so is less steep than the tangent line to the curve at D; the average cost at this point is less than the marginal cost.

54. The cost in dollars of producing alarm clocks is $C(x) = 50x^3 - 750x^2 + 3740x + 3750$ where x is in units of 1000.

(a) Calculate the average cost at $x = 4, 6, 8$, and 10.

(b) Use the graphical interpretation of average cost to find the production level x_0 at which average cost is lowest. What is the relation between average cost and marginal cost at x_0 (see Figure 16)?

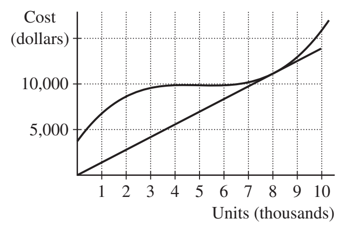


FIGURE 16 Cost function $C(x) = 50x^3 - 750x^2 + 3740x + 3750$.

SOLUTION Let $C(x) = 50x^3 - 750x^2 + 3740x + 3750$.

(a) The slope of the line through the origin and the point $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{\text{av}}(x),$$

the average cost.

x	4	6	8	10
$C(x)$	9910	9990	11270	16150
$C_{\text{av}}(x)$	2477.5	1665	1408.75	1615

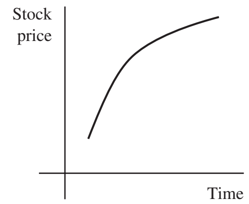
(b) The average cost is lowest at the point P_0 where the angle between the x-axis and the line through the origin and P_0 is lowest. This is at the point $(8, 11270)$, where the line through the origin and the graph of $C(x)$ meet in the figure above. You can see that the line is also tangent to the graph of $C(x)$, so the average cost and the marginal cost are equal at this point.

3.5 Higher Derivatives

Preliminary Questions

1. On September 4, 2003, the *Wall Street Journal* printed the headline "Stocks Go Higher, Though the Pace of Their Gains Slows." Rephrase this headline as a statement about the first and second time derivatives of stock prices and sketch a possible graph.

SOLUTION Because stocks are going higher, stock prices are increasing and the first derivative of stock prices must therefore be positive. On the other hand, because the pace of gains is slowing, the second derivative of stock prices must be negative.



2. True or false? The third derivative of position with respect to time is zero for an object falling to earth under the influence of gravity. Explain.

SOLUTION This statement is true. The acceleration of an object falling to earth under the influence of gravity is constant; hence, the second derivative of position with respect to time is constant. Because the third derivative is just the derivative of the second derivative and the derivative of a constant is zero, it follows that the third derivative is zero.

3. Which type of polynomial satisfies $f'''(x) = 0$ for all x?

SOLUTION The third derivative of all quadratic polynomials (polynomials of the form $ax^2 + bx + c$ for some constants a, b and c) is equal to 0 for all x.

4. What is the millionth derivative of $f(x) = e^x$?

SOLUTION Every derivative of $f(x) = e^x$ is e^x.

Exercises

In Exercises 1–16, calculate y'' and y'''.

1. $y = 14x^2$

SOLUTION Let $y = 14x^2$. Then $y' = 28x$, $y'' = 28$, and $y''' = 0$.

2. $y = 7 - 2x$

SOLUTION Let $y = 7 - 2x$. Then $y' = -2$, $y'' = 0$, and $y''' = 0$.

3. $y = x^4 - 25x^2 + 2x$

SOLUTION Let $y = x^4 - 25x^2 + 2x$. Then $y' = 4x^3 - 50x + 2$, $y'' = 12x^2 - 50$, and $y''' = 24x$.

4. $y = 4t^3 - 9t^2 + 7$

SOLUTION Let $y = 4t^3 - 9t^2 + 7$. Then $y' = 12t^2 - 18t$, $y'' = 24t - 18$, and $y''' = 24$.

5. $y = \dfrac{4}{3}\pi r^3$

SOLUTION Let $y = \frac{4}{3}\pi r^3$. Then $y' = 4\pi r^2$, $y'' = 8\pi r$, and $y''' = 8\pi$.

6. $y = \sqrt{x}$

SOLUTION Let $y = \sqrt{x} = x^{1/2}$. Then $y' = \frac{1}{2}x^{-1/2}$, $y'' = -\frac{1}{4}x^{-3/2}$, and $y''' = \frac{3}{8}x^{-5/2}$.

7. $y = 20t^{4/5} - 6t^{2/3}$

SOLUTION Let $y = 20t^{4/5} - 6t^{2/3}$. Then $y' = 16t^{-1/5} - 4t^{-1/3}$, $y'' = -\frac{16}{5}t^{-6/5} + \frac{4}{3}t^{-4/3}$, and $y''' = \frac{96}{25}t^{-11/15} - \frac{16}{9}t^{-7/3}$.

8. $y = x^{-9/5}$

SOLUTION Let $y = x^{-9/5}$. Then $y' = -\frac{9}{5}x^{-14/5}$, $y'' = \frac{126}{25}x^{-19/5}$, and $y''' = -\frac{2394}{125}x^{-24/5}$.

9. $y = z - \dfrac{4}{z}$

SOLUTION Let $y = z - 4z^{-1}$. Then $y' = 1 + 4z^{-2}$, $y'' = -8z^{-3}$, and $y''' = 24z^{-4}$.

10. $y = 5t^{-3} + 7t^{-8/3}$

SOLUTION Let $y = 5t^{-3} + 7t^{-8/3}$. Then $y' = -15t^{-4} - \frac{56}{3}t^{-11/3}$, $y'' = 60t^{-5} + \frac{616}{9}t^{-14/3}$, and $y''' = -300t^{-6} - \frac{8624}{27}t^{-17/3}$.

11. $y = \theta^2(2\theta + 7)$

SOLUTION Let $y = \theta^2(2\theta + 7) = 2\theta^3 + 7\theta^2$. Then $y' = 6\theta^2 + 14\theta$, $y'' = 12\theta + 14$, and $y''' = 12$.

12. $y = (x^2 + x)(x^3 + 1)$

SOLUTION Since we don't want to apply the product rule to an ever growing list of products, we multiply through first. Let $y = (x^2 + x)(x^3 + 1) = x^5 + x^4 + x^2 + x$. Then $y' = 5x^4 + 4x^3 + 2x + 1$, $y'' = 20x^3 + 12x^2 + 2$, and $y''' = 60x^2 + 24x$.

13. $y = \dfrac{x - 4}{x}$

SOLUTION Let $y = \frac{x-4}{x} = 1 - 4x^{-1}$. Then $y' = 4x^{-2}$, $y'' = -8x^{-3}$, and $y''' = 24x^{-4}$.

14. $y = \dfrac{1}{1 - x}$

SOLUTION Let $y = \frac{1}{1-x}$. Applying the quotient rule:

$$y' = \frac{(1-x)(0) - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} = \frac{1}{1 - 2x + x^2}$$

$$y'' = \frac{(1 - 2x + x^2)(0) - (1)(-2 + 2x)}{(1 - 2x + x^2)^2} = \frac{2 - 2x}{(1-x)^4} = \frac{2}{(1-x)^3} = \frac{2}{1 - 3x + 3x^2 - x^3}$$

$$y''' = \frac{(1 - 3x + 3x^2 - x^3)(0) - 2(-3 + 6x - 3x^2)}{(1 - 3x + 3x^2 - x^3)^2} = \frac{6(x^2 - 2x + 1)}{(1-x)^6} = \frac{6}{(1-x)^4}.$$

15. $y = x^5 e^x$

SOLUTION Let $y = x^5 e^x$. Then

$$\begin{aligned} y' &= x^5 e^x + 5x^4 e^x = (x^5 + 5x^4)e^x \\ y'' &= (x^5 + 5x^4)e^x + (5x^4 + 20x^3)e^x = (x^5 + 10x^4 + 20x^3)e^x \\ y''' &= (x^5 + 10x^4 + 20x^3)e^x + (5x^4 + 40x^3 + 60x^2)e^x = (x^5 + 15x^4 + 60x^3 + 60x^2)e^x. \end{aligned}$$

16. $y = \dfrac{e^x}{x}$

SOLUTION Let $y = \frac{e^x}{x} = x^{-1}e^x$. Then

$$\begin{aligned} y' &= x^{-1}e^x + e^x(-x^{-2}) = (x^{-1} - x^{-2})e^x \\ y'' &= (x^{-1} - x^{-2})e^x + e^x(-x^{-2} + 2x^{-3}) = (x^{-1} - 2x^{-2} + 2x^{-3})e^x \\ y''' &= (x^{-1} - 2x^{-2} + 2x^{-3})e^x + e^x(-x^{-2} + 4x^{-3} - 6x^{-4}) = (x^{-1} - 3x^{-2} + 6x^{-3} - 6x^{-4})e^x. \end{aligned}$$

In Exercises 17–26, calculate the derivative indicated.

17. $f^{(4)}(1)$, $\quad f(x) = x^4$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'''(x) = 24x$, and $f^{(4)}(x) = 24$. Thus $f^{(4)}(1) = 24$.

18. $g'''(-1)$, $\quad g(t) = -4t^{-5}$

SOLUTION Let $g(t) = -4t^{-5}$. Then $g'(t) = 20t^{-6}$, $g''(t) = -120t^{-7}$, and $g'''(t) = 840t^{-8}$. Hence $g'''(-1) = 840$.

19. $\left.\dfrac{d^2y}{dt^2}\right|_{t=1}$, $\quad y = 4t^{-3} + 3t^2$

SOLUTION Let $y = 4t^{-3} + 3t^2$. Then $\frac{dy}{dt} = -12t^{-4} + 6t$ and $\frac{d^2y}{dt^2} = 48t^{-5} + 6$. Hence

$$\left.\frac{d^2y}{dt^2}\right|_{t=1} = 48(1)^{-5} + 6 = 54.$$

20. $\left.\dfrac{d^4 f}{dt^4}\right|_{t=1}$, $\quad f(t) = 6t^9 - 2t^5$

SOLUTION Let $f(t) = 6t^9 - 2t^5$. Then $\frac{df}{dt} = 54t^8 - 10t^4$, $\frac{d^2 f}{dt^2} = 432t^7 - 40t^3$, $\frac{d^3 f}{dt^3} = 3024t^6 - 120t^2$, and $\frac{d^4 f}{dt^4} = 18144t^5 - 240t$. Therefore,

$$\left.\frac{d^4 f}{dt^4}\right|_{t=1} = 17904.$$

21. $\left.\dfrac{d^4 x}{dt^4}\right|_{t=16}$, $\quad x = t^{-3/4}$

SOLUTION Let $x(t) = t^{-3/4}$. Then $\frac{dx}{dt} = -\frac{3}{4}t^{-7/4}$, $\frac{d^2x}{dt^2} = \frac{21}{16}t^{-11/4}$, $\frac{d^3x}{dt^3} = -\frac{231}{64}t^{-15/4}$, and $\frac{d^4x}{dt^4} = \frac{3465}{256}t^{-19/4}$. Thus

$$\left.\frac{d^4x}{dt^4}\right|_{t=16} = \frac{3465}{256}16^{-19/4} = \frac{3465}{134217728}.$$

22. $f'''(4)$, $\quad f(t) = 2t^2 - t$

SOLUTION Since $f(t) = 2t^2 - t$, $f'(t) = 4t - 1$, $f''(t) = 4$, and $f'''(t) = 0$ for all t. In particular, $f'''(4) = 0$.

23. $f'''(-3)$, $\quad f(x) = 4e^x - x^3$

SOLUTION Let $f(x) = 4e^x - x^3$. Then $f'(x) = 4e^x - 3x^2$, $f''(x) = 4e^x - 6x$, $f'''(x) = 4e^x - 6$, and $f'''(-3) = 4e^{-3} - 6$.

24. $f''(1)$, $\quad f(t) = \dfrac{t}{t+1}$

SOLUTION Let $f(t) = \dfrac{t}{t+1}$. Then

$$f'(t) = \frac{(t+1)(1) - (t)(1)}{(t+1)^2} = \frac{1}{(t+1)^2} = \frac{1}{t^2 + 2t + 1}$$

and

$$f''(t) = \frac{(t^2 + 2t + 1)(0) - 1(2t+2)}{(t^2 + 2t + 1)^2} = -\frac{2(t+1)}{(t+1)^4} = -\frac{2}{(t+1)^3}.$$

Thus, $f''(1) = -\dfrac{1}{4}$.

25. $h''(1)$, $\quad h(w) = \sqrt{w}e^w$

SOLUTION Let $h(w) = \sqrt{w}e^w = w^{1/2}e^w$. Then

$$h'(w) = w^{1/2}e^w + e^w\left(\frac{1}{2}w^{-1/2}\right) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w$$

and

$$h''(w) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w + e^w\left(\frac{1}{2}w^{-1/2} - \frac{1}{4}w^{-3/2}\right) = \left(w^{1/2} + w^{-1/2} - \frac{1}{4}w^{-3/2}\right)e^w.$$

Thus, $h''(1) = \dfrac{7}{4}e$.

26. $g''(0)$, $\quad g(s) = \dfrac{e^s}{s+1}$

SOLUTION Let $g(s) = \dfrac{e^s}{s+1}$. Then

$$g'(s) = \frac{(s+1)e^s - e^s(1)}{(s+1)^2} = \frac{se^s}{s^2 + 2s + 1}$$

and

$$g''(s) = \frac{(s^2 + 2s + 1)(se^s + e^s) - se^s(2s+2)}{(s^2 + 2s + 1)^2} = \frac{(s^2+1)e^s}{(s+1)^3}.$$

Thus, $g''(0) = 1$.

27. Calculate $y^{(k)}(0)$ for $0 \le k \le 5$, where $y = x^4 + ax^3 + bx^2 + cx + d$ (with a, b, c, d the constants).

SOLUTION Applying the power, constant multiple, and sum rules at each stage, we get (note $y^{(0)}$ is y by convention):

k	$y^{(k)}$
0	$x^4 + ax^3 + bx^2 + cx + d$
1	$4x^3 + 3ax^2 + 2bx + c$
2	$12x^2 + 6ax + 2b$
3	$24x + 6a$
4	24
5	0

from which we get $y^{(0)}(0) = d$, $y^{(1)}(0) = c$, $y^{(2)}(0) = 2b$, $y^{(3)}(0) = 6a$, $y^{(4)}(0) = 24$, and $y^{(5)}(0) = 0$.

28. Which of the following satisfy $f^{(k)}(x) = 0$ for all $k \ge 6$?

(a) $f(x) = 7x^4 + 4 + x^{-1}$ **(b)** $f(x) = x^3 - 2$

(c) $f(x) = \sqrt{x}$ **(d)** $f(x) = 1 - x^6$

(e) $f(x) = x^{9/5}$ **(f)** $f(x) = 2x^2 + 3x^5$

SOLUTION Equations (b) and (f) go to zero after the sixth derivative. We don't have to take the derivatives to see this.

- Look at (a). $f'(x) = 28x^3 - x^{-2}$. Every time we take higher derivatives of $f(x)$, the negative exponent will keep decreasing, and will never become zero.
- In the case of (b), we see that every derivative decreases the degree (the highest exponent) of the polynomial by one, so that $f^{(4)}(x) = 0$.
- For (c), $f'(x) = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$. Every further derivative of $f(x)$ is going to make the exponent more negative, so that it will never go to zero.
- In the case of (d), like (b), the highest exponent will decrease with every derivative, but 6 derivatives will leave the exponent zero, $f^{(6)}(x)$ will be $-6!$. This is easy to verify.
- (e) is like (c). Since the exponent is not a whole number, successive derivatives will make the exponent "pass over" zero, and go to negative infinity.
- In the case of (f), $f^{(5)}(x)$ is constant, so that $f^{(6)}(x) = 0$ for all x.

29. Use the result in Example 3 to find $\dfrac{d^6}{dx^6}\,x^{-1}$.

SOLUTION The equation in Example 3 indicates that

$$\frac{d^6}{dx^6}\,x^{-1} = (-1)^6 6! x^{-6-1}.$$

$(-1)^6 = 1$ and $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, so

$$\frac{d^6}{dx^6}\,x^{-1} = 720x^{-7}.$$

30. Calculate the first five derivatives of $f(x) = \sqrt{x}$.

(a) Show that $f^{(n)}(x)$ is a multiple of $x^{-n+1/2}$.

(b) Show that $f^{(n)}(x)$ alternates in sign as $(-1)^{n-1}$ for $n \ge 1$.

(c) Find a formula for $f^{(n)}(x)$ for $n \ge 2$. *Hint:* Verify that the coefficient is $\pm 1 \cdot 3 \cdot 5 \cdots \dfrac{2n-3}{2^n}$.

SOLUTION We use the Power Rule:

$$\frac{df}{dx} = \frac{1}{2}x^{-1/2} \qquad \frac{d^4 f}{dx^4} = -\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-7/2}$$

$$\frac{d^2 f}{dx^2} = -\frac{1}{2}\left(\frac{1}{2}\right)x^{-3/2} \qquad \frac{d^5 f}{dx^5} = \frac{7}{2}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-9/2}$$

$$\frac{d^3 f}{dx^3} = \frac{3}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-5/2} \qquad \frac{d^6 f}{dx^6} = -\frac{9}{2}\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-11/2}$$

The pattern we see here is that the nth derivative is a multiple of $\pm x^{-n+\frac{1}{2}}$. Which multiple? The coefficient is the product of the odd numbers up to $2n-3$ divided by 2^n. Therefore we can write a general formula for the nth derivative as follows:

$$f^{(n)}(x) = (-1)^{n-1}\,\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}\,x^{-n+\frac{1}{2}}$$

In Exercises 31–36, find a general formula for $f^{(n)}(x)$.

31. $f(x) = x^{-2}$

SOLUTION $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, $f^{(4)}(x) = 5\cdot 24x^{-6}$, ... From this we can conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n(n+1)!x^{-(n+2)}$.

32. $f(x) = (x+2)^{-1}$

SOLUTION Let $f(x) = (x+2)^{-1} = \frac{1}{x+2}$. Then $f'(x) = -1(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$, $f^{(4)}(x) = 24(x+2)^{-5}$, ... From this we conclude that the nth derivative can be written as

$$f^{(n)}(x) = (-1)^n n!(x+2)^{-(n+1)}.$$

33. $f(x) = x^{-1/2}$

SOLUTION $f'(x) = \frac{-1}{2}x^{-3/2}$. We will avoid simplifying numerators and denominators to find the pattern:

$$\begin{aligned}
f''(x) &= \frac{-3}{2}\frac{-1}{2}x^{-5/2} = (-1)^2\frac{3\times 1}{2^2}x^{-5/2}\\
f'''(x) &= -\frac{5}{2}\frac{3\times 1}{2^2}x^{-7/2} = (-1)^3\frac{5\times 3\times 1}{2^3}x^{-7/2}\\
&\vdots\\
f^{(n)}(x) &= (-1)^n\frac{(2n-1)\times(2n-3)\times\ldots\times 1}{2^n}\,x^{-(2n+1)/2}.
\end{aligned}$$

34. $f(x) = x^{-3/2}$

SOLUTION $f'(x) = \frac{-3}{2}x^{-5/2}$. We will avoid simplifying numerators and denominators to find the pattern:

$$\begin{aligned}
f''(x) &= \frac{-5}{2}\frac{-3}{2}x^{-7/2} = (-1)^2\frac{5\times 3}{2^2}x^{-7/2}\\
f'''(x) &= -\frac{7}{2}\frac{5\times 3}{2^2}x^{-9/2} = (-1)^3\frac{7\times 5\times 3}{2^3}x^{-9/2}\\
&\vdots\\
f^{(n)}(x) &= (-1)^n\frac{(2n+1)\times(2n-1)\times\ldots\times 3}{2^n}\,x^{-(2n+3)/2}.
\end{aligned}$$

35. $f(x) = xe^{-x}$

SOLUTION Let $f(x) = xe^{-x}$. Then

$$\begin{aligned}
f'(x) &= x(-e^{-x}) + e^{-x} = (1-x)e^{-x} = -(x-1)e^{-x}\\
f''(x) &= (1-x)(-e^{-x}) - e^{-x} = (x-2)e^{-x}\\
f'''(x) &= (x-2)(-e^{-x}) + e^{-x} = (3-x)e^{-x} = -(x-3)e^{-x}
\end{aligned}$$

From this we conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.

36. $f(x) = x^2e^x$

SOLUTION Let $f(x) = x^2e^x$. Then

$$\begin{aligned}
f'(x) &= x^2e^x + 2xe^x = (x^2+2x)e^x\\
f''(x) &= (x^2+2x)e^x + e^x(2x+2) = (x^2+4x+2)e^x\\
f'''(x) &= (x^2+4x+2)e^x + e^x(2x+4) = (x^2+6x+6)e^x\\
f^{(4)}(x) &= (x^2+6x+6)e^x + e^x(2x+6) = (x^2+8x+12)e^x
\end{aligned}$$

From this we conclude that the nth derivative can be written as $f^{(n)}(x) = (x^2+2nx+n(n-1))e^x$.

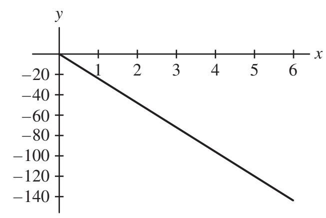
37. **(a)** Find the acceleration at time $t = 5$ min of a helicopter whose height is $s(t) = 300t - 4t^3$ m.

(b) Plot the acceleration $h''(t)$ for $0 \le t \le 6$. How does this graph show that the helicopter is slowing down during this time interval?

SOLUTION

(a) Let $s(t) = 300t - 4t^3$, with t in minutes and s in meters. The velocity is $v(t) = s'(t) = 300 - 12t^2$ and acceleration is $a(t) = s''(t) = -24t$. Thus $a(5) = -120$ m/min^2.

(b) The acceleration of the helicopter for $0 \le t \le 6$ is shown in the figure below. As the acceleration of the helicopter is negative, the velocity of the helicopter must be decreasing. Because the velocity is positive for $0 \le t \le 6$, the helicopter is slowing down.

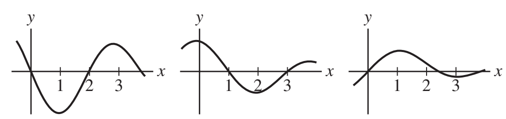


38. Find an equation of the tangent to the graph of $y = f'(x)$ at $x = 3$, where $f(x) = x^4$.

SOLUTION Let $f(x) = x^4$ and $g(x) = f'(x) = 4x^3$. Then $g'(x) = 12x^2$. The tangent line to g at $x = 3$ is given by

$$y = g'(3)(x - 3) + g(3) = 108(x - 3) + 108 = 108x - 216.$$

39. Figure 5 shows f, f', and f''. Determine which is which.



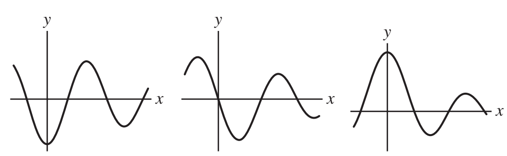
(A) (B) (C)

FIGURE 5

SOLUTION (a) f'' (b) f' (c) f.

The tangent line to (c) is horizontal at $x = 1$ and $x = 3$, where (b) has roots. The tangent line to (b) is horizontal at $x = 2$ and $x = 0$, where (a) has roots.

40. The second derivative f'' is shown in Figure 6. Which of (A) or (B) is the graph of f and which is f'?



$f''(x)$ (A) (B)

FIGURE 6

SOLUTION $f'(x) = A$ and $f(x) = B$.

41. Figure 7 shows the graph of the position s of an object as a function of time t. Determine the intervals on which the acceleration is positive.

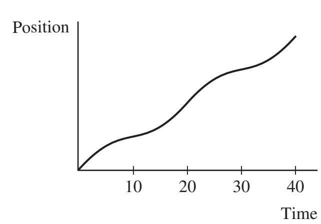


FIGURE 7

SOLUTION Roughly from time 10 to time 20 and from time 30 to time 40. The acceleration is positive over the same intervals over which the graph is bending upward.

42. Find a polynomial $f(x)$ that satisfies the equation $xf''(x) + f(x) = x^2$.

SOLUTION Since $xf''(x) + f(x) = x^2$, and x^2 is a polynomial, it seems reasonable to assume that $f(x)$ is a polynomial of some degree, call it n. The degree of $f''(x)$ is $n-2$, so the degree of $xf''(x)$ is $n-1$, and the degree of $xf''(x) + f(x)$ is n. Hence, $n = 2$, since the degree of x^2 is 2. Therefore, let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Substituting into the equation $xf''(x) + f(x) = x^2$ yields $ax^2 + (2a+b)x + c = x^2$, an identity in x. Equating coefficients, we have $a = 1$, $2a + b = 0$, $c = 0$. Therefore, $b = -2$ and $f(x) = x^2 - 2x$.

43. Find a value of n such that $y = x^n e^x$ satisfies the equation $xy' = (x-3)y$.

SOLUTION Let $y = x^n e^x$. Then

$$y' = x^n e^x + nx^{n-1}e^x = x^{n-1}e^x(x+n),$$

and

$$xy' = x^n e^x(x+n) = (x+n)y.$$

Thus, $y = x^n e^x$ satisfies the equation $xy' = (x-3)y$ for $n = -3$.

44. Which of the following descriptions could *not* apply to Figure 8? Explain.

(a) Graph of acceleration when velocity is constant

(b) Graph of velocity when acceleration is constant

(c) Graph of position when acceleration is zero

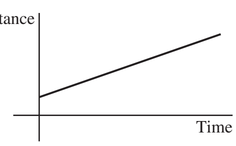


FIGURE 8

SOLUTION

(a) Does NOT apply to the figure because if $v(t) = C$ where C is a constant, then $a(t) = v'(t) = 0$, which is the horizontal line going through the origin.

(b) Can apply because the graph has a constant slope.

(c) Can apply because if we took this as a position graph, the velocity graph would be a horizontal line and thus, acceleration would be zero.

45. According to one model that takes into account air resistance, the acceleration $a(t)$ (in m/s^2) of a skydiver of mass m in free fall satisfies

$$a(t) = -9.8 + \frac{k}{m}v(t)^2$$

where $v(t)$ is velocity (negative since the object is falling) and k is a constant. Suppose that $m = 75$ kg and $k = 14$ kg/m.

(a) What is the object's velocity when $a(t) = -4.9$?

(b) What is the object's velocity when $a(t) = 0$? This velocity is the object's terminal velocity.

SOLUTION Solving $a(t) = -9.8 + \frac{k}{m}v(t)^2$ for the velocity and taking into account that the velocity is negative since the object is falling, we find

$$v(t) = -\sqrt{\frac{m}{k}(a(t) + 9.8)} = -\sqrt{\frac{75}{14}(a(t) + 9.8)}.$$

(a) Substituting $a(t) = -4.9$ into the above formula for the velocity, we find

$$v(t) = -\sqrt{\frac{75}{14}(4.9)} = -\sqrt{26.25} = -5.12 \text{ m/s}.$$

(b) When $a(t) = 0$,

$$v(t) = -\sqrt{\frac{75}{14}(9.8)} = -\sqrt{52.5} = -7.25 \text{ m/s}.$$

46. According to one model that attempts to account for air resistance, the distance $s(t)$ (in meters) traveled by a falling raindrop satisfies

$$\frac{d^2s}{dt^2} = g - \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

where D is the raindrop diameter and $g = 9.8$ m/s^2. Terminal velocity v_{term} is defined as the velocity at which the drop has zero acceleration (one can show that velocity approaches v_{term} as time proceeds).

(a) Show that $v_{\text{term}} = \sqrt{2000gD}$.

(b) Find v_{term} for drops of diameter 10^{-3} m and 10^{-4} m.

(c) In this model, do raindrops accelerate more rapidly at higher or lower velocities?

SOLUTION

(a) v_{term} is found by setting $\frac{d^2s}{dt^2} = 0$, and solving for $\frac{ds}{dt} = v$.

$$0 = g - \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

$$g = \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

$$\frac{ds}{dt} = \sqrt{g\frac{D}{0.0005}} = \sqrt{2000gD} = v^{1/2}.$$

(b) If $D = 0.003$ ft,

$$v_{\text{term}} = \sqrt{2000g(0.003)} = \sqrt{58.8} = 7.668 \text{ m/s}.$$

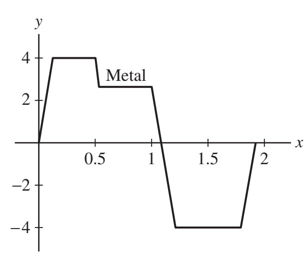
If $D = 0.0003$ ft,

$$v_{\text{term}} = \sqrt{2000g(0.0003)} = \sqrt{5.88} = 2.425 \text{ m/s}.$$

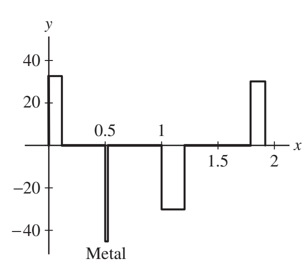
(c) The greater the velocity, the more gets subtracted from g in the formula for acceleration. Therefore, assuming velocity is less than v_{term}, greater velocities correspond to *lower* acceleration.

47. A servomotor controls the vertical movement of a drill bit that will drill a pattern of holes in sheet metal. The maximum vertical speed of the drill bit is 4 in./s, and while drilling the hole, it must move no more than 2.6 in./s to avoid warping the metal. During a cycle, the bit begins and ends at rest, quickly approaches the sheet metal, and quickly returns to its initial position after the hole is drilled. Sketch possible graphs of the drill bit's vertical velocity and acceleration. Label the point where the bit enters the sheet metal.

SOLUTION There will be multiple cycles, each of which will be more or less identical. Let $v(t)$ be the *downward* vertical velocity of the drill bit, and let $a(t)$ be the vertical acceleration. From the narrative, we see that $v(t)$ can be no greater than 4 and no greater than 2.6 while drilling is taking place. During each cycle, $v(t) = 0$ initially, $v(t)$ goes to 4 quickly. When the bit hits the sheet metal, $v(t)$ goes down to 2.6 quickly, at which it stays until the sheet metal is drilled through. As the drill pulls out, it reaches maximum non-drilling upward speed ($v(t) = -4$) quickly, and maintains this speed until it returns to rest. A possible plot follows:



A graph of the acceleration is extracted from this graph:



In Exercises 48 and 49, refer to the following. In a 1997 study, Boardman and Lave related the traffic speed S on a two-lane road to traffic density Q (number of cars per mile of road) by the formula

$$S = 2882Q^{-1} - 0.052Q + 31.73$$

for $60 \le Q \le 400$ *(Figure 9).*

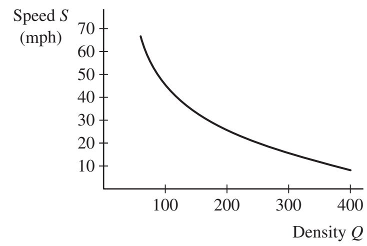


FIGURE 9 Speed as a function of traffic density.

48. Calculate dS/dQ and d^2S/dQ^2.

SOLUTION

$$dS/dQ = -2882Q^{-2} - 0.052$$
$$d^2S/dQ^2 = 5764Q^{-3}.$$

49. **(a)** Explain intuitively why we should expect that $dS/dQ < 0$.

(b) Show that $d^2S/dQ^2 > 0$. Then use the fact that $dS/dQ < 0$ and $d^2S/dQ^2 > 0$ to justify the following statement: *A one-unit increase in traffic density slows down traffic more when Q is small than when Q is large.*

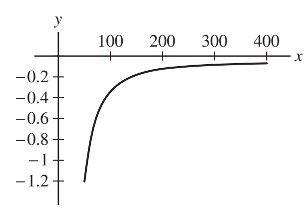
(c) GU Plot dS/dQ. Which property of this graph shows that $d^2S/dQ^2 > 0$?

SOLUTION

(a) Traffic speed must be reduced when the road gets more crowded so we expect dS/dQ to be negative. This is indeed the case since $dS/dQ = -0.052 - 2882/Q^2 < 0$.

(b) The decrease in speed due to a one-unit increase in density is approximately dS/dQ (a negative number). Since $d^2S/dQ^2 = 5764Q^{-3} > 0$ is positive, this tells us that dS/dQ gets larger as Q increases—and a negative number which gets larger is getting closer to zero. So the decrease in speed is smaller when Q is larger, that is, a one-unit increase in traffic density has a smaller effect when Q is large.

(c) dS/dQ is plotted below. The fact that this graph is increasing shows that $d^2S/dQ^2 > 0$.



50. CAS Use a computer algebra system to compute $f^{(k)}(x)$ for $k = 1, 2, 3$ for the following functions.

(a) $f(x) = (1 + x^3)^{5/3}$

(b) $f(x) = \dfrac{1 - x^4}{1 - 5x - 6x^2}$

SOLUTION

(a) Let $f(x) = (1 + x^3)^{5/3}$. Using a computer algebra system,

$$f'(x) = 5x^2(1 + x^3)^{2/3};$$
$$f''(x) = 10x(1 + x^3)^{2/3} + 10x^4(1 + x^3)^{-1/3}; \text{ and}$$
$$f'''(x) = 10(1 + x^3)^{2/3} + 60x^3(1 + x^3)^{-1/3} - 10x^6(1 + x^3)^{-4/3}.$$

(b) Let $f(x) = \dfrac{1 - x^4}{1 - 5x - 6x^2}$. Using a computer algebra system,

$$f'(x) = \frac{12x^3 - 9x^2 + 2x + 5}{(6x - 1)^2};$$

$$f''(x) = \frac{2(36x^3 - 18x^2 + 3x - 31)}{(6x-1)^3}; \text{ and}$$

$$f'''(x) = \frac{1110}{(6x-1)^4}.$$

51. CAS Let $f(x) = \dfrac{x+2}{x-1}$. Use a computer algebra system to compute the $f^{(k)}(x)$ for $1 \le k \le 4$. Can you find a general formula for $f^{(k)}(x)$?

SOLUTION Let $f(x) = \dfrac{x+2}{x-1}$. Using a computer algebra system,

$$f'(x) = -\frac{3}{(x-1)^2} = (-1)^1 \frac{3 \cdot 1}{(x-1)^{1+1}};$$

$$f''(x) = \frac{6}{(x-1)^3} = (-1)^2 \frac{3 \cdot 2 \cdot 1}{(x-1)^{2+1}};$$

$$f'''(x) = -\frac{18}{(x-1)^4} = (-1)^3 \frac{3 \cdot 3!}{(x-1)^{3+1}}; \text{ and}$$

$$f^{(4)}(x) = \frac{72}{(x-1)^5} = (-1)^4 \frac{3 \cdot 4!}{(x-1)^{4+1}}.$$

From the pattern observed above, we conjecture

$$f^{(k)}(x) = (-1)^k \frac{3 \cdot k!}{(x-1)^{k+1}}.$$

Further Insights and Challenges

52. Find the 100th derivative of

$$p(x) = (x + x^5 + x^7)^{10}(1 + x^2)^{11}(x^3 + x^5 + x^7)$$

SOLUTION This is a polynomial of degree $70 + 22 + 7 = 99$, so its 100th derivative is zero.

53. What is $p^{(99)}(x)$ for $p(x)$ as in Exercise 52?

SOLUTION First note that for any integer $n \le 98$,

$$\frac{d^{99}}{dx^{99}} x^n = 0.$$

Now, if we expand $p(x)$, we find

$$p(x) = x^{99} + \text{ terms of degree at most 98;}$$

therefore,

$$\frac{d^{99}}{dx^{99}} p(x) = \frac{d^{99}}{dx^{99}}(x^{99} + \text{ terms of degree at most 98}) = \frac{d^{99}}{dx^{99}} x^{99}$$

Using logic similar to that used to compute the derivative in Example (3), we compute:

$$\frac{d^{99}}{dx^{99}}(x^{99}) = 99 \times 98 \times \ldots 1,$$

so that $\frac{d^{99}}{dx^{99}} p(x) = 99!$.

54. Use the Product Rule twice to find a formula for $(fg)''$ in terms of f and g and their first and second derivatives.

SOLUTION Let $h = fg$. Then $h' = fg' + gf' = f'g + fg'$ and

$$h'' = f'g' + gf'' + fg'' + g'f' = f''g + 2f'g' + fg''.$$

55. Use the Product Rule to find a formula for $(fg)'''$ and compare your result with the expansion of $(a+b)^3$. Then try to guess the general formula for $(fg)^{(n)}$.

SOLUTION Continuing from Exercise 54, we have

$$h''' = f''g' + gf''' + 2(f'g'' + g'f'') + fg''' + g''f' = f'''g + 3f''g' + 3f'g'' + fg'''$$

The binomial theorem gives

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

and more generally

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficients are given by

$$\binom{n}{k} = \frac{k(k-1)\cdots(k-n+1)}{n!}.$$

Accordingly, the general formula for $(fg)^{(n)}$ is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $p^{(k)}$ is the kth derivative of p (or p itself when $k = 0$).

56. Compute

$$\Delta f(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

for the following functions:

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

Based on these examples, what do you think the limit Δf represents?

SOLUTION For $f(x) = x$, we have

$$f(x+h) + f(x-h) - 2f(x) = (x+h) + (x-h) - 2x = 0.$$

Hence, $\Delta(x) = 0$. For $f(x) = x^2$,

$$\begin{aligned} f(x+h) + f(x-h) - 2f(x) &= (x+h)^2 + (x-h)^2 - 2x^2 \\ &= x^2 + 2xh + h^2 + x^2 - 2xh + h^2 - 2x^2 = 2h^2, \end{aligned}$$

so $\Delta(x^2) = 2$. Working in a similar fashion, we find $\Delta(x^3) = 6x$. One can prove that for twice differentiable functions, $\Delta f = f''$. It is an interesting fact of more advanced mathematics that there are functions f for which Δf exists at all points, but the function is not differentiable.

3.6 Trigonometric Functions

Preliminary Questions

1. Determine the sign ($+$ or $-$) that yields the correct formula for the following:

(a) $\dfrac{d}{dx}(\sin x + \cos x) = \pm \sin x \pm \cos x$

(b) $\dfrac{d}{dx} \sec x = \pm \sec x \tan x$

(c) $\dfrac{d}{dx} \cot x = \pm \csc^2 x$

SOLUTION The correct formulas are

(a) $\dfrac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x$

(b) $\dfrac{d}{dx}\sec x = \sec x \tan x$

(c) $\dfrac{d}{dx}\cot x = -\csc^2 x$

2. Which of the following functions can be differentiated using the rules we have covered so far?

(a) $y = 3\cos x \cot x$ **(b)** $y = \cos(x^2)$ **(c)** $y = e^x \sin x$

SOLUTION

(a) $3\cos x \cot x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

(b) $\cos(x^2)$ is a composition of the functions $\cos x$ and x^2. We have not yet discussed how to differentiate composite functions.

(c) $x^2 \cos x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

3. Compute $\frac{d}{dx}(\sin^2 x + \cos^2 x)$ without using the derivative formulas for $\sin x$ and $\cos x$.

SOLUTION Recall that $\sin^2 x + \cos^2 x = 1$ for all x. Thus,

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}1 = 0.$$

4. How is the addition formula used in deriving the formula $(\sin x)' = \cos x$?

SOLUTION The difference quotient for the function $\sin x$ involves the expression $\sin(x+h)$. The addition formula for the sine function is used to expand this expression as $\sin(x+h) = \sin x \cos h + \sin h \cos x$.

Exercises

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. $y = \sin x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right).$$

2. $y = \cos x, \quad x = \frac{\pi}{3}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2} = -\frac{\sqrt{3}}{2}x + \frac{1}{2} + \frac{\pi\sqrt{3}}{6}.$$

3. $y = \tan x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\left(x - \frac{\pi}{4}\right) + 1 = 2x + 1 - \frac{\pi}{2}.$$

4. $y = \sec x, \quad x = \frac{\pi}{6}$

SOLUTION Let $f(x) = \sec x$. Then $f'(x) = \sec x \tan x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right) = \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}} = \frac{2}{3}x + \frac{2\sqrt{3}}{3} + \frac{\pi}{9}.$$

In Exercises 5–24, compute the derivative.

5. $f(x) = \sin x \cos x$

SOLUTION Let $f(x) = \sin x \cos x$. Then

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = -\sin^2 x + \cos^2 x.$$

6. $f(x) = x^2 \cos x$

SOLUTION Let $f(x) = x^2 \cos x$. Then

$$f'(x) = x^2 (-\sin x) + (\cos x)(2x) = 2x \cos x - x^2 \sin x.$$

7. $f(x) = \sin^2 x$

SOLUTION Let $f(x) = \sin^2 x = \sin x \sin x$. Then

$$f'(x) = \sin x(\cos x) + \sin x(\cos x) = 2 \sin x \cos x.$$

8. $f(x) = 9 \sec x + 12 \cot x$

SOLUTION Let $f(x) = 9 \sec x + 12 \cot x$. Then $f'(x) = 9 \sec x \tan x - 12 \csc^2 x$.

9. $H(t) = \sin t \sec^2 t$

SOLUTION Let $H(t) = \sin t \sec^2 t$. Then

$$\begin{aligned} H'(t) &= \sin t \frac{d}{dt}(\sec t \cdot \sec t) + \sec^2 t(\cos t) \\ &= \sin t(\sec t \sec t \tan t + \sec t \sec t \tan t) + \sec t \\ &= 2 \sin t \sec^2 t \tan t + \sec t. \end{aligned}$$

10. $h(t) = 9 \csc t + t \cot t$

SOLUTION Let $h(t) = 9 \csc t + t \cot t$. Then

$$h'(t) = 9(-\csc t \cot t) + t(-\csc^2 t) + \cot t = \cot t - 9 \csc t \cot t - t \csc^2 t.$$

11. $f(\theta) = \tan\theta \sec\theta$

SOLUTION Let $f(\theta) = \tan\theta \sec\theta$. Then

$$f'(\theta) = \tan\theta \sec\theta \tan\theta + \sec\theta \sec^2\theta = \sec\theta \tan^2\theta + \sec^3\theta = \left(\tan^2\theta + \sec^2\theta\right)\sec\theta.$$

12. $k(\theta) = \theta^2 \sin^2\theta$

SOLUTION Let $k(\theta) = \theta^2 \sin^2\theta$. Then

$$k'(\theta) = \theta^2 (2 \sin\theta \cos\theta) + 2\theta \sin^2\theta = 2\theta^2 \sin\theta \cos\theta + 2\theta \sin^2\theta.$$

Here we used the result from Exercise 7.

13. $f(x) = (2x^4 - 4x^{-1}) \sec x$

SOLUTION Let $f(x) = (2x^4 - 4x^{-1}) \sec x$. Then

$$f'(x) = (2x^4 - 4x^{-1}) \sec x \tan x + \sec x(8x^3 + 4x^{-2}).$$

14. $f(z) = z \tan z$

SOLUTION Let $f(z) = z \tan z$. Then $f'(z) = z(\sec^2 z) + \tan z$.

15. $y = \dfrac{\sec\theta}{\theta}$

SOLUTION Let $y = \dfrac{\sec\theta}{\theta}$. Then

$$y' = \frac{\theta \sec\theta \tan\theta - \sec\theta}{\theta^2}.$$

16. $G(z) = \dfrac{1}{\tan z - \cot z}$

SOLUTION Let $G(z) = \dfrac{1}{\tan z - \cot z}$. Then

$$G'(z) = \frac{(\tan z - \cot z)(0) - 1(\sec^2 z + \csc^2 z)}{(\tan z - \cot z)^2} = -\frac{\sec^2 z + \csc^2 z}{(\tan z - \cot z)^2}.$$

17. $R(y) = \dfrac{3\cos y - 4}{\sin y}$

SOLUTION Let $R(y) = \dfrac{3\cos y - 4}{\sin y}$. Then

$$R'(y) = \frac{\sin y(-3\sin y) - (3\cos y - 4)(\cos y)}{\sin^2 y} = \frac{4\cos y - 3(\sin^2 y + \cos^2 y)}{\sin^2 y} = \frac{4\cos y - 3}{\sin^2 y}.$$

18. $f(x) = \dfrac{x}{\sin x + 2}$

SOLUTION Let $f(x) = \dfrac{x}{2 + \sin x}$. Then

$$f'(x) = \frac{(2 + \sin x)(1) - x\cos x}{(2 + \sin x)^2} = \frac{2 + \sin x - x\cos x}{(2 + \sin x)^2}.$$

19. $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$

SOLUTION Let $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$. Then

$$f'(x) = \frac{(1 - \tan x)\sec^2 x - (1 + \tan x)\left(-\sec^2 x\right)}{(1 - \tan x)^2} = \frac{2\sec^2 x}{(1 - \tan x)^2}.$$

20. $f(\theta) = \theta \tan\theta \sec\theta$

SOLUTION Let $f(\theta) = \theta\tan\theta\sec\theta$. Then

$$\begin{aligned} f'(\theta) &= \theta\frac{d}{d\theta}(\tan\theta\sec\theta) + \tan\theta\sec\theta \\ &= \theta(\tan\theta\sec\theta\tan\theta + \sec\theta\sec^2\theta) + \tan\theta\sec\theta \\ &= \theta\tan^2\theta\sec\theta + \theta\sec^3\theta + \tan\theta\sec\theta. \end{aligned}$$

21. $f(x) = e^x \sin x$

SOLUTION Let $f(x) = e^x\sin x$. Then $f'(x) = e^x\cos x + \sin x e^x = e^x(\cos x + \sin x)$.

22. $h(t) = e^t \csc t$

SOLUTION Let $h(t) = e^t\csc t$. Then $h'(t) = e^t(-\csc t\cot t) + \csc t e^t = e^t\csc t(1 - \cot t)$.

23. $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$

SOLUTION Let $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$. Then

$$\begin{aligned} f'(\theta) &= e^\theta(5\cos\theta - 4\sec^2\theta) + e^\theta(5\sin\theta - 4\tan\theta) \\ &= e^\theta(5\sin\theta + 5\cos\theta - 4\tan\theta - 4\sec^2\theta). \end{aligned}$$

24. $f(x) = xe^x\cos x$

SOLUTION Let $f(x) = xe^x\cos x$. Then

$$\begin{aligned} f'(x) &= x\frac{d}{dx}(e^x\cos x) + e^x\cos x = x(e^x(-\sin x) + \cos x e^x) + e^x\cos x \\ &= e^x(x\cos x - x\sin x + \cos x). \end{aligned}$$

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. $y = x^3 + \cos x, \quad x = 0$

SOLUTION Let $f(x) = x^3 + \cos x$. Then $f'(x) = 3x^2 - \sin x$ and $f'(0) = 0$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = 0(x) + 1 = 1.$$

26. $y = \tan\theta, \quad \theta = \frac{\pi}{6}$

SOLUTION Let $f(\theta) = \tan\theta$. Then $f'(\theta) = \sec^2\theta$ and $f'(\frac{\pi}{6}) = \frac{4}{3}$. The tangent line at $x = \frac{\pi}{6}$ is

$$y = f'\left(\frac{\pi}{6}\right)\left(\theta - \frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right) = \frac{4}{3}\left(\theta - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{3} = \frac{4}{3}\theta + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.$$

27. $y = \sin x + 3\cos x, \quad x = 0$

SOLUTION Let $f(x) = \sin x + 3\cos x$. Then $f'(x) = \cos x - 3\sin x$ and $f'(0) = 1$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = x + 3.$$

28. $y = \dfrac{\sin t}{1 + \cos t}, \quad t = \frac{\pi}{3}$

SOLUTION Let $f(t) = \frac{\sin t}{1+\cos t}$. Then

$$f'(t) = \frac{(1 + \cos t)(\cos t) - \sin t(-\sin t)}{(1 + \cos t)^2} = \frac{1 + \cos t}{(1 + \cos t)^2} = \frac{1}{1 + \cos t},$$

and

$$f'\left(\frac{\pi}{3}\right) = \frac{1}{1 + 1/2} = \frac{2}{3}.$$

The tangent line at $x = \frac{\pi}{3}$ is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{2}{3}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{3} = \frac{2}{3}x + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.$$

29. $y = 2(\sin\theta + \cos\theta), \quad \theta = \frac{\pi}{3}$

SOLUTION Let $f(\theta) = 2(\sin\theta + \cos\theta)$. Then $f'(\theta) = 2(\cos\theta - \sin\theta)$ and $f'(\frac{\pi}{3}) = 1 - \sqrt{3}$. The tangent line at $x = \frac{\pi}{3}$ is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = (1 - \sqrt{3})\left(x - \frac{\pi}{3}\right) + 1 + \sqrt{3}.$$

30. $y = \csc x - \cot x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \csc x - \cot x$. Then

$$f'(x) = \csc^2 x - \csc x \cot x$$

and

$$f'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \cdot 1 = 2 - \sqrt{2}.$$

Hence the tangent line is

$$\begin{aligned} y &= f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \left(2 - \sqrt{2}\right)\left(x - \frac{\pi}{4}\right) + \left(\sqrt{2} - 1\right) \\ &= \left(2 - \sqrt{2}\right)x + \sqrt{2} - 1 + \frac{\pi}{4}\left(\sqrt{2} - 2\right). \end{aligned}$$

31. $y = e^x \cos x, \quad x = 0$

SOLUTION Let $f(x) = e^x \cos x$. Then

$$f'(x) = e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x),$$

and $f'(0) = e^0(\cos 0 - \sin 0) = 1$. Thus, the equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = x + 1.$$

32. $y = e^x \cos^2 x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = e^x \cos^2 x$. Then

$$f'(x) = e^x \frac{d}{dx}(\cos x \cdot \cos x) + e^x \cos^2 x = e^x(\cos x(-\sin x) + \cos x(-\sin x)) + e^x \cos^2 x$$
$$= e^x(\cos^2 x - 2\sin x \cos x),$$

and

$$f'\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{2} - 1\right) = -\frac{1}{2}e^{\pi/4}.$$

The tangent line at $x = \frac{\pi}{4}$ is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = -\frac{1}{2}e^{\pi/4}\left(x - \frac{\pi}{4}\right) + \frac{1}{2}e^{\pi/4}.$$

33. $y = e^t(1 - \cos t), \quad t = \frac{\pi}{2}$

SOLUTION Let $f(t) = e^t(1 - \cos t)$. Then

$$f'(t) = e^t \sin t + e^t(1 - \cos t) = e^t(1 + \sin t - \cos t),$$

and $f'(\frac{\pi}{2}) = 2e^{\pi/2}$. The tangent line at $x = \frac{\pi}{2}$ is

$$y = f'\left(\frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 2e^{\pi/2}\left(t - \frac{\pi}{2}\right) + e^{\pi/2}.$$

34. $y = e^\theta \sec\theta, \quad \theta = \frac{\pi}{4}$

SOLUTION Let $f(\theta) = e^\theta \sec\theta$. Then

$$f'(\theta) = e^\theta \sec\theta \tan\theta + e^\theta \sec\theta = e^\theta \sec\theta(\tan\theta + 1),$$

and

$$f'\left(\frac{\pi}{4}\right) = e^{\pi/4} \sec\frac{\pi}{4}\left(\tan\frac{\pi}{4} + 1\right) = 2\sqrt{2}e^{\pi/4}.$$

Thus, the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\sqrt{2}e^{\pi/4}\left(x - \frac{\pi}{4}\right) + \sqrt{2}e^{\pi/4}.$$

In Exercises 35–37, use Theorem 1 to verify the formula.

35. $\frac{d}{dx}\cot x = -\csc^2 x$

SOLUTION $\cot x = \frac{\cos x}{\sin x}$. Using the quotient rule and the derivative formulas, we compute:

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

36. $\frac{d}{dx}\sec x = \sec x \tan x$

SOLUTION Since $\sec x = \frac{1}{\cos x}$, we can apply the quotient rule and the known derivatives to get:

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \tan x \sec x.$$

37. $\frac{d}{dx}\csc x = -\csc x \cot x$

SOLUTION Since $\csc x = \frac{1}{\sin x}$, we can apply the quotient rule and the two known derivatives to get:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\cot x \csc x.$$

38. Show that both $y = \sin x$ and $y = \cos x$ satisfy $y'' = -y$.

SOLUTION Let $y = \sin x$. Then $y' = \cos x$ and $y'' = -\sin x = -y$. Similarly, if we let $y = \cos x$, then $y' = -\sin x$ and $y'' = -\cos x = -y$.

In Exercises 39–42, calculate the higher derivative.

39. $f''(\theta), \quad f(\theta) = \theta \sin\theta$

SOLUTION Let $f(\theta) = \theta \sin\theta$. Then

$$f'(\theta) = \theta\cos\theta + \sin\theta$$
$$f''(\theta) = \theta(-\sin\theta) + \cos\theta + \cos\theta = -\theta\sin\theta + 2\cos\theta.$$

40. $\dfrac{d^2}{dt^2}\cos^2 t$

SOLUTION

$$\frac{d}{dt}\cos^2 t = \frac{d}{dt}(\cos t \cdot \cos t) = \cos t(-\sin t) + \cos t(-\sin t) = -2\sin t\cos t$$
$$\frac{d^2}{dt^2}\cos^2 t = \frac{d}{dt}(-2\sin t\cos t) = -2(\sin t(-\sin t) + \cos t(\cos t)) = -2(\cos^2 t - \sin^2 t).$$

41. $y'', \quad y''', \quad y = \tan x$

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$ and by the Chain Rule,

$$y'' == \frac{d}{dx}\sec^2 x = 2(\sec x)(\sec x\tan x) = 2\sec^2 x\tan x$$
$$y''' = 2\sec^2 x(\sec^2 x) + (2\sec^2 x\tan x)\tan x = 2\sec^4 + 4\sec^4 x\tan^2 x$$

42. $y'', \quad y''', \quad y = e^t \sin t$

SOLUTION Let $y = e^t \sin t$. Then

$$y' = e^t\cos t + e^t\sin t = e^t(\cos t + \sin t)$$
$$y'' = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t$$
$$y''' = 2e^t(-\sin t) + 2e^t\cos t = 2e^t(\cos t - \sin t).$$

43. Calculate the first five derivatives of $f(x) = \cos x$. Then determine $f^{(8)}$ and $f^{(37)}$.

SOLUTION Let $f(x) = \cos x$.

- Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and $f^{(5)}(x) = -\sin x$.
- Accordingly, the successive derivatives of f cycle among

  $$\{-\sin x, -\cos x, \sin x, \cos x\}$$

  in that order. Since 8 is a multiple of 4, we have $f^{(8)}(x) = \cos x$.
- Since 36 is a multiple of 4, we have $f^{(36)}(x) = \cos x$. Therefore, $f^{(37)}(x) = -\sin x$.

44. Find $y^{(157)}$, where $y = \sin x$.

SOLUTION Let $f(x) = \sin x$. Then the successive derivatives of f cycle among

$$\{\cos x, -\sin x, -\cos x, \sin x\}$$

in that order. Since 156 is a multiple of 4, we have $f^{(156)}(x) = \sin x$. Therefore, $f^{(157)}(x) = \cos x$.

45. Find the values of x between 0 and 2π where the tangent line to the graph of $y = \sin x\cos x$ is horizontal.

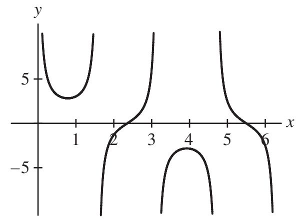
SOLUTION Let $y = \sin x\cos x$. Then

$$y' = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x.$$

When $y' = 0$, we have $\sin x = \pm\cos x$. In the interval $[0, 2\pi]$, this occurs when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

46. GU Plot the graph $f(\theta) = \sec\theta + \csc\theta$ over $[0, 2\pi]$ and determine the number of solutions to $f'(\theta) = 0$ in this interval graphically. Then compute $f'(\theta)$ and find the solutions.

SOLUTION The graph of $f(\theta) = \sec\theta + \csc\theta$ over $[0, 2\pi]$ is given below. From the graph, it appears that there are two locations where the tangent line would be horizontal; that is, there appear to be two solutions to $f'(\theta) = 0$. Now $f'(\theta) = \sec\theta\tan\theta - \csc\theta\cot\theta$. Setting $\sec\theta\tan\theta - \csc\theta\cot\theta = 0$ and then multiplying by $\sin\theta\tan\theta$ and rearranging terms yields $\tan^3\theta = 1$. Thus, on the interval $[0, 2\pi]$, there are two solution of $f'(\theta) = 0$: $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.



47. GU Let $g(t) = t - \sin t$.

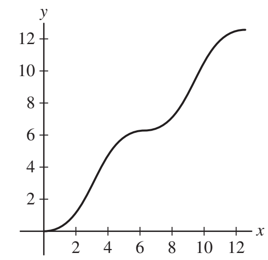
(a) Plot the graph of g with a graphing utility for $0 \le t \le 4\pi$.

(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.

(c) For which values of t in the given range is the tangent line horizontal?

SOLUTION Let $g(t) = t - \sin t$.

(a) Here is a graph of g over the interval $[0, 4\pi]$.



(b) Since $g'(t) = 1 - \cos t \ge 0$ for all t, the slope of the tangent line to g is always nonnegative.

(c) In the interval $[0, 4\pi]$, the tangent line is horizontal when $t = 0, 2\pi, 4\pi$.

48. CAS Let $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$.

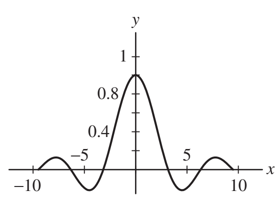
(a) Plot $f(x)$ on $[-3\pi, 3\pi]$.

(b) Show that $f'(c) = 0$ if $c = \tan c$. Use the numerical root finder on a computer algebra system to find a good approximation to the smallest *positive* value c_0 such that $f'(c_0) = 0$.

(c) Verify that the horizontal line $y = f(c_0)$ is tangent to the graph of $y = f(x)$ at $x = c_0$ by plotting them on the same set of axes.

SOLUTION

(a) Here is the graph of $f(x)$ over $[-3\pi, 3\pi]$.



(b) Let $f(x) = \dfrac{\sin x}{x}$. Then

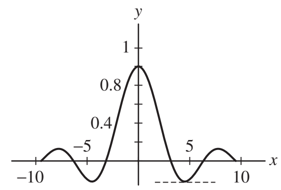
$$f'(x) = \frac{x\cos x - \sin x}{x^2}.$$

To have $f'(c) = 0$, it follows that $c\cos c - \sin c = 0$, or

$$\tan c = c.$$

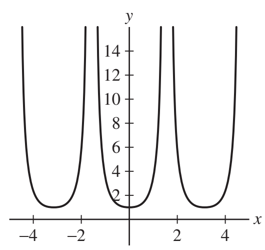
Using a computer algebra system, we find that the smallest positive value c_0 such that $f'(c_0) = 0$ is $c_0 = 4.493409$.

(c) The horizontal line $y = f(c_0) = -0.217234$ and the function $y = f(x)$ are both plotted below. The horizontal line is clearly tangent to the graph of $f(x)$.



49. Show that no tangent line to the graph of $f(x) = \tan x$ has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of $(\tan x)'$.

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$. Note that $f'(x) = \frac{1}{\cos^2 x}$ has numerator 1; the equation $f'(x) = 0$ therefore has no solution. Because the maximum value of $\cos^2 x$ is 1, the minimum value of $f'(x) = \frac{1}{\cos^2 x}$ is 1. Hence, the least slope for a tangent line to $\tan x$ is 1. Here is a graph of f'.



50. The height at time t (in seconds) of a mass, oscillating at the end of a spring, is $s(t) = 300 + 40 \sin t$ cm. Find the velocity and acceleration at $t = \frac{\pi}{3}$ s.

SOLUTION Let $s(t) = 300 + 40 \sin t$ be the height. Then the velocity is

$$v(t) = s'(t) = 40 \cos t$$

and the acceleration is

$$a(t) = v'(t) = -40 \sin t.$$

At $t = \frac{\pi}{3}$, the velocity is $v\left(\frac{\pi}{3}\right) = 20$ cm/sec and the acceleration is $a\left(\frac{\pi}{3}\right) = -20\sqrt{3}$ cm/sec^2.

51. The horizontal range R of a projectile launched from ground level at an angle θ and initial velocity v_0 m/s is $R = (v_0^2/9.8) \sin\theta \cos\theta$. Calculate $dR/d\theta$. If $\theta = 7\pi/24$, will the range increase or decrease if the angle is increased slightly? Base your answer on the sign of the derivative.

SOLUTION Let $R(\theta) = (v_0^2/9.8) \sin\theta \cos\theta$.

$$\frac{dR}{d\theta} = R'(\theta) = (v_0^2/9.8)(-\sin^2\theta + \cos^2\theta).$$

If $\theta = 7\pi/24$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, so $|\sin\theta| > |\cos\theta|$, and $dR/d\theta < 0$ (numerically, $dR/d\theta = -0.0264101 v_0^2$). At this point, increasing the angle will *decrease* the range.

52. Show that if $\frac{\pi}{2} < \theta < \pi$, then the distance along the x-axis between θ and the point where the tangent line intersects the x-axis is equal to $|\tan\theta|$ (Figure 4).

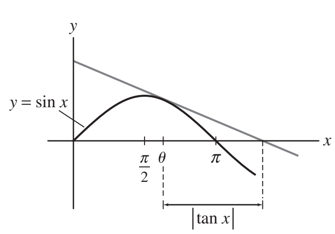


FIGURE 4

SOLUTION Let $f(x) = \sin x$. Since $f'(x) = \cos x$, this means that the tangent line at $(\theta, \sin\theta)$ is $y = \cos\theta(x - \theta) + \sin\theta$. When $y = 0$, $x = \theta - \tan\theta$. The distance from x to θ is then

$$|\theta - (\theta - \tan\theta)| = |\tan\theta|.$$

Further Insights and Challenges

53. Use the limit definition of the derivative and the addition law for the cosine function to prove that $(\cos x)' = -\sin x$.

SOLUTION Let $f(x) = \cos x$. Then

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0} \left((-\sin x) \frac{\sin h}{h} + (\cos x) \frac{\cos h - 1}{h} \right) = (-\sin x)\cdot 1 + (\cos x)\cdot 0 = -\sin x.$$

54. Use the addition formula for the tangent

$$\tan(x+h) = \frac{\tan x + \tan h}{1 + \tan x \tan h}$$

to compute $(\tan x)'$ directly as a limit of the difference quotients. You will also need to show that $\lim_{h\to 0} \frac{\tan h}{h} = 1$.

SOLUTION First note that

$$\lim_{h\to 0} \frac{\tan h}{h} = \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos h} = 1(1) = 1.$$

Now, using the addition formula for tangent,

$$\frac{\tan(x+h) - \tan x}{h} = \frac{\frac{\tan x + \tan h}{1+\tan x \tan h} - \tan x}{h}$$
$$= \frac{\tan h(1 - \tan^2 x)}{h(1 + \tan x \tan h)} = \frac{\tan h}{h} \cdot \frac{\sec^2 x}{1 + \tan x \tan h}.$$

Therefore,

$$\frac{d}{dx} \tan x = \lim_{h\to 0} \frac{\tan h}{h} \cdot \frac{\sec^2 x}{1 + \tan x \tan h}$$
$$= \lim_{h\to 0} \frac{\tan h}{h} \cdot \lim_{h\to 0} \frac{\sec^2 x}{1 + \tan x \tan h}$$
$$= 1(\sec^2 x) = \sec^2 x.$$

55. Verify the following identity and use it to give another proof of the formula $(\sin x)' = \cos x$.

$$\sin(x+h) - \sin x = 2\cos\left(x + \tfrac{1}{2}h\right) \sin\left(\tfrac{1}{2}h\right)$$

Hint: Use the addition formula to prove that $\sin(a+b) - \sin(a-b) = 2\cos a \sin b$.

SOLUTION Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b.$$

Subtracting the second identity from the first yields

$$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

If we now set $a = x + \frac{h}{2}$ and $b = \frac{h}{2}$, then the previous equation becomes

$$\sin(x+h) - \sin x = 2\cos\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right).$$

Finally, we use the limit definition of the derivative of $\sin x$ to obtain

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$
$$= \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \cos x \cdot 1 = \cos x.$$

In other words, $\frac{d}{dx}(\sin x) = \cos x$.

56. Show that a nonzero polynomial function $y = f(x)$ *cannot* satisfy the equation $y'' = -y$. Use this to prove that neither $\sin x$ nor $\cos x$ is a polynomial. Can you think of another way to reach this conclusion by considering limits as $x \to \infty$?

SOLUTION

- Let p be a nonzero polynomial of degree n and *assume* that p satisfies the differential equation $y'' + y = 0$. Then $p'' + p = 0$ for all x. There are exactly three cases.

  (a) If $n = 0$, then p is a constant polynomial and thus $p'' = 0$. Hence $0 = p'' + p = p$ or $p \equiv 0$ (i.e., p is equal to 0 for all x or p is identically 0). This is a contradiction, since p is a *non*zero polynomial.

  (b) If $n = 1$, then p is a linear polynomial and thus $p'' = 0$. Once again, we have $0 = p'' + p = p$ or $p \equiv 0$, a contradiction since p is a nonzero polynomial.

  (c) If $n \geq 2$, then p is at least a quadratic polynomial and thus p'' is a polynomial of degree $n - 2 \geq 0$. Thus $q = p'' + p$ is a polynomial of degree $n \geq 2$. By assumption, however, $p'' + p = 0$. Thus $q \equiv 0$, a polynomial of degree 0. This is a contradiction, since the degree of q is $n \geq 2$.

  CONCLUSION: In all cases, we have reached a contradiction. Therefore the *assumption* that p satisfies the differential equation $y'' + y = 0$ is *false*. Accordingly, a nonzero polynomial *cannot* satisfy the stated differential equation.

- Let $y = \sin x$. Then $y' = \cos x$ and $y'' = -\sin x$. Therefore, $y'' = -y$. Now, let $y = \cos x$. Then $y' = -\sin x$ and $y'' = -\cos x$. Therefore, $y'' = -y$. Because $\sin x$ and $\cos x$ are nonzero functions that satisfy $y'' = -y$, it follows that neither $\sin x$ nor $\cos x$ is a polynomial.

57. Let $f(x) = x \sin x$ and $g(x) = x \cos x$.

(a) Show that $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Verify that $f''(x) = -f(x) + 2\cos x$ and
$g''(x) = -g(x) - 2\sin x$.

(c) By further experimentation, try to find formulas for all higher derivatives of f and g. *Hint:* The kth derivative depends on whether $k = 4n, 4n + 1, 4n + 2$, or $4n + 3$.

SOLUTION Let $f(x) = x \sin x$ and $g(x) = x \cos x$.

(a) We examine first derivatives: $f'(x) = x \cos x + (\sin x) \cdot 1 = g(x) + \sin x$ and $g'(x) = (x)(-\sin x) + (\cos x) \cdot 1 = -f(x) + \cos x$; i.e., $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Now look at second derivatives: $f''(x) = g'(x) + \cos x = -f(x) + 2\cos x$ and $g''(x) = -f'(x) - \sin x = -g(x) - 2\sin x$; i.e., $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c)
- The third derivatives are $f'''(x) = -f'(x) - 2\sin x = -g(x) - 3\sin x$ and $g'''(x) = -g'(x) - 2\cos x = f(x) - 3\cos x$; i.e., $f'''(x) = -g(x) - 3\sin x$ and $g'''(x) = f(x) - 3\cos x$.
- The fourth derivatives are $f^{(4)}(x) = -g'(x) - 3\cos x = f(x) - 4\cos x$ and $g^{(4)}(x) = f'(x) + 3\sin x = g(x) + 4\sin x$; i.e., $f^{(4)} = f(x) - 4\cos x$ and $g^{(4)}(x) = g(x) + 4\sin x$.
- We can now see the pattern for the derivatives, which are summarized in the following table. Here $n = 0, 1, 2, \ldots$

k	$4n$	$4n + 1$	$4n + 2$	$4n + 3$
$f^{(k)}(x)$	$f(x) - k\cos x$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$
$g^{(k)}(x)$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$	$f(x) - k\cos x$

58. Figure 5 shows the geometry behind the derivative formula $(\sin\theta)' = \cos\theta$. Segments $\overline{BA}$ and $\overline{BD}$ are parallel to the x- and y-axes. Let $\Delta \sin\theta = \sin(\theta + h) - \sin\theta$. Verify the following statements.

(a) $\Delta \sin\theta = BC$

(b) $\angle BDA = \theta$ *Hint:* $\overline{OA} \perp AD$.

(c) $BD = (\cos\theta)AD$

Now explain the following intuitive argument: If h is small, then $BC \approx BD$ and $AD \approx h$, so $\Delta \sin\theta \approx (\cos\theta)h$ and $(\sin\theta)' = \cos\theta$.

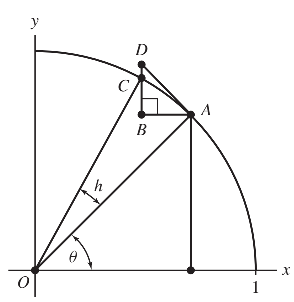


FIGURE 5

SOLUTION

(a) We note that $\sin(\theta + h)$ is the y-coordinate of the point C and $\sin\theta$ is the y-coordinate of the point A, and therefore also of the point B. Now, $\Delta \sin\theta = \sin(\theta + h) - \sin\theta$ can be interpreted as the difference between the y-coordinates of the points B and C; that is, $\Delta \sin\theta = BC$.

(b) From the figure, we note that $\angle OAB = \theta$, so $\angle BAD = \pi - \theta$ and $\angle BDA = \theta$.

(c) Using part (b), it follows that

$$\cos\theta = \frac{BD}{AD} \quad \text{or} \quad BD = (\cos\theta)AD.$$

For h "small," $BC \approx BD$ and AD is roughly the length of the arc subtended from A to C; that is, $AD \approx 1(h) = h$. Thus, using (a) and (c),

$$\Delta \sin\theta = BC \approx BD = (\cos\theta)AD \approx (\cos\theta)h.$$

In the limit as $h \to 0$,

$$\frac{\Delta \sin\theta}{h} \to (\sin\theta)',$$

so $(\sin\theta)' = \cos\theta$.

3.7 The Chain Rule

Preliminary Questions

1. Identify the outside and inside functions for each of these composite functions.

(a) $y = \sqrt{4x + 9x^2}$ **(b)** $y = \tan(x^2 + 1)$

(c) $y = \sec^5 x$ **(d)** $y = (1 + e^x)^4$

SOLUTION

(a) The outer function is $\sqrt{x}$, and the inner function is $4x + 9x^2$.

(b) The outer function is $\tan x$, and the inner function is $x^2 + 1$.

(c) The outer function is x^5, and the inner function is $\sec x$.

(d) The outer function is x^4, and the inner function is $1 + e^x$.

2. Which of the following can be differentiated easily *without* using the Chain Rule?

(a) $y = \tan(7x^2 + 2)$ **(b)** $y = \dfrac{x}{x+1}$

(c) $y = \sqrt{x} \cdot \sec x$ **(d)** $y = \sqrt{x \cos x}$

(e) $y = xe^x$ **(f)** $y = e^{\sin x}$

SOLUTION The function $\frac{x}{x+1}$ can be differentiated using the Quotient Rule, and the functions $\sqrt{x} \cdot \sec x$ and xe^x can be differentiated using the Product Rule. The functions $\tan(7x^2 + 2)$, $\sqrt{x \cos x}$ and $e^{\sin x}$ require the Chain Rule.

3. Which is the derivative of $f(5x)$?

(a) $5f'(x)$ **(b)** $5f'(5x)$ **(c)** $f'(5x)$

SOLUTION The correct answer is **(b)**: $5f'(5x)$.

4. Suppose that $f'(4) = g(4) = g'(4) = 1$. Do we have enough information to compute $F'(4)$, where $F(x) = f(g(x))$? If not, what is missing?

SOLUTION If $F(x) = f(g(x))$, then $F'(x) = f'(g(x))g'(x)$ and $F'(4) = f'(g(4))g'(4)$. Thus, we do not have enough information to compute $F'(4)$. We are missing the value of $f'(1)$.

Exercises

In Exercises 1–4, fill in a table of the following type:

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$

1. $f(u) = u^{3/2}, \quad g(x) = x^4 + 1$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(x^4 + 1)^{3/2}$	$\frac{3}{2}u^{1/2}$	$\frac{3}{2}(x^4 + 1)^{1/2}$	$4x^3$	$6x^3(x^4 + 1)^{1/2}$

2. $f(u) = u^3, \quad g(x) = 3x + 5$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(3x+5)^3$	$3u^2$	$3(3x+5)^2$	3	$9(3x+5)^2$

3. $f(u) = \tan u, \quad g(x) = x^4$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$\tan(x^4)$	$\sec^2 u$	$\sec^2(x^4)$	$4x^3$	$4x^3 \sec^2(x^4)$

4. $f(u) = u^4 + u, \quad g(x) = \cos x$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(\cos x)^4 + \cos x$	$4u^3 + 1$	$4(\cos x)^3 + 1$	$-\sin x$	$-4 \sin x \cos^3 x - \sin x$

In Exercises 5 and 6, write the function as a composite $f(g(x))$ and compute the derivative using the Chain Rule.

5. $y = (x + \sin x)^4$

SOLUTION Let $f(x) = x^4$, $g(x) = x + \sin x$, and $y = f(g(x)) = (x + \sin x)^4$. Then

$$\frac{dy}{dx} = f'(g(x))g'(x) = 4(x + \sin x)^3(1 + \cos x).$$

6. $y = \cos(x^3)$

SOLUTION Let $f(x) = \cos x$, $g(x) = x^3$, and $y = f(g(x)) = \cos(x^3)$. Then

$$\frac{dy}{dx} = f'(g(x))g'(x) = -3x^2 \sin(x^3).$$

7. Calculate $\frac{d}{dx} \cos u$ for the following choices of $u(x)$:

(a) $u = 9 - x^2$ **(b)** $u = x^{-1}$ **(c)** $u = \tan x$

SOLUTION

(a) $\cos(u(x)) = \cos(9 - x^2)$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(9 - x^2)(-2x) = 2x \sin(9 - x^2).$$

(b) $\cos(u(x)) = \cos(x^{-1})$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(x^{-1})\left(-\frac{1}{x^2}\right) = \frac{\sin(x^{-1})}{x^2}.$$

(c) $\cos(u(x)) = \cos(\tan x)$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(\tan x)(\sec^2 x) = -\sec^2 x \sin(\tan x).$$

8. Calculate $\frac{d}{dx} f(x^2 + 1)$ for the following choices of $f(u)$:

(a) $f(u) = \sin u$ **(b)** $f(u) = 3u^{3/2}$ **(c)** $f(u) = u^2 - u$

SOLUTION

(a) Let $\sin(u) = \sin(x^2 + 1)$. Then

$$\frac{d}{dx}\sin(x^2 + 1) = \cos(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) = \cos(x^2 + 1)2x = 2x \cos(x^2 + 1).$$

(b) Let $3u^{3/2} = 3(x^2+1)^{3/2}$. Then

$$\frac{d}{dx}3(x^2+1)^{3/2} = 3\cdot\frac{3}{2}(x^2+1)^{1/2}\frac{d}{dx}(x^2+1) = \frac{9}{2}(x^2+1)^{1/2}(2x) = 9x(x^2+1)^{1/2}.$$

(c) Let $u^2 - u = (x^2+1)^2 - (x^2+1)$. Then

$$\frac{d}{dx}\left((x^2+1)^2-(x^2+1)\right) = [2(x^2+1)-1]\frac{d}{dx}(x^2+1) = [2(x^2+1)-1](2x) = 4x^3+2x.$$

9. Compute $\frac{df}{dx}$ if $\frac{df}{du} = 2$ and $\frac{du}{dx} = 6$.

SOLUTION Assuming f is a function of u, which is in turn a function of x,

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = 2(6) = 12.$$

10. Compute $\left.\frac{df}{dx}\right|_{x=2}$ if $f(u) = u^2$, $u(2) = -5$, and $u'(2) = -5$.

SOLUTION Because $f(u) = u^2$, it follows that $f'(u) = 2u$. Therefore,

$$\left.\frac{df}{dx}\right|_{x=2} = f'(u(2))u'(2) = 2u(2)u'(2) = 2(-5)(-5) = 50.$$

In Exercises 11–22, use the General Power Rule or the Shifting and Scaling Rule to compute the derivative.

11. $y = (x^4+5)^3$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^4+5)^3 = 3(x^4+5)^2\frac{d}{dx}(x^4+5) = 3(x^4+5)^2(4x^3) = 12x^3(x^4+5)^2.$$

12. $y = (8x^4+5)^3$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(8x^4+5)^3 = 3(8x^4+5)^2\frac{d}{dx}(8x^4+5) = 3(8x^4+5)^2(32x^3) = 96x^3(8x^4+5)^2.$$

13. $y = \sqrt{7x-3}$

SOLUTION Using the Shifting and Scaling Rule

$$\frac{d}{dx}\sqrt{7x-3} = \frac{d}{dx}(7x-3)^{1/2} = \frac{1}{2}(7x-3)^{-1/2}(7) = \frac{7}{2\sqrt{7x-3}}.$$

14. $y = (4-2x-3x^2)^5$

SOLUTION Using the General Power Rule,

$$\begin{aligned}\frac{d}{dx}(4-2x-3x^2)^5 &= 5(4-2x-3x^2)^4\frac{d}{dx}(4-2x-3x^2) = 5(4-2x-3x^2)^4(-2-6x)\\ &= -10(1+3x)(4-2x-3x^2)^4.\end{aligned}$$

15. $y = (x^2+9x)^{-2}$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^2+9x)^{-2} = -2(x^2+9x)^{-3}\frac{d}{dx}(x^2+9x) = -2(x^2+9x)^{-3}(2x+9).$$

16. $y = (x^3+3x+9)^{-4/3}$

SOLUTION Using the General Power Rule,

$$\begin{aligned}\frac{d}{dx}(x^3+3x+9)^{-4/3} &= -\frac{4}{3}(x^3+3x+9)^{-7/3}\frac{d}{dx}(x^3+3x+9) = -\frac{4}{3}(x^3+3x+9)^{-7/3}(3x^2+3)\\ &= -4(x^2+1)(x^3+3x+9)^{-7/3}.\end{aligned}$$

17. $y = \cos^4 \theta$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta} \cos^4 \theta = 4 \cos^3 \theta \frac{d}{d\theta} \cos \theta = -4 \cos^3 \theta \sin \theta.$$

18. $y = \cos(9\theta + 41)$

SOLUTION Using the Shifting and Scaling Rule

$$\frac{d}{d\theta} \cos(9\theta + 41) = -9 \sin(9\theta + 41).$$

19. $y = (2 \cos \theta + 5 \sin \theta)^9$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta}(2 \cos \theta + 5 \sin \theta)^9 = 9(2 \cos \theta + 5 \sin \theta)^8 \frac{d}{d\theta}(2 \cos \theta + 5 \sin \theta) = 9(2 \cos \theta + 5 \sin \theta)^8 (5 \cos \theta - 2 \sin \theta).$$

20. $y = \sqrt{9 + x + \sin x}$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}\sqrt{9 + x + \sin x} = \frac{1}{2}(9 + x + \sin x)^{-1/2} \frac{d}{dx}(9 + x + \sin x) = \frac{1 + \cos x}{2\sqrt{9 + x + \sin x}}.$$

21. $y = e^{x-12}$

SOLUTION Using the Shifting and Scaling Rule,

$$\frac{d}{dx} e^{x-12} = (1)e^{x-12} = e^{x-12}.$$

22. $y = e^{8x+9}$

SOLUTION Using the Shifting and Scaling Rule,

$$\frac{d}{dx} e^{8x+9} = 8e^{8x+9}.$$

In Exercises 23–26, compute the derivative of $f \circ g$.

23. $f(u) = \sin u, \quad g(x) = 2x + 1$

SOLUTION Let $h(x) = f(g(x)) = \sin(2x + 1)$. Then, applying the shifting and scaling rule, $h'(x) = 2 \cos(2x + 1)$. Alternately,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = \cos(2x + 1) \cdot 2 = 2 \cos(2x + 1).$$

24. $f(u) = 2u + 1, \quad g(x) = \sin x$

SOLUTION Let $h(x) = f(g(x)) = 2(\sin x) + 1$. Then $h'(x) = 2 \cos x$. Alternately,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = 2 \cos x.$$

25. $f(u) = e^u, \quad g(x) = x + x^{-1}$

SOLUTION Let $h(x) = f(g(x)) = e^{x+x^{-1}}$. Then

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = e^{x+x^{-1}} \left(1 - x^{-2}\right).$$

26. $f(u) = \dfrac{u}{u-1}, \quad g(x) = \csc x$

SOLUTION Let $h(x) = f(g(x))$. Then, applying the quotient rule:

$$h'(x) = \frac{(\csc x - 1)(-\csc x \cot x) - (\csc x)(-\csc x \cot x)}{(\csc x - 1)^2} = \frac{\csc x \cot x}{(\csc x - 1)^2}.$$

Alternately,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = -\frac{1}{(\csc x - 1)^2}(-\csc x \cot x) = \frac{\csc x \cot x}{(\csc x - 1)^2},$$

where we have used the quotient rule to calculate $f'(u) = -\frac{1}{(u-1)^2}$.

In Exercises 27 and 28, find the derivatives of $f(g(x))$ and $g(f(x))$.

27. $f(u) = \cos u, \quad u = g(x) = x^2 + 1$

SOLUTION

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = -\sin(x^2 + 1)(2x) = -2x \sin(x^2 + 1).$$

$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = 2(\cos x)(-\sin x) = -2 \sin x \cos x.$$

28. $f(u) = u^3, \quad u = g(x) = \dfrac{1}{x+1}$

SOLUTION The derivative of $\frac{1}{x+1}$ is taken using the shifting and scaling rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = 3\left(\frac{1}{x+1}\right)^2 \left(-\frac{1}{(x+1)^2}\right) = -\frac{3}{(x+1)^4}.$$

$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = -\frac{1}{(x^3+1)^2}(3x^2) = -\frac{3x^2}{(x^3+1)^2}.$$

In Exercises 29–42, use the Chain Rule to find the derivative.

29. $y = \sin(x^2)$

SOLUTION Let $y = \sin\left(x^2\right)$. Then $y' = \cos\left(x^2\right) \cdot 2x = 2x \cos\left(x^2\right)$.

30. $y = \sin^2 x$

SOLUTION Let $y = \sin^2 x = (\sin x)^2$. Then $y' = 2 \sin x (\cos x)$.

31. $y = \sqrt{t^2 + 9}$

SOLUTION Let $y = \sqrt{t^2 + 9} = (t^2 + 9)^{1/2}$. Then

$$y' = \frac{1}{2}(t^2 + 9)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 9}}.$$

32. $y = (t^2 + 3t + 1)^{-5/2}$

SOLUTION Let $y = \left(t^2 + 3t + 1\right)^{-5/2}$. Then

$$y' = -\frac{5}{2}\left(t^2 + 3t + 1\right)^{-7/2}(2t + 3) = -\frac{5\,(2t + 3)}{2\left(t^2 + 3t + 1\right)^{7/2}}.$$

33. $y = (x^4 - x^3 - 1)^{2/3}$

SOLUTION Let $y = \left(x^4 - x^3 - 1\right)^{2/3}$. Then

$$y' = \frac{2}{3}\left(x^4 - x^3 - 1\right)^{-1/3}\left(4x^3 - 3x^2\right).$$

34. $y = (\sqrt{x+1} - 1)^{3/2}$

SOLUTION Let $y = \left((x+1)^{1/2} - 1\right)^{3/2}$. Here, we note that calculating the derivative of the inside function, $\sqrt{x+1} - 1$, requires the chain rule (in the form of the scaling and shifting rule). After two applications of the chain rule, we have

$$y' = \frac{3}{2}\left((x+1)^{1/2} - 1\right)^{1/2} \cdot \left(\frac{1}{2}(x+1)^{-1/2} \cdot 1\right) = \frac{3\sqrt{\sqrt{x+1} - 1}}{4\sqrt{x+1}}.$$

35. $y = \left(\frac{x+1}{x-1}\right)^4$

SOLUTION Let $y = \left(\frac{x+1}{x-1}\right)^4$. Then

$$y' = 4\left(\frac{x+1}{x-1}\right)^3 \cdot \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} = -\frac{8(x+1)^3}{(x-1)^5} = \frac{8(1+x)^3}{(1-x)^5}.$$

36. $y = \cos^3(12\theta)$

SOLUTION After two applications of the chain rule,

$$y' = 3\cos^2(12\theta)(-\sin(12\theta))(12) = -36\cos^2(12\theta)\sin(12\theta).$$

37. $y = \sec\frac{1}{x}$

SOLUTION Let $f(x) = \sec\left(x^{-1}\right)$. Then

$$f'(x) = \sec\left(x^{-1}\right)\tan\left(x^{-1}\right) \cdot \left(-x^{-2}\right) = -\frac{\sec(1/x)\tan(1/x)}{x^2}.$$

38. $y = \tan(\theta^2 - 4\theta)$

SOLUTION Let $y = \tan(\theta^2 - 4\theta)$. Then

$$y' = \sec^2(\theta^2 - 4\theta) \cdot (2\theta - 4) = (2\theta - 4)\sec^2(\theta^2 - 4\theta).$$

39. $y = \tan(\theta + \cos\theta)$

SOLUTION Let $y = \tan(\theta + \cos\theta)$. Then

$$y' = \sec^2(\theta + \cos\theta) \cdot (1 - \sin\theta) = (1 - \sin\theta)\sec^2(\theta + \cos\theta).$$

40. $y = e^{2x^2}$

SOLUTION Let $y = e^{2x^2}$. Then

$$y' = e^{2x^2}(4x) = 4xe^{2x^2}.$$

41. $y = e^{2-9t^2}$

SOLUTION Let $y = e^{2-9t^2}$. Then

$$y' = e^{2-9t^2}(-18t) = -18te^{2-9t^2}.$$

42. $y = \cos^3(e^{4\theta})$

SOLUTION Let $y = \cos^3(e^{4\theta})$. After two applications of the chain rule, we have

$$y' = 3\cos^2(e^{4\theta})(-\sin(e^{4\theta}))(4e^{4\theta}) = -12e^{4\theta}\cos^2(e^{4\theta})\sin(e^{4\theta}).$$

In Exercises 43–72, find the derivative using the appropriate rule or combination of rules.

43. $y = \tan(x^2 + 4x)$

SOLUTION Let $y = \tan(x^2 + 4x)$. By the chain rule,

$$y' = \sec^2(x^2 + 4x) \cdot (2x + 4) = (2x + 4)\sec^2(x^2 + 4x).$$

44. $y = \sin(x^2 + 4x)$

SOLUTION Let $y = \sin(x^2 + 4x)$. By the chain rule,

$$\frac{dy}{dx} = (2x + 4)\cos(x^2 + 4x).$$

45. $y = x\cos(1 - 3x)$

SOLUTION Let $y = x\cos(1 - 3x)$. Applying the product rule and then the scaling and shifting rule,

$$y' = x(-\sin(1 - 3x)) \cdot (-3) + \cos(1 - 3x) \cdot 1 = 3x\sin(1 - 3x) + \cos(1 - 3x).$$

46. $y = \sin(x^2)\cos(x^2)$

SOLUTION We start by using a trig identity to rewrite

$$y = \sin(x^2)\cos(x^2) = \frac{1}{2}\sin(2x^2).$$

Then, by the chain rule,

$$y' = \frac{1}{2}\cos(2x^2) \cdot 4x = 2x\cos(2x^2).$$

47. $y = (4t + 9)^{1/2}$

SOLUTION Let $y = (4t + 9)^{1/2}$. By the shifting and scaling rule,

$$\frac{dy}{dt} = 4\left(\frac{1}{2}\right)(4t + 9)^{-1/2} = 2(4t + 9)^{-1/2}.$$

48. $y = (z + 1)^4(2z - 1)^3$

SOLUTION Let $y = (z + 1)^4(2z - 1)^3$. Applying the product rule and the general power rule,

$$\begin{aligned}\frac{dy}{dz} &= (z + 1)^4(3(2z - 1)^2)(2) + (2z - 1)^3(4(z + 1)^3)(1) = (z + 1)^3(2z - 1)^2(6(z + 1) + 4(2z - 1)) \\ &= (z + 1)^3(2z - 1)^2(14z + 2).\end{aligned}$$

49. $y = (x^3 + \cos x)^{-4}$

SOLUTION Let $y = (x^3 + \cos x)^{-4}$. By the general power rule,

$$y' = -4(x^3 + \cos x)^{-5}(3x^2 - \sin x) = 4(\sin x - 3x^2)(x^3 + \cos x)^{-5}.$$

50. $y = \sin(\cos(\sin x))$

SOLUTION Let $y = \sin(\cos(\sin x))$. Applying the chain rule twice,

$$y' = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x = -\cos x\sin(\sin x)\cos(\cos(\sin x)).$$

51. $y = \sqrt{\sin x\cos x}$

SOLUTION We start by using a trig identity to rewrite

$$y = \sqrt{\sin x\cos x} = \sqrt{\frac{1}{2}\sin 2x} = \frac{1}{\sqrt{2}}(\sin 2x)^{1/2}.$$

Then, after two applications of the chain rule,

$$y' = \frac{1}{\sqrt{2}} \cdot \frac{1}{2}(\sin 2x)^{-1/2} \cdot \cos 2x \cdot 2 = \frac{\cos 2x}{\sqrt{2\sin 2x}}.$$

52. $y = (9 - (5 - 2x^4)^7)^3$

SOLUTION Let $y = (9 - (5 - 2x^4)^7)^3$. Applying the chain rule twice, we find

$$y' = 3(9 - (5 - 2x^4)^7)^2(-7(5 - 2x^4)^6)(-8x^3) = 168x^3(5 - 2x^4)^6(9 - (5 - 2x^4)^7)^2.$$

53. $y = (\cos 6x + \sin x^2)^{1/2}$

SOLUTION Let $y = (\cos 6x + \sin(x^2))^{1/2}$. Applying the general power rule followed by both the scaling and shifting rule and the chain rule,

$$y' = \frac{1}{2}\left(\cos 6x + \sin(x^2)\right)^{-1/2}\left(-\sin 6x \cdot 6 + \cos(x^2) \cdot 2x\right) = \frac{x\cos(x^2) - 3\sin 6x}{\sqrt{\cos 6x + \sin(x^2)}}.$$

54. $y = \dfrac{(x+1)^{1/2}}{x+2}$

SOLUTION Let $y = \frac{(x+1)^{1/2}}{x+2}$. Applying the quotient rule and the shifting and scaling rule, we get

$$\frac{dy}{dx} = \frac{(x+2)\frac{1}{2}(x+1)^{-1/2} - (x+1)^{1/2}}{(x+2)^2} = \frac{1}{2\sqrt{x+1}}\frac{(x+2) - 2(x+1)}{(x+2)^2} = -\frac{1}{2\sqrt{x+1}}\frac{x}{(x+2)^2}.$$

55. $y = \tan^3 x + \tan(x^3)$

SOLUTION Let $y = \tan^3 x + \tan(x^3) = (\tan x)^3 + \tan(x^3)$. Applying the general power rule to the first term and the chain rule to the second term,

$$y' = 3(\tan x)^2 \sec^2 x + \sec^2(x^3) \cdot 3x^2 = 3\left(x^2 \sec^2(x^3) + \sec^2 x \tan^2 x\right).$$

56. $y = \sqrt{4 - 3\cos x}$

SOLUTION Let $y = (4 - 3\cos x)^{1/2}$. By the general power rule,

$$y' = \frac{1}{2}(4 - 3\cos x)^{-1/2} \cdot 3\sin x = \frac{3\sin x}{2\sqrt{4 - 3\cos x}}.$$

57. $y = \sqrt{\dfrac{z+1}{z-1}}$

SOLUTION Let $y = \left(\dfrac{z+1}{z-1}\right)^{1/2}$. Applying the general power rule followed by the quotient rule,

$$\frac{dy}{dz} = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{-1/2} \cdot \frac{(z-1)\cdot 1 - (z+1)\cdot 1}{(z-1)^2} = \frac{-1}{\sqrt{z+1}\,(z-1)^{3/2}}.$$

58. $y = (\cos^3 x + 3\cos x + 7)^9$

SOLUTION Let $y = \left(\cos^3 x + 3\cos x + 7\right)^9$. Applying the general power rule followed by the sum rule, with the first term requiring the general power rule,

$$\begin{aligned}\frac{dy}{dx} &= 9\left(\cos^3 x + 3\cos x + 7\right)^8\left(3\cos^2 x \cdot (-\sin x) - 3\sin x\right)\\ &= -27\sin x\left(\cos^3 x + 3\cos x + 7\right)^8\left(1 + \cos^2 x\right).\end{aligned}$$

59. $y = \dfrac{\cos(1+x)}{1+\cos x}$

SOLUTION Let

$$y = \frac{\cos(1+x)}{1+\cos x}.$$

Then, applying the quotient rule and the shifting and scaling rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{-(1+\cos x)\sin(1+x) + \cos(1+x)\sin x}{(1+\cos x)^2} = \frac{\cos(1+x)\sin x - \cos x\sin(1+x) - \sin(1+x)}{(1+\cos x)^2}\\ &= \frac{\sin(-1) - \sin(1+x)}{(1+\cos x)^2}.\end{aligned}$$

The last line follows from the identity

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

with $A = x$ and $B = 1 + x$.

60. $y = \sec(\sqrt{t^2 - 9})$

SOLUTION Let $y = \sec\left(\sqrt{t^2 - 9}\right)$. Applying the chain rule followed by the general power rule,

$$\frac{dy}{dt} = \sec\left(\sqrt{t^2 - 9}\right)\tan\left(\sqrt{t^2 - 9}\right) \cdot \frac{1}{2}\left(t^2 - 9\right)^{-1/2} \cdot 2t = \frac{t\sec\left(\sqrt{t^2 - 9}\right)\tan\left(\sqrt{t^2 - 9}\right)}{\sqrt{t^2 - 9}}.$$

61. $y = \cot^7(x^5)$

SOLUTION Let $y = \cot^7\left(x^5\right)$. Applying the general power rule followed by the chain rule,

$$\frac{dy}{dx} = 7\cot^6\left(x^5\right) \cdot \left(-\csc^2\left(x^5\right)\right) \cdot 5x^4 = -35x^4\cot^6\left(x^5\right)\csc^2\left(x^5\right).$$

62. $y = \dfrac{\cos(1/x)}{1 + x^2}$

SOLUTION Let $y = \frac{\cos(1/x)}{1+x^2} = \frac{\cos(x^{-1})}{1+x^2}$. Then, applying the quotient rule and the chain rule, we get:

$$\frac{dy}{dx} = \frac{(1 + x^2)(x^{-2}\sin(x^{-1})) - \cos(x^{-1})(2x)}{(1 + x^2)^2} = \frac{\sin(x^{-1}) - 2x\cos(x^{-1}) + x^{-2}\sin(x^{-1})}{(1 + x^2)^2}.$$

63. $y = \left(1 + \cot^5(x^4 + 1)\right)^9$

SOLUTION Let $y = \left(1 + \cot^5\left(x^4 + 1\right)\right)^9$. Applying the general power rule, the chain rule, and the general power rule in succession,

$$\begin{aligned}\frac{dy}{dx} &= 9\left(1 + \cot^5\left(x^4 + 1\right)\right)^8 \cdot 5\cot^4\left(x^4 + 1\right) \cdot \left(-\csc^2\left(x^4 + 1\right)\right) \cdot 4x^3 \\ &= -180x^3\cot^4\left(x^4 + 1\right)\csc^2\left(x^4 + 1\right)\left(1 + \cot^5\left(x^4 + 1\right)\right)^8.\end{aligned}$$

64. $y = 4e^{-x} + 7e^{-2x}$

SOLUTION Let $y = 4e^{-x} + 7e^{-2x}$. Using the chain rule twice, once for each exponential function, we obtain

$$\frac{dy}{dx} = -4e^{-x} - 14e^{-2x}.$$

65. $y = (2e^{3x} + 3e^{-2x})^4$

SOLUTION Let $y = (2e^{3x} + 3e^{-2x})^4$. Applying the general power rule followed by two applications of the chain rule, one for each exponential function, we find

$$\frac{dy}{dx} = 4(2e^{3x} + 3e^{-2x})^3(6e^{3x} - 6e^{-2x}) = 24(2e^{3x} + 3e^{-2x})^3(e^{3x} - e^{-2x}).$$

66. $y = \cos(te^{-2t})$

SOLUTION Let $y = \cos(te^{-2t})$. Applying the chain rule and the product rule, we have

$$\frac{dy}{dt} = -\sin(te^{-2t})\left(-2te^{-2t} + e^{-2t}\right) = e^{-2t}(2t - 1)\sin(te^{-2t}).$$

67. $y = e^{(x^2+2x+3)^2}$

SOLUTION Let $y = e^{(x^2+2x+3)^2}$. By the chain rule and the general power rule, we obtain

$$\frac{dy}{dx} = e^{(x^2+2x+3)^2} \cdot 2(x^2 + 2x + 3)(2x + 2) = 4(x + 1)(x^2 + 2x + 3)e^{(x^2+2x+3)^2}.$$

68. $y = e^{e^x}$

SOLUTION Let $y = e^{e^x}$. Applying the chain rule, we have

$$\frac{dy}{dx} = e^{e^x} e^x.$$

69. $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

SOLUTION Let $y = \left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{1/2}$. Applying the general power rule twice,

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}\left(1 + x^{1/2}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}.$$

70. $y = \sqrt{\sqrt{x+1} + 1}$

SOLUTION Let $y = \left(1 + (x+1)^{1/2}\right)^{1/2}$. Applying the general power rule twice,

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + (x+1)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}(x+1)^{-1/2} \cdot 1 = \frac{1}{4\sqrt{x+1}\sqrt{1+\sqrt{x+1}}}.$$

71. $y = (kx + b)^{-1/3}$; k and b any constants

SOLUTION Let $y = (kx + b)^{-1/3}$, where b and k are constants. By the scaling and shifting rule,

$$y' = -\frac{1}{3}(kx + b)^{-4/3} \cdot k = -\frac{k}{3}(kx + b)^{-4/3}.$$

72. $y = \dfrac{1}{\sqrt{kt^4 + b}}$; k, b constants, not both zero

SOLUTION Let $y = \left(kt^4 + b\right)^{-1/2}$, where b and k are constants. By the chain rule,

$$y' = -\frac{1}{2}\left(kt^4 + b\right)^{-3/2} \cdot 4kt^3 = -\frac{2kt^3}{\left(kt^4 + b\right)^{3/2}}.$$

In Exercises 73–76, compute the higher derivative.

73. $\dfrac{d^2}{dx^2}\sin(x^2)$

SOLUTION Let $f(x) = \sin\left(x^2\right)$. Then, by the chain rule, $f'(x) = 2x\cos\left(x^2\right)$ and, by the product rule and the chain rule,

$$f''(x) = 2x\left(-\sin\left(x^2\right) \cdot 2x\right) + 2\cos\left(x^2\right) = 2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right).$$

74. $\dfrac{d^2}{dx^2}(x^2 + 9)^5$

SOLUTION Let $f(x) = (x^2 + 9)^5$. Then, by the general power rule,

$$f'(x) = 5(x^2 + 9)^4 \cdot 2x = 10x(x^2 + 9)^4$$

and, by the product rule and the general power rule,

$$f''(x) = 10x \cdot 4(x^2 + 9)^3 \cdot 2x + 10(x^2 + 9)^4 = 80x^2(x^2 + 9)^3 + 10(x^2 + 9)^4.$$

75. $\dfrac{d^3}{dx^3}(9-x)^8$

SOLUTION Let $f(x) = (9-x)^8$. Then, by repeated use of the scaling and shifting rule,

$$f'(x) = 8(9-x)^7 \cdot (-1) = -8(9-x)^7$$
$$f''(x) = -56(9-x)^6 \cdot (-1) = 56(9-x)^6,$$
$$f'''(x) = 336(9-x)^5 \cdot (-1) = -336(9-x)^5.$$

76. $\dfrac{d^3}{dx^3}\sin(2x)$

SOLUTION Let $f(x) = \sin(2x)$. Then, by repeated use of the scaling and shifting rule,

$$f'(x) = 2\cos(2x)$$
$$f''(x) = -4\sin(2x)$$
$$f'''(x) = -8\cos(2x).$$

77. The average molecular velocity v of a gas in a certain container is given by $v = 29\sqrt{T}$ m/s, where T is the temperature in kelvins. The temperature is related to the pressure (in atmospheres) by $T = 200P$. Find $\left.\dfrac{dv}{dP}\right|_{P=1.5}$.

SOLUTION First note that when $P = 1.5$ atmospheres, $T = 200(1.5) = 300$K. Thus,

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \left.\frac{dv}{dT}\right|_{T=300} \cdot \left.\frac{dT}{dP}\right|_{P=1.5} = \frac{29}{2\sqrt{300}} \cdot 200 = \frac{290\sqrt{3}}{3} \frac{\text{m}}{\text{s} \cdot \text{atmospheres}}.$$

Alternately, substituting $T = 200P$ into the equation for v gives $v = 290\sqrt{2P}$. Therefore,

$$\frac{dv}{dP} = \frac{290\sqrt{2}}{2\sqrt{P}} = \frac{290}{\sqrt{2P}},$$

so

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \frac{290}{\sqrt{3}} = \frac{290\sqrt{3}}{3} \frac{\text{m}}{\text{s} \cdot \text{atmospheres}}.$$

78. The power P in a circuit is $P = Ri^2$, where R is the resistance and i is the current. Find dP/dt at $t = \frac{1}{3}$ if $R = 1000\ \Omega$ and i varies according to $i = \sin(4\pi t)$ (time in seconds).

SOLUTION $\left.\dfrac{d}{dt}\left(Ri^2\right)\right|_{t=1/3} = 2Ri\left.\dfrac{di}{dt}\right|_{t=2} = 2(1000)4\pi\sin(4\pi t)\cos(4\pi t)|_{t=1/3} = 2000\pi\sqrt{3}.$

79. An expanding sphere has radius $r = 0.4t$ cm at time t (in seconds). Let V be the sphere's volume. Find dV/dt when (a) $r = 3$ and (b) $t = 3$.

SOLUTION Let $r = 0.4t$, where t is in seconds (s) and r is in centimeters (cm). With $V = \frac{4}{3}\pi r^3$, we have

$$\frac{dV}{dr} = 4\pi r^2.$$

Thus

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \cdot (0.4) = 1.6\pi r^2.$$

(a) When $r = 3$, $\dfrac{dV}{dt} = 1.6\pi(3)^2 \approx 45.24$ cm/s.

(b) When $t = 3$, we have $r = 1.2$. Hence $\dfrac{dV}{dt} = 1.6\pi(1.2)^2 \approx 7.24$ cm/s.

80. A 2005 study by the Fisheries Research Services in Aberdeen, Scotland, suggests that the average length of the species *Clupea harengus* (Atlantic herring) as a function of age t (in years) can be modeled by $L(t) = 32(1 - e^{-0.37t})$ cm for $0 \le t \le 13$. See Figure 2.

(a) How fast is the length changing at age $t = 6$ years?

(b) At what age is the length changing at a rate of 5 cm/yr?

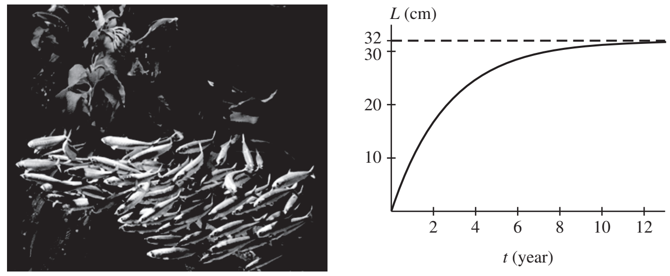


FIGURE 2 Average length of the species *Clupea harengus*

SOLUTION Let $L(t) = 32(1 - e^{-0.37t})$. Then

$$L'(t) = 32(0.37)e^{-0.37t} = 11.84e^{-0.37t}.$$

(a) At age $t = 6$,

$$L'(t) = 11.84e^{-0.37(6)} = 11.84e^{-2.22} \approx 1.29 \text{ cm/yr}.$$

(b) The length will be changing at a rate of 5 cm/yr when

$$11.84e^{-0.37t} = 5.$$

Solving for t yields

$$t = -\frac{1}{0.37}\ln\frac{5}{11.84} \approx 2.33 \text{ years}.$$

81. A 1999 study by Starkey and Scarnecchia developed the following model for the average weight (in kilograms) at age t (in years) of channel catfish in the Lower Yellowstone River (Figure 3):

$$W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$$

Find the rate at which weight is changing at age $t = 10$.

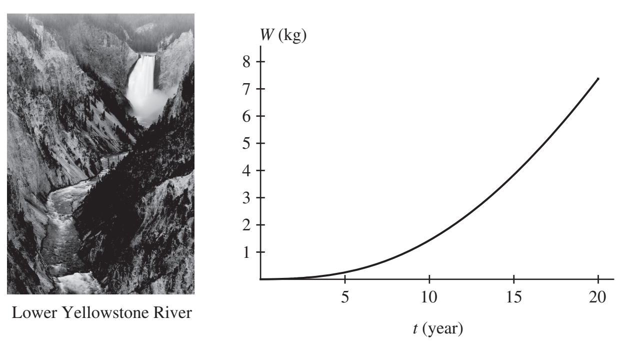


FIGURE 3 Average weight of channel catfish at age t

SOLUTION Let $W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$. Then

$$\begin{aligned} W'(t) &= 3.4026(3.46293 - 3.32173e^{-0.03456t})^{2.4026}(3.32173)(0.03456)e^{-0.03456t} \\ &= 0.3906(3.46293 - 3.32173e^{-0.03456t})^{2.4026}e^{-0.03456t}. \end{aligned}$$

At age $t = 10$,

$$W'(10) = 0.3906(1.1118)^{2.4026}(0.7078) \approx 0.3566 \text{ kg/yr}.$$

82. The functions in Exercises 80 and 81 are examples of the **von Bertalanffy growth function**

$$M(t) = \left(a + (b - a)e^{kmt}\right)^{1/m} \qquad (m \neq 0)$$

introduced in the 1930s by Austrian-born biologist Karl Ludwig von Bertalanffy. Calculate $M'(0)$ in terms of the constants a, b, k and m.

SOLUTION Let

$$M(t) = \left(a + (b-a)e^{kmt}\right)^{1/m} \qquad (m \neq 0).$$

Then

$$M'(t) = \frac{1}{m}(a + (b-a)e^{kmt})^{1/m-1} km(b-a)e^{kmt} = k(b-a)e^{kmt}(a + (b-a)e^{kmt})^{1/m-1},$$

and

$$M'(0) = k(b-a)e^{0}(a + (b-a)e^{0})^{1/m-1} = k(b-a)b^{1/m-1}.$$

83. With notation as in Example 7, calculate

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ}$ **(b)** $\left.\dfrac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\right|_{\theta=45^\circ}$

SOLUTION

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ} = \left.\dfrac{d}{d\theta} \sin\left(\dfrac{\pi}{180}\theta\right)\right|_{\theta=60^\circ} = \left(\dfrac{\pi}{180}\right)\cos\left(\dfrac{\pi}{180}(60)\right) = \dfrac{\pi}{180}\dfrac{1}{2} = \dfrac{\pi}{360}.$

(b) $\left.\dfrac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\right|_{\theta=45^\circ} = \left.\dfrac{d}{d\theta}\left(\theta + \tan\left(\dfrac{\pi}{180}\theta\right)\right)\right|_{\theta=45^\circ} = 1 + \dfrac{\pi}{180}\sec^2\left(\dfrac{\pi}{4}\right) = 1 + \dfrac{\pi}{90}.$

84. Assume that

$$f(0) = 2, \quad f'(0) = 3, \quad h(0) = -1, \quad h'(0) = 7$$

Calculate the derivatives of the following functions at $x = 0$:

(a) $(f(x))^3$ **(b)** $f(7x)$ **(c)** $f(4x)h(5x)$

SOLUTION

(a) Let $g(x) = (f(x))^3$. Then

$$g'(0) = 3(f(0))^2(f'(0)) = 12(3) = 36.$$

(b) Let $g(x) = f(7x)$. Then

$$g'(0) = 7f'(7(0)) = 21.$$

(c) Let $F(x) = f(4x)h(5x)$. Then $F'(x) = 4f'(4x)h(5x) + 5f(4x)h'(5x)$ and

$$F'(0) = 4(3)(-1) + 5(2)(7) = 58.$$

85. Compute the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$, assuming that $h'(0.5) = 10$.

SOLUTION Let $u = \sin x$ and suppose that $h'(0.5) = 10$. Then

$$\frac{d}{dx}(h(u)) = \frac{dh}{du}\frac{du}{dx} = \frac{dh}{du}\cos x.$$

When $x = \frac{\pi}{6}$, we have $u = .5$. Accordingly, the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$ is $10\cos\left(\frac{\pi}{6}\right) = 5\sqrt{3}$.

86. Let $F(x) = f(g(x))$, where the graphs of f and g are shown in Figure 4. Estimate $g'(2)$ and $f'(g(2))$ and compute $F'(2)$.

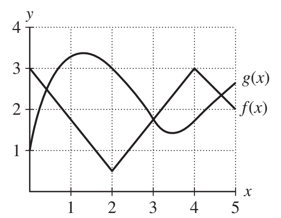
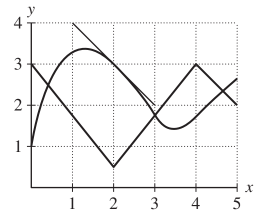


FIGURE 4

SOLUTION After sketching an approximate tangent line to g at $x = 2$ (see the figure below), we estimate $g'(2) = -1$. It appears from the graph that $g(2) = 3$ and $f'(3) = \frac{5}{4}$ (since between $x = 2$ and $x = 4$ the graph of f appears to be linear with slope $\frac{5}{4}$). Thus,

$$F'(2) = f'(g(2))g'(2) = \frac{5}{4}(-1) = -1.25.$$



In Exercises 87–90, use the table of values to calculate the derivative of the function at the given point.

x	1	4	6
$f(x)$	4	0	6
$f'(x)$	5	7	4
$g(x)$	4	1	6
$g'(x)$	5	$\frac{1}{2}$	3

87. $f(g(x)), \quad x = 6$

SOLUTION $\left.\frac{d}{dx} f(g(x))\right|_{x=6} = f'(g(6))g'(6) = f'(6)g'(6) = 4 \times 3 = 12.$

88. $e^{f(x)}, \quad x = 4$

SOLUTION $\left.\frac{d}{dx} e^{f(x)}\right|_{x=4} = e^{f(4)} f'(4) = e^0(7) = 7.$

89. $g(\sqrt{x}), \quad x = 16$

SOLUTION $\left.\frac{d}{dx} g(\sqrt{x})\right|_{x=16} = g'(4)\left(\frac{1}{2}\right)(1/\sqrt{16}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{16}.$

90. $f(2x + g(x)), \quad x = 1$

SOLUTION $\left.\frac{d}{dx} f(2x + g(x))\right|_{x=1} = f'(2(1) + g(1))(2 + g'(1)) = f'(2 + 4)(7) = 4(7) = 28.$

91. The price (in dollars) of a computer component is $P = 2C - 18C^{-1}$, where C is the manufacturer's cost to produce it. Assume that cost at time t (in years) is $C = 9 + 3t^{-1}$. Determine the rate of change of price with respect to time at $t = 3$.

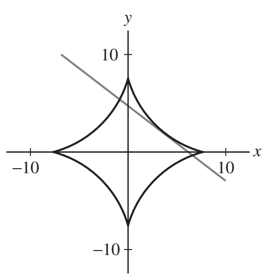
SOLUTION $\frac{dC}{dt} = -3t^{-2}$. $C(3) = 10$ and $C'(3) = -\frac{1}{3}$, so we compute:

$$\left.\frac{dP}{dt}\right|_{t=3} = 2C'(3) + \frac{18}{(C(3))^2} C'(3) = -\frac{2}{3} + \frac{18}{100}\left(-\frac{1}{3}\right) = -0.727 \ \frac{\text{dollars}}{\text{year}}.$$

92. $\boxed{\text{GU}}$ Plot the "astroid" $y = (4 - x^{2/3})^{3/2}$ for $0 \le x \le 8$. Show that the part of every tangent line in the first quadrant has a constant length 8.

SOLUTION

- Here is a graph of the astroid.



- Let $f(x) = (4 - x^{2/3})^{3/2}$. Then

$$f'(x) = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\sqrt{4 - x^{2/3}}}{x^{1/3}},$$

and the tangent line to f at $x = a$ is

$$y = -\frac{\sqrt{4 - a^{2/3}}}{a^{1/3}}(x - a) + \left(4 - a^{2/3}\right)^{3/2}.$$

The y-intercept of this line is the point $P = \left(0, 4\sqrt{4 - a^{2/3}}\right)$, its x-intercept is the point $Q = \left(4a^{1/3}, 0\right)$, and the distance between P and Q is 8.

93. According to the U.S. standard atmospheric model, developed by the National Oceanic and Atmospheric Administration for use in aircraft and rocket design, atmospheric temperature T (in degrees Celsius), pressure P (kPa $= 1{,}000$ pascals), and altitude h (in meters) are related by these formulas (valid in the troposphere $h \le 11{,}000$):

$$T = 15.04 - 0.000649h, \qquad P = 101.29 + \left(\frac{T + 273.1}{288.08}\right)^{5.256}$$

Use the Chain Rule to calculate dP/dh. Then estimate the change in P (in pascals, Pa) per additional meter of altitude when $h = 3{,}000$.

SOLUTION

$$\frac{dP}{dT} = 5.256\left(\frac{T + 273.1}{288.08}\right)^{4.256}\left(\frac{1}{288.08}\right) = 6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}$$

and $\frac{dT}{dh} = -0.000649°\text{C/m}$. $\frac{dP}{dh} = \frac{dP}{dT}\frac{dT}{dh}$, so

$$\frac{dP}{dh} = \left(6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}\right)(-0.000649) = -4.03366 \times 10^{-16}\,(288.14 - 0.000649\,h)^{4.256}.$$

When $h = 3000$,

$$\frac{dP}{dh} = -4.03366 \times 10^{-16}(286.193)^{4.256} = -1.15 \times 10^{-5}\ \text{kPa/m};$$

therefore, for each additional meter of altitude,

$$\Delta P \approx -1.15 \times 10^{-5}\ \text{kPa} = -1.15 \times 10^{-2}\ \text{Pa}.$$

94. Climate scientists use the **Stefan-Boltzmann Law** $R = \sigma T^4$ to estimate the change in the earth's average temperature T (in kelvins) caused by a change in the radiation R (in joules per square meter per second) that the earth receives from the sun. Here $\sigma = 5.67 \times 10^{-8}\ \text{Js}^{-1}\text{m}^{-2}\text{K}^{-4}$. Calculate dR/dt, assuming that $T = 283$ and $\frac{dT}{dt} = 0.05$ K/yr. What are the units of the derivative?

SOLUTION By the Chain Rule,

$$\frac{dR}{dt} = \frac{dR}{dT} \cdot \frac{dT}{dt} = 4\sigma T^3 \frac{dT}{dt}.$$

Assuming $T = 283$ K and $\frac{dT}{dt} = 0.05$ K/yr, it follows that

$$\frac{dR}{dt} = 4\sigma(283^3)(0.05) \approx 0.257\ \text{Js}^{-1}\text{m}^{-2}/\text{yr}$$

95. In the setting of Exercise 94, calculate the yearly rate of change of T if $T = 283$ K and R increases at a rate of 0.5 $\text{Js}^{-1}\text{m}^{-2}$ per year.

SOLUTION By the Chain Rule,

$$\frac{dR}{dt} = \frac{dR}{dT} \cdot \frac{dT}{dt} = 4\sigma T^3 \frac{dT}{dt}.$$

Assuming $T = 283$ K and $\frac{dR}{dt} = 0.5\ \text{Js}^{-1}\text{m}^{-2}$ per year, it follows that author:

$$0.5 = 4\sigma(283)^3\frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{0.5}{4\sigma(283)^3} \approx 0.0973\ \text{kelvins/yr}$$

96. CAS Use a computer algebra system to compute $f^{(k)}(x)$ for $k = 1, 2, 3$ for the following functions:

(a) $f(x) = \cot(x^2)$ **(b)** $f(x) = \sqrt{x^3 + 1}$

SOLUTION

(a) Let $f(x) = \cot(x^2)$. Using a computer algebra system,

$$f'(x) = -2x\csc^2(x^2);$$
$$f''(x) = 2\csc^2(x^2)(4x^2\cot(x^2) - 1); \text{ and}$$
$$f'''(x) = -8x\csc^2(x^2)\left(6x^2\cot^2(x^2) - 3\cot(x^2) + 2x^2\right).$$

(b) Let $f(x) = \sqrt{x^3 + 1}$. Using a computer algebra system,

$$f'(x) = \frac{3x^2}{2\sqrt{x^3+1}};$$
$$f''(x) = \frac{3x(x^3+4)}{4(x^3+1)^{3/2}}; \text{ and}$$
$$f'''(x) = -\frac{3(x^6+20x^3-8)}{8(x^3+1)^{5/2}}.$$

97. Use the Chain Rule to express the second derivative of $f \circ g$ in terms of the first and second derivatives of f and g.

SOLUTION Let $h(x) = f(g(x))$. Then

$$h'(x) = f'(g(x))g'(x)$$

and

$$h''(x) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x) = f'(g(x))g''(x) + f''(g(x))\left(g'(x)\right)^2.$$

98. Compute the second derivative of $\sin(g(x))$ at $x = 2$, assuming that $g(2) = \frac{\pi}{4}$, $g'(2) = 5$, and $g''(2) = 3$.

SOLUTION Let $f(x) = \sin(g(x))$. Then $f'(x) = \cos(g(x))g'(x)$ and

$$f''(x) = \cos(g(x))g''(x) + g'(x)(-\sin(g(x)))g'(x) = \cos(g(x))g''(x) - (g'(x))^2\sin(g(x)).$$

Therefore,

$$f''(2) = g''(2)\cos(g(2)) - \left(g'(2)\right)^2\sin(g(2)) = 3\cos\left(\tfrac{\pi}{4}\right) - (5)^2\sin\left(\tfrac{\pi}{4}\right) = -22\cdot\frac{\sqrt{2}}{2} = -11\sqrt{2}$$

Further Insights and Challenges

99. Show that if f, g, and h are differentiable, then

$$[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x)$$

SOLUTION Let f, g, and h be differentiable. Let $u = h(x)$, $v = g(u)$, and $w = f(v)$. Then

$$\frac{dw}{dx} = \frac{df}{dv}\frac{dv}{dx} = \frac{df}{dv}\frac{dg}{du}\frac{du}{dx} = f'(g(h(x))g'(h(x))h'(x)$$

100. Show that differentiation reverses parity: If f is even, then f' is odd, and if f is odd, then f' is even. *Hint:* Differentiate $f(-x)$.

SOLUTION A function is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$. By the chain rule, $\frac{d}{dx}f(-x) = -f'(-x)$. Now suppose that f is even. Then $f(-x) = f(x)$ and

$$\frac{d}{dx}f(-x) = \frac{d}{dx}f(x) = f'(x).$$

Hence, when f is even, $-f'(-x) = f'(x)$ or $f'(-x) = -f'(x)$ and f' is odd. On the other hand, suppose f is odd. Then $f(-x) = -f(x)$ and

$$\frac{d}{dx}f(-x) = -\frac{d}{dx}f(x) = -f'(x).$$

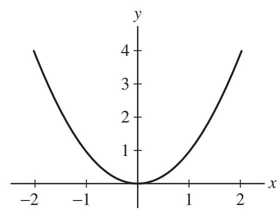
Hence, when f is odd, $-f'(-x) = -f'(x)$ or $f'(-x) = f'(x)$ and f' is even.

101. (a) Sketch a graph of any even function $f(x)$ and explain graphically why $f'(x)$ is odd.

(b) Suppose that $f'(x)$ is even. Is $f(x)$ necessarily odd? *Hint:* Check whether this is true for linear functions.

SOLUTION

(a) The graph of an even function is symmetric with respect to the y-axis. Accordingly, its image in the left half-plane is a mirror reflection of that in the right half-plane through the y-axis. If at $x = a \geq 0$, the slope of f exists and is equal to m, then by reflection its slope at $x = -a \leq 0$ is $-m$. That is, $f'(-a) = -f'(a)$. *Note:* This means that if $f'(0)$ exists, then it equals 0.



(b) Suppose that f' is even. Then f is not necessarily odd. Let $f(x) = 4x + 7$. Then $f'(x) = 4$, an even function. But f is not odd. For example, $f(2) = 15$, $f(-2) = -1$, but $f(-2) \neq -f(2)$.

102. Power Rule for Fractional Exponents Let $f(u) = u^q$ and $g(x) = x^{p/q}$. Assume that $g(x)$ is differentiable.

(a) Show that $f(g(x)) = x^p$ (recall the laws of exponents).

(b) Apply the Chain Rule and the Power Rule for whole-number exponents to show that $f'(g(x))\,g'(x) = px^{p-1}$.

(c) Then derive the Power Rule for $x^{p/q}$.

SOLUTION

(a) Let $f(u) = u^q$ and $g(x) = x^{p/q}$, where q is a positive integer and p is an integer. Then

$$f(g(x)) = f\left(x^{p/q}\right) = \left(x^{p/q}\right)^q = x^p.$$

(b) Differentiating both sides of the final expression in part (a), applying the chain rule on the left and the power rule for whole number exponents on the right, it follows that

$$f'(g(x))g'(x) = px^{p-1}.$$

(c) Thus

$$g'(x) = \frac{px^{p-1}}{f'(g(x))} = \frac{px^{p-1}}{q\left(x^{p/q}\right)^{q-1}} = \frac{px^{p-1}}{qx^{p-p/q}} = \frac{p}{q}x^{p/q-1}.$$

103. Prove that for all whole numbers $n \geq 1$,

$$\frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right)$$

Hint: Use the identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

SOLUTION We will proceed by induction on n. For $n = 1$, we find

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

as required. Now, suppose that for some positive integer k,

$$\frac{d^k}{dx^k}\sin x = \sin\left(x + \frac{k\pi}{2}\right).$$

Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\sin x &= \frac{d}{dx}\sin\left(x + \frac{k\pi}{2}\right)\\ &= \cos\left(x + \frac{k\pi}{2}\right) = \sin\left(x + \frac{(k+1)\pi}{2}\right).\end{aligned}$$

104. A Discontinuous Derivative Use the limit definition to show that $g'(0)$ exists but $g'(0) \neq \lim_{x\to 0} g'(x)$, where

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

SOLUTION Using the limit definition,

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0,$$

where we have used the squeeze theorem in the last step. Now, for $x \neq 0$,

$$g'(x) = x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Although the first term in g' has a limit of 0 as $x \to 0$ (by the squeeze theorem), the limit as $x \to 0$ of the second term does not exist. Hence, $\lim_{x\to 0} g'(x)$ does not exist, so $g'(0) \neq \lim_{x\to 0} g'(x)$.

105. Chain Rule This exercise proves the Chain Rule without the special assumption made in the text. For any number b, define a new function

$$F(u) = \frac{f(u) - f(b)}{u - b} \quad \text{for all } u \neq b$$

(a) Show that if we define $F(b) = f'(b)$, then $F(u)$ is continuous at $u = b$.

(b) Take $b = g(a)$. Show that if $x \neq a$, then for all u,

$$\frac{f(u) - f(g(a))}{x - a} = F(u)\frac{u - g(a)}{x - a} \tag{2}$$

Note that both sides are zero if $u = g(a)$.

(c) Substitute $u = g(x)$ in Eq. (2) to obtain

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x))\,\frac{g(x) - g(a)}{x - a}$$

Derive the Chain Rule by computing the limit of both sides as $x \to a$.

SOLUTION For any differentiable function f and any number b, define

$$F(u) = \frac{f(u) - f(b)}{u - b}$$

for all $u \neq b$.

(a) Define $F(b) = f'(b)$. Then

$$\lim_{u\to b} F(u) = \lim_{u\to b} \frac{f(u) - f(b)}{u - b} = f'(b) = F(b),$$

i.e., $\lim_{u\to b} F(u) = F(b)$. Therefore, F is continuous at $u = b$.

(b) Let g be a differentiable function and take $b = g(a)$. Let x be a number distinct from a. If we substitute $u = g(a)$ into Eq. (2), both sides evaluate to 0, so equality is satisfied. On the other hand, if $u \neq g(a)$, then

$$\frac{f(u) - f(g(a))}{x - a} = \frac{f(u) - f(g(a))}{u - g(a)}\,\frac{u - g(a)}{x - a} = \frac{f(u) - f(b)}{u - b}\,\frac{u - g(a)}{x - a} = F(u)\,\frac{u - g(a)}{x - a}.$$

(c) Hence for all u, we have

$$\frac{f(u) - f(g(a))}{x - a} = F(u)\,\frac{u - g(a)}{x - a}.$$

(d) Substituting $u = g(x)$ in Eq. (2), we have

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x))\,\frac{g(x) - g(a)}{x - a}.$$

Letting $x \to a$ gives

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} = \lim_{x\to a} \left(F(g(x)) \frac{g(x)-g(a)}{x-a} \right) = F(g(a))g'(a) = F(b)g'(a) = f'(b)g'(a)$$
$$= f'(g(a))g'(a)$$

Therefore $(f \circ g)'(a) = f'(g(a))g'(a)$, which is the Chain Rule.

3.8 Derivatives of Inverse Functions

Preliminary Questions

1. What is the slope of the line obtained by reflecting the line $y = \frac{x}{2}$ through the line $y = x$?

SOLUTION The line obtained by reflecting the line $y = x/2$ through the line $y = x$ has slope 2.

2. Suppose that $P = (2, 4)$ lies on the graph of $f(x)$ and that the slope of the tangent line through P is $m = 3$. Assuming that $f^{-1}(x)$ exists, what is the slope of the tangent line to the graph of $f^{-1}(x)$ at the point $Q = (4, 2)$?

SOLUTION The tangent line to the graph of $f^{-1}(x)$ at the point $Q = (4, 2)$ has slope $\frac{1}{3}$.

3. Which inverse trigonometric function $g(x)$ has the derivative $g'(x) = \dfrac{1}{x^2+1}$?

SOLUTION $g(x) = \tan^{-1} x$ has the derivative $g'(x) = \dfrac{1}{x^2+1}$.

4. What does the following identity tell us about the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$?

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

SOLUTION Angles whose sine and cosine are x are complementary.

Exercises

1. Find the inverse $g(x)$ of $f(x) = \sqrt{x^2+9}$ with domain $x \ge 0$ and calculate $g'(x)$ in two ways: using Theorem 1 and by direct calculation.

SOLUTION To find a formula for $g(x) = f^{-1}(x)$, solve $y = \sqrt{x^2+9}$ for x. This yields $x = \pm\sqrt{y^2-9}$. Because the domain of f was restricted to $x \ge 0$, we must choose the positive sign in front of the radical. Thus

$$g(x) = f^{-1}(x) = \sqrt{x^2-9}.$$

Because $x^2 + 9 \ge 9$ for all x, it follows that $f(x) \ge 3$ for all x. Thus, the domain of $g(x) = f^{-1}(x)$ is $x \ge 3$. The range of g is the restricted domain of f: $y \ge 0$.

By Theorem 1,

$$g'(x) = \frac{1}{f'(g(x))}.$$

With

$$f'(x) = \frac{x}{\sqrt{x^2+9}},$$

it follows that

$$f'(g(x)) = \frac{\sqrt{x^2-9}}{\sqrt{(\sqrt{x^2-9})^2+9}} = \frac{\sqrt{x^2-9}}{\sqrt{x^2}} = \frac{\sqrt{x^2-9}}{x}$$

since the domain of g is $x \ge 3$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{x}{\sqrt{x^2-9}}.$$

This agrees with the answer we obtain by differentiating directly:

$$g'(x) = \frac{2x}{2\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}.$$

2. Let $g(x)$ be the inverse of $f(x) = x^3 + 1$. Find a formula for $g(x)$ and calculate $g'(x)$ in two ways: using Theorem 1 and then by direct calculation.

SOLUTION To find $g(x)$, we solve $y = x^3 + 1$ for x:

$$y - 1 = x^3$$
$$x = (y-1)^{1/3}$$

Therefore, the inverse is $g(x) = (x-1)^{1/3}$.

We have $f'(x) = 3x^2$. According to Theorem 1,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3(x-1)^{2/3}} = \frac{1}{3}(x-1)^{-2/3}$$

This agrees with the answer we obtain by differentiating directly:

$$\frac{d}{dx}(x-1)^{1/3} = \tfrac{1}{3}(x-1)^{-2/3}.$$

In Exercises 3–8, use Theorem 1 to calculate $g'(x)$, where $g(x)$ is the inverse of $f(x)$.

3. $f(x) = 7x + 6$

SOLUTION Let $f(x) = 7x + 6$ then $f'(x) = 7$. Solving $y = 7x + 6$ for x and switching variables, we obtain the inverse $g(x) = (x-6)/7$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{7}.$$

4. $f(x) = \sqrt{3-x}$

SOLUTION Let $f(x) = (3-x)^{1/2}$. Then

$$f'(x) = \frac{1}{2}(3-x)^{-1/2}(-1) = \frac{-1}{2(3-x)^{1/2}}.$$

Solving $y = \sqrt{3-x}$ for x and switching variables, we obtain the inverse $g(x) = 3 - x^2$. Thus,

$$g'(x) = 1\Bigg/\frac{-1}{2(3-3+x^2)^{1/2}} = -2x,$$

where we have used the fact that the domain of g is $x \ge 0$ to write $\sqrt{x^2} = x$.

5. $f(x) = x^{-5}$

SOLUTION Let $f(x) = x^{-5}$, then $f'(x) = -5x^{-6}$. Solving $y = x^{-5}$ for x and switching variables, we obtain the inverse $g(x) = x^{-1/5}$. Thus,

$$g'(x) = \frac{1}{-5(x^{-1/5})^{-6}} = -\frac{1}{5}x^{-6/5}.$$

6. $f(x) = 4x^3 - 1$

SOLUTION Let $f(x) = 4x^3 - 1$, then $f'(x) = 12x^2$. Solving $y = 4x^3 - 1$ for x and switching variables, we obtain the inverse $g(x) = (\frac{x+1}{4})^{1/3}$. Thus,

$$g'(x) = \frac{1}{12}\left(\frac{x+1}{4}\right)^{-2/3}$$

7. $f(x) = \dfrac{x}{x+1}$

SOLUTION Let $f(x) = \frac{x}{x+1}$, then

$$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

Solving $y = \frac{x}{x+1}$ for x and switching variables, we obtain the inverse $g(x) = \frac{x}{1-x}$. Thus

$$g'(x) = 1\Bigg/\frac{1}{(x/(1-x)+1)^2} = \frac{1}{(1-x)^2}.$$

8. $f(x) = 2 + x^{-1}$

SOLUTION Let $f(x) = 2 + x^{-1}$, then $f'(x) = -1/x^2$. Solving $y = 2 + x^{-1}$ for x and switching variables, we obtain the inverse $g(x) = 1/(x-2)$. Thus,

$$g'(x) = 1 \Big/ \frac{-1}{1/(x-2)^2} = -\frac{1}{(x-2)^2}.$$

9. Let $g(x)$ be the inverse of $f(x) = x^3 + 2x + 4$. Calculate $g(7)$ [without finding a formula for $g(x)$], and then calculate $g'(7)$.

SOLUTION Let $g(x)$ be the inverse of $f(x) = x^3 + 2x + 4$. Because

$$f(1) = 1^3 + 2(1) + 4 = 7,$$

it follows that $g(7) = 1$. Moreover, $f'(x) = 3x^2 + 2$, and

$$g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

10. Find $g'\left(-\frac{1}{2}\right)$, where $g(x)$ is the inverse of $f(x) = \dfrac{x^3}{x^2+1}$.

SOLUTION Let $g(x)$ be the inverse of $f(x) = \dfrac{x^3}{x^2+1}$. Because

$$f(-1) = \frac{(-1)^3}{(-1)^2+1} = -\frac{1}{2},$$

it follows that $g(-\frac{1}{2}) = -1$. Moreover,

$$f'(x) = \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2},$$

and

$$g'\left(-\frac{1}{2}\right) = \frac{1}{f'(g(-\frac{1}{2}))} = \frac{1}{f'(-1)} = 1.$$

In Exercises 11–16, calculate $g(b)$ and $g'(b)$, where g is the inverse of f (in the given domain, if indicated).

11. $f(x) = x + \cos x, \quad b = 1$

SOLUTION $f(0) = 1$, so $g(1) = 0$. $f'(x) = 1 - \sin x$ so $f'(g(1)) = f'(0) = 1 - \sin 0 = 1$. Thus, $g'(1) = 1/1 = 1$.

12. $f(x) = 4x^3 - 2x, \quad b = -2$

SOLUTION $f(-1) = -2$, so $g(-2) = -1$. $f'(x) = 12x^2 - 2$ so $f'(g(-2)) = f'(-1) = 12 - 2 = 10$. Thus, $g'(-2) = 1/10$.

13. $f(x) = \sqrt{x^2+6x}$ for $x \ge 0, \quad b = 4$

SOLUTION To determine $g(4)$, we solve $f(x) = \sqrt{x^2+6x} = 4$ for x. This yields:

$$\begin{aligned} x^2 + 6x &= 16 \\ x^2 + 6x - 16 &= 0 \\ (x+8)(x-2) &= 0 \end{aligned}$$

or $x = -8, 2$. Because the domain of f has been restricted to $x \ge 0$, we have $g(4) = 2$. With

$$f'(x) = \frac{x+3}{\sqrt{x^2+6x}},$$

it then follows that

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(2)} = \frac{4}{5}.$$

14. $f(x) = \sqrt{x^2 + 6x}$ for $x \le -6$, $\quad b = 4$

SOLUTION To determine $g(4)$, we solve $f(x) = \sqrt{x^2 + 6x} = 4$ for x. This yields:

$$x^2 + 6x = 16$$
$$x^2 + 6x - 16 = 0$$
$$(x+8)(x-2) = 0$$

or $x = -8, 2$. Because the domain of f has been restricted to $x \le -6$, we have $g(4) = -8$. With

$$f'(x) = \frac{x+3}{\sqrt{x^2+6x}},$$

it then follows that

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(-8)} = -\frac{4}{5}.$$

15. $f(x) = \dfrac{1}{x+1}$, $\quad b = \dfrac{1}{4}$

SOLUTION $f(3) = 1/4$, so $g(1/4) = 3$. $f'(x) = \frac{-1}{(x+1)^2}$ so $f'(g(1/4)) = f'(3) = \frac{-1}{(3+1)^2} = -1/16$. Thus, $g'(1/4) = -16$.

16. $f(x) = e^x$, $\quad b = e$

SOLUTION $f(1) = e$ so $g(e) = 1$. $f'(x) = e^x$ so $f'(g(e)) = f'(1) = e$. Thus, $g'(x) = 1/e$.

17. Let $f(x) = x^n$ and $g(x) = x^{1/n}$. Compute $g'(x)$ using Theorem 1 and check your answer using the Power Rule.

SOLUTION Note that $g(x) = f^{-1}(x)$. Therefore,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(g(x))^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n(x^{1-1/n})} = \frac{x^{1/n-1}}{n} = \frac{1}{n}(x^{1/n-1})$$

which agrees with the Power Rule.

18. Show that $f(x) = \dfrac{1}{1+x}$ and $g(x) = \dfrac{1-x}{x}$ are inverses. Then compute $g'(x)$ directly and verify that $g'(x) = 1/f'(g(x))$.

SOLUTION Let $f(x) = \dfrac{1}{1+x}$ and $g(x) = \dfrac{1-x}{x}$. Then

$$f(g(x)) = \frac{1}{1 + \frac{1-x}{x}} = \frac{x}{x+1-x} = x,$$

and

$$g(f(x)) = \frac{1 - \frac{1}{1+x}}{\frac{1}{1+x}} = \frac{1+x-1}{1} = x;$$

consequently, f and g are inverses. Rewriting $g(x) = x^{-1} - 1$, we see that $g'(x) = -x^{-2}$. Moreover, $f'(x) = -(1+x)^{-2}$, so

$$f'(g(x)) = -\left(1 + \frac{1-x}{x}\right)^{-2} = -(x^{-1})^{-2} = -x^2,$$

and

$$\frac{1}{f'(g(x))} = -x^{-2} = g'(x).$$

In Exercises 19–22, compute the derivative at the point indicated without using a calculator.

19. $y = \sin^{-1} x$, $\quad x = \frac{3}{5}$

SOLUTION Let $y = \sin^{-1} x$. Then $y' = \frac{1}{\sqrt{1-x^2}}$ and

$$y'\left(\frac{3}{5}\right) = \frac{1}{\sqrt{1 - 9/25}} = \frac{1}{4/5} = \frac{5}{4}.$$

20. $y = \tan^{-1} x, \quad x = \frac{1}{2}$

SOLUTION Let $y = \tan^{-1} x$. Then $y' = \frac{1}{x^2+1}$ and

$$y'\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{4}+1} = \frac{4}{5}.$$

21. $y = \sec^{-1} x, \quad x = 4$

SOLUTION Let $y = \sec^{-1} x$. Then $y' = \frac{1}{|x|\sqrt{x^2-1}}$ and

$$y'(4) = \frac{1}{4\sqrt{15}}.$$

22. $y = \arccos(4x), \quad x = \frac{1}{5}$

SOLUTION Let $y = \cos^{-1}(4x)$. Then $y' = \frac{-4}{\sqrt{1-16x^2}}$ and

$$y'\left(\frac{1}{5}\right) = \frac{-4}{\sqrt{1-\frac{16}{25}}} = \frac{-4}{\frac{3}{5}} = -\frac{20}{3}.$$

In Exercises 23–36, find the derivative.

23. $y = \sin^{-1}(7x)$

SOLUTION $\displaystyle \frac{d}{dx}\sin^{-1}(7x) = \frac{1}{\sqrt{1-(7x)^2}} \cdot \frac{d}{dx}7x = \frac{7}{\sqrt{1-(7x)^2}}.$

24. $y = \arctan\left(\frac{x}{3}\right)$

SOLUTION $\displaystyle \frac{d}{dx}\tan^{-1}\left(\frac{x}{3}\right) = \frac{1}{(x/3)^2+1} \cdot \frac{d}{dx}\left(\frac{x}{3}\right) = \frac{1}{3} \cdot \frac{1}{\left(\frac{x}{3}\right)^2+1} = \frac{1}{(x^2/3)+3}.$

25. $y = \cos^{-1}(x^2)$

SOLUTION $\displaystyle \frac{d}{dx}\cos^{-1}(x^2) = \frac{-1}{\sqrt{1-x^4}} \cdot \frac{d}{dx}x^2 = \frac{-2x}{\sqrt{1-x^4}}.$

26. $y = \sec^{-1}(t+1)$

SOLUTION $\displaystyle \frac{d}{dt}\sec^{-1}(t+1) = \frac{1}{|t+1|\sqrt{(t+1)^2-1}} = \frac{1}{|t+1|\sqrt{t^2+2t}}.$

27. $y = x\tan^{-1} x$

SOLUTION $\displaystyle \frac{d}{dx}x\tan^{-1} x = x\left(\frac{1}{1+x^2}\right) + \tan^{-1} x.$

28. $y = e^{\cos^{-1} x}$

SOLUTION $\displaystyle \frac{d}{dx}e^{\cos^{-1} x} = e^{\cos^{-1} x}\frac{d}{dx}\cos^{-1} x = \frac{-e^{\cos^{-1} x}}{\sqrt{1-x^2}}.$

29. $y = \arcsin(e^x)$

SOLUTION $\displaystyle \frac{d}{dx}\sin^{-1}(e^x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot \frac{d}{dx}e^x = \frac{e^x}{\sqrt{1-e^{2x}}}.$

30. $y = \csc^{-1}(x^{-1})$

SOLUTION $\displaystyle \frac{d}{dx}\csc^{-1}(x^{-1}) = \frac{-1}{|1/x|\sqrt{1/x^2-1}}\left(\frac{-1}{x^2}\right) = \frac{1}{x^2|1/x|\sqrt{1/x^2-1}} = \frac{1}{\sqrt{1-x^2}}.$

31. $y = \sqrt{1-t^2} + \sin^{-1} t$

SOLUTION $\displaystyle \frac{d}{dt}\left(\sqrt{1-t^2} + \sin^{-1} t\right) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) + \frac{1}{\sqrt{1-t^2}} = \frac{-t}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} = \frac{1-t}{\sqrt{1-t^2}}.$

32. $y = \tan^{-1}\left(\dfrac{1+t}{1-t}\right)$

SOLUTION $\dfrac{d}{dx}\tan^{-1}\left(\dfrac{1+t}{1-t}\right) = \dfrac{1}{\left(\frac{1+t}{1-t}\right)^2+1}\cdot\left(\dfrac{(1-t)-(1+t)(-1)}{(1-t)^2}\right) = \dfrac{2}{(1+t)^2+(1-t)^2} = \dfrac{1}{t^2+1}.$

33. $y = (\tan^{-1} x)^3$

SOLUTION $\dfrac{d}{dx}\left((\tan^{-1} x)^3\right) = 3(\tan^{-1} x)^2 \dfrac{d}{dx}\tan^{-1} x = \dfrac{3(\tan^{-1} x)^2}{x^2+1}.$

34. $y = \dfrac{\cos^{-1} x}{\sin^{-1} x}$

SOLUTION $\dfrac{d}{dx}\left(\dfrac{\cos^{-1} x}{\sin^{-1} x}\right) = \dfrac{\sin^{-1} x\left(\frac{-1}{\sqrt{1-x^2}}\right) - \cos^{-1} x\left(\frac{1}{\sqrt{1-x^2}}\right)}{(\sin^{-1} x)^2} = -\dfrac{\pi}{2\sqrt{1-x^2}(\sin^{-1} x)^2}.$

35. $y = \cos^{-1} t^{-1} - \sec^{-1} t$

SOLUTION

$$\frac{d}{dx}(\cos^{-1} t^{-1} - \sec^{-1} t) = \frac{-1}{\sqrt{1-(1/t)^2}}\left(\frac{-1}{t^2}\right) - \frac{1}{|t|\sqrt{t^2-1}}$$
$$= \frac{1}{\sqrt{t^4-t^2}} - \frac{1}{|t|\sqrt{t^2-1}} = \frac{1}{|t|\sqrt{t^2-1}} - \frac{1}{|t|\sqrt{t^2-1}} = 0.$$

Alternately, let $t = \sec\theta$. Then $t^{-1} = \cos\theta$ and $\cos^{-1} t^{-1} - \sec^{-1} t = \theta - \theta = 0$. Consequently,

$$\frac{d}{dx}(\cos^{-1} t^{-1} - \sec^{-1} t) = 0.$$

36. $y = \cos^{-1}(x + \sin^{-1} x)$

SOLUTION $\dfrac{d}{dx}\cos^{-1}(x+\sin^{-1} x) = \dfrac{-1}{\sqrt{1-(x+\sin^{-1} x)^2}}\left(1+\dfrac{1}{\sqrt{1-x^2}}\right).$

37. Use Figure 5 to prove that $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$.

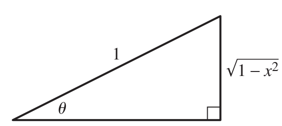


FIGURE 5 Right triangle with $\theta = \cos^{-1} x$.

SOLUTION Let $\theta = \cos^{-1} x$. Then $\cos\theta = x$ and

$$-\sin\theta\frac{d\theta}{dx} = 1 \quad\text{or}\quad \frac{d\theta}{dx} = -\frac{1}{\sin\theta} = -\frac{1}{\sin(\cos^{-1} x)}.$$

From Figure 5, we see that $\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}$; hence,

$$\frac{d}{dx}\cos^{-1} x = \frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.$$

38. Show that $(\tan^{-1} x)' = \cos^2(\tan^{-1} x)$ and then use Figure 6 to prove that $(\tan^{-1} x)' = (x^2+1)^{-1}$.

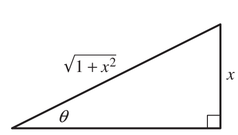


FIGURE 6 Right triangle with $\theta = \tan^{-1} x$.

SOLUTION Let $\theta = \tan^{-1} x$. Then $x = \tan\theta$ and

$$1 = \sec^2\theta \frac{d\theta}{dx} \quad \text{or} \quad \frac{d\theta}{dx} = \frac{1}{\sec^2\theta} = \cos^2\theta = \cos^2(\tan^{-1} x).$$

From Figure 6, $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, thus $\cos^2\theta = \frac{1}{1+x^2}$ and

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

39. Let $\theta = \sec^{-1} x$. Show that $\tan\theta = \sqrt{x^2-1}$ if $x \ge 1$ and that $\tan\theta = -\sqrt{x^2-1}$ if $x \le -1$. *Hint:* $\tan\theta \ge 0$ on $\left(0, \frac{\pi}{2}\right)$ and $\tan\theta \le 0$ on $\left(\frac{\pi}{2}, \pi\right)$.

SOLUTION In general, $1 + \tan^2\theta = \sec^2\theta$, so $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$. With $\theta = \sec^{-1} x$, it follows that $\sec\theta = x$, so $\tan\theta = \pm\sqrt{x^2-1}$. Finally, if $x \ge 1$ then $\theta = \sec^{-1} x \in [0, \pi/2)$ so $\tan\theta$ is positive; on the other hand, if $x \le 1$ then $\theta = \sec^{-1} x \in (-\pi/2, 0]$ so $\tan\theta$ is negative.

40. Use Exercise 39 to verify the formula

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

SOLUTION Let $\theta = \sec^{-1} x$. Then $\sec\theta = x$ and

$$\sec\theta\tan\theta \frac{d\theta}{dx} = 1 \quad \text{or} \quad \frac{d\theta}{dx} = \frac{1}{\sec\theta\tan\theta} = \frac{1}{x\tan(\sec^{-1} x)}.$$

By Exercise 39, $\tan(\sec^{-1} x) = \sqrt{x^2-1}$ for $x > 1$ and $\tan(\sec^{-1} x) = -\sqrt{x^2-1}$ for $x < -1$. Hence,

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Further Insights and Challenges

41. Let $g(x)$ be the inverse of $f(x)$. Show that if $f'(x) = f(x)$, then $g'(x) = x^{-1}$. We will apply this in the next section to show that the inverse of $f(x) = e^x$ (the natural logarithm) has the derivative $f'(x) = x^{-1}$.

SOLUTION

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f(f^{-1}(x))} = \frac{1}{x}.$$

3.9 Derivatives of General Exponential and Logarithmic Functions

Preliminary Questions

1. What is the slope of the tangent line to $y = 4^x$ at $x = 0$?

SOLUTION The slope of the tangent line to $y = 4^x$ at $x = 0$ is

$$\frac{d}{dx}4^x\bigg|_{x=0} = 4^x \ln 4\bigg|_{x=0} = \ln 4.$$

2. What is the rate of change of $y = \ln x$ at $x = 10$?

SOLUTION The rate of change of $y = \ln x$ at $x = 10$ is

$$\frac{d}{dx}\ln x\bigg|_{x=10} = \frac{1}{x}\bigg|_{x=10} = \frac{1}{10}.$$

3. What is $b > 0$ if the tangent line to $y = b^x$ at $x = 0$ has slope 2?

SOLUTION The tangent line to $y = b^x$ at $x = 0$ has slope

$$\frac{d}{dx}b^x\bigg|_{x=0} = b^x \ln b\bigg|_{x=0} = \ln b.$$

This slope will be equal to 2 when

$$\ln b = 2 \quad \text{or} \quad b = e^2.$$

4. What is b if $(\log_b x)' = \dfrac{1}{3x}$?

SOLUTION $(\log_b x)' = \left(\dfrac{\ln x}{\ln b}\right)' = \dfrac{1}{x \ln b}$. This derivative will equal $\frac{1}{3x}$ when

$$\ln b = 3 \quad \text{or} \quad b = e^3.$$

5. What are $y^{(100)}$ and $y^{(101)}$ for $y = \cosh x$?

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$, $y'' = \cosh x$, and this pattern repeats indefinitely. Thus, $y^{(100)} = \cosh x$ and $y^{(101)} = \sinh x$.

Exercises

In Exercises 1–20, find the derivative.

1. $y = x \ln x$

SOLUTION $\dfrac{d}{dx} x \ln x = \ln x + \dfrac{x}{x} = \ln x + 1.$

2. $y = t \ln t - t$

SOLUTION $\dfrac{d}{dt}(t \ln t - t) = t\left(\dfrac{1}{t}\right) + \ln t - 1 = \ln t.$

3. $y = (\ln x)^2$

SOLUTION $\dfrac{d}{dx}(\ln x)^2 = (2 \ln x)\,\dfrac{1}{x} = \dfrac{2}{x}\ln x.$

4. $y = \ln(x^5)$

SOLUTION $\dfrac{d}{dx}(\ln x^5) = \dfrac{1}{x^5}(5x^4) = \dfrac{5}{x}.$

5. $y = \ln(9x^2 - 8)$

SOLUTION $\dfrac{d}{dx}\ln(9x^2 - 8) = \dfrac{18x}{9x^2 - 8}.$

6. $y = \ln(t5^t)$

SOLUTION Using the rules for logarithms, we write

$$y = \ln(t5^t) = \ln t + \ln(5^t) = \ln t + t \ln 5.$$

Then,

$$\frac{d}{dt}\ln(t5^t) = \frac{1}{t} + \ln 5.$$

7. $y = \ln(\sin t + 1)$

SOLUTION $\dfrac{d}{dt}\ln(\sin t + 1) = \dfrac{\cos t}{\sin t + 1}.$

8. $y = x^2 \ln x$

SOLUTION $\dfrac{d}{dx} x^2 \ln x = 2x \ln x + \dfrac{x^2}{x} = 2x \ln x + x.$

9. $y = \dfrac{\ln x}{x}$

SOLUTION $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{\frac{1}{x}(x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$

10. $y = e^{(\ln x)^2}$

SOLUTION $\dfrac{d}{dx} e^{(\ln x)^2} = e^{(\ln x)^2} \cdot 2 \cdot \dfrac{\ln x}{x}.$

11. $y = \ln(\ln x)$

SOLUTION $\dfrac{d}{dx}\ln(\ln x) = \dfrac{1}{x \ln x}.$

12. $y = \ln(\cot x)$

SOLUTION $\dfrac{d}{dx}\ln(\cot x) = \dfrac{1}{\cot x}(-\csc^2 x) = -\dfrac{1}{\sin x \cos x}.$

13. $y = \big(\ln(\ln x)\big)^3$

SOLUTION $\dfrac{d}{dx}(\ln(\ln x))^3 = 3(\ln(\ln x))^2\left(\dfrac{1}{\ln x}\right)\left(\dfrac{1}{x}\right) = \dfrac{3(\ln(\ln x))^2}{x\ln x}.$

14. $y = \ln\big((\ln x)^3\big)$

SOLUTION $\dfrac{d}{dx}\ln((\ln x)^3) = \dfrac{3(\ln x)^2}{x(\ln x)^3} = \dfrac{3}{x\ln x}.$

Alternately, because $\ln((\ln x)^3) = 3\ln(\ln x)$,

$$\frac{d}{dx}\ln((\ln x)^3) = 3\frac{d}{dx}\ln(\ln x) = 3\cdot\frac{1}{x\ln x}.$$

15. $y = \ln\big((x+1)(2x+9)\big)$

SOLUTION

$$\frac{d}{dx}\ln((x+1)(2x+9)) = \frac{1}{(x+1)(2x+9)}\cdot((x+1)2+(2x+9)) = \frac{4x+11}{(x+1)(2x+9)}.$$

Alternately, because $\ln((x+1)(2x+9)) = \ln(x+1)+\ln(2x+9)$,

$$\frac{d}{dx}\ln((x+1)(2x+9)) = \frac{1}{x+1}+\frac{2}{2x+9} = \frac{4x+11}{(x+1)(2x+9)}.$$

16. $y = \ln\left(\dfrac{x+1}{x^3+1}\right)$

SOLUTION

$$\frac{d}{dx}\ln\left(\frac{x+1}{x^3+1}\right) = \frac{d}{dx}\ln\left(\frac{1}{x^2-x+1}\right) = -\frac{d}{dx}\ln(x^2-x+1) = -\frac{2x-1}{x^2-x+1}.$$

17. $y = 11^x$

SOLUTION $\dfrac{d}{dx}11^x = \ln 11\cdot 11^x.$

18. $y = 7^{4x-x^2}$

SOLUTION $\dfrac{d}{dx}7^{4x-x^2} = \ln 7(4-2x)7^{4x-x^2}.$

19. $y = \dfrac{2^x-3^{-x}}{x}$

SOLUTION $\dfrac{d}{dx}\dfrac{2^x-3^{-x}}{x} = \dfrac{x(2^x\ln 2+3^{-x}\ln 3)-(2^x-3^{-x})}{x^2}.$

20. $y = 16^{\sin x}$

SOLUTION $\dfrac{d}{dx}16^{\sin x} = \ln 16(\cos x)16^{\sin x}.$

In Exercises 21–24, compute the derivative.

21. $f'(x)$, $f(x) = \log_2 x$

SOLUTION $f(x) = \log_2 x = \dfrac{\ln x}{\ln 2}$. Thus, $f'(x) = \dfrac{1}{x}\cdot\dfrac{1}{\ln 2}.$

22. $f'(3)$, $f(x) = \log_5 x$

SOLUTION $f(x) = \dfrac{\ln x}{\ln 5}$, so $f'(x) = \dfrac{1}{x\ln 5}$. Thus, $f'(3) = \dfrac{1}{3\ln 5}.$

23. $\dfrac{d}{dt}\log_3(\sin t)$

SOLUTION $\dfrac{d}{dt}\log_3(\sin t) = \dfrac{d}{dt}\left(\dfrac{\ln(\sin t)}{\ln 3}\right) = \dfrac{1}{\ln 3}\cdot\dfrac{1}{\sin t}\cdot\cos t = \dfrac{\cot t}{\ln 3}.$

24. $\dfrac{d}{dt}\log_{10}(t+2^t)$

SOLUTION $\dfrac{d}{dt}\log_{10}(t+2^t) = \dfrac{d}{dt}\left(\dfrac{\ln(t+2^t)}{\ln 10}\right) = \dfrac{1}{\ln 10}\cdot\dfrac{1+2^t\ln 2}{t+2^t}.$

In Exercises 25–36, find an equation of the tangent line at the point indicated.

25. $f(x) = 6^x, \quad x = 2$

SOLUTION Let $f(x) = 6^x$. Then $f(2) = 36$, $f'(x) = 6^x \ln 6$ and $f'(2) = 36\ln 6$. The equation of the tangent line is therefore $y = 36\ln 6(x-2)+36$.

26. $y = (\sqrt{2})^x, \quad x = 8$

SOLUTION Let $y = (\sqrt{2})^x$. Then $y(8) = 16$, $y'(x) = (\sqrt{2})^x \ln\sqrt{2}$ and $y'(8) = 16\ln\sqrt{2} = 8\ln 2$. The equation of the tangent line is therefore $y = 8\ln 2(x-8)+16$.

27. $s(t) = 3^{9t}, \quad t = 2$

SOLUTION Let $s(t) = 3^{9t}$. Then $s(2) = 3^{18}$, $s'(t) = 3^{9t}9\ln 3$, and $s'(2) = 3^{18}\cdot 9\ln 3 = 3^{20}\ln 3$. The equation of the tangent line is therefore $y = 3^{20}\ln 3(t-2)+3^{18}$.

28. $y = \pi^{5x-2}, \quad x = 1$

SOLUTION Let $y = \pi^{5x-2}$. Then $y(1) = \pi^3$, $y'(x) = \pi^{5x-2}5\ln\pi$, and $y'(1) = 5\pi^3\ln\pi$. The equation of the tangent line is therefore $y = 5\pi^3\ln\pi(x-1)+\pi^3$.

29. $f(x) = 5^{x^2-2x}, \quad x = 1$

SOLUTION Let $f(x) = 5^{x^2-2x}$. Then $f(1) = 5^{-1}$, $f'(x) = \ln 5\cdot 5^{x^2-2x}(2x-2)$, and $f'(1) = \ln 5(0) = 0$. Therefore, the equation of the tangent line is $y = 5^{-1}$.

30. $s(t) = \ln t, \quad t = 5$

SOLUTION Let $s(t) = \ln t$. Then $s(5) = \ln 5$. $s'(t) = 1/t$, so $s'(5) = 1/5$. Therefore the equation of the tangent line is $y = (1/5)(t-5)+\ln 5$.

31. $s(t) = \ln(8-4t), \quad t = 1$

SOLUTION Let $s(t) = \ln(8-4t)$. Then $s(1) = \ln(8-4) = \ln 4$. $s'(t) = \frac{-4}{8-4t}$, so $s'(1) = -4/4 = -1$. Therefore the equation of the tangent line is $y = -1(t-1)+\ln 4$.

32. $f(x) = \ln(x^2), \quad x = 4$

SOLUTION Let $f(x) = \ln x^2 = 2\ln x$. Then $f(4) = 2\ln 4$. $f'(x) = 2/x$, so $f'(4) = 1/2$. Therefore the equation of the tangent line is $y = (1/2)(x-4)+2\ln 4$.

33. $R(z) = \log_5(2z^2+7), \quad z = 3$

SOLUTION Let $R(z) = \log_5(2z^2+7)$. Then $R(3) = \log_5(25) = 2$,

$$R'(z) = \frac{4z}{(2z^2+7)\ln 5}, \quad \text{and} \quad R'(3) = \frac{12}{25\ln 5}.$$

The equation of the tangent line is therefore

$$y = \frac{12}{25\ln 5}(z-3)+2.$$

34. $y = \ln(\sin x), \quad x = \dfrac{\pi}{4}$

SOLUTION Let $f(x) = \ln\sin x$. Then $f(\pi/4) = \ln(\sqrt{2}/2)$. $f'(x) = \cos x/\sin x = \cot x$, so $f'(\pi/4) = 1$. Therefore the equation of the tangent line is $y = (x-\pi/4)+\ln(\sqrt{2}/2)$.

35. $f(w) = \log_2 w, \quad w = \frac{1}{8}$

SOLUTION Let $f(w) = \log_2 w$. Then

$$f\left(\frac{1}{8}\right) = \log_2\frac{1}{8} = \log_2 2^{-3} = -3,$$

$f'(w) = \frac{1}{w\ln 2}$, and

$$f'\left(\frac{1}{8}\right) = \frac{8}{\ln 2}.$$

The equation of the tangent line is therefore

$$y = \frac{8}{\ln 2}\left(w-\frac{1}{8}\right)-3.$$

36. $y = \log_2(1 + 4x^{-1})$, $\quad x = 4$

SOLUTION Let $y = \log_2(1 + 4x^{-1})$. Then $y(4) = \log_2(1 + 1) = 1$,

$$y'(x) = -\frac{4x^{-2}}{(1 + 4x^{-1})\ln 2}, \quad \text{and} \quad y'(4) = -\frac{1}{8\ln 2}.$$

The equation of the tangent line is therefore

$$y = -\frac{1}{8\ln 2}(x - 4) - 1.$$

In Exercises 37–44, find the derivative using logarithmic differentiation as in Example 5.

37. $y = (x + 5)(x + 9)$

SOLUTION Let $y = (x + 5)(x + 9)$. Then $\ln y = \ln((x + 5)(x + 9)) = \ln(x + 5) + \ln(x + 9)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x + 5} + \frac{1}{x + 9}$$

or

$$y' = (x + 5)(x + 9)\left(\frac{1}{x + 5} + \frac{1}{x + 9}\right) = (x + 9) + (x + 5) = 2x + 14.$$

38. $y = (3x + 5)(4x + 9)$

SOLUTION Let $y = (3x + 5)(4x + 9)$. Then $\ln y = \ln((3x + 5)(4x + 9)) = \ln(3x + 5) + \ln(4x + 9)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{3}{3x + 5} + \frac{4}{4x + 9}$$

or

$$y' = (3x + 5)(4x + 9)\left(\frac{3}{3x + 5} + \frac{4}{4x + 9}\right) = (12x + 27) + (12x + 20) = 24x + 47.$$

39. $y = (x - 1)(x - 12)(x + 7)$

SOLUTION Let $y = (x - 1)(x - 12)(x + 7)$. Then $\ln y = \ln(x - 1) + \ln(x - 12) + \ln(x + 7)$. By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x - 1} + \frac{1}{x - 12} + \frac{1}{x + 7}$$

or

$$y' = (x - 12)(x + 7) + (x - 1)(x + 7) + (x - 1)(x - 12) = 3x^2 - 12x + 79.$$

40. $y = \dfrac{x(x + 1)^3}{(3x - 1)^2}$

SOLUTION Let $y = \frac{x(x+1)^3}{(3x-1)^2}$. Then $\ln y = \ln x + 3\ln(x + 1) - 2\ln(3x - 1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{3}{x + 1} - \frac{6}{3x - 1},$$

so

$$y' = \frac{(x + 1)^3}{(3x - 1)^2} + \frac{3x(x + 1)^2}{(3x - 1)^2} - \frac{6x(x + 1)^3}{(3x - 1)^3}.$$

41. $y = \dfrac{x(x^2 + 1)}{\sqrt{x + 1}}$

SOLUTION Let $y = \frac{x(x^2+1)}{\sqrt{x+1}}$. Then $\ln y = \ln x + \ln(x^2 + 1) - \frac{1}{2}\ln(x + 1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{2x}{x^2 + 1} - \frac{1}{2(x + 1)},$$

so

$$y' = \frac{x(x^2 + 1)}{\sqrt{x + 1}}\left(\frac{1}{x} + \frac{2x}{x^2 + 1} - \frac{1}{2(x + 1)}\right).$$

42. $y = (2x+1)(4x^2)\sqrt{x-9}$

SOLUTION Let $y = (2x+1)(4x^2)\sqrt{x-9}$. Then

$$\ln y = \ln(2x+1) + \ln 4x^2 + \ln(x-9)^{1/2} = \ln(2x+1) + \ln 4 + 2\ln x + \frac{1}{2}\ln(x-9).$$

By logarithmic differentiation

$$\frac{y'}{y} = \frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)},$$

so

$$y' = (2x+1)(4x^2)\sqrt{x-9}\left(\frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)}\right).$$

43. $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x+2)}}$

SOLUTION Let $y = \sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}}$. Then $\ln y = \frac{1}{2}[\ln(x) + \ln(x+2) - \ln(2x+1) - \ln(3x+2)]$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right),$$

so

$$y' = \frac{1}{2}\sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}} \cdot \left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right).$$

44. $y = (x^3+1)(x^4+2)(x^5+3)^2$

SOLUTION Let $y = (x^3+1)(x^4+2)(x^5+3)^2$. Then $\ln y = \ln(x^3+1) + \ln(x^4+2) + 2\ln(x^5+3)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{4x^3}{x^4+2} + \frac{10x^4}{x^5+3},$$

so

$$y' = (x^3+1)(x^4+2)(x^5+3)^2\left(\frac{3x^2}{x^3+1} + \frac{4x^3}{x^4+2} + \frac{10x^4}{x^5+3}\right).$$

In Exercises 45–50, find the derivative using either method of Example 6.

45. $f(x) = x^{3x}$

SOLUTION Method 1: $x^{3x} = e^{3x\ln x}$, so

$$\frac{d}{dx}x^{3x} = e^{3x\ln x}(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

Method 2: Let $y = x^{3x}$. Then, $\ln y = 3x\ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3x \cdot \frac{1}{x} + 3\ln x,$$

so

$$y' = y(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

46. $f(x) = x^{\cos x}$

SOLUTION Method 1: $x^{\cos x} = e^{\cos x \ln x}$, so

$$\frac{d}{dx}x^{\cos x} = e^{\cos x \ln x}\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Method 2: Let $y = x^{\cos x}$. Then $\ln y = \cos x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = \cos x \frac{1}{x} + \ln x(-\sin x),$$

so

$$y' = y\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

47. $f(x) = x^{e^x}$

SOLUTION Method 1: $x^{e^x} = e^{e^x \ln x}$, so

$$\frac{d}{dx}x^{e^x} = e^{e^x \ln x}\left(\frac{e^x}{x} + e^x \ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right).$$

Method 2: Let $y = x^{e^x}$. Then $\ln y = e^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = e^x \cdot \frac{1}{x} + e^x \ln x,$$

so

$$y' = y\left(\frac{e^x}{x} + e^x \ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right).$$

48. $f(x) = x^{x^2}$

SOLUTION Method 1: $x^{x^2} = e^{x^2 \ln x}$, so

$$\frac{d}{dx}x^{x^2} = e^{x^2 \ln x}(x + 2x \ln x) = x^{x^2}(x + 2x \ln x) = x^{x^2+1}(1 + 2\ln x).$$

Method 2: Let $y = x^{x^2}$. Then $\ln y = x^2 \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = x + 2x \ln x,$$

so

$$y' = x^{x^2}(x + 2x \ln x) = x^{x^2+1}(1 + 2\ln x).$$

49. $f(x) = x^{3^x}$

SOLUTION Method 1: $x^{3^x} = e^{3^x \ln x}$, so

$$\frac{d}{dx}x^{3^x} = e^{3^x \ln x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right) = x^{3^x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right).$$

Method 2: Let $y = x^{3^x}$. Then $\ln y = 3^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3^x \frac{1}{x} + (\ln x)(\ln 3)3^x,$$

so

$$y' = x^{3^x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right).$$

50. $f(x) = e^{x^x}$

SOLUTION Method 1:

$$\frac{d}{dx}e^{x^x} = e^{x^x}\frac{d}{dx}x^x = e^{x^x} \cdot x^x(1 + \ln x),$$

by Example 6 from the text.

Method 2: Let $y = e^{x^x}$. Then $\ln y = x^x \ln e = x^x$. By logarithmic differentiation and Example 6

$$\frac{y'}{y} = x^x(1 + \ln x), \quad \text{so} \quad y' = e^{x^x}(x^x)(1 + \ln x).$$

In Exercises 51–74, calculate the derivative.

51. $y = \sinh(9x)$

SOLUTION $\frac{d}{dx}\sinh(9x) = 9\cosh(9x)$.

52. $y = \sinh(x^2)$

SOLUTION $\frac{d}{dx}\sinh(x^2) = 2x\cosh(x^2)$.

53. $y = \cosh^2(9 - 3t)$

SOLUTION $\frac{d}{dt}\cosh^2(9 - 3t) = 2\cosh(9 - 3t) \cdot (-3\sinh(9 - 3t)) = -6\cosh(9 - 3t)\sinh(9 - 3t)$.

54. $y = \tanh(t^2 + 1)$

SOLUTION $\frac{d}{dt}\tanh(t^2 + 1) = 2t\,\mathrm{sech}^2(t^2 + 1)$.

55. $y = \sqrt{\cosh x + 1}$

SOLUTION $\frac{d}{dx}\sqrt{\cosh x + 1} = \frac{1}{2}(\cosh x + 1)^{-1/2}\sinh x$.

56. $y = \sinh x \tanh x$

SOLUTION $\frac{d}{dx}\sinh x \tanh x = \cosh x \tanh x + \sinh x\,\mathrm{sech}^2 x = \sinh x + \tanh x\,\mathrm{sech}\, x$.

57. $y = \frac{\coth t}{1 + \tanh t}$

SOLUTION $\frac{d}{dt}\frac{\coth t}{1 + \tanh t} = \frac{-\mathrm{csch}^2 t(1 + \tanh t) - \coth t(\mathrm{sech}^2 t)}{(1 + \tanh t)^2} = \frac{1 + \cosh t}{(1 + \cosh t)^2} = \frac{1}{1 + \cosh t}$.

58. $y = (\ln(\cosh x))^5$

SOLUTION $\frac{d}{dx}(\ln(\cosh x))^5 = 5(\ln\cosh x)^4\frac{\sinh x}{\cosh x} = 5(\ln\cosh x)^4\tanh x$.

59. $y = \sinh(\ln x)$

SOLUTION $\frac{d}{dx}\sinh(\ln x) = \frac{\cosh(\ln x)}{x}$.

60. $y = e^{\coth x}$

SOLUTION $\frac{d}{dx}e^{\coth x} = -\mathrm{csch}^2 x \cdot e^{\coth x}$.

61. $y = \tanh(e^x)$

SOLUTION $\frac{d}{dx}\tanh(e^x) = e^x\,\mathrm{sech}^2(e^x)$.

62. $y = \sinh(\cosh^3 x)$

SOLUTION $\frac{d}{dx}\sinh(\cosh^3 x) = \cosh(\cosh^3 x)(3\cosh^2 x\sinh x)$.

63. $y = \mathrm{sech}(\sqrt{x})$

SOLUTION $\frac{d}{dx}\mathrm{sech}(\sqrt{x}) = -\frac{1}{2}x^{-1/2}\,\mathrm{sech}\sqrt{x}\tanh\sqrt{x}$.

64. $y = \ln(\coth x)$

SOLUTION $\frac{d}{dx}\ln(\coth x) = \frac{-\mathrm{csch}^2 x}{\coth x} = \frac{-1}{\sinh^2 x(\frac{\cosh x}{\sinh x})} = \frac{-1}{\sinh x\cosh x}$.

65. $y = \operatorname{sech} x \coth x$

SOLUTION $\dfrac{d}{dx} \operatorname{sech} x \coth x = \dfrac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$.

66. $y = x^{\sinh x}$

SOLUTION

$$\frac{d}{dx} x^{\sinh x} = \frac{d}{dx} e^{\ln x \sinh x} = \left(\cosh x \ln x + \frac{\sinh x}{x}\right) e^{\sinh x \ln x} = x^{\sinh x}\left(\cosh x \ln x + \frac{\sinh x}{x}\right).$$

67. $y = \cosh^{-1}(3x)$

SOLUTION $\dfrac{d}{dx} \cosh^{-1}(3x) = \dfrac{3}{\sqrt{9x^2 - 1}}$.

68. $y = \tanh^{-1}(e^x + x^2)$

SOLUTION $\dfrac{d}{dx} \tanh^{-1}(e^x + x^2) = \dfrac{e^x + 2x}{1 - (e^x + x^2)^2}$.

69. $y = (\sinh^{-1}(x^2))^3$

SOLUTION $\dfrac{d}{dx} (\sinh^{-1}(x^2))^3 = 3(\sinh^{-1}(x^2))^2 \dfrac{2x}{\sqrt{x^4 + 1}}$.

70. $y = (\operatorname{csch}^{-1} 3x)^4$

SOLUTION $\dfrac{d}{dx} (\operatorname{csch}^{-1} 3x)^4 = 4(\operatorname{csch}^{-1} 3x)^3 \left(\dfrac{-1}{|3x|\sqrt{1 + 9x^2}}\right)(3) = \dfrac{-4(\operatorname{csch}^{-1} 3x)^3}{|x|\sqrt{1 + 9x^2}}$.

71. $y = e^{\cosh^{-1} x}$

SOLUTION $\dfrac{d}{dx} e^{\cosh^{-1} x} = e^{\cosh^{-1} x}\left(\dfrac{1}{\sqrt{x^2 - 1}}\right)$.

72. $y = \sinh^{-1}(\sqrt{x^2 + 1})$

SOLUTION $\dfrac{d}{dx} \sinh^{-1}(\sqrt{x^2 + 1}) = \dfrac{1}{\sqrt{x^2 + 1 + 1}} \left(\dfrac{1}{2\sqrt{x^2 + 1}}\right)(2x) = \dfrac{x}{\sqrt{x^2 + 2} \cdot \sqrt{x^2 + 1}}$.

73. $y = \tanh^{-1}(\ln t)$

SOLUTION $\dfrac{d}{dt} \tanh^{-1}(\ln t) = \dfrac{1}{t(1 - (\ln t)^2)}$.

74. $y = \ln(\tanh^{-1} x)$

SOLUTION $\dfrac{d}{dx} \ln(\tanh^{-1} x) = \dfrac{1}{\tanh^{-1} x}\left(\dfrac{1}{1 - x^2}\right)$.

In Exercises 75–77, prove the formula.

75. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

SOLUTION $\dfrac{d}{dx} \coth x = \dfrac{d}{dx} \dfrac{\cosh x}{\sinh x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \dfrac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$.

76. $\dfrac{d}{dt} \sinh^{-1} t = \dfrac{1}{\sqrt{t^2 + 1}}$

SOLUTION Let $x = \sinh^{-1} t$. Then $t = \sinh x$ and

$$1 = \cosh x \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{\cosh x}.$$

Thus,

$$\frac{d}{dt} \sinh^{-1} t = \frac{1}{\cosh x},$$

where $\sinh x = t$. Working from the identity $\cosh^2 x - \sinh^2 x = 1$, we find $\cosh x = \pm\sqrt{\sinh^2 x + 1}$. Because the hyperbolic cosine is always positive, we know to choose the positive square root. Hence, $\cosh x = \sqrt{\sinh^2 x + 1} = \sqrt{t^2 + 1}$, and

$$\frac{d}{dt} \sinh^{-1} t = \frac{1}{\cosh x} = \frac{1}{\sqrt{t^2 + 1}}.$$

77. $\dfrac{d}{dt}\cosh^{-1} t = \dfrac{1}{\sqrt{t^2-1}}$ for $t > 1$

SOLUTION Let $x = \cosh^{-1} t$. Then $x \geq 0$, $t = \cosh x$ and

$$1 = \sinh x \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{\sinh x}.$$

Thus, for $t > 1$,

$$\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sinh x},$$

where $\cosh x = t$. Working from the identity $\cosh^2 x - \sinh^2 x = 1$, we find $\sinh x = \pm\sqrt{\cosh^2 x - 1}$. Because $\sinh w \geq 0$ for $w \geq 0$, we know to choose the positive square root. Hence, $\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{t^2-1}$, and

$$\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sinh x} = \frac{1}{\sqrt{t^2-1}}.$$

78. Use the formula $(\ln f(x))' = f'(x)/f(x)$ to show that $\ln x$ and $\ln(2x)$ have the same derivative. Is there a simpler explanation of this result?

SOLUTION Observe

$$(\ln x)' = \frac{1}{x} \quad \text{and} \quad (\ln 2x)' = \frac{2}{2x} = \frac{1}{x}.$$

As an alternative explanation, note that $\ln(2x) = \ln 2 + \ln x$. Hence, $\ln x$ and $\ln(2x)$ differ by a constant, which implies the two functions have the same derivative.

79. According to one simplified model, the purchasing power of a dollar in the year $2000 + t$ is equal to $P(t) = 0.68(1.04)^{-t}$ (in 1983 dollars). Calculate the predicted rate of decline in purchasing power (in cents per year) in the year 2020.

SOLUTION First, note that

$$P'(t) = -0.68(1.04)^{-t}\ln 1.04;$$

thus, the rate of change in the year 2020 is

$$P'(20) = -0.68(1.04)^{-20}\ln 1.04 = -0.0122.$$

That is, the rate of decline is 1.22 cents per year.

80. The energy E (in joules) radiated as seismic waves by an earthquake of Richter magnitude M satisfies $\log_{10} E = 4.8 + 1.5M$.

(a) Show that when M increases by 1, the energy increases by a factor of approximately 31.5.

(b) Calculate dE/dM.

SOLUTION Solving $\log_{10} E = 4.8 + 1.5M$ for E yields

$$E = 10^{4.8+1.5M}.$$

(a) We find

$$E(M+1) = 10^{4.8+1.5(M+1)} = 10^{1.5}10^{4.8+1.5M} \approx 31.6E(M).$$

(b)

$$\frac{dE}{dM} = (1.5\ln 10)10^{4.8+1.5M}.$$

81. Show that for any constants M, k, and a, the function

$$y(t) = \frac{1}{2}M\left(1 + \tanh\left(\frac{k(t-a)}{2}\right)\right)$$

satisfies the **logistic equation**: $\dfrac{y'}{y} = k\left(1 - \dfrac{y}{M}\right)$.

SOLUTION Let

$$y(t) = \frac{1}{2}M\left(1 + \tanh\left(\frac{k(t-a)}{2}\right)\right).$$

Then

$$1 - \frac{y(t)}{M} = \frac{1}{2}\left(1 - \tanh\left(\frac{k(t-a)}{2}\right)\right),$$

and

$$\begin{aligned} ky(t)\left(1 - \frac{y(t)}{M}\right) &= \frac{1}{4}Mk\left(1 - \tanh^2\left(\frac{k(t-a)}{2}\right)\right) \\ &= \frac{1}{4}Mk\,\mathrm{sech}^2\left(\frac{k(t-a)}{2}\right). \end{aligned}$$

Finally,

$$y'(t) = \frac{1}{4}Mk\,\mathrm{sech}^2\left(\frac{k(t-a)}{2}\right) = ky(t)\left(1 - \frac{y(t)}{M}\right).$$

82. Show that $V(x) = 2\ln(\tanh(x/2))$ satisfies the **Poisson-Boltzmann** equation $V''(x) = \sinh(V(x))$, which is used to describe electrostatic forces in certain molecules.

SOLUTION Let $V(x) = 2\ln(\tanh(x/2))$. Then

$$V'(x) = 2\frac{1}{\tanh(x/2)} \cdot \frac{1}{2}\,\mathrm{sech}^2(x/2) = \frac{1}{\sinh(x/2)\cosh(x/2)}$$

and

$$V''(x) = -\frac{1}{2}\frac{\sinh^2(x/2) + \cosh^2(x/2)}{\sinh^2(x/2)\cosh^2(x/2)} = -\frac{1}{2}\left(\mathrm{sech}^2(x/2) + \mathrm{csch}^2(x/2)\right).$$

On the other hand,

$$\begin{aligned} \sinh(V(x)) &= \frac{e^{2\ln(\tanh(x/2))} - e^{-2\ln(\tanh(x/2))}}{2} \\ &= \frac{\tanh^2(x/2) - \coth^2(x/2)}{2} \\ &= \frac{(1 - \mathrm{sech}^2(x/2)) - (1 + \mathrm{csch}^2(x/2))}{2} = -\frac{1}{2}\left(\mathrm{sech}^2(x/2) + \mathrm{csch}^2(x/2)\right). \end{aligned}$$

Thus, $V''(x) = \sinh(V(x))$.

83. The Palermo Technical Impact Hazard Scale P is used to quantify the risk associated with the impact of an asteroid colliding with the earth:

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right)$$

where p_i is the probability of impact, T is the number of years until impact, and E is the energy of impact (in megatons of TNT). The risk is greater than a random event of similar magnitude if $P > 0$.

(a) Calculate dP/dT, assuming that $p_i = 2 \times 10^{-5}$ and $E = 2$ megatons.

(b) Use the derivative to estimate the change in P if T increases from 8 to 9 years.

SOLUTION
(a) Observe that

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right) = \log_{10}\left(\frac{p_i E^{0.8}}{0.03}\right) - \log_{10} T,$$

so

$$\frac{dP}{dT} = -\frac{1}{T \ln 10}.$$

(b) If T increases to 9 years from 8 years, then

$$\Delta P \approx \left.\frac{dP}{dT}\right|_{T=8} \cdot \Delta T = -\frac{1}{(8 \text{ yr}) \ln 10} \cdot (1 \text{ yr}) = -0.054$$

Further Insights and Challenges

84. **(a)** Show that if f and g are differentiable, then

$$\frac{d}{dx} \ln(f(x)g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad \boxed{4}$$

(b) Give a new proof of the Product Rule by observing that the left-hand side of Eq. (4) is equal to $\dfrac{(f(x)g(x))'}{f(x)g(x)}$.

SOLUTION
(a) $\dfrac{d}{dx} \ln f(x)g(x) = \dfrac{d}{dx}(\ln f(x) + \ln g(x)) = \dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}$.
(b) By part (a),

$$\frac{d}{dx} \ln f(x)g(x) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)}.$$

Alternately,

$$\frac{d}{dx} \ln f(x)g(x) = \frac{(f(x)g(x))'}{f(x)g(x)}.$$

Thus,

$$\frac{(f(x)g(x))'}{f(x)g(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)},$$

or

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

85. Use the formula $\log_b x = \dfrac{\log_a x}{\log_a b}$ for $a, b > 0$ to verify the formula

$$\frac{d}{dx} \log_b x = \frac{1}{(\ln b)x}$$

SOLUTION $\dfrac{d}{dx} \log_b x = \dfrac{d}{dx} \dfrac{\ln x}{\ln b} = \dfrac{1}{(\ln b)x}$.

3.10 Implicit Differentiation

Preliminary Questions

1. Which differentiation rule is used to show $\dfrac{d}{dx} \sin y = \cos y \dfrac{dy}{dx}$?

SOLUTION The chain rule is used to show that $\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$.

2. One of (a)–(c) is incorrect. Find and correct the mistake.

(a) $\dfrac{d}{dy} \sin(y^2) = 2y \cos(y^2)$ **(b)** $\dfrac{d}{dx} \sin(x^2) = 2x \cos(x^2)$ **(c)** $\dfrac{d}{dx} \sin(y^2) = 2y \cos(y^2)$

SOLUTION

(a) This is correct. Note that the differentiation is with respect to the variable y.
(b) This is correct. Note that the differentiation is with respect to the variable x.
(c) This is incorrect. Because the differentiation is with respect to the variable x, the chain rule is needed to obtain

$$\frac{d}{dx}\sin(y^2) = 2y\cos(y^2)\frac{dy}{dx}.$$

3. On an exam, Jason was asked to differentiate the equation

$$x^2 + 2xy + y^3 = 7$$

Find the errors in Jason's answer: $2x + 2xy' + 3y^2 = 0$

SOLUTION There are two mistakes in Jason's answer. First, Jason should have applied the product rule to the second term to obtain

$$\frac{d}{dx}(2xy) = 2x\frac{dy}{dx} + 2y.$$

Second, he should have applied the general power rule to the third term to obtain

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}.$$

4. Which of (a) or (b) is equal to $\frac{d}{dx}(x\sin t)$?

(a) $(x\cos t)\frac{dt}{dx}$ **(b)** $(x\cos t)\frac{dt}{dx} + \sin t$

SOLUTION Using the product rule and the chain rule we see that

$$\frac{d}{dx}(x\sin t) = x\cos t\frac{dt}{dx} + \sin t,$$

so the correct answer is **(b)**.

Exercises

1. Show that if you differentiate both sides of $x^2 + 2y^3 = 6$, the result is $2x + 6y^2\frac{dy}{dx} = 0$. Then solve for dy/dx and evaluate it at the point $(2, 1)$.

SOLUTION

$$\begin{aligned}
\frac{d}{dx}(x^2 + 2y^3) &= \frac{d}{dx}6\\
2x + 6y^2\frac{dy}{dx} &= 0\\
2x + 6y^2\frac{dy}{dx} &= 0\\
6y^2\frac{dy}{dx} &= -2x\\
\frac{dy}{dx} &= \frac{-2x}{6y^2}.
\end{aligned}$$

At $(2, 1)$, $\frac{dy}{dx} = \frac{-4}{6} = -\frac{2}{3}$.

2. Show that if you differentiate both sides of $xy + 4x + 2y = 1$, the result is $(x+2)\frac{dy}{dx} + y + 4 = 0$. Then solve for dy/dx and evaluate it at the point $(1, -1)$.

SOLUTION Applying the product rule

$$\begin{aligned}
\frac{d}{dx}(xy + 4x + 2y) &= \frac{d}{dx}1\\
x\frac{dy}{dx} + y + 4 + 2\frac{dy}{dx} &= 0\\
(x+2)\frac{dy}{dx} &= -(y+4)\\
\frac{dy}{dx} &= -\frac{y+4}{x+2}.
\end{aligned}$$

At $(1, -1)$, $dy/dx = -3/3 = -1$.

In Exercises 3–8, differentiate the expression with respect to x, assuming that $y = f(x)$.

3. x^2y^3

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(x^2y^3\right) = x^2 \cdot 3y^2y' + y^3 \cdot 2x = 3x^2y^2y' + 2xy^3.$$

4. $\dfrac{x^3}{y^2}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\frac{x^3}{y^2}\right) = \frac{y^2(3x^2) - x^3 2yy'}{y^4} = \frac{3x^2}{y^2} - \frac{2x^3y'}{y^3}.$$

5. $(x^2 + y^2)^{3/2}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\left(x^2 + y^2\right)^{3/2}\right) = \frac{3}{2}\left(x^2 + y^2\right)^{1/2}\left(2x + 2yy'\right) = 3\left(x + yy'\right)\sqrt{x^2 + y^2}.$$

6. $\tan(xy)$

SOLUTION Assuming that y depends on x, then $\dfrac{d}{dx}(\tan(xy)) = \left(xy' + y\right)\sec^2(xy)$.

7. $\dfrac{y}{y+1}$

SOLUTION Assuming that y depends on x, then $\dfrac{d}{dx}\dfrac{y}{y+1} = \dfrac{(y+1)y' - yy'}{(y+1)^2} = \dfrac{y'}{(y+1)^2}$.

8. $e^{y/t}$

SOLUTION Assuming that y depends on t, then

$$\frac{d}{dt}e^{y/t} = e^{y/t}\left(\frac{ty' - y}{t^2}\right).$$

In Exercises 9–26, calculate the derivative with respect to x.

9. $3y^3 + x^2 = 5$

SOLUTION Let $3y^3 + x^2 = 5$. Then $9y^2y' + 2x = 0$, and $y' = -\dfrac{2x}{9y^2}$.

10. $y^4 - 2y = 4x^3 + x$

SOLUTION Let $y^4 - 2y = 4x^3 + x$. Then

$$\begin{aligned}
\frac{d}{dx}(y^4 - 2y) &= \frac{d}{dx}(4x^3 + x)\\
4y^3y' - 2y' &= 12x^2 + 1\\
y'(4y^3 - 2) &= 12x^2 + 1\\
y' &= \frac{12x^2 + 1}{4y^3 - 2}
\end{aligned}$$

11. $x^2y + 2x^3y = x + y$

SOLUTION Let $x^2y + 2x^3y = x + y$. Then

$$\begin{aligned}
x^2y' + 2xy + 2x^3y' + 6x^2y &= 1 + y'\\
x^2y' + 2x^3y' - y' &= 1 - 2xy - 6x^2y\\
y' &= \frac{1 - 2xy - 6x^2y}{x^2 + 2x^3 - 1}.
\end{aligned}$$

12. $xy^2 + x^2y^5 - x^3 = 3$

SOLUTION Let $xy^2 + x^2y^5 - x^3 = 3$. Then

$$\begin{aligned} 2xyy' + y^2 + 5x^2y^4y' + 2xy^5 - 3x^2 &= 0 \\ (2xy + 5x^2y^4)y' &= 3x^2 - y^2 - 2xy^5 \\ y' &= \frac{3x^2 - y^2 - 2xy^5}{2xy + 5x^2y^4} \end{aligned}$$

13. $x^3R^5 = 1$

SOLUTION Let $x^3R^5 = 1$. Then $x^3 \cdot 5R^4R' + R^5 \cdot 3x^2 = 0$, and $R' = -\frac{3x^2R^5}{5x^3R^4} = -\frac{3R}{5x}$.

14. $x^4 + z^4 = 1$

SOLUTION Let $x^4 + z^4 = 1$. Then $4x^3 + 4z^3z' = 0$, and $z' = -x^3/z^3$.

15. $\frac{y}{x} + \frac{x}{y} = 2y$

SOLUTION Let

$$\frac{y}{x} + \frac{x}{y} = 2y.$$

Then

$$\begin{aligned} \frac{xy' - y}{x^2} + \frac{y - xy'}{y^2} &= 2y' \\ \left(\frac{1}{x} - \frac{x}{y^2} - 2\right)y' &= \frac{y}{x^2} - \frac{1}{y} \\ \frac{y^2 - x^2 - 2xy^2}{xy^2}y' &= \frac{y^2 - x^2}{x^2y} \\ y' &= \frac{y(y^2 - x^2)}{x(y^2 - x^2 - 2xy^2)}. \end{aligned}$$

16. $\sqrt{x+s} = \frac{1}{x} + \frac{1}{s}$

SOLUTION Let $(x+s)^{1/2} = x^{-1} + s^{-1}$. Then

$$\frac{1}{2}(x+s)^{-1/2}\left(1 + s'\right) = -x^{-2} - s^{-2}s'.$$

Multiplying by $2x^2s^2\sqrt{x+s}$ and then solving for s' gives

$$\begin{aligned} x^2s^2\left(1 + s'\right) &= -2s^2\sqrt{x+s} - 2x^2s'\sqrt{x+s} \\ x^2s^2s' + 2x^2s'\sqrt{x+s} &= -2s^2\sqrt{x+s} - x^2s^2 \\ x^2\left(s^2 + 2\sqrt{x+s}\right)s' &= -s^2\left(x^2 + 2\sqrt{x+s}\right) \\ s' &= -\frac{s^2\left(x^2 + 2\sqrt{x+s}\right)}{x^2\left(s^2 + 2\sqrt{x+s}\right)}. \end{aligned}$$

17. $y^{-2/3} + x^{3/2} = 1$

SOLUTION Let $y^{-2/3} + x^{3/2} = 1$. Then

$$-\frac{2}{3}y^{-5/3}y' + \frac{3}{2}x^{1/2} = 0 \quad \text{or} \quad y' = \frac{9}{4}x^{1/2}y^{5/3}.$$

18. $x^{1/2} + y^{2/3} = -4y$

SOLUTION Let $x^{1/2} + y^{2/3} = y^{-4}$. Then $\frac{1}{2}x^{-1/2} + \frac{2}{3}y^{-1/3}y' = -4y^{-5}y'$, and

$$y' = -\frac{\frac{1}{2}x^{-1/2}}{\frac{2}{3}y^{-1/3} + 4y^{-5}}.$$

19. $y + \frac{1}{y} = x^2 + x$

SOLUTION Let $y + \frac{1}{y} = x^2 + x$. Then

$$y' - \frac{1}{y^2} y' = 2x + 1 \quad \text{or} \quad y' = \frac{2x+1}{1 - y^{-2}} = \frac{(2x+1)y^2}{y^2 - 1}.$$

20. $\sin(xt) = t$

SOLUTION In what follows, $t' = \frac{dt}{dx}$. Applying the chain rule and the product rule, we get:

$$\begin{aligned}
\frac{d}{dx}\sin(xt) &= \frac{d}{dx}t \\
\cos(xt)(xt' + t) &= t' \\
x\cos(xt)t' + t\cos(xt) &= t' \\
x\cos(xt)t' - t' &= -t\cos(xt) \\
t'(x\cos(xt) - 1) &= -t\cos(xt) \\
t' &= \frac{-t\cos(xt)}{x\cos(xt) - 1}.
\end{aligned}$$

21. $\sin(x + y) = x + \cos y$

SOLUTION Let $\sin(x + y) = x + \cos y$. Then

$$\begin{aligned}
(1 + y')\cos(x + y) &= 1 - y'\sin y \\
\cos(x + y) + y'\cos(x + y) &= 1 - y'\sin y \\
(\cos(x + y) + \sin y)\, y' &= 1 - \cos(x + y) \\
y' &= \frac{1 - \cos(x + y)}{\cos(x + y) + \sin y}.
\end{aligned}$$

22. $\tan(x^2 y) = (x + y)^3$

SOLUTION Let $\tan\left(x^2 y\right) = (x + y)^3$. Then

$$\begin{aligned}
\sec^2(x^2 y) \cdot (x^2 y' + 2xy) &= 3(x + y)^2(1 + y') \\
x^2\sec^2(x^2 y)y' + 2xy\sec^2(x^2 y) &= 3(x + y)^2 + 3(x + y)^2 y' \\
\left(x^2\sec^2(x^2 y) - 3(x + y)^2\right) y' &= 3(x + y)^2 - 2xy\sec^2(x^2 y) \\
y' &= \frac{3(x + y)^2 - 2xy\sec^2\left(x^2 y\right)}{x^2\sec^2\left(x^2 y\right) - 3(x + y)^2}.
\end{aligned}$$

23. $xe^y = 2xy + y^3$

SOLUTION Let $xe^y = 2xy + y^3$. Then $xy'e^y + e^y = 2xy' + 2y + 3y^2 y'$, whence

$$y' = \frac{e^y - 2y}{2x + 3y^2 - xe^y}.$$

24. $e^{xy} = \sin(y^2)$

SOLUTION Let $e^{xy} = \sin(y^2)$. Then $e^{xy}\left(xy' + y\right) = 2y\cos(y^2)y'$, whence

$$y' = \frac{ye^{xy}}{2y\cos(y^2) - xe^{xy}}.$$

25. $\ln x + \ln y = x - y$

SOLUTION Let $\ln x + \ln y = x - y$. Then

$$\frac{1}{x} + \frac{y'}{y} = 1 - y' \quad \text{or} \quad y' = \frac{1 - \frac{1}{x}}{1 + \frac{1}{y}} = \frac{xy - y}{xy + x}.$$

26. $\ln(x^2 + y^2) = x + 4$

SOLUTION Let $\ln(x^2 + y^2) = x + 4$. Then

$$\frac{2x + 2yy'}{x^2 + y^2} = 1 \quad \text{or} \quad y' = \frac{x^2 + y^2 - 2x}{2y}.$$

27. Show that $x + yx^{-1} = 1$ and $y = x - x^2$ define the same curve (except that $(0, 0)$ is not a solution of the first equation) and that implicit differentiation yields $y' = yx^{-1} - x$ and $y' = 1 - 2x$. Explain why these formulas produce the same values for the derivative.

SOLUTION Multiply the first equation by x and then isolate the y term to obtain

$$x^2 + y = x \quad \Rightarrow \quad y = x - x^2.$$

Implicit differentiation applied to the first equation yields

$$1 - yx^{-2} + x^{-1}y' = 0 \quad \text{or} \quad y' = yx^{-1} - x.$$

From the first equation, we find $yx^{-1} = 1 - x$; upon substituting this expression into the previous derivative, we find

$$y' = 1 - x - x = 1 - 2x,$$

which is the derivative of the second equation.

28. Use the method of Example 4 to compute $\left.\frac{dy}{dx}\right|_P$ at $P = (2, 1)$ on the curve $y^2x^3 + y^3x^4 - 10x + y = 5$.

SOLUTION Implicit differentiation yields

$$3x^2y^2 + 2x^3yy' + 4x^3y^3 + 3x^4y^2y' - 10 + y' = 0 \quad \text{or} \quad y' = \frac{10 - 3x^2y^2 - 4x^3y^3}{2x^3y + 3x^4y^2 + 1}.$$

Thus, at $P = (2, 1)$,

$$\left.\frac{dy}{dx}\right|_P = \frac{10 - 3(2)^2(1)^2 - 4(2)^3(1)^3}{2(2)^3(1) + 3(2)^4(1)^2 + 1} = -\frac{34}{65}.$$

In Exercises 29 and 30, find dy/dx at the given point.

29. $(x + 2)^2 - 6(2y + 3)^2 = 3, \quad (1, -1)$

SOLUTION By the scaling and shifting rule,

$$2(x + 2) - 24(2y + 3)y' = 0.$$

If $x = 1$ and $y = -1$, then

$$2(3) - 24(1)y' = 0.$$

so that $24y' = 6$, or $y' = \frac{1}{4}$.

30. $\sin^2(3y) = x + y, \quad \left(\frac{2-\pi}{4}, \frac{\pi}{4}\right)$

SOLUTION Taking the derivative of both sides of $\sin^2(3y) = x + y$ yields

$$2\sin(3y)\cos(3y)(3y') = 1 + y'.$$

If $x = \frac{2-\pi}{4}$ and $y = \frac{\pi}{4}$, we get

$$6\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)y' = 1 + y'.$$

Using

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

we find

$$\begin{aligned} -6\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)y' &= 1 + y' \\ -3y' &= 1 + y' \\ y' &= -\frac{1}{4}. \end{aligned}$$

In Exercises 31–38, find an equation of the tangent line at the given point.

31. $xy + x^2y^2 = 5, \quad (2, 1)$

SOLUTION Taking the derivative of both sides of $xy + x^2y^2 = 5$ yields

$$xy' + y + 2xy^2 + 2x^2yy' = 0.$$

Substituting $x = 2$, $y = 1$, we find

$$2y' + 1 + 4 + 8y' = 0 \quad \text{or} \quad y' = -\frac{1}{2}.$$

Hence, the equation of the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

32. $x^{2/3} + y^{2/3} = 2, \quad (1, 1)$

SOLUTION Taking the derivative of both sides of $x^{2/3} + y^{2/3} = 2$ yields

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0.$$

Substituting $x = 1$, $y = 1$ yields $\frac{2}{3} + \frac{2}{3}y' = 0$, so that $1 + y' = 0$, or $y' = -1$. Hence, the equation of the tangent line at $(1, 1)$ is $y - 1 = -(x - 1)$, or $y = 2 - x$.

33. $x^2 + \sin y = xy^2 + 1, \quad (1, 0)$

SOLUTION Taking the derivative of both sides of $x^2 + \sin y = xy^2 + 1$ yields

$$2x + \cos y y' = y^2 + 2xyy'.$$

Substituting $x = 1$, $y = 0$, we find

$$2 + y' = 0 \quad \text{or} \quad y' = -2.$$

Hence, the equation of the tangent line is $y - 0 = -2(x - 1)$ or $y = -2x + 2$.

34. $\sin(x - y) = x\cos\left(y + \frac{\pi}{4}\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

SOLUTION Taking the derivative of both sides of $\sin(x - y) = x\cos\left(y + \frac{\pi}{4}\right)$ yields

$$\cos(x - y)(1 - y') = \cos\left(y + \frac{\pi}{4}\right) - x\sin\left(y + \frac{\pi}{4}\right)y'.$$

Substituting $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$, we find

$$1(1 - y') = 0 - \frac{\pi}{4}y' \quad \text{or} \quad y' = \frac{4}{4 + \pi}.$$

Hence, the equation of the tangent line is

$$y - \frac{\pi}{4} = \frac{4}{4+\pi}\left(x - \frac{\pi}{4}\right).$$

35. $2x^{1/2} + 4y^{-1/2} = xy, \quad (1, 4)$

SOLUTION Taking the derivative of both sides of $2x^{1/2} + 4y^{-1/2} = xy$ yields

$$x^{-1/2} - 2y^{-3/2}y' = xy' + y.$$

Substituting $x = 1$, $y = 4$, we find

$$1 - 2\left(\frac{1}{8}\right)y' = y' + 4 \quad \text{or} \quad y' = -\frac{12}{5}.$$

Hence, the equation of the tangent line is $y - 4 = -\frac{12}{5}(x - 1)$ or $y = -\frac{12}{5}x + \frac{32}{5}$.

36. $x^2e^y + ye^x = 4, \quad (2, 0)$

SOLUTION Taking the derivative of both sides of $x^2e^y + ye^x = 4$ yields

$$x^2e^yy' + 2xe^y + ye^x + e^xy' = 0.$$

Substituting $x = 2$, $y = 0$, we find

$$4y' + 4 + 0 + e^2y' = 0 \quad \text{or} \quad y' = -\frac{4}{4+e^2}.$$

Hence, the equation of the tangent line is

$$y = -\frac{4}{4+e^2}(x - 2).$$

37. $e^{2x-y} = \dfrac{x^2}{y}, \quad (2, 4)$

SOLUTION taking the derivative of both sides of $e^{2x-y} = \dfrac{x^2}{y}$ yields

$$e^{2x-y}(2 - y') = \frac{2xy - x^2y'}{y^2}.$$

Substituting $x = 2$, $y = 4$, we find

$$e^0(2 - y') = \frac{16 - 4y'}{16} \quad \text{or} \quad y' = \frac{4}{3}.$$

Hence, the equation of the tangent line is $y - 4 = \frac{4}{3}(x - 2)$ or $y = \frac{4}{3}x + \frac{4}{3}$.

38. $y^2e^{x^2-16} - xy^{-1} = 2, \quad (4, 2)$

SOLUTION Taking the derivative of both sides of $y^2e^{x^2-16} - xy^{-1} = 2$ yields

$$2xy^2e^{x^2-16} + 2yy'e^{x^2-16} + xy^{-2}y' - y^{-1} = 0.$$

Substituting $x = 4$, $y = 2$, we find

$$32e^0 + 4y'e^0 + y' - \frac{1}{2} = 0 \quad \text{or} \quad y' = -\frac{63}{10}.$$

Hence, the equation of the tangent line is $y - 2 = -\frac{63}{10}(x - 4)$ or $y = -\frac{63}{10}x + \frac{136}{5}$.

39. Find the points on the graph of $y^2 = x^3 - 3x + 1$ (Figure 6) where the tangent line is horizontal.

(a) First show that $2yy' = 3x^2 - 3$, where $y' = dy/dx$.

(b) Do not solve for y'. Rather, set $y' = 0$ and solve for x. This yields two values of x where the slope may be zero.

(c) Show that the positive value of x does not correspond to a point on the graph.

(d) The negative value corresponds to the two points on the graph where the tangent line is horizontal. Find their coordinates.

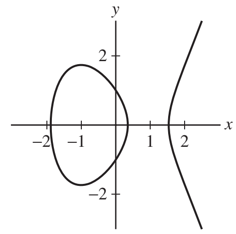


FIGURE 6 Graph of $y^2 = x^3 - 3x + 1$.

SOLUTION

(a) Applying implicit differentiation to $y^2 = x^3 - 3x + 1$, we have

$$2y\frac{dy}{dx} = 3x^2 - 3.$$

(b) Setting $y' = 0$ we have $0 = 3x^2 - 3$, so $x = 1$ or $x = -1$.

(c) If we return to the equation $y^2 = x^3 - 3x + 1$ and substitute $x = 1$, we obtain the equation $y^2 = -1$, which has no real solutions.

(d) Substituting $x = -1$ into $y^2 = x^3 - 3x + 1$ yields

$$y^2 = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3,$$

so $y = \sqrt{3}$ or $-\sqrt{3}$. The tangent is horizontal at the points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

40. Show, by differentiating the equation, that if the tangent line at a point (x, y) on the curve $x^2y - 2x + 8y = 2$ is horizontal, then $xy = 1$. Then substitute $y = x^{-1}$ in $x^2y - 2x + 8y = 2$ to show that the tangent line is horizontal at the points $\left(2, \frac{1}{2}\right)$ and $\left(-4, -\frac{1}{4}\right)$.

SOLUTION Taking the derivative on both sides of the equation $x^2y - 2x + 8y = 2$ yields

$$x^2y' + 2xy - 2 + 8y' = 0 \quad \text{or} \quad y' = \frac{2(1 - xy)}{x^2 + 8}.$$

Thus, if the tangent line to the given curve is horizontal, it must be that $1 - xy = 0$, or $xy = 1$. Substituting $y = x^{-1}$ into $x^2y - 2x + 8y = 2$ then yields

$$x - 2x + \frac{8}{x} = 2 \quad \text{or} \quad x^2 + 2x - 8 = (x + 4)(x - 2) = 0.$$

Hence, the given curve has a horizontal tangent line when $x = 2$ and when $x = -4$. The corresponding points on the curve are thus $\left(2, \frac{1}{2}\right)$ and $\left(-4, -\frac{1}{4}\right)$.

41. Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal (Figure 7).

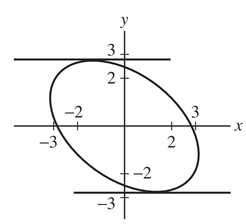


FIGURE 7 Graph of $3x^2 + 4y^2 + 3xy = 24$.

SOLUTION Differentiating the equation $3x^2 + 4y^2 + 3xy = 24$ implicitly yields

$$6x + 8yy' + 3xy' + 3y = 0,$$

so

$$y' = -\frac{6x + 3y}{8y + 3x}.$$

Setting $y' = 0$ leads to $6x + 3y = 0$, or $y = -2x$. Substituting $y = -2x$ into the equation $3x^2 + 4y^2 + 3xy = 24$ yields

$$3x^2 + 4(-2x)^2 + 3x(-2x) = 24,$$

or $13x^2 = 24$. Thus, $x = \pm 2\sqrt{78}/13$, and the coordinates of the two points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal are

$$\left(\frac{2\sqrt{78}}{13}, -\frac{4\sqrt{78}}{13}\right) \qquad \text{and} \qquad \left(-\frac{2\sqrt{78}}{13}, \frac{4\sqrt{78}}{13}\right).$$

42. Show that no point on the graph of $x^2 - 3xy + y^2 = 1$ has a horizontal tangent line.

SOLUTION Let the implicit curve $x^2 - 3xy + y^2 = 1$ be given. Then

$$2x - 3xy' - 3y + 2yy' = 0,$$

so

$$y' = \frac{2x - 3y}{3x - 2y}.$$

Setting $y' = 0$ leads to $y = \frac{2}{3}x$. Substituting $y = \frac{2}{3}x$ into the equation of the implicit curve gives

$$x^2 - 3x\left(\frac{2}{3}x\right) + \left(\frac{2}{3}x\right)^2 = 1,$$

or $-\frac{5}{9}x^2 = 1$, which has *no* real solutions. Accordingly, there are *no* points on the implicit curve where the tangent line has slope zero.

43. Figure 1 shows the graph of $y^4 + xy = x^3 - x + 2$. Find dy/dx at the two points on the graph with x-coordinate 0 and find an equation of the tangent line at $(1, 1)$.

SOLUTION Consider the equation $y^4 + xy = x^3 - x + 2$. Then $4y^3y' + xy' + y = 3x^2 - 1$, and

$$y' = \frac{3x^2 - y - 1}{x + 4y^3}.$$

- Substituting $x = 0$ into $y^4 + xy = x^3 - x + 2$ gives $y^4 = 2$, which has two real solutions, $y = \pm 2^{1/4}$. When $y = 2^{1/4}$, we have

  $$y' = \frac{-2^{1/4} - 1}{4\left(2^{3/4}\right)} = -\frac{\sqrt{2} + \sqrt[4]{2}}{8} \approx -.3254.$$

  When $y = -2^{1/4}$, we have

  $$y' = \frac{2^{1/4} - 1}{-4\left(2^{3/4}\right)} = -\frac{\sqrt{2} - \sqrt[4]{2}}{8} \approx -.02813.$$

- At the point $(1, 1)$, we have $y' = \frac{1}{5}$. At this point the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

44. Folium of Descartes The curve $x^3 + y^3 = 3xy$ (Figure 8) was first discussed in 1638 by the French philosopher-mathematician René Descartes, who called it the folium (meaning "leaf"). Descartes's scientific colleague Gilles de Roberval called it the jasmine flower. Both men believed incorrectly that the leaf shape in the first quadrant was repeated in each quadrant, giving the appearance of petals of a flower. Find an equation of the tangent line at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.

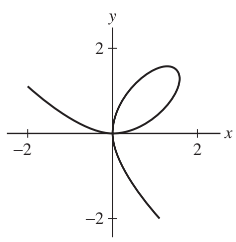


FIGURE 8 Folium of Descartes: $x^3 + y^3 = 3xy$.

SOLUTION Let $x^3 + y^3 = 3xy$. Then $3x^2 + 3y^2y' = 3xy' + 3y$, and $y' = \dfrac{x^2 - y}{x - y^2}$. At the point $\left(\frac{2}{3}, \frac{4}{3}\right)$, we have

$$y' = \frac{\frac{4}{9} - \frac{4}{3}}{\frac{2}{3} - \frac{16}{9}} = \frac{-\frac{8}{9}}{-\frac{10}{9}} = \frac{4}{5}.$$

The tangent line at P is thus $y - \frac{4}{3} = \frac{4}{5}\left(x - \frac{2}{3}\right)$ or $y = \frac{4}{5}x + \frac{4}{5}$.

45. Find a point on the folium $x^3 + y^3 = 3xy$ other than the origin at which the tangent line is horizontal.

SOLUTION Using implicit differentiation, we find

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(3xy)$$
$$3x^2 + 3y^2 y' = 3(xy' + y)$$

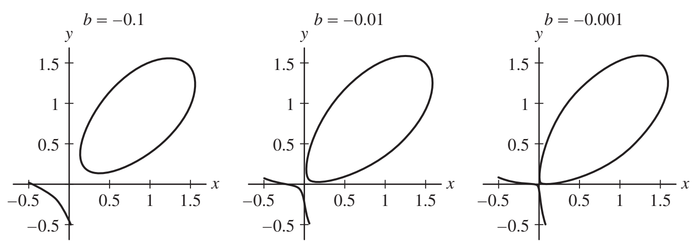
Setting $y' = 0$ in this equation yields $3x^2 = 3y$ or $y = x^2$. If we substitute this expression into the original equation $x^3 + y^3 = 3xy$, we obtain:

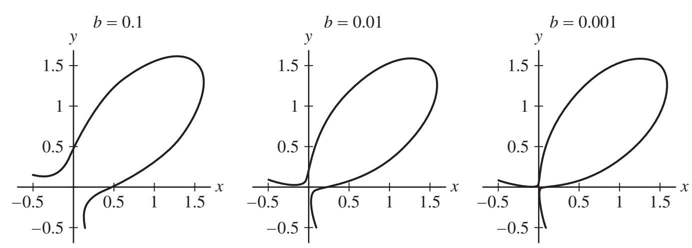
$$x^3 + x^6 = 3x(x^2) = 3x^3 \qquad \text{or} \qquad x^3(x^3 - 2) = 0.$$

One solution of this equation is $x = 0$ and the other is $x = 2^{1/3}$. Thus, the two points on the folium $x^3 + y^3 = 3xy$ at which the tangent line is horizontal are $(0, 0)$ and $(2^{1/3}, 2^{2/3})$.

46. GU Plot $x^3 + y^3 = 3xy + b$ for several values of b and describe how the graph changes as $b \to 0$. Then compute dy/dx at the point $(b^{1/3}, 0)$. How does this value change as $b \to \infty$? Do your plots confirm this conclusion?

SOLUTION Consider the first row of figures below. When $b < 0$, the graph of $x^3 + y^3 = 3xy + b$ consists of two pieces. As $b \to 0-$, the two pieces move closer to intersecting at the origin. From the second row of figures, we see that the graph of $x^3 + y^3 = 3xy + b$ when $b > 0$ consists of a single piece that has a "loop" in the first quadrant. As $b \to 0+$, the loop comes closer to "pinching off" at the origin.





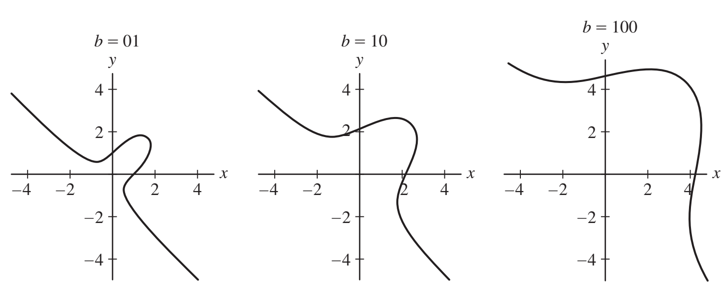
Differentiating the equation $x^3 + y^3 = 3xy + b$ with respect to x yields $3x^2 + 3y^2 y' = 3xy' + 3y$, so

$$y' = \frac{y - x^2}{y^2 - x}.$$

At $(b^{1/3}, 0)$, we have

$$y' = \frac{0 - x^2}{0^2 - x} = x = \sqrt[3]{b}.$$

Consequently, as $b \to \infty$, $y' \to \infty$ at the point on the graph where $y = 0$. This conclusion is supported by the figures shown below, which correspond to $b = 1$, $b = 10$, and $b = 100$.



47. Find the x-coordinates of the points where the tangent line is horizontal on the *trident curve* $xy = x^3 - 5x^2 + 2x - 1$, so named by Isaac Newton in his treatise on curves published in 1710 (Figure 9).
Hint: $2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1)$.

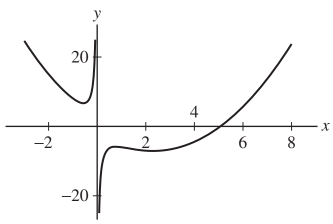


FIGURE 9 Trident curve: $xy = x^3 - 5x^2 + 2x - 1$.

SOLUTION Take the derivative of the equation of a trident curve:

$$xy = x^3 - 5x^2 + 2x - 1$$

to obtain

$$xy' + y = 3x^2 - 10x + 2.$$

Setting $y' = 0$ gives $y = 3x^2 - 10x + 2$. Substituting this into the equation of the trident, we have

$$xy = x(3x^2 - 10x + 2) = x^3 - 5x^2 + 2x - 1$$

or

$$3x^3 - 10x^2 + 2x = x^3 - 5x^2 + 2x - 1$$

Collecting like terms and setting to zero, we have

$$0 = 2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1).$$

Hence, $x = \frac{1}{2}, 1 \pm \sqrt{2}$.

48. Find an equation of the tangent line at each of the four points on the curve $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$ where $x = 1$. This curve (Figure 10) is an example of a *limaçon of Pascal*, named after the father of the French philosopher Blaise Pascal, who first described it in 1650.

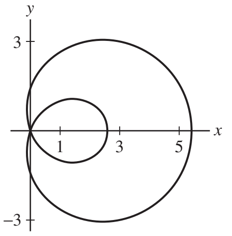


FIGURE 10 Limaçon: $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$.

SOLUTION Plugging $x = 1$ into the equation for the limaçon and solving for y, we find that the points on the curve where $x = 1$ are: $(1, 1)$, $(1, -1)$, $(1, \sqrt{7})$, $(1, -\sqrt{7})$. Using implicit differentiation, we obtain

$$2(x^2 + y^2 - 4x)(2x + 2yy' - 4) = 2(2x + 2yy').$$

We plug in $x = 1$ and get

$$2(1 + y^2 - 4)(2 + 2yy' - 4) = 2(2 + 2yy')$$

or

$$(2y^2 - 6)(2yy' - 2) = 4 + 4yy'.$$

After collecting like terms and solving for y', we have

$$y' = \frac{-2 + y^2}{y^3 - 4y}.$$

At the point (1, 1) the slope of the tangent is $\frac{1}{3}$ and the tangent line is

$$y - 1 = \frac{1}{3}(x-1) \quad \text{or} \quad y = \frac{1}{3}x + \frac{2}{3}.$$

At the point (1, −1) the slope of the tangent is $-\frac{1}{3}$ and the tangent line is

$$y + 1 = -\frac{1}{3}(x-1) \quad \text{or} \quad y = -\frac{1}{3}x - \frac{2}{3}.$$

At the point $(1, \sqrt{7})$ the slope of the tangent is $5/3\sqrt{7}$ and the tangent line is

$$y - \sqrt{7} = \frac{5}{3\sqrt{7}}(x-1) \quad \text{or} \quad y = \frac{5}{3\sqrt{7}}x + \sqrt{7} - \frac{5}{3\sqrt{7}}.$$

At the point $(1, -\sqrt{7})$ the slope of the tangent is $-5/3\sqrt{7}$ and the tangent line is

$$y + \sqrt{7} = -\frac{5}{3\sqrt{7}}(x-1) \quad \text{or} \quad y = -\frac{5}{3\sqrt{7}}x + \frac{5}{3\sqrt{7}} - \sqrt{7}.$$

49. Find the derivative at the points where $x = 1$ on the folium $(x^2 + y^2)^2 = \frac{25}{4}xy^2$. See Figure 11.

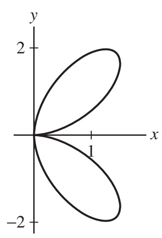


FIGURE 11 Folium curve: $(x^2 + y^2)^2 = \dfrac{25}{4}xy^2$

SOLUTION First, find the points $(1, y)$ on the curve. Setting $x = 1$ in the equation $(x^2 + y^2)^2 = \frac{25}{4}xy^2$ yields

$$\begin{aligned}
(1 + y^2)^2 &= \frac{25}{4}y^2 \\
y^4 + 2y^2 + 1 &= \frac{25}{4}y^2 \\
4y^4 + 8y^2 + 4 &= 25y^2 \\
4y^4 - 17y^2 + 4 &= 0 \\
(4y^2 - 1)(y^2 - 4) &= 0 \\
y^2 &= \frac{1}{4} \text{ or } y^2 = 4
\end{aligned}$$

Hence $y = \pm\frac{1}{2}$ or $y = \pm 2$. Taking $\frac{d}{dx}$ of both sides of the original equation yields

$$\begin{aligned}
2(x^2 + y^2)(2x + 2yy') &= \frac{25}{4}y^2 + \frac{25}{2}xyy' \\
4(x^2 + y^2)x + 4(x^2 + y^2)yy' &= \frac{25}{4}y^2 + \frac{25}{2}xyy' \\
(4(x^2 + y^2) - \frac{25}{2}x)yy' &= \frac{25}{4}y^2 - 4(x^2 + y^2)x \\
y' &= \frac{\frac{25}{4}y^2 - 4(x^2 + y^2)x}{y(4(x^2 + y^2) - \frac{25}{2}x)}
\end{aligned}$$

- At (1, 2), $x^2 + y^2 = 5$, and

$$y' = \frac{\frac{25}{4}2^2 - 4(5)(1)}{2(4(5) - \frac{25}{2}(1))} = \frac{1}{3}.$$

- At $(1, -2)$, $x^2 + y^2 = 5$ as well, and

$$y' = \frac{\frac{25}{4}(-2)^2 - 4(5)(1)}{-2(4(5) - \frac{25}{2}(1))} = -\frac{1}{3}.$$

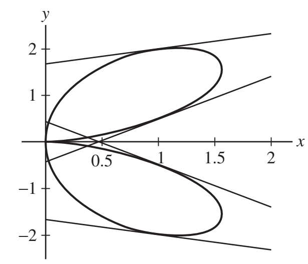
- At $(1, \frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

$$y' = \frac{\frac{25}{4}\left(\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = \frac{11}{12}.$$

- At $(1, -\frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

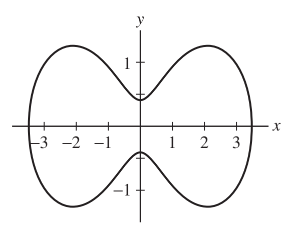
$$y' = \frac{\frac{25}{4}\left(-\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{-\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = -\frac{11}{12}.$$

The folium and its tangent lines are plotted below:



50. CAS Plot $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ for $-4 \le x \le 4$, $4 \le y \le 4$ using a computer algebra system. How many horizontal tangent lines does the curve appear to have? Find the points where these occur.

SOLUTION A plot of the curve $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ is shown below. From this plot, it appears that the curve has a horizontal tangent line at six different locations.



Differentiating the equation $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ with respect to x yields

$$2(x^2 + y^2)(2x + 2yy') = 12(2x - 2yy'),$$

so

$$y' = \frac{x(6 - x^2 - y^2)}{y(x^2 + y^2 + 6)}.$$

Thus, horizontal tangent lines occur when $x = 0$ and when $x^2 + y^2 = 6$. Substituting $x = 0$ into the equation for the curve leaves $y^4 + 12y^2 - 2 = 0$, from which it follows that $y^2 = \sqrt{38} - 6$ or $y = \pm\sqrt{\sqrt{38} - 6}$. Substituting $x^2 + y^2 = 6$ into the equation for the curve leaves $x^2 - y^2 = \frac{17}{6}$. From here, it follows that

$$x = \pm\frac{\sqrt{159}}{6} \qquad \text{and} \qquad y = \pm\frac{\sqrt{57}}{6}.$$

The six points at which horizontal tangent lines occur are therefore

$$\left(0, \sqrt{\sqrt{38} - 6}\right), \left(0, -\sqrt{\sqrt{38} - 6}\right)$$

$$\left(\frac{\sqrt{159}}{6}, \frac{\sqrt{57}}{6}\right), \left(\frac{\sqrt{159}}{6}, -\frac{\sqrt{57}}{6}\right), \left(-\frac{\sqrt{159}}{6}, \frac{\sqrt{57}}{6}\right), \left(-\frac{\sqrt{159}}{6}, -\frac{\sqrt{57}}{6}\right)$$

Exercises 51–53: If the derivative dx/dy (instead of $dy/dx = 0$) exists at a point and $dx/dy = 0$, then the tangent line at that point is vertical.

51. Calculate dx/dy for the equation $y^4 + 1 = y^2 + x^2$ and find the points on the graph where the tangent line is vertical.

SOLUTION Let $y^4 + 1 = y^2 + x^2$. Differentiating this equation with respect to y yields

$$4y^3 = 2y + 2x\frac{dx}{dy},$$

so

$$\frac{dx}{dy} = \frac{4y^3 - 2y}{2x} = \frac{y(2y^2 - 1)}{x}.$$

Thus, $\frac{dx}{dy} = 0$ when $y = 0$ and when $y = \pm\frac{\sqrt{2}}{2}$. Substituting $y = 0$ into the equation $y^4 + 1 = y^2 + x^2$ gives $1 = x^2$, so $x = \pm 1$. Substituting $y = \pm\frac{\sqrt{2}}{2}$, gives $x^2 = 3/4$, so $x = \pm\frac{\sqrt{3}}{2}$. Thus, there are six points on the graph of $y^4 + 1 = y^2 + x^2$ where the tangent line is vertical:

$$(1, 0), (-1, 0), \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right).$$

52. Show that the tangent lines at $x = 1 \pm \sqrt{2}$ to the *conchoid* with equation $(x - 1)^2(x^2 + y^2) = 2x^2$ are vertical (Figure 12).

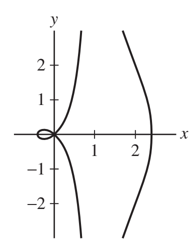


FIGURE 12 Conchoid: $(x - 1)^2(x^2 + y^2) = 2x^2$.

SOLUTION Consider the equation of a conchoid:

$$(x - 1)^2 \left(x^2 + y^2\right) = 2x^2.$$

Taking the derivative of both sides of this equation gives

$$(x - 1)^2 \left(2x\frac{dx}{dy} + 2y\right) + \left(x^2 + y^2\right) \cdot 2\,(x - 1)\,\frac{dx}{dy} = 4x\frac{dx}{dy},$$

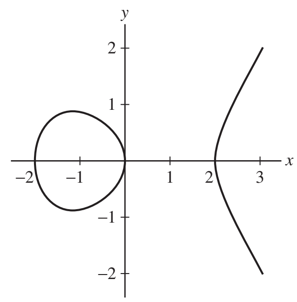
so that

$$\frac{dx}{dy} = \frac{(x - 1)^2\, y}{2x + (1 - x)\left(x^2 + y^2\right) - x\,(x - 1)^2}.$$

Setting $dx/dy = 0$ yields $x = 1$ or $y = 0$. We can't have $x = 1$, lest $0 = 2$ in the conchoid's equation. Plugging $y = 0$ into the equation gives $(x - 1)^2\, x^2 = 2x^2$ or $x^2\left((x - 1)^2 - 2\right) = 0$, which implies $x = 0$ (a double root) or $x = 1 \pm \sqrt{2}$. [Plugging $x = 0$ into the conchoid's equation gives $y^2 = 0$ or $y = 0$. At $(x, y) = (0, 0)$ the expression for dx/dy is undefined $(0/0)$. Via an alternative parametric analysis, the slopes of the tangent lines at the origin turn out to be $\pm\sqrt{3}$.] Accordingly, the tangent lines to the conchoid are vertical at $(x, y) = (1 \pm \sqrt{2}, 0)$.

53. CAS Use a computer algebra system to plot $y^2 = x^3 - 4x$ for $-4 \le x \le 4$, $4 \le y \le 4$. Show that if $dx/dy = 0$, then $y = 0$. Conclude that the tangent line is vertical at the points where the curve intersects the x-axis. Does your plot confirm this conclusion?

SOLUTION A plot of the curve $y^2 = x^3 - 4x$ is shown below.



Differentiating the equation $y^2 = x^3 - 4x$ with respect to y yields

$$2y = 3x^2 \frac{dx}{dy} - 4\frac{dx}{dy},$$

or

$$\frac{dx}{dy} = \frac{2y}{3x^2 - 4}.$$

From here, it follows that $\frac{dx}{dy} = 0$ when $y = 0$, so the tangent line to this curve is vertical at the points where the curve intersects the x-axis. This conclusion is confirmed by the plot of the curve shown above.

54. Show that for all points P on the graph in Figure 13, the segments $\overline{OP}$ and $\overline{PR}$ have equal length.

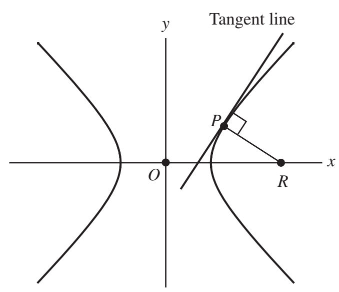


FIGURE 13 Graph of $x^2 - y^2 = a^2$.

SOLUTION Because of the symmetry of the graph, we may restrict attention to any point P in the first quadrant. Suppose P has coordinates $(p, \sqrt{p^2 - a^2})$. Taking the derivative of both sides of the equation $x^2 - y^2 = a^2$ yields $2x - 2yy' = 0$, or $y' = x/y$. It follows that the slope of the line tangent to the graph at P has slope

$$\frac{p}{\sqrt{p^2 - a^2}}$$

and the slope of the normal line is

$$-\frac{\sqrt{p^2 - a^2}}{p}.$$

Thus, the equation of the normal line is

$$y - \sqrt{p^2 - a^2} = -\frac{\sqrt{p^2 - a^2}}{p}(x - p),$$

and the coordinates of the point R are $(2p, 0)$. Finally, the length of the line segment $\overline{OP}$ is

$$\sqrt{p^2 + p^2 - a^2} = \sqrt{2p^2 - a^2},$$

while the length of the segment $\overline{PR}$ is

$$\sqrt{(2p - p)^2 + p^2 - a^2} = \sqrt{2p^2 - a^2}.$$

In Exercises 55–58, use implicit differentiation to calculate higher derivatives.

55. Consider the equation $y^3 - \frac{3}{2}x^2 = 1$.

(a) Show that $y' = x/y^2$ and differentiate again to show that

$$y'' = \frac{y^2 - 2xyy'}{y^4}$$

(b) Express y'' in terms of x and y using part (a).

SOLUTION

(a) Let $y^3 - \frac{3}{2}x^2 = 1$. Then $3y^2y' - 3x = 0$, and $y' = x/y^2$. Therefore,

$$y'' = \frac{y^2 \cdot 1 - x \cdot 2yy'}{y^4} = \frac{y^2 - 2xyy'}{y^4}.$$

(b) Substituting the expression for y' into the result for y'' gives

$$y'' = \frac{y^2 - 2xy\left(x/y^2\right)}{y^4} = \frac{y^3 - 2x^2}{y^5}.$$

56. Use the method of the previous exercise to show that $y'' = -y^{-3}$ on the circle $x^2 + y^2 = 1$.

SOLUTION Let $x^2 + y^2 = 1$. Then $2x + 2yy' = 0$, and $y' = -\frac{x}{y}$. Thus

$$y'' = -\frac{y \cdot 1 - xy'}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3} = -y^{-3}.$$

57. Calculate y'' at the point $(1, 1)$ on the curve $xy^2 + y - 2 = 0$ by the following steps:

(a) Find y' by implicit differentiation and calculate y' at the point $(1, 1)$.

(b) Differentiate the expression for y' found in (a). Then compute y'' at $(1, 1)$ by substituting $x = 1$, $y = 1$, and the value of y' found in (a).

SOLUTION Let $xy^2 + y - 2 = 0$.

(a) Then $x \cdot 2yy' + y^2 \cdot 1 + y' = 0$, and $y' = -\frac{y^2}{2xy + 1}$. At $(x, y) = (1, 1)$, we have $y' = -\frac{1}{3}$.

(b) Therefore,

$$y'' = -\frac{(2xy + 1)\left(2yy'\right) - y^2\left(2xy' + 2y\right)}{(2xy + 1)^2} = -\frac{(3)\left(-\frac{2}{3}\right) - (1)\left(-\frac{2}{3} + 2\right)}{3^2} = -\frac{-6 + 2 - 6}{27} = \frac{10}{27}$$

given that $(x, y) = (1, 1)$ and $y' = -\frac{1}{3}$.

58. Use the method of the previous exercise to compute y'' at the point $(1, 1)$ on the curve $x^3 + y^3 = 3x + y - 2$.

SOLUTION Let $x^3 + y^3 = 3x + y - 2$. Then $3x^2 + 3y^2y' = 3 + y'$, and $y' = \frac{3(1 - x^2)}{3y^2 - 1}$. At $(x, y) = (1, 1)$, we find

$$y' = \frac{3(1 - 1)}{3(1) - 1} = 0.$$

Similarly,

$$y'' = \frac{\left(3y^2 - 1\right)(-6x) - \left(3 - 3x^2\right)\left(6yy'\right)}{\left(3y^2 - 1\right)^2} = -3$$

when $(x, y) = (1, 1)$ and $y' = 0$.

In Exercises 59–61, x and y are functions of a variable t and use implicit differentiation to relate dy/dt and dx/dt.

59. Differentiate $xy = 1$ with respect to t and derive the relation $\dfrac{dy}{dt} = -\dfrac{y}{x}\dfrac{dx}{dt}$.

SOLUTION Let $xy = 1$. Then $x\dfrac{dy}{dt} + y\dfrac{dx}{dt} = 0$, and $\dfrac{dy}{dt} = -\dfrac{y}{x}\dfrac{dx}{dt}$.

60. Differentiate $x^3 + 3xy^2 = 1$ with respect to t and express dy/dt in terms of dx/dt, as in Exercise 59.

SOLUTION Let $x^3 + 3xy^2 = 1$. Then

$$3x^2\frac{dx}{dt} + 6xy\frac{dy}{dt} + 3y^2\frac{dx}{dt} = 0,$$

and

$$\frac{dy}{dt} = -\frac{x^2 + y^2}{2xy}\frac{dx}{dt}.$$

61. Calculate dy/dt in terms of dx/dt.

(a) $x^3 - y^3 = 1$

(b) $y^4 + 2xy + x^2 = 0$

SOLUTION

(a) Taking the derivative of both sides of the equation $x^3 - y^3 = 1$ with respect to t yields

$$3x^2\frac{dx}{dt} - 3y^2\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = \frac{x^2}{y^2}\frac{dx}{dt}.$$

(b) Taking the derivative of both sides of the equation $y^4 + 2xy + x^2 = 0$ with respect to t yields

$$4y^3\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} + 2x\frac{dx}{dt} = 0,$$

or

$$\frac{dy}{dt} = -\frac{x + y}{2y^3 + x}\frac{dx}{dt}.$$

62. The volume V and pressure P of gas in a piston (which vary in time t) satisfy $PV^{3/2} = C$, where C is a constant. Prove that

$$\frac{dP/dt}{dV/dt} = -\frac{3}{2}\frac{P}{V}$$

The ratio of the derivatives is negative. Could you have predicted this from the relation $PV^{3/2} = C$?

SOLUTION Let $PV^{3/2} = C$, where C is a constant. Then

$$P \cdot \frac{3}{2}V^{1/2}\frac{dV}{dt} + V^{3/2}\frac{dP}{dt} = 0, \quad \text{so} \quad \frac{dP/dt}{dV/dt} = -\frac{3}{2}\frac{P}{V}.$$

If P is increasing (respectively, decreasing), then $V = (C/P)^{2/3}$ is decreasing (respectively, increasing). Hence the ratio of the derivatives ($+/-$ or $-/+$) is negative.

Further Insights and Challenges

63. Show that if P lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$ (c, d constants), then the tangents to the curves at P are perpendicular.

SOLUTION Let $C1$ be the curve described by $x^2 - y^2 = c$, and let $C2$ be the curve described by $xy = d$. Suppose that $P = (x_0, y_0)$ lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$. Since $x^2 - y^2 = c$, the chain rule gives us $2x - 2yy' = 0$, so that $y' = \frac{2x}{2y} = \frac{x}{y}$. The slope to the tangent line to $C1$ is $\frac{x_0}{y_0}$. On the curve $C2$, since $xy = d$, the product rule yields that $xy' + y = 0$, so that $y' = -\frac{y}{x}$. Therefore the slope to the tangent line to $C2$ is $-\frac{y_0}{x_0}$. The two slopes are negative reciprocals of one another, hence the tangents to the two curves are perpendicular.

64. The *lemniscate curve* $(x^2 + y^2)^2 = 4(x^2 - y^2)$ was discovered by Jacob Bernoulli in 1694, who noted that it is "shaped like a figure 8, or a knot, or the bow of a ribbon." Find the coordinates of the four points at which the tangent line is horizontal (Figure 14).

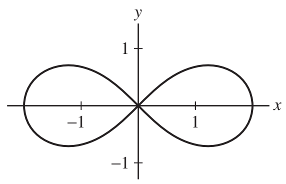


FIGURE 14 Lemniscate curve: $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

SOLUTION Consider the equation of a lemniscate curve: $\left(x^2+y^2\right)^2 = 4\left(x^2-y^2\right)$. Taking the derivative of both sides of this equation, we have

$$2\left(x^2+y^2\right)\left(2x+2yy'\right) = 4\left(2x-2yy'\right).$$

Therefore,

$$y' = \frac{8x - 4x\left(x^2+y^2\right)}{8y+4y\left(x^2+y^2\right)} = -\frac{\left(x^2+y^2-2\right)x}{\left(x^2+y^2+2\right)y}.$$

If $y'=0$, then either $x=0$ or $x^2+y^2=2$.

- If $x=0$ in the lemniscate curve, then $y^4=-4y^2$ or $y^2\left(y^2+4\right)=0$. If y is real, then $y=0$. The formula for y' in (a) is not defined at the origin (0/0). An alternative parametric analysis shows that the slopes of the tangent lines to the curve at the origin are ± 1.
- If $x^2+y^2=2$ or $y^2=2-x^2$, then plugging this into the lemniscate equation gives $4=4\left(2x^2-2\right)$ which yields $x=\pm\sqrt{\frac{3}{2}}=\pm\frac{\sqrt{6}}{2}$. Thus $y=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$. Accordingly, the four points at which the tangent lines to the lemniscate curve are horizontal are $\left(-\frac{\sqrt{6}}{2},-\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{6}}{2},\frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{6}}{2},-\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{6}}{2},\frac{\sqrt{2}}{2}\right)$.

65. Divide the curve in Figure 15 into five branches, each of which is the graph of a function. Sketch the branches.

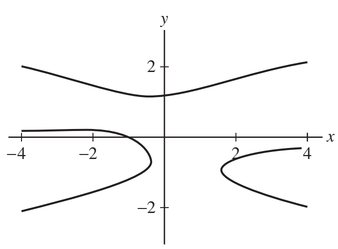
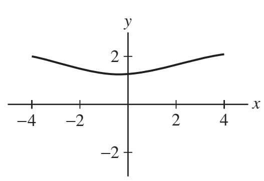


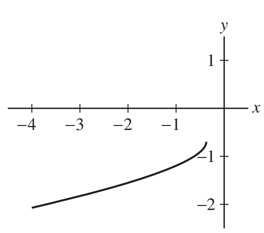
FIGURE 15 Graph of $y^5 - y = x^2y + x + 1$.

SOLUTION The branches are:

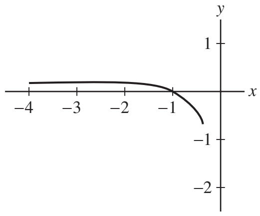
- Upper branch:



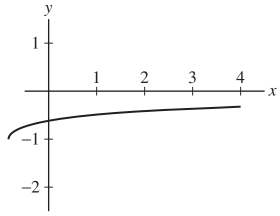
- Lower part of lower left curve:



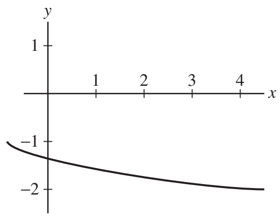
- Upper part of lower left curve:



- Upper part of lower right curve:



- Lower part of lower right curve:



3.11 Related Rates

Preliminary Questions

1. Assign variables and restate the following problem in terms of known and unknown derivatives (but do not solve it): How fast is the volume of a cube increasing if its side increases at a rate of 0.5 cm/s?

SOLUTION Let s and V denote the length of the side and the corresponding volume of a cube, respectively. Determine $\frac{dV}{dt}$ if $\frac{ds}{dt} = 0.5$ cm/s.

2. What is the relation between dV/dt and dr/dt if $V = \left(\frac{4}{3}\right)\pi r^3$?

SOLUTION Applying the general power rule, we find $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Therefore, the ratio is $4\pi r^2$.

In Questions 3 and 4, water pours into a cylindrical glass of radius 4 cm. Let V and h denote the volume and water level respectively, at time t.

3. Restate this question in terms of dV/dt and dh/dt: How fast is the water level rising if water pours in at a rate of 2 cm^3/min?

SOLUTION Determine $\frac{dh}{dt}$ if $\frac{dV}{dt} = 2$ cm^3/min.

4. Restate this question in terms of dV/dt and dh/dt: At what rate is water pouring in if the water level rises at a rate of 1 cm/min?

SOLUTION Determine $\frac{dV}{dt}$ if $\frac{dh}{dt} = 1$ cm/min.

Exercises

In Exercises 1 and 2, consider a rectangular bathtub whose base is 18 ft^2.

1. How fast is the water level rising if water is filling the tub at a rate of 0.7 ft^3/min?

SOLUTION Let h be the height of the water in the tub and V be the volume of the water. Then $V = 18h$ and $\frac{dV}{dt} = 18\frac{dh}{dt}$. Thus

$$\frac{dh}{dt} = \frac{1}{18}\frac{dV}{dt} = \frac{1}{18}(0.7) \approx 0.039 \text{ ft/min}.$$

2. At what rate is water pouring into the tub if the water level rises at a rate of 0.8 ft/min?

SOLUTION Let h be the height of the water in the tub and V its volume. Then $V = 18h$ and

$$\frac{dV}{dt} = 18\frac{dh}{dt} = 18\,(0.8) = 14.4\ \text{ft}^3/\text{min}.$$

3. The radius of a circular oil slick expands at a rate of 2 m/min.
(a) How fast is the area of the oil slick increasing when the radius is 25 m?
(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 3 min?

SOLUTION Let r be the radius of the oil slick and A its area.
(a) Then $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Substituting $r = 25$ and $\frac{dr}{dt} = 2$, we find

$$\frac{dA}{dt} = 2\pi\,(25)\,(2) = 100\pi \approx 314.16\ \text{m}^2/\text{min}.$$

(b) Since $\frac{dr}{dt} = 2$ and $r(0) = 0$, it follows that $r(t) = 2t$. Thus, $r(3) = 6$ and

$$\frac{dA}{dt} = 2\pi\,(6)\,(2) = 24\pi \approx 75.40\ \text{m}^2/\text{min}.$$

4. At what rate is the diagonal of a cube increasing if its edges are increasing at a rate of 2 cm/s?

SOLUTION Let s be the length of an edge of the cube and q the length of its diagonal. Two applications of the Pythagorean Theorem (or the distance formula) yield $q = \sqrt{3}s$. Thus $\frac{dq}{dt} = \sqrt{3}\frac{ds}{dt}$. Using $\frac{ds}{dt} = 2$,

$$\frac{dq}{dt} = \sqrt{3} \times 2 = 2\sqrt{3} \approx 3.46\ \text{cm/s}.$$

In Exercises 5–8, assume that the radius r of a sphere is expanding at a rate of 30 cm/min. *The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $4\pi r^2$. Determine the given rate.*

5. Volume with respect to time when $r = 15$ cm.

SOLUTION As the radius is expanding at 30 centimeters per minute, we know that $\frac{dr}{dt} = 30$ cm/min. Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}.$$

Substituting $r = 15$ and $\frac{dr}{dt} = 30$ yields

$$\frac{dV}{dt} = 4\pi(15)^2(30) = 27000\pi\ \text{cm}^3/\text{min}.$$

6. Volume with respect to time at $t = 2$ min, assuming that $r = 0$ at $t = 0$.

SOLUTION Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Since $\frac{dr}{dt} = 30$ and $r(0) = 0$, it follows that $r(t) = 30t$. From this, $r(2) = 60$, so

$$\frac{dV}{dt} = 4\pi(60^2)(30) = 432000\pi\ \text{cm}^3/\text{min}.$$

7. Surface area with respect to time when $r = 40$ cm.

SOLUTION Taking the derivative of both sides of $A = 4\pi r^2$ with respect to t yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. $\frac{dr}{dt} = 30$, so

$$\frac{dA}{dt} = 8\pi(40)(30) = 9600\pi\ \text{cm}^2/\text{min}.$$

8. Surface area with respect to time at $t = 2$ min, assuming that $r = 10$ at $t = 0$.

SOLUTION Taking $\frac{d}{dt}$ of both sides of $A = 4\pi r^2$ yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. Since $r = 10$ at $t = 0$ and $\frac{dr}{dt} = 30$, $r = 30t + 10$. Hence, at $t = 2$,

$$\frac{dA}{dt} = 8\pi(30 \cdot 2 + 10)(30) = 16800\pi\ \text{cm}^2/\text{min}.$$

In Exercises 9–12, refer to a 5-meter ladder sliding down a wall, as in Figures 1 and 2. The variable h is the height of the ladder's top at time t, and x is the distance from the wall to the ladder's bottom.

9. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is 1.5 m from the wall at $t = 0$ s.

SOLUTION Let x denote the distance from the base of the ladder to the wall, and h denote the height of the top of the ladder from the floor. The ladder is 5 m long, so $h^2 + x^2 = 5^2$. At any time t, $x = 1.5 + 0.8t$. Therefore, at time $t = 2$, the base is $x = 1.5 + 0.8(2) = 3.1$ m from the wall. Furthermore, we have

$$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0 \quad \text{so} \quad \frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}.$$

Substituting $x = 3.1$, $h = \sqrt{5^2 - 3.1^2}$ and $\frac{dx}{dt} = 0.8$, we obtain

$$\frac{dh}{dt} = -\frac{3.1}{\sqrt{5^2 - 3.1^2}}(0.8) \approx -0.632 \text{ m/s}.$$

10. Suppose that the top is sliding down the wall at a rate of 1.2 m/s. Calculate dx/dt when $h = 3$ m.

SOLUTION Let h be the height of the ladder's top and x the distance from the wall of the ladder's bottom. Then $h^2 + x^2 = 5^2$. Thus $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, and $\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$. With $h = 3$, $x = \sqrt{5^2 - 3^2} = 4$, and $\frac{dh}{dt} = -1.2$, we find

$$\frac{dx}{dt} = -\frac{3}{4}(-1.2) = 0.9 \text{ m/s}.$$

11. Suppose that $h(0) = 4$ and the top slides down the wall at a rate of 1.2 m/s. Calculate x and dx/dt at $t = 2$ s.

SOLUTION Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. After 2 seconds, $h = 4 + 2(-1.2) = 1.6$ m. Since $h^2 + x^2 = 5^2$,

$$x = \sqrt{5^2 - 1.6^2} = 4.737 \text{ m}.$$

Furthermore, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, so that $\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$. Substituting $h = 1.6$, $x = 4.737$, and $\frac{dh}{dt} = -1.2$, we find

$$\frac{dx}{dt} = -\frac{1.6}{4.737}(-1.2) \approx 0.405 \text{ m/s}.$$

12. What is the relation between h and x at the moment when the top and bottom of the ladder move at the same speed?

SOLUTION Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. When the top and the bottom of the ladder are moving at the same *speed* (say $s > 0$), their *velocities* satisfy $\frac{dh}{dt} = -\frac{dx}{dt} = -s$. Since $h^2 + x^2 = 16^2$, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$ or $-hs + xs = 0$. This implies $h = x$.

13. A conical tank has height 3 m and radius 2 m at the top. Water flows in at a rate of 2 m^3/min. How fast is the water level rising when it is 2 m?

SOLUTION Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Therefore,

$$\frac{dV}{dt} = \tfrac{4}{9}\pi h^2 \frac{dh}{dt},$$

and

$$\frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}.$$

Substituting $h = 2$ and $\frac{dV}{dt} = 2$ yields

$$\frac{dh}{dt} = \frac{9}{4\pi\,(2)^2} \times 2 = \frac{9}{8\pi} \approx -0.36 \text{ m/min}.$$

14. Follow the same set-up as Exercise 13, but assume that the water level is rising at a rate of 0.3 m/min when it is 2 m. At what rate is water flowing in?

SOLUTION Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Accordingly,

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}.$$

Substituting $h = 2$ and $\frac{dh}{dt} = 0.3$ yields

$$\frac{dV}{dt} = \frac{4}{9}\pi\,(2)^2\,(0.3) \approx 1.68\ \text{m}^3/\text{min}.$$

15. The radius r and height h of a circular cone change at a rate of 2 cm/s. How fast is the volume of the cone increasing when $r = 10$ and $h = 20$?

SOLUTION Let r be the radius, h be the height, and V be the volume of a right circular cone. Then $V = \frac{1}{3}\pi r^2 h$, and

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right).$$

When $r = 10$, $h = 20$, and $\frac{dr}{dt} = \frac{dh}{dt} = 2$, we find

$$\frac{dV}{dt} = \frac{\pi}{3}\left(10^2 \cdot 2 + 2 \cdot 20 \cdot 10 \cdot 2\right) = \frac{1000\pi}{3} \approx 1047.20\ \text{cm}^3/\text{s}.$$

16. A road perpendicular to a highway leads to a farmhouse located 2 km away (Figure 8). An automobile travels past the farmhouse at a speed of 80 km/h. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 6 km past the intersection of the highway and the road?

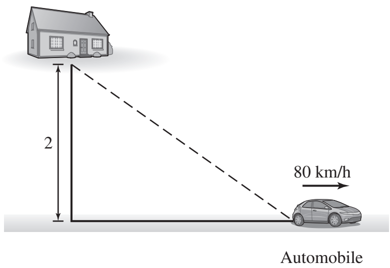


FIGURE 8

SOLUTION Let l denote the distance between the automobile and the farmhouse, and let s denote the distance past the intersection of the highway and the road. Then $l^2 = 2^2 + s^2$. Taking the derivative of both sides of this equation yields $2l\frac{dl}{dt} = 2s\frac{ds}{dt}$, so

$$\frac{dl}{dt} = \frac{s}{l}\frac{ds}{dt}.$$

When the auto is 6 km past the intersection, we have

$$\frac{dl}{dt} = \frac{6 \cdot 80}{\sqrt{2^2 + 6^2}} = \frac{480}{\sqrt{40}} = 24\sqrt{10} \approx 75.89\ \text{km/h}.$$

17. A man of height 1.8 meters walks away from a 5-meter lamppost at a speed of 1.2 m/s (Figure 9). Find the rate at which his shadow is increasing in length.

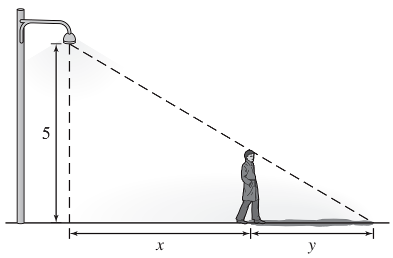


FIGURE 9

SOLUTION Since the man is moving at a rate of 1.2 m/s, his distance from the light post at any given time is $x = 1.2t$. Knowing the man is 1.8 meters tall and that the length of his shadow is denoted by y, we set up a proportion of similar triangles from the diagram:

$$\frac{y}{1.8} = \frac{1.2t + y}{5}.$$

Clearing fractions and solving for y yields

$$y = 0.675t.$$

Thus, $dy/dt = 0.675$ meters per second is the rate at which the length of the shadow is increasing.

18. As Claudia walks away from a 264-cm lamppost, the tip of her shadow moves twice as fast as she does. What is Claudia's height?

SOLUTION Let L be the distance from the base of the lamppost to the tip of Claudia's shadow, let x denote the distance from the base of the lamppost to Claudia's feet, and let h denote Claudia's height. The right triangle with legs $L - x$ and h (formed by Claudia and her shadow) and the right triangle with legs L and 264 (formed by the lamppost and the total distance L) are similar. By similarity

$$\frac{L - x}{h} = L264.$$

h is constant, so taking the derivative of both sides of this equation yields

$$\frac{dL/dt - dx/dt}{h} = \frac{dL/dt}{264}.$$

The problem states that $\frac{dL}{dt} = 2\frac{dx}{dt}$, so

$$264\left(2\frac{dx}{dt} - \frac{dx}{dt}\right) = 2h\frac{dx}{dt} \quad \text{or} \quad 264 = 2h.$$

Hence, $h = 132$ cm.

19. At a given moment, a plane passes directly above a radar station at an altitude of 6 km.

(a) The plane's speed is 800 km/h. How fast is the distance between the plane and the station changing half an hour later?

(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

SOLUTION Let x be the distance of the plane from the station along the ground and h the distance through the air.

(a) By the Pythagorean Theorem, we have

$$h^2 = x^2 + 6^2 = x^2 + 36.$$

Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, and $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}$. After an half hour, $x = \frac{1}{2} \times 800 = 400$ kilometers. With $x = 400$, $h = \sqrt{400^2 + 36}$, and $\frac{dx}{dt} = 800$,

$$\frac{dh}{dt} = \frac{400}{\sqrt{400^2 + 36}} \times 800 \approx 799.91 \text{ km/h}.$$

(b) When the plane is directly above the station, $x = 0$, so the distance between the plane and the station is not changing, for at this instant we have

$$\frac{dh}{dt} = \frac{0}{6} \times 800 = 0 \text{ km/h}.$$

20. In the setting of Exercise 19, let θ be the angle that the line through the radar station and the plane makes with the horizontal. How fast is θ changing 12 min after the plane passes over the radar station?

SOLUTION Let the distance x and angle θ be defined as in the figure below. Then

$$\tan\theta = \frac{6}{x} \qquad \text{and} \qquad \sec^2\theta\frac{d\theta}{dt} = -\frac{6}{x^2}\frac{dx}{dt}.$$

Because the plane is traveling at 800 km/h, 12 minutes after the plane passes over the radar station,

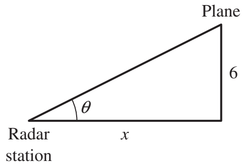
$$x = 160 \qquad \text{and} \qquad \tan\theta = \frac{3}{80}.$$

Furthermore,

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{3^2}{80^2}.$$

Finally,

$$\frac{d\theta}{dt} = -\frac{6}{160^2}\frac{1}{1+\frac{3^2}{80^2}}800 = -\frac{1200}{6409} = -0.187 \text{ rad/hour.}$$



21. A hot air balloon rising vertically is tracked by an observer located 4 km from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{5}$, and it is changing at a rate of 0.2 rad/min. How fast is the balloon rising at this moment?

SOLUTION Let y be the height of the balloon (in miles) and θ the angle between the line-of-sight and the horizontal. Via trigonometry, we have $\tan\theta = \dfrac{y}{4}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{4}\frac{dy}{dt},$$

and

$$\frac{dy}{dt} = 4\frac{d\theta}{dt}\sec^2\theta.$$

Using $\frac{d\theta}{dt} = 0.2$ and $\theta = \frac{\pi}{5}$ yields

$$\frac{dy}{dt} = 4\,(0.2)\,\frac{1}{\cos^2(\pi/5)} \approx 1.22 \text{ km/min.}$$

22. A laser pointer is placed on a platform that rotates at a rate of 20 revolutions per minute. The beam hits a wall 8 m away, producing a dot of light that moves horizontally along the wall. Let θ be the angle between the beam and the line through the searchlight perpendicular to the wall (Figure 10). How fast is this dot moving when $\theta = \frac{\pi}{6}$?

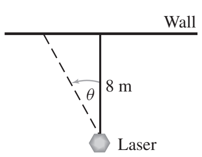


FIGURE 10

SOLUTION Let y be the distance between the dot of light and the point of intersection of the wall and the line through the searchlight perpendicular to the wall. Let θ be the angle between the beam of light and the line. Using trigonometry, we have $\tan\theta = \frac{y}{8}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{8}\frac{dy}{dt},$$

and

$$\frac{dy}{dt} = 8\frac{d\theta}{dt}\sec^2\theta.$$

With $\theta = \frac{\pi}{6}$ and $\frac{d\theta}{dt} = 40\pi$, we find

$$\frac{dy}{dt} = 8\,(40\pi)\,\frac{1}{\cos^2(\pi/6)} = \frac{1280}{3}\pi \approx 1340.4 \text{ m/min.}$$

23. A rocket travels vertically at a speed of 1200 km/h. The rocket is tracked through a telescope by an observer located 16 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after lift-off.

SOLUTION Let y be the height of the rocket and θ the angle between the telescope and the ground. Using trigonometry, we have $\tan\theta = \frac{y}{16}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{16}\frac{dy}{dt},$$

and

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{16}\frac{dy}{dt}.$$

After the rocket has traveled for 3 minutes (or $\frac{1}{20}$ hour), its height is $\frac{1}{20} \times 1200 = 60$ km. At this instant, $\tan\theta = 60/16 = 15/4$ and thus

$$\cos\theta = \frac{4}{\sqrt{15^2+4^2}} = \frac{4}{\sqrt{241}}.$$

Finally,

$$\frac{d\theta}{dt} = \frac{16/241}{16}(1200) = \frac{1200}{241} \approx 4.98 \text{ rad/hr}.$$

24. Using a telescope, you track a rocket that was launched 4 km away, recording the angle θ between the telescope and the ground at half-second intervals. Estimate the velocity of the rocket if $\theta(10) = 0.205$ and $\theta(10.5) = 0.225$.

SOLUTION Let h be the height of the vertically ascending rocket. Using trigonometry, $\tan\theta = \frac{h}{4}$, so

$$\frac{dh}{dt} = 4\sec^2\theta \cdot \frac{d\theta}{dt}.$$

We are given $\theta(10) = 0.205$, and we can estimate

$$\left.\frac{d\theta}{dt}\right|_{t=10} \approx \frac{\theta(10.5) - \theta(10)}{0.5} = 0.04.$$

Thus,

$$\frac{dh}{dt} = 4\sec^2(0.205)\cdot(0.04) \approx 0.166 \text{ km/s},$$

or roughly 600 km/h.

25. A police car traveling south toward Sioux Falls at 160 km/h pursues a truck traveling east away from Sioux Falls, Iowa, at 140 km/h (Figure 11). At time $t = 0$, the police car is 20 km north and the truck is 30 km east of Sioux Falls. Calculate the rate at which the distance between the vehicles is changing:

(a) At time $t = 0$

(b) 5 minutes later

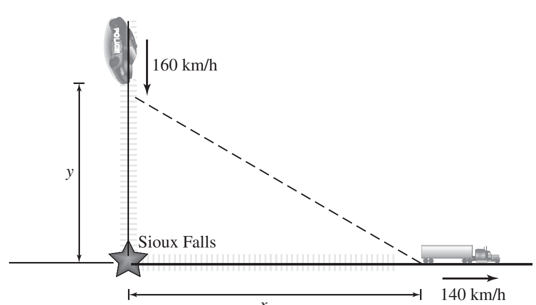


FIGURE 11

SOLUTION Let y denote the distance the police car is north of Sioux Falls and x the distance the truck is east of Sioux Falls. Then $y = 20 - 160t$ and $x = 30 + 140t$. If ℓ denotes the distance between the police car and the truck, then

$$\ell^2 = x^2 + y^2 = (30 + 140t)^2 + (20 - 160t)^2$$

and

$$\ell\frac{d\ell}{dt} = 140(30 + 140t) - 160(20 - 160t) = 1000 + 45200t.$$

(a) At $t = 0$, $\ell = \sqrt{30^2 + 20^2} = 10\sqrt{13}$, so

$$\frac{d\ell}{dt} = \frac{1000}{10\sqrt{13}} = \frac{100\sqrt{13}}{13} \approx 27.735 \text{ km/h}.$$

(b) At $t = 5$ minutes $= \frac{1}{12}$ hour,

$$\ell = \sqrt{\left(30 + 140 \cdot \frac{1}{12}\right)^2 + \left(20 - 160 \cdot \frac{1}{12}\right)^2} \approx 42.197 \text{ km},$$

and

$$\frac{d\ell}{dt} = \frac{1000 + 45200 \cdot \frac{1}{12}}{42.197} \approx 112.962 \text{ km/h}.$$

26. A car travels down a highway at 25 m/s. An observer stands 150 m from the highway.

(a) How fast is the distance from the observer to the car increasing when the car passes in front of the observer? Explain your answer without making any calculations.

(b) How fast is the distance increasing 20 s later?

SOLUTION Let x be the distance (in feet) along the road that the car has traveled and h be the distance (in feet) between the car and the observer.

(a) Before the car passes the observer, we have $dh/dt < 0$; after it passes, we have $dh/dt > 0$. So at the instant it passes we have $dh/dt = 0$, given that dh/dt varies continuously since the car travels at a constant velocity.

(b) By the Pythagorean Theorem, we have $h^2 = x^2 + 150^2$. Thus

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt},$$

and

$$\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}.$$

The car travels at 25 m/s, so after 20 seconds, $x = 25(20) = 500$ meters. Therefore,

$$\frac{dh}{dt} = \frac{500}{\sqrt{500^2 + 125^2}}(25) \approx 24.25 \text{ m/s}.$$

27. In the setting of Example 5, at a certain moment, the tractor's speed is 3 m/s and the bale is rising at 2 m/s. How far is the tractor from the bale at this moment?

SOLUTION From Example 5, we have the equation

$$\frac{x\frac{dx}{dt}}{\sqrt{x^2 + 4.5^2}} = \frac{dh}{dt},$$

where x denote the distance from the tractor to the bale and h denotes the height of the bale. Given

$$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dh}{dt} = 2,$$

it follows that

$$\frac{3x}{\sqrt{4.5^2 + x^2}} = 2,$$

which yields $x = \sqrt{16.2} \approx 4.025$ m.

28. Placido pulls a rope attached to a wagon through a pulley at a rate of q m/s. With dimensions as in Figure 12:

(a) Find a formula for the speed of the wagon in terms of q and the variable x in the figure.

(b) Find the speed of the wagon when $x = 0.6$ if $q = 0.5$ m/s.

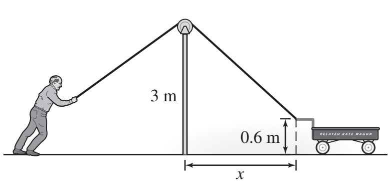


FIGURE 12

SOLUTION Let h be the distance from the pulley to the loop on the wagon. Using the Pythagorean Theorem, we have $h^2 = x^2 + (3 - 0.6)^2 = x^2 + 2.4^2$.

(a) Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, and $\frac{dx}{dt} = \frac{h}{x}\frac{dh}{dt}$. Given $dh/dt = q$, it follows that

$$\frac{dx}{dt} = \frac{\sqrt{x^2 + 2.4^2}}{x}q.$$

(b) As Placido pulls the rope at the rate of $q = 0.5$ m/s and $x = 0.6$

$$\frac{dx}{dt} = \frac{\sqrt{0.6^2 + 2.4^2}}{0.6}(0.5) \approx 2.06 \text{ m/s}.$$

29. Julian is jogging around a circular track of radius 50 m. In a coordinate system with origin at the center of the track, Julian's x-coordinate is changing at a rate of -1.25 m/s when his coordinates are (40, 30). Find dy/dt at this moment.

SOLUTION We have $x^2 + y^2 = 50^2$, so

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Given $x = 40$, $y = 30$ and $dx/dt = -1.25$, we find

$$\frac{dy}{dt} = -\frac{40}{30}(-1.25) = \frac{5}{3} \text{ m/s}.$$

30. A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$ (Figure 13).

(a) In which of the four quadrants is $dx/dt > 0$? Explain.

(b) Find a relation between dx/dt and dy/dt.

(c) At what rate is the x-coordinate changing when the particle passes the point (1, 1) if its y-coordinate is increasing at a rate of 6 m/s?

(d) Find dy/dt when the particle is at the top and bottom of the ellipse.

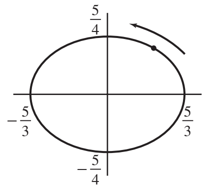


FIGURE 13

SOLUTION A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$.

(a) The derivative dx/dt is positive in quadrants 3 and 4 since the particle is moving to the right.

(b) From $9x^2 + 16y^2 = 25$ we have $18x\frac{dx}{dt} + 32y\frac{dy}{dt} = 0$.

(c) From (b), we have $\frac{dx}{dt} = -\frac{16y}{9x}\frac{dy}{dt}$. With $x = y = 1$ and $\frac{dy}{dt} = 6$,

$$\frac{dx}{dt} = -\frac{16 \cdot 1}{9 \cdot 1}(6) = -\frac{32}{3} \text{ m/s}.$$

(d) From (b), we have $\frac{dy}{dt} = -\frac{9x}{16y}\frac{dx}{dt}$. When $(x, y) = \left(0, \pm\frac{5}{4}\right)$, it follows that $\frac{dy}{dt} = 0$.

In Exercises 31 and 32, assume that the pressure P (in kilopascals) and volume V (in cubic centimeters) of an expanding gas are related by $PV^b = C$, where b and C are constants (this holds in an adiabatic *expansion, without heat gain or loss).*

31. Find dP/dt if $b = 1.2$, $P = 8$ kPa, $V = 100$ cm^2, and $dV/dt = 20$ cm^3/min.

SOLUTION Let $PV^b = C$. Then

$$PbV^{b-1}\frac{dV}{dt} + V^b\frac{dP}{dt} = 0,$$

and

$$\frac{dP}{dt} = -\frac{Pb}{V}\frac{dV}{dt}.$$

Substituting $b = 1.2$, $P = 8$, $V = 100$, and $\frac{dV}{dt} = 20$, we find

$$\frac{dP}{dt} = -\frac{(8)(1.2)}{100}(20) = -1.92 \text{ kPa/min}.$$

32. Find b if $P = 25$ kPa, $dP/dt = 12$ kPa/min, $V = 100$ cm^2, and $dV/dt = 20$ cm^3/min.

SOLUTION Let $PV^b = C$. Then

$$PbV^{b-1}\frac{dV}{dt} + V^b\frac{dP}{dt} = 0,$$

and

$$b = -\frac{V}{P}\,\frac{dP/dt}{dV/dt}.$$

With $P = 25$, $V = 100$, $\frac{dP}{dt} = 12$, and $\frac{dV}{dt} = 20$, we have

$$b = -\frac{100}{25} \times \frac{12}{20} = -\frac{12}{5}.$$

(*Note:* If instead we have $\frac{dP}{dt} = -12$ kPa/min, then $b = \frac{12}{5}$.)

33. The base x of the right triangle in Figure 14 increases at a rate of 5 cm/s, while the height remains constant at $h = 20$. How fast is the angle θ changing when $x = 20$?

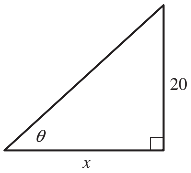


FIGURE 14

SOLUTION We have $\cot\theta = \dfrac{x}{20}$, from which

$$-\csc^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{20}\,\frac{dx}{dt}$$

and thus

$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{20}\,\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 5$ and when $x = h = 20$, $\theta = \frac{\pi}{4}$. Hence,

$$\frac{d\theta}{dt} = -\frac{\sin^2\left(\frac{\pi}{4}\right)}{20}(5) = -\frac{1}{8}\text{ rad/s}.$$

34. Two parallel paths 15 m apart run east-west through the woods. Brooke jogs east on one path at 10 km/h, while Jamail walks west on the other path at 6 km/h. If they pass each other at time $t = 0$, how far apart are they 3 s later, and how fast is the distance between them changing at that moment?

SOLUTION Brooke jogs at 10 km/h $= \frac{25}{9}$ m/s and Jamail walks at 6 km/h $= \frac{5}{3}$ m/s. At time zero, consider Brooke to be at the origin $(0, 0)$ and (without loss of generality) Jamail to be at $(0, 15)$; i.e., due north of Brooke. Then at time t, the position of Brooke is $\left(\frac{25}{9}t, 0\right)$ and that of Jamail is $\left(-\frac{5}{3}t, 15\right)$. The distance between them is

$$L = \sqrt{\left(\frac{25}{9}t + \frac{5}{3}t\right)^2 + (15)^2} = \left(\left(\frac{40}{9}t\right)^2 + 15^2\right)^{1/2}.$$

- When $t = 3$ seconds, the distance between them is

$$L = \sqrt{\left(\frac{40}{3}\right)^2 + 15^2} = \frac{5}{3}\sqrt{145} \approx 20.07\text{ m}.$$

- The distance between them is changing at the rate

$$\frac{dL}{dt} = \frac{1}{2}\left(\left(\frac{40}{9}t\right)^2 + 15^2\right)^{-1/2}\left(2\left(\frac{40}{9}t\right)\frac{40}{9}\right).$$

When $t = 3$, we then have

$$\frac{dL}{dt} = \frac{\frac{1}{9}(40)^2}{\sqrt{40^2 + 45^2}} \approx 2.95\text{ m/s}$$

35. A particle travels along a curve $y = f(x)$ as in Figure 15. Let $L(t)$ be the particle's distance from the origin.

(a) Show that $\dfrac{dL}{dt} = \left(\dfrac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\dfrac{dx}{dt}$ if the particle's location at time t is $P = (x, f(x))$.

(b) Calculate $L'(t)$ when $x = 1$ and $x = 2$ if $f(x) = \sqrt{3x^2 - 8x + 9}$ and $dx/dt = 4$.

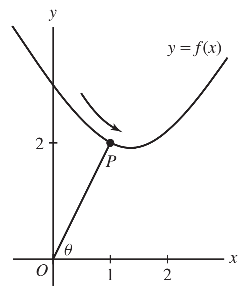


FIGURE 15

SOLUTION

(a) If the particle's location at time t is $P = (x, f(x))$, then

$$L(t) = \sqrt{x^2 + f(x)^2}.$$

Thus,

$$\frac{dL}{dt} = \frac{1}{2}(x^2 + f(x)^2)^{-1/2}\left(2x\frac{dx}{dt} + 2f(x)f'(x)\frac{dx}{dt}\right) = \left(\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\frac{dx}{dt}.$$

(b) Given $f(x) = \sqrt{3x^2 - 8x + 9}$, it follows that

$$f'(x) = \frac{3x - 4}{\sqrt{3x^2 - 8x + 9}}.$$

Let's start with $x = 1$. Then $f(1) = 2$, $f'(1) = -\frac{1}{2}$ and

$$\frac{dL}{dt} = \left(\frac{1 - 1}{\sqrt{1^2 + 2^2}}\right)(4) = 0.$$

With $x = 2$, $f(2) = \sqrt{5}$, $f'(2) = 2/\sqrt{5}$ and

$$\frac{dL}{dt} = \frac{2 + 2}{\sqrt{2^2 + \sqrt{5}^2}}(4) = \frac{16}{3}.$$

36. Let θ be the angle in Figure 15, where $P = (x, f(x))$. In the setting of the previous exercise, show that

$$\frac{d\theta}{dt} = \left(\frac{xf'(x) - f(x)}{x^2 + f(x)^2}\right)\frac{dx}{dt}$$

Hint: Differentiate $\tan\theta = f(x)/x$ and observe that $\cos\theta = x/\sqrt{x^2 + f(x)^2}$.

SOLUTION If the particle's location at time t is $P = (x, f(x))$, then $\tan\theta = f(x)/x$ and

$$\sec^2\theta\frac{d\theta}{dx} = \frac{xf'(x) - f(x)}{x^2}.$$

Now

$$\cos\theta = \frac{x}{\sqrt{x^2 + f(x)^2}} \quad \text{so} \quad \sec^2\theta = \frac{x^2 + f(x)^2}{x^2}.$$

Finally,

$$\frac{d\theta}{dx} = \frac{xf'(x) - f(x)}{x^2 + f(x)^2}.$$

Exercises 37 and 38 refer to the baseball diamond (a square of side 90 ft*) in Figure 16.*

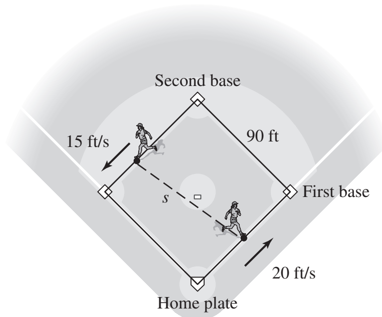


FIGURE 16

37. A baseball player runs from home plate toward first base at 20 ft/s. How fast is the player's distance from second base changing when the player is halfway to first base?

SOLUTION Let x be the distance of the player from home plate and h the player's distance from second base. Using the Pythagorean theorem, we have $h^2 = 90^2 + (90 - x)^2$. Therefore,

$$2h\frac{dh}{dt} = 2\,(90 - x)\left(-\frac{dx}{dt}\right),$$

and

$$\frac{dh}{dt} = -\frac{90 - x}{h}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 20$. When the player is halfway to first base, $x = 45$ and $h = \sqrt{90^2 + 45^2}$, so

$$\frac{dh}{dt} = -\frac{45}{\sqrt{90^2 + 45^2}}\,(20) = -4\sqrt{5} \approx -8.94 \text{ ft/s}.$$

38. Player 1 runs to first base at a speed of 20 ft/s while Player 2 runs from second base to third base at a speed of 15 ft/s. Let s be the distance between the two players. How fast is s changing when Player 1 is 30 ft from home plate and Player 2 is 60 ft from second base?

SOLUTION Let x denote the distance from home plate to Player 1 and y denote the distance from second base to Player 2, both distances measured along the base path. Then

$$s(t) = \sqrt{(90 - x - y)^2 + 90^2},$$

and

$$\frac{ds}{dt} = -\frac{90 - x - y}{\sqrt{(90 - x - y)^2 + 90^2}}\left(\frac{dx}{dt} + \frac{dy}{dt}\right).$$

With $x = 30$ and $y = 60$, it follows that

$$\frac{ds}{dt} = 0.$$

39. The conical watering pail in Figure 17 has a grid of holes. Water flows out through the holes at a rate of kA m^3/min, where k is a constant and A is the surface area of the part of the cone in contact with the water. This surface area is $A = \pi r\sqrt{h^2 + r^2}$ and the volume is $V = \frac{1}{3}\pi r^2 h$. Calculate the rate dh/dt at which the water level changes at $h = 0.3$ m, assuming that $k = 0.25$ m.

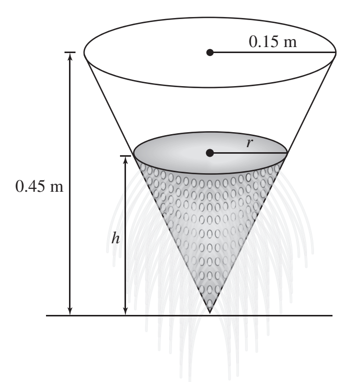


FIGURE 17

SOLUTION By similar triangles, we have

$$\frac{r}{h} = \frac{0.15}{0.45} = \frac{1}{3} \quad \text{so} \quad r = \frac{1}{3}h.$$

Substituting this expression for r into the formula for V yields

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3.$$

From here and the problem statement, it follows that

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt} = -kA = -0.25\pi r\sqrt{h^2 + r^2}.$$

Solving for dh/dt gives

$$\frac{dh}{dt} = -\frac{9}{4}\frac{r}{h^2}\sqrt{h^2 + r^2}.$$

When $h = 0.3$, $r = 0.1$ and

$$\frac{dh}{dt} = -\frac{9}{4}\frac{0.1}{0.3^2}\sqrt{0.3^2 + 0.1^2} = -0.79 \text{ m/min}.$$

Further Insights and Challenges

40. A bowl contains water that evaporates at a rate proportional to the surface area of water exposed to the air (Figure 18). Let $A(h)$ be the cross-sectional area of the bowl at height h.

(a) Explain why $V(h + \Delta h) - V(h) \approx A(h)\Delta h$ if Δh is small.

(b) Use (a) to argue that $\dfrac{dV}{dh} = A(h)$.

(c) Show that the water level h decreases at a constant rate.

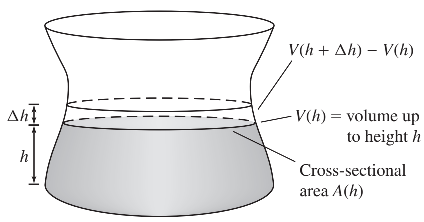


FIGURE 18

SOLUTION

(a) Consider a thin horizontal slice of the water in the cup of thickness Δh at height h. Assuming the cross-sectional area of the cup is roughly constant across this slice, it follows that

$$V(h + \Delta h) - V(h) \approx A(h)\Delta h.$$

(b) If we take the expression from part (a), divide by Δh and pass to the limit as $\Delta h \to 0$, we find

$$\frac{dV}{dh} = A(h).$$

(c) If we take the expression from part (b) and multiply by dh/dt, recognizing that

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt},$$

we find that

$$\frac{dV}{dt} = A(h)\frac{dh}{dt}.$$

We are told that the water in the bowl evaporates at a rate proportional to the surface area exposed to the air; translated into mathematics, this means

$$\frac{dV}{dt} = -kA(h),$$

where k is a positive constant of proportionality. Combining the last two equations yields

$$\frac{dh}{dt} = -k;$$

that is, the water level decreases at a constant rate.

41. A roller coaster has the shape of the graph in Figure 19. Show that when the roller coaster passes the point $(x, f(x))$, the vertical velocity of the roller coaster is equal to $f'(x)$ times its horizontal velocity.

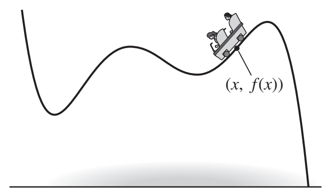


FIGURE 19 Graph of $f(x)$ as a roller coaster track.

SOLUTION Let the equation $y = f(x)$ describe the shape of the roller coaster track. Taking $\frac{d}{dt}$ of both sides of this equation yields $\frac{dy}{dt} = f'(x)\frac{dx}{dt}$. In other words, the vertical velocity of a car moving along the track, $\frac{dy}{dt}$, is equal to $f'(x)$ times the horizontal velocity, $\frac{dx}{dt}$.

42. Two trains leave a station at $t = 0$ and travel with constant velocity v along straight tracks that make an angle θ.

(a) Show that the trains are separating from each other at a rate $v\sqrt{2 - 2\cos\theta}$.

(b) What does this formula give for $\theta = \pi$?

SOLUTION Choose a coordinate system such that

- the origin is the point of departure of the trains;
- the first train travels along the positive x-axis;
- the second train travels along the ray emanating from the origin at an angle of $\theta > 0$.

(a) At time t, the position of the first train is $(vt, 0)$, while that of the second is $(vt\cos\theta, vt\sin\theta)$. The distance between the trains is

$$L = \sqrt{(vt\,(1 - \cos\theta))^2 + (vt\sin\theta)^2} = vt\sqrt{2 - 2\cos\theta}.$$

Thus $dL/dt = v\sqrt{2 - 2\cos\theta}$.

(b) When $\theta = \pi$, we have $dL/dt = 2v$. This is obviously correct since at this angle the trains travel in opposite directions at the same constant speed, having started from the same point.

43. As the wheel of radius r cm in Figure 20 rotates, the rod of length L attached at point P drives a piston back and forth in a straight line. Let x be the distance from the origin to point Q at the end of the rod, as shown in the figure.

(a) Use the Pythagorean Theorem to show that

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta \qquad \boxed{6}$$

(b) Differentiate Eq. (6) with respect to t to prove that

$$2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0$$

(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 revolutions per minute.

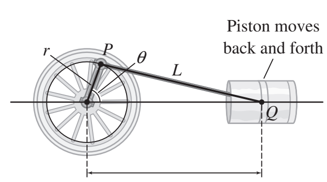


FIGURE 20

SOLUTION From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

(a) The distance formula gives

$$L = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}.$$

Thus,

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Note that L (the length of the fixed rod) and r (the radius of the wheel) are constants.

(b) From (a) we have

$$0 = 2\,(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}.$$

(c) Solving for dx/dt in (b) gives

$$\frac{dx}{dt} = \frac{r^2\sin\theta\cos\theta\frac{d\theta}{dt}}{r\cos\theta - x} - r\sin\theta\frac{d\theta}{dt} = \frac{rx\sin\theta\frac{d\theta}{dt}}{r\cos\theta - x}.$$

With $\theta = \frac{\pi}{2}$, $r = 10$, $L = 30$, and $\frac{d\theta}{dt} = 8\pi$,

$$\frac{dx}{dt} = \frac{(10)\,(x)\left(\sin\frac{\pi}{2}\right)(8\pi)}{(10)\,(0) - x} = -80\pi \approx -251.33 \text{ cm/min}$$

44. A spectator seated 300 m away from the center of a circular track of radius 100 m watches an athlete run laps at a speed of 5 m/s. How fast is the distance between the spectator and athlete changing when the runner is approaching the spectator and the distance between them is 250 m? *Hint:* The diagram for this problem is similar to Figure 20, with $r = 100$ and $x = 300$.

SOLUTION From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

- The distance formula gives

$$L = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}.$$

Thus,

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Note that x (the distance of the spectator from the center of the track) and r (the radius of the track) are constants.

- Differentiating with respect to t gives

$$2L\frac{dL}{dt} = 2\,(x - r\cos\theta)\,r\sin\theta\frac{d\theta}{dt} + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}.$$

Thus,

$$\frac{dL}{dt} = \frac{rx}{L}\sin\theta\frac{d\theta}{dt}.$$

- Recall the relation between arc length s and angle θ, namely $s = r\theta$. Thus $\dfrac{d\theta}{dt} = \dfrac{1}{r}\dfrac{ds}{dt}$. Given $r = 100$ and $\frac{ds}{dt} = -5$, we have

$$\frac{d\theta}{dt} = \frac{1}{100}\,(-5) = -\frac{1}{20}\text{ rad/s}.$$

(*Note:* In this scenario, the runner traverses the track in a *clockwise* fashion and approaches the spectator from Quadrant 1.)

- Next, the Law of Cosines gives $L^2 = r^2 + x^2 - 2rx\cos\theta$, so

$$\cos\theta = \frac{r^2 + x^2 - L^2}{2rx} = \frac{100^2 + 300^2 - 250^2}{2\,(100)\,(300)} = \frac{5}{8}.$$

Accordingly,

$$\sin\theta = \sqrt{1 - \left(\frac{5}{8}\right)^2} = \frac{\sqrt{39}}{8}.$$

- Finally

$$\frac{dL}{dt} = \frac{(300)\,(100)}{250}\left(\frac{\sqrt{39}}{8}\right)\left(-\frac{1}{20}\right) = -\frac{3\sqrt{39}}{4} \approx -4.68 \text{ m/s}.$$

45. A cylindrical tank of radius R and length L lying horizontally as in Figure 21 is filled with oil to height h.

(a) Show that the volume $V(h)$ of oil in the tank is

$$V(h) = L\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right) - (R-h)\sqrt{2hR-h^2}\right)$$

(b) Show that $\frac{dV}{dh} = 2L\sqrt{h(2R-h)}$.

(c) Suppose that $R = 1.5$ m and $L = 10$ m and that the tank is filled at a constant rate of 0.6 m^3/min. How fast is the height h increasing when $h = 0.5$?

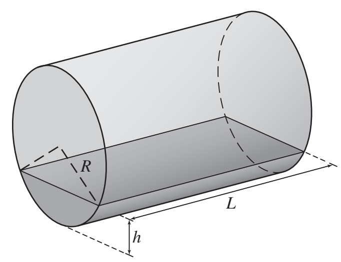


FIGURE 21 Oil in the tank has level h.

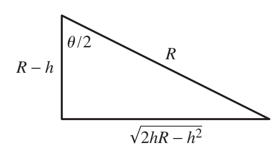
SOLUTION

(a) From Figure 21, we see that the volume of oil in the tank, $V(h)$, is equal to L times $A(h)$, the area of that portion of the circular cross section occupied by the oil. Now,

$$A(h) = \text{area of sector} - \text{area of triangle} = \frac{R^2\theta}{2} - \frac{R^2\sin\theta}{2},$$

where θ is the central angle of the sector. Referring to the diagram below,

$$\cos\frac{\theta}{2} = \frac{R-h}{R} \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{\sqrt{2hR-h^2}}{R}.$$



Thus,

$$\theta = 2\cos^{-1}\left(1-\frac{h}{R}\right),$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{(R-h)\sqrt{2hR-h^2}}{R^2},$$

and

$$V(h) = L\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right) - (R-h)\sqrt{2hR-h^2}\right).$$

(b) Recalling that $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-x^2}}\frac{du}{dx}$,

$$\frac{dV}{dh} = L\left(\frac{d}{dh}\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right)\right) - \frac{d}{dh}\left((R-h)\sqrt{2hR-h^2}\right)\right)$$

$$= L\left(-R\frac{-1}{\sqrt{1-(1-(h/R))^2}} + \sqrt{2hR-h^2} - \frac{(R-h)^2}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{R^2}{\sqrt{2hR-h^2}} + \sqrt{2hR-h^2} - \frac{R^2-2Rh+h^2}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{R^2+(2hR-h^2)-(R^2-2Rh+h^2)}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{4hR-2h^2}{\sqrt{2hR-h^2}}\right) = L\left(\frac{2(2hR-h^2)}{\sqrt{2hR-h^2}}\right) = 2L\sqrt{2hR-h^2}.$$

(c) $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{dV/dh}\frac{dV}{dt}$. From part (b) with $R = 1.5$, $L = 10$ and $h = 0.5$,

$$\frac{dV}{dh} = 2(10)\sqrt{2(0.5)(1.5)-0.5^2} = 10\sqrt{5}\ \text{m}^2.$$

Thus,

$$\frac{dh}{dt} = \frac{1}{10\sqrt{5}}(0.6) = \frac{3\sqrt{5}}{2500} \approx 0.0027\ \text{m/min}.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

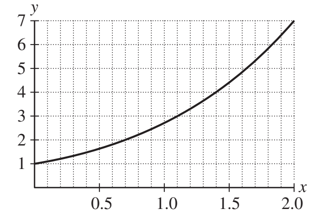


FIGURE 1

1. Compute the average rate of change of $f(x)$ over [0, 2]. What is the graphical interpretation of this average rate?

SOLUTION The average rate of change of $f(x)$ over [0, 2] is

$$\frac{f(2)-f(0)}{2-0} = \frac{7-1}{2-0} = 3.$$

Graphically, this average rate of change represents the slope of the secant line through the points (2, 7) and (0, 1) on the graph of $f(x)$.

2. For which value of h is $\frac{f(0.7+h)-f(0.7)}{h}$ equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$?

SOLUTION Because $1.1 = 0.7 + 0.4$, the difference quotient

$$\frac{f(0.7+h)-f(0.7)}{h}$$

is equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$ for $h = 0.4$.

3. Estimate $\frac{f(0.7+h)-f(0.7)}{h}$ for $h = 0.3$. Is this number larger or smaller than $f'(0.7)$?

SOLUTION For $h = 0.3$,

$$\frac{f(0.7+h)-f(0.7)}{h} = \frac{f(1)-f(0.7)}{0.3} \approx \frac{2.8-2}{0.3} = \frac{8}{3}.$$

Because the curve is concave up, the slope of the secant line is larger than the slope of the tangent line, so the value of the difference quotient should be larger than the value of the derivative.

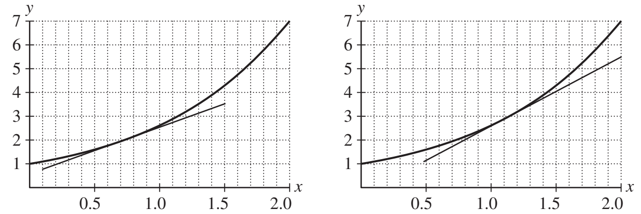
4. Estimate $f'(0.7)$ and $f'(1.1)$.

SOLUTION The tangent line sketched in the graph below at the left appears to pass through the points (0.2, 1) and (1.5, 3.5). Thus,

$$f'(0.7) \approx \frac{3.5-1}{1.5-0.2} = \frac{2.5}{1.3} = 1.923.$$

The tangent line sketched in the graph below at the right appears to pass through the points (0.8, 2) and (2, 5.5). Thus,

$$f'(1.1) \approx \frac{5.5-2}{2-0.8} = \frac{3.5}{1.2} = 2.917.$$



In Exercises 5–8, compute $f'(a)$ using the limit definition and find an equation of the tangent line to the graph of $f(x)$ at $x = a$.

5. $f(x) = x^2 - x, \quad a = 1$

SOLUTION Let $f(x) = x^2 - x$ and $a = 1$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - (1+h) - (1^2 - 1)}{h}$$
$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1 - h}{h} = \lim_{h\to 0}(1+h) = 1$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 1(x-1) + 0 = x - 1.$$

6. $f(x) = 5 - 3x, \quad a = 2$

SOLUTION Let $f(x) = 5 - 3x$ and $a = 2$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{5 - 3(2+h) - (5-6)}{h} = \lim_{h\to 0} -3 = -3$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -3(x-2) - 1 = -3x + 5.$$

7. $f(x) = x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x^{-1}$ and $a = 4$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{4 - (4+h)}{4h(4+h)}$$
$$= \lim_{h\to 0} \frac{-1}{4(4+h)} = -\frac{1}{4(4+0)} = -\frac{1}{16}$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-4) + \frac{1}{4} = -\frac{1}{16}x + \frac{1}{2}.$$

8. $f(x) = x^3, \quad a = -2$

SOLUTION Let $f(x) = x^3$ and $a = -2$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(-2+h)^3 - (-2)^3}{h} = \lim_{h\to 0} \frac{-8 + 12h - 6h^2 + h^3 + 8}{h}$$
$$= \lim_{h\to 0}(12 - 6h + h^2) = 12 - 6(0) + 0^2 = 12$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x - a) + f(a) = 12(x + 2) - 8 = 12x + 16.$$

In Exercises 9–12, compute dy/dx using the limit definition.

9. $y = 4 - x^2$

SOLUTION Let $y = 4 - x^2$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h} = \lim_{h\to 0} \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} = \lim_{h\to 0}(-2x - h) = -2x - 0 = -2x.$$

10. $y = \sqrt{2x+1}$

SOLUTION Let $y = \sqrt{2x+1}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} = \lim_{h\to 0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$
$$= \lim_{h\to 0} \frac{(2x+2h+1) - (2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

11. $y = \dfrac{1}{2-x}$

SOLUTION Let $y = \dfrac{1}{2-x}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h} = \lim_{h\to 0} \frac{(2-x) - (2-x-h)}{h(2-x-h)(2-x)} = \lim_{h\to 0} \frac{1}{(2-x-h)(2-x)} = \frac{1}{(2-x)^2}.$$

12. $y = \dfrac{1}{(x-1)^2}$

SOLUTION Let $y = \dfrac{1}{(x-1)^2}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{(x+h-1)^2} - \frac{1}{(x-1)^2}}{h} = \lim_{h\to 0} \frac{(x-1)^2 - (x+h-1)^2}{h(x+h-1)^2(x-1)^2}$$
$$= \lim_{h\to 0} \frac{x^2 - 2x + 1 - (x^2 + 2xh + h^2 - 2x - 2h + 1)}{h(x+h-1)^2(x-1)^2} = \lim_{h\to 0} \frac{-2x - h + 2}{(x+h-1)^2(x-1)^2}$$
$$= \frac{-2x+2}{(x-1)^4} = -\frac{2}{(x-1)^3}.$$

In Exercises 13–16, express the limit as a derivative.

13. $\displaystyle\lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h}$

SOLUTION Let $f(x) = \sqrt{x}$. Then

$$\lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = f'(1).$$

14. $\displaystyle\lim_{x\to -1} \frac{x^3 + 1}{x + 1}$

SOLUTION Let $f(x) = x^3$. Then

$$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = f'(-1).$$

15. $\lim_{t\to\pi} \frac{\sin t \cos t}{t-\pi}$

SOLUTION Let $f(t) = \sin t \cos t$ and note that $f(\pi) = \sin\pi \cos\pi = 0$. Then

$$\lim_{t\to\pi} \frac{\sin t \cos t}{t-\pi} = \lim_{t\to\pi} \frac{f(t) - f(\pi)}{t-\pi} = f'(\pi).$$

16. $\lim_{\theta\to\pi} \frac{\cos\theta - \sin\theta + 1}{\theta-\pi}$

SOLUTION Let $f(\theta) = \cos\theta - \sin\theta$ and note that $f(\pi) = -1$. Then

$$\lim_{\theta\to\pi} \frac{\cos\theta - \sin\theta + 1}{\theta-\pi} = \lim_{\theta\to\pi} \frac{f(\theta) - f(\pi)}{\theta-\pi} = f'(\pi).$$

17. Find $f(4)$ and $f'(4)$ if the tangent line to the graph of $f(x)$ at $x = 4$ has equation $y = 3x - 14$.

SOLUTION The equation of the tangent line to the graph of $f(x)$ at $x = 4$ is $y = f'(4)(x-4) + f(4) = f'(4)x + (f(4) - 4f'(4))$. Matching this to $y = 3x - 14$, we see that $f'(4) = 3$ and $f(4) - 4(3) = -14$, so $f(4) = -2$.

18. Each graph in Figure 2 shows the graph of a function $f(x)$ and its derivative $f'(x)$. Determine which is the function and which is the derivative.

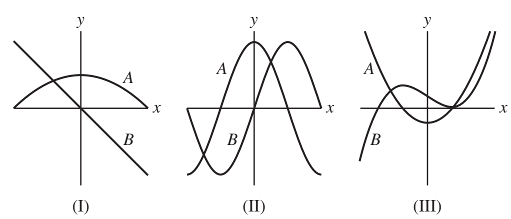


FIGURE 2 Graph of $f(x)$.

SOLUTION

- In (I), the graph labeled A is increasing when the graph labeled B is positive and is decreasing when the graph labeled B is negative. Therefore, the graph labeled A is the function $f(x)$ and the graph labeled B is the derivative $f'(x)$.
- In (II), the graph labeled B is increasing when the graph labeled A is positive and is decreasing when the graph labeled A is negative. Therefore, the graph labeled B is the function $f(x)$ and the graph labeled A is the derivative $f'(x)$.
- In (III), the graph labeled B has horizontal tangent lines at the locations the graph labeled A is zero. Therefore, the graph labeled B is the function $f(x)$ and the graph labeled A is the derivative $f'(x)$.

19. Is (A), (B), or (C) the graph of the derivative of the function $f(x)$ shown in Figure 3?

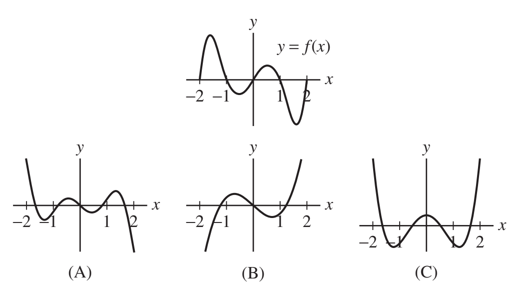


FIGURE 3

SOLUTION The graph of $f(x)$ has four horizontal tangent lines on $[-2, 2]$, so the graph of its derivative must have four x-intercepts on $[-2, 2]$. This eliminates (B). Moreover, $f(x)$ is increasing at both ends of the interval, so its derivative must be positive at both ends. This eliminates (A) and identifies (C) as the graph of $f'(x)$.

20. Let $N(t)$ be the percentage of a state population infected with a flu virus on week t of an epidemic. What percentage is likely to be infected in week 4 if $N(3) = 8$ and $N'(3) = 1.2$?

SOLUTION Because $N(4) - N(3) \approx N'(3)$, we estimate that

$$N(4) \approx N(3) + N'(3) = 8 + 1.2 = 9.2.$$

Thus, 9.2% of the population is likely infected in week 4.

21. A girl's height $h(t)$ (in centimeters) is measured at time t (in years) for $0 \le t \le 14$:

52, 75.1, 87.5, 96.7, 104.5, 111.8, 118.7, 125.2,
131.5, 137.5, 143.3, 149.2, 155.3, 160.8, 164.7

(a) What is the average growth rate over the 14-year period?

(b) Is the average growth rate larger over the first half or the second half of this period?

(c) Estimate $h'(t)$ (in centimeters per year) for $t = 3, 8$.

SOLUTION

(a) The average growth rate over the 14-year period is

$$\frac{164.7 - 52}{14} = 8.05 \text{ cm/year.}$$

(b) Over the first half of the 14-year period, the average growth rate is

$$\frac{125.2 - 52}{7} \approx 10.46 \text{ cm/year,}$$

which is larger than the average growth rate over the second half of the 14-year period:

$$\frac{164.7 - 125.2}{7} \approx 5.64 \text{ cm/year.}$$

(c) For $t = 3$,

$$h'(3) \approx \frac{h(4) - h(3)}{4 - 3} = \frac{104.5 - 96.7}{1} = 7.8 \text{ cm/year;}$$

for $t = 8$,

$$h'(8) \approx \frac{h(9) - h(8)}{9 - 8} = \frac{137.5 - 131.5}{1} = 6.0 \text{ cm/year.}$$

22. A planet's period P (number of days to complete one revolution around the sun) is approximately $0.199A^{3/2}$, where A is the average distance (in millions of kilometers) from the planet to the sun.

(a) Calculate P and dP/dA for Earth using the value $A = 150$.

(b) Estimate the increase in P if A is increased to 152.

SOLUTION

(a) Let $P = 0.199A^{3/2}$. Then $\frac{dP}{dA} = 0.2985A^{1/2}$. For $A = 150$,

$$P = 0.199(150)^{3/2} \approx 365.6 \text{ days; and}$$

$$\frac{dP}{dA} = 0.2985(150)^{1/2} \approx 3.656 \text{ days/millions of kilometers.}$$

(b) If A is increased to 150, then

$$P(152) - P(150) \approx \left.\frac{dP}{dA}\right|_{A=150} = 3.656 \text{ days.}$$

In Exercises 23 and 24, use the following table of values for the number $A(t)$ of automobiles (in millions) manufactured in the United States in year t.

t	1970	1971	1972	1973	1974	1975	1976
$A(t)$	6.55	8.58	8.83	9.67	7.32	6.72	8.50

23. What is the interpretation of $A'(t)$? Estimate $A'(1971)$. Does $A'(1974)$ appear to be positive or negative?

SOLUTION Because $A(t)$ measures the number of automobiles manufactured in the United States in year t, $A'(t)$ measures the rate of change in automobile production in the United States. For $t = 1971$,

$$A'(1971) \approx \frac{A(1972) - A(1971)}{1972 - 1971} = \frac{8.83 - 8.58}{1} = 0.25 \text{ million automobiles/year.}$$

Because $A(t)$ decreases from 1973 to 1974 and from 1974 to 1975, it appears that $A'(1974)$ would be negative.

24. Given the data, which of (A)–(C) in Figure 4 could be the graph of the derivative $A'(t)$? Explain.

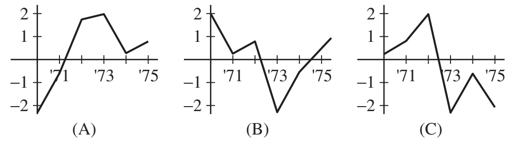


FIGURE 4

SOLUTION The values of $A(t)$ increase, then decrease and finally increase. Thus $A'(t)$ should transition from positive to negative and back to positive. This describes the graph in (B).

25. Which of the following is equal to $\frac{d}{dx}2^x$?

(a) 2^x **(b)** $(\ln 2)2^x$ **(c)** $x2^{x-1}$ **(d)** $\frac{1}{\ln 2}2^x$

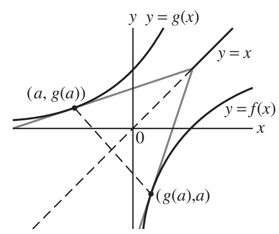
SOLUTION The derivative of $f(x) = 2^x$ is

$$\frac{d}{dx}2^x = 2^x \ln 2.$$

Hence, the correct answer is **(b)**.

26. Describe the graphical interpretation of the relation $g'(x) = 1/f'(g(x))$, where $f(x)$ and $g(x)$ are inverses of each other.

SOLUTION Suppose $f(x)$ and $g(x)$ are inverse functions. Consider a point on the graph of $y = f(x)$ – say (a, b) – and the point on the graph of $y = g(x)$ symmetric with respect to the line $y = x$ – that is, (b, a). The relation $g'(x) = 1/f'(g(x))$ indicates that the lines tangent to the two graphs at these symmetric points have slopes that are reciprocals of one another.



27. Show that if $f(x)$ is a function satisfying $f'(x) = f(x)^2$, then its inverse $g(x)$ satisfies $g'(x) = x^{-2}$.

SOLUTION

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))^2} = \frac{1}{x^2} = x^{-2}.$$

28. Find $g'(8)$, where $g(x)$ is the inverse of a differentiable function $f(x)$ such that $f(-1) = 8$ and $f'(-1) = 12$.

SOLUTION The Theorem on the derivative of an inverse function states

$$g'(x) = \frac{1}{f'(g(x))}.$$

Setting $x = 8$, we obtain

$$g'(8) = \frac{1}{f'(g(8))}.$$

Because $f(-1) = 8$, it follows that $g(8) = -1$. Thus,

$$g'(8) = \frac{1}{f'(-1)} = \frac{1}{12}.$$

In Exercises 29–80, compute the derivative.

29. $y = 3x^5 - 7x^2 + 4$

SOLUTION Let $y = 3x^5 - 7x^2 + 4$. Then

$$\frac{dy}{dx} = 15x^4 - 14x.$$

30. $y = 4x^{-3/2}$

SOLUTION Let $y = 4x^{-3/2}$. Then

$$\frac{dy}{dx} = -6x^{-5/2}.$$

31. $y = t^{-7.3}$

SOLUTION Let $y = t^{-7.3}$. Then

$$\frac{dy}{dt} = -7.3t^{-8.3}.$$

32. $y = 4x^2 - x^{-2}$

SOLUTION Let $y = 4x^2 - x^{-2}$. Then

$$\frac{dy}{dx} = 8x + 2x^{-3}.$$

33. $y = \dfrac{x+1}{x^2+1}$

SOLUTION Let $y = \dfrac{x+1}{x^2+1}$. Then

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}.$$

34. $y = \dfrac{3t-2}{4t-9}$

SOLUTION Let $y = \dfrac{3t-2}{4t-9}$. Then

$$\frac{dy}{dt} = \frac{(4t-9)(3) - (3t-2)(4)}{(4t-9)^2} = -\frac{19}{(4t-9)^2}.$$

35. $y = (x^4 - 9x)^6$

SOLUTION Let $y = (x^4 - 9x)^6$. Then

$$\frac{dy}{dx} = 6(x^4 - 9x)^5 \frac{d}{dx}(x^4 - 9x) = 6(4x^3 - 9)(x^4 - 9x)^5.$$

36. $y = (3t^2 + 20t^{-3})^6$

SOLUTION Let $y = (3t^2 + 20t^{-3})^6$. Then

$$\frac{dy}{dt} = 6(3t^2 + 20t^{-3})^5 \frac{d}{dt}(3t^2 + 20t^{-3}) = 6(6t - 60t^{-4})(3t^2 + 20t^{-3})^5.$$

37. $y = (2 + 9x^2)^{3/2}$

SOLUTION Let $y = (2 + 9x^2)^{3/2}$. Then

$$\frac{dy}{dx} = \frac{3}{2}(2 + 9x^2)^{1/2} \frac{d}{dx}(2 + 9x^2) = 27x(2 + 9x^2)^{1/2}.$$

38. $y = (x+1)^3(x+4)^4$

SOLUTION Let $y = (x+1)^3(x+4)^4$. Then

$$\begin{aligned}\frac{dy}{dx} &= 4(x+1)^3(x+4)^3 + 3(x+1)^2(x+4)^4 = (x+1)^2(x+4)^3(4x + 4 + 3x + 12)\\ &= (7x+16)(x+1)^2(x+4)^3.\end{aligned}$$

39. $y = \dfrac{z}{\sqrt{1-z}}$

SOLUTION Let $y = \dfrac{z}{\sqrt{1-z}}$. Then

$$\frac{dy}{dz} = \frac{\sqrt{1-z} - (-\frac{z}{2})\frac{1}{\sqrt{1-z}}}{1-z} = \frac{1 - z + \frac{z}{2}}{(1-z)^{3/2}} = \frac{2-z}{2(1-z)^{3/2}}.$$

40. $y = \left(1 + \frac{1}{x}\right)^3$

SOLUTION Let $y = \left(1 + \frac{1}{x}\right)^3$. Then

$$\frac{dy}{dx} = 3\left(1 + \frac{1}{x}\right)^2 \frac{d}{dx}\left(1 + \frac{1}{x}\right) = -\frac{3}{x^2}\left(1 + \frac{1}{x}\right)^2.$$

41. $y = \frac{x^4 + \sqrt{x}}{x^2}$

SOLUTION Let

$$y = \frac{x^4 + \sqrt{x}}{x^2} = x^2 + x^{-3/2}.$$

Then

$$\frac{dy}{dx} = 2x - \frac{3}{2}x^{-5/2}.$$

42. $y = \frac{1}{(1-x)\sqrt{2-x}}$

SOLUTION Let $y = \frac{1}{(1-x)\sqrt{2-x}} = \left((1-x)\sqrt{2-x}\right)^{-1}$. Then

$$\frac{dy}{dx} = -\left((1-x)\sqrt{2-x}\right)^{-2} \frac{d}{dx}\left((1-x)\sqrt{2-x}\right) = -\left((1-x)\sqrt{2-x}\right)^{-2}\left(-\frac{1-x}{2\sqrt{2-x}} - \sqrt{2-x}\right)$$
$$= \frac{5-3x}{2(1-x)^2(2-x)^{3/2}}.$$

43. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

SOLUTION Let $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$. Then

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right)$$
$$= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x}\right)\right)$$
$$= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right).$$

44. $h(z) = \left(z + (z+1)^{1/2}\right)^{-3/2}$

SOLUTION

$$\frac{d}{dz}\left(z + (z+1)^{1/2}\right)^{-3/2} = -\frac{3}{2}\left(z + (z+1)^{1/2}\right)^{-5/2} \frac{d}{dz}\left(z + (z+1)^{1/2}\right)$$
$$= -\frac{3}{2}\left(z + (z+1)^{1/2}\right)^{-5/2}\left(1 + \frac{1}{2}(z+1)^{-1/2}\right).$$

45. $y = \tan(t^{-3})$

SOLUTION Let $y = \tan(t^{-3})$. Then

$$\frac{dy}{dt} = \sec^2(t^{-3})\frac{d}{dt}t^{-3} = -3t^{-4}\sec^2(t^{-3}).$$

46. $y = 4\cos(2 - 3x)$

SOLUTION Let $y = 4\cos(2 - 3x)$. Then

$$\frac{dy}{dx} = -4\sin(2 - 3x)\frac{d}{dx}(2 - 3x) = 12\sin(2 - 3x).$$

47. $y = \sin(2x)\cos^2 x$

SOLUTION Let $y = \sin(2x)\cos^2 x = 2\sin x\cos^3 x$. Then

$$\frac{dy}{dx} = -6\sin^2 x\cos^2 x + 2\cos^4 x.$$

48. $y = \sin\left(\frac{4}{\theta}\right)$

SOLUTION Let $y = \sin\left(\frac{4}{\theta}\right)$. Then

$$\frac{dy}{d\theta} = \cos\left(\frac{4}{\theta}\right)\frac{d}{d\theta}\left(\frac{4}{\theta}\right) = -\frac{4}{\theta^2}\cos\left(\frac{4}{\theta}\right).$$

49. $y = \dfrac{t}{1+\sec t}$

SOLUTION Let $y = \dfrac{t}{1+\sec t}$. Then

$$\frac{dy}{dt} = \frac{1+\sec t - t\sec t\tan t}{(1+\sec t)^2}.$$

50. $y = z\csc(9z+1)$

SOLUTION Let $y = z\csc(9z+1)$. Then

$$\frac{dy}{dz} = -9z\csc(9z+1)\cot(9z+1) + \csc(9z+1).$$

51. $y = \dfrac{8}{1+\cot\theta}$

SOLUTION Let $y = \dfrac{8}{1+\cot\theta} = 8(1+\cot\theta)^{-1}$. Then

$$\frac{dy}{d\theta} = -8(1+\cot\theta)^{-2}\frac{d}{d\theta}(1+\cot\theta) = \frac{8\csc^2\theta}{(1+\cot\theta)^2}.$$

52. $y = \tan(\cos x)$

SOLUTION Let $y = \tan(\cos x)$. Then

$$\frac{dy}{dx} = \sec^2(\cos x)\frac{d}{dx}\cos x = -\sin x\sec^2(\cos x).$$

53. $y = \tan(\sqrt{1+\csc\theta})$

SOLUTION

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\sqrt{1+\csc\theta})\frac{d}{dx}\sqrt{1+\csc\theta}\\ &= \sec^2(\sqrt{1+\csc\theta})\cdot\frac{1}{2}(1+\csc\theta)^{-1/2}\frac{d}{dx}(1+\csc\theta)\\ &= -\frac{\sec^2(\sqrt{1+\csc\theta})\csc\theta\cot\theta}{2(\sqrt{1+\csc\theta})}.\end{aligned}$$

54. $y = \cos(\cos(\cos(\theta)))$

SOLUTION Let $y = \cos(\cos(\cos(\theta)))$. Then

$$\begin{aligned}\frac{dy}{d\theta} &= -\sin(\cos(\cos(\theta)))\frac{d}{d\theta}\cos(\cos(\theta)) = \sin(\cos(\cos(\theta)))\sin(\cos(\theta))\frac{d}{d\theta}\cos(\theta)\\ &= -\sin(\cos(\cos(\theta)))\sin(\cos(\theta))\sin(\theta).\end{aligned}$$

55. $f(x) = 9e^{-4x}$

SOLUTION $\dfrac{d}{dx}9e^{-4x} = -36e^{-4x}$.

56. $f(x) = \dfrac{e^{-x}}{x}$

SOLUTION $\dfrac{d}{dx}\left(\dfrac{e^{-x}}{x}\right) = \dfrac{-xe^{-x} - e^{-x}}{x^2} = -\dfrac{e^{-x}(x+1)}{x^2}.$

57. $g(t) = e^{4t-t^2}$

SOLUTION $\dfrac{d}{dt}e^{4t-t^2} = (4-2t)e^{4t-t^2}.$

58. $g(t) = t^2 e^{1/t}$

SOLUTION $\dfrac{d}{dt}t^2e^{1/t} = 2te^{1/t} + t^2\left(-\dfrac{1}{t^2}\right)e^{1/t} = (2t-1)e^{1/t}.$

59. $f(x) = \ln(4x^2+1)$

SOLUTION $\dfrac{d}{dx}\ln(4x^2+1) = \dfrac{8x}{4x^2+1}.$

60. $f(x) = \ln(e^x - 4x)$

SOLUTION $\dfrac{d}{dx}\ln(e^x-4x) = \dfrac{e^x-4}{e^x-4x}.$

61. $G(s) = (\ln(s))^2$

SOLUTION $\dfrac{d}{ds}(\ln s)^2 = \dfrac{2\ln s}{s}.$

62. $G(s) = \ln(s^2)$

SOLUTION $\dfrac{d}{ds}\ln(s^2) = 2\dfrac{d}{ds}\ln s = \dfrac{2}{s}.$

63. $f(\theta) = \ln(\sin\theta)$

SOLUTION $\dfrac{d}{d\theta}\ln(\sin\theta) = \dfrac{\cos\theta}{\sin\theta} = \cot\theta.$

64. $f(\theta) = \sin(\ln\theta)$

SOLUTION $\dfrac{d}{d\theta}\sin(\ln\theta) = \dfrac{\cos(\ln\theta)}{\theta}.$

65. $h(z) = \sec(z + \ln z)$

SOLUTION $\dfrac{d}{dz}\sec(z+\ln z) = \sec(z+\ln z)\tan(z+\ln z)\left(1+\dfrac{1}{z}\right).$

66. $f(x) = e^{\sin^2 x}$

SOLUTION $\dfrac{d}{dx}e^{\sin^2 x} = 2\sin x\cos x e^{\sin^2 x} = \sin 2x e^{\sin^2 x}.$

67. $f(x) = 7^{-2x}$

SOLUTION $\dfrac{d}{dx}7^{-2x} = \big(-2\ln 7\big)\big(7^{-2x}\big).$

68. $h(y) = \dfrac{1+e^y}{1-e^y}$

SOLUTION $\dfrac{d}{dy}\left(\dfrac{1+e^y}{1-e^y}\right) = \dfrac{(1-e^y)e^y - (1+e^y)(-e^y)}{(1-e^y)^2} = \dfrac{e^y(1-e^y+1+e^y)}{(1-e^y)^2} = \dfrac{2e^y}{(1-e^y)^2}.$

69. $g(x) = \tan^{-1}(\ln x)$

SOLUTION $\dfrac{d}{dx}\tan^{-1}(\ln x) = \dfrac{1}{1+(\ln x)^2}\cdot\dfrac{1}{x}.$

70. $G(s) = \cos^{-1}(s^{-1})$

SOLUTION $\dfrac{d}{ds}\cos^{-1}(s^{-1}) = \dfrac{-1}{\sqrt{1-\left(\frac{1}{s}\right)^2}}\left(-\dfrac{1}{s^2}\right) = \dfrac{1}{\sqrt{s^4-s^2}}.$

71. $f(x) = \ln(\csc^{-1} x)$

SOLUTION $\dfrac{d}{dx}\ln(\csc^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}\csc^{-1} x}.$

72. $f(x) = e^{\sec^{-1} x}$

SOLUTION $\dfrac{d}{dx}e^{\sec^{-1} x} = \dfrac{1}{|x|\sqrt{x^2-1}}e^{\sec^{-1} x}.$

73. $R(s) = s^{\ln s}$

SOLUTION Rewrite

$$R(s) = \left(e^{\ln s}\right)^{\ln s} = e^{(\ln s)^2}.$$

Then

$$\frac{dR}{ds} = e^{(\ln s)^2}\cdot 2\ln s\cdot\frac{1}{s} = \frac{2\ln s}{s}s^{\ln s}.$$

Alternately, $R(s) = s^{\ln s}$ implies that $\ln R = \ln\left(s^{\ln s}\right) = (\ln s)^2$. Thus,

$$\frac{1}{R}\frac{dR}{ds} = 2\ln s\cdot\frac{1}{s} \quad\text{or}\quad \frac{dR}{ds} = \frac{2\ln s}{s}s^{\ln s}.$$

74. $f(x) = (\cos^2 x)^{\cos x}$

SOLUTION Rewrite

$$f(x) = \left(e^{\ln\cos^2 x}\right)^{\cos x} = e^{2\cos x\ln\cos x}.$$

Then

$$\begin{aligned}\frac{df}{dx} &= e^{2\cos x\ln\cos x}\left(2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\right)\\ &= -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).\end{aligned}$$

Alternately, $f(x) = (\cos^2 x)^{\cos x}$ implies that $\ln f = \cos x\ln\cos^2 x = 2\cos x\ln\cos x$. Thus,

$$\begin{aligned}\frac{1}{f}\frac{df}{dx} &= 2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\\ &= -2\sin x(1+\ln\cos x),\end{aligned}$$

and

$$\frac{df}{dx} = -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).$$

75. $G(t) = (\sin^2 t)^t$

SOLUTION Rewrite

$$G(t) = \left(e^{\ln\sin^2 t}\right)^t = e^{2t\ln\sin t}.$$

Then

$$\frac{dG}{dt} = e^{2t\ln\sin t}\left(2t\cdot\frac{\cos t}{\sin t} + 2\ln\sin t\right) = 2(\sin^2 t)^t(t\cot t + \ln\sin t).$$

Alternately, $G(t) = (\sin^2 t)^t$ implies that $\ln G = t \ln \sin^2 t = 2t \ln \sin t$. Thus,

$$\frac{1}{G}\frac{dG}{dt} = 2t \cdot \frac{\cos t}{\sin t} + 2 \ln \sin t,$$

and

$$\frac{dG}{dt} = 2(\sin^2 t)^t (t \cot t + \ln \sin t).$$

76. $h(t) = t^{(t^t)}$

SOLUTION Let $h(t) = t^{(t^t)}$. Then $\ln h = t^t \ln t$ and

$$\begin{aligned}\ln(\ln h) &= \ln(t^t \ln t) = \ln t^t + \ln(\ln t)\\ &= t \ln t + \ln(\ln t).\end{aligned}$$

Thus,

$$\frac{1}{h \ln h}\frac{dh}{dt} = t \cdot \frac{1}{t} + \ln t + \frac{1}{t \ln t} = 1 + \ln t + \frac{1}{t \ln t},$$

and

$$\frac{dh}{dt} = t^{(t^t)} t^t \ln t \left(1 + \ln t + \frac{1}{t \ln t}\right).$$

77. $g(t) = \sinh(t^2)$

SOLUTION $\dfrac{d}{dt}\sinh(t^2) = 2t\cosh(t^2)$.

78. $h(y) = y \tanh(4y)$

SOLUTION $\dfrac{d}{dy} y \tanh(4y) = \tanh(4y) + 4y \operatorname{sech}^2(4y)$.

79. $g(x) = \tanh^{-1}(e^x)$

SOLUTION $\dfrac{d}{dx}\tanh^{-1}(e^x) = \dfrac{1}{1-(e^x)^2}e^x = \dfrac{e^x}{1-e^{2x}}$.

80. $g(t) = \sqrt{t^2-1}\sinh^{-1} t$

SOLUTION $\dfrac{d}{dt}\sqrt{t^2-1}\sinh^{-1} t = \dfrac{t}{\sqrt{t^2-1}}\sinh^{-1} t + \sqrt{t^2-1} \cdot \dfrac{1}{\sqrt{t^2+1}} = \dfrac{t\sinh^{-1} t}{\sqrt{t^2-1}} + \sqrt{\dfrac{t^2-1}{t^2+1}}$.

81. For which values of α is $f(x) = |x|^\alpha$ differentiable at $x = 0$?

SOLUTION Let $f(x) = |x|^\alpha$. If $\alpha < 0$, then $f(x)$ is not continuous at $x = 0$ and therefore cannot be differentiable at $x = 0$. If $\alpha = 0$, then the function reduces to $f(x) = 1$, which is differentiable at $x = 0$. Now, suppose $\alpha > 0$ and consider the limit

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{|x|^\alpha}{x}.$$

If $0 < \alpha < 1$, then

$$\lim_{x\to 0-}\frac{|x|^\alpha}{x} = -\infty \qquad \text{while} \qquad \lim_{x\to 0+}\frac{|x|^\alpha}{x} = \infty$$

and $f'(0)$ does not exist. If $\alpha = 1$, then

$$\lim_{x\to 0-}\frac{|x|}{x} = -1 \qquad \text{while} \qquad \lim_{x\to 0+}\frac{|x|}{x} = 1$$

and $f'(0)$ again does not exist. Finally, if $\alpha > 1$, then

$$\lim_{x\to 0}\frac{|x|^\alpha}{x} = 0,$$

so $f'(0)$ does exist.

In summary, $f(x) = |x|^\alpha$ is differentiable at $x = 0$ when $\alpha = 0$ and when $\alpha > 1$.

82. Find $f'(2)$ if $f(g(x)) = e^{x^2}$, $g(1) = 2$, and $g'(1) = 4$.

SOLUTION We differentiate both sides of the equation $f(g(x)) = e^{x^2}$ to obtain,

$$f'(g(x))\, g'(x) = 2xe^{x^2}.$$

Setting $x = 1$ yields

$$f'(g(1))\, g'(1) = 2e.$$

Since $g(1) = 2$ and $g'(1) = 4$, we find

$$f'(2) \cdot 4 = 2e,$$

or

$$f'(2) = \frac{e}{2}.$$

In Exercises 83 and 84, let $f(x) = xe^{-x}$.

83. Show that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ be this inverse. Find the domain and range of $g(x)$ and compute $g'(2e^{-2})$.

SOLUTION Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x}(1 - x)$. On $[1, \infty)$, $f'(x) < 0$, so $f(x)$ is decreasing and therefore one-to-one. It follows that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ denote this inverse. Because $f(1) = e^{-1}$ and $f(x) \to 0$ as $x \to \infty$, the domain of $g(x)$ is $(0, e^{-1}]$, and the range is $[1, \infty)$.

To determine $g'(2e^{-2})$, we use the formula $g'(x) = 1/f'(g(x))$. Because $f(2) = 2e^{-2}$, it follows that $g(2e^{-2}) = 2$. Then,

$$g'(2e^{-2}) = \frac{1}{f'(g(2e^{-2}))} = \frac{1}{f'(2)} = \frac{1}{-e^{-2}} = -e^2.$$

84. Show that $f(x) = c$ has two solutions if $0 < c < e^{-1}$.

SOLUTION First note that $f(x) < 0$ for $x < 0$, so we only need to examine $(0, \infty)$ for solutions to $f(x) = c$ when $c > 0$. Next, because $f'(x) = e^{-x}(1 - x)$, f is decreasing on $(1, \infty)$ and increasing on $(0, 1)$. Therefore, f is one-to-one on each of these intervals, which guarantees that the equation $f(x) = c$ can have at most one solution on each of these intervals for any value of c.

Now, let $0 < c < e^{-1}$ and consider the interval $[1, \infty)$. Because

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = 0,$$

it follows that there exists a $d \in (1, \infty)$ such that $f(d) < c$. With $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[1, \infty)$. Next, consider the interval $[0, 1]$. With $f(0) = 0 < c$ and $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[0, 1]$.

In summary, the equation $f(x) = c$ has exactly two solutions for any c between 0 and e^{-1}.

In Exercises 85–90, use the following table of values to calculate the derivative of the given function at $x = 2$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	5	4	−3	9
4	3	2	−2	3

85. $S(x) = 3f(x) - 2g(x)$

SOLUTION Let $S(x) = 3f(x) - 2g(x)$. Then $S'(x) = 3f'(x) - 2g'(x)$ and

$$S'(2) = 3f'(2) - 2g'(2) = 3(-3) - 2(9) = -27.$$

86. $H(x) = f(x)g(x)$

SOLUTION Let $H(x) = f(x)g(x)$. Then $H'(x) = f(x)g'(x) + f'(x)g(x)$ and

$$H'(2) = f(2)g'(2) + f'(2)g(2) = 5(9) + (-3)(4) = 33.$$

87. $R(x) = \dfrac{f(x)}{g(x)}$

SOLUTION Let $R(x) = f(x)/g(x)$. Then

$$R'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

and

$$R'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{4(-3) - 5(9)}{4^2} = -\frac{57}{16}.$$

88. $G(x) = f(g(x))$

SOLUTION Let $G(x) = f(g(x))$. Then $G'(x) = f'(g(x))g'(x)$ and

$$G'(2) = f'(g(2))g'(2) = f'(4)g'(2) = -2(9) = -18.$$

89. $F(x) = f(g(2x))$

SOLUTION Let $F(x) = f(g(2x))$. Then $F'(x) = 2f'(g(2x))g'(2x)$ and

$$F'(2) = 2f'(g(4))g'(4) = 2f'(2)g'(4) = 2(-3)(3) = -18.$$

90. $K(x) = f(x^2)$

SOLUTION Let $K(x) = f(x^2)$. Then $K'(x) = 2xf'(x^2)$ and

$$K'(2) = 2(2)f'(4) = 4(-2) = -8.$$

91. Find the points on the graph of $f(x) = x^3 - 3x^2 + x + 4$ where the tangent line has slope 10.

SOLUTION Let $f(x) = x^3 - 3x^2 + x + 4$. Then $f'(x) = 3x^2 - 6x + 1$. The tangent line to the graph of $f(x)$ will have slope 10 when $f'(x) = 10$. Solving the quadratic equation $3x^2 - 6x + 1 = 10$ yields $x = -1$ and $x = 3$. Thus, the points on the graph of $f(x)$ where the tangent line has slope 10 are $(-1, -1)$ and $(3, 7)$.

92. Find the points on the graph of $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1.

SOLUTION Suppose $x^{2/3} + y^{2/3} = 1$. Differentiating with respect to x leads to

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}.$$

Tangents to the curve therefore have slope 1 when $y = -x$. Substituting $y = -x$ into the equation for the curve yields $2x^{2/3} = 1$, so $x = \pm\frac{\sqrt{2}}{4}$. Thus, the points along the curve $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1 are:

$$\left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right).$$

93. Find a such that the tangent lines $y = x^3 - 2x^2 + x + 1$ at $x = a$ and $x = a + 1$ are parallel.

SOLUTION Let $f(x) = x^3 - 2x^2 + x + 1$. Then $f'(x) = 3x^2 - 4x + 1$ and the slope of the tangent line at $x = a$ is $f'(a) = 3a^2 - 4a + 1$, while the slope of the tangent line at $x = a + 1$ is

$$f'(a + 1) = 3(a + 1)^2 - 4(a + 1) + 1 = 3(a^2 + 2a + 1) - 4a - 4 + 1 = 3a^2 + 2a.$$

In order for the tangent lines at $x = a$ and $x = a + 1$ to have the same slope, we must have $f'(a) = f'(a + 1)$, or

$$3a^2 - 4a + 1 = 3a^2 + 2a.$$

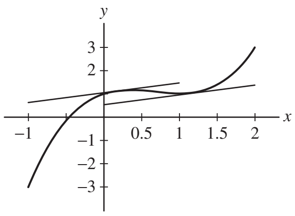
The only solution to this equation is $a = \frac{1}{6}$. The equation of the tangent line at $x = \frac{1}{6}$ is

$$y = f'\left(\frac{1}{6}\right)\left(x - \frac{1}{6}\right) + f\left(\frac{1}{6}\right) = \frac{5}{12}\left(x - \frac{1}{6}\right) + \frac{241}{216} = \frac{5}{12}x + \frac{113}{108},$$

and the equation of the tangent line at $x = \frac{7}{6}$ is

$$y = f'\left(\frac{7}{6}\right)\left(x - \frac{7}{6}\right) + f\left(\frac{7}{6}\right) = \frac{5}{12}\left(x - \frac{7}{6}\right) + \frac{223}{216} = \frac{5}{12}x + \frac{59}{108}.$$

The graphs of $f(x)$ and the two tangent lines appear below.



94. Use the table to compute the average rate of change of Candidate A's percentage of votes over the intervals from day 20 to day 15, day 15 to day 10, and day 10 to day 5. If this trend continues over the last 5 days before the election, will Candidate A win?

Days Before Election	**20**	**15**	**10**	**5**
Candidate A	44.8%	46.8%	48.3%	49.3%
Candidate B	55.2%	53.2%	51.7%	50.7%

SOLUTION The average rate of change of A's percentage for the period from day 20 to day 15 is

$$\frac{46.8 - 44.8}{5} = 0.4\%/\text{day}.$$

For the period from day 15 to day 10, the average rate of change is

$$\frac{48.3 - 46.8}{5} = 0.3\%/\text{day}.$$

Finally, for the period from day 10 to day 5, the average rate of change is

$$\frac{49.3 - 48.3}{5} = 0.2\%/\text{day}.$$

If this trend continues over the last five days before the election, the average rate of change will drop to 0.1 %/day, so A's percentage will increase another 0.5% to 49.8%. Accordingly, A will *not* win the election.

In Exercises 95–100, calculate y''.

95. $y = 12x^3 - 5x^2 + 3x$

SOLUTION Let $y = 12x^3 - 5x^2 + 3x$. Then

$$y' = 36x^2 - 10x + 3 \quad \text{and} \quad y'' = 72x - 10.$$

96. $y = x^{-2/5}$

SOLUTION Let $y = x^{-2/5}$. Then

$$y' = -\frac{2}{5}x^{-7/5} \quad \text{and} \quad y'' = \frac{14}{25}x^{-12/5}.$$

97. $y = \sqrt{2x + 3}$

SOLUTION Let $y = \sqrt{2x + 3} = (2x + 3)^{1/2}$. Then

$$y' = \frac{1}{2}(2x + 3)^{-1/2}\frac{d}{dx}(2x + 3) = (2x + 3)^{-1/2} \quad \text{and} \quad y'' = -\frac{1}{2}(2x + 3)^{-3/2}\frac{d}{dx}(2x + 3) = -(2x + 3)^{-3/2}.$$

98. $y = \dfrac{4x}{x + 1}$

SOLUTION Let $y = \dfrac{4x}{x + 1}$. Then

$$y' = \frac{(x + 1)(4) - 4x}{(x + 1)^2} = \frac{4}{(x + 1)^2} \quad \text{and} \quad y'' = -\frac{8}{(x + 1)^3}.$$

99. $y = \tan(x^2)$

SOLUTION Let $y = \tan(x^2)$. Then

$$y' = 2x\sec^2(x^2) \quad \text{and}$$
$$y'' = 2x\left(2\sec(x^2)\frac{d}{dx}\sec(x^2)\right) + 2\sec^2(x^2) = 8x^2\sec^2(x^2)\tan(x^2) + 2\sec^2(x^2).$$

100. $y = \sin^2(4x+9)$

SOLUTION Let $y = \sin^2(4x+9)$. Then

$$y' = 8\sin(4x+9)\cos(4x+9) = 4\sin(8x+18) \quad \text{and} \quad y'' = 32\cos(8x+18).$$

In Exercises 101–106, compute $\frac{dy}{dx}$.

101. $x^3 - y^3 = 4$

SOLUTION Consider the equation $x^3 - y^3 = 4$. Differentiating with respect to x yields

$$3x^2 - 3y^2\frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

102. $4x^2 - 9y^2 = 36$

SOLUTION Consider the equation $4x^2 - 9y^2 = 36$. Differentiating with respect to x yields

$$8x - 18y\frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{4x}{9y}.$$

103. $y = xy^2 + 2x^2$

SOLUTION Consider the equation $y = xy^2 + 2x^2$. Differentiating with respect to x yields

$$\frac{dy}{dx} = 2xy\frac{dy}{dx} + y^2 + 4x.$$

Therefore,

$$\frac{dy}{dx} = \frac{y^2 + 4x}{1 - 2xy}.$$

104. $\frac{y}{x} = x + y$

SOLUTION Solving $\frac{y}{x} = x + y$ for y yields

$$y = \frac{x^2}{1-x}.$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}.$$

105. $y = \sin(x+y)$

SOLUTION Consider the equation $y = \sin(x+y)$. Differentiating with respect to x yields

$$\frac{dy}{dx} = \cos(x+y)\left(1 + \frac{dy}{dx}\right).$$

Therefore,

$$\frac{dy}{dx} = \frac{\cos(x+y)}{1 - \cos(x+y)}.$$

106. $\tan(x + y) = xy$

SOLUTION Consider the equation $\tan(x + y) = xy$. Differentiating with respect to x yields

$$\sec^2(x + y)\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y.$$

Therefore,

$$\frac{dy}{dx} = \frac{y - \sec^2(x + y)}{\sec^2(x + y) - x}.$$

107. In Figure 5, label the graphs f, f', and f''.

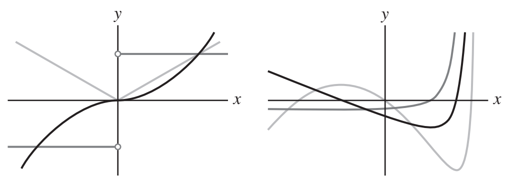


FIGURE 5

SOLUTION First consider the plot on the left. Observe that the green curve is nonnegative whereas the red curve is increasing, suggesting that the green curve is the derivative of the red curve. Moreover, the green curve is linear with negative slope for $x < 0$ and linear with positive slope for $x > 0$ while the blue curve is a negative constant for $x < 0$ and a positive constant for $x > 0$, suggesting the blue curve is the derivative of the green curve. Thus, the red, green and blue curves, respectively, are the graphs of f, f' and f''.

Now consider the plot on the right. Because the red curve is decreasing when the blue curve is negative and increasing when the blue curve is positive and the green curve is decreasing when the red curve is negative and increasing when the red curve is positive, it follows that the green, red and blue curves, respectively, are the graphs of f, f' and f''.

108. Let $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists for all x (including $x = 0$) but that $f'(x)$ is not continuous at $x = 0$ (Figure 6).

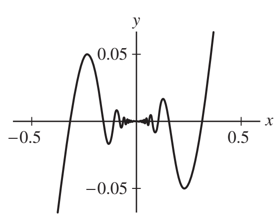


FIGURE 6 Graph of $f(x) = x^2 \sin(x^{-1})$.

SOLUTION Let $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. For $x \neq 0$, the product rule and the chain rule give

$$f'(x) = 2x \sin(x^{-1}) - x^2 \cos(x^{-1})(x^{-2}) = 2x \sin(x^{-1}) - \cos(x^{-1}),$$

which exists for all $x \neq 0$. At $x = 0$ we use the limit definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1}{h}(h^2 \sin(h^{-1})) = \lim_{h\to 0} h \sin(h^{-1}) = 0,$$

by the Squeeze Theorem, since $-h \le h \sin \frac{1}{h} \le h$. Thus, $f'(x)$ exists for all x. However,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x \sin(x^{-1}) - \cos(x^{-1})\right)$$

does not exist, so $f'(x)$ is not continuous at $x = 0$.

In Exercises 109–114, use logarithmic differentiation to find the derivative.

109. $y = \dfrac{(x + 1)^3}{(4x - 2)^2}$

SOLUTION Let $y = \dfrac{(x + 1)^3}{(4x - 2)^2}$. Then

$$\ln y = \ln\left(\frac{(x + 1)^3}{(4x - 2)^2}\right) = \ln (x + 1)^3 - \ln (4x - 2)^2 = 3 \ln(x + 1) - 2 \ln(4x - 2).$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{3}{x+1} - \frac{2}{4x-2} \cdot 4 = \frac{3}{x+1} - \frac{4}{2x-1},$$

so

$$y' = \frac{(x+1)^3}{(4x-2)^2}\left(\frac{3}{x+1} - \frac{4}{2x-1}\right).$$

110. $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$

SOLUTION Let $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$. Then

$$\begin{aligned}\ln y &= \ln\left((x+1)(x+2)^2\right) - \ln\left((x+3)(x+4)\right)\\ &= \ln(x+1) + 2\ln(x+2) - \ln(x+3) - \ln(x+4).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4},$$

so

$$y' = \frac{(x+1)(x+2)^2}{(x+3)(x+4)}\left(\frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}\right).$$

111. $y = e^{(x-1)^2}e^{(x-3)^2}$

SOLUTION Let $y = e^{(x-1)^2}e^{(x-3)^2}$. Then

$$\ln y = \ln\left(e^{(x-1)^2}e^{(x-3)^2}\right) = \ln\left(e^{(x-1)^2+(x-3)^2}\right) = (x-1)^2 + (x-3)^2.$$

By logarithmic differentiation,

$$\frac{y'}{y} = 2(x-1) + 2(x-3) = 4x - 8,$$

so

$$y' = 4e^{(x-1)^2}e^{(x-3)^2}(x-2).$$

112. $y = \dfrac{e^x \sin^{-1} x}{\ln x}$

SOLUTION Let $y = \dfrac{e^x \sin^{-1} x}{\ln x}$. Then

$$\begin{aligned}\ln y &= \ln\left(\frac{e^x \sin^{-1} x}{\ln x}\right) = \ln(e^x \sin^{-1} x) - \ln(\ln x)\\ &= \ln\left(e^x\right) + \ln(\sin^{-1} x) - \ln(\ln x) = x + \ln(\sin^{-1} x) - \ln(\ln x).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = 1 + \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} - \frac{1}{\ln x} \cdot \frac{1}{x},$$

so

$$y' = \frac{e^x \sin^{-1} x}{\ln x}\left(1 + \frac{1}{\sqrt{1-x^2}\sin^{-1} x} - \frac{1}{x\ln x}\right).$$

113. $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$

SOLUTION Let $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$. Then

$$\begin{aligned}\ln y = \ln\left(\frac{e^{3x}(x-2)^2}{(x+1)^2}\right) &= \ln e^{3x} + \ln(x-2)^2 - \ln(x+1)^2\\ &= 3x + 2\ln(x-2) - 2\ln(x+1).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = 3 + \frac{2}{x-2} - \frac{2}{x+1},$$

so

$$y = \frac{e^{3x}(x-2)^2}{(x+1)^2}\left(3 + \frac{2}{x-2} - \frac{2}{x+1}\right).$$

114. $y = x^{\sqrt{x}}(x^{\ln x})$

SOLUTION Let $y = x^{\sqrt{x}}(x^{\ln x})$. Then

$$\ln y = \sqrt{x}\ln x + (\ln x)^2$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} + 2(\ln x)\cdot\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2\ln x}{x},$$

so

$$y' = x^{\sqrt{x}}(x^{\ln x})\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2\ln x}{x}\right).$$

*Exercises 115–117: Let q be the number of units of a product (cell phones, barrels of oil, etc.) that can be sold at the price p. The **price elasticity of demand** E is defined as the percentage rate of change of q with respect to p. In terms of derivatives,*

$$E = \frac{p}{q}\frac{dq}{dp} = \lim_{\Delta p\to 0}\frac{(100\Delta q)/q}{(100\Delta p)/p}$$

115. Show that the total revenue $R = pq$ satisfies $\dfrac{dR}{dp} = q(1+E)$.

SOLUTION Let $R = pq$. Then

$$\frac{dR}{dp} = p\frac{dq}{dp} + q = q\frac{p}{q}\frac{dq}{dp} + q = q(E+1).$$

116. A commercial bakery can sell q chocolate cakes per week at price \$$p$, where $q = 50p(10-p)$ for $5 < p < 10$.

(a) Show that $E(p) = \dfrac{2p-10}{p-10}$.

(b) Show, by computing $E(8)$, that if $p = \$8$, then a 1% increase in price reduces demand by approximately 3%.

SOLUTION

(a) Let $q = 50p(10-p) = 500p - 50p^2$. Then $q'(p) = 500 - 100p$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{p}{50p(10-p)}(500-100p) = \frac{10-2p}{10-p} = \frac{2p-10}{p-10}.$$

(b) From part (a),

$$E(8) = \frac{2(8)-10}{8-10} = -3.$$

Thus, with the price set at \$8, a 1% increase in price results in a 3% decrease in demand.

117. The monthly demand (in thousands) for flights between Chicago and St. Louis at the price p is $q = 40 - 0.2p$. Calculate the price elasticity of demand when $p = \$150$ and estimate the percentage increase in number of additional passengers if the ticket price is lowered by 1%.

SOLUTION Let $q = 40 - 0.2p$. Then $q'(p) = -0.2$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{0.2p}{0.2p - 40}.$$

For $p = 150$,

$$E(150) = \frac{0.2(150)}{0.2(150) - 40} = -3,$$

so a 1% decrease in price increases demand by 3%. The demand when $p = 150$ is $q = 40 - 0.2(150) = 10$, or 10000 passengers. Therefore, a 1% increase in demand translates to 300 additional passengers.

118. How fast does the water level rise in the tank in Figure 7 when the water level is $h = 4$ m and water pours in at 20 m^3/min?

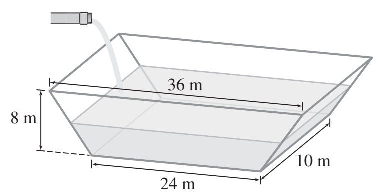


FIGURE 7

SOLUTION When the water level is at height h, the length of the upper surface of the water is $24 + \frac{3}{2}h$ and the volume of water in the trough is

$$V = \frac{1}{2}h\left(24 + 24 + \frac{3}{2}h\right)(10) = 240h + \frac{15}{2}h^2.$$

Therefore,

$$\frac{dV}{dt} = (240 + 15h)\frac{dh}{dt} = 20 \text{ m}^3/\text{min}.$$

When $h = 4$, we have

$$\frac{dh}{dt} = \frac{20}{240 + 15(4)} = \frac{1}{15} \text{ m/min}.$$

119. The minute hand of a clock is 8 cm long, and the hour hand is 5 cm long. How fast is the distance between the tips of the hands changing at 3 o'clock?

SOLUTION Let S be the distance between the tips of the two hands. By the law of cosines

$$S^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5\cos(\theta),$$

where θ is the angle between the hands. Thus

$$2S\frac{dS}{dt} = 80\sin(\theta)\frac{d\theta}{dt}.$$

At three o'clock $\theta = \pi/2$, $S = \sqrt{89}$, and

$$\frac{d\theta}{dt} = \left(\frac{\pi}{360} - \frac{\pi}{30}\right) \text{ rad/min} = -\frac{11\pi}{360} \text{ rad/min},$$

so

$$\frac{dS}{dt} = \frac{1}{2\sqrt{89}}(80)(1)\frac{-11\pi}{360} \approx -0.407 \text{ cm/min}.$$

120. Chloe and Bao are in motorboats at the center of a lake. At time $t = 0$, Chloe begins traveling south at a speed of 50 km/h. One minute later, Bao takes off, heading east at a speed of 40 km/h. At what rate is the distance between them increasing at $t = 12$ min?

SOLUTION Take the center of the lake to be origin of our coordinate system. Because Chloe travels at 50 km/h = $\frac{5}{6}$ km/min due south, her position at time $t > 0$ is $(0, \frac{5}{6}t)$; because Bao travels at 40 km/h = $\frac{2}{3}$ km/min due east, her position at time $t > 1$ is $(\frac{2}{3}(t-1), 0)$. Thus, the distance between the two motorboats at time $t > 1$ is

$$s = \sqrt{\frac{4}{9}(t-1)^2 + \frac{25}{36}t^2} = \frac{1}{6}\sqrt{41t^2 - 32t + 16},$$

and

$$\frac{ds}{dt} = \frac{41t - 16}{6\sqrt{41t^2 - 32t + 16}}.$$

At $t = 12$, it follows that

$$\frac{ds}{dt} = \frac{476}{6\sqrt{5536}} \approx 1.066 \text{ km/min}.$$

121. A bead slides down the curve $xy = 10$. Find the bead's horizontal velocity at time $t = 2$ s if its height at time t seconds is $y = 400 - 16t^2$ cm.

SOLUTION Let $xy = 10$. Then $x = 10/y$ and

$$\frac{dx}{dt} = -\frac{10}{y^2}\frac{dy}{dt}.$$

If $y = 400 - 16t^2$, then $\frac{dy}{dt} = -32t$ and

$$\frac{dx}{dt} = -\frac{10}{(400 - 16t^2)^2}(-32t) = \frac{320t}{(400 - 16t^2)^2}.$$

Thus, at $t = 2$,

$$\frac{dx}{dt} = \frac{640}{(336)^2} \approx 0.00567 \text{ cm/s}.$$

122. In Figure 8, x is increasing at 2 cm/s, y is increasing at 3 cm/s, and θ is decreasing such that the area of the triangle has the constant value 4 cm^2.

(a) How fast is θ decreasing when $x = 4$, $y = 4$?

(b) How fast is the distance between P and Q changing when $x = 4$, $y = 4$?

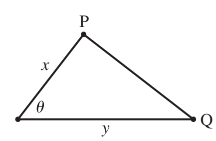


FIGURE 8

SOLUTION

(a) The area of the triangle is

$$A = \frac{1}{2}xy\sin\theta = 4.$$

Differentiating with respect to t, we obtain

$$\frac{dA}{dt} = \frac{1}{2}xy\cos\theta\frac{d\theta}{dt} + \frac{1}{2}y\sin\theta\frac{dx}{dt} + x\sin\theta\frac{dy}{dt} = 0.$$

When $x = y = 4$, we have $\frac{1}{2}(4)(4)\sin\theta = 4$, so $\sin\theta = \frac{1}{2}$. Thus, $\theta = \frac{\pi}{6}$ and

$$\frac{1}{2}(4)(4)\frac{\sqrt{3}}{2}\frac{d\theta}{dt} + \frac{1}{2}(4)\left(\frac{1}{2}\right)(2) + \frac{1}{2}(4)\left(\frac{1}{2}\right)(3) = 0.$$

Solving for $d\theta/dt$, we find

$$\frac{d\theta}{dt} = -\frac{5}{4\sqrt{3}} \approx -0.72 \text{ rad/s}.$$

(b) By the Law of Cosines, the distance D between P and Q satisfies

$$D^2 = x^2 + y^2 - 2xy\cos\theta,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2xy\sin\theta\frac{d\theta}{dt} - 2x\cos\theta\frac{dy}{dt} - 2y\cos\theta\frac{dx}{dt}.$$

With $x = y = 4$ and $\theta = \frac{\pi}{6}$,

$$D = \sqrt{4^2 + 4^2 - 2(4)(4)\frac{\sqrt{3}}{2}} = 4\sqrt{2-\sqrt{3}}.$$

Therefore,

$$\frac{dD}{dt} = \frac{16 + 24 - \frac{20}{\sqrt{3}} - 12\sqrt{3} - 8\sqrt{3}}{8\sqrt{2-\sqrt{3}}} \approx -1.50 \text{ cm/s}.$$

123. A light moving at 0.8 m/s approaches a man standing 4 m from a wall (Figure 9). The light is 1 m above the ground. How fast is the tip P of the man's shadow moving when the light is 7 m from the wall?

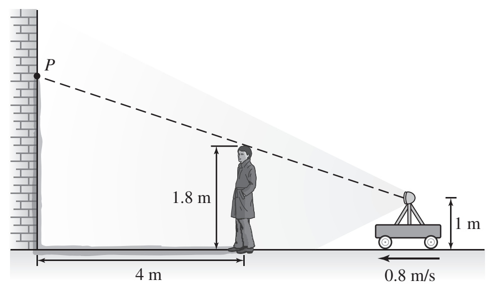


FIGURE 9

SOLUTION Let x denote the distance between the man and the light. Using similar triangles, we find

$$\frac{0.8}{x} = \frac{P-1}{4+x} \qquad \text{or} \qquad P = \frac{3.2}{x} + 1.8.$$

Therefore,

$$\frac{dP}{dt} = -\frac{3.2}{x^2}\frac{dx}{dt}.$$

When the light is 7 feet from the wall, $x = 3$. With $\frac{dx}{dt} = -0.8$, we have

$$\frac{dP}{dt} = -\frac{3.2}{3^2}(-0.8) = 0.284 \text{ m/s}.$$

4 APPLICATIONS OF THE DERIVATIVE

4.1 Linear Approximation and Applications

Preliminary Questions

1. True or False? The Linear Approximation says that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

SOLUTION This statement is true. The linear approximation does say that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

2. Estimate $g(1.2) - g(1)$ if $g'(1) = 4$.

SOLUTION Using the Linear Approximation,

$$g(1.2) - g(1) \approx g'(1)(1.2 - 1) = 4(0.2) = 0.8.$$

3. Estimate $f(2.1)$ if $f(2) = 1$ and $f'(2) = 3$.

SOLUTION Using the Linearization,

$$f(2.1) \approx f(2) + f'(2)(2.1 - 2) = 1 + 3(0.1) = 1.3$$

4. Complete the sentence: The Linear Approximation shows that up to a small error, the change in output Δf is directly proportional to ….

SOLUTION The Linear Approximation tells us that up to a small error, the change in output Δf is directly proportional to the change in input Δx when Δx is small.

Exercises

In Exercises 1–6, use Eq. (1) to estimate $\Delta f = f(3.02) - f(3)$.

1. $f(x) = x^2$

SOLUTION Let $f(x) = x^2$. Then $f'(x) = 2x$ and $\Delta f \approx f'(3)\Delta x = 6(0.02) = 0.12$.

2. $f(x) = x^4$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$ and $\Delta f \approx f'(3)\Delta x = 4(27)(0.02) = 2.16$.

3. $f(x) = x^{-1}$

SOLUTION Let $f(x) = x^{-1}$. Then $f'(x) = -x^{-2}$ and $\Delta f \approx f'(3)\Delta x = -\frac{1}{9}(0.02) = -0.00222$.

4. $f(x) = \dfrac{1}{x+1}$

SOLUTION Let $f(x) = (x+1)^{-1}$. Then $f'(x) = -(x+1)^{-2}$ and $\Delta f \approx f'(3)\Delta x = -\frac{1}{16}(0.02) = -0.00125$.

5. $f(x) = \sqrt{x+6}$

SOLUTION Let $f(x) = \sqrt{x+6}$. Then $f'(x) = \frac{1}{2}(x+6)^{-1/2}$ and

$$\Delta f \approx f'(3)\Delta x = \frac{1}{2}9^{-1/2}(0.02) = 0.003333.$$

6. $f(x) = \tan \dfrac{\pi x}{3}$

SOLUTION Let $f(x) = \tan \frac{\pi x}{3}$. Then $f'(x) = \frac{\pi}{3}\sec^2 \frac{\pi x}{3}$ and

$$\Delta f \approx f'(3)\Delta x = \frac{\pi}{3}(0.02) = 0.020944.$$

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

SOLUTION Let $f(x) = x^{1/3}$, $a = 27$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(27) = \frac{1}{27}$. The Linear Approximation is

$$\Delta f \approx f'(a)\Delta x = \frac{1}{27}(0.2) = 0.0074074$$

8. Estimate $\ln(e^3 + 0.1) - \ln(e^3)$ using differentials.

SOLUTION Let $f(x) = \ln x$, $a = e^3$, and $\Delta x = 0.1$. Then $f'(x) = x^{-1}$ and $f'(a) = e^{-3}$. Thus,

$$\ln(e^3 + 0.1) - \ln(e^3) = \Delta f \approx f'(a)\Delta x = e^{-3}(0.1) = 0.00498.$$

In Exercises 9–12, use Eq. (1) to estimate Δf. Use a calculator to compute both the error and the percentage error.

9. $f(x) = \sqrt{1+x}, \quad a = 3, \quad \Delta x = 0.2$

SOLUTION Let $f(x) = (1+x)^{1/2}$, $a = 3$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f'(a) = f'(3) = \frac{1}{4}$ and $\Delta f \approx f'(a)\Delta x = \frac{1}{4}(0.2) = 0.05$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(3.2) - f(3) = \sqrt{4.2} - 2 \approx 0.049390.$$

The error in the Linear Approximation is therefore $|0.049390 - 0.05| = 0.000610$; in percentage terms, the error is

$$\frac{0.000610}{0.049390} \times 100\% \approx 1.24\%.$$

10. $f(x) = 2x^2 - x, \quad a = 5, \quad \Delta x = -0.4$

SOLUTION Let $f(x) = 2x^2 - x$, $a = 5$ and $\Delta x = -0.4$. Then $f'(x) = 4x - 1$, $f'(a) = 19$ and $\Delta f \approx f'(a)\Delta x = 19(-0.4) = -7.6$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(4.6) - f(5) = 37.72 - 45 = -7.28.$$

The error in the Linear Approximation is therefore $|-7.28 - (-7.6)| = 0.32$; in percentage terms, the error is

$$\frac{0.32}{7.28} \times 100\% \approx 4.40\%.$$

11. $f(x) = \dfrac{1}{1+x^2}, \quad a = 3, \quad \Delta x = 0.5$

SOLUTION Let $f(x) = \frac{1}{1+x^2}$, $a = 3$, and $\Delta x = .5$. Then $f'(x) = -\frac{2x}{(1+x^2)^2}$, $f'(a) = f'(3) = -0.06$ and $\Delta f \approx f'(a)\Delta x = -0.06(0.5) = -0.03$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(3.5) - f(3) \approx -0.0245283.$$

The error in the Linear Approximation is therefore $|-0.0245283 - (-0.03)| = 0.0054717$; in percentage terms, the error is

$$\left|\frac{0.0054717}{-0.0245283}\right| \times 100\% \approx 22.31\%$$

12. $f(x) = \ln(x^2 + 1), \quad a = 1, \quad \Delta x = 0.1$

SOLUTION Let $f(x) = \ln(x^2 + 1)$, $a = 1$, and $\Delta x = 0.1$. Then $f'(x) = \frac{2x}{x^2+1}$, $f'(a) = f'(1) = 1$, and $\Delta f \approx f'(a)\Delta x = 1(0.1) = 0.1$. The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(1.1) - f(1) = 0.099845.$$

The error in the Linear Approximation is therefore $|0.099845 - 0.1| = 0.000155$; in percentage terms, the error is

$$\frac{0.000155}{0.099845} \times 100\% \approx 0.16\%.$$

In Exercises 13–16, estimate Δy using differentials [Eq. (3)].

13. $y = \cos x, \quad a = \frac{\pi}{6}, \quad dx = 0.014$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and

$$\Delta y \approx dy = f'(a)dx = -\sin\left(\frac{\pi}{6}\right)(0.014) = -0.007.$$

14. $y = \tan^2 x, \quad a = \frac{\pi}{4}, \quad dx = -0.02$

SOLUTION Let $f(x) = \tan^2 x$. Then $f'(x) = 2\tan x \sec^2 x$ and

$$\Delta y \approx dy = f'(a)dx = 2\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}(-0.02) = -0.08.$$

15. $y = \dfrac{10 - x^2}{2 + x^2}, \quad a = 1, \quad dx = 0.01$

SOLUTION Let $f(x) = \dfrac{10 - x^2}{2 + x^2}$. Then

$$f'(x) = \frac{(2 + x^2)(-2x) - (10 - x^2)(2x)}{(2 + x^2)^2} = -\frac{24x}{(2 + x^2)^2}$$

and

$$\Delta y \approx dy = f'(a)dx = -\frac{24}{9}(0.01) = -0.026667.$$

16. $y = x^{1/3}e^{x-1}, \quad a = 1, \quad dx = 0.1$

SOLUTION Let $y = x^{1/3}e^{x-1}$, $a = 1$, and $dx = 0.1$. Then $y'(x) = \frac{1}{3}x^{-2/3}e^{x-1}(3x + 1)$, $y'(a) = y'(1) = \frac{4}{3}$, and $\Delta y \approx dy = y'(a)\,dx = \frac{4}{3}(0.1) = 0.133333$.

In Exercises 17–24, estimate using the Linear Approximation and find the error using a calculator.

17. $\sqrt{26} - \sqrt{25}$

SOLUTION Let $f(x) = \sqrt{x}$, $a = 25$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(25) = \frac{1}{10}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{10}(1) = 0.1$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(26) - f(25) \approx 0.0990195$.
- The error in this estimate is $|0.0990195 - 0.1| = 0.000980486$.

18. $16.5^{1/4} - 16^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 16$, and $\Delta x = .5$. Then $f'(x) = \frac{1}{4}x^{-3/4}$ and $f'(a) = f'(16) = \frac{1}{32}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{32}(0.5) = 0.015625$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(16.5) - f(16) \approx 2.015445 - 2 = 0.015445$$

- The error in this estimate is $|0.015625 - 0.015445| \approx 0.00018$.

19. $\dfrac{1}{\sqrt{101}} - \dfrac{1}{10}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}}$, $a = 100$, and $\Delta x = 1$. Then $f'(x) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = -\frac{1}{2}(\frac{1}{1000}) = -0.0005$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = -0.0005(1) = -0.0005$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = \frac{1}{\sqrt{101}} - \frac{1}{10} = -0.000496281.$$

- The error in this estimate is $|-0.0005 - (-0.000496281)| = 3.71902 \times 10^{-6}$.

20. $\dfrac{1}{\sqrt{98}} - \dfrac{1}{10}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}}$, $a = 100$, and $\Delta x = -2$. Then $f'(x) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = -\frac{1}{2}(\frac{1}{1000}) = -0.0005$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = -0.0005(-2) = 0.001$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(98) - f(100) = 0.00101525$.
- The error in this estimate is $|0.001 - 0.00101525| \approx 0.00001525$.

21. $9^{1/3} - 2$

SOLUTION Let $f(x) = x^{1/3}$, $a = 8$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(8) = \frac{1}{12}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{12}(1) = 0.083333$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(9) - f(8) = 0.080084$.
- The error in this estimate is $|0.080084 - 0.083333| \approx 3.25 \times 10^{-3}$.

22. $\tan^{-1}(1.05) - \frac{\pi}{4}$

SOLUTION Let $f(x) = \tan^{-1} x$, $a = 1$, and $\Delta x = 0.05$. Then $f'(x) = (1 + x^2)^{-1}$ and $f'(a) = f'(1) = \frac{1}{2}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{2}(0.05) = 0.025$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(1.05) - f(1) = 0.024385$.
- The error in this estimate is $|0.024385 - 0.025| \approx 6.15 \times 10^{-4}$.

23. $e^{-0.1} - 1$

SOLUTION Let $f(x) = e^x$, $a = 0$, and $\Delta x = -0.1$. Then $f'(x) = e^x$ and $f'(a) = f'(0) = 1$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = 1(-0.1) = -0.1$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(-0.1) - f(0) = -0.095163$.
- The error in this estimate is $|-0.095163 - (-0.1)| \approx 4.84 \times 10^{-3}$.

24. $\ln(0.97)$

SOLUTION Let $f(x) = \ln x$, $a = 1$, and $\Delta x = -0.03$. Then $f'(x) = \frac{1}{x}$ and $f'(a) = f'(1) = 1$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = (1)(-0.03) = -0.03$, so $\ln(0.97) \approx \ln 1 - 0.03 = -0.03$.
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(0.97) - f(1) \approx -0.030459 - 0 = -0.030459.$$

- The error is $|\Delta f - f'(a)\Delta x| \approx 0.000459$.

25. Estimate $f(4.03)$ for $f(x)$ as in Figure 8.

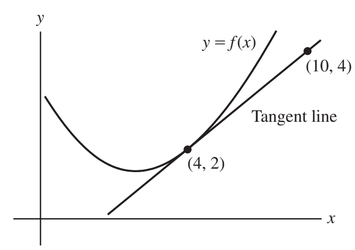


FIGURE 8

SOLUTION Using the Linear Approximation, $f(4.03) \approx f(4) + f'(4)(0.03)$. From the figure, we find that $f(4) = 2$ and

$$f'(4) = \frac{4 - 2}{10 - 4} = \frac{1}{3}.$$

Thus,

$$f(4.03) \approx 2 + \frac{1}{3}(0.03) = 2.01.$$

26. At a certain moment, an object in linear motion has velocity 100 m/s. Estimate the distance traveled over the next quarter-second, and explain how this is an application of the Linear Approximation.

SOLUTION Because the velocity is 100 m/s, we estimate the object will travel

$$\left(100\ \frac{\text{m}}{\text{s}}\right)\left(\frac{1}{4}\ \text{s}\right) = 25\ \text{m}$$

in the next quarter-second. Recall that velocity is the derivative of position, so we have just estimated the change in position, Δs, using the product $s'\Delta t$, which is just the Linear Approximation.

27. Which is larger: $\sqrt{2.1} - \sqrt{2}$ or $\sqrt{9.1} - \sqrt{9}$? Explain using the Linear Approximation.

SOLUTION Let $f(x) = \sqrt{x}$, and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and the Linear Approximation at $x = a$ gives

$$\Delta f = \sqrt{a + 0.1} - \sqrt{a} \approx f'(a)(0.1) = \frac{1}{2}a^{-1/2}(0.1) = \frac{0.05}{\sqrt{a}}$$

We see that Δf decreases as a increases. In particular

$$\sqrt{2.1} - \sqrt{2} \approx \frac{0.05}{\sqrt{2}} \qquad \text{is larger than} \qquad \sqrt{9.1} - \sqrt{9} \approx \frac{0.05}{3}$$

28. Estimate $\sin 61^\circ - \sin 60^\circ$ using the Linear Approximation. *Hint:* Express $\Delta\theta$ in radians.

SOLUTION Let $f(x) = \sin x$, $a = \frac{\pi}{3}$, and $\Delta x = \frac{\pi}{180}$. Then $f'(x) = \cos x$ and $f'(a) = f'(\frac{\pi}{3}) = \frac{1}{2}$. Finally, the Linear Approximation is

$$\Delta f \approx f'(a)\Delta x = \frac{1}{2}\left(\frac{\pi}{180}\right) = \frac{\pi}{360} \approx 0.008727$$

29. Box office revenue at a multiplex cinema in Paris is $R(p) = 3600p - 10p^3$ euros per showing when the ticket price is p euros. Calculate $R(p)$ for $p = 9$ and use the Linear Approximation to estimate ΔR if p is raised or lowered by 0.5 euros.

SOLUTION Let $R(p) = 3600p - 10p^3$. Then $R(9) = 3600(9) - 10(9)^3 = 25110$ euros. Moreover, $R'(p) = 3600 - 30p^2$, so by the Linear Approximation,

$$\Delta R \approx R'(9)\Delta p = 1170\Delta p.$$

If p is raised by 0.5 euros, then $\Delta R \approx 585$ euros; on the other hand, if p is lowered by 0.5 euros, then $\Delta R \approx -585$ euros.

30. The *stopping distance* for an automobile is $F(s) = 1.1s + 0.054s^2$ ft, where s is the speed in mph. Use the Linear Approximation to estimate the change in stopping distance per additional mph when $s = 35$ and when $s = 55$.

SOLUTION Let $F(s) = 1.1s + 0.054s^2$.

- The Linear Approximation at $s = 35$ mph is

$$\Delta F \approx F'(35)\Delta s = (1.1 + 0.108 \times 35)\Delta s = 4.88\Delta s \text{ ft}$$

  The change in stopping distance per additional mph for $s = 35$ mph is approximately 4.88 ft.
- The Linear Approximation at $s = 55$ mph is

$$\Delta F \approx F'(55)\Delta s = (1.1 + 0.108 \times 55)\Delta s = 7.04\Delta s \text{ ft}$$

  The change in stopping distance per additional mph for $s = 55$ mph is approximately 7.04 ft.

31. A thin silver wire has length $L = 18$ cm when the temperature is $T = 30^\circ$C. Estimate ΔL when T decreases to 25°C if the coefficient of thermal expansion is $k = 1.9 \times 10^{-5}{}^\circ\text{C}^{-1}$ (see Example 3).

SOLUTION We have

$$\frac{dL}{dT} = kL = (1.9 \times 10^{-5})(18) = 3.42 \times 10^{-4} \text{ cm}/^\circ\text{C}$$

The change in temperature is $\Delta T = -5^\circ$ C, so by the Linear Approximation, the change in length is approximately

$$\Delta L \approx 3.42 \times 10^{-4}\Delta T = (3.42 \times 10^{-4})(-5) = -0.00171 \text{ cm}$$

At $T = 25^\circ$ C, the length of the wire is approximately 17.99829 cm.

32. At a certain moment, the temperature in a snake cage satisfies $dT/dt = 0.008^\circ$C/s. Estimate the rise in temperature over the next 10 seconds.

SOLUTION Using the Linear Approximation, the rise in temperature over the next 10 seconds will be

$$\Delta T \approx \frac{dT}{dt}\Delta t = 0.008(10) = 0.08^\circ\text{C}.$$

33. The atmospheric pressure at altitude h (kilometers) for $11 \le h \le 25$ is approximately

$$P(h) = 128e^{-0.157h} \text{ kilopascals.}$$

(a) Estimate ΔP at $h = 20$ when $\Delta h = 0.5$.

(b) Compute the actual change, and compute the percentage error in the Linear Approximation.

SOLUTION

(a) Let $P(h) = 128e^{-0.157h}$. Then $P'(h) = -20.096e^{-0.157h}$. Using the Linear Approximation,

$$\Delta P \approx P'(h)\Delta h = P'(20)(0.5) = -0.434906 \text{ kilopascals.}$$

(b) The actual change in pressure is

$$P(20.5) - P(20) = -0.418274 \text{ kilopascals.}$$

The percentage error in the Linear Approximation is

$$\left|\frac{-0.434906 - (-0.418274)}{-0.418274}\right| \times 100\% \approx 3.98\%.$$

34. The resistance R of a copper wire at temperature $T = 20°\text{C}$ is $R = 15\ \Omega$. Estimate the resistance at $T = 22°\text{C}$, assuming that $dR/dT\big|_{T=20} = 0.06\ \Omega/°\text{C}$.

SOLUTION $\Delta T = 2°\text{C}$. The Linear Approximation gives us:

$$R(22) - R(20) \approx dR/dT\Big|_{T=20} \Delta T = 0.06\ \Omega/°\text{C}(2°\text{C}) = 0.12\ \Omega.$$

Therefore, $R(22) \approx 15\ \Omega + 0.12\ \Omega = 15.12\ \Omega$.

35. Newton's Law of Gravitation shows that if a person weighs w pounds on the surface of the earth, then his or her weight at distance x from the center of the earth is

$$W(x) = \frac{wR^2}{x^2} \qquad (\text{for } x \geq R)$$

where $R = 3960$ miles is the radius of the earth (Figure 9).

(a) Show that the weight lost at altitude h miles above the earth's surface is approximately $\Delta W \approx -(0.0005w)h$. *Hint:* Use the Linear Approximation with $dx = h$.

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

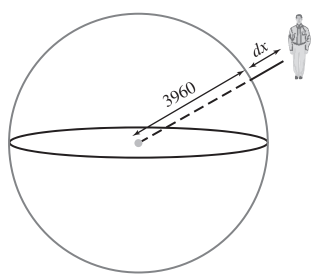


FIGURE 9 The distance to the center of the earth is $3960 + h$ miles.

SOLUTION

(a) Using the Linear Approximation

$$\Delta W \approx W'(R)\Delta x = -\frac{2wR^2}{R^3}h = -\frac{2wh}{R} \approx -0.0005wh.$$

(b) Substitute $w = 200$ and $h = 7$ into the result from part (a) to obtain

$$\Delta W \approx -0.0005(200)(7) = -0.7 \text{ pounds.}$$

36. Using Exercise 35(a), estimate the altitude at which a 130-lb pilot would weigh 129.5 lb.

SOLUTION From Exercise 35(a), the weight loss ΔW at altitude h (in miles) for a person weighing w at the surface of the earth is approximately

$$\Delta W \approx -0.0005wh$$

If $w = 130$ pounds, then $\Delta W \approx -0.065h$. Accordingly, the pilot loses approximately 0.065 pounds per mile of altitude gained. The pilot will weigh 129.5 pounds at the altitude h such that $-0.065h = -0.5$, or $h = 0.5/0.065 \approx 7.7$ miles.

37. A stone tossed vertically into the air with initial velocity v cm/s reaches a maximum height of $h = v^2/1960$ cm.

(a) Estimate Δh if $v = 700$ cm/s and $\Delta v = 1$ cm/s.

(b) Estimate Δh if $v = 1{,}000$ cm/s and $\Delta v = 1$ cm/s.

(c) In general, does a 1 cm/s increase in v lead to a greater change in h at low or high initial velocities? Explain.

SOLUTION A stone tossed vertically with initial velocity v cm/s attains a maximum height of $h(v) = v^2/1960$ cm. Thus, $h'(v) = v/980$.

(a) If $v = 700$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{980}(700)(1) \approx 0.71$ cm.

(b) If $v = 1000$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{980}(1000)(1) = 1.02$ cm.

(c) A one centimeter per second increase in initial velocity v increases the maximum height by approximately $v/980$ cm. Accordingly, there is a bigger effect at higher velocities.

38. The side s of a square carpet is measured at 6 m. Estimate the maximum error in the area A of the carpet if s is accurate to within 2 centimeters.

SOLUTION Let s be the length in meters of the side of the square carpet. Then $A(s) = s^2$ is the area of the carpet. With $a = 6$ and $\Delta s = 0.02$ (note that 1 cm equals 0.01 m), an estimate of the size of the error in the area is given by the Linear Approximation:

$$\Delta A \approx A'(6)\Delta s = 12\,(0.02) = 0.24 \text{ m}^2$$

In Exercises 39 and 40, use the following fact derived from Newton's Laws: An object released at an angle θ with initial velocity v ft/s *travels a horizontal distance*

$$s = \frac{1}{32}v^2 \sin 2\theta \text{ ft} \quad \text{(Figure 10)}$$

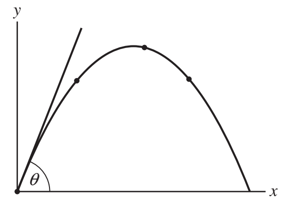


FIGURE 10 Trajectory of an object released at an angle θ.

39. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle $\theta = 34^\circ$ and initial velocity $v = 25$ ft/s.)

(a) Show that $\Delta s \approx 0.255\Delta\theta$ ft for a small change of $\Delta\theta$.

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

SOLUTION Using Newton's laws and the given initial velocity of $v = 25$ ft/s, the shot travels $s = \frac{1}{32}v^2 \sin 2t = \frac{625}{32}\sin 2t$ ft, where t is in radians.

(a) If $\theta = 34^\circ$ (i.e., $t = \frac{17}{90}\pi$), then

$$\Delta s \approx s'(t)\Delta t = \frac{625}{16}\cos\left(\frac{17}{45}\pi\right)\Delta t = \frac{625}{16}\cos\left(\frac{17}{45}\pi\right)\Delta\theta \cdot \frac{\pi}{180} \approx 0.255\Delta\theta.$$

(b) If $\Delta\theta = 2^\circ$, this gives $\Delta s \approx 0.51$ ft, in which case the shot would not have been successful, having been off half a foot.

40. Estimate Δs if $\theta = 34^\circ$, $v = 25$ ft/s, and $\Delta v = 2$.

SOLUTION Using Newton's laws and the fixed angle of $\theta = 34^\circ = \frac{17}{90}\pi$, the shot travels

$$s = \frac{1}{32}v^2 \sin\frac{17}{45}\pi.$$

With $v = 25$ ft/s and $\Delta v = 2$ ft/s, we find

$$\Delta s \approx s'(v)\Delta v = \frac{1}{16}(25)\sin\frac{17\pi}{45} \cdot 2 = 2.897 \text{ ft}.$$

41. The radius of a spherical ball is measured at $r = 25$ cm. Estimate the maximum error in the volume and surface area if r is accurate to within 0.5 cm.

SOLUTION The volume and surface area of the sphere are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, respectively. If $r = 25$ and $\Delta r = \pm 0.5$, then

$$\Delta V \approx V'(25)\Delta r = 4\pi(25)^2(0.5) \approx 3927 \text{ cm}^3,$$

and

$$\Delta S \approx S'(25)\Delta r = 8\pi(25)(0.5) \approx 314.2 \text{ cm}^2.$$

42. The dosage D of diphenhydramine for a dog of body mass w kg is $D = 4.7w^{2/3}$ mg. Estimate the maximum allowable error in w for a cocker spaniel of mass $w = 10$ kg if the percentage error in D must be less than 3%.

SOLUTION We have $D = kw^{2/3}$ where $k = 4.7$. The Linear Approximation yields

$$\Delta D \approx \frac{2}{3}kw^{-1/3}\Delta w,$$

so

$$\frac{\Delta D}{D} \approx \frac{\frac{2}{3}kw^{-1/3}\Delta w}{kw^{2/3}} = \frac{2}{3} \cdot \frac{\Delta w}{w}$$

If the percentage error in D must be less than 3%, we estimate the maximum allowable error in w to be

$$\Delta w \approx \frac{3w}{2} \cdot \frac{\Delta D}{D} = \frac{3(10)}{2}(.03) = 0.45 \text{ kg}$$

43. The volume (in liters) and pressure P (in atmospheres) of a certain gas satisfy $PV = 24$. A measurement yields $V = 4$ with a possible error of ± 0.3 L. Compute P and estimate the maximum error in this computation.

SOLUTION Given $PV = 24$ and $V = 4$, it follows that $P = 6$ atmospheres. Solving $PV = 24$ for P yields $P = 24V^{-1}$. Thus, $P' = -24V^{-2}$ and

$$\Delta P \approx P'(4)\Delta V = -24(4)^{-2}(\pm 0.3) = \pm 0.45 \text{ atmospheres.}$$

44. In the notation of Exercise 43, assume that a measurement yields $V = 4$. Estimate the maximum allowable error in V if P must have an error of less than 0.2 atm.

SOLUTION From Exercise 43, with $V = 4$, we have

$$\Delta P \approx -\frac{3}{2}\Delta V \quad \text{or} \quad \Delta V = -\frac{2}{3}\Delta P.$$

If we require $|\Delta P| \le 0.2$, then we must have

$$|\Delta V| \le \frac{2}{3}(0.2) = 0.133333 \text{ L.}$$

In Exercises 45–54, find the linearization at $x = a$.

45. $f(x) = x^4, \quad a = 1$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$. The linearization at $a = 1$ is

$$L(x) = f'(a)(x - a) + f(a) = 4(x - 1) + 1 = 4x - 3.$$

46. $f(x) = \dfrac{1}{x}, \quad a = 2$

SOLUTION Let $f(x) = \frac{1}{x} = x^{-1}$. Then $f'(x) = -x^{-2}$. The linearization at $a = 2$ is

$$L(x) = f'(a)(x - a) + f(a) = -\frac{1}{4}(x - 2) + \frac{1}{2} = -\frac{1}{4}x + 1.$$

47. $f(\theta) = \sin^2\theta, \quad a = \frac{\pi}{4}$

SOLUTION Let $f(\theta) = \sin^2\theta$. Then $f'(\theta) = 2\sin\theta\cos\theta = \sin 2\theta$. The linearization at $a = \frac{\pi}{4}$ is

$$L(\theta) = f'(a)(\theta - a) + f(a) = 1\left(\theta - \frac{\pi}{4}\right) + \frac{1}{2} = \theta - \frac{\pi}{4} + \frac{1}{2}.$$

48. $g(x) = \dfrac{x^2}{x - 3}, \quad a = 4$

SOLUTION Let $g(x) = \frac{x^2}{x-3}$. Then

$$g'(x) = \frac{(x - 3)(2x) - x^2}{(x - 3)^2} = \frac{x^2 - 6x}{(x - 3)^2}.$$

The linearization at $a = 4$ is

$$L(x) = g'(a)(x - a) + g(a) = -8(x - 4) + 16 = -8x + 48.$$

49. $y = (1+x)^{-1/2}$, $a = 0$

SOLUTION Let $f(x) = (1+x)^{-1/2}$. Then $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$. The linearization at $a = 0$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{2}x + 1.$$

50. $y = (1+x)^{-1/2}$, $a = 3$

SOLUTION Let $f(x) = (1+x)^{-1/2}$. Then $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$, $f(a) = 4^{-1/2} = \frac{1}{2}$, and $f'(a) = -\frac{1}{2}(4^{-3/2}) = -\frac{1}{16}$, so the linearization at $a = 3$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-3) + \frac{1}{2} = -\frac{1}{16}x + \frac{11}{16}.$$

51. $y = (1+x^2)^{-1/2}$, $a = 0$

SOLUTION Let $f(x) = (1+x^2)^{-1/2}$. Then $f'(x) = -x(1+x^2)^{-3/2}$, $f(a) = 1$ and $f'(a) = 0$, so the linearization at a is

$$L(x) = f'(a)(x-a) + f(a) = 1.$$

52. $y = \tan^{-1} x$, $a = 1$

SOLUTION Let $f(x) = \tan^{-1} x$. Then

$$f'(x) = \frac{1}{1+x^2}, \quad f(a) = \frac{\pi}{4}, \text{ and } f'(a) = \frac{1}{2},$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{2}(x-1) + \frac{\pi}{4}.$$

53. $y = e^{\sqrt{x}}$, $a = 1$

SOLUTION Let $f(x) = e^{\sqrt{x}}$. Then

$$f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}, \quad f(a) = e, \text{ and } f'(a) = \frac{1}{2}e,$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{2}e(x-1) + e = \frac{1}{2}e(x+1).$$

54. $y = e^x \ln x$, $a = 1$

SOLUTION Let $f(x) = e^x \ln x$. Then

$$f'(x) = \frac{e^x}{x} + e^x \ln x, \quad f(a) = 0, \text{ and } f'(a) = e,$$

so the linearization of $f(x)$ at a is

$$L(x) = f'(a)(x-a) + f(a) = e(x-1).$$

55. What is $f(2)$ if the linearization of $f(x)$ at $a = 2$ is $L(x) = 2x + 4$?

SOLUTION $f(2) = L(2) = 2(2) + 4 = 8$.

56. Compute the linearization of $f(x) = 3x - 4$ at $a = 0$ and $a = 2$. Prove more generally that a linear function coincides with its linearization at $x = a$ for all a.

SOLUTION Let $f(x) = 3x - 4$. Then $f'(x) = 3$. With $a = 0$, $f(a) = -4$ and $f'(a) = 3$, so the linearization of $f(x)$ at $a = 0$ is

$$L(x) = -4 + 3(x-0) = 3x - 4 = f(x).$$

With $a = 2$, $f(a) = 2$ and $f'(a) = 3$, so the linearization of $f(x)$ at $a = 2$ is

$$L(x) = 2 + 3(x-2) = 2 + 3x - 6 = 3x - 4 = f(x).$$

More generally, let $g(x) = bx + c$ be any linear function. The linearization $L(x)$ of $g(x)$ at $x = a$ is

$$L(x) = g'(a)(x-a) + g(a) = b(x-a) + ba + c = bx + c = g(x);$$

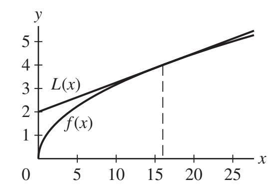
i.e., $L(x) = g(x)$.

57. Estimate $\sqrt{16.2}$ using the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 16$. Plot $f(x)$ and $L(x)$ on the same set of axes and determine whether the estimate is too large or too small.

SOLUTION Let $f(x) = x^{1/2}$, $a = 16$, and $\Delta x = 0.2$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(16) = \frac{1}{8}$. The linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{8}(x-16) + 4 = \frac{1}{8}x + 2.$$

Thus, we have $\sqrt{16.2} \approx L(16.2) = 4.025$. Graphs of $f(x)$ and $L(x)$ are shown below. Because the graph of $L(x)$ lies above the graph of $f(x)$, we expect that the estimate from the Linear Approximation is too large.



58. $\boxed{\text{GU}}$ Estimate $1/\sqrt{15}$ using a suitable linearization of $f(x) = 1/\sqrt{x}$. Plot $f(x)$ and $L(x)$ on the same set of axes and determine whether the estimate is too large or too small. Use a calculator to compute the percentage error.

SOLUTION The nearest perfect square to 15 is 16. Let $f(x) = \frac{1}{\sqrt{x}}$ and $a = 16$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = f'(16) = -\frac{1}{128}$. The linearization is

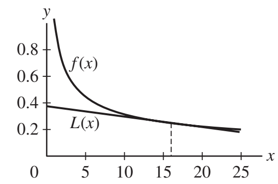
$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{128}(x-16) + \frac{1}{4}.$$

Then

$$\frac{1}{\sqrt{15}} \approx L(15) = -\frac{1}{128}(-1) + \frac{1}{4} = \frac{33}{128} = 0.257813.$$

Graphs of $f(x)$ and $L(x)$ are shown below. Because the graph of $L(x)$ lies below the graph of $f(x)$, we expect that the estimate from the Linear Approximation is too small. The percentage error in the estimate is

$$\left|\frac{\frac{1}{\sqrt{15}} - 0.257813}{\frac{1}{\sqrt{15}}}\right| \times 100\% \approx 0.15\%$$



In Exercises 59–67, approximate using linearization and use a calculator to compute the percentage error.

59. $\dfrac{1}{\sqrt{17}}$

SOLUTION Let $f(x) = x^{-1/2}$, $a = 16$, and $\Delta x = 1$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(a) = f'(16) = -\frac{1}{128}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = -\frac{1}{128}(x-16) + \frac{1}{4} = -\frac{1}{128}x + \frac{3}{8}.$$

Thus, we have $\frac{1}{\sqrt{17}} \approx L(17) \approx 0.24219$. The percentage error in this estimate is

$$\left|\frac{\frac{1}{\sqrt{17}} - 0.24219}{\frac{1}{\sqrt{17}}}\right| \times 100\% \approx 0.14\%$$

60. $\dfrac{1}{101}$

SOLUTION Let $f(x) = x^{-1}$, $a = 100$ and $\Delta x = 1$. Then $f'(x) = -x^{-2}$, $f'(a) = f'(100) = -0.0001$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = -0.0001(x-100) + 0.01 = -0.0001x + 0.02.$$

Thus, we have

$$\frac{1}{101} \approx L(101) = -0.0001(101) + 0.02 = 0.0099.$$

The percentage error in this estimate is

$$\left|\frac{\frac{1}{101} - 0.0099}{\frac{1}{101}}\right| \times 100\% \approx 0.01\%$$

61. $\dfrac{1}{(10.03)^2}$

SOLUTION Let $f(x) = x^{-2}$, $a = 10$ and $\Delta x = 0.03$. Then $f'(x) = -2x^{-3}$, $f'(a) = f'(10) = -0.002$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = -0.002(x-10) + 0.01 = -0.002x + 0.03.$$

Thus, we have

$$\frac{1}{(10.03)^2} \approx L(10.03) = -0.002(10.03) + 0.03 = 0.00994.$$

The percentage error in this estimate is

$$\left|\frac{\frac{1}{(10.03)^2} - 0.00994}{\frac{1}{(10.03)^2}}\right| \times 100\% \approx 0.0027\%$$

62. $(17)^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 16$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(a) = f'(16) = \frac{1}{32}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{32}(x-16) + 2 = \frac{1}{32}x + \frac{3}{2}.$$

Thus, we have $(17)^{1/4} \approx L(17) = 2.03125$. The percentage error in this estimate is

$$\left|\frac{(17)^{1/4} - 2.03125}{(17)^{1/4}}\right| \times 100\% \approx 0.035\%$$

63. $(64.1)^{1/3}$

SOLUTION Let $f(x) = x^{1/3}$, $a = 64$, and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(a) = f'(64) = \frac{1}{48}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{48}(x-64) + 4 = \frac{1}{48}x + \frac{8}{3}.$$

Thus, we have $(64.1)^{1/3} \approx L(64.1) \approx 4.002083$. The percentage error in this estimate is

$$\left|\frac{(64.1)^{1/3} - 4.002083}{(64.1)^{1/3}}\right| \times 100\% \approx 0.000019\%$$

64. $(1.2)^{5/3}$

SOLUTION Let $f(x) = (1+x)^{5/3}$ and $a = 0$. Then $f'(x) = \frac{5}{3}(1+x)^{2/3}$, $f'(a) = f'(0) = \frac{5}{3}$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x-a) + f(a) = \frac{5}{3}x + 1.$$

Thus, we have $(1.2)^{5/3} \approx L(0.2) = \frac{5}{3}(0.2) + 1 = 1.3333$. The percentage error in this estimate is

$$\left| \frac{(1.2)^{5/3} - 1.3333}{(1.2)^{5/3}} \right| \times 100\% \approx 1.61\%$$

65. $\cos^{-1}(0.52)$

SOLUTION Let $f(x) = \cos^{-1} x$ and $a = 0.5$. Then

$$f'(x) = -\frac{1}{\sqrt{1 - x^2}}, \quad f'(a) = f'(0) = -\frac{2\sqrt{3}}{3},$$

and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = -\frac{2\sqrt{3}}{3}(x - 0.5) + \frac{\pi}{3}.$$

Thus, we have $\cos^{-1}(0.52) \approx L(0.02) = 1.024104$. The percentage error in this estimate is

$$\left| \frac{\cos^{-1}(0.52) - 1.024104}{\cos^{-1}(0.52)} \right| \times 100\% \approx 0.015\%.$$

66. $\ln 1.07$

SOLUTION Let $f(x) = \ln(1 + x)$ and $a = 0$. Then $f'(x) = \frac{1}{1+x}$, $f'(a) = f'(0) = 1$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = x.$$

Thus, we have $\ln 1.07 \approx L(0.07) = 0.07$. The percentage error in this estimate is

$$\left| \frac{\ln 1.07 - 0.07}{\ln 1.07} \right| \times 100\% \approx 3.46\%.$$

67. $e^{-0.012}$

SOLUTION Let $f(x) = e^x$ and $a = 0$. Then $f'(x) = e^x$, $f'(a) = f'(0) = 1$ and the linearization to $f(x)$ is

$$L(x) = f'(a)(x - a) + f(a) = 1(x - 0) + 1 = x + 1.$$

Thus, we have $e^{-0.012} \approx L(-0.012) = 1 - 0.012 = 0.988$. The percentage error in this estimate is

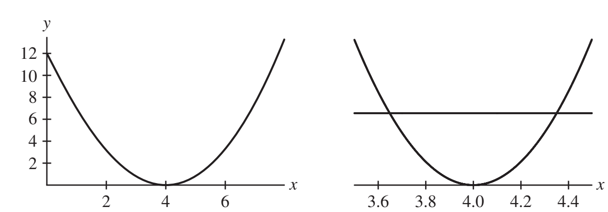
$$\left| \frac{e^{-0.012} - 0.988}{e^{-0.012}} \right| \times 100\% \approx 0.0073\%.$$

68. GU Compute the linearization $L(x)$ of $f(x) = x^2 - x^{3/2}$ at $a = 4$. Then plot $f(x) - L(x)$ and find an interval I around $a = 4$ such that $|f(x) - L(x)| \le 0.1$ for $x \in I$.

SOLUTION Let $f(x) = x^2 - x^{3/2}$ and $a = 4$. Then $f'(x) = 2x - \frac{3}{2}x^{1/2}$, $f'(4) = 5$ and

$$L(x) = f(a) + f'(a)(x - a) = 8 + 5(x - 4) = 5x - 12.$$

The graph of $y = f(x) - L(x)$ is shown below at the left, and portions of the graphs of $y = f(x) - L(x)$ and $y = 0.1$ are shown below at the right. From the graph on the right, we see that $|f(x) - L(x)| < 0.1$ roughly for $3.6 < x < 4.4$.



69. Show that the Linear Approximation to $f(x) = \sqrt{x}$ at $x = 9$ yields the estimate $\sqrt{9 + h} - 3 \approx \frac{1}{6}h$. Set $K = 0.01$ and show that $|f''(x)| \le K$ for $x \ge 9$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \le n \le 4$.

SOLUTION Let $f(x) = \sqrt{x}$. Then $f(9) = 3$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(9) = \frac{1}{6}$. Therefore, by the Linear Approximation,

$$f(9+h) - f(9) = \sqrt{9+h} - 3 \approx \frac{1}{6}h.$$

Moreover, $f''(x) = -\frac{1}{4}x^{-3/2}$, so $|f''(x)| = \frac{1}{4}x^{-3/2}$. Because this is a decreasing function, it follows that for $x \geq 9$,

$$K = \max |f''(x)| \leq |f''(9)| = \frac{1}{108} < 0.01.$$

From the following table, we see that for $h = 10^{-n}$, $1 \leq n \leq 4$, $E \leq \frac{1}{2}Kh^2$.

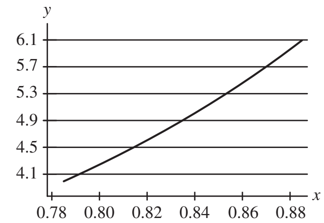
h	$E = \lvert\sqrt{9+h} - 3 - \frac{1}{6}h\rvert$	$\frac{1}{2}Kh^2$
10^{-1}	4.604×10^{-5}	5.00×10^{-5}
10^{-2}	4.627×10^{-7}	5.00×10^{-7}
10^{-3}	4.629×10^{-9}	5.00×10^{-9}
10^{-4}	4.627×10^{-11}	5.00×10^{-11}

70. GU The Linear Approximation to $f(x) = \tan x$ at $x = \frac{\pi}{4}$ yields the estimate $\tan\left(\frac{\pi}{4} + h\right) - 1 \approx 2h$. Set $K = 6.2$ and show, using a plot, that $|f''(x)| \leq K$ for $x \in [\frac{\pi}{4}, \frac{\pi}{4} + 0.1]$. Then verify numerically that the error E satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$.

SOLUTION Let $f(x) = \tan x$. Then $f(\frac{\pi}{4}) = 1$, $f'(x) = \sec^2 x$ and $f'(\frac{\pi}{4}) = 2$. Therefore, by the Linear Approximation,

$$f\left(\frac{\pi}{4} + h\right) - f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4} + h\right) - 1 \approx 2h.$$

Moreover, $f''(x) = 2\sec^2 x \tan x$. The graph of the second derivative over the interval $[\pi/4, \pi/4 + 0.1]$ is shown below. From this graph we see that $K = \max |f''(x)| \approx 6.1 < 6.2$.



Finally, from the following table, we see that for $h = 10^{-n}$, $1 \leq n \leq 4$, $E \leq \frac{1}{2}Kh^2$.

h	$E = \lvert\tan(\frac{\pi}{4} + h) - 1 - 2h\rvert$	$\frac{1}{2}Kh^2$
10^{-1}	2.305×10^{-2}	3.10×10^{-2}
10^{-2}	2.027×10^{-4}	3.10×10^{-4}
10^{-3}	2.003×10^{-6}	3.10×10^{-6}
10^{-4}	2.000×10^{-8}	3.10×10^{-8}

Further Insights and Challenges

71. Compute dy/dx at the point $P = (2, 1)$ on the curve $y^3 + 3xy = 7$ and show that the linearization at P is $L(x) = -\frac{1}{3}x + \frac{5}{3}$. Use $L(x)$ to estimate the y-coordinate of the point on the curve where $x = 2.1$.

SOLUTION Differentiating both sides of the equation $y^3 + 3xy = 7$ with respect to x yields

$$3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0,$$

so

$$\frac{dy}{dx} = -\frac{y}{y^2 + x}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{1}{1^2 + 2} = -\frac{1}{3},$$

and the linearization at $P = (2, 1)$ is

$$L(x) = 1 - \frac{1}{3}(x - 2) = -\frac{1}{3}x + \frac{5}{3}.$$

Finally, when $x = 2.1$, we estimate that the y-coordinate of the point on the curve is

$$y \approx L(2.1) = -\frac{1}{3}(2.1) + \frac{5}{3} = 0.967.$$

72. Apply the method of Exercise 71 to $P = (0.5, 1)$ on $y^5 + y - 2x = 1$ to estimate the y-coordinate of the point on the curve where $x = 0.55$.

SOLUTION Differentiating both sides of the equation $y^5 + y - 2x = 1$ with respect to x yields

$$5y^4\frac{dy}{dx} + \frac{dy}{dx} - 2 = 0,$$

so

$$\frac{dy}{dx} = \frac{2}{5y^4 + 1}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(0.5,1)} = \frac{2}{5(1)^2 + 1} = \frac{1}{3},$$

and the linearization at $P = (0.5, 1)$ is

$$L(x) = 1 + \frac{1}{3}\left(x - \frac{1}{2}\right) = \frac{1}{3}x + \frac{5}{6}.$$

Finally, when $x = 0.55$, we estimate that the y-coordinate of the point on the curve is

$$y \approx L(0.55) = \frac{1}{3}(0.55) + \frac{5}{6} = 1.017.$$

73. Apply the method of Exercise 71 to $P = (-1, 2)$ on $y^4 + 7xy = 2$ to estimate the solution of $y^4 - 7.7y = 2$ near $y = 2$.

SOLUTION Differentiating both sides of the equation $y^4 + 7xy = 2$ with respect to x yields

$$4y^3\frac{dy}{dx} + 7x\frac{dy}{dx} + 7y = 0,$$

so

$$\frac{dy}{dx} = -\frac{7y}{4y^3 + 7x}.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = -\frac{7(2)}{4(2)^3 + 7(-1)} = -\frac{14}{25},$$

and the linearization at $P = (-1, 2)$ is

$$L(x) = 2 - \frac{14}{25}(x + 1) = -\frac{14}{25}x + \frac{36}{25}.$$

Finally, the equation $y^4 - 7.7y = 2$ corresponds to $x = -1.1$, so we estimate the solution of this equation near $y = 2$ is

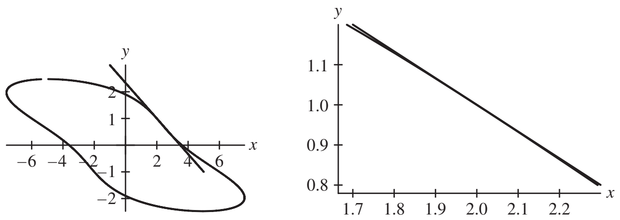
$$y \approx L(-1.1) = -\frac{14}{25}(-1.1) + \frac{36}{25} = 2.056.$$

74. Show that for any real number k, $(1 + \Delta x)^k \approx 1 + k\Delta x$ for small Δx. Estimate $(1.02)^{0.7}$ and $(1.02)^{-0.3}$.

SOLUTION Let $f(x) = (1 + x)^k$. Then for small Δx, we have

$$f(\Delta x) \approx L(\Delta x) = f'(0)(\Delta x - 0) + f(0) = k(1 + 0)^{k-1}(\Delta x - 0) + 1 = 1 + k\Delta x$$

- Let $k = 0.7$ and $\Delta x = 0.02$. Then $L(0.02) = 1 + (0.7)(0.02) = 1.014$.
- Let $k = -0.3$ and $\Delta x = 0.02$. Then $L(0.02) = 1 + (-0.3)(0.02) = 0.994$.



75. Let $\Delta f = f(5+h) - f(5)$, where $f(x) = x^2$. Verify directly that $E = |\Delta f - f'(5)h|$ satisfies (5) with $K = 2$.

SOLUTION Let $f(x) = x^2$. Then

$$\Delta f = f(5+h) - f(5) = (5+h)^2 - 5^2 = h^2 + 10h$$

and

$$E = |\Delta f - f'(5)h| = |h^2 + 10h - 10h| = h^2 = \frac{1}{2}(2)h^2 = \frac{1}{2}Kh^2.$$

76. Let $\Delta f = f(1+h) - f(1)$ where $f(x) = x^{-1}$. Show directly that $E = |\Delta f - f'(1)h|$ is equal to $h^2/(1+h)$. Then prove that $E \le 2h^2$ if $-\frac{1}{2} \le h \le \frac{1}{2}$. *Hint:* In this case, $\frac{1}{2} \le 1+h \le \frac{3}{2}$.

SOLUTION Let $f(x) = x^{-1}$. Then

$$\Delta f = f(1+h) - f(1) = \frac{1}{1+h} - 1 = -\frac{h}{1+h}$$

and

$$E = |\Delta f - f'(1)h| = \left| -\frac{h}{1+h} + h \right| = \frac{h^2}{1+h}.$$

If $-\frac{1}{2} \le h \le \frac{1}{2}$, then $\frac{1}{2} \le 1+h \le \frac{3}{2}$ and $\frac{2}{3} \le \frac{1}{1+h} \le 2$. Thus, $E \le 2h^2$ for $-\frac{1}{2} \le h \le \frac{1}{2}$.

4.2 Extreme Values

Preliminary Questions

1. What is the definition of a critical point?

SOLUTION A critical point is a value of the independent variable x in the domain of a function f at which either $f'(x) = 0$ or $f'(x)$ does not exist.

In Questions 2 and 3, choose the correct conclusion.

2. If $f(x)$ is not continuous on [0, 1], then

(a) $f(x)$ has no extreme values on [0, 1].

(b) $f(x)$ might not have any extreme values on [0, 1].

SOLUTION The correct response is **(b)**: $f(x)$ might not have any extreme values on [0, 1]. Although [0, 1] is closed, because f is not continuous, the function is not guaranteed to have any extreme values on [0, 1].

3. If $f(x)$ is continuous but has no critical points in [0, 1], then

(a) $f(x)$ has no min or max on [0, 1].

(b) Either $f(0)$ or $f(1)$ is the minimum value on [0, 1].

SOLUTION The correct response is **(b)**: either $f(0)$ or $f(1)$ is the minimum value on [0, 1]. Remember that extreme values occur either at critical points or endpoints. If a continuous function on a closed interval has no critical points, the extreme values must occur at the endpoints.

4. Fermat's Theorem *does not* claim that if $f'(c) = 0$, then $f(c)$ is a local extreme value (this is false). What *does* Fermat's Theorem assert?

SOLUTION Fermat's Theorem claims: If $f(c)$ is a local extreme value, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Exercises

1. The following questions refer to Figure 15.

(a) How many critical points does $f(x)$ have on [0, 8]?
(b) What is the maximum value of $f(x)$ on [0, 8]?
(c) What are the local maximum values of $f(x)$?
(d) Find a closed interval on which both the minimum and maximum values of $f(x)$ occur at critical points.
(e) Find an interval on which the minimum value occurs at an endpoint.

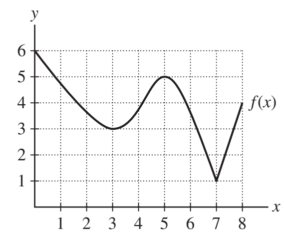


FIGURE 15

SOLUTION

(a) $f(x)$ has three critical points on the interval [0, 8]: at $x = 3$, $x = 5$ and $x = 7$. Two of these, $x = 3$ and $x = 5$, are where the derivative is zero and one, $x = 7$, is where the derivative does not exist.

(b) The maximum value of $f(x)$ on [0, 8] is 6; the function takes this value at $x = 0$.

(c) $f(x)$ achieves a local maximum of 5 at $x = 5$.

(d) Answers may vary. One example is the interval [4, 8]. Another is [2, 6].

(e) Answers may vary. The easiest way to ensure this is to choose an interval on which the graph takes no local minimum. One example is [0, 2].

2. State whether $f(x) = x^{-1}$ (Figure 16) has a minimum or maximum value on the following intervals:

(a) $(0, 2)$ **(b)** $(1, 2)$ **(c)** $[1, 2]$

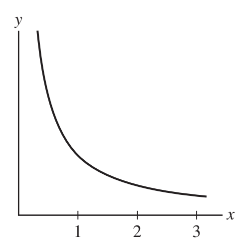


FIGURE 16 Graph of $f(x) = x^{-1}$.

SOLUTION $f(x)$ has no local minima or maxima. Hence, $f(x)$ only takes minimum and maximum values on an interval if it takes them at the endpoints.

(a) $f(x)$ takes no minimum or maximum value on this interval, since the interval does not contain its endpoints.

(b) $f(x)$ takes no minimum or maximum value on this interval, since the interval does not contain its endpoints.

(c) The function is decreasing on the whole interval [1, 2]. Hence, $f(x)$ takes on its maximum value of 1 at $x = 1$ and $f(x)$ takes on its minimum value of $\frac{1}{2}$ at $x = 2$.

In Exercises 3–20, find all critical points of the function.

3. $f(x) = x^2 - 2x + 4$

SOLUTION Let $f(x) = x^2 - 2x + 4$. Then $f'(x) = 2x - 2 = 0$ implies that $x = 1$ is the lone critical point of f.

4. $f(x) = 7x - 2$

SOLUTION Let $f(x) = 7x - 2$. Then $f'(x) = 7$, which is never zero, so $f(x)$ has no critical points.

5. $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$

SOLUTION Let $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$. Then $f'(x) = 3x^2 - 9x - 54 = 3(x + 3)(x - 6) = 0$ implies that $x = -3$ and $x = 6$ are the critical points of f.

6. $f(t) = 8t^3 - t^2$

SOLUTION Let $f(t) = 8t^3 - t^2$. Then $f'(t) = 24t^2 - 2t = 2t(12t - 1) = 0$ implies that $t = 0$ and $t = \frac{1}{12}$ are the critical points of f.

7. $f(x) = x^{-1} - x^{-2}$

SOLUTION Let $f(x) = x^{-1} - x^{-2}$. Then

$$f'(x) = -x^{-2} + 2x^{-3} = \frac{2-x}{x^3} = 0$$

implies that $x = 2$ is the only critical point of f. Though $f'(x)$ does not exist at $x = 0$, this is not a critical point of f because $x = 0$ is not in the domain of f.

8. $g(z) = \dfrac{1}{z-1} - \dfrac{1}{z}$

SOLUTION Let

$$g(z) = \frac{1}{z-1} - \frac{1}{z} = \frac{z-(z-1)}{z(z-1)} = \frac{1}{z^2-z}.$$

Then

$$g'(z) = -\frac{1}{(z^2-z)^2}(2z-1) = -\frac{2z-1}{(z^2-z)^2} = 0$$

implies that $z = 1/2$ is the only critical point of g. Though $g'(z)$ does not exist at either $z = 0$ or $z = 1$, neither is a critical point of g because neither is in the domain of g.

9. $f(x) = \dfrac{x}{x^2+1}$

SOLUTION Let $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} = 0$ implies that $x = \pm 1$ are the critical points of f.

10. $f(x) = \dfrac{x^2}{x^2-4x+8}$

SOLUTION Let $f(x) = \dfrac{x^2}{x^2-4x+8}$. Then

$$f'(x) = \frac{(x^2-4x+8)(2x) - x^2(2x-4)}{(x^2-4x+8)^2} = \frac{4x(4-x)}{(x^2-4x+8)^2} = 0$$

implies that $x = 0$ and $x = 4$ are the critical points of f.

11. $f(t) = t - 4\sqrt{t+1}$

SOLUTION Let $f(t) = t - 4\sqrt{t+1}$. Then

$$f'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0$$

implies that $t = 3$ is a critical point of f. Because $f'(t)$ does not exist at $t = -1$, this is another critical point of f.

12. $f(t) = 4t - \sqrt{t^2+1}$

SOLUTION Let $f(t) = 4t - \sqrt{t^2+1}$. Then

$$f'(t) = 4 - \frac{t}{(t^2+1)^{1/2}} = \frac{4(t^2+1)^{1/2} - t}{(t^2+1)^{1/2}} = 0$$

implies that there are no critical points of f since neither the numerator nor denominator equals 0 for any value of t.

13. $f(x) = x^2\sqrt{1-x^2}$

SOLUTION Let $f(x) = x^2\sqrt{1-x^2}$. Then

$$f'(x) = -\frac{x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2} = \frac{2x-3x^3}{\sqrt{1-x^2}}.$$

This derivative is 0 when $x = 0$ and when $x = \pm\sqrt{2/3}$; the derivative does not exist when $x = \pm 1$. All five of these values are critical points of f

14. $f(x) = x + |2x + 1|$

SOLUTION Removing the absolute values, we see that

$$f(x) = \begin{cases} -x - 1, & x < -\frac{1}{2} \\ 3x + 1, & x \ge -\frac{1}{2} \end{cases}$$

Thus,

$$f'(x) = \begin{cases} -1, & x < -\frac{1}{2} \\ 3, & x \ge -\frac{1}{2} \end{cases}$$

and we see that $f'(0)$ is never equal to 0. However, $f'(-1/2)$ does not exist, so $x = -1/2$ is the only critical point of f.

15. $g(\theta) = \sin^2 \theta$

SOLUTION Let $g(\theta) = \sin^2 \theta$. Then $g'(\theta) = 2 \sin \theta \cos \theta = \sin 2\theta = 0$ implies that

$$\theta = \frac{n\pi}{2}$$

is a critical value of g for all integer values of n.

16. $R(\theta) = \cos \theta + \sin^2 \theta$

SOLUTION Let $R(\theta) = \cos \theta + \sin^2 \theta$. Then

$$R'(\theta) = -\sin \theta + 2 \sin \theta \cos \theta = \sin \theta (2 \cos \theta - 1) = 0$$

implies that $\theta = n\pi$,

$$\theta = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} + 2n\pi$$

are critical points of R for all integer values of n.

17. $f(x) = x \ln x$

SOLUTION Let $f(x) = x \ln x$. Then $f'(x) = 1 + \ln x = 0$ implies that $x = e^{-1} = \frac{1}{e}$ is the only critical point of f.

18. $f(x) = xe^{2x}$

SOLUTION Let $f(x) = xe^{2x}$. Then $f'(x) = (2x + 1)e^{2x} = 0$ implies that $x = -\frac{1}{2}$ is the only critical point of f.

19. $f(x) = \sin^{-1} x - 2x$

SOLUTION Let $f(x) = \sin^{-1} x - 2x$. Then

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0$$

implies that $x = \pm\frac{\sqrt{3}}{2}$ are the critical points of f.

20. $f(x) = \sec^{-1} x - \ln x$

SOLUTION Let $f(x) = \sec^{-1} x - \ln x$. Then

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}} - \frac{1}{x}.$$

This derivative is equal to zero when $\sqrt{x^2 - 1} = 1$, or when $x = \pm\sqrt{2}$. Moreover, the derivative does not exist at $x = 0$ and at $x = \pm 1$. Among these numbers, $x = 1$ and $x = \sqrt{2}$ are the only critical points of f. $x = -\sqrt{2}$, $x = -1$, and $x = 0$ are not critical points of f because none are in the domain of f.

21. Let $f(x) = x^2 - 4x + 1$.

(a) Find the critical point c of $f(x)$ and compute $f(c)$.

(b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 4]$.

(c) Determine the min and max of $f(x)$ on $[0, 4]$.

(d) Find the extreme values of $f(x)$ on $[0, 1]$.

SOLUTION Let $f(x) = x^2 - 4x + 1$.

(a) Then $f'(c) = 2c - 4 = 0$ implies that $c = 2$ is the sole critical point of f. We have $f(2) = -3$.

(b) $f(0) = f(4) = 1$.

(c) Using the results from (a) and (b), we find the maximum value of f on $[0, 4]$ is 1 and the minimum value is -3.

(d) We have $f(1) = -2$. Hence the maximum value of f on $[0, 1]$ is 1 and the minimum value is -2.

22. Find the extreme values of $f(x) = 2x^3 - 9x^2 + 12x$ on [0, 3] and [0, 2].

SOLUTION Let $f(x) = 2x^3 - 9x^2 + 12x$. First, we find the critical points. Setting $f'(x) = 6x^2 - 18x + 12 = 0$ yields $x^2 - 3x + 2 = 0$ so that $x = 2$ or $x = 1$. Next, we compare: first, for [0, 3]:

x-value	Value of f	
1 (critical point)	$f(1) = 5$	
2 (critical point)	$f(2) = 4$	
0 (endpoint)	$f(0) = 0$	min
3 (endpoint)	$f(3) = 9$	max

Then, for [0, 2]:

x-value	Value of f	
1 (critical point)	$f(1) = 5$	max
2 (endpoint)	$f(2) = 4$	
0 (endpoint)	$f(0) = 0$	min

23. Find the critical points of $f(x) = \sin x + \cos x$ and determine the extreme values on $\left[0, \frac{\pi}{2}\right]$.

SOLUTION

- Let $f(x) = \sin x + \cos x$. Then on the interval $\left[0, \frac{\pi}{2}\right]$, we have $f'(x) = \cos x - \sin x = 0$ at $x = \frac{\pi}{4}$, the only critical point of f in this interval.
- Since $f(\frac{\pi}{4}) = \sqrt{2}$ and $f(0) = f(\frac{\pi}{2}) = 1$, the maximum value of f on $\left[0, \frac{\pi}{2}\right]$ is $\sqrt{2}$, while the minimum value is 1.

24. Compute the critical points of $h(t) = (t^2 - 1)^{1/3}$. Check that your answer is consistent with Figure 17. Then find the extreme values of $h(t)$ on [0, 1] and [0, 2].

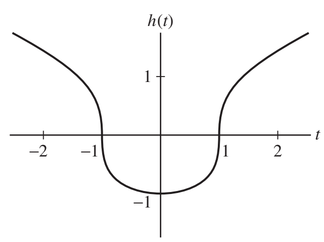


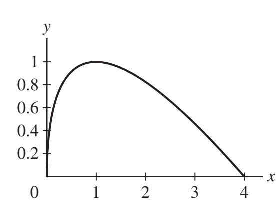
FIGURE 17 Graph of $h(t) = (t^2 - 1)^{1/3}$.

SOLUTION

- Let $h(t) = (t^2 - 1)^{1/3}$. Then $h'(t) = \dfrac{2t}{3(t^2 - 1)^{2/3}} = 0$ implies critical points at $t = 0$ and $t = \pm 1$. These results are consistent with Figure 17 which shows a horizontal tangent at $t = 0$ and vertical tangents at $t = \pm 1$.
- Since $h(0) = -1$ and $h(1) = 0$, the maximum value on [0, 1] is $h(1) = 0$ and the minimum is $h(0) = -1$. Similarly, the minimum on [0, 2] is $h(0) = -1$ and the maximum is $h(2) = 3^{1/3} \approx 1.44225$.

25. $\boxed{\text{GU}}$ Plot $f(x) = 2\sqrt{x} - x$ on [0, 4] and determine the maximum value graphically. Then verify your answer using calculus.

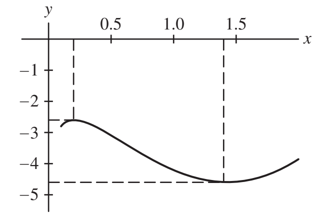
SOLUTION The graph of $y = 2\sqrt{x} - x$ over the interval [0, 4] is shown below. From the graph, we see that at $x = 1$, the function achieves its maximum value of 1.



To verify the information obtained from the plot, let $f(x) = 2\sqrt{x} - x$. Then $f'(x) = x^{-1/2} - 1$. Solving $f'(x) = 0$ yields the critical points $x = 0$ and $x = 1$. Because $f(0) = f(4) = 0$ and $f(1) = 1$, we see that the maximum value of f on $[0, 4]$ is 1.

26. GU Plot $f(x) = \ln x - 5\sin x$ on $[0.1, 2]$ and approximate both the critical points and the extreme values.

SOLUTION The graph of $f(x) = \ln x - 5\sin x$ is shown below. From the graph, we see that critical points occur at approximately $x = 0.2$ and $x = 1.4$. The maximum value of approximately -2.6 occurs at $x \approx 0.2$; the minimum value of approximately -4.6 occurs at $x \approx 1.4$.



27. CAS Approximate the critical points of $g(x) = x\cos^{-1} x$ and estimate the maximum value of $g(x)$.

SOLUTION $g'(x) = \frac{-x}{\sqrt{1-x^2}} + \cos^{-1} x$, so $g'(x) = 0$ when $x \approx 0.652185$. Evaluating g at the endpoints of its domain, $x = \pm 1$, and at the critical point $x \approx 0.652185$, we find $g(-1) = -\pi$, $g(0.652185) \approx 0.561096$, and $g(1) = 0$. Hence, the maximum value of $g(x)$ is approximately 0.561096.

28. CAS Approximate the critical points of $g(x) = 5e^x - \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

SOLUTION Let $g(x) = 5e^x - \tan x$. Then $g'(x) = 5e^x - \sec^2 x$. The derivative is defined for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and is equal to 0 for $x \approx 1.339895$ and $x \approx -0.82780$. Hence, the critical points of g are $x \approx 1.339895$ and $x \approx -0.82780$.

In Exercises 29–58, find the min and max of the function on the given interval by comparing values at the critical points and endpoints.

29. $y = 2x^2 + 4x + 5$, $[-2, 2]$

SOLUTION Let $f(x) = 2x^2 + 4x + 5$. Then $f'(x) = 4x + 4 = 0$ implies that $x = -1$ is the only critical point of f. The minimum of f on the interval $[-2, 2]$ is $f(-1) = 3$, whereas its maximum is $f(2) = 21$. (*Note:* $f(-2) = 5$.)

30. $y = 2x^2 + 4x + 5$, $[0, 2]$

SOLUTION Let $f(x) = 2x^2 + 4x + 5$. Then $f'(x) = 4x + 4 = 0$ implies that $x = -1$ is the only critical point of f. The minimum of f on the interval $[0, 2]$ is $f(0) = 5$, whereas its maximum is $f(2) = 21$. (*Note:* The critical point $x = -1$ is not on the interval $[0, 2]$.)

31. $y = 6t - t^2$, $[0, 5]$

SOLUTION Let $f(t) = 6t - t^2$. Then $f'(t) = 6 - 2t = 0$ implies that $t = 3$ is the only critical point of f. The minimum of f on the interval $[0, 5]$ is $f(0) = 0$, whereas the maximum is $f(3) = 9$. (*Note:* $f(5) = 5$.)

32. $y = 6t - t^2$, $[4, 6]$

SOLUTION Let $f(t) = 6t - t^2$. Then $f'(t) = 6 - 2t = 0$ implies that $t = 3$ is the only critical point of f. The minimum of f on the interval $[4, 6]$ is $f(6) = 0$, whereas the maximum is $f(4) = 8$. (*Note:* The critical point $t = 3$ is not on the interval $[4, 6]$.)

33. $y = x^3 - 6x^2 + 8$, $[1, 6]$

SOLUTION Let $f(x) = x^3 - 6x^2 + 8$. Then $f'(x) = 3x^2 - 12x = 3x(x - 4) = 0$ implies that $x = 0$ and $x = 4$ are the critical points of f. The minimum of f on the interval $[1, 6]$ is $f(4) = -24$, whereas the maximum is $f(6) = 8$. (*Note:* $f(1) = 3$ and the critical point $x = 0$ is not in the interval $[1, 6]$.)

34. $y = x^3 + x^2 - x$, $[-2, 2]$

SOLUTION Let $f(x) = x^3 + x^2 - x$. Then $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1) = 0$ implies that $x = 1/3$ and $x = -1$ are critical points of f. The minimum of f on the interval $[-2, 2]$ is $f(-2) = -2$, whereas the maximum is $f(2) = 10$. (*Note:* $f(-1) = 1$ and $f(1/3) = -5/27$.)

35. $y = 2t^3 + 3t^2$, $[1, 2]$

SOLUTION Let $f(t) = 2t^3 + 3t^2$. Then $f'(t) = 6t^2 + 6t = 6t(t + 1) = 0$ implies that $t = 0$ and $t = -1$ are the critical points of f. The minimum of f on the interval $[1, 2]$ is $f(1) = 5$, whereas the maximum is $f(2) = 28$. (*Note:* Neither critical points are in the interval $[1, 2]$.)

36. $y = x^3 - 12x^2 + 21x$, $[0, 2]$

SOLUTION Let $f(x) = x^3 - 12x^2 + 21x$. Then $f'(x) = 3x^2 - 24x + 21 = 3(x - 7)(x - 1) = 0$ implies that $x = 1$ and $x = 7$ are the critical points of f. The minimum of f on the interval $[0, 2]$ is $f(0) = 0$, whereas its maximum is $f(1) = 10$. (*Note:* $f(2) = 2$ and the critical point $x = 7$ is not in the interval $[0, 2]$.)

37. $y = z^5 - 80z$, $[-3, 3]$

SOLUTION Let $f(z) = z^5 - 80z$. Then $f'(z) = 5z^4 - 80 = 5(z^4 - 16) = 5(z^2 + 4)(z + 2)(z - 2) = 0$ implies that $z = \pm 2$ are the critical points of f. The minimum value of f on the interval $[-3, 3]$ is $f(2) = -128$, whereas the maximum is $f(-2) = 128$. (*Note:* $f(-3) = 3$ and $f(3) = -3$.)

38. $y = 2x^5 + 5x^2$, $[-2, 2]$

SOLUTION Let $f(x) = 2x^5 + 5x^2$. Then $f'(x) = 10x^4 + 10x = 10x(x^3 + 1) = 0$ implies that $x = 0$ and $x = -1$ are critical points of f. The minimum value of f on the interval $[-2, 2]$ is $f(-2) = -44$, whereas the maximum is $f(2) = 84$. (*Note:* $f(-1) = 3$ and $f(0) = 0$.)

39. $y = \dfrac{x^2 + 1}{x - 4}$, $[5, 6]$

SOLUTION Let $f(x) = \dfrac{x^2 + 1}{x - 4}$. Then

$$f'(x) = \frac{(x - 4) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 4)^2} = \frac{x^2 - 8x - 1}{(x - 4)^2} = 0$$

implies $x = 4 \pm \sqrt{17}$ are critical points of f. $x = 4$ is not a critical point because $x = 4$ is not in the domain of f. On the interval $[5, 6]$, the minimum of f is $f(6) = \frac{37}{2} = 18.5$, whereas the maximum of f is $f(5) = 26$. (*Note:* The critical points $x = 4 \pm \sqrt{17}$ are not in the interval $[5, 6]$.)

40. $y = \dfrac{1 - x}{x^2 + 3x}$, $[1, 4]$

SOLUTION Let $f(x) = \dfrac{1 - x}{x^2 + 3x}$. Then

$$f'(x) = \frac{-(x^2 + 3x) - (1 - x)(2x + 3)}{(x^2 + 3x)^2} = \frac{(x - 3)(x + 1)}{(x^2 + 3x)^2} = 0$$

implies that $x = 3$ and $x = -1$ are critical points. Neither $x = 0$ nor $x = -3$ is a critical point because neither is in the domain of f. On the interval $[1, 4]$, the maximum value is $f(1) = 0$ and the minimum value is $f(3) = -\frac{1}{9}$. (*Note:* The critical point $x = -1$ is not in the interval $[1, 4]$.)

41. $y = x - \dfrac{4x}{x + 1}$, $[0, 3]$

SOLUTION Let $f(x) = x - \dfrac{4x}{x + 1}$. Then

$$f'(x) = 1 - \frac{4}{(x + 1)^2} = \frac{(x - 1)(x + 3)}{(x + 1)^2} = 0$$

implies that $x = 1$ and $x = -3$ are critical points of f. $x = -1$ is not a critical point because $x = -1$ is not in the domain of f. The minimum of f on the interval $[0, 3]$ is $f(1) = -1$, whereas the maximum is $f(0) = f(3) = 0$. (*Note:* The critical point $x = -3$ is not in the interval $[0, 3]$.)

42. $y = 2\sqrt{x^2 + 1} - x$, $[0, 2]$

SOLUTION Let $f(x) = 2\sqrt{x^2 + 1} - x$. Then

$$f'(x) = \frac{2x}{\sqrt{x^2 + 1}} - 1 = 0$$

implies that $x = \pm\sqrt{\frac{1}{3}}$ are critical points of f. On the interval $[0, 2]$, the minimum is $f\left(\sqrt{\frac{1}{3}}\right) = \sqrt{3}$ and the maximum is $f(2) = 2\sqrt{5} - 2$. (*Note:* The critical point $x = -\sqrt{\frac{1}{3}}$ is not in the interval $[0, 2]$.)

43. $y = (2+x)\sqrt{2+(2-x)^2}, \quad [0,2]$

SOLUTION Let $f(x) = (2+x)\sqrt{2+(2-x)^2}$. Then

$$f'(x) = \sqrt{2+(2-x)^2} - (2+x)(2+(2-x)^2)^{-1/2}(2-x) = \frac{2(x-1)^2}{\sqrt{2+(2-x)^2}} = 0$$

implies that $x = 1$ is the critical point of f. On the interval $[0, 2]$, the minimum is $f(0) = 2\sqrt{6} \approx 4.9$ and the maximum is $f(2) = 4\sqrt{2} \approx 5.66$. (*Note:* $f(1) = 3\sqrt{3} \approx 5.2$.)

44. $y = \sqrt{1+x^2} - 2x, \quad [0,1]$

SOLUTION Let $f(x) = \sqrt{1+x^2} - 2x$. Then

$$f'(x) = \frac{x}{\sqrt{1+x^2}} - 2 = 0$$

implies that f has no critical points. The minimum value of f on the interval $[0, 1]$ is $f(1) = \sqrt{2} - 2$, whereas the maximum is $f(0) = 1$.

45. $y = \sqrt{x+x^2} - 2\sqrt{x}, \quad [0,4]$

SOLUTION Let $f(x) = \sqrt{x+x^2} - 2\sqrt{x}$. Then

$$f'(x) = \frac{1}{2}(x+x^2)^{-1/2}(1+2x) - x^{-1/2} = \frac{1+2x-2\sqrt{1+x}}{2\sqrt{x}\sqrt{1+x}} = 0$$

implies that $x = 0$ and $x = \frac{\sqrt{3}}{2}$ are the critical points of f. Neither $x = -1$ nor $x = -\frac{\sqrt{3}}{2}$ is a critical point because neither is in the domain of f. On the interval $[0, 4]$, the minimum of f is $f\left(\frac{\sqrt{3}}{2}\right) \approx -0.589980$ and the maximum is $f(4) \approx 0.472136$. (*Note:* $f(0) = 0$.)

46. $y = (t-t^2)^{1/3}, \quad [-1,2]$

SOLUTION Let $s(t) = (t-t^2)^{1/3}$. Then $s'(t) = \frac{1}{3}(t-t^2)^{-2/3}(1-2t) = 0$ at $t = \frac{1}{2}$, a critical point of s. Other critical points of s are $t = 0$ and $t = 1$, where the derivative of s does not exist. Therefore, on the interval $[-1, 2]$, the minimum of s is $s(-1) = s(2) = -2^{1/3} \approx -1.26$ and the maximum is $s(\frac{1}{2}) = (\frac{1}{4})^{1/3} \approx 0.63$. (*Note:* $s(0) = s(1) = 0$.)

47. $y = \sin x \cos x, \quad \left[0, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x$. On the interval $\left[0, \frac{\pi}{2}\right]$, $f'(x) = \cos 2x = 0$ when $x = \frac{\pi}{4}$. The minimum of f on this interval is $f(0) = f(\frac{\pi}{2}) = 0$, whereas the maximum is $f(\frac{\pi}{4}) = \frac{1}{2}$.

48. $y = x + \sin x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = x + \sin x$. Then $f'(x) = 1 + \cos x = 0$ implies that $x = \pi$ is the only critical point on $[0, 2\pi]$. The minimum value of f on the interval $[0, 2\pi]$ is $f(0) = 0$, whereas the maximum is $f(2\pi) = 2\pi$. (*Note:* $f(\pi) = \pi - 1$.)

49. $y = \sqrt{2}\,\theta - \sec\theta, \quad \left[0, \frac{\pi}{3}\right]$

SOLUTION Let $f(\theta) = \sqrt{2}\theta - \sec\theta$. On the interval $[0, \frac{\pi}{3}]$, $f'(\theta) = \sqrt{2} - \sec\theta\tan\theta = 0$ at $\theta = \frac{\pi}{4}$. The minimum value of f on this interval is $f(0) = -1$, whereas the maximum value over this interval is $f(\frac{\pi}{4}) = \sqrt{2}(\frac{\pi}{4} - 1) \approx -0.303493$. (*Note:* $f(\frac{\pi}{3}) = \sqrt{2}\frac{\pi}{3} - 2 \approx -.519039$.)

50. $y = \cos\theta + \sin\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \cos\theta + \sin\theta$. On the interval $[0, 2\pi]$, $f'(\theta) = -\sin\theta + \cos\theta = 0$ where $\sin\theta = \cos\theta$, which is at the two points $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. The minimum value on the interval is $f(\frac{5\pi}{4}) = -\sqrt{2}$, whereas the maximum value on the interval is $f(\frac{\pi}{4}) = \sqrt{2}$. (*Note:* $f(0) = f(2\pi) = 1$.)

51. $y = \theta - 2\sin\theta, \quad [0, 2\pi]$

SOLUTION Let $g(\theta) = \theta - 2\sin\theta$. On the interval $[0, 2\pi]$, $g'(\theta) = 1 - 2\cos\theta = 0$ at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5}{3}\pi$. The minimum of g on this interval is $g(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3} \approx -.685$ and the maximum is $g(\frac{5}{3}\pi) = \frac{5}{3}\pi + \sqrt{3} \approx 6.968$. (*Note:* $g(0) = 0$ and $g(2\pi) = 2\pi \approx 6.283$.)

52. $y = 4\sin^3\theta - 3\cos^2\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = 4\sin^3\theta - 3\cos^2\theta$. Then

$$\begin{aligned} f'(\theta) &= 12\sin^2\theta\cos\theta + 6\cos\theta\sin\theta \\ &= 6\cos\theta\sin\theta(2\sin\theta + 1) = 0 \end{aligned}$$

yields $\theta = 0, \pi/2, \pi, 7\pi/6, 3\pi/2, 11\pi/6, 2\pi$ as critical points of f. The minimum value of f on the interval $[0, 2\pi]$ is $f(3\pi/2) = -4$, whereas the maximum is $f(\pi/2) = 4$. (*Note:* $f(0) = f(\pi) = f(2\pi) = -3$ and $f(7\pi/6) = f(11\pi/6) = -11/4$.)

53. $y = \tan x - 2x, \quad [0, 1]$

SOLUTION Let $f(x) = \tan x - 2x$. Then on the interval $[0, 1]$, $f'(x) = \sec^2 x - 2 = 0$ at $x = \frac{\pi}{4}$. The minimum of f is $f(\frac{\pi}{4}) = 1 - \frac{\pi}{2} \approx -0.570796$ and the maximum is $f(0) = 0$. (*Note:* $f(1) = \tan 1 - 2 \approx -0.442592$.)

54. $y = xe^{-x}, \quad [0, 2]$

SOLUTION Let $f(x) = xe^{-x}$. Then, on the interval $[0, 2]$, $f'(x) = -xe^{-x} + e^{-x} = (1 - x)e^{-x} = 0$ at $x = 1$. The minimum of f on this interval is $f(0) = 0$ and the maximum is $f(1) = e^{-1} \approx 0.367879$. (*Note:* $f(2) = 2e^{-2} \approx 0.270671$.)

55. $y = \dfrac{\ln x}{x}, \quad [1, 3]$

SOLUTION Let $f(x) = \frac{\ln x}{x}$. Then, on the interval $[1, 3]$,

$$f'(x) = \frac{1 - \ln x}{x^2} = 0$$

at $x = e$. The minimum of f on this interval is $f(1) = 0$ and the maximum is $f(e) = e^{-1} \approx 0.367879$. (*Note:* $f(3) = \frac{1}{3} \ln 3 \approx 0.366204$.)

56. $y = 3e^x - e^{2x}, \quad \left[-\frac{1}{2}, 1\right]$

SOLUTION Let $f(x) = 3e^x - e^{2x}$. Then, on the interval $\left[-\frac{1}{2}, 1\right]$, $f'(x) = 3e^x - 2e^{2x} = e^x(3 - 2e^x) = 0$ at $x = \ln(3/2)$. The minimum of f on this interval is $f(1) = 3e - e^2 \approx 0.765789$ and the maximum is $f(\ln(3/2)) = 2.25$. (*Note:* $f(-1/2) = 3e^{-1/2} - e^{-1} \approx 1.451713$.)

57. $y = 5\tan^{-1} x - x, \quad [1, 5]$

SOLUTION Let $f(x) = 5\tan^{-1} x - x$. Then, on the interval $[1, 5]$,

$$f'(x) = 5\frac{1}{1 + x^2} - 1 = 0$$

at $x = 2$. The minimum of f on this interval is $f(5) = 5\tan^{-1} 5 - 5 \approx 1.867004$ and the maximum is $f(2) = 5\tan^{-1} 2 - 2 \approx 3.535744$. (*Note:* $f(1) = \frac{5\pi}{4} - 1 \approx 2.926991$.)

58. $y = x^3 - 24\ln x, \quad \left[\frac{1}{2}, 3\right]$

SOLUTION Let $f(x) = x^3 - 24\ln x$. Then, on the interval $\left[\frac{1}{2}, 3\right]$,

$$f'(x) = 3x^2 - \frac{24}{x} = 0$$

at $x = 2$. The minimum of f on this interval is $f(2) = 8 - 24\ln 2 \approx -8.635532$ and the maximum is $f(1/2) = \frac{1}{8} + 24\ln 2 \approx 16.760532$. (*Note:* $f(3) = 27 - 24\ln 2 \approx 0.633305$.)

59. Let $f(\theta) = 2\sin 2\theta + \sin 4\theta$.

(a) Show that θ is a critical point if $\cos 4\theta = -\cos 2\theta$.

(b) Show, using a unit circle, that $\cos\theta_1 = -\cos\theta_2$ if and only if $\theta_1 = \pi \pm \theta_2 + 2\pi k$ for an integer k.

(c) Show that $\cos 4\theta = -\cos 2\theta$ if and only if $\theta = \frac{\pi}{2} + \pi k$ or $\theta = \frac{\pi}{6} + \left(\frac{\pi}{3}\right)k$.

(d) Find the six critical points of $f(\theta)$ on $[0, 2\pi]$ and find the extreme values of $f(\theta)$ on this interval.

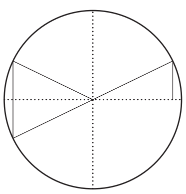
(e) $\boxed{\text{GU}}$ Check your results against a graph of $f(\theta)$.

SOLUTION $f(\theta) = 2\sin 2\theta + \sin 4\theta$ is differentiable at all θ, so the way to find the critical points is to find all points such that $f'(\theta) = 0$.

(a) $f'(\theta) = 4\cos 2\theta + 4\cos 4\theta$. If $f'(\theta) = 0$, then $4\cos 4\theta = -4\cos 2\theta$, so $\cos 4\theta = -\cos 2\theta$.

(b) Given the point $(\cos\theta, \sin\theta)$ at angle θ on the unit circle, there are two points with x coordinate $-\cos\theta$. The graphic shows these two points, which are:

- The point $(\cos(\theta + \pi), \sin(\theta + \pi))$ on the opposite end of the unit circle.
- The point $(\cos(\pi - \theta), \sin(\theta - \pi))$ obtained by reflecting through the y axis.



If we include all angles representing these points on the circle, we find that $\cos\theta_1 = -\cos\theta_2$ if and only if $\theta_1 = (\pi+\theta_2)+2\pi k$ or $\theta_1 = (\pi-\theta_2)+2\pi k$ for integers k.

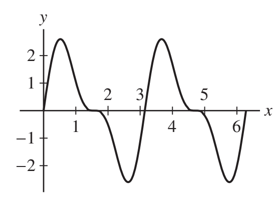
(c) Using (b), we recognize that $\cos 4\theta = -\cos 2\theta$ if $4\theta = 2\theta+\pi+2\pi k$ or $4\theta = \pi - 2\theta + 2\pi k$. Solving for θ, we obtain $\theta = \frac{\pi}{2}+k\pi$ or $\theta = \frac{\pi}{6}+\frac{\pi}{3}k$.

(d) To find all θ, $0 \le \theta < 2\pi$ indicated by (c), we use the following table:

k	0	1	2	3	4	5
$\frac{\pi}{2}+k\pi$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$				
$\frac{\pi}{6}+\frac{\pi}{3}k$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$

The critical points in the range $[0, 2\pi]$ are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$. On this interval, the maximum value is $f(\frac{\pi}{6}) = f(\frac{7\pi}{6}) = \frac{3\sqrt{3}}{2}$ and the minimum value is $f(\frac{5\pi}{6}) = f(\frac{11\pi}{6}) = -\frac{3\sqrt{3}}{2}$.

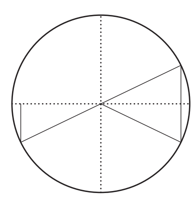
(e) The graph of $f(\theta) = 2\sin 2\theta + \sin 4\theta$ is shown here:



We can see that there are six flat points on the graph between 0 and 2π, as predicted. There are 4 local extrema, and two points at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ where the graph has neither a local maximum nor a local minimum.

60. GU Find the critical points of $f(x) = 2\cos 3x + 3\cos 2x$ in $[0, 2\pi]$. Check your answer against a graph of $f(x)$.

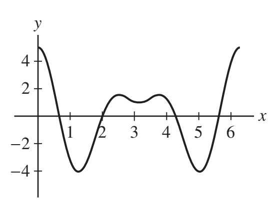
SOLUTION $f(x)$ is differentiable for all x, so we are looking for points where $f'(x) = 0$ only. Setting $f'(x) = -6\sin 3x - 6\sin 2x$, we get $\sin 3x = -\sin 2x$. Looking at a unit circle, we find the relationship between angles y and x such that $\sin y = -\sin x$. This technique is also used in Exercise 59.



From the diagram, we see that $\sin y = -\sin x$ if y is either (i.) the point antipodal to x ($y = \pi + x + 2\pi k$) or (ii.) the point obtained by reflecting x through the horizontal axis ($y = -x + 2\pi k$).

Since $\sin 3x = -\sin 2x$, we get either $3x = \pi + 2x + 2\pi k$ or $3x = -2x + 2\pi k$. Solving each of these equations for x yields $x = \pi + 2\pi k$ and $x = \frac{2\pi}{5}k$, respectively. The values of x between 0 and 2π are $0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{8\pi}{5}$, and 2π.

The graph is shown below. As predicted, it has horizontal tangent lines at $\frac{2\pi}{5}k$ and at $x = \frac{\pi}{2}$. Each of these points is a local extremum.



In Exercises 61–64, find the critical points and the extreme values on $[0, 4]$. *In Exercises 63 and 64, refer to Figure 18.*

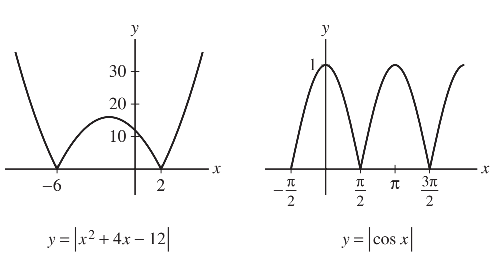
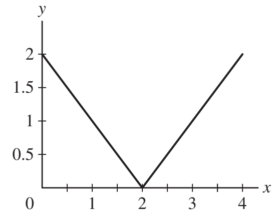


FIGURE 18

61. $y = |x - 2|$

SOLUTION Let $f(x) = |x - 2|$. For $x < 2$, we have $f'(x) = -1$. For $x > 2$, we have $f'(x) = 1$. Now as $x \to 2-$, we have $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{(2 - x) - 0}{x - 2} \to -1$; whereas as $x \to 2+$, we have $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{(x - 2) - 0}{x - 2} \to 1$. Therefore, $f'(2) = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ does not exist and the lone critical point of f is $x = 2$. Alternately, we examine the graph of $f(x) = |x - 2|$ shown below.

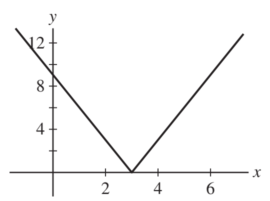
To find the extremum, we check the values of $f(x)$ at the critical point and the endpoints. $f(0) = 2$, $f(4) = 2$, and $f(2) = 0$. $f(x)$ takes its minimum value of 0 at $x = 2$, and its maximum of 2 at $x = 0$ and at $x = 4$.



62. $y = |3x - 9|$

SOLUTION Let $f(x) = |3x - 9| = 3|x - 3|$. For $x < 3$, we have $f'(x) = -3$. For $x > 3$, we have $f'(x) = 3$. Now as $x \to 3-$, we have $\dfrac{f(x) - f(3)}{x - 3} = \dfrac{3(3 - x) - 0}{x - 3} \to -3$; whereas as $x \to 3+$, we have $\dfrac{f(x) - f(3)}{x - 3} = \dfrac{3(x - 3) - 0}{x - 3} \to 3$. Therefore, $f'(3) = \lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$ does not exist and the lone critical point of f is $x = 3$. Alternately, we examine the graph of $f(x) = |3x - 9|$ shown below.

To find the extrema of $f(x)$ on $[0, 4]$, we test the values of $f(x)$ at the critical point and the endpoints. $f(0) = 9$, $f(3) = 0$ and $f(4) = 3$, so $f(x)$ takes its minimum value of 0 at $x = 3$, and its maximum value of 9 at $x = 0$.



63. $y = |x^2 + 4x - 12|$

SOLUTION Let $f(x) = |x^2 + 4x - 12| = |(x + 6)(x - 2)|$. From the graph of f in Figure 18, we see that $f'(x)$ does not exist at $x = -6$ and at $x = 2$, so these are critical points of f. There is also a critical point between $x = -6$ and $x = 2$ at which $f'(x) = 0$. For $-6 < x < 2$, $f(x) = -x^2 - 4x + 12$, so $f'(x) = -2x - 4 = 0$ when $x = -2$. On the interval $[0, 4]$ the minimum value of f is $f(2) = 0$ and the maximum value is $f(4) = 20$. (*Note:* $f(0) = 12$ and the critical points $x = -6$ and $x = -2$ are not in the interval.)

64. $y = |\cos x|$

SOLUTION Let $f(x) = |\cos x|$. There are two types of critical points: points of the form πn where the derivative is zero and points of the form $n\pi + \pi/2$ where the derivative does not exist. Only two of these, $x = \frac{\pi}{2}$ and $x = \pi$ are in the interval $[0, 4]$. Now, $f(0) = f(\pi) = 1$, $f(4) = |\cos 4| \approx 0.6536$, and $f(\frac{\pi}{2}) = 0$, so $f(x)$ takes its maximum value of 1 at $x = 0$ and $x = \pi$ and its minimum of 0 at $x = \frac{\pi}{2}$.

In Exercises 65–68, verify Rolle's Theorem for the given interval.

65. $f(x) = x + x^{-1}$, $\left[\frac{1}{2}, 2\right]$

SOLUTION Because f is continuous on $[\frac{1}{2}, 2]$, differentiable on $(\frac{1}{2}, 2)$ and

$$f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{5}{2} = 2 + \frac{1}{2} = f(2),$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{1}{2}, 2)$ at which $f'(c) = 0$. Here, $f'(x) = 1 - x^{-2} = \frac{x^2 - 1}{x^2}$, so we may take $c = 1$.

66. $f(x) = \sin x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

SOLUTION Because f is continuous on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, differentiable on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and

$$f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{\pi}{4}, \frac{3\pi}{4})$ at which $f'(c) = 0$. Here, $f'(x) = \cos x$, so we may take $c = \frac{\pi}{2}$.

67. $f(x) = \dfrac{x^2}{8x - 15}, \quad [3, 5]$

SOLUTION Because f is continuous on $[3, 5]$, differentiable on $(3, 5)$ and $f(3) = f(5) = 1$, we may conclude from Rolle's Theorem that there exists a $c \in (3, 5)$ at which $f'(c) = 0$. Here,

$$f'(x) = \frac{(8x - 15)(2x) - 8x^2}{(8x - 15)^2} = \frac{2x(4x - 15)}{(8x - 15)^2},$$

so we may take $c = \frac{15}{4}$.

68. $f(x) = \sin^2 x - \cos^2 x, \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

SOLUTION Because f is continuous on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, differentiable on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and

$$f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = 0,$$

we may conclude from Rolle's Theorem that there exists a $c \in (\frac{\pi}{4}, \frac{3\pi}{4})$ at which $f'(c) = 0$. Here,

$$f'(x) = 2\sin x \cos x - 2\cos x(-\sin x) = 2\sin 2x,$$

so we may take $c = \frac{\pi}{2}$.

69. Prove that $f(x) = x^5 + 2x^3 + 4x - 12$ has precisely one real root.

SOLUTION Let's first establish the $f(x) = x^5 + 2x^3 + 4x - 12$ has at least one root. Because f is a polynomial, it is continuous for all x. Moreover, $f(0) = -12 < 0$ and $f(2) = 44 > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, 2)$ such that $f(c) = 0$.

Next, we prove that this is the only root. We will use proof by contradiction. Suppose $f(x) = x^5 + 2x^3 + 4x - 12$ has two real roots, $x = a$ and $x = b$. Then $f(a) = f(b) = 0$ and Rolle's Theorem guarantees that there exists a $c \in (a, b)$ at which $f'(c) = 0$. However, $f'(x) = 5x^4 + 6x^2 + 4 \geq 4$ for all x, so there is no $c \in (a, b)$ at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^5 + 2x^3 + 4x - 12$ cannot have more than one real root. Finally, f must have precisely one real root.

70. Prove that $f(x) = x^3 + 3x^2 + 6x$ has precisely one real root.

SOLUTION First, note that $f(0) = 0$, so f has at least one real root. We will proceed by contradiction to establish that $x = 0$ is the only real root. Suppose there exists another real root, say $x = a$. Because the polynomial f is continuous and differentiable for all real x, it follows by Rolle's Theorem that there exists a real number c between 0 and a such that $f'(c) = 0$. However, $f'(x) = 3x^2 + 6x + 6 = 3(x + 1)^2 + 3 \geq 3$ for all x. Thus, there is no c between 0 and a at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^3 + 3x^2 + 6x$ cannot have more than one real root. Finally, f must have precisely one real root.

71. Prove that $f(x) = x^4 + 5x^3 + 4x$ has no root c satisfying $c > 0$. *Hint:* Note that $x = 0$ is a root and apply Rolle's Theorem.

SOLUTION We will proceed by contradiction. Note that $f(0) = 0$ and suppose that there exists a $c > 0$ such that $f(c) = 0$. Then $f(0) = f(c) = 0$ and Rolle's Theorem guarantees that there exists a $d \in (0, c)$ such that $f'(d) = 0$. However, $f'(x) = 4x^3 + 15x^2 + 4 > 4$ for all $x > 0$, so there is no $d \in (0, c)$ such that $f'(d) = 0$. Based on this contradiction, we conclude that $f(x) = x^4 + 5x^3 + 4x$ has no root c satisfying $c > 0$.

72. Prove that $c = 4$ is the largest root of $f(x) = x^4 - 8x^2 - 128$.

SOLUTION First, note that $f(4) = 4^4 - 8(4)^2 - 128 = 256 - 128 - 128 = 0$, so $c = 4$ is a root of f. We will proceed by contradiction to establish that $c = 4$ is the largest real root. Suppose there exists real root, say $x = a$, where $a > 4$. Because the polynomial f is continuous and differentiable for all real x, it follows by Rolle's Theorem that there exists a real number $c \in (4, a)$ such that $f'(c) = 0$. However, $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) > 0$ for all $x > 4$. Thus, there is no $c \in (4, a)$ at which $f'(c) = 0$. Based on this contradiction, we conclude that $f(x) = x^4 - 8x^2 - 128$ has no real root larger than 4.

73. The position of a mass oscillating at the end of a spring is $s(t) = A\sin\omega t$, where A is the amplitude and ω is the angular frequency. Show that the speed $|v(t)|$ is at a maximum when the acceleration $a(t)$ is zero and that $|a(t)|$ is at a maximum when $v(t)$ is zero.

SOLUTION Let $s(t) = A\sin\omega t$. Then

$$v(t) = \frac{ds}{dt} = A\omega\cos\omega t$$

and

$$a(t) = \frac{dv}{dt} = -A\omega^2\sin\omega t.$$

Thus, the speed

$$|v(t)| = |A\omega\cos\omega t|$$

is a maximum when $|\cos\omega t| = 1$, which is precisely when $\sin\omega t = 0$; that is, the speed $|v(t)|$ is at a maximum when the acceleration $a(t)$ is zero. Similarly,

$$|a(t)| = |A\omega^2\sin\omega t|$$

is a maximum when $|\sin\omega t| = 1$, which is precisely when $\cos\omega t = 0$; that is, $|a(t)|$ is at a maximum when $v(t)$ is zero.

74. The concentration $C(t)$ (in mg/cm^3) of a drug in a patient's bloodstream after t hours is

$$C(t) = \frac{0.016t}{t^2+4t+4}$$

Find the maximum concentration in the time interval [0, 8] and the time at which it occurs.

SOLUTION

$$C'(t) = \frac{0.016(t^2+4t+4) - (0.016t(2t+4))}{(t^2+4t+4)^2} = 0.016\frac{-t^2+4}{(t^2+4t+4)^2} = 0.016\frac{2-t}{(t+2)^3}.$$

$C'(t)$ exists for all $t \geq 0$, so we are looking for points where $C'(t) = 0$. $C'(t) = 0$ when $t = 2$. Using a calculator, we find that $C(2) = 0.002\frac{\text{mg}}{\text{cm}^3}$. On the other hand, $C(0) = 0$ and $C(10) \approx 0.001$. Hence, the maximum concentration occurs at $t = 2$ hours and is equal to $.002\frac{\text{mg}}{\text{cm}^3}$.

75. CAS **Antibiotic Levels** A study shows that the concentration $C(t)$ (in micrograms per milliliter) of antibiotic in a patient's blood serum after t hours is $C(t) = 120(e^{-0.2t} - e^{-bt})$, where $b \geq 1$ is a constant that depends on the particular combination of antibiotic agents used. Solve numerically for the value of b (to two decimal places) for which maximum concentration occurs at $t = 1$ h. You may assume that the maximum occurs at a critical point as suggested by Figure 19.

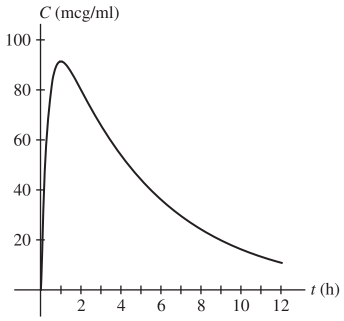


FIGURE 19 Graph of $C(t) = 120(e^{-0.2t} - e^{-bt})$ with b chosen so that the maximum occurs at $t = 1$ h.

SOLUTION Answer is $b = 2.86$. The max of $C(t)$ occurs at $t = \ln(5b)/(b-0.2)$ so we solve $\ln(5b)/(b-0.1) = 1$ numerically.

Let $C(t) = 120(e^{-0.2t} - e^{-bt})$. Then $C'(t) = 120(-0.2e^{-0.2t} + be^{-bt}) = 0$ when

$$t = \frac{\ln 5b}{b-0.2}.$$

Substituting $t = 1$ and solving for b numerically yields $b \approx 2.86$.

76. CAS In the notation of Exercise 75, find the value of b (to two decimal places) for which the maximum value of $C(t)$ is equal to 100 mcg/ml.

SOLUTION From the previous exercise, we know that $C(t)$ achieves its maximum when

$$t = \frac{\ln 5b}{b - 0.2}.$$

Substituting this expression into the formula for $C(t)$, setting the resulting expression equal to 100 and solving for b yields $b \approx 4.75$.

77. In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is around 59%. If wind enters a turbine with speed v_1 and exits with speed v_2, then the power extracted is the difference in kinetic energy per unit time:

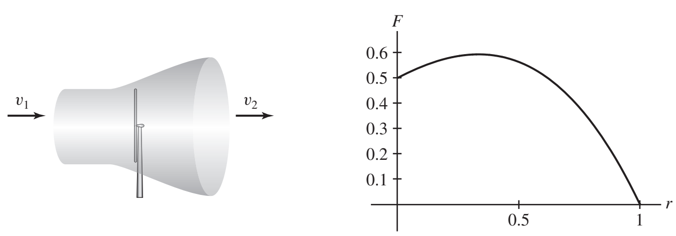
$$P = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 \quad \text{watts}$$

where m is the mass of wind flowing through the rotor per unit time (Figure 20). Betz assumed that $m = \rho A(v_1 + v_2)/2$, where ρ is the density of air and A is the area swept out by the rotor. Wind flowing undisturbed through the same area A would have mass per unit time $\rho A v_1$ and power $P_0 = \frac{1}{2}\rho A v_1^3$. The fraction of power extracted by the turbine is $F = P/P_0$.

(a) Show that F depends only on the ratio $r = v_2/v_1$ and is equal to $F(r) = \frac{1}{2}(1 - r^2)(1 + r)$, where $0 \le r \le 1$.

(b) Show that the maximum value of $F(r)$, called the **Betz Limit**, is $16/27 \approx 0.59$.

(c) Explain why Betz's formula for $F(r)$ is not meaningful for r close to zero. *Hint:* How much wind would pass through the turbine if v_2 were zero? Is this realistic?



(A) Wind flowing through a turbine.

(B) F is the fraction of energy extracted by the turbine as a function of $r = v_2/v_1$.

FIGURE 20

SOLUTION

(a) We note that

$$F = \frac{P}{P_0} = \frac{\frac{1}{2}\frac{\rho A(v_1+v_2)}{2}(v_1^2 - v_2^2)}{\frac{1}{2}\rho A v_1^3}$$

$$= \frac{1}{2}\frac{v_1^2 - v_2^2}{v_1^2} \cdot \frac{v_1 + v_2}{v_1}$$

$$= \frac{1}{2}\left(1 - \frac{v_2^2}{v_1^2}\right)\left(1 + \frac{v_2}{v_1}\right)$$

$$= \frac{1}{2}(1 - r^2)(1 + r).$$

(b) Based on part (a),

$$F'(r) = \frac{1}{2}(1 - r^2) - r(1 + r) = -\frac{3}{2}r^2 - r + \frac{1}{2}.$$

The roots of this quadratic are $r = -1$ and $r = \frac{1}{3}$. Now, $F(0) = \frac{1}{2}$, $F(1) = 0$ and

$$F\left(\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{8}{9} \cdot \frac{4}{3} = \frac{16}{27} \approx 0.59.$$

Thus, the Betz Limit is $16/27 \approx 0.59$.

(c) If v_2 were 0, then no air would be passing through the turbine, which is not realistic.

78. GU The **Bohr radius** a_0 of the hydrogen atom is the value of r that minimizes the energy

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

where $\hbar$, m, e, and ϵ_0 are physical constants. Show that $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$. Assume that the minimum occurs at a critical point, as suggested by Figure 21.

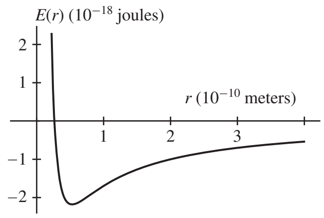


FIGURE 21

SOLUTION Let

$$E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}.$$

Then

$$\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0$$

implies

$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

Thus,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

79. The response of a circuit or other oscillatory system to an input of frequency ω ("omega") is described by the function

$$\phi(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}$$

Both ω_0 (the natural frequency of the system) and D (the damping factor) are positive constants. The graph of ϕ is called a **resonance curve**, and the positive frequency $\omega_r > 0$, where ϕ takes its maximum value, if it exists, is called the **resonant frequency**. Show that $\omega_r = \sqrt{\omega_0^2 - 2D^2}$ if $0 < D < \omega_0/\sqrt{2}$ and that no resonant frequency exists otherwise (Figure 22).

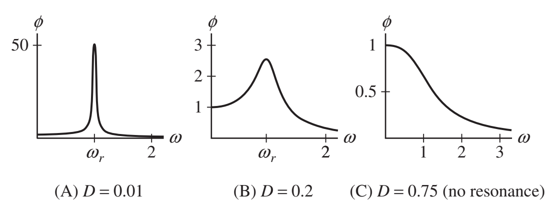


FIGURE 22 Resonance curves with $\omega_0 = 1$.

SOLUTION Let $\phi(\omega) = ((\omega_0^2 - \omega^2)^2 + 4D^2\omega^2)^{-1/2}$. Then

$$\phi'(\omega) = \frac{2\omega((\omega_0^2 - \omega^2) - 2D^2)}{((\omega_0^2 - \omega^2)^2 + 4D^2\omega^2)^{3/2}}$$

and the non-negative critical points are $\omega = 0$ and $\omega = \sqrt{\omega_0^2 - 2D^2}$. The latter critical point is positive if and only if $\omega_0^2 - 2D^2 > 0$, and since we are given $D > 0$, this is equivalent to $0 < D < \omega_0/\sqrt{2}$.

Define $\omega_r = \sqrt{\omega_0^2 - 2D^2}$. Now, $\phi(0) = 1/\omega_0^2$ and $\phi(\omega) \to 0$ as $\omega \to \infty$. Finally,

$$\phi(\omega_r) = \frac{1}{2D\sqrt{\omega_0^2 - D^2}},$$

which, for $0 < D < \omega_0/\sqrt{2}$, is larger than $1/\omega_0^2$. Hence, the point $\omega = \sqrt{\omega_0^2 - 2D^2}$, if defined, is a local maximum.

80. Bees build honeycomb structures out of cells with a hexagonal base and three rhombus-shaped faces on top, as in Figure 23. We can show that the surface area of this cell is

$$A(\theta) = 6hs + \frac{3}{2}s^2(\sqrt{3}\ \csc\theta - \cot\theta)$$

with h, s, and θ as indicated in the figure. Remarkably, bees "know" which angle θ minimizes the surface area (and therefore requires the least amount of wax).

(a) Show that $\theta \approx 54.7^\circ$ (assume h and s are constant). *Hint:* Find the critical point of $A(\theta)$ for $0 < \theta < \pi/2$.

(b) $\boxed{\text{GU}}$ Confirm, by graphing $f(\theta) = \sqrt{3}\csc\theta - \cot\theta$, that the critical point indeed minimizes the surface area.

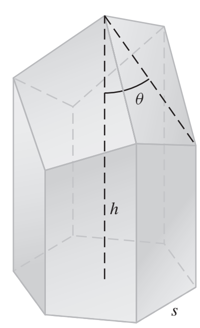
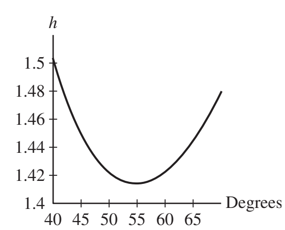


FIGURE 23 A cell in a honeycomb constructed by bees.

SOLUTION

(a) Because h and s are constant relative to θ, we have $A'(\theta) = \frac{3}{2}s^2(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta) = 0$. From this, we get $\sqrt{3}\csc\theta\cot\theta = \csc^2\theta$, or $\cos\theta = \frac{1}{\sqrt{3}}$, whence $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.955317$ radians = 54.736°.

(b) The plot of $\sqrt{3}\csc\theta - \cot\theta$, where θ is given in degrees, is given below. We can see that the minimum occurs just below 55°.



81. Find the maximum of $y = x^a - x^b$ on [0, 1] where $0 < a < b$. In particular, find the maximum of $y = x^5 - x^{10}$ on [0, 1].

SOLUTION

- Let $f(x) = x^a - x^b$. Then $f'(x) = ax^{a-1} - bx^{b-1}$. Since $a < b$, $f'(x) = x^{a-1}(a - bx^{b-a}) = 0$ implies critical points $x = 0$ and $x = (\frac{a}{b})^{1/(b-a)}$, which is in the interval [0, 1] as $a < b$ implies $\frac{a}{b} < 1$ and consequently $x = (\frac{a}{b})^{1/(b-a)} < 1$. Also, $f(0) = f(1) = 0$ and $a < b$ implies $x^a > x^b$ on the interval [0, 1], which gives $f(x) > 0$ and thus the maximum value of f on [0, 1] is

$$f\left(\left(\frac{a}{b}\right)^{1/(b-a)}\right) = \left(\frac{a}{b}\right)^{a/(b-a)} - \left(\frac{a}{b}\right)^{b/(b-a)}.$$

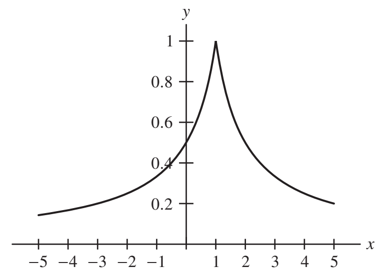
- Let $f(x) = x^5 - x^{10}$. Then by part (a), the maximum value of f on [0, 1] is

$$f\left(\left(\frac{1}{2}\right)^{1/5}\right) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

In Exercises 82–84, plot the function using a graphing utility and find its critical points and extreme values on $[-5, 5]$.

82. GU $y = \dfrac{1}{1 + |x - 1|}$

SOLUTION Let $f(x) = \frac{1}{1+|x-1|}$. The plot follows:



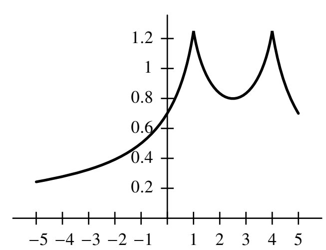
We can see on the plot that the only critical point of $f(x)$ lies at $x = 1$. With $f(-5) = \frac{1}{7}$, $f(1) = 1$ and $f(5) = \frac{1}{5}$, it follows that the maximum value of $f(x)$ on $[-5, 5]$ is $f(1) = 1$ and the minimum value is $f(-5) = \frac{1}{7}$.

83. GU $y = \dfrac{1}{1 + |x - 1|} + \dfrac{1}{1 + |x - 4|}$

SOLUTION Let

$$f(x) = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}.$$

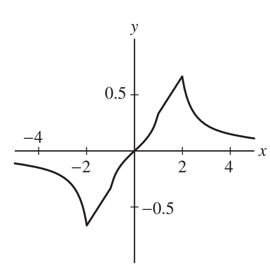
The plot follows:



We can see on the plot that the critical points of $f(x)$ lie at the cusps at $x = 1$ and $x = 4$ and at the location of the horizontal tangent line at $x = \frac{5}{2}$. With $f(-5) = \frac{17}{70}$, $f(1) = f(4) = \frac{5}{4}$, $f(\frac{5}{2}) = \frac{4}{5}$ and $f(5) = \frac{7}{10}$, it follows that the maximum value of $f(x)$ on $[-5, 5]$ is $f(1) = f(4) = \frac{5}{4}$ and the minimum value is $f(-5) = \frac{17}{70}$.

84. GU $y = \dfrac{x}{|x^2 - 1| + |x^2 - 4|}$

SOLUTION Let $f(x) = \frac{x}{|x^2-1|+|x^2-4|}$. The cusps of the graph of a function containing $|g(x)|$ are likely to lie where $g(x) = 0$, so we choose a plot range that includes $x = \pm 2$ and $x = \pm 1$:



As we can see from the graph, the function has cusps at $x = \pm 2$ and sharp corners at $x = \pm 1$. The cusps at $(2, \frac{2}{3})$ and $(-2, -\frac{2}{3})$ are the locations of the maximum and minimum values of $f(x)$, respectively.

85. **(a)** Use implicit differentiation to find the critical points on the curve $27x^2 = (x^2 + y^2)^3$.
(b) GU Plot the curve and the horizontal tangent lines on the same set of axes.

SOLUTION

(a) Differentiating both sides of the equation $27x^2 = (x^2 + y^2)^3$ with respect to x yields

$$54x = 3(x^2 + y^2)^2 \left(2x + 2y\frac{dy}{dx}\right).$$

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{27x - 3x(x^2 + y^2)^2}{3y(x^2 + y^2)^2} = \frac{x(9 - (x^2 + y^2)^2)}{y(x^2 + y^2)^2}.$$

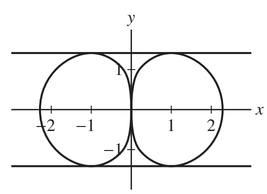
Thus, the derivative is zero when $x^2 + y^2 = 3$. Substituting into the equation for the curve, this yields $x^2 = 1$, or $x = \pm 1$. There are therefore four points at which the derivative is zero:

$$(-1, -\sqrt{2}),\ (-1, \sqrt{2}),\ (1, -\sqrt{2}),\ (1, \sqrt{2}).$$

There are also critical points where the derivative does not exist. This occurs when $y = 0$ and gives the following points with vertical tangents:

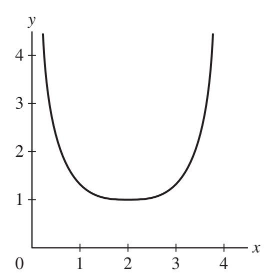
$$(0, 0),\ (\pm\sqrt[4]{27}, 0).$$

(b) The curve $27x^2 = (x^2 + y^2)^3$ and its horizontal tangents are plotted below.



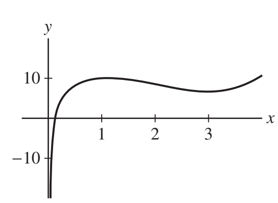
86. Sketch the graph of a continuous function on $(0, 4)$ with a minimum value but no maximum value.

SOLUTION Here is the graph of a function f on $(0, 4)$ with a minimum value [at $x = 2$] but no maximum value [since $f(x) \to \infty$ as $x \to 0+$ and as $x \to 4-$].



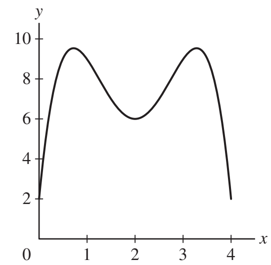
87. Sketch the graph of a continuous function on $(0, 4)$ having a local minimum but no absolute minimum.

SOLUTION Here is the graph of a function f on $(0, 4)$ with a local minimum value [between $x = 2$ and $x = 4$] but no absolute minimum [since $f(x) \to -\infty$ as $x \to 0+$].



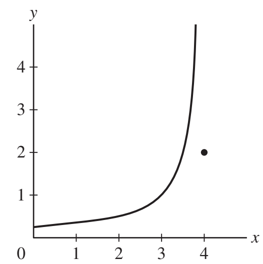
88. Sketch the graph of a function on $[0, 4]$ having
(a) Two local maxima and one local minimum.
(b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point.

SOLUTION Here is the graph of a function on $[0, 4]$ that (a) has two local maxima and one local minimum and (b) has an absolute minimum that occurs at an endpoint (at $x = 0$ or $x = 4$) and has an absolute maximum that occurs at a critical point.



89. Sketch the graph of a function $f(x)$ on $[0, 4]$ with a discontinuity such that $f(x)$ has an absolute minimum but no absolute maximum.

SOLUTION Here is the graph of a function f on $[0, 4]$ that (a) has a discontinuity [at $x = 4$] and (b) has an absolute minimum [at $x = 0$] but no absolute maximum [since $f(x) \to \infty$ as $x \to 4-$].



90. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 24. The angle between the incoming and reflected rays is $\theta = 4r - 2i$, where the angle of incidence i and refraction r are related by Snell's Law $\sin i = n \sin r$ with $n \approx 1.33$ (the index of refraction for air and water).

(a) Use Snell's Law to show that $\dfrac{dr}{di} = \dfrac{\cos i}{n \cos r}$.

(b) Show that the maximum value $\theta_{\max}$ of θ occurs when i satisfies $\cos i = \sqrt{\dfrac{n^2 - 1}{3}}$. *Hint:* Show that $\dfrac{d\theta}{di} = 0$ if $\cos i = \dfrac{n}{2} \cos r$. Then use Snell's Law to eliminate r.

(c) Show that $\theta_{\max} \approx 59.58°$.

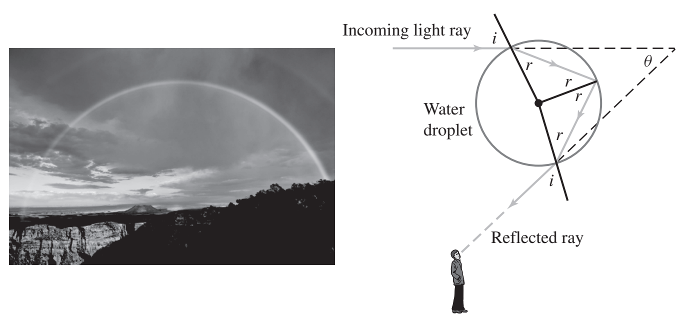


FIGURE 24

SOLUTION

(a) Differentiating Snell's Law with respect to i yields

$$\cos i = n \cos r \frac{dr}{di} \quad \text{or} \quad \frac{dr}{di} = \frac{\cos i}{n \cos r}.$$

(b) Differentiating the formula for θ with respect to i yields

$$\frac{d\theta}{di} = 4\frac{dr}{di} - 2 = 4\frac{\cos i}{n \cos r} - 2$$

by part (a). Thus,

$$\frac{d\theta}{di} = 0 \quad \text{when} \quad \cos i = \frac{n}{2} \cos r.$$

Squaring both sides of this last equation gives

$$\cos^2 i = \frac{n^2}{4}\cos^2 r,$$

while squaring both sides of Snell's Law gives

$$\sin^2 i = n^2 \sin^2 r \quad \text{or} \quad 1 - \cos^2 i = n^2(1 - \cos^2 r).$$

Solving this equation for $\cos^2 r$ gives

$$\cos^2 r = 1 - \frac{1 - \cos^2 i}{n^2};$$

Combining these last two equations and solving for $\cos i$ yields

$$\cos i = \sqrt{\frac{n^2 - 1}{3}}.$$

(c) With $n = 1.33$,

$$\cos i = \sqrt{\frac{(1.33)^2 - 1}{3}} = 0.5063$$

and

$$\cos r = \frac{2}{1.33}\cos i = 0.7613.$$

Thus, $r = 40.42°$, $i = 59.58°$ and

$$\theta_{\text{max}} = 4r - 2i = 42.53°.$$

Further Insights and Challenges

91. Show that the extreme values of $f(x) = a\sin x + b\cos x$ are $\pm\sqrt{a^2 + b^2}$.

SOLUTION If $f(x) = a\sin x + b\cos x$, then $f'(x) = a\cos x - b\sin x$, so that $f'(x) = 0$ implies $a\cos x - b\sin x = 0$. This implies $\tan x = \frac{a}{b}$. Then,

$$\sin x = \frac{\pm a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos x = \frac{\pm b}{\sqrt{a^2 + b^2}}.$$

Therefore

$$f(x) = a\sin x + b\cos x = a\frac{\pm a}{\sqrt{a^2 + b^2}} + b\frac{\pm b}{\sqrt{a^2 + b^2}} = \pm\frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \pm\sqrt{a^2 + b^2}.$$

92. Show, by considering its minimum, that $f(x) = x^2 - 2x + 3$ takes on only positive values. More generally, find the conditions on r and s under which the quadratic function $f(x) = x^2 + rx + s$ takes on only positive values. Give examples of r and s for which f takes on both positive and negative values.

SOLUTION

- Observe that $f(x) = x^2 - 2x + 3 = (x - 1)^2 + 2 > 0$ for all x. Let $f(x) = x^2 + rx + s$. Completing the square, we note that $f(x) = (x + \frac{1}{2}r)^2 + s - \frac{1}{4}r^2 > 0$ for all x provided that $s > \frac{1}{4}r^2$.
- Let $f(x) = x^2 - 4x + 3 = (x - 1)(x - 3)$. Then f takes on both positive and negative values. Here, $r = -4$ and $s = 3$.

93. Show that if the quadratic polynomial $f(x) = x^2 + rx + s$ takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

SOLUTION Let $f(x) = x^2 + rx + s$ and suppose that $f(x)$ takes on both positive and negative values. This will guarantee that f has two real roots. By the quadratic formula, the roots of f are

$$x = \frac{-r \pm \sqrt{r^2 - 4s}}{2}.$$

Observe that the midpoint between these roots is

$$\frac{1}{2}\left(\frac{-r+\sqrt{r^2-4s}}{2}+\frac{-r-\sqrt{r^2-4s}}{2}\right)=-\frac{r}{2}.$$

Next, $f'(x)=2x+r=0$ when $x=-\frac{r}{2}$ and, because the graph of $f(x)$ is an upward opening parabola, it follows that $f(-\frac{r}{2})$ is a minimum. Thus, f takes on its minimum value at the midpoint between the two roots.

94. Generalize Exercise 93: Show that if the horizontal line $y=c$ intersects the graph of $f(x)=x^2+rx+s$ at two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then $f(x)$ takes its minimum value at the midpoint $M=\dfrac{x_1+x_2}{2}$ (Figure 25).

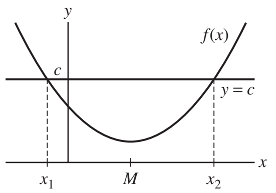


FIGURE 25

SOLUTION Suppose that a horizontal line $y=c$ intersects the graph of a quadratic function $f(x)=x^2+rx+s$ in two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Then of course $f(x_1)=f(x_2)=c$. Let $g(x)=f(x)-c$. Then $g(x_1)=g(x_2)=0$. By Exercise 93, g takes on its minimum value at $x=\frac{1}{2}(x_1+x_2)$. Hence so does $f(x)=g(x)+c$.

95. A cubic polynomial may have a local min and max, or it may have neither (Figure 26). Find conditions on the coefficients a and b of

$$f(x)=\frac{1}{3}x^3+\frac{1}{2}ax^2+bx+c$$

that ensure that f has neither a local min nor a local max. *Hint:* Apply Exercise 92 to $f'(x)$.

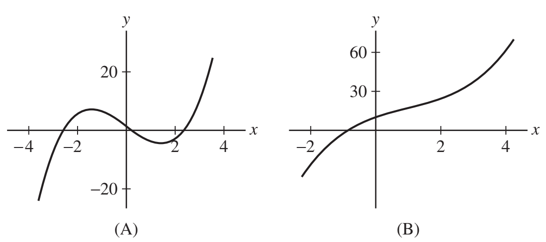


FIGURE 26 Cubic polynomials

SOLUTION Let $f(x)=\frac{1}{3}x^3+\frac{1}{2}ax^2+bx+c$. Using Exercise 92, we have $g(x)=f'(x)=x^2+ax+b>0$ for all x provided $b>\frac{1}{4}a^2$, in which case f has no critical points and hence no local extrema. (Actually $b\ge\frac{1}{4}a^2$ will suffice, since in this case [as we'll see in a later section] f has an inflection point but no local extrema.)

96. Find the min and max of

$$f(x)=x^p(1-x)^q \quad \text{on } [0,1],$$

where $p, q>0$.

SOLUTION Let $f(x)=x^p(1-x)^q$, $0\le x\le 1$, where p and q are positive numbers. Then

$$\begin{aligned} f'(x)&=x^pq(1-x)^{q-1}(-1)+(1-x)^qpx^{p-1}\\ &=x^{p-1}(1-x)^{q-1}(p(1-x)-qx)=0 \quad \text{at} \quad x=0,1,\frac{p}{p+q}\end{aligned}$$

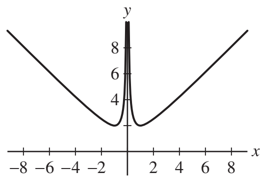
The minimum value of f on $[0,1]$ is $f(0)=f(1)=0$, whereas its maximum value is

$$f\left(\frac{p}{p+q}\right)=\frac{p^pq^q}{(p+q)^{p+q}}.$$

97. Prove that if f is continuous and $f(a)$ and $f(b)$ are local minima where $a<b$, then there exists a value c between a and b such that $f(c)$ is a local maximum. (*Hint:* Apply Theorem 1 to the interval $[a,b]$.) Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

SOLUTION

- Let $f(x)$ be a continuous function with $f(a)$ and $f(b)$ local minima on the interval $[a, b]$. By Theorem 1, $f(x)$ must take on both a minimum and a maximum on $[a, b]$. Since local minima occur at $f(a)$ and $f(b)$, the maximum must occur at some other point in the interval, call it c, where $f(c)$ is a local maximum.
- The function graphed here is discontinuous at $x = 0$.



4.3 The Mean Value Theorem and Monotonicity

Preliminary Questions

1. For which value of m is the following statement correct? If $f(2) = 3$ and $f(4) = 9$, and $f(x)$ is differentiable, then f has a tangent line of slope m.

SOLUTION The Mean Value Theorem guarantees that the function has a tangent line with slope equal to

$$\frac{f(4) - f(2)}{4 - 2} = \frac{9 - 3}{4 - 2} = 3.$$

Hence, $m = 3$ makes the statement correct.

2. Assume f is differentiable. Which of the following statements does *not* follow from the MVT?

(a) If f has a secant line of slope 0, then f has a tangent line of slope 0.

(b) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in (5, 9)$.

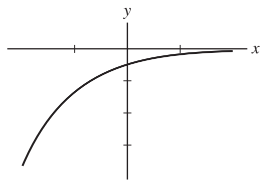
(c) If f has a tangent line of slope 0, then f has a secant line of slope 0.

(d) If $f'(x) > 0$ for all x, then every secant line has positive slope.

SOLUTION Conclusion **(c)** does not follow from the Mean Value Theorem. As a counterexample, consider the function $f(x) = x^3$. Note that $f'(0) = 0$, but no secant line has zero slope.

3. Can a function that takes on only negative values have a positive derivative? If so, sketch an example.

SOLUTION Yes. The figure below displays a function that takes on only negative values but has a positive derivative.



4. For $f(x)$ with derivative as in Figure 12:

(a) Is $f(c)$ a local minimum or maximum?

(b) Is $f(x)$ a decreasing function?

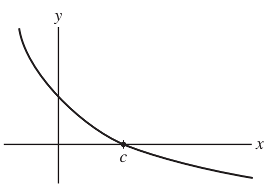


FIGURE 12 Graph of derivative $f'(x)$.

SOLUTION

(a) To the left of $x = c$, the derivative is positive, so f is increasing; to the right of $x = c$, the derivative is negative, so f is decreasing. Consequently, $f(c)$ must be a local maximum.

(b) No. The derivative is a decreasing function, but as noted in part (a), $f(x)$ is increasing for $x < c$ and decreasing for $x > c$.

Exercises

In Exercises 1–8, find a point c satisfying the conclusion of the MVT for the given function and interval.

1. $y = x^{-1}$, $[2, 8]$

SOLUTION Let $f(x) = x^{-1}$, $a = 2$, $b = 8$. Then $f'(x) = -x^{-2}$, and by the MVT, there exists a $c \in (2, 8)$ such that

$$-\frac{1}{c^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{8} - \frac{1}{2}}{8 - 2} = -\frac{1}{16}.$$

Thus $c^2 = 16$ and $c = \pm 4$. Choose $c = 4 \in (2, 8)$.

2. $y = \sqrt{x}$, $[9, 25]$

SOLUTION Let $f(x) = x^{1/2}$, $a = 9$, $b = 25$. Then $f'(x) = \frac{1}{2}x^{-1/2}$, and by the MVT, there exists a $c \in (9, 25)$ such that

$$\frac{1}{2}c^{-1/2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{5 - 3}{25 - 9} = \frac{1}{8}.$$

Thus $\frac{1}{\sqrt{c}} = \frac{1}{4}$ and $c = 16 \in (9, 25)$.

3. $y = \cos x - \sin x$, $[0, 2\pi]$

SOLUTION Let $f(x) = \cos x - \sin x$, $a = 0$, $b = 2\pi$. Then $f'(x) = -\sin x - \cos x$, and by the MVT, there exists a $c \in (0, 2\pi)$ such that

$$-\sin c - \cos c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{2\pi - 0} = 0.$$

Thus $-\sin c = \cos c$. Choose either $c = \frac{3\pi}{4}$ or $c = \frac{7\pi}{4} \in (0, 2\pi)$.

4. $y = \dfrac{x}{x + 2}$, $[1, 4]$

SOLUTION Let $f(x) = x/(x + 2)$, $a = 1$, $b = 4$. Then $f'(x) = \frac{2}{(x+2)^2}$, and by the MVT, there exists a $c \in (1, 4)$ such that

$$\frac{2}{(c + 2)^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{1}{9}.$$

Thus $(c + 2)^2 = 18$ and $c = -2 \pm 3\sqrt{2}$. Choose $c = 3\sqrt{2} - 2 \approx 2.24 \in (1, 4)$.

5. $y = x^3$, $[-4, 5]$

SOLUTION Let $f(x) = x^3$, $a = -4$, $b = 5$. Then $f'(x) = 3x^2$, and by the MVT, there exists a $c \in (-4, 5)$ such that

$$3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{189}{9} = 21.$$

Solving for c yields $c^2 = 7$, so $c = \pm\sqrt{7}$. Both of these values are in the interval $[-4, 5]$, so either value can be chosen.

6. $y = x \ln x$, $[1, 2]$

SOLUTION Let $f(x) = x \ln x$, $a = 1$, $b = 2$. Then $f'(x) = 1 + \ln x$, and by the MVT, there exists a $c \in (1, 2)$ such that

$$1 + \ln c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{2 \ln 2}{1} = 2 \ln 2.$$

Solving for c yields $c = e^{2ln2-1} = \frac{4}{e} \approx 1.4715 \in (1, 2)$.

7. $y = e^{-2x}$, $[0, 3]$

SOLUTION Let $f(x) = e^{-2x}$, $a = 0$, $b = 3$. Then $f'(x) = -2e^{-2x}$, and by the MVT, there exists a $c \in (0, 3)$ such that

$$-2e^{-2c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{e^{-6} - 1}{3 - 0} = \frac{e^{-6} - 1}{3}.$$

Solving for c yields

$$c = -\frac{1}{2} \ln\left(\frac{1 - e^{-6}}{6}\right) \approx 0.8971 \in (0, 3).$$

8. $y = e^x - x, \quad [-1, 1]$

SOLUTION Let $f(x) = e^x - x$, $a = -1$, $b = 1$. Then $f'(x) = e^x - 1$, and by the MVT, there exists a $c \in (-1, 1)$ such that

$$e^c - 1 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{(e - 1) - (e^{-1} + 1)}{1 - (-1)} = \frac{1}{2}(e - e^{-1}) - 1.$$

Solving for c yields

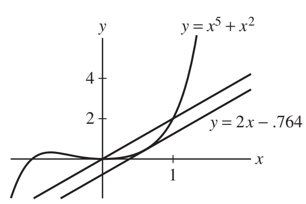
$$c = \ln\left(\frac{e - e^{-1}}{2}\right) \approx 0.1614 \in (-1, 1).$$

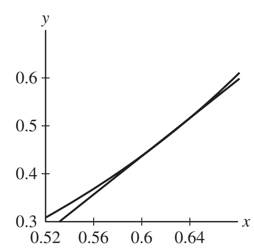
9. GU Let $f(x) = x^5 + x^2$. The secant line between $x = 0$ and $x = 1$ has slope 2 (check this), so by the MVT, $f'(c) = 2$ for some $c \in (0, 1)$. Plot $f(x)$ and the secant line on the same axes. Then plot $y = 2x + b$ for different values of b until the line becomes tangent to the graph of f. Zoom in on the point of tangency to estimate x-coordinate c of the point of tangency.

SOLUTION Let $f(x) = x^5 + x^2$. The slope of the secant line between $x = 0$ and $x = 1$ is

$$\frac{f(1) - f(0)}{1 - 0} = \frac{2 - 0}{1} = 2.$$

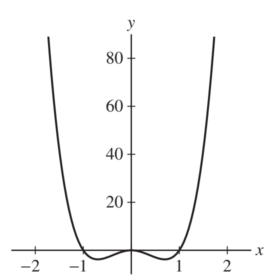
A plot of $f(x)$, the secant line between $x = 0$ and $x = 1$, and the line $y = 2x - 0.764$ is shown below at the left. The line $y = 2x - 0.764$ appears to be tangent to the graph of $y = f(x)$. Zooming in on the point of tangency (see below at the right), it appears that the x-coordinate of the point of tangency is approximately 0.62.

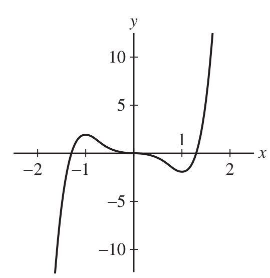




10. GU Plot the derivative of $f(x) = 3x^5 - 5x^3$. Describe its sign changes and use this to determine the local extreme values of $f(x)$. Then graph $f(x)$ to confirm your conclusions.

SOLUTION Let $f(x) = 3x^5 - 5x^3$. Then $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$. The graph of $f'(x)$ is shown below at the left. Because $f'(x)$ changes from positive to negative at $x = -1$, $f(x)$ changes from increasing to decreasing and therefore has a local maximum at $x = -1$. At $x = 1$, $f'(x)$ changes from negative to positive, so $f(x)$ changes from decreasing to increasing and therefore has a local minimum. Though $f'(x) = 0$ at $x = 0$, $f'(x)$ does not change sign at $x = 0$, so $f(x)$ has neither a local maximum nor a local minimum at $x = 0$. The graph of $f(x)$, shown below at the right, confirms each of these conclusions.





11. Determine the intervals on which $f'(x)$ is positive and negative, assuming that Figure 13 is the graph of $f(x)$.

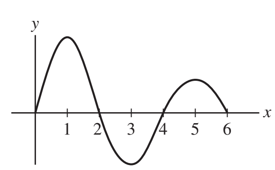


FIGURE 13

SOLUTION The derivative of f is positive on the intervals $(0, 1)$ and $(3, 5)$ where f is increasing; it is negative on the intervals $(1, 3)$ and $(5, 6)$ where f is decreasing.

12. Determine the intervals on which $f(x)$ is increasing or decreasing, assuming that Figure 13 is the graph of $f'(x)$.

SOLUTION $f(x)$ is increasing on every interval (a, b) over which $f'(x) > 0$, and is decreasing on every interval over which $f'(x) < 0$. If the graph of $f'(x)$ is given in Figure 13, then $f(x)$ is increasing on the intervals $(0, 2)$ and $(4, 6)$, and is decreasing on the interval $(2, 4)$.

13. State whether $f(2)$ and $f(4)$ are local minima or local maxima, assuming that Figure 13 is the graph of $f'(x)$.

SOLUTION

- $f'(x)$ makes a transition from positive to negative at $x = 2$, so $f(2)$ is a local maximum.
- $f'(x)$ makes a transition from negative to positive at $x = 4$, so $f(4)$ is a local minimum.

14. Figure 14 shows the graph of the derivative $f'(x)$ of a function $f(x)$. Find the critical points of $f(x)$ and determine whether they are local minima, local maxima, or neither.

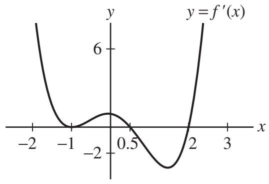


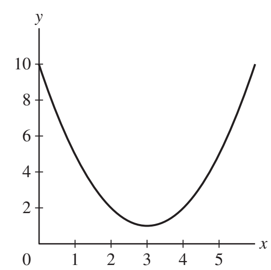
FIGURE 14

SOLUTION Since $f'(x) = 0$ when $x = -1$, $x = \frac{1}{2}$ and $x = 2$, these are the critical points of f. At $x = -1$, there is no sign transition in f', so $f(-1)$ is neither a local maximum nor a local minimum. At $x = \frac{1}{2}$, f' transitions from $+$ to $-$, so $f(\frac{1}{2})$ is a local maximum. Finally, at $x = 2$, f' transitions from $-$ to $+$, so $f(2)$ is a local minimum.

In Exercises 15–18, sketch the graph of a function $f(x)$ whose derivative $f'(x)$ has the given description.

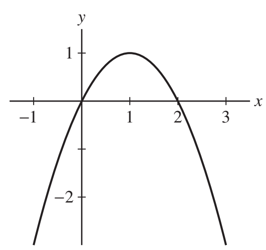
15. $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$

SOLUTION Here is the graph of a function f for which $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$.



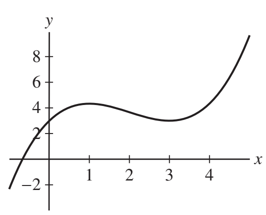
16. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$

SOLUTION Here is the graph of a function f for which $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.



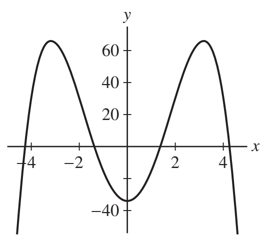
17. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.

SOLUTION Here is the graph of a function f for which $f'(x)$ is negative on $(1, 3)$ and positive elsewhere.



18. $f'(x)$ makes the sign transitions $+, -, +, -$.

SOLUTION Here is the graph of a function f for which f' makes the sign transitions $+, -, +, -$.



In Exercises 19–22, find all critical points of f and use the First Derivative Test to determine whether they are local minima or maxima.

19. $f(x) = 4 + 6x - x^2$

SOLUTION Let $f(x) = 4 + 6x - x^2$. Then $f'(x) = 6 - 2x = 0$ implies that $x = 3$ is the only critical point of f. As x increases through 3, $f'(x)$ makes the sign transition $+, -$. Therefore, $f(3) = 13$ is a local maximum.

20. $f(x) = x^3 - 12x - 4$

SOLUTION Let $f(x) = x^3 - 12x - 4$. Then, $f'(x) = 3x^2 - 12 = 3(x-2)(x+2) = 0$ implies that $x = \pm 2$ are critical points of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-2)$ is a local maximum. On the other hand, as x increases through 2, $f'(x)$ makes the sign transition $-, +$; therefore, $f(2)$ is a local minimum.

21. $f(x) = \dfrac{x^2}{x+1}$

SOLUTION Let $f(x) = \dfrac{x^2}{x+1}$. Then

$$f'(x) = \frac{x(x+2)}{(x+1)^2} = 0$$

implies that $x = 0$ and $x = -2$ are critical points. Note that $x = -1$ is not a critical point because it is not in the domain of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$ so $f(-2) = -4$ is a local maximum. As x increases through 0, $f'(x)$ makes the sign transition $-, +$ so $f(0) = 0$ is a local minimum.

22. $f(x) = x^3 + x^{-3}$

SOLUTION Let $f(x) = x^3 + x^{-3}$. Then

$$f'(x) = 3x^2 - 3x^{-4} = \frac{3}{x^4}(x^6 - 1) = \frac{3}{x^4}(x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1) = 0$$

implies that $x = \pm 1$ are critical points of f. Though $f'(x)$ does not exist at $x = 0$, $x = 0$ is not a critical point of f because it is not in the domain of f. As x increases through -1, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-1)$ is a local maximum. On the other hand, as x increases through 1, $f'(x)$ makes the sign transition $-, +$; therefore, $f(1)$ is a local minimum.

In Exercises 23–52, find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point is a local min or max (or neither).

SOLUTION *Here is a table legend for Exercises 23–44.*

SYMBOL	*MEANING*
$-$	*The entity is negative on the given interval.*
0	*The entity is zero at the specified point.*
$+$	*The entity is positive on the given interval.*
U	*The entity is undefined at the specified point.*
↗	*f is increasing on the given interval.*
↘	*f is decreasing on the given interval.*
M	*f has a local maximum at the specified point.*
m	*f has a local minimum at the specified point.*
¬	*There is no local extremum here.*

23. $y = -x^2 + 7x - 17$

SOLUTION Let $f(x) = -x^2 + 7x - 17$. Then $f'(x) = 7 - 2x = 0$ yields the critical point $c = \frac{7}{2}$.

x	$\left(-\infty, \frac{7}{2}\right)$	$7/2$	$\left(\frac{7}{2}, \infty\right)$
f'	$+$	0	$-$
f	↗	M	↘

24. $y = 5x^2 + 6x - 4$

SOLUTION Let $f(x) = 5x^2 + 6x - 4$. Then $f'(x) = 10x + 6 = 0$ yields the critical point $c = -\frac{3}{5}$.

x	$\left(-\infty, -\frac{3}{5}\right)$	$-3/5$	$\left(-\frac{3}{5}, \infty\right)$
f'	$-$	0	$+$
f	↘	m	↗

25. $y = x^3 - 12x^2$

SOLUTION Let $f(x) = x^3 - 12x^2$. Then $f'(x) = 3x^2 - 24x = 3x(x - 8) = 0$ yields critical points $c = 0, 8$.

x	$(-\infty, 0)$	0	$(0, 8)$	8	$(8, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

26. $y = x(x - 2)^3$

SOLUTION Let $f(x) = x\,(x - 2)^3$. Then

$$f'(x) = x \cdot 3\,(x - 2)^2 + (x - 2)^3 \cdot 1 = (4x - 2)\,(x - 2)^2 = 0$$

yields critical points $c = 2, \frac{1}{2}$.

x	$(-\infty, 1/2)$	$1/2$	$(1/2, 2)$	2	$(2, \infty)$
f'	$-$	0	$+$	0	$+$
f	↘	m	↗	¬	↗

27. $y = 3x^4 + 8x^3 - 6x^2 - 24x$

SOLUTION Let $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x$. Then

$$\begin{aligned} f'(x) &= 12x^3 + 24x^2 - 12x - 24 \\ &= 12x^2(x + 2) - 12(x + 2) = 12(x + 2)(x^2 - 1) \\ &= 12\,(x - 1)\,(x + 1)\,(x + 2) = 0 \end{aligned}$$

yields critical points $c = -2, -1, 1$.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$	0	$-$	0	$+$
f	↘	m	↗	M	↘	m	↗

28. $y = x^2 + (10 - x)^2$

SOLUTION Let $f(x) = x^2 + (10 - x)^2$. Then $f'(x) = 2x + 2\,(10 - x)\,(-1) = 4x - 20 = 0$ yields the critical point $c = 5$.

x	$(-\infty, 5)$	5	$(5, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

29. $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$

SOLUTION Let $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$. Then $f'(x) = x^2 + 3x + 2 = (x+1)(x+2) = 0$ yields critical points $c = -2, -1$.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

30. $y = x^4 + x^3$

SOLUTION Let $f(x) = x^4 + x^3$. Then $f'(x) = 4x^3 + 3x^2 = x^2(4x + 3)$ yields critical points $c = 0, -\frac{3}{4}$.

x	$\left(-\infty, -\frac{3}{4}\right)$	$-\frac{3}{4}$	$\left(-\frac{3}{4}, 0\right)$	0	$(0, \infty)$
f'	$-$	0	$+$	0	$+$
f	↘	m	↗	¬	↗

31. $y = x^5 + x^3 + 1$

SOLUTION Let $f(x) = x^5 + x^3 + 1$. Then $f'(x) = 5x^4 + 3x^2 = x^2(5x^2 + 3)$ yields a single critical point: $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$+$
f	↗	¬	↗

32. $y = x^5 + x^3 + x$

SOLUTION Let $f(x) = x^5 + x^3 + x$. Then $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ for all x. Thus, f has no critical points and is always increasing.

33. $y = x^4 - 4x^{3/2} \quad (x > 0)$

SOLUTION Let $f(x) = x^4 - 4x^{3/2}$ for $x > 0$. Then $f'(x) = 4x^3 - 6x^{1/2} = 2x^{1/2}(2x^{5/2} - 3) = 0$, which gives us the critical point $c = (\frac{3}{2})^{2/5}$. (*Note:* $c = 0$ is not in the interval under consideration.)

x	$\left(0, \left(\frac{3}{2}\right)^{2/5}\right)$	$\frac{3}{2}^{2/5}$	$\left(\left(\frac{3}{2}\right)^{2/5}, \infty\right)$
f'	$-$	0	$+$
f	↘	m	↗

34. $y = x^{5/2} - x^2 \quad (x > 0)$

SOLUTION Let $f(x) = x^{5/2} - x^2$. Then $f'(x) = \frac{5}{2}x^{3/2} - 2x = x(\frac{5}{2}x^{1/2} - 2) = 0$, so the critical point is $c = \frac{16}{25}$. (*Note:* $c = 0$ is not in the interval under consideration.)

x	$\left(0, \frac{16}{25}\right)$	$\frac{16}{25}$	$\left(\frac{16}{25}, \infty\right)$
f'	$-$	0	$+$
f	↘	m	↗

35. $y = x + x^{-1} \quad (x > 0)$

SOLUTION Let $f(x) = x + x^{-1}$ for $x > 0$. Then $f'(x) = 1 - x^{-2} = 0$ yields the critical point $c = 1$. (*Note:* $c = -1$ is not in the interval under consideration.)

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

36. $y = x^{-2} - 4x^{-1} \quad (x > 0)$

SOLUTION Let $f(x) = x^{-2} - 4x^{-1}$. Then $f'(x) = -2x^{-3} + 4x^{-2} = 0$ yields $-2 + 4x = 0$. Thus, $2x = 1$, and $x = \frac{1}{2}$.

x	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

37. $y = \dfrac{1}{x^2 + 1}$

SOLUTION Let $f(x) = \left(x^2+1\right)^{-1}$. Then $f'(x) = -2x\left(x^2+1\right)^{-2} = 0$ yields critical point $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$
f	↗	M	↘

38. $y = \dfrac{2x + 1}{x^2 + 1}$

SOLUTION Let $f(x) = \dfrac{2x+1}{x^2+1}$. Then

$$f'(x) = \frac{\left(x^2+1\right)(2) - (2x+1)(2x)}{\left(x^2+1\right)^2} = \frac{-2\left(x^2+x-1\right)}{\left(x^2+1\right)^2} = 0$$

yields critical points $c = \dfrac{-1 \pm \sqrt{5}}{2}$.

x	$\left(-\infty, \frac{-1-\sqrt{5}}{2}\right)$	$\frac{-1-\sqrt{5}}{2}$	$\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$	$\frac{-1+\sqrt{5}}{2}$	$\left(\frac{-1+\sqrt{5}}{2}, \infty\right)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

39. $y = \dfrac{x^3}{x^2 + 1}$

SOLUTION Let $f(x) = \dfrac{x^3}{x^2+1}$. Then

$$f'(x) = \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2} = 0$$

yields the single critical point $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$+$
f	↗	¬	↗

40. $y = \dfrac{x^3}{x^2 - 3}$

SOLUTION Let $f(x) = \dfrac{x^3}{x^2-3}$. Then

$$f'(x) = \frac{(x^2-3)(3x^2) - x^3(2x)}{(x^2-3)^2} = \frac{x^2(x^2-9)}{(x^2-3)^2} = 0$$

yields the critical points $c = 0$ and $c = \pm 3$. $c = \pm\sqrt{3}$ are not critical points because they are not in the domain of f.

x	$(-\infty, -3)$	-3	$(-3, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, 3)$	3	$(3, \infty)$
f'	$+$	0	$-$	∞	$-$	0	$-$	∞	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	¬	$\searrow$	¬	$\searrow$	¬	$\searrow$	m	$\nearrow$

41. $y = \theta + \sin\theta + \cos\theta$

SOLUTION Let $f(\theta) = \theta + \sin\theta + \cos\theta$. Then $f'(\theta) = 1 + \cos\theta - \sin\theta = 0$ yields the critical points $c = \frac{\pi}{2}$ and $c = \pi$.

θ	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \pi\right)$	π	$(\pi, 2\pi)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

42. $y = \sin\theta + \sqrt{3}\cos\theta$

SOLUTION Let $f(\theta) = \sin\theta + \sqrt{3}\cos\theta$. Then $f'(\theta) = \cos\theta - \sqrt{3}\sin\theta = 0$ yields the critical points $c = \frac{\pi}{6}$ and $c = \frac{7\pi}{6}$.

θ	$\left(0, \frac{\pi}{6}\right)$	$\frac{\pi}{6}$	$\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

43. $y = \sin^2\theta + \sin\theta$

SOLUTION Let $f(\theta) = \sin^2\theta + \sin\theta$. Then $f'(\theta) = 2\sin\theta\cos\theta + \cos\theta = \cos\theta(2\sin\theta + 1) = 0$ yields the critical points $c = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$.

θ	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$	M	$\searrow$	m	$\nearrow$

44. $y = \theta - 2\cos\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta - 2\cos\theta$. Then $f'(\theta) = 1 + 2\sin\theta = 0$, which yields $c = \frac{7\pi}{6}, \frac{11\pi}{6}$ on the interval $[0, 2\pi]$.

θ	$\left(0, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

45. $y = x + e^{-x}$

SOLUTION Let $f(x) = x + e^{-x}$. Then $f'(x) = 1 - e^{-x}$, which yields $c = 0$ as the only critical point.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$-$	0	$+$
f	$\searrow$	m	$\nearrow$

46. $y = \dfrac{e^x}{x} \quad (x > 0)$

SOLUTION Let $f(x) = \frac{e^x}{x}$. Then

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2},$$

which yields $c = 1$ as the only critical point.

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	$\searrow$	m	$\nearrow$

47. $y = e^{-x}\cos x, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = e^{-x}\cos x$. Then

$$f'(x) = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x),$$

which yields $c = -\frac{\pi}{4}$ as the only critical point on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

x	$\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}, \frac{\pi}{2}\right]$
f'	$+$	0	$-$
f	↗	M	↘

48. $y = x^2e^x$

SOLUTION Let $f(x) = x^2e^x$. Then $f'(x) = x^2e^x + 2xe^x = xe^x(x+2)$, which yields $c = -2$ and $c = 0$ as critical points.

x	$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

49. $y = \tan^{-1} x - \frac{1}{2}x$

SOLUTION Let $f(x) = \tan^{-1} x - \frac{1}{2}x$. Then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{2},$$

which yields $c = \pm 1$ as critical points.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

50. $y = (x^2 - 2x)e^x$

SOLUTION Let $f(x) = (x^2 - 2x)e^x$. Then

$$f'(x) = (x^2 - 2x)e^x + (2x - 2)e^x = (x^2 - 2)e^x,$$

which yields $c = \pm\sqrt{2}$ as critical points.

x	$(-\infty, \sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2}, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2}, \infty)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

51. $y = x - \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x - \ln x$. Then $f'(x) = 1 - x^{-1}$, which yields $c = 1$ as the only critical point.

x	$(0, 1)$	1	$(1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

52. $y = \dfrac{\ln x}{x} \quad (x > 0)$

SOLUTION Let $f(x) = \frac{\ln x}{x}$. Then

$$f'(x) = \frac{1 - \ln x}{x^2},$$

which yields $c = e$ as the only critical point.

x	$(0, e)$	e	(e, ∞)
f'	$+$	0	$-$
f	$\nearrow$	M	$\searrow$

53. Find the minimum value of $f(x) = x^x$ for $x > 0$.

SOLUTION Let $f(x) = x^x$. By logarithmic differentiation, we know that $f'(x) = x^x(1 + \ln x)$. Thus, $x = \frac{1}{e}$ is the only critical point. Because $f'(x) < 0$ for $0 < x < \frac{1}{e}$ and $f'(x) > 0$ for $x > \frac{1}{e}$,

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} \approx 0.692201$$

is the minimum value.

54. Show that $f(x) = x^2 + bx + c$ is decreasing on $\left(-\infty, -\frac{b}{2}\right)$ and increasing on $\left(-\frac{b}{2}, \infty\right)$.

SOLUTION Let $f(x) = x^2 + bx + c$. Then $f'(x) = 2x + b = 0$ yields the critical point $c = -\frac{b}{2}$.

- For $x < -\frac{b}{2}$, we have $f'(x) < 0$, so f is decreasing on $\left(-\infty, -\frac{b}{2}\right)$.
- For $x > -\frac{b}{2}$, we have $f'(x) > 0$, so f is increasing on $\left(-\frac{b}{2}, \infty\right)$.

55. Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function. *Hint:* Find the minimum value of $f'(x)$.

SOLUTION Let $f(x) = x^3 - 2x^2 + 2x$. For all x, we have

$$f'(x) = 3x^2 - 4x + 2 = 3\left(x - \frac{2}{3}\right)^2 + \frac{2}{3} \geq \frac{2}{3} > 0.$$

Since $f'(x) > 0$ for all x, the function f is everywhere increasing.

56. Find conditions on a and b that ensure that $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$.

SOLUTION Let $f(x) = x^3 + ax + b$.

- If $a > 0$, then $f'(x) = 3x^2 + a > 0$ and f is increasing for all x.
- If $a = 0$, then

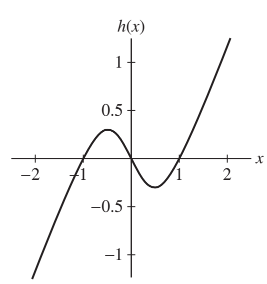
  $$f(x_2) - f(x_1) = (3x_2^3 + b) - (3x_1^3 + b) = 3(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2) > 0$$

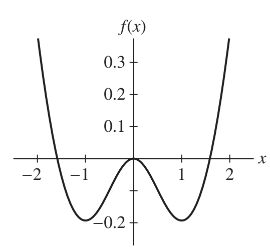
  whenever $x_2 > x_1$. Thus, f is increasing for all x.
- If $a < 0$, then $f'(x) = 3x^2 + a < 0$ and f is decreasing for $|x| < \sqrt{-\frac{a}{3}}$.

In summary, $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$ whenever $a \geq 0$.

57. $\boxed{\text{GU}}$ Let $h(x) = \dfrac{x(x^2 - 1)}{x^2 + 1}$ and suppose that $f'(x) = h(x)$. Plot $h(x)$ and use the plot to describe the local extrema and the increasing/decreasing behavior of $f(x)$. Sketch a plausible graph for $f(x)$ itself.

SOLUTION The graph of $h(x)$ is shown below at the left. Because $h(x)$ is negative for $x < -1$ and for $0 < x < 1$, it follows that $f(x)$ is decreasing for $x < -1$ and for $0 < x < 1$. Similarly, $f(x)$ is increasing for $-1 < x < 0$ and for $x > 1$ because $h(x)$ is positive on these intervals. Moreover, $f(x)$ has local minima at $x = -1$ and $x = 1$ and a local maximum at $x = 0$. A plausible graph for $f(x)$ is shown below at the right.





58. Sam made two statements that Deborah found dubious.

(a) "The average velocity for my trip was 70 mph; at no point in time did my speedometer read 70 mph."

(b) "A policeman clocked me going 70 mph, but my speedometer never read 65 mph."

In each case, which theorem did Deborah apply to prove Sam's statement false: the Intermediate Value Theorem or the Mean Value Theorem? Explain.

SOLUTION

(a) Deborah is applying the Mean Value Theorem here. Let $s(t)$ be Sam's distance, in miles, from his starting point, let a be the start time for Sam's trip, and let b be the end time of the same trip. Sam is claiming that at no point was

$$s'(t) = \frac{s(b) - s(a)}{b - a}.$$

This violates the MVT.

(b) Deborah is applying the Intermediate Value Theorem here. Let $v(t)$ be Sam's velocity in miles per hour. Sam started out at rest, and reached a velocity of 70 mph. By the IVT, he should have reached a velocity of 65 mph at some point.

59. Determine where $f(x) = (1000 - x)^2 + x^2$ is decreasing. Use this to decide which is larger: $800^2 + 200^2$ or $600^2 + 400^2$.

SOLUTION If $f(x) = (1000 - x)^2 + x^2$, then $f'(x) = -2(1000 - x) + 2x = 4x - 2000$. $f'(x) < 0$ as long as $x < 500$. Therefore, $800^2 + 200^2 = f(200) > f(400) = 600^2 + 400^2$.

60. Show that $f(x) = 1 - |x|$ satisfies the conclusion of the MVT on $[a, b]$ if both a and b are positive or negative, but not if $a < 0$ and $b > 0$.

SOLUTION Let $f(x) = 1 - |x|$.

- If a and b (where $a < b$) are both positive (or both negative), then f is continuous on $[a, b]$ and differentiable on (a, b). Accordingly, the hypotheses of the MVT are met and the theorem does apply. Indeed, in these cases, any point $c \in (a, b)$ satisfies the conclusion of the MVT (since f' is constant on $[a, b]$ in these instances).
- For $a = -2$ and $b = 1$, we have $\dfrac{f(b) - f(a)}{b - a} = \dfrac{0 - (-1)}{1 - (-2)} = \dfrac{1}{3}$. Yet there is no point $c \in (-2, 1)$ such that $f'(c) = \frac{1}{3}$. Indeed, $f'(x) = 1$ for $x < 0$, $f'(x) = -1$ for $x > 0$, and $f'(0)$ is undefined. The MVT does not apply in this case, since f is not differentiable on the open interval $(-2, 1)$.

61. Which values of c satisfy the conclusion of the MVT on the interval $[a, b]$ if $f(x)$ is a linear function?

SOLUTION Let $f(x) = px + q$, where p and q are constants. Then the slope of *every* secant line and tangent line of f is p. Accordingly, considering the interval $[a, b]$, *every* point $c \in (a, b)$ satisfies $f'(c) = p = \dfrac{f(b) - f(a)}{b - a}$, the conclusion of the MVT.

62. Show that if $f(x)$ is any quadratic polynomial, then the midpoint $c = \dfrac{a + b}{2}$ satisfies the conclusion of the MVT on $[a, b]$ for any a and b.

SOLUTION Let $f(x) = px^2 + qx + r$ with $p \neq 0$ and consider the interval $[a, b]$. Then $f'(x) = 2px + q$, and by the MVT we have

$$2pc + q = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\left(pb^2 + qb + r\right) - \left(pa^2 + qa + r\right)}{b - a}$$
$$= \frac{(b - a)\,(p\,(b + a) + q)}{b - a} = p\,(b + a) + q$$

Thus $2pc + q = p(a + b) + q$, and $c = \dfrac{a + b}{2}$.

63. Suppose that $f(0) = 2$ and $f'(x) \le 3$ for $x > 0$. Apply the MVT to the interval $[0, 4]$ to prove that $f(4) \le 14$. Prove more generally that $f(x) \le 2 + 3x$ for all $x > 0$.

SOLUTION The MVT, applied to the interval $[0, 4]$, guarantees that there exists a $c \in (0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} \qquad \text{or} \qquad f(4) - f(0) = 4f'(c).$$

Because $c > 0$, $f'(c) \le 3$, so $f(4) - f(0) \le 12$. Finally, $f(4) \le f(0) + 12 = 14$.

More generally, let $x > 0$. The MVT, applied to the interval $[0, x]$, guarantees there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \qquad \text{or} \qquad f(x) - f(0) = f'(c)x.$$

Because $c > 0$, $f'(c) \le 3$, so $f(x) - f(0) \le 3x$. Finally, $f(x) \le f(0) + 3x = 3x + 2$.

64. Show that if $f(2) = -2$ and $f'(x) \ge 5$ for $x > 2$, then $f(4) \ge 8$.

SOLUTION The MVT, applied to the interval $[2, 4]$, guarantees there exists a $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \qquad \text{or} \qquad f(4) - f(2) = 2f'(c).$$

Because $f'(x) \ge 5$, it follows that $f(4) - f(2) \ge 10$, or $f(4) \ge f(2) + 10 = 8$.

65. Show that if $f(2) = 5$ and $f'(x) \geq 10$ for $x > 2$, then $f(x) \geq 10x - 15$ for all $x > 2$.

SOLUTION Let $x > 2$. The MVT, applied to the interval $[2, x]$, guarantees there exists a $c \in (2, x)$ such that

$$f'(c) = \frac{f(x) - f(2)}{x - 2} \qquad \text{or} \qquad f(x) - f(2) = (x - 2) f'(c).$$

Because $f'(x) \geq 10$, it follows that $f(x) - f(2) \geq 10(x - 2)$, or $f(x) \geq f(2) + 10(x - 2) = 10x - 15$.

Further Insights and Challenges

66. Show that a cubic function $f(x) = x^3 + ax^2 + bx + c$ is increasing on $(-\infty, \infty)$ if $b > a^2/3$.

SOLUTION Let $f(x) = x^3 + ax^2 + bx + c$. Then $f'(x) = 3x^2 + 2ax + b = 3\left(x + \frac{a}{3}\right)^2 - \frac{a^2}{3} + b > 0$ for all x if $b - \frac{a^2}{3} > 0$. Therefore, if $b > a^2/3$, then $f(x)$ is increasing on $(-\infty, \infty)$.

67. Prove that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. *Hint:* Show that $f(x) - g(x)$ is nonincreasing.

SOLUTION Let $h(x) = f(x) - g(x)$. By the sum rule, $h'(x) = f'(x) - g'(x)$. Since $f'(x) \leq g'(x)$ for all $x \geq 0$, $h'(x) \leq 0$ for all $x \geq 0$. This implies that h is nonincreasing. Since $h(0) = f(0) - g(0) = 0$, $h(x) \leq 0$ for all $x \geq 0$ (as h is nonincreasing, it cannot climb above zero). Hence $f(x) - g(x) \leq 0$ for all $x \geq 0$, and so $f(x) \leq g(x)$ for $x \geq 0$.

68. Use Exercise 67 to prove that $x \leq \tan x$ for $0 \leq x < \frac{\pi}{2}$.

SOLUTION Let $f(x) = x$ and $g(x) = \tan x$. Then $f(0) = g(0) = 0$ and $f'(x) = 1 \leq \sec^2 x = g'(x)$ for $0 \leq x < \frac{\pi}{2}$. Apply the result of Exercise 67 to conclude that $x \leq \tan x$ for $0 \leq x < \frac{\pi}{2}$.

69. Use Exercise 67 and the inequality $\sin x \leq x$ for $x \geq 0$ (established in Theorem 3 of Section 2.6) to prove the following assertions for all $x \geq 0$ (each assertion follows from the previous one).

(a) $\cos x \geq 1 - \frac{1}{2}x^2$

(b) $\sin x \geq x - \frac{1}{6}x^3$

(c) $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

(d) Can you guess the next inequality in the series?

SOLUTION

(a) We prove this using Exercise 67: Let $g(x) = \cos x$ and $f(x) = 1 - \frac{1}{2}x^2$. Then $f(0) = g(0) = 1$ and $g'(x) = -\sin x \geq -x = f'(x)$ for $x \geq 0$ by Exercise 68. Now apply Exercise 67 to conclude that $\cos x \geq 1 - \frac{1}{2}x^2$ for $x \geq 0$.

(b) Let $g(x) = \sin x$ and $f(x) = x - \frac{1}{6}x^3$. Then $f(0) = g(0) = 0$ and $g'(x) = \cos x \geq 1 - \frac{1}{2}x^2 = f'(x)$ for $x \geq 0$ by part (a). Now apply Exercise 67 to conclude that $\sin x \geq x - \frac{1}{6}x^3$ for $x \geq 0$.

(c) Let $g(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $f(x) = \cos x$. Then $f(0) = g(0) = 1$ and $g'(x) = -x + \frac{1}{6}x^3 \geq -\sin x = f'(x)$ for $x \geq 0$ by part (b). Now apply Exercise 67 to conclude that $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ for $x \geq 0$.

(d) The next inequality in the series is $\sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, valid for $x \geq 0$. To construct (d) from (c), we note that the derivative of $\sin x$ is $\cos x$, and look for a polynomial (which we currently must do by educated guess) whose derivative is $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. We know the derivative of x is 1, and that a term whose derivative is $-\frac{1}{2}x^2$ should be of the form Cx^3. $\frac{d}{dx}Cx^3 = 3Cx^2 = -\frac{1}{2}x^2$, so $C = -\frac{1}{6}$. A term whose derivative is $\frac{1}{24}x^4$ should be of the form Dx^5. From this, $\frac{d}{dx}Dx^5 = 5Dx^4 = \frac{1}{24}x^4$, so that $5D = \frac{1}{24}$, or $D = \frac{1}{120}$.

70. Let $f(x) = e^{-x}$. Use the method of Exercise 69 to prove the following inequalities for $x \geq 0$.

(a) $e^{-x} \geq 1 - x$

(b) $e^{-x} \leq 1 - x + \frac{1}{2}x^2$

(c) $e^{-x} \geq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

Can you guess the next inequality in the series?

SOLUTION

(a) Let $f(x) = 1 - x$ and $g(x) = e^{-x}$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -1 \leq -e^{-x} = g'(x).$$

Thus, by Exercise 67 we conclude that $e^{-x} \geq 1 - x$ for $x \geq 0$.

(b) Let $f(x) = e^{-x}$ and $g(x) = 1 - x + \frac{1}{2}x^2$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -e^{-x} \leq x - 1 = g'(x)$$

by the result from part (a). Thus, by Exercise 67 we conclude that $e^{-x} \leq 1 - x + \dfrac{1}{2}x^2$ for $x \geq 0$.

(c) Let $f(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ and $g(x) = e^{-x}$. Then $f(0) = g(0) = 1$ and, for $x \geq 0$,

$$f'(x) = -1 + x - \frac{1}{2}x^2 \leq -e^{-x} = g'(x)$$

by the result from part (b). Thus, by Exercise 67 we conclude that $e^{-x} \geq 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3$ for $x \geq 0$.

The next inequality in the series is $e^{-x} \leq 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$ for $x \geq 0$

71. Assume that f'' exists and $f''(x) = 0$ for all x. Prove that $f(x) = mx + b$, where $m = f'(0)$ and $b = f(0)$.

SOLUTION

- Let $f''(x) = 0$ for all x. Then $f'(x) =$ constant for all x. Since $f'(0) = m$, we conclude that $f'(x) = m$ for all x.
- Let $g(x) = f(x) - mx$. Then $g'(x) = f'(x) - m = m - m = 0$ which implies that $g(x) =$ constant for all x and consequently $f(x) - mx =$ constant for all x. Rearranging the statement, $f(x) = mx +$ constant. Since $f(0) = b$, we conclude that $f(x) = mx + b$ for all x.

72. Define $f(x) = x^3 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.
(a) Show that $f'(x)$ is continuous at $x = 0$ and that $x = 0$ is a critical point of f.
(b) GU Examine the graphs of $f(x)$ and $f'(x)$. Can the First Derivative Test be applied?
(c) Show that $f(0)$ is neither a local min nor a local max.

SOLUTION

(a) Let $f(x) = x^3 \sin(\frac{1}{x})$. Then

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) + x^3 \cos\left(\frac{1}{x}\right)(-x^{-2}) = x\left(3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right).$$

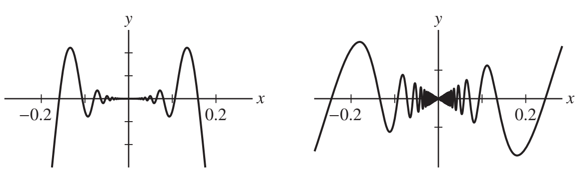
This formula is not defined at $x = 0$, but its limit is. Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all x,

$$|f'(x)| = |x|\left|3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \leq |x|\left(\left|3x \sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right|\right) \leq |x|(3|x| + 1)$$

so, by the Squeeze Theorem, $\lim\limits_{x\to 0} |f'(x)| = 0$. But does $f'(0) = 0$? We check using the limit definition of the derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Thus $f'(x)$ is continuous at $x = 0$, and $x = 0$ is a critical point of f.
(b) The figure below at the left shows $f(x)$, and the figure below at the right shows $f'(x)$. Note how the two functions oscillate near $x = 0$, which implies that the First Derivative Test cannot be applied.



(c) As x approaches 0 from either direction, $f(x)$ alternates between positive and negative arbitrarily close to $x = 0$. This means that $f(0)$ cannot be a local minimum (since $f(x)$ gets lower than $f(0)$ arbitrarily close to 0), nor can $f(0)$ be a local maximum (since $f(x)$ takes values higher than $f(0)$ arbitrarily close to $x = 0$). Therefore $f(0)$ is neither a local minimum nor a local maximum of f.

73. Suppose that $f(x)$ satisfies the following equation (an example of a **differential equation**):

$$f''(x) = -f(x) \qquad \boxed{1}$$

(a) Show that $f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2$ for all x. *Hint:* Show that the function on the left has zero derivative.
(b) Verify that $\sin x$ and $\cos x$ satisfy Eq. (1), and deduce that $\sin^2 x + \cos^2 x = 1$.

SOLUTION

(a) Let $g(x) = f(x)^2 + f'(x)^2$. Then

$$g'(x) = 2f(x)f'(x) + 2f'(x)f''(x) = 2f(x)f'(x) + 2f'(x)(-f(x)) = 0,$$

where we have used the fact that $f''(x) = -f(x)$. Because $g'(0) = 0$ for all x, $g(x) = f(x)^2 + f'(x)^2$ must be a constant function. In other words, $f(x)^2 + f'(x)^2 = C$ for some constant C. To determine the value of C, we can substitute any number for x. In particular, for this problem, we want to substitute $x = 0$ and find $C = f(0)^2 + f'(0)^2$. Hence,

$$f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2.$$

(b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and $f''(x) = -\sin x$, so $f''(x) = -f(x)$. Next, let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, and we again have $f''(x) = -f(x)$. Finally, if we take $f(x) = \sin x$, the result from part (a) guarantees that

$$\sin^2 x + \cos^2 x = \sin^2 0 + \cos^2 0 = 0 + 1 = 1.$$

74. Suppose that functions f and g satisfy Eq. (1) and have the same initial values—that is, $f(0) = g(0)$ and $f'(0) = g'(0)$. Prove that $f(x) = g(x)$ for all x. *Hint:* Apply Exercise 73(a) to $f - g$.

SOLUTION Let $h(x) = f(x) - g(x)$. Then

$$h''(x) = f''(x) - g''(x) = -f(x) - (-g(x)) = -(f(x) - g(x)) = -h(x).$$

Furthermore, $h(0) = f(0) - g(0) = 0$ and $h'(0) = f'(0) - g'(0) = 0$. Thus, by part (a) of Exercise 73, $h(x)^2 + h'(x)^2 = 0$. This can only happen if $h(x) = 0$ for all x, or, equivalently, $f(x) = g(x)$ for all x.

75. Use Exercise 74 to prove: $f(x) = \sin x$ is the unique solution of Eq. (1) such that $f(0) = 0$ and $f'(0) = 1$; and $g(x) = \cos x$ is the unique solution such that $g(0) = 1$ and $g'(0) = 0$. This result can be used to develop all the properties of the trigonometric functions "analytically"—that is, without reference to triangles.

SOLUTION In part (b) of Exercise 73, it was shown that $f(x) = \sin x$ satisfies Eq. (1), and we can directly calculate that $f(0) = \sin 0 = 0$ and $f'(0) = \cos 0 = 1$. Suppose there is another function, call it $F(x)$, that satisfies Eq. (1) with the same initial conditions: $F(0) = 0$ and $F'(0) = 1$. By Exercise 74, it follows that $F(x) = \sin x$ for all x. Hence, $f(x) = \sin x$ is the unique solution of Eq. (1) satisfying $f(0) = 0$ and $f'(0) = 1$. The proof that $g(x) = \cos x$ is the unique solution of Eq. (1) satisfying $g(0) = 1$ and $g'(0) = 0$ is carried out in a similar manner.

4.4 The Shape of a Graph

Preliminary Questions

1. If f is concave up, then f' is (choose one):

(a) increasing **(b)** decreasing

SOLUTION The correct response is **(a)**: increasing. If the function is concave up, then f'' is positive. Since f'' is the derivative of f', it follows that the derivative of f' is positive and f' must therefore be increasing.

2. What conclusion can you draw if $f'(c) = 0$ and $f''(c) < 0$?

SOLUTION If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.

3. True or False? If $f(c)$ is a local min, then $f''(c)$ must be positive.

SOLUTION False. $f''(c)$ could be zero.

4. True or False? If $f''(x)$ changes from $+$ to $-$ at $x = c$, then f has a point of inflection at $x = c$.

SOLUTION False. f will have a point of inflection at $x = c$ only if $x = c$ is in the domain of f.

Exercises

1. Match the graphs in Figure 13 with the description:

(a) $f''(x) < 0$ for all x.

(b) $f''(x)$ goes from $+$ to $-$.

(c) $f''(x) > 0$ for all x.

(d) $f''(x)$ goes from $-$ to $+$.

(A) (B) (C) (D)

FIGURE 13

SOLUTION

(a) In C, we have $f''(x) < 0$ for all x.

(b) In A, $f''(x)$ goes from $+$ to $-$.

(c) In B, we have $f''(x) > 0$ for all x.

(d) In D, $f''(x)$ goes from $-$ to $+$.

2. Match each statement with a graph in Figure 14 that represents company profits as a function of time.
(a) The outlook is great: The growth rate keeps increasing.
(b) We're losing money, but not as quickly as before.
(c) We're losing money, and it's getting worse as time goes on.
(d) We're doing well, but our growth rate is leveling off.
(e) Business had been cooling off, but now it's picking up.
(f) Business had been picking up, but now it's cooling off.

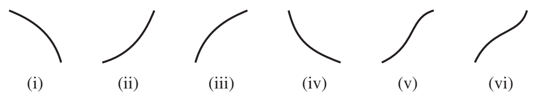


FIGURE 14

SOLUTION
(a) **(ii)** An increasing growth rate implies an increasing f', and so a graph that is concave up.
(b) **(iv)** "Losing money" implies a downward curve. "Not as fast" implies that f' is becoming less negative, so that $f''(x) > 0$.
(c) **(i)** "Losing money" implies a downward curve. "Getting worse" implies that f' is becoming more negative, so the curve is concave down.
(d) **(iii)** "We're doing well" implies that f is increasing, but "the growth rate is leveling off" implies that f' is decreasing, so that the graph is concave down.
(e) **(vi)** "Cooling off" generally means increasing at a decreasing rate. The use of "had" implies that only the beginning of the graph is that way. The phrase "...now it's picking up" implies that the end of the graph is concave up.
(f) **(v)** "Business had been picking up" implies that the graph started out concave up. The phrase "…but now it's cooling off" implies that the graph ends up concave down.

In Exercises 3–18, determine the intervals on which the function is concave up or down and find the points of inflection.

3. $y = x^2 - 4x + 3$

SOLUTION Let $f(x) = x^2 - 4x + 3$. Then $f'(x) = 2x - 4$ and $f''(x) = 2 > 0$ for all x. Therefore, f is concave up everywhere, and there are no points of inflection.

4. $y = t^3 - 6t^2 + 4$

SOLUTION Let $f(t) = t^3 - 6t^2 + 4$. Then $f'(t) = 3t^2 - 12t$ and $f''(t) = 6t - 12 = 0$ at $t = 2$. Now, f is concave up on $(2, \infty)$, since $f''(t) > 0$ there. Moreover, f is concave down on $(-\infty, 2)$, since $f''(t) < 0$ there. Finally, because $f''(t)$ changes sign at $t = 2$, $f(t)$ has a point of inflection at $t = 2$.

5. $y = 10x^3 - x^5$

SOLUTION Let $f(x) = 10x^3 - x^5$. Then $f'(x) = 30x^2 - 5x^4$ and $f''(x) = 60x - 20x^3 = 20x(3 - x^2)$. Now, f is concave up for $x < -\sqrt{3}$ and for $0 < x < \sqrt{3}$ since $f''(x) > 0$ there. Moreover, f is concave down for $-\sqrt{3} < x < 0$ and for $x > \sqrt{3}$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$ and at $x = \pm\sqrt{3}$, $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm\sqrt{3}$.

6. $y = 5x^2 + x^4$

SOLUTION Let $f(x) = 5x^2 + x^4$. Then $f'(x) = 10x + 4x^3$ and $f''(x) = 10 + 12x^2 > 10$ for all x. Thus, f is concave up for all x and has no points of inflection.

7. $y = \theta - 2\sin\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta - 2\sin\theta$. Then $f'(\theta) = 1 - 2\cos\theta$ and $f''(\theta) = 2\sin\theta$. Now, f is concave up for $0 < \theta < \pi$ since $f''(\theta) > 0$ there. Moreover, f is concave down for $\pi < \theta < 2\pi$ since $f''(\theta) < 0$ there. Finally, because $f''(\theta)$ changes sign at $\theta = \pi$, $f(\theta)$ has a point of inflection at $\theta = \pi$.

8. $y = \theta + \sin^2\theta, \quad [0, \pi]$

SOLUTION Let $f(\theta) = \theta + \sin^2\theta$. Then $f'(\theta) = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta$ and $f''(\theta) = 2\cos 2\theta$. Now, f is concave up for $0 < \theta < \pi/4$ and for $3\pi/4 < \theta < \pi$ since $f''(\theta) > 0$ there. Moreover, f is concave down for $\pi/4 < \theta < 3\pi/4$ since $f''(\theta) < 0$ there. Finally, because $f''(\theta)$ changes sign at $\theta = \pi/4$ and at $\theta = 3\pi/4$, $f(\theta)$ has a point of inflection at $\theta = \pi/4$ and at $\theta = 3\pi/4$.

9. $y = x(x - 8\sqrt{x}) \quad (x \geq 0)$

SOLUTION Let $f(x) = x(x - 8\sqrt{x}) = x^2 - 8x^{3/2}$. Then $f'(x) = 2x - 12x^{1/2}$ and $f''(x) = 2 - 6x^{-1/2}$. Now, f is concave down for $0 < x < 9$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 9$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 9$, $f(x)$ has a point of inflection at $x = 9$.

10. $y = x^{7/2} - 35x^2$

SOLUTION Let $f(x) = x^{7/2} - 35x^2$. Then

$$f'(x) = \frac{7}{2}x^{5/2} - 70x \quad \text{and} \quad f''(x) = \frac{35}{4}x^{3/2} - 70.$$

Now, f is concave down for $0 < x < 4$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 4$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 4$, $f(x)$ has a point of inflection at $x = 4$.

11. $y = (x-2)(1-x^3)$

SOLUTION Let $f(x) = (x-2)\left(1-x^3\right) = x - x^4 - 2 + 2x^3$. Then $f'(x) = 1 - 4x^3 + 6x^2$ and $f''(x) = 12x - 12x^2 = 12x(1-x) = 0$ at $x = 0$ and $x = 1$. Now, f is concave up on $(0, 1)$ since $f''(x) > 0$ there. Moreover, f is concave down on $(-\infty, 0) \cup (1, \infty)$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at both $x = 0$ and $x = 1$, $f(x)$ has a point of inflection at both $x = 0$ and $x = 1$.

12. $y = x^{7/5}$

SOLUTION Let $f(x) = x^{7/5}$. Then $f'(x) = \frac{7}{5}x^{2/5}$ and $f''(x) = \frac{14}{25}x^{-3/5}$. Now, f is concave down for $x < 0$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > 0$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$, $f(x)$ has a point of inflection at $x = 0$.

13. $y = \dfrac{1}{x^2+3}$

SOLUTION Let $f(x) = \dfrac{1}{x^2+3}$. Then $f'(x) = -\dfrac{2x}{(x^2+3)^2}$ and

$$f''(x) = -\frac{2(x^2+3)^2 - 8x^2(x^2+3)}{(x^2+3)^4} = \frac{6x^2-6}{(x^2+3)^3}.$$

Now, f is concave up for $|x| > 1$ since $f''(x) > 0$ there. Moreover, f is concave down for $|x| < 1$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at both $x = -1$ and $x = 1$, $f(x)$ has a point of inflection at both $x = -1$ and $x = 1$.

14. $y = \dfrac{x}{x^2+9}$

SOLUTION Let $f(x) = \frac{x}{x^2+9}$. Then

$$f'(x) = \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$$

and

$$f''(x) = \frac{(x^2+9)^2(-2x) - (9-x^2)(2)(x^2+9)(2x)}{(x^2+9)^4} = \frac{2x(x^2-27)}{(x^2+9)^3}.$$

Now, f is concave up for $-3\sqrt{3} < x < 0$ and for $x > 3\sqrt{3}$ since $f''(x) > 0$ there. Moreover, f is concave down for $x < -3\sqrt{3}$ and for $0 < x < 3\sqrt{3}$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = 0$ and at $x = \pm 3\sqrt{3}$, $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm 3\sqrt{3}$.

15. $y = xe^{-3x}$

SOLUTION Let $f(x) = xe^{-3x}$. Then $f'(x) = -3xe^{-3x} + e^{-3x} = (1-3x)e^{-3x}$ and $f''(x) = -3(1-3x)e^{-3x} - 3e^{-3x} = (9x-6)e^{-3x}$. Now, f is concave down for $x < \frac{2}{3}$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > \frac{2}{3}$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = \frac{2}{3}$, $x = \frac{2}{3}$ is a point of inflection.

16. $y = (x^2-7)e^x$

SOLUTION Let $f(x) = (x^2-7)e^x$. Then $f'(x) = (x^2-7)e^x + 2xe^x = (x^2 + 2x07)e^x$ and $f''(x) = (x^2 + 2x - 7)e^x + (2x+2)e^x = (x+5)(x-1)e^x$. Now, f is concave up for $x < -5$ and for $x > 1$ since $f''(x) > 0$ there. Moreover, f is concave down for $-5 < x < 1$ since $f''(x) < 0$ there. Finally, because $f''(x)$ changes sign at $x = -5$ and at $x = 1$, f has a point of inflection at $x = -5$ and at $x = 1$.

17. $y = 2x^2 + \ln x \quad (x > 0)$

SOLUTION Let $f(x) = 2x^2 + \ln x$. Then $f'(x) = 4x + x^{-1}$ and $f''(x) = 4 - x^{-2}$. Now, f is concave down for $x < \frac{1}{2}$ since $f''(x) < 0$ there. Moreover, f is concave up for $x > \frac{1}{2}$ since $f''(x) > 0$ there. Finally, because $f''(x)$ changes sign at $x = \frac{1}{2}$, f has a point of inflection at $x = \frac{1}{2}$.

18. $y = x - \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x - \ln x$. Then $f'(x) = 1 - 1/x$ and $f''(x) = x^{-2} > 0$ for all $x > 0$. Thus, f is concave up for all $x > 0$ and has no points of inflection.

19. The growth of a sunflower during the first 100 days after sprouting is modeled well by the *logistic curve* $y = h(t)$ shown in Figure 15. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of $h(t)$.

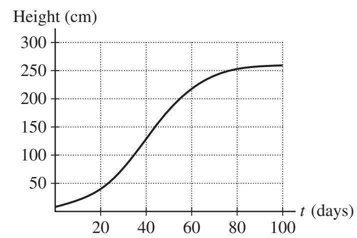
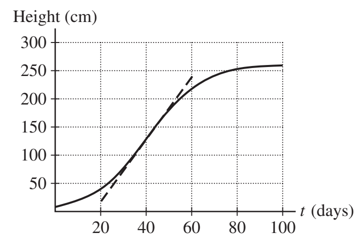


FIGURE 15

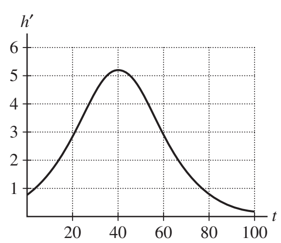
SOLUTION The point of inflection in Figure 15 appears to occur at $t = 40$ days. The graph below shows the logistic curve with an approximate tangent line drawn at $t = 40$. The approximate tangent line passes roughly through the points (20, 20) and (60, 240). The growth rate at the point of inflection is thus

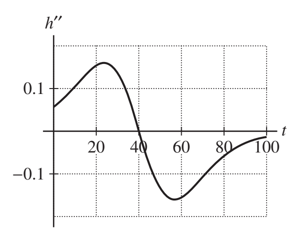
$$\frac{240 - 20}{60 - 20} = \frac{220}{40} = 5.5 \text{ cm/day}.$$

Because the logistic curve changes from concave up to concave down at $t = 40$, the growth rate at this point is the maximum growth rate for the sunflower plant.



Sketches of the first and second derivative of $h(t)$ are shown below at the left and at the right, respectively.





20. Assume that Figure 16 is the graph of $f(x)$. Where do the points of inflection of $f(x)$ occur, and on which interval is $f(x)$ concave down?

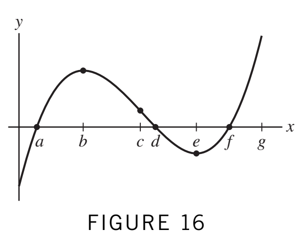


FIGURE 16

SOLUTION The function in Figure 16 changes concavity at $x = c$; therefore, there is a single point of inflection at $x = c$. The graph is concave down for $x < c$.

21. Repeat Exercise 20 but assume that Figure 16 is the graph of the *derivative* $f'(x)$.

SOLUTION Points of inflection occur when $f''(x)$ changes sign. Consequently, points of inflection occur when $f'(x)$ changes from increasing to decreasing or from decreasing to increasing. In Figure 16, this occurs at $x = b$ and at $x = e$; therefore, $f(x)$ has an inflection point at $x = b$ and another at $x = e$. The function $f(x)$ will be concave down when $f''(x) < 0$ or when $f'(x)$ is decreasing. Thus, $f(x)$ is concave down for $b < x < e$.

22. Repeat Exercise 20 but assume that Figure 16 is the graph of the *second derivative* $f''(x)$.

SOLUTION Inflection points occur when $f''(x)$ changes sign; therefore, $f(x)$ has inflection points at $x = a$, $x = d$ and $x = f$. The function $f(x)$ is concave down for $x < a$ and for $d < x < f$.

23. Figure 17 shows the *derivative* $f'(x)$ on [0, 1.2]. Locate the points of inflection of $f(x)$ and the points where the local minima and maxima occur. Determine the intervals on which $f(x)$ has the following properties:

(a) Increasing
(b) Decreasing
(c) Concave up
(d) Concave down

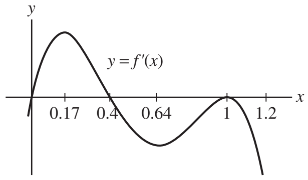


FIGURE 17

SOLUTION Recall that the graph is that of f', *not* f. The inflection points of f occur where f' changes from increasing to decreasing or vice versa because it is at these points that the sign of f'' changes. From the graph we conclude that f has points of inflection at $x = 0.17$, $x = 0.64$, and $x = 1$. The local extrema of f occur where f' changes sign. This occurs at $x = 0.4$. Because the sign of f' changes from + to −, $f(0.4)$ is a local maximum. There are no local minima.

(a) f is increasing when f' is positive. Hence, f is increasing on $(0, 0.4)$.

(b) f is decreasing when f' is negative. Hence, f is decreasing on $(0.4, 1) \cup (1, 1.2)$.

(c) Now f is concave up where f' is increasing. This occurs on $(0, 0.17) \cup (0.64, 1)$.

(d) Moreover, f is concave down where f' is decreasing. This occurs on $(0.17, 0.64) \cup (1, 1.2)$.

24. Leticia has been selling solar-powered laptop chargers through her website, with monthly sales as recorded below. In a report to investors, she states, "Sales reached a point of inflection when I started using pay-per-click advertising." In which month did that occur? Explain.

Month	1	2	3	4	5	6	7	8
Sales	2	30	50	60	90	150	230	340

SOLUTION Note that in successive months, sales increased by 28, 20, 10, 30, 60, 80 and 110. Until month 5, the rate of increase in sales was decreasing. After month 5, the rate of increase in sales increased. Thus, Leticia began using pay-per-click advertising in month 5.

In Exercises 25–38, find the critical points and apply the Second Derivative Test.

25. $f(x) = x^3 - 12x^2 + 45x$

SOLUTION Let $f(x) = x^3 - 12x^2 + 45x$. Then $f'(x) = 3x^2 - 24x + 45 = 3(x-3)(x-5)$, and the critical points are $x = 3$ and $x = 5$. Moreover, $f''(x) = 6x - 24$, so $f''(3) = -6 < 0$ and $f''(5) = 6 > 0$. Therefore, by the Second Derivative Test, $f(3) = 54$ is a local maximum, and $f(5) = 50$ is a local minimum.

26. $f(x) = x^4 - 8x^2 + 1$

SOLUTION Let $f(x) = x^4 - 8x^2 + 1$. Then $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$, and the critical points are $x = 0$ and $x = \pm 2$. Moreover, $f''(x) = 12x^2 - 16$, so $f''(-2) = f''(2) = 32 > 0$ and $f''(0) = -16 < 0$. Therefore, by the second derivative test, $f(-2) = -15$ and $f(2) = -15$ are local minima, and $f(0) = 1$ is a local maximum.

27. $f(x) = 3x^4 - 8x^3 + 6x^2$

SOLUTION Let $f(x) = 3x^4 - 8x^3 + 6x^2$. Then $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2 = 0$ at $x = 0, 1$ and $f''(x) = 36x^2 - 48x + 12$. Thus, $f''(0) > 0$, which implies $f(0)$ is a local minimum; however, $f''(1) = 0$, which is inconclusive.

28. $f(x) = x^5 - x^3$

SOLUTION Let $f(x) = x^5 - x^3$. Then $f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3) = 0$ at $x = 0$, $x = \pm\sqrt{\frac{3}{5}}$ and $f''(x) = 20x^3 - 6x = x(20x^2 - 6)$. Thus, $f''\left(\sqrt{\frac{3}{5}}\right) > 0$, which implies $f\left(\sqrt{\frac{3}{5}}\right)$ is a local minimum, and $f''\left(-\sqrt{\frac{3}{5}}\right) < 0$, which implies that $f\left(-\sqrt{\frac{3}{5}}\right)$ is a local maximum; however, $f''(0) = 0$, which is inconclusive.

29. $f(x) = \dfrac{x^2 - 8x}{x + 1}$

SOLUTION Let $f(x) = \dfrac{x^2 - 8x}{x + 1}$. Then

$$f'(x) = \frac{x^2 + 2x - 8}{(x + 1)^2} \quad \text{and} \quad f''(x) = \frac{2(x + 1)^2 - 2(x^2 + 2x - 8)}{(x + 1)^3}.$$

Thus, the critical points are $x = -4$ and $x = 2$. Moreover, $f''(-4) < 0$ and $f''(2) > 0$. Therefore, by the second derivative test, $f(-4) = -16$ is a local maximum and $f(2) = -4$ is a local minimum.

30. $f(x) = \dfrac{1}{x^2 - x + 2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2 - x + 2}$. Then $f'(x) = \dfrac{-2x + 1}{(x^2 - x + 2)^2} = 0$ at $x = \frac{1}{2}$ and

$$f''(x) = \frac{-2(x^2 - x + 2) + 2(2x - 1)^2}{(x^2 - x + 2)^3}.$$

Thus $f''\left(\frac{1}{2}\right) < 0$, which implies that $f\left(\frac{1}{2}\right)$ is a local maximum.

31. $y = 6x^{3/2} - 4x^{1/2}$

SOLUTION Let $f(x) = 6x^{3/2} - 4x^{1/2}$. Then $f'(x) = 9x^{1/2} - 2x^{-1/2} = x^{-1/2}(9x - 2)$, so there are two critical points: $x = 0$ and $x = \frac{2}{9}$. Now,

$$f''(x) = \frac{9}{2}x^{-1/2} + x^{-3/2} = \frac{1}{2}x^{-3/2}(9x + 2).$$

Thus, $f''\left(\frac{2}{9}\right) > 0$, which implies $f\left(\frac{2}{9}\right)$ is a local minimum. $f''(x)$ is undefined at $x = 0$, so the Second Derivative Test cannot be applied there.

32. $y = 9x^{7/3} - 21x^{1/2}$

SOLUTION Let $f(x) = 9x^{7/3} - 21x^{1/2}$. Then $f'(x) = 21x^{4/3} - \frac{21}{2}x^{-1/2} = 0$ when

$$x = \left(\frac{1}{2}\right)^{6/11},$$

and $f''(x) = 28x^{1/3} + \frac{21}{4}x^{-3/2}$. Thus,

$$f''\left(\left(\frac{1}{2}\right)^{6/11}\right) > 0,$$

which implies $f\left(\left(\frac{1}{2}\right)^{6/11}\right)$ is a local minimum.

33. $f(x) = \sin^2 x + \cos x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin^2 x + \cos x$. Then $f'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)$. On the interval $[0, \pi]$, $f'(x) = 0$ at $x = 0$, $x = \frac{\pi}{3}$ and $x = \pi$. Now,

$$f''(x) = 2\cos^2 x - 2\sin^2 x - \cos x.$$

Thus, $f''(0) > 0$, so $f(0)$ is a local minimum. On the other hand, $f''(\frac{\pi}{3}) < 0$, so $f(\frac{\pi}{3})$ is a local maximum. Finally, $f''(\pi) > 0$, so $f(\pi)$ is a local minimum.

34. $y = \dfrac{1}{\sin x + 4}, \quad [0, 2\pi]$

SOLUTION Let $f(x) = (\sin x + 4)^{-1}$. Then

$$f'(x) = -\frac{\cos x}{(\sin x + 4)^2} \quad \text{and} \quad f''(x) = \frac{2\cos^2 x + \sin^2 x + 4\sin x}{(\sin x + 4)^3}.$$

Now, $f'(x) = 0$ when $x = \pi/2$ and when $x = 3\pi/2$. Since $f''(\pi/2) > 0$, it follows that $f(\pi/2)$ is a local minimum. On the other hand, $f''(3\pi/2) < 0$, so $f(3\pi/2)$ is a local maximum.

35. $f(x) = xe^{-x^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. Then $f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1-2x^2)e^{-x^2}$, so there are two critical points: $x = \pm\frac{\sqrt{2}}{2}$. Now,

$$f''(x) = (4x^3 - 2x)e^{-x^2} - 4xe^{-x^2} = (4x^3 - 6x)e^{-x^2}.$$

Thus, $f''\left(\frac{\sqrt{2}}{2}\right) < 0$, so $f\left(\frac{\sqrt{2}}{2}\right)$ is a local maximum. On the other hand, $f''\left(-\frac{\sqrt{2}}{2}\right) > 0$, so $f\left(-\frac{\sqrt{2}}{2}\right)$ is a local minimum.

36. $f(x) = e^{-x} - 4e^{-2x}$

SOLUTION Let $f(x) = e^{-x} - 4e^{-2x}$. Then $f'(x) = -e^{-x} + 8e^{-2x} = 0$ when $x = 3\ln 2$. Now, $f''(x) = e^{-x} - 16e^{-2x}$, so $f''(3\ln 2) < 0$. Thus, $f(3\ln 2)$ is a local maximum.

37. $f(x) = x^3 \ln x \quad (x > 0)$

SOLUTION Let $f(x) = x^3 \ln x$. Then $f'(x) = x^2 + 3x^2 \ln x = x^2(1 + 3\ln x)$, so there is only one critical point: $x = e^{-1/3}$. Now,

$$f''(x) = 3x + 2x(1 + 3\ln x) = x(5 + 6lnx).$$

Thus, $f''\left(e^{-1/3}\right) > 0$, so $f\left(e^{-1/3}\right)$ is a local minimum.

38. $f(x) = \ln x + \ln(4 - x^2), \quad (0, 2)$

SOLUTION Let $f(x) = \ln x + \ln(4 - x^2)$. Then

$$f'(x) = \frac{1}{x} - \frac{2x}{4 - x^2},$$

so there is only one critical point on the interval $0 < x < 2$: $x = \frac{2\sqrt{3}}{3}$. Now,

$$f''(x) = -\frac{1}{x^2} - \frac{(4 - x^2)(2) - 2x(-2x)}{(4 - x^2)^2} = -\frac{1}{x^2} - \frac{8 + 2x^2}{(4 - x^2)^2}.$$

Thus, $f''\left(\frac{2\sqrt{3}}{3}\right) < 0$, so $f\left(\frac{2\sqrt{3}}{3}\right)$ is a local maximum.

In Exercises 39–52, find the intervals on which f is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

SOLUTION Here is a table legend for Exercises 39–49.

SYMBOL	MEANING
−	The entity is negative on the given interval.
0	The entity is zero at the specified point.
+	The entity is positive on the given interval.
U	The entity is undefined at the specified point.
↗	The function (f, g, etc.) is increasing on the given interval.
↘	The function (f, g, etc.) is decreasing on the given interval.
⌣	The function (f, g, etc.) is concave up on the given interval.
⌢	The function (f, g, etc.) is concave down on the given interval.
M	The function (f, g, etc.) has a local maximum at the specified point.
m	The function (f, g, etc.) has a local minimum at the specified point.
I	The function (f, g, etc.) has an inflection point here.
¬	There is no local extremum or inflection point here.

39. $f(x) = x^3 - 2x^2 + x$

SOLUTION Let $f(x) = x^3 - 2x^2 + x$.

- Then $f'(x) = 3x^2 - 4x + 1 = (x-1)(3x-1) = 0$ yields $x = 1$ and $x = \frac{1}{3}$ as candidates for extrema.
- Moreover, $f''(x) = 6x - 4 = 0$ gives a candidate for a point of inflection at $x = \frac{2}{3}$.

x	$\left(-\infty, \frac{1}{3}\right)$	$\frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$	1	$(1, \infty)$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

x	$\left(-\infty, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \infty\right)$
f''	−	0	+
f	⌢	I	⌣

40. $f(x) = x^2(x-4)$

SOLUTION Let $f(x) = x^2(x-4) = x^3 - 4x^2$.

- Then $f'(x) = 3x^2 - 8x = x(3x-8) = 0$ yields $x = 0$ and $x = \frac{8}{3}$ as candidates for extrema.
- Moreover, $f''(x) = 6x - 8 = 0$ gives a candidate for a point of inflection at $x = \frac{4}{3}$.

x	$(-\infty, 0)$	0	$\left(0, \frac{8}{3}\right)$	$\frac{8}{3}$	$\left(\frac{8}{3}, \infty\right)$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

x	$\left(-\infty, \frac{4}{3}\right)$	$\frac{4}{3}$	$\left(\frac{4}{3}, \infty\right)$
f''	−	0	+
f	⌢	I	⌣

41. $f(t) = t^2 - t^3$

SOLUTION Let $f(t) = t^2 - t^3$.

- Then $f'(t) = 2t - 3t^2 = t(2-3t) = 0$ yields $t = 0$ and $t = \frac{2}{3}$ as candidates for extrema.
- Moreover, $f''(t) = 2 - 6t = 0$ gives a candidate for a point of inflection at $t = \frac{1}{3}$.

t	$(-\infty, 0)$	0	$\left(0, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \infty\right)$
f'	−	0	+	0	−
f	↘	m	↗	M	↘

t	$\left(-\infty, \frac{1}{3}\right)$	$\frac{1}{3}$	$\left(\frac{1}{3}, \infty\right)$
f''	+	0	−
f	⌣	I	⌢

42. $f(x) = 2x^4 - 3x^2 + 2$

SOLUTION Let $f(x) = 2x^4 - 3x^2 + 2$.

- Then $f'(x) = 8x^3 - 6x = 2x(4x^2 - 3) = 0$ yields $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$ as candidates for extrema.
- Moreover, $f''(x) = 24x^2 - 6 = 6(4x^2 - 1) = 0$ gives candidates for a point of inflection at $x = \pm\frac{1}{2}$.

x	$\left(-\infty, -\frac{\sqrt{3}}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$\left(-\frac{\sqrt{3}}{2}, 0\right)$	0	$\left(0, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\sqrt{3}}{2}, \infty\right)$
f'	−	0	+	0	−	0	+
f	↘	m	↗	M	↘	m	↗

x	$\left(-\infty, -\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, \infty\right)$
f''	+	0	−	0	+
f	⌣	I	⌢	I	⌣

43. $f(x) = x^2 - 8x^{1/2} \quad (x \geq 0)$

SOLUTION Let $f(x) = x^2 - 8x^{1/2}$. Note that the domain of f is $x \geq 0$.

- Then $f'(x) = 2x - 4x^{-1/2} = x^{-1/2}\left(2x^{3/2} - 4\right) = 0$ yields $x = 0$ and $x = (2)^{2/3}$ as candidates for extrema.
- Moreover, $f''(x) = 2 + 2x^{-3/2} > 0$ for all $x \geq 0$, which means there are no inflection points.

x	0	$\left(0, (2)^{2/3}\right)$	$(2)^{2/3}$	$\left((2)^{2/3}, \infty\right)$
f'	U	$-$	0	$+$
f	M	↘	m	↗

44. $f(x) = x^{3/2} - 4x^{-1/2} \quad (x > 0)$

SOLUTION Let $f(x) = x^{3/2} - 4x^{-1/2}$. Then

$$f'(x) = \frac{3}{2}x^{1/2} + 2x^{-3/2} > 0$$

for all $x > 0$. Thus, f is always increasing and there are no local extrema. Now,

$$f''(x) = \frac{3}{4}x^{-1/2} - 3x^{-5/2}$$

so $x = 2$ is a candidate point of inflection.

x	$(0, 2)$	2	$(2, \infty)$
f''	$-$	0	$+$
f	⌢	I	⌣

45. $f(x) = \dfrac{x}{x^2 + 27}$

SOLUTION Let $f(x) = \dfrac{x}{x^2 + 27}$.

- Then $f'(x) = \dfrac{27 - x^2}{\left(x^2 + 27\right)^2} = 0$ yields $x = \pm 3\sqrt{3}$ as candidates for extrema.
- Moreover, $f''(x) = \dfrac{-2x\left(x^2 + 27\right)^2 - (27 - x^2)(2)\left(x^2 + 27\right)(2x)}{(x^2 + 27)^4} = \dfrac{2x\left(x^2 - 81\right)}{\left(x^2 + 27\right)^3} = 0$ gives candidates for a point of inflection at $x = 0$ and at $x = \pm 9$.

x	$\left(-\infty, -3\sqrt{3}\right)$	$-3\sqrt{3}$	$\left(-3\sqrt{3}, 3\sqrt{3}\right)$	$3\sqrt{3}$	$\left(3\sqrt{3}, \infty\right)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

x	$(-\infty, -9)$	-9	$(-9, 0)$	0	$(0, 9)$	9	$(9, \infty)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	⌢	I	⌣	I	⌢	I	⌣

46. $f(x) = \dfrac{1}{x^4 + 1}$

SOLUTION Let $f(x) = \dfrac{1}{x^4 + 1}$.

- Then $f'(x) = -\dfrac{4x^3}{\left(x^4 + 1\right)^2} = 0$ yields $x = 0$ as a candidate for an extremum.
- Moreover,

$$f''(x) = \frac{\left(x^4 + 1\right)^2\left(-12x^2\right) - (-4x^3) \cdot 2\left(x^4 + 1\right)\left(4x^3\right)}{\left(x^2 + 1\right)^4} = \frac{4x^2\left(5x^4 - 3\right)}{\left(x^4 + 1\right)^3} = 0$$

gives candidates for a point of inflection at $x = 0$ and at $x = \pm\left(\frac{3}{5}\right)^{1/4}$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$
f	$\nearrow$	M	$\searrow$

x	$\left(-\infty, -\left(\frac{3}{5}\right)^{1/4}\right)$	$-\left(\frac{3}{5}\right)^{1/4}$	$\left(-\left(\frac{3}{5}\right)^{1/4}, 0\right)$	0	$\left(0, \left(\frac{3}{5}\right)^{1/4}\right)$	$\left(\frac{3}{5}\right)^{1/4}$	$\left(\left(\frac{3}{5}\right)^{1/4}, \infty\right)$
f''	$+$	0	$-$	0	$-$	0	$+$
f	$\smile$	I	$\frown$	$\neg$	$\frown$	I	$\smile$

47. $f(\theta) = \theta + \sin\theta, \quad [0, 2\pi]$

SOLUTION Let $f(\theta) = \theta + \sin\theta$ on $[0, 2\pi]$.

- Then $f'(\theta) = 1 + \cos\theta = 0$ yields $\theta = \pi$ as a candidate for an extremum.
- Moreover, $f''(\theta) = -\sin\theta = 0$ gives candidates for a point of inflection at $\theta = 0$, at $\theta = \pi$, and at $\theta = 2\pi$.

θ	$(0, \pi)$	π	$(\pi, 2\pi)$
f'	$+$	0	$+$
f	$\nearrow$	$\neg$	$\nearrow$

θ	0	$(0, \pi)$	π	$(\pi, 2\pi)$	2π
f''	0	$-$	0	$+$	0
f	$\neg$	$\frown$	I	$\smile$	$\neg$

48. $f(x) = \cos^2 x, \quad [0, \pi]$

SOLUTION Let $f(x) = \cos^2 x$. Then $f'(x) = -2\cos x \sin x = -2\sin 2x = 0$ when $x = 0$, $x = \pi/2$ and $x = \pi$. All three are candidates for extrema. Moreover, $f''(x) = -4\cos 2x = 0$ when $x = \pi/4$ and $x = 3\pi/4$. Both are candidates for a point of inflection.

x	0	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \pi\right)$	π
f'	0	$-$	0	$+$	0
f	M	$\searrow$	m	$\nearrow$	M

x	$\left(0, \frac{\pi}{4}\right)$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4}, \pi\right)$
f''	$-$	0	$+$	0	$-$
f	$\frown$	I	$\smile$	I	$\frown$

49. $f(x) = \tan x, \quad \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$

SOLUTION Let $f(x) = \tan x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

- Then $f'(x) = \sec^2 x \geq 1 > 0$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.
- Moreover, $f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x = 0$ gives a candidate for a point of inflection at $x = 0$.

x	$\left(-\frac{\pi}{4}, \frac{\pi}{3}\right)$
f'	$+$
f	$\nearrow$

x	$\left(-\frac{\pi}{4}, 0\right)$	0	$\left(0, \frac{\pi}{3}\right)$
f''	$-$	0	$+$
f	$\frown$	I	$\smile$

50. $f(x) = e^{-x}\cos x, \quad \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

SOLUTION Let $f(x) = e^{-x}\cos x$ on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

- Then, $f'(x) = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x) = 0$ gives $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ as candidates for extrema.
- Moreover,

$$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(\sin x + \cos x) = 2e^{-x}\sin x = 0$$

gives $x = 0$ and $x = \pi$ as inflection point candidates.

x	$\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4}, \frac{3\pi}{2}\right)$
f'	$+$	0	$-$	0	$+$
f	↗	M	↘	m	↗

x	$\left(-\frac{\pi}{2}, 0\right)$	0	$(0, \pi)$	π	$\left(\pi, \frac{3\pi}{2}\right)$
f''	$-$	0	$+$	0	$-$
f	$\frown$	I	$\smile$	I	$\frown$

51. $y = (x^2 - 2)e^{-x} \quad (x > 0)$

SOLUTION Let $f(x) = (x^2 - 2)e^{-x}$.

- Then $f'(x) = -(x^2 - 2x - 2)e^{-x} = 0$ gives $x = 1 + \sqrt{3}$ as a candidate for an extrema.
- Moreover, $f''(x) = (x^2 - 4x)e^{-x} = 0$ gives $x = 4$ as a candidate for a point of inflection.

x	$\left(0, 1+\sqrt{3}\right)$	$1+\sqrt{3}$	$\left(1+\sqrt{3}, \infty\right)$
f'	$+$	0	$-$
f	↗	M	↘

x	$(0, 4)$	4	$(4, \infty)$
f''	$-$	0	$+$
f	$\frown$	I	$\smile$

52. $y = \ln(x^2 + 2x + 5)$

SOLUTION Let $f(x) = \ln(x^2 + 2x + 5)$. Then

$$f'(x) = \frac{2x + 2}{x^2 + 2x + 5} = 0$$

when $x = -1$. This is the only critical point. Moreover,

$$f''(x) = -\frac{2(x-1)(x+3)}{(x^2 + 2x + 5)^2},$$

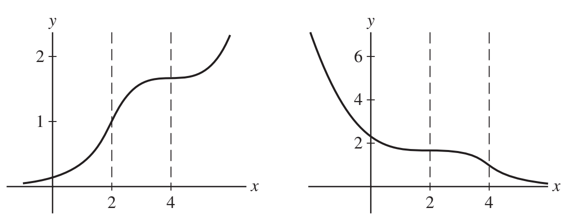
so $x = 1$ and $x = -3$ are candidates for inflection points.

x	$(-\infty, -1)$	-1	$(-1, \infty)$
f'	$-$	0	$+$
f	↘	m	↗

x	$(-\infty, -3)$	-3	$(-3, 1)$	1	$(1, \infty)$
f''	$-$	0	$+$	0	$-$
f	$\frown$	I	$\smile$	I	$\frown$

53. Sketch the graph of an increasing function such that $f''(x)$ changes from $+$ to $-$ at $x = 2$ and from $-$ to $+$ at $x = 4$. Do the same for a decreasing function.

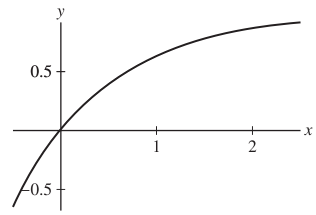
SOLUTION The graph shown below at the left is an increasing function which changes from concave up to concave down at $x = 2$ and from concave down to concave up at $x = 4$. The graph shown below at the right is a decreasing function which changes from concave up to concave down at $x = 2$ and from concave down to concave up at $x = 4$.



In Exercises 54–56, sketch the graph of a function $f(x)$ satisfying all of the given conditions.

54. $f'(x) > 0$ and $f''(x) < 0$ for all x.

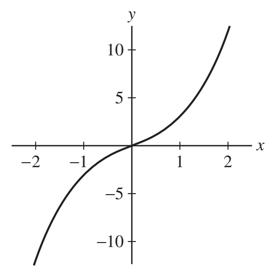
SOLUTION Here is the graph of a function $f(x)$ satisfying $f'(x) > 0$ for all x and $f''(x) < 0$ for all x.



55. **(i)** $f'(x) > 0$ for all x, and

(ii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

SOLUTION Here is the graph of a function $f(x)$ satisfying **(i)** $f'(x) > 0$ for all x and **(ii)** $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.



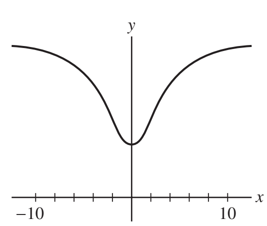
56. **(i)** $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, and

(ii) $f''(x) < 0$ for $|x| > 2$, and $f''(x) > 0$ for $|x| < 2$.

SOLUTION

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Direction	↘	↘	↗	↗
Concavity	$\frown$	$\smile$	$\smile$	$\frown$

One potential graph with this shape is the following:



57. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If $R(t)$ is the number of individuals infected at time t, describe the concavity of the graph of R near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that $R(t)$ has a point of inflection.

SOLUTION

(a) Near the beginning of the epidemic, the graph of R is concave up. Near the epidemic's end, R is concave down.

(b) "Epidemic subsiding: number of new cases declining."

58. Water is pumped into a sphere at a constant rate (Figure 18). Let $h(t)$ be the water level at time t. Sketch the graph of $h(t)$ (approximately, but with the correct concavity). Where does the point of inflection occur?

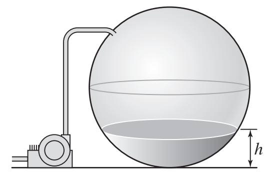
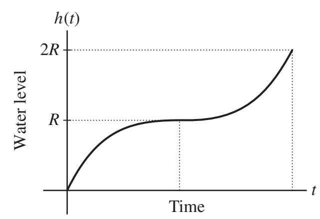


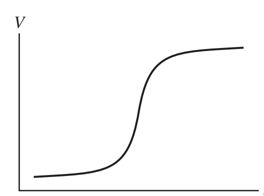
FIGURE 18

SOLUTION Because water is entering the sphere at a constant rate, we expect the water level to rise more rapidly near the bottom and top of the sphere where the sphere is not as "wide" and to rise more slowly near the middle of the sphere. The graph of $h(t)$ should therefore start concave down and end concave up, with an inflection point when the sphere is half full; that is, when the water level is equal to the radius of the sphere. A possible graph of $h(t)$ is shown below.



59. Water is pumped into a sphere of radius R at a variable rate in such a way that the water level rises at a constant rate (Figure 18). Let $V(t)$ be the volume of water in the tank at time t. Sketch the graph $V(t)$ (approximately, but with the correct concavity). Where does the point of inflection occur?

SOLUTION Because water is entering the sphere in such a way that the water level rises at a constant rate, we expect the volume to increase more slowly near the bottom and top of the sphere where the sphere is not as "wide" and to increase more rapidly near the middle of the sphere. The graph of $V(t)$ should therefore start concave up and change to concave down when the sphere is half full; that is, the point of inflection should occur when the water level is equal to the radius of the sphere. A possible graph of $V(t)$ is shown below.



60. (Continuation of Exercise 59) If the sphere has radius R, the volume of water is $V = \pi\left(Rh^2 - \frac{1}{3}h^3\right)$ where h is the water level. Assume the level rises at a constant rate of 1 (that is, $h = t$).

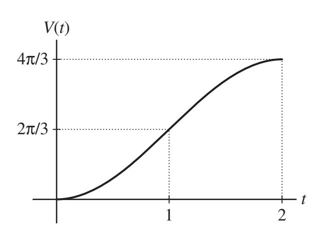
(a) Find the inflection point of $V(t)$. Does this agree with your conclusion in Exercise 59?

(b) GU Plot $V(t)$ for $R = 1$.

SOLUTION

(a) With $h = t$ and $V(t) = \pi(Rt^2 - \frac{1}{3}t^3)$. Then, $V'(t) = \pi(2Rt - t^2)$ and $V''(t) = \pi(2R - 2t)$. Therefore, $V(t)$ is concave up for $t < R$, concave down for $t > R$ and has an inflection point at $t = R$. In other words, $V(t)$ has an inflection point when the water level is equal to the radius of the sphere, in agreement with the conclusion of Exercise 59.

(b) With $h = t$ and $R = 1$, $V(t) = \pi(t^2 - \frac{1}{3}t^3)$. The graph of $V(t)$ is shown below.



61. Image Processing The intensity of a pixel in a digital image is measured by a number u between 0 and 1. Often, images can be enhanced by rescaling intensities (Figure 19), where pixels of intensity u are displayed with intensity $g(u)$ for a suitable function $g(u)$. One common choice is the **sigmoidal correction**, defined for constants a, b by

$$g(u) = \frac{f(u) - f(0)}{f(1) - f(0)} \quad \text{where} \quad f(u) = \left(1 + e^{b(a-u)}\right)^{-1}$$

Figure 20 shows that $g(u)$ reduces the intensity of low-intensity pixels (where $g(u) < u$) and increases the intensity of high-intensity pixels.

(a) Verify that $f'(u) > 0$ and use this to show that $g(u)$ increases from 0 to 1 for $0 \le u \le 1$.
(b) Where does $g'(u)$ have a point of inflection?

Original Sigmoidal correction

FIGURE 19

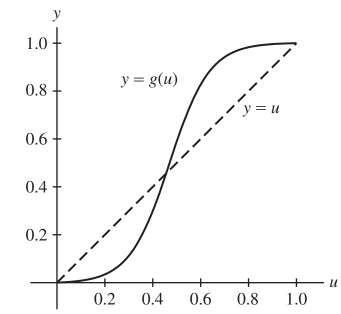


FIGURE 20 Sigmoidal correction with $a = 0.47$, $b = 12$.

SOLUTION

(a) With $f(u) = (1 + e^{b(a-u)})^{-1}$, it follows that

$$f'(u) = -(1 + e^{b(a-u)})^{-2} \cdot -be^{b(a-u)} = \frac{be^{b(a-u)}}{(1 + e^{b(a-u)})^2} > 0$$

for all u. Next, observe that

$$g(0) = \frac{f(0) - f(0)}{f(1) - f(0)} = 0, \quad g(1) = \frac{f(1) - f(0)}{f(1) - f(0)} = 1,$$

and

$$g'(u) = \frac{1}{f(1) - f(0)} f'(u) > 0$$

for all u. Thus, $g(u)$ increases from 0 to 1 for $0 \le u \le 1$.

(b) Working from part (a), we find

$$f''(u) = \frac{b^2 e^{b(a-u)}(2e^{b(a-u)} - 1)}{(1 + e^{b(a-u)})^3}.$$

Because

$$g''(u) = \frac{1}{f(1) - f(0)} f''(u),$$

it follows that $g(u)$ has a point of inflection when

$$2e^{b(a-u)} - 1 = 0 \quad \text{or} \quad u = a + \frac{1}{b} \ln 2.$$

62. Use graphical reasoning to determine whether the following statements are true or false. If false, modify the statement to make it correct.

(a) If $f(x)$ is increasing, then $f^{-1}(x)$ is decreasing.
(b) If $f(x)$ is decreasing, then $f^{-1}(x)$ is decreasing.
(c) If $f(x)$ is concave up, then $f^{-1}(x)$ is concave up.
(d) If $f(x)$ is concave down, then $f^{-1}(x)$ is concave up.

SOLUTION

(a) False. Should be: If $f(x)$ is increasing, then $f^{-1}(x)$ is increasing.
(b) True.
(c) False. Should be: If $f(x)$ is concave up, then $f^{-1}(x)$ is concave down.
(d) True.

Further Insights and Challenges

In Exercises 63–65, assume that $f(x)$ is differentiable.

63. Proof of the Second Derivative Test Let c be a critical point such that $f''(c) > 0$ (the case $f''(c) < 0$ is similar).

(a) Show that $f''(c) = \lim_{h\to 0} \dfrac{f'(c+h)}{h}$.

(b) Use (a) to show that there exists an open interval (a, b) containing c such that $f'(x) < 0$ if $a < x < c$ and $f'(x) > 0$ if $c < x < b$. Conclude that $f(c)$ is a local minimum.

SOLUTION

(a) Because c is a critical point, either $f'(c) = 0$ or $f'(c)$ does not exist; however, $f''(c)$ exists, so $f'(c)$ must also exist. Therefore, $f'(c) = 0$. Now, from the definition of the derivative, we have

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}.$$

(b) We are given that $f''(c) > 0$. By part (a), it follows that

$$\lim_{h\to 0} \frac{f'(c+h)}{h} > 0;$$

in other words, for sufficiently small h,

$$\frac{f'(c+h)}{h} > 0.$$

Now, if h is sufficiently small but negative, then $f'(c+h)$ must also be negative (so that the ratio $f'(c+h)/h$ will be positive) and $c + h < c$. On the other hand, if h is sufficiently small but positive, then $f'(c+h)$ must also be positive and $c + h > c$. Thus, there exists an open interval (a, b) containing c such that $f'(x) < 0$ for $a < x < c$ and $f'(c) > 0$ for $c < x < b$. Finally, because $f'(x)$ changes from negative to positive at $x = c$, $f(c)$ must be a local minimum.

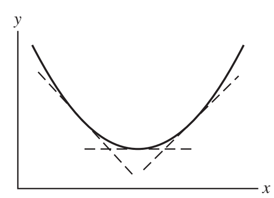
64. Prove that if $f''(x)$ exists and $f''(x) > 0$ for all x, then the graph of $f(x)$ "sits above" its tangent lines.

(a) For any c, set $G(x) = f(x) - f'(c)(x - c) - f(c)$. It is sufficient to prove that $G(x) \geq 0$ for all c. Explain why with a sketch.

(b) Show that $G(c) = G'(c) = 0$ and $G''(x) > 0$ for all x. Conclude that $G'(x) < 0$ for $x < c$ and $G'(x) > 0$ for $x > c$. Then deduce, using the MVT, that $G(x) > G(c)$ for $x \neq c$.

SOLUTION

(a) Let c be any number. Then $y = f'(c)(x - c) + f(c)$ is the equation of the line tangent to the graph of $f(x)$ at $x = c$ and $G(x) = f(x) - f'(c)(x - c) - f(c)$ measures the amount by which the value of the function exceeds the value of the tangent line (see the figure below). Thus, to prove that the graph of $f(x)$ "sits above" its tangent lines, it is sufficient to prove that $G(x) \geq 0$ for all c.



(b) Note that $G(c) = f(c) - f'(c)(c - c) - f(c) = 0$, $G'(x) = f'(x) - f'(c)$ and $G'(c) = f'(c) - f'(c) = 0$. Moreover, $G''(x) = f''(x) > 0$ for all x. Now, because $G'(c) = 0$ and $G'(x)$ is increasing, it must be true that $G'(x) < 0$ for $x < c$ and that $G'(x) > 0$ for $x > c$. Therefore, $G(x)$ is decreasing for $x < c$ and increasing for $x > c$. This implies that $G(c) = 0$ is a minimum; consequently $G(x) > G(c) = 0$ for $x \neq c$.

65. Assume that $f''(x)$ exists and let c be a point of inflection of $f(x)$.

(a) Use the method of Exercise 64 to prove that the tangent line at $x = c$ *crosses the graph* (Figure 21). *Hint:* Show that $G(x)$ changes sign at $x = c$.

(b) GU Verify this conclusion for $f(x) = \dfrac{x}{3x^2+1}$ by graphing $f(x)$ and the tangent line at each inflection point on the same set of axes.

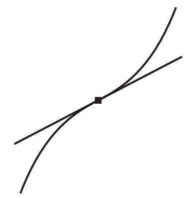


FIGURE 21 Tangent line crosses graph at point of inflection.

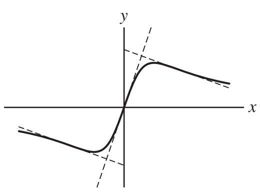
SOLUTION

(a) Let $G(x) = f(x) - f'(c)(x-c) - f(c)$. Then, as in Exercise 63, $G(c) = G'(c) = 0$ and $G''(x) = f''(x)$. If $f''(x)$ changes from positive to negative at $x = c$, then so does $G''(x)$ and $G'(x)$ is increasing for $x < c$ and decreasing for $x > c$. This means that $G'(x) < 0$ for $x < c$ and $G'(x) < 0$ for $x > c$. This in turn implies that $G(x)$ is decreasing, so $G(x) > 0$ for $x < c$ but $G(x) < 0$ for $x > c$. On the other hand, if $f''(x)$ changes from negative to positive at $x = c$, then so does $G''(x)$ and $G'(x)$ is decreasing for $x < c$ and increasing for $x > c$. Thus, $G'(x) > 0$ for $x < c$ and $G'(x) > 0$ for $x > c$. This in turn implies that $G(x)$ is increasing, so $G(x) < 0$ for $x < c$ and $G(x) > 0$ for $x > c$. In either case, $G(x)$ changes sign at $x = c$, and the tangent line at $x = c$ crosses the graph of the function.

(b) Let $f(x) = \dfrac{x}{3x^2+1}$. Then

$$f'(x) = \frac{1-3x^2}{(3x^2+1)^2} \qquad \text{and} \qquad f''(x) = \frac{-18x(1-x^2)}{(3x^2+1)^3}.$$

Therefore $f(x)$ has a point of inflection at $x = 0$ and at $x = \pm 1$. The figure below shows the graph of $y = f(x)$ and its tangent lines at each of the points of inflection. It is clear that each tangent line crosses the graph of $f(x)$ at the inflection point.



66. Let $C(x)$ be the cost of producing x units of a certain good. Assume that the graph of $C(x)$ is concave up.

(a) Show that the average cost $A(x) = C(x)/x$ is minimized at the production level x_0 such that average cost equals marginal cost—that is, $A(x_0) = C'(x_0)$.

(b) Show that the line through $(0, 0)$ and $(x_0, C(x_0))$ is tangent to the graph of $C(x)$.

SOLUTION Let $C(x)$ be the cost of producing x units of a commodity. Assume the graph of C is concave up.

(a) Let $A(x) = C(x)/x$ be the average cost and let x_0 be the production level at which average cost is minimized. Then $A'(x_0) = \dfrac{x_0C'(x_0) - C(x_0)}{x_0^2} = 0$ implies $x_0C'(x_0) - C(x_0) = 0$, whence $C'(x_0) = C(x_0)/x_0 = A(x_0)$. In other words, $A(x_0) = C'(x_0)$ or average cost equals marginal cost at production level x_0. To confirm that x_0 corresponds to a local minimum of A, we use the Second Derivative Test. We find

$$A''(x_0) = \frac{x_0^2C''(x_0) - 2(x_0C'(x_0) - C(x_0))}{x_0^3} = \frac{C''(x_0)}{x_0} > 0$$

because C is concave up. Hence, x_0 corresponds to a local minimum.

(b) The line between $(0, 0)$ and $(x_0, C(x_0))$ is

$$\begin{aligned}\frac{C(x_0)-0}{x_0-0}(x-x_0) + C(x_0) &= \frac{C(x_0)}{x_0}(x-x_0) + C(x_0) = A(x_0)(x-x_0) + C(x_0)\\ &= C'(x_0)(x-x_0) + C(x_0)\end{aligned}$$

which is the tangent line to C at x_0.

67. Let $f(x)$ be a polynomial of degree $n \geq 2$. Show that $f(x)$ has at least one point of inflection if n is odd. Then give an example to show that $f(x)$ need not have a point of inflection if n is even.

SOLUTION Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree n. Then $f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$ and $f''(x) = n(n-1)a_nx^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \cdots + 6a_3x + 2a_2$. If $n \geq 3$ and is odd, then $n-2$ is also odd and $f''(x)$ is a polynomial of odd degree. Therefore $f''(x)$ must take on both positive and negative values. It follows that $f''(x)$ has at least one root c such that $f''(x)$ changes sign at c. The function $f(x)$ will then have a point of inflection at $x = c$. On the other hand, the functions $f(x) = x^2, x^4$ and x^8 are polynomials of even degree that do not have any points of inflection.

68. Critical and Inflection Points If $f'(c) = 0$ and $f(c)$ is neither a local min nor a local max, must $x = c$ be a point of inflection? This is true for "reasonable" functions (including the functions studied in this text), but it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Use the limit definition of the derivative to show that $f'(0)$ exists and $f'(0) = 0$.

(b) Show that $f(0)$ is neither a local min nor a local max.

(c) Show that $f'(x)$ changes sign infinitely often near $x = 0$. Conclude that $x = 0$ is not a point of inflection.

SOLUTION Let $f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$.

(a) Now $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$ by the Squeeze Theorem: as $x \to 0$ we have

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right| \to 0,$$

since $|\sin u| \le 1$.

(b) Since $\sin(\frac{1}{x})$ oscillates through every value between -1 and 1 with increasing frequency as $x \to 0$, in any open interval $(-\delta, \delta)$ there are points a and b such that $f(a) = a^2 \sin(\frac{1}{a}) < 0$ and $f(b) = b^2 \sin(\frac{1}{b}) > 0$. Accordingly, $f(0) = 0$ can neither be a local minimum value nor a local maximum value of f.

(c) In part (a) it was shown that $f'(0) = 0$. For $x \neq 0$, we have

$$f'(x) = x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

As $x \to 0$, $f'(x)$ oscillates increasingly rapidly; consequently, $f'(x)$ changes sign infinitely often near $x = 0$. From this we conclude that $f(x)$ does not have a point of inflection at $x = 0$.

4.5 L'Hôpital's Rule

Preliminary Questions

1. What is wrong with applying L'Hôpital's Rule to $\lim_{x\to 0} \dfrac{x^2 - 2x}{3x - 2}$?

SOLUTION As $x \to 0$,

$$\frac{x^2 - 2x}{3x - 2}$$

is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so L'Hôpital's Rule cannot be used.

2. Does L'Hôpital's Rule apply to $\lim_{x\to a} f(x)g(x)$ if $f(x)$ and $g(x)$ both approach ∞ as $x \to a$?

SOLUTION No. L'Hôpital's Rule only applies to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Exercises

In Exercises 1–10, use L'Hôpital's Rule to evaluate the limit, or state that L'Hôpital's Rule does not apply.

1. $\lim_{x\to 3} \dfrac{2x^2 - 5x - 3}{x - 4}$

SOLUTION Because the quotient is not indeterminate at $x = 3$,

$$\left.\frac{2x^2 - 5x - 3}{x - 4}\right|_{x=3} = \frac{18 - 15 - 3}{3 - 4} = \frac{0}{-1},$$

L'Hôpital's Rule does not apply.

2. $\lim_{x\to -5} \dfrac{x^2 - 25}{5 - 4x - x^2}$

SOLUTION The functions $x^2 - 25$ and $5 - 4x - x^2$ are differentiable, but the quotient is indeterminate at $x = -5$,

$$\left.\frac{x^2 - 25}{5 - 4x - x^2}\right|_{x=-5} = \frac{25 - 25}{5 + 20 - 25} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to -5} \frac{x^2 - 25}{5 - 4x - x^2} = \lim_{x\to -5} \frac{2x}{-4 - 2x} = \frac{-10}{-4 + 10} = -\frac{5}{3}.$$

3. $\lim_{x\to 4} \dfrac{x^3 - 64}{x^2 + 16}$

SOLUTION Because the quotient is not indeterminate at $x = 4$,

$$\left.\frac{x^3 - 64}{x^2 + 16}\right|_{x=4} = \frac{64 - 64}{16 + 16} = \frac{0}{32},$$

L'Hôpital's Rule does not apply.

4. $\lim_{x\to -1} \dfrac{x^4 + 2x + 1}{x^5 - 2x - 1}$

SOLUTION The functions $x^4 + 2x + 1$ and $x^5 - 2x - 1$ are differentiable, but the quotient is indeterminate at $x = -1$,

$$\left.\frac{x^4 + 2x + 1}{x^5 - 2x - 1}\right|_{x=-1} = \frac{1 - 2 + 1}{-1 + 2 - 1} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to -1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1} = \lim_{x\to -1} \frac{4x^3 + 2}{5x^4 - 2} = \frac{-4 + 2}{5 - 2} = -\frac{2}{3}.$$

5. $\lim_{x\to 9} \dfrac{x^{1/2} + x - 6}{x^{3/2} - 27}$

SOLUTION Because the quotient is not indeterminate at $x = 9$,

$$\left.\frac{x^{1/2} + x - 6}{x^{3/2} - 27}\right|_{x=9} = \frac{3 + 9 - 6}{27 - 27} = \frac{6}{0},$$

L'Hôpital's Rule does not apply.

6. $\lim_{x\to 3} \dfrac{\sqrt{x+1} - 2}{x^3 - 7x - 6}$

SOLUTION The functions $\sqrt{x+1} - 2$ and $x^3 - 7x - 6$ are differentiable, but the quotient is indeterminate at $x = 3$,

$$\left.\frac{\sqrt{x+1} - 2}{x^3 - 7x - 6}\right|_{x=3} = \frac{2 - 2}{27 - 21 - 6} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to 3} \frac{\sqrt{x+1} - 2}{x^3 - 7x - 6} = \frac{\frac{1}{2\sqrt{x+1}}}{3x^2 - 7} = \frac{\frac{1}{4}}{20} = \frac{1}{80}.$$

7. $\lim_{x\to 0} \dfrac{\sin 4x}{x^2 + 3x + 1}$

SOLUTION Because the quotient is not indeterminate at $x = 0$,

$$\left.\frac{\sin 4x}{x^2 + 3x + 1}\right|_{x=0} = \frac{0}{0 + 0 + 1} = \frac{0}{1},$$

L'Hôpital's Rule does not apply.

8. $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

SOLUTION The functions x^3 and $\sin x - x$ are differentiable, but the quotient is indeterminate at $x = 0$,

$$\left.\frac{x^3}{\sin x - x}\right|_{x=0} = \frac{0}{0-0} = \frac{0}{0},$$

so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule three times to find

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = -6.$$

9. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x}$

SOLUTION The functions $\cos 2x - 1$ and $\sin 5x$ are differentiable, but the quotient is indeterminate at $x = 0$,

$$\left.\frac{\cos 2x - 1}{\sin 5x}\right|_{x=0} = \frac{1-1}{0} = \frac{0}{0},$$

so L'Hôpital's Rule applies. We find

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x} = \lim_{x\to 0} \frac{-2\sin 2x}{5\cos 5x} = \frac{0}{5} = 0.$$

10. $\lim_{x\to 0} \frac{\cos x - \sin^2 x}{\sin x}$

SOLUTION Because the quotient is not indeterminate at $x = 0$,

$$\left.\frac{\cos x - \sin^2 x}{\sin x}\right|_{x=0} = \frac{1-0}{0} = \frac{1}{0},$$

L'Hôpital's Rule does not apply.

In Exercises 11–16, show that L'Hôpital's Rule is applicable to the limit as $x \to \pm\infty$ and evaluate.

11. $\lim_{x\to\infty} \frac{9x+4}{3-2x}$

SOLUTION As $x \to \infty$, the quotient $\frac{9x+4}{3-2x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to\infty} \frac{9x+4}{3-2x} = \lim_{x\to\infty} \frac{9}{-2} = -\frac{9}{2}.$$

12. $\lim_{x\to -\infty} x \sin \frac{1}{x}$

SOLUTION As $x \to \infty$, $x \sin \frac{1}{x}$ is of the form $\infty \cdot 0$, so L'Hôpital's Rule does not immediately apply. If we rewrite $x \sin \frac{1}{x}$ as $\frac{\sin(1/x)}{1/x}$, the rewritten expression is of the form $\frac{0}{0}$ as $x \to \infty$, so L'Hôpital's Rule now applies. We find

$$\lim_{x\to\infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos(1/x) = \cos 0 = 1.$$

13. $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$

SOLUTION As $x \to \infty$, the quotient $\frac{\ln x}{x^{1/2}}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1}{2x^{1/2}} = 0.$$

14. $\lim_{x\to\infty} \frac{x}{e^x}$

SOLUTION As $x \to \infty$, the quotient $\frac{x}{e^x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. We find

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

15. $\lim_{x\to-\infty} \frac{\ln(x^4+1)}{x}$

SOLUTION As $x \to \infty$, the quotient $\frac{\ln(x^4+1)}{x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule twice to find

$$\lim_{x\to\infty} \frac{\ln(x^4+1)}{x} = \lim_{x\to\infty} \frac{\frac{4x^3}{x^4+1}}{1} = \lim_{x\to\infty} \frac{12x^2}{4x^3} = \lim_{x\to\infty} \frac{3}{x} = 0.$$

16. $\lim_{x\to\infty} \frac{x^2}{e^x}$

SOLUTION As $x \to \infty$, the quotient $\frac{x^2}{e^x}$ is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule twice to find

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

In Exercises 17–54, evaluate the limit.

17. $\lim_{x\to 1} \frac{\sqrt{8+x} - 3x^{1/3}}{x^2 - 3x + 2}$

SOLUTION $\lim_{x\to 1} \frac{\sqrt{8+x} - 3x^{1/3}}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{\frac{1}{2}(8+x)^{-1/2} - x^{-2/3}}{2x-3} = \frac{\frac{1}{6} - 1}{-1} = \frac{5}{6}.$

18. $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right]$

SOLUTION $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right] = \lim_{x\to 4} \left[\frac{\sqrt{x}+2}{x-4} - \frac{4}{x-4}\right] = \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{4}.$

19. $\lim_{x\to-\infty} \frac{3x-2}{1-5x}$

SOLUTION $\lim_{x\to-\infty} \frac{3x-2}{1-5x} = \lim_{x\to-\infty} \frac{3}{-5} = -\frac{3}{5}.$

20. $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x}$

SOLUTION $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{3}{x^{2/3}}}{1 - \frac{1}{x^{2/3}}} = \frac{0+0}{1-0} = 0.$

21. $\lim_{x\to-\infty} \frac{7x^2+4x}{9-3x^2}$

SOLUTION $\lim_{x\to-\infty} \frac{7x^2+4x}{9-3x^2} = \lim_{x\to-\infty} \frac{14x+4}{-6x} = \lim_{x\to-\infty} \frac{14}{-6} = -\frac{7}{3}.$

22. $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7}$

SOLUTION $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7} = \lim_{x\to\infty} \frac{9x^2+8x}{12x^2} = \lim_{x\to\infty} \frac{18x+8}{24x} = \frac{18}{24} = \frac{3}{4}.$

23. $\lim_{x\to 1} \frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2}$

SOLUTION Apply L'Hôpital's Rule once:

$$\lim_{x\to 1} \frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2} = \lim_{x\to 1} \frac{\frac{3}{2}(1+3x)^{-1/2}}{\frac{7}{3}(1+7x)^{-2/3}}$$

$$= \frac{(\frac{3}{2})\frac{1}{2}}{(\frac{7}{3})(\frac{1}{4})} = \frac{9}{7}$$

24. $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2}$

SOLUTION $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2} = \lim_{x\to 8} \frac{\frac{5}{3}x^{2/3}-2}{\frac{1}{3}x^{-2/3}} = \frac{\frac{20}{3}-2}{\frac{1}{12}} = 56.$

25. $\lim_{x\to 0} \frac{\sin 2x}{\sin 7x}$

SOLUTION $\lim_{x\to 0} \frac{\sin 2x}{\sin 7x} = \lim_{x\to 0} \frac{2\cos 2x}{7\cos 7x} = \frac{2}{7}.$

26. $\lim_{x\to\pi/2} \frac{\tan 4x}{\tan 5x}$

SOLUTION

$$\lim_{x\to\pi/2} \frac{\tan 4x}{\tan 5x} = \lim_{x\to\pi/2} \frac{4\sec^2 4x}{5\sec^2 5x} = \frac{4}{5}\lim_{x\to\pi/2} \frac{\cos^2 5x}{\cos^2 4x}$$

$$= \frac{4}{5}\lim_{x\to\pi/2} \frac{-10\sin 5x\cos 5x}{-8\sin 4x\cos 4x} = \lim_{x\to\pi/2} \frac{\sin 10x}{\sin 8x}$$

$$= \lim_{x\to\pi/2} \frac{10\cos 10x}{8\cos 8x} = -\frac{5}{4}.$$

27. $\lim_{x\to 0} \frac{\tan x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1.$

28. $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

SOLUTION

$$\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x} = \lim_{x\to 0} \frac{-x\sin x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{-x\cos x - x}{-x\sin x + \cos x + \cos x} = \frac{0}{2} = 0.$$

29. $\lim_{x\to 0} \frac{\sin x - x\cos x}{x - \sin x}$

SOLUTION

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x-\sin x} = \lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x + \cos x - x\sin x}{\cos x} = 2.$$

30. $\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right)\tan x$

SOLUTION

$$\lim_{x\to\pi/2} \left(x-\frac{\pi}{2}\right)\tan x = \lim_{x\to\pi/2} \frac{x-\pi/2}{1/\tan x} = \lim_{x\to\pi/2} \frac{x-\pi/2}{\cot x} = \lim_{x\to\pi/2} \frac{1}{-\csc^2 x} = \lim_{x\to\pi/2} -\sin^2 x = -1.$$

31. $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x}$

SOLUTION $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x} = \lim_{x\to 0} \frac{-\sin(x+\frac{\pi}{2})}{\cos x} = -1.$

32. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

SOLUTION $\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{2x}{\sin x} = \lim_{x\to 0} \frac{2}{\cos x} = 2.$

33. $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)}$

SOLUTION $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)} = \lim_{x\to \pi/2} \frac{-\sin x}{2\cos(2x)} = \frac{1}{2}.$

34. $\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right)$

SOLUTION

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right) &= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \\
&= \lim_{x\to 0} \frac{2\sin x\cos x - 2x}{2x^2 \sin x \cos x + 2x\sin^2 x} = \lim_{x\to 0} \frac{\sin 2x - 2x}{x^2 \sin 2x + 2x \sin^2 x} \\
&= \lim_{x\to 0} \frac{2\cos 2x - 2}{2x^2\cos 2x + 2x\sin 2x + 4x\sin x\cos x + 2\sin^2 x} = \lim_{x\to 0} \frac{\cos 2x - 1}{x^2 \cos 2x + 2x\sin 2x + \sin^2 x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{-2x^2\sin 2x + 2x\cos 2x + 4x\cos 2x + 2\sin 2x + 2\sin x\cos x} \\
&= \lim_{x\to 0} \frac{-2\sin 2x}{(3-2x^2)\sin 2x + 6x\cos 2x} \\
&= \lim_{x\to 0} \frac{-4\cos 2x}{2(3-2x^2)\cos 2x - 4x\sin 2x + -12x\sin 2x + 6\cos 2x} = -\frac{1}{3}.
\end{aligned}$$

35. $\lim_{x\to \pi/2} (\sec x - \tan x)$

SOLUTION

$$\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x}\right) = 0.$$

36. $\lim_{x\to 2} \frac{e^{x^2} - e^4}{x-2}$

SOLUTION $\lim_{x\to 2} \frac{e^{x^2} - e^4}{x-2} = \lim_{x\to 2} \frac{2xe^{x^2}}{1} = 4e^4.$

37. $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right) \ln x$

SOLUTION $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right)\ln x = \lim_{x\to 1} \frac{\ln x}{\cot(\frac{\pi x}{2})} = \lim_{x\to 1} \frac{\frac{1}{x}}{-\frac{\pi}{2}\csc^2(\frac{\pi x}{2})} = \lim_{x\to 1} \frac{-2}{\pi x} \sin^2\left(\frac{\pi}{2}x\right) = -\frac{2}{\pi}.$

38. $\lim_{x\to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x}$

SOLUTION

$$\lim_{x\to 1} \frac{x(\ln x - 1)+1}{(x-1)\ln x} = \lim_{x\to 1} \frac{x(\frac{1}{x}) + (\ln x - 1)}{(x-1)(\frac{1}{x}) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}.$$

39. $\lim_{x\to 0} \frac{e^x - 1}{\sin x}$

SOLUTION $\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = 1.$

40. $\lim_{x\to 1} \frac{e^x - e}{\ln x}$

SOLUTION $\lim_{x\to 1} \frac{e^x - e}{\ln x} = \lim_{x\to 1} \frac{e^x}{x^{-1}} = \frac{e}{1} = e.$

41. $\lim_{x\to 0} \frac{e^{2x} - 1 - x}{x^2}$

SOLUTION $\lim_{x\to 0} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x\to 0} \frac{2e^{2x} - 1}{2x}$ which does not exist.

42. $\lim_{x\to\infty} \frac{e^{2x} - 1 - x}{x^2}$

SOLUTION

$$\lim_{x\to\infty} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x\to\infty} \frac{2e^{2x} - 1}{2x}$$
$$= \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty.$$

43. $\lim_{t\to 0+} (\sin t)(\ln t)$

SOLUTION

$$\lim_{t\to 0+} (\sin t)(\ln t) = \lim_{t\to 0+} \frac{\ln t}{\csc t} = \lim_{t\to 0+} \frac{\frac{1}{t}}{-\csc t \cot t} = \lim_{t\to 0+} \frac{-\sin^2 t}{t\cos t} = \lim_{t\to 0+} \frac{-2\sin t \cos t}{\cos t - t \sin t} = 0.$$

44. $\lim_{x\to\infty} e^{-x}(x^3 - x^2 + 9)$

SOLUTION

$$\lim_{x\to\infty} e^{-x}(x^3 - x^2 + 9) = \lim_{x\to\infty} \frac{x^3 - x^2 + 9}{e^x} = \lim_{x\to\infty} \frac{3x^2 - 2x}{e^x} = \lim_{x\to\infty} \frac{6x - 2}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

45. $\lim_{x\to 0} \frac{a^x - 1}{x} \quad (a > 0)$

SOLUTION $\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{\ln a \cdot a^x}{1} = \ln a.$

46. $\lim_{x\to\infty} x^{1/x^2}$

SOLUTION $\lim_{x\to\infty} \ln x^{1/x^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$ Hence,

$$\lim_{x\to\infty} x^{1/x^2} = \lim_{x\to\infty} e^{\ln x^{1/x^2}} = e^0 = 1.$$

47. $\lim_{x\to 1} (1 + \ln x)^{1/(x-1)}$

SOLUTION $\lim_{x\to 1} \ln(1 + \ln x)^{1/(x-1)} = \lim_{x\to 1} \frac{\ln(1 + \ln x)}{x - 1} = \lim_{x\to 1} \frac{1}{x(1 + \ln x)} = 1.$ Hence,

$$\lim_{x\to 1} (1 + \ln x)^{1/(x-1)} = \lim_{x\to 1} e^{(1+\ln x)^{1/(x-1)}} = e.$$

48. $\lim_{x\to 0+} x^{\sin x}$

SOLUTION

$$\lim_{x\to 0+} \ln(x^{\sin x}) = \lim_{x\to 0+} \sin x(\ln x) = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\cos x(\sin x)^{-2}}$$
$$= \lim_{x\to 0+} -\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0+} -\frac{2\sin x\cos x}{-x\sin x + \cos x} = 0.$$

Hence, $\lim_{x\to 0+} x^{\sin x} = \lim_{x\to 0+} e^{\ln(x^{\sin x})} = e^0 = 1.$

49. $\lim_{x\to 0}(\cos x)^{3/x^2}$

SOLUTION

$$\lim_{x\to 0}\ln(\cos x)^{3/x^2} = \lim_{x\to 0}\frac{3\ln\cos x}{x^2}$$
$$= \lim_{x\to 0} -\frac{3\tan x}{2x}$$
$$= \lim_{x\to 0} -\frac{3\sec^2 x}{2} = -\frac{3}{2}.$$

Hence, $\lim_{x\to 0}(\cos x)^{3/x^2} = e^{-3/2}$.

50. $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$

SOLUTION

$$\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{1/x} = \lim_{x\to\infty}\frac{\left(\frac{x+1}{x}\right)\left(\frac{1}{(x+1)^2}\right)}{-1/x^2} = \lim_{x\to\infty} -\frac{x}{x+1} = -1.$$

Hence,

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \frac{1}{e}.$$

51. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

SOLUTION $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = \lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$

52. $\lim_{x\to 0}\frac{\tan^{-1}x}{\sin^{-1}x}$

SOLUTION $\lim_{x\to 0}\frac{\tan^{-1}x}{\sin^{-1}x} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = 1.$

53. $\lim_{x\to 1}\frac{\tan^{-1}x - \frac{\pi}{4}}{\tan\frac{\pi}{4}x - 1}$

SOLUTION $\lim_{x\to 1}\frac{\tan^{-1}x - \frac{\pi}{4}}{\tan(\pi x/4) - 1} = \lim_{x\to 1}\frac{\frac{1}{1+x^2}}{\frac{\pi}{4}\sec^2(\pi x/4)} = \frac{\frac{1}{2}}{\frac{\pi}{2}} = \frac{1}{\pi}.$

54. $\lim_{x\to 0+}\ln x\tan^{-1}x$

SOLUTION Let $h(x) = \ln x\tan^{-1}x$. $\lim_{x\to 0}h(x) = -\infty\cdot 0$, so we apply L'Hôpital's rule to $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = \tan^{-1}(x)$ and $g(x) = \frac{1}{\ln x}$.

$$f'(x) = \frac{1}{1+x^2}$$
$$\lim_{x\to 0}f'(x) = 1$$
$$g'(x) = -\frac{1}{x(\ln x)^2}$$
$$\lim_{x\to 0}g'(x) = -\infty$$

Hence, L'Hôpital's rule yields:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\lim_{x\to 0}f'(x)}{\lim_{x\to 0}g'(x)} = -\frac{1}{\infty} = 0.$$

55. Evaluate $\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx}$, where $m, n \neq 0$ are integers.

SOLUTION Suppose m and n are even. Then there exist integers k and l such that $m = 2k$ and $n = 2l$ and

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \frac{\cos k\pi}{\cos l\pi} = (-1)^{k-l}.$$

Now, suppose m is even and n is odd. Then

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx}$$

does not exist (from one side the limit tends toward $-\infty$, while from the other side the limit tends toward $+\infty$). Third, suppose m is odd and n is even. Then

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = 0.$$

Finally, suppose m and n are odd. This is the only case when the limit is indeterminate. Then there exist integers k and l such that $m = 2k+1$, $n = 2l+1$ and, by L'Hôpital's Rule,

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \lim_{x\to\pi/2}\frac{-m\sin mx}{-n\sin nx} = (-1)^{k-l}\frac{m}{n}.$$

To summarize,

$$\lim_{x\to\pi/2}\frac{\cos mx}{\cos nx} = \begin{cases} (-1)^{(m-n)/2}, & m, n \text{ even} \\ \text{does not exist}, & m \text{ even}, n \text{ odd} \\ 0 & m \text{ odd}, n \text{ even} \\ (-1)^{(m-n)/2}\frac{m}{n}, & m, n \text{ odd} \end{cases}$$

56. Evaluate $\lim_{x\to 1}\frac{x^m - 1}{x^n - 1}$ for any numbers $m, n \neq 0$.

SOLUTION $\lim_{x\to 1}\frac{x^m - 1}{x^n - 1} = \lim_{x\to 1}\frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n}.$

57. Prove the following limit formula for e:

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Then find a value of x such that $|(1+x)^{1/x} - e| \le 0.001$.

SOLUTION Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1.$$

Thus,

$$\lim_{x\to 0}\ln\left((1+x)^{1/x}\right) = \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1,$$

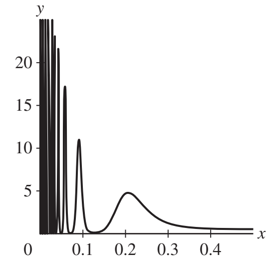
and $\lim_{x\to 0}(1+x)^{1/x} = e^1 = e$. For $x = 0.0005$,

$$\left|(1+x)^{1/x} - e\right| = |(1.0005)^{2000} - e| \approx 6.79 \times 10^{-4} < 0.001.$$

58. $\boxed{\text{GU}}$ Can L'Hôpital's Rule be applied to $\lim_{x\to 0+} x^{\sin(1/x)}$? Does a graphical or numerical investigation suggest that the limit exists?

SOLUTION Since $\sin(1/x)$ oscillates as $x \to 0+$, L'Hôpital's Rule cannot be applied. Both numerical and graphical investigations suggest that the limit does not exist due to the oscillation.

x	1	0.1	0.01	0.001	0.0001	0.00001
$x^{\sin(1/x)}$	1	3.4996	10.2975	0.003316	16.6900	0.6626



59. Let $f(x) = x^{1/x}$ for $x > 0$.

(a) Calculate $\lim_{x\to 0+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the maximum value of $f(x)$, and determine the intervals on which $f(x)$ is increasing or decreasing.

SOLUTION

(a) Let $f(x) = x^{1/x}$. Note that $\lim_{x\to 0+} x^{1/x}$ is not indeterminate. As $x \to 0+$, the base of the function tends toward 0 and the exponent tends toward $+\infty$. Both of these factors force $x^{1/x}$ toward 0. Thus, $\lim_{x\to 0+} f(x) = 0$. On the other hand, $\lim_{x\to\infty} f(x)$ is indeterminate. We calculate this limit as follows:

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

so $\lim_{x\to\infty} f(x) = e^0 = 1$.

(b) Again, let $f(x) = x^{1/x}$, so that $\ln f(x) = \frac{1}{x}\ln x$. To find the derivative f', we apply the derivative to both sides:

$$\begin{aligned}
\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\left(\frac{1}{x}\ln x\right)\\
\frac{1}{f(x)} f'(x) &= -\frac{\ln x}{x^2} + \frac{1}{x^2}\\
f'(x) &= f(x)\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right) = \frac{x^{1/x}}{x^2}(1 - \ln x)
\end{aligned}$$

Thus, f is increasing for $0 < x < e$, is decreasing for $x > e$ and has a maximum at $x = e$. The maximum value is $f(e) = e^{1/e} \approx 1.444668$.

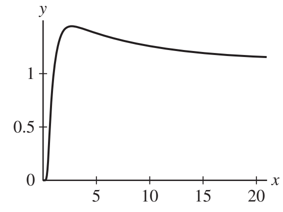
60. **(a)** Use the results of Exercise 59 to prove that $x^{1/x} = c$ has a unique solution if $0 < c \le 1$ or $c = e^{1/e}$, two solutions if $1 < c < e^{1/e}$, and no solutions if $c > e^{1/e}$.

(b) GU Plot the graph of $f(x) = x^{1/x}$ and verify that it confirms the conclusions of (a).

SOLUTION

(a) Because $(e, e^{1/e})$ is the only maximum, no solution exists for $c > e^{1/e}$ and only one solution exists for $c = e^{1/e}$. Moreover, because $f(x)$ increases from 0 to $e^{1/e}$ as x goes from 0 to e and then decreases from $e^{1/e}$ to 1 as x goes from e to $+\infty$, it follows that there are two solutions for $1 < c < e^{1/e}$, but only one solution for $0 < c \le 1$.

(b) Observe that if we sketch the horizontal line $y = c$, this line will intersect the graph of $y = f(x)$ only once for $0 < c \le 1$ and $c = e^{1/e}$ and will intersect the graph of $y = f(x)$ twice for $1 < c < e^{1/e}$. There are no points of intersection for $c > e^{1/e}$.



61. Determine whether $f << g$ or $g << f$ (or neither) for the functions $f(x) = \log_{10} x$ and $g(x) = \ln x$.

SOLUTION Because

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log_{10} x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 10}}{\ln x} = \frac{1}{\ln 10},$$

neither $f << g$ or $g << f$ is satisfied.

62. Show that $(\ln x)^2 << \sqrt{x}$ and $(\ln x)^4 << x^{1/10}$.

SOLUTION

- $(\ln x)^2 << \sqrt{x}$:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x}\ln x} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty.$$

- $(\ln x)^4 << x^{1/10}$:

$$\lim_{x\to\infty} \frac{x^{1/10}}{(\ln x)^4} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{4}{x}(\ln x)^3} = \lim_{x\to\infty} \frac{x^{1/10}}{40(\ln x)^3} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{120}{x}(\ln x)^2} = \lim_{x\to\infty} \frac{x^{1/10}}{1200(\ln x)^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{2400}{x}(\ln x)} = \lim_{x\to\infty} \frac{x^{1/10}}{24000\ln x} = \lim_{x\to\infty} \frac{\frac{1}{10x^{9/10}}}{\frac{24000}{x}} = \lim_{x\to\infty} \frac{x^{1/10}}{240000} = \infty.$$

63. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that $\ln x << x^a$ for all $a > 0$.

SOLUTION Using L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{x^{-1}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{a}x^{-a} = 0;$$

hence, $\ln x << (x^a)$.

64. Show that $(\ln x)^N << x^a$ for all N and all $a > 0$.

SOLUTION

$$\lim_{x\to\infty} \frac{x^a}{(\ln x)^N} = \lim_{x\to\infty} \frac{ax^{a-1}}{\frac{N}{x}(\ln x)^{N-1}} = \lim_{x\to\infty} \frac{ax^a}{N(\ln x)^{N-1}} = \cdots$$

If we continue in this manner, L'Hôpital's Rule will give a factor of x^a in the numerator, but the power on $\ln x$ in the denominator will eventually be zero. Thus,

$$\lim_{x\to\infty} \frac{x^a}{(\ln x)^N} = \infty,$$

so $(\ln x)^N << x^a$ for all N and for all $a > 0$.

65. Determine whether $\sqrt{x} << e^{\sqrt{\ln x}}$ or $e^{\sqrt{\ln x}} << \sqrt{x}$. *Hint:* Use the substitution $u = \ln x$ instead of L'Hôpital's Rule.

SOLUTION Let $u = \ln x$, then $x = e^u$, and as $x \to \infty$, $u \to \infty$. So

$$\lim_{x\to\infty} \frac{e^{\sqrt{\ln x}}}{\sqrt{x}} = \lim_{u\to\infty} \frac{e^{\sqrt{u}}}{e^{u/2}} = \lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}}.$$

We need to examine $\lim_{u\to\infty}(\sqrt{u} - \frac{u}{2})$. Since

$$\lim_{u\to\infty} \frac{u/2}{\sqrt{u}} = \lim_{u\to\infty} \frac{\frac{1}{2}}{\frac{1}{2\sqrt{u}}} = \lim_{u\to\infty} \sqrt{u} = \infty,$$

$\sqrt{u} = o(u/2)$ and $\lim_{u\to\infty}\left(\sqrt{u} - \dfrac{u}{2}\right) = -\infty$. Thus

$$\lim_{u\to\infty} e^{\sqrt{u}-\frac{u}{2}} = e^{-\infty} = 0 \quad \text{so} \quad \lim_{x\to\infty} \frac{e^{\sqrt{\ln x}}}{\sqrt{x}} = 0$$

and $e^{\sqrt{\ln x}} << \sqrt{x}$.

66. Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all whole numbers $n > 0$.

SOLUTION

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$$
$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$
$$\vdots$$
$$= \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

67. Assumptions Matter Let $f(x) = x(2+\sin x)$ and $g(x) = x^2+1$.

(a) Show directly that $\lim_{x\to\infty} f(x)/g(x) = 0$.

(b) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, but $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

Do (a) and (b) contradict L'Hôpital's Rule? Explain.

SOLUTION

(a) $1 \le 2+\sin x \le 3$, so

$$\frac{x}{x^2+1} \le \frac{x(2+\sin x)}{x^2+1} \le \frac{3x}{x^2+1}.$$

Since,

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{3x}{x^2+1} = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to\infty} \frac{x(2+\sin x)}{x^2+1} = 0.$$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(2+\sin x) \ge \lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (x^2+1) = \infty$, but

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{x(\cos x) + (2+\sin x)}{2x}$$

does not exist since $\cos x$ oscillates. This does not violate L'Hôpital's Rule since the theorem clearly states

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

"provided the limit on the right exists."

68. Let $H(b) = \lim_{x\to\infty} \dfrac{\ln(1+b^x)}{x}$ for $b > 0$.

(a) Show that $H(b) = \ln b$ if $b \ge 1$

(b) Determine $H(b)$ for $0 < b \le 1$.

SOLUTION

(a) Suppose $b \ge 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \frac{b^x \ln b}{b^x} = \ln b.$$

(b) Now, suppose $0 < b < 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \frac{0}{1} = 0.$$

69. Let $G(b) = \lim_{x\to\infty} (1+b^x)^{1/x}$.

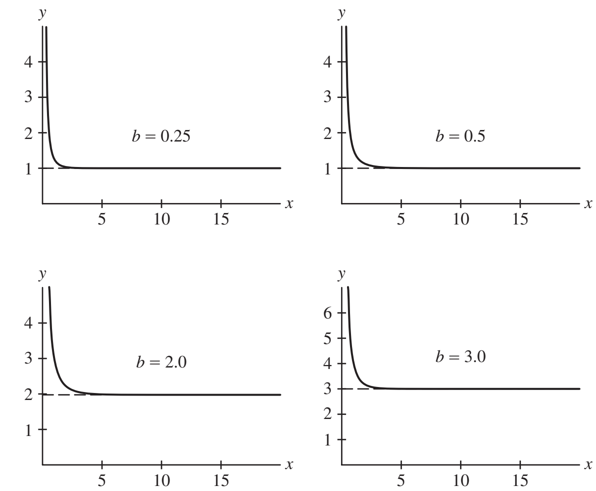
(a) Use the result of Exercise 68 to evaluate $G(b)$ for all $b > 0$.

(b) $\boxed{\text{GU}}$ Verify your result graphically by plotting $y = (1+b^x)^{1/x}$ together with the horizontal line $y = G(b)$ for the values $b = 0.25, 0.5, 2, 3$.

SOLUTION

(a) Using Exercise 68, we see that $G(b) = e^{H(b)}$. Thus, $G(b) = 1$ if $0 \le b \le 1$ and $G(b) = b$ if $b > 1$.

(b)



70. Show that $\lim\limits_{t\to\infty} t^k e^{-t^2} = 0$ for all k. *Hint:* Compare with $\lim\limits_{t\to\infty} t^k e^{-t} = 0$.

SOLUTION Because we are interested in the limit as $t \to +\infty$, we will restrict attention to $t > 1$. Then, for all k,

$$0 \le t^k e^{-t^2} \le t^k e^{-t}.$$

As $\lim\limits_{t\to\infty} t^k e^{-t} = 0$, it follows from the Squeeze Theorem that

$$\lim_{t\to\infty} t^k e^{-t^2} = 0.$$

In Exercises 71–73, let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

These exercises show that $f(x)$ has an unusual property: All of its derivatives at $x = 0$ exist and are equal to zero.

71. Show that $\lim\limits_{x\to 0} \dfrac{f(x)}{x^k} = 0$ for all k. *Hint:* Let $t = x^{-1}$ and apply the result of Exercise 70.

SOLUTION $\lim\limits_{x\to 0} \dfrac{f(x)}{x^k} = \lim\limits_{x\to 0} \dfrac{1}{x^k e^{1/x^2}}$. Let $t = 1/x$. As $x \to 0$, $t \to \infty$. Thus,

$$\lim_{x\to 0} \frac{1}{x^k e^{1/x^2}} = \lim_{t\to\infty} \frac{t^k}{e^{t^2}} = 0$$

by Exercise 70.

72. Show that $f'(0)$ exists and is equal to zero. Also, verify that $f''(0)$ exists and is equal to zero.

SOLUTION Working from the definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 0$$

by the previous exercise. Thus, $f'(0)$ exists and is equal to 0. Moreover,

$$f'(x) = \begin{cases} e^{-1/x^2}\left(\frac{2}{x^3}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Now,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} e^{-1/x^2}\left(\frac{2}{x^4}\right) = 2\lim_{x\to 0} \frac{f(x)}{x^4} = 0$$

by the previous exercise. Thus, $f''(0)$ exists and is equal to 0.

73. Show that for $k \geq 1$ and $x \neq 0$,

$$f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$$

for some polynomial $P(x)$ and some exponent $r \geq 1$. Use the result of Exercise 71 to show that $f^{(k)}(0)$ exists and is equal to zero for all $k \geq 1$.

SOLUTION For $x \neq 0$, $f'(x) = e^{-1/x^2}\left(\frac{2}{x^3}\right)$. Here $P(x) = 2$ and $r = 3$. Assume $f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$. Then

$$f^{(k+1)}(x) = e^{-1/x^2}\left(\frac{x^3P'(x) + (2 - rx^2)P(x)}{x^{r+3}}\right)$$

which is of the form desired.

Moreover, from Exercise 72, $f'(0) = 0$. Suppose $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{P(x)e^{-1/x^2}}{x^{r+1}} = P(0)\lim_{x\to 0}\frac{f(x)}{x^{r+1}} = 0.$$

Further Insights and Challenges

74. Show that L'Hôpital's Rule applies to $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ but that it does not help. Then evaluate the limit directly.

SOLUTION Both the numerator $f(x) = x$ and the denominator $g(x) = \sqrt{x^2+1}$ tend to infinity as $x \to \infty$, and $g'(x) = x/\sqrt{x^2+1}$ is nonzero for $x > 0$. Therefore, L'Hôpital's Rule applies:

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{1}{x(x^2+1)^{-1/2}} = \lim_{x\to\infty}\frac{(x^2+1)^{1/2}}{x}$$

We may apply L'Hôpital's Rule again: $\lim_{x\to\infty}\frac{(x^2+1)^{1/2}}{x} = \lim_{x\to\infty}\frac{x(x^2+1)^{-1/2}}{1} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$. This takes us back to the original limit, so L'Hôpital's Rule is ineffective. However, we can evaluate the limit directly by observing that

$$\frac{x}{\sqrt{x^2+1}} = \frac{x^{-1}(x)}{x^{-1}\sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^{-2}}} \quad \text{and hence} \quad \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+x^{-2}}} = 1.$$

75. The Second Derivative Test for critical points fails if $f''(c) = 0$. This exercise develops a **Higher Derivative Test** based on the sign of the first nonzero derivative. Suppose that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0$$

(a) Show, by applying L'Hôpital's Rule n times, that

$$\lim_{x\to c}\frac{f(x) - f(c)}{(x-c)^n} = \frac{1}{n!}f^{(n)}(c)$$

where $n! = n(n-1)(n-2)\cdots(2)(1)$.

(b) Use (a) to show that if n is even, then $f(c)$ is a local minimum if $f^{(n)}(c) > 0$ and is a local maximum if $f^{(n)}(c) < 0$. *Hint:* If n is even, then $(x-c)^n > 0$ for $x \neq a$, so $f(x) - f(c)$ must be positive for x near c if $f^{(n)}(c) > 0$.

(c) Use (a) to show that if n is odd, then $f(c)$ is neither a local minimum nor a local maximum.

SOLUTION

(a) Repeated application of L'Hôpital's rule yields

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{(x-c)^n} &= \lim_{x\to c}\frac{f'(x)}{n(x-c)^{n-1}}\\ &= \lim_{x\to c}\frac{f''(x)}{n(n-1)(x-c)^{n-2}}\\ &= \lim_{x\to c}\frac{f'''(x)}{n(n-1)(n-2)(x-c)^{n-3}}\\ &= \cdots\\ &= \frac{1}{n!}f^{(n)}(c)\end{aligned}$$

(b) Suppose n is even. Then $(x-c)^n > 0$ for all $x \neq c$. If $f^{(n)}(c) > 0$, it follows that $f(x) - f(c)$ must be positive for x near c. In other words, $f(x) > f(c)$ for x near c and $f(c)$ is a local minimum. On the other hand, if $f^{(n)}(c) < 0$, it follows that $f(x) - f(c)$ must be negative for x near c. In other words, $f(x) < f(c)$ for x near c and $f(c)$ is a local maximum.

(c) If n is odd, then $(x-c)^n > 0$ for $x > c$ but $(x-c)^n < 0$ for $x < c$. If $f^{(n)}(c) > 0$, it follows that $f(x) - f(c)$ must be positive for x near c and $x > c$ but is negative for x near c and $x < c$. In other words, $f(x) > f(c)$ for x near c and $x > c$ but $f(x) < f(c)$ for x near c and $x < c$. Thus, $f(c)$ is neither a local minimum nor a local maximum. We obtain a similar result if $f^{(n)}(c) < 0$.

76. When a spring with natural frequency $\lambda/2\pi$ is driven with a sinusoidal force $\sin(\omega t)$ with $\omega \neq \lambda$, it oscillates according to

$$y(t) = \frac{1}{\lambda^2 - \omega^2}\big(\lambda \sin(\omega t) - \omega \sin(\lambda t)\big)$$

Let $y_0(t) = \lim_{\omega \to \lambda} y(t)$.

(a) Use L'Hôpital's Rule to determine $y_0(t)$.

(b) Show that $y_0(t)$ ceases to be periodic and that its amplitude $|y_0(t)|$ tends to ∞ as $t \to \infty$ (the system is said to be in **resonance**; eventually, the spring is stretched beyond its limits).

(c) CAS Plot $y(t)$ for $\lambda = 1$ and $\omega = 0.8, 0.9, 0.99$, and 0.999. Do the graphs confirm your conclusion in (b)?

SOLUTION

(a)

$$\lim_{\omega \to \lambda} y(t) = \lim_{\omega \to \lambda} \frac{\lambda \sin(\omega t) - \omega \sin(\lambda t)}{\lambda^2 - \omega^2} = \lim_{\omega \to \lambda} \frac{\frac{d}{d\omega}(\lambda \sin(\omega t) - \omega \sin(\lambda t))}{\frac{d}{d\omega}(\lambda^2 - \omega^2)}$$

$$= \lim_{\omega \to \lambda} \frac{\lambda t \cos(\omega t) - \sin(\lambda t)}{-2\omega} = \frac{\lambda t \cos(\lambda t) - \sin(\lambda t)}{-2\lambda}$$

(b) From part (a)

$$y_0(t) = \lim_{\omega \to \lambda} y(t) = \frac{\lambda t \cos(\lambda t) - \sin(\lambda t)}{-2\lambda}.$$

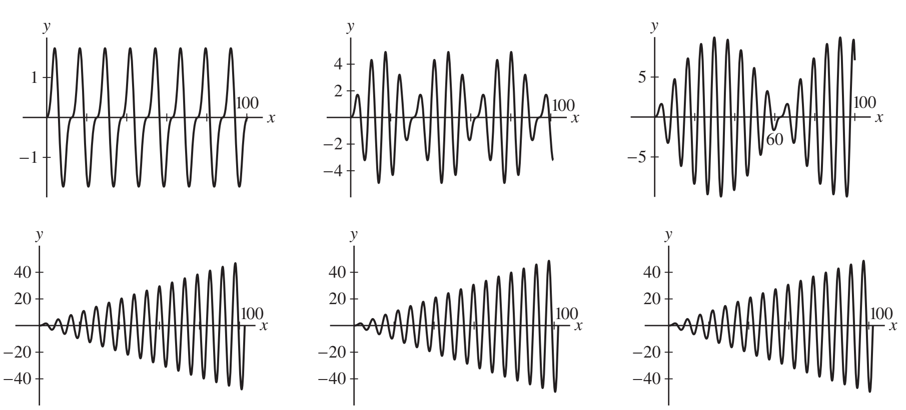
This may be rewritten as

$$y_0(t) = \frac{\sqrt{\lambda^2 t^2 + 1}}{-2\lambda} \cos(\lambda t + \phi),$$

where $\cos\phi = \dfrac{\lambda t}{\sqrt{\lambda^2 t^2 + 1}}$ and $\sin\phi = \dfrac{1}{\sqrt{\lambda^2 t^2 + 1}}$. Since the amplitude varies with t, $y_0(t)$ is not periodic. Also note that

$$\frac{\sqrt{\lambda^2 t^2 + 1}}{-2\lambda} \to \infty \quad \text{as} \quad t \to \infty.$$

(c) The graphs below were produced with $\lambda = 1$. Moving from left to right and from top to bottom, $\omega = 0.5, 0.8, 0.9, 0.99, 0.999, 1$.



77. We expended a lot of effort to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ in Chapter 2. Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve *circular reasoning*.

SOLUTION $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$. To use L'Hôpital's Rule to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$, we must know that the derivative of $\sin x$ is $\cos x$, but to determine the derivative of $\sin x$, we must be able to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

78. By a fact from algebra, if f, g are polynomials such that $f(a) = g(a) = 0$, then there are polynomials f_1, g_1 such that

$$f(x) = (x-a)f_1(x), \qquad g(x) = (x-a)g_1(x)$$

Use this to verify L'Hôpital's Rule directly for $\lim_{x\to a} f(x)/g(x)$.

SOLUTION As in the problem statement, let $f(x)$ and $g(x)$ be two polynomials such that $f(a) = g(a) = 0$, and let $f_1(x)$ and $g_1(x)$ be the polynomials such that $f(x) = (x-a)f_1(x)$ and $g(x) = (x-a)g_1(x)$. By the product rule, we have the following facts,

$$f'(x) = (x-a)f_1'(x) + f_1(x)$$
$$g'(x) = (x-a)g_1'(x) + g_1(x)$$

so

$$\lim_{x\to a} f'(x) = f_1(a) \quad \text{and} \quad \lim_{x\to a} g'(x) = g_1(a).$$

L'Hôpital's Rule stated for f and g is: if $\lim_{x\to a} g'(x) \neq 0$, so that $g_1(a) \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f_1(a)}{g_1(a)}.$$

Suppose $g_1(a) \neq 0$. Then, by direct computation,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)f_1(x)}{(x-a)g_1(x)} = \lim_{x\to a} \frac{f_1(x)}{g_1(x)} = \frac{f_1(a)}{g_1(a)},$$

exactly as predicted by L'Hôpital's Rule.

79. Patience Required Use L'Hôpital's Rule to evaluate and check your answers numerically:

(a) $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2}$ **(b)** $\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

SOLUTION

(a) We start by evaluating

$$\lim_{x\to 0+} \ln\left(\frac{\sin x}{x}\right)^{1/x^2} = \lim_{x\to 0+} \frac{\ln(\sin x) - \ln x}{x^2}.$$

Repeatedly using L'Hôpital's Rule, we find

$$\begin{aligned}\lim_{x\to 0+} \ln\left(\frac{\sin x}{x}\right)^{1/x^2} &= \lim_{x\to 0+} \frac{\cot x - x^{-1}}{2x} = \lim_{x\to 0+} \frac{x\cos x - \sin x}{2x^2 \sin x} = \lim_{x\to 0+} \frac{-x\sin x}{2x^2\cos x + 4x\sin x} \\ &= \lim_{x\to 0+} \frac{-x\cos x - \sin x}{8x\cos x + 4\sin x - 2x^2\sin x} = \lim_{x\to 0+} \frac{-2\cos x + x\sin x}{12\cos x - 2x^2\cos x - 12x\sin x} \\ &= -\frac{2}{12} = -\frac{1}{6}.\end{aligned}$$

Therefore, $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$. Numerically we find:

x	1	0.1	0.01
$\left(\frac{\sin x}{x}\right)^{1/x^2}$	0.841471	0.846435	0.846481

Note that $e^{-1/6} \approx 0.846481724$.

(b) Repeatedly using L'Hôpital's Rule and simplifying, we find

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{x^2(2\sin x\cos x)+2x\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin 2x}{x^2\sin 2x+2x\sin^2 x}$$
$$=\lim_{x\to 0}\frac{2-2\cos 2x}{2x^2\cos 2x+2x\sin 2x+4x\sin x\cos x+2\sin^2 x}$$
$$=\lim_{x\to 0}\frac{2-2\cos 2x}{2x^2\cos 2x+4x\sin 2x+2\sin^2 x}$$
$$=\lim_{x\to 0}\frac{4\sin 2x}{-4x^2\sin 2x+4x\cos 2x+8x\cos 2x+4\sin 2x+4\sin x\cos x}$$
$$=\lim_{x\to 0}\frac{4\sin 2x}{(6-4x^2)\sin 2x+12x\cos 2x}$$
$$=\lim_{x\to 0}\frac{8\cos 2x}{(12-8x^2)\cos 2x-8x\sin 2x+12\cos 2x-24x\sin 2x}=\frac{1}{3}.$$

Numerically we find:

x	1	0.1	0.01
$\frac{1}{\sin^2 x}-\frac{1}{x^2}$	0.412283	0.334001	0.333340

80. In the following cases, check that $x=c$ is a critical point and use Exercise 75 to determine whether $f(c)$ is a local minimum or a local maximum.
(a) $f(x)=x^5-6x^4+14x^3-16x^2+9x+12 \quad (c=1)$
(b) $f(x)=x^6-x^3 \quad (c=0)$

SOLUTION

(a) Let $f(x)=x^5-6x^4+14x^3-16x^2+9x+12$. Then $f'(x)=5x^4-24x^3+42x^2-32x+9$, so $f'(1)=5-24+42-32+9=0$ and $c=1$ is a critical point. Now,

$$f''(x)=20x^3-72x^2+84x-32 \text{ so } f''(1)=0;$$
$$f'''(x)=60x^2-144x+84 \text{ so } f'''(1)=0;$$
$$f^{(4)}(x)=120x-144 \text{ so } f^{(4)}(1)=-24\neq 0.$$

Thus, $n=4$ is even and $f^{(4)}<0$, so $f(1)$ is a local maximum.
(b) Let $f(x)=x^6-x^3$. Then, $f'(x)=6x^5-3x^2$, so $f'(0)=0$ and $c=0$ is a critical point. Now,

$$f''(x)=30x^4-6x \text{ so } f''(0)=0;$$
$$f'''(x)=120x-6 \text{ so } f'''(0)=-6\neq 0.$$

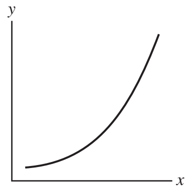
Thus, $n=3$ is odd, so $f(0)$ is neither a local minimum nor a local maximum.

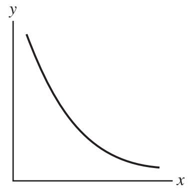
4.6 Graph Sketching and Asymptotes

Preliminary Questions

1. Sketch an arc where f' and f'' have the sign combination $++$. Do the same for $-+$.

SOLUTION An arc with the sign combination $++$ (increasing, concave up) is shown below at the left. An arc with the sign combination $-+$ (decreasing, concave up) is shown below at the right.





2. If the sign combination of f' and f'' changes from $++$ to $+-$ at $x = c$, then (choose the correct answer):

(a) $f(c)$ is a local min

(b) $f(c)$ is a local max

(c) c is a point of inflection

SOLUTION Because the sign of the second derivative changes at $x = c$, the correct response is **(c)**: c is a point of inflection.

3. The second derivative of the function $f(x) = (x-4)^{-1}$ is $f''(x) = 2(x-4)^{-3}$. Although $f''(x)$ changes sign at $x = 4$, $f(x)$ does not have a point of inflection at $x = 4$. Why not?

SOLUTION The function f does not have a point of inflection at $x = 4$ because $x = 4$ is not in the domain of f.

Exercises

1. Determine the sign combinations of f' and f'' for each interval A–G in Figure 16.

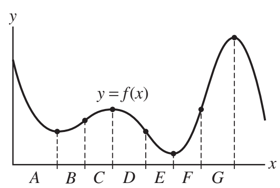


FIGURE 16

SOLUTION

- In A, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In B, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In C, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.
- In D, f is decreasing and concave down, so $f' < 0$ and $f'' < 0$.
- In E, f is decreasing and concave up, so $f' < 0$ and $f'' > 0$.
- In F, f is increasing and concave up, so $f' > 0$ and $f'' > 0$.
- In G, f is increasing and concave down, so $f' > 0$ and $f'' < 0$.

2. State the sign change at each transition point A–G in Figure 17. Example: $f'(x)$ goes from $+$ to $-$ at A.

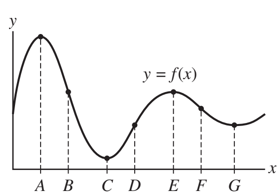


FIGURE 17

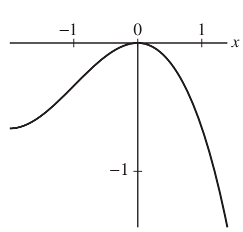
SOLUTION

- At B, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At C, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.
- At D, the graph changes from concave up to concave down, so f'' goes from $+$ to $-$.
- At E, the graph changes from increasing to decreasing, so f' goes from $+$ to $-$.
- At F, the graph changes from concave down to concave up, so f'' goes from $-$ to $+$.
- At G, the graph changes from decreasing to increasing, so f' goes from $-$ to $+$.

In Exercises 3–6, draw the graph of a function for which f' and f'' take on the given sign combinations.

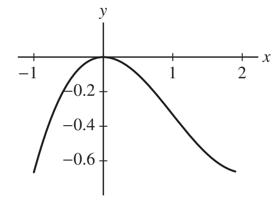
3. $++$, $+-$, $--$

SOLUTION This function changes from concave up to concave down at $x = -1$ and from increasing to decreasing at $x = 0$.



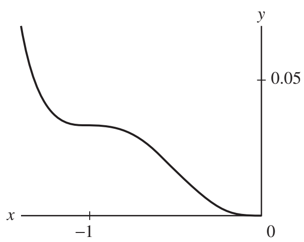
4. $+-$, $--$, $-+$

SOLUTION This function changes from increasing to decreasing at $x = 0$ and from concave down to concave up at $x = 1$.



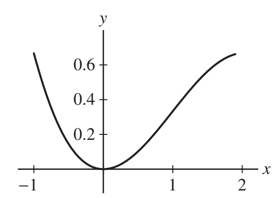
5. $-+$, $--$, $-+$

SOLUTION The function is decreasing everywhere and changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = -\frac{1}{2}$.



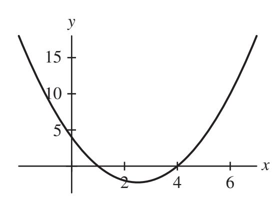
6. $-+$, $++$, $+-$

SOLUTION This function changes from decreasing to increasing at $x = 0$ and from concave up to concave down at $x = 1$.



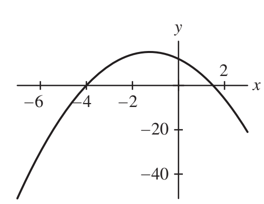
7. Sketch the graph of $y = x^2 - 5x + 4$.

SOLUTION Let $f(x) = x^2 - 5x + 4$. Then $f'(x) = 2x - 5$ and $f''(x) = 2$. Hence f is decreasing for $x < 5/2$, is increasing for $x > 5/2$, has a local minimum at $x = 5/2$ and is concave up everywhere.



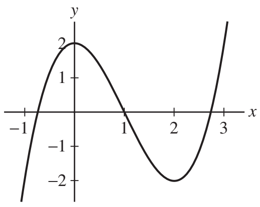
8. Sketch the graph of $y = 12 - 5x - 2x^2$.

SOLUTION Let $f(x) = 12 - 5x - 2x^2$. Then $f'(x) = -5 - 4x$ and $f''(x) = -4$. Hence f is increasing for $x < -5/4$, is decreasing for $x > -5/4$, has a local maximum at $x = -5/4$ and is concave down everywhere.



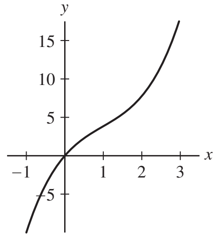
9. Sketch the graph of $f(x) = x^3 - 3x^2 + 2$. Include the zeros of $f(x)$, which are $x = 1$ and $1 \pm \sqrt{3}$ (approximately -0.73, 2.73).

SOLUTION Let $f(x) = x^3 - 3x^2 + 2$. Then $f'(x) = 3x^2 - 6x = 3x(x-2) = 0$ yields $x = 0, 2$ and $f''(x) = 6x - 6$. Thus f is concave down for $x < 1$, is concave up for $x > 1$, has an inflection point at $x = 1$, is increasing for $x < 0$ and for $x > 2$, is decreasing for $0 < x < 2$, has a local maximum at $x = 0$, and has a local minimum at $x = 2$.



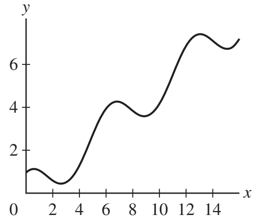
10. Show that $f(x) = x^3 - 3x^2 + 6x$ has a point of inflection but no local extreme values. Sketch the graph.

SOLUTION Let $f(x) = x^3 - 3x^2 + 6x$. Then $f'(x) = 3x^2 - 6x + 6 = 3\left((x-1)^2 + 1\right) > 0$ for all values of x and $f''(x) = 6x - 6$. Hence f is everywhere increasing and has an inflection point at $x = 1$. It is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.



11. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ in Example 4 to the interval $[0, 5\pi]$.

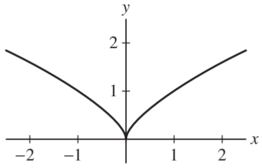
SOLUTION Let $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} = 0$ yields critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$, and $\frac{29\pi}{6}$. Moreover, $f''(x) = -\cos x$ so there are points of inflection at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and $\frac{9\pi}{2}$.



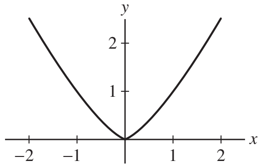
12. Sketch the graphs of $y = x^{2/3}$ and $y = x^{4/3}$.

SOLUTION

- Let $f(x) = x^{2/3}$. Then $f'(x) = \frac{2}{3}x^{-1/3}$ and $f''(x) = -\frac{2}{9}x^{-4/3}$, neither of which exist at $x = 0$. Thus f is decreasing and concave down for $x < 0$ and increasing and concave down for $x > 0$.



- Let $f(x) = x^{4/3}$. Then $f'(x) = \frac{4}{3}x^{1/3}$ and $f''(x) = \frac{4}{9}x^{-2/3}$. Thus f is decreasing and concave up for $x < 0$ and increasing and concave up for $x > 0$.



In Exercises 13–34, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

13. $y = x^3 + 24x^2$

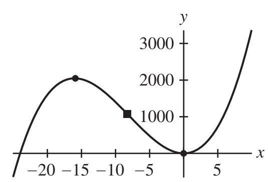
SOLUTION Let $f(x) = x^3 + 24x^2$. Then $f'(x) = 3x^2 + 48x = 3x(x + 16)$ and $f''(x) = 6x + 48$. This shows that f has critical points at $x = 0$ and $x = -16$ and a candidate for an inflection point at $x = -8$.

Interval	$(-\infty, -16)$	$(-16, -8)$	$(-8, 0)$	$(0, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, there is a local maximum at $x = -16$, a local minimum at $x = 0$, and an inflection point at $x = -8$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with these transition points highlighted as in the graphs in the textbook.



14. $y = x^3 - 3x + 5$

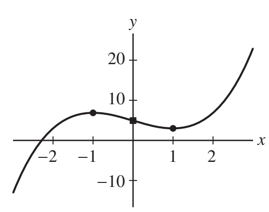
SOLUTION Let $f(x) = x^3 - 3x + 5$. Then $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Critical points are at $x = \pm 1$ and the sole candidate point of inflection is at $x = 0$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Signs of f' and f''	$+-$	$--$	$-+$	$++$

Thus, $f(-1)$ is a local maximum, $f(1)$ is a local minimum, and there is a point of inflection at $x = 0$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

Here is the graph of f with the transition points highlighted as in the textbook.



15. $y = x^2 - 4x^3$

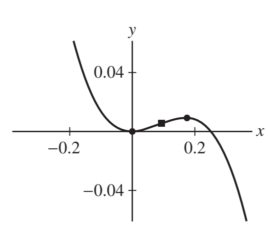
SOLUTION Let $f(x) = x^2 - 4x^3$. Then $f'(x) = 2x - 12x^2 = 2x(1 - 6x)$ and $f''(x) = 2 - 24x$. Critical points are at $x = 0$ and $x = \frac{1}{6}$, and the sole candidate point of inflection is at $x = \frac{1}{12}$.

Interval	$(-\infty, 0)$	$(0, \frac{1}{12})$	$(\frac{1}{12}, \frac{1}{6})$	$(\frac{1}{6}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$

Thus, $f(0)$ is a local minimum, $f(\frac{1}{6})$ is a local maximum, and there is a point of inflection at $x = \frac{1}{12}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is the graph of f with transition points highlighted as in the textbook:

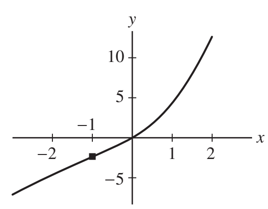


16. $y = \frac{1}{3}x^3 + x^2 + 3x$

SOLUTION Let $f(x) = \frac{1}{3}x^3 + x^2 + 3x$. Then $f'(x) = x^2 + 2x + 3$, and $f''(x) = 2x + 2 = 0$ if $x = -1$. Sign analysis shows that $f'(x) = (x+1)^2 + 2 > 0$ for all x (so that $f(x)$ has no critical points and is always increasing), and that $f''(x)$ changes from negative to positive at $x = -1$, implying that the graph of $f(x)$ has an inflection point at $(-1, f(-1))$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \lim_{x\to\infty} f(x) = \infty.$$

A graph with the inflection point indicated appears below:



17. $y = 4 - 2x^2 + \frac{1}{6}x^4$

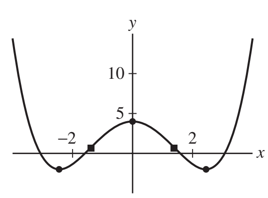
SOLUTION Let $f(x) = \frac{1}{6}x^4 - 2x^2 + 4$. Then $f'(x) = \frac{2}{3}x^3 - 4x = \frac{2}{3}x\left(x^2 - 6\right)$ and $f''(x) = 2x^2 - 4$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and has candidates for points of inflection at $x = \pm\sqrt{2}$.

Interval	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, \sqrt{6})$	$(\sqrt{6}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x = \pm\sqrt{6}$, a local maximum at $x = 0$, and inflection points at $x = \pm\sqrt{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.



18. $y = 7x^4 - 6x^2 + 1$

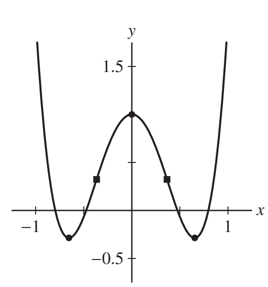
SOLUTION Let $f(x) = 7x^4 - 6x^2 + 1$. Then $f'(x) = 28x^3 - 12x = 4x\left(7x^2 - 3\right)$ and $f''(x) = 84x^2 - 12$. This shows that f has critical points at $x = 0$ and $x = \pm\frac{\sqrt{21}}{7}$ and candidates for points of inflection at $x = \pm\frac{\sqrt{7}}{7}$.

Interval	$(-\infty, -\frac{\sqrt{21}}{7})$	$(-\frac{\sqrt{21}}{7}, -\frac{\sqrt{7}}{7})$	$(-\frac{\sqrt{7}}{7}, 0)$	$(0, \frac{\sqrt{7}}{7})$	$(\frac{\sqrt{7}}{7}, \frac{\sqrt{21}}{7})$	$(\frac{\sqrt{21}}{7}, \infty)$
Signs of f' and f''	$-+$	$++$	$+-$	$--$	$-+$	$++$

Thus, f has local minima at $x = \pm\frac{\sqrt{21}}{7}$, a local maximum at $x = 0$, and inflection points at $x = \pm\frac{\sqrt{7}}{7}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

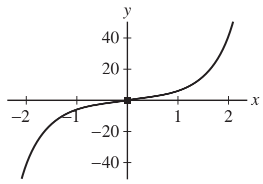


19. $y = x^5 + 5x$

SOLUTION Let $f(x) = x^5 + 5x$. Then $f'(x) = 5x^4 + 5 = 5(x^4 + 1)$ and $f''(x) = 20x^3$. $f'(x) > 0$ for all x, so the graph has no critical points and is always increasing. $f''(x) = 0$ at $x = 0$. Sign analyses reveal that $f''(x)$ changes from negative to positive at $x = 0$, so that the graph of $f(x)$ has an inflection point at $(0, 0)$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

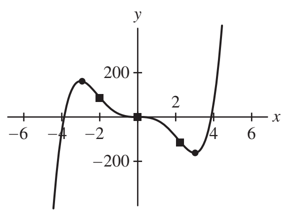


20. $y = x^5 - 15x^3$

SOLUTION Let $f(x) = x^5 - 15x^3$. Then $f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$ and $f''(x) = 20x^3 - 90x = 10x(2x^2 - 9)$. This shows that f has critical points at $x = 0$ and $x = \pm 3$ and candidate inflection points at $x = 0$ and $x = \pm 3\sqrt{2}/2$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = -3$, is negative on either side of $x = 0$ and changes from negative to positive at $x = 3$. The graph therefore has a local maximum at $x = -3$ and a local minimum at $x = 3$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = 0$ and from negative to positive at $x = \pm 3\sqrt{2}/2$. The graph therefore has points of inflection at $x = 0$ and $x = \pm 3\sqrt{2}/2$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

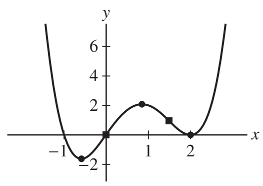


21. $y = x^4 - 3x^3 + 4x$

SOLUTION Let $f(x) = x^4 - 3x^3 + 4x$. Then $f'(x) = 4x^3 - 9x^2 + 4 = (4x^2 - x - 2)(x - 2)$ and $f''(x) = 12x^2 - 18x = 6x(2x - 3)$. This shows that f has critical points at $x = 2$ and $x = \dfrac{1 \pm \sqrt{33}}{8}$ and candidate points of inflection at $x = 0$ and $x = \frac{3}{2}$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = \frac{1-\sqrt{33}}{8}$, from positive to negative at $x = \frac{1+\sqrt{33}}{8}$, and again from negative to positive at $x = 2$. Therefore, $f(\frac{1-\sqrt{33}}{8})$ and $f(2)$ are local minima of $f(x)$, and $f(\frac{1+\sqrt{33}}{8})$ is a local maximum. Further sign analyses reveal that $f''(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = \frac{3}{2}$, so that there are points of inflection both at $x = 0$ and $x = \frac{3}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.



22. $y = x^2(x - 4)^2$

SOLUTION Let $f(x) = x^2(x - 4)^2$. Then

$$f'(x) = 2x(x - 4)^2 + 2x^2(x - 4) = 2x(x - 4)(x - 4 + x) = 4x(x - 4)(x - 2)$$

and

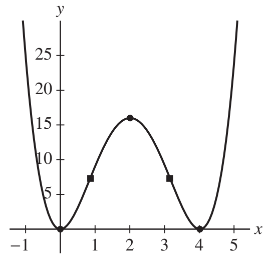
$$f''(x) = 12x^2 - 48x + 32 = 4(3x^2 - 12x + 8).$$

Critical points are therefore at $x = 0$, $x = 4$, and $x = 2$. Candidate inflection points are at solutions of $4(3x^2 - 12x + 8) = 0$, which, from the quadratic formula, are at $2 \pm \frac{\sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = 0$ and $x = 4$, and from positive to negative at $x = 2$. Therefore, $f(0)$ and $f(4)$ are local minima, and $f(2)$ a local maximum, of $f(x)$. Also, $f''(x)$ changes from positive to negative at $2 - \frac{2\sqrt{3}}{3}$ and from negative to positive at $2 + \frac{2\sqrt{3}}{3}$. Therefore there are points of inflection at both $x = 2 \pm \frac{2\sqrt{3}}{3}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with transition points highlighted.

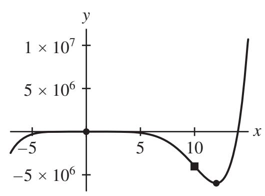


23. $y = x^7 - 14x^6$

SOLUTION Let $f(x) = x^7 - 14x^6$. Then $f'(x) = 7x^6 - 84x^5 = 7x^5(x - 12)$ and $f''(x) = 42x^5 - 420x^4 = 42x^4(x - 10)$. Critical points are at $x = 0$ and $x = 12$, and candidate inflection points are at $x = 0$ and $x = 10$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = 12$. Therefore $f(0)$ is a local maximum and $f(12)$ is a local minimum. Also, $f''(x)$ changes from negative to positive at $x = 10$. Therefore, there is a point of inflection at $x = 10$. Moreover,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

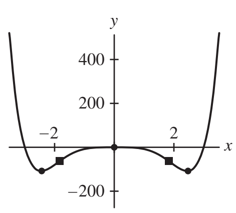


24. $y = x^6 - 9x^4$

SOLUTION Let $f(x) = x^6 - 9x^4$. Then $f'(x) = 6x^5 - 36x^3 = 6x^3(x^2 - 6)$ and $f''(x) = 30x^4 - 108x^2 = 6x^2(5x^2 - 18)$. This shows that f has critical points at $x = 0$ and $x = \pm\sqrt{6}$ and candidate inflection points at $x = 0$ and $x = \pm 3\sqrt{10}/5$. Sign analyses reveal that $f'(x)$ changes from negative to positive at $x = -\sqrt{6}$, from positive to negative at $x = 0$ and from negative to positive at $x = \sqrt{6}$. The graph therefore has a local maximum at $x = 0$ and local minima at $x = \pm\sqrt{6}$. Further sign analyses show that $f''(x)$ transitions from positive to negative at $x = -3\sqrt{10}/5$ and from negative to positive at $x = 3\sqrt{10}/5$. The graph therefore has points of inflection at $x = \pm 3\sqrt{10}/5$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.

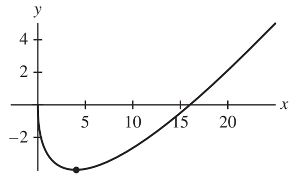


25. $y = x - 4\sqrt{x}$

SOLUTION Let $f(x) = x - 4\sqrt{x} = x - 4x^{1/2}$. Then $f'(x) = 1 - 2x^{-1/2}$. This shows that f has critical points at $x = 0$ (where the derivative does not exist) and at $x = 4$ (where the derivative is zero). Because $f'(x) < 0$ for $0 < x < 4$ and $f'(x) > 0$ for $x > 4$, $f(4)$ is a local minimum. Now $f''(x) = x^{-3/2} > 0$ for all $x > 0$, so the graph is always concave up. Moreover,

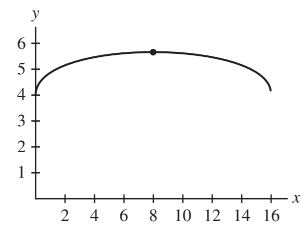
$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with transition points highlighted.



26. $y = \sqrt{x} + \sqrt{16 - x}$

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{16 - x} = x^{1/2} + (16 - x)^{1/2}$. Note that the domain of f is [0, 16]. Now, $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(16 - x)^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2} - \frac{1}{4}(16 - x)^{-3/2}$. Thus, the critical points are $x = 0$, $x = 8$ and $x = 16$. Sign analysis reveals that $f'(x) > 0$ for $0 < x < 8$ and $f'(x) < 0$ for $8 < x < 16$, so f has a local maximum at $x = 9$. Further, $f''(x) < 0$ on (0, 16), so the graph is always concave down. Here is a graph of f with the transition point highlighted.



27. $y = x(8 - x)^{1/3}$

SOLUTION Let $f(x) = x(8 - x)^{1/3}$. Then

$$f'(x) = x \cdot \tfrac{1}{3}(8 - x)^{-2/3}(-1) + (8 - x)^{1/3} \cdot 1 = \frac{24 - 4x}{3(8 - x)^{2/3}}$$

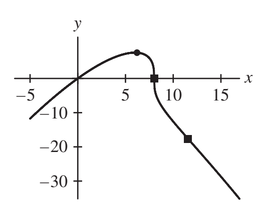
and similarly

$$f''(x) = \frac{4x - 48}{9(8 - x)^{5/3}}.$$

Critical points are at $x = 8$ and $x = 6$, and candidate inflection points are $x = 8$ and $x = 12$. Sign analyses reveal that $f'(x)$ changes from positive to negative at $x = 6$ and $f'(x)$ remains negative on either side of $x = 8$. Moreover, $f''(x)$ changes from negative to positive at $x = 8$ and from positive to negative at $x = 12$. Therefore, f has a local maximum at $x = 6$ and inflection points at $x = 8$ and $x = 12$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



28. $y = (x^2 - 4x)^{1/3}$

SOLUTION Let $f(x) = (x^2 - 4x)^{1/3}$. Then

$$f'(x) = \frac{2}{3}(x-2)(x^2-4x)^{-2/3}$$

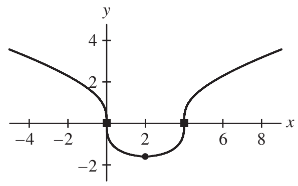
and

$$\begin{aligned} f''(x) &= \frac{2}{3}\left((x^2-4x)^{-2/3} - \frac{4}{3}(x-2)^2(x^2-4x)^{-5/3}\right) \\ &= \frac{2}{9}(x^2-4x)^{-5/3}\left(3(x^2-4x) - 4(x-2)^2\right) = -\frac{2}{9}(x^2-4x)^{-5/3}(x^2-4x+16). \end{aligned}$$

Critical points of $f(x)$ are $x = 2$ (where the derivative is zero) an $x = 0$ and $x = 4$ (where the derivative does not exist); candidate points of inflection are $x = 0$ and $x = 4$. Sign analyses reveal that $f''(x) < 0$ for $x < 0$ and for $x > 4$, while $f''(x) > 0$ for $0 < x < 4$. Therefore, the graph of $f(x)$ has points of inflection at $x = 0$ and $x = 4$. Since $(x^2 - x)^{-2/3}$ is positive wherever it is defined, the sign of $f'(x)$ depends solely on the sign of $x - 2$. Hence, $f'(x)$ does not change sign at $x = 0$ or $x = 4$, and goes from negative to positive at $x = 2$. $f(2)$ is, in that case, a local minimum. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of $f(x)$ with the transition points indicated.



29. $y = xe^{-x^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. Then

$$f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1-2x^2)e^{-x^2},$$

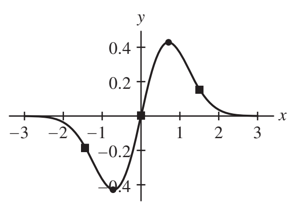
and

$$f''(x) = (4x^3 - 2x)e^{-x^2} - 4xe^{-x^2} = 2x(2x^2-3)e^{-x^2}.$$

There are critical points at $x = \pm\frac{\sqrt{2}}{2}$, and $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = -\frac{\sqrt{2}}{2}$ and from positive to negative at $x = \frac{\sqrt{2}}{2}$. Moreover, $f''(x)$ changes from negative to positive at both $x = \pm\frac{\sqrt{3}}{2}$ and from positive to negative at $x = 0$. Therefore, f has a local minimum at $x = -\frac{\sqrt{2}}{2}$, a local maximum at $x = \frac{\sqrt{2}}{2}$ and inflection points at $x = 0$ and at $x = \pm\frac{\sqrt{3}}{2}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



30. $y = (2x^2 - 1)e^{-x^2}$

SOLUTION Let $f(x) = (2x^2 - 1)e^{-x^2}$. Then

$$f'(x) = (2x - 4x^3)e^{-x^2} + 4xe^{-x^2} = 2x(3-2x^2)e^{-x^2},$$

and

$$f''(x) = (8x^4 - 12x^2)e^{-x^2} + (6 - 12x^2)e^{-x^2} = 2(4x^4 - 12x^2 + 3)e^{-x^2}.$$

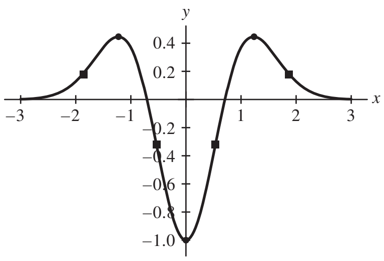
There are critical points at $x = 0$ and $x = \pm\frac{\sqrt{3}}{2}$, and

$$x = -\sqrt{\frac{3+\sqrt{6}}{2}}, \quad x = -\sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3-\sqrt{6}}{2}}, \quad x = \sqrt{\frac{3+\sqrt{6}}{2}}$$

are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = \pm\frac{\sqrt{3}}{2}$ and from negative to positive at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{3+\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3-\sqrt{6}}{2}}$ and from negative to positive at $x = -\sqrt{\frac{3-\sqrt{6}}{2}}$ and at $x = \sqrt{\frac{3+\sqrt{6}}{2}}$. Therefore, f has local maxima at $x = \pm\frac{\sqrt{3}}{2}$, a local minimum at $x = 0$ and points of inflection at $x = \pm\sqrt{\frac{3\pm\sqrt{6}}{2}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

so the graph has a horizontal asymptote at $y = 0$. Here is a graph of f with the transition points highlighted.



31. $y = x - 2\ln x$

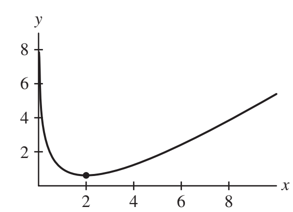
SOLUTION Let $f(x) = x - 2\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 1 - \frac{2}{x} \quad \text{and} \quad f''(x) = \frac{2}{x^2}.$$

The only critical point is $x = 2$. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = 2$, so $f(2)$ is a local minimum. Further, $f''(x) > 0$ for $x > 0$, so the graph is always concave up. Moreover,

$$\lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



32. $y = x(4 - x) - 3\ln x$

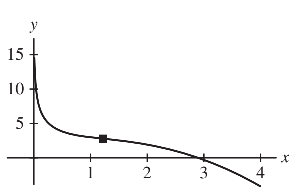
SOLUTION Let $f(x) = x(4 - x) - 3\ln x$. Note that the domain of f is $x > 0$. Now,

$$f'(x) = 4 - 2x - \frac{3}{x} \quad \text{and} \quad f''(x) = -2 + \frac{3}{x^2}.$$

Because $f'(x) < 0$ for all $x > 0$, the graph is always decreasing. On the other hand, $f''(x)$ changes from positive to negative at $x = \sqrt{\frac{3}{2}}$, so there is a point of inflection at $x = \sqrt{\frac{3}{2}}$. Moreover,

$$\lim_{x\to 0+} f(x) = \infty \quad \text{and} \quad \lim_{x\to infty} f(x) = -\infty,$$

so f has a vertical asymptote at $x = 0$. Here is a graph of f with the transition points highlighted.

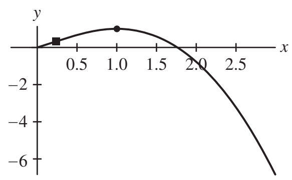


33. $y = x - x^2 \ln x$

SOLUTION Let $f(x) = x - x^2 \ln x$. Then $f'(x) = 1 - x - 2x \ln x$ and $f''(x) = -3 - 2\ln x$. There is a critical point at $x = 1$, and $x = e^{-3/2} \approx 0.223$ is a candidate inflection point. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 1$ and that $f''(x)$ changes from positive to negative at $x = e^{-3/2}$. Therefore, f has a local maximum at $x = 1$ and a point of inflection at $x = e^{-3/2}$. Moreover,

$$\lim_{x\to\infty} f(x) = -\infty.$$

Here is a graph of f with the transition points highlighted.



34. $y = x - 2\ln(x^2 + 1)$

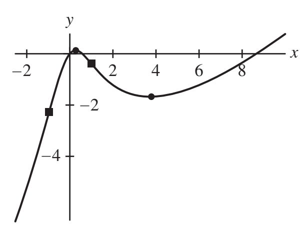
SOLUTION Let $f(x) = x - 2\ln(x^2 + 1)$. Then $f'(x) = 1 - \dfrac{4x}{x^2 + 1}$, and

$$f''(x) = -\frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(x^2 - 1)}{(x^2 + 1)^2}.$$

There are critical points at $x = 2 \pm \sqrt{3}$, and $x = \pm 1$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3}$ and from negative to positive at $x = 2 + \sqrt{3}$. Moreover, $f''(x)$ changes from positive to negative at $x = -1$ and from negative to positive at $x = 1$. Therefore, f has a local maximum at $x = 2 - \sqrt{3}$, a local minimum at $x = 2 + \sqrt{3}$ and points of inflection at $x = \pm 1$. Finally,

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.



35. Sketch the graph of $f(x) = 18(x - 3)(x - 1)^{2/3}$ using the formulas

$$f'(x) = \frac{30\left(x - \frac{9}{5}\right)}{(x - 1)^{1/3}}, \qquad f''(x) = \frac{20\left(x - \frac{3}{5}\right)}{(x - 1)^{4/3}}$$

SOLUTION

$$f'(x) = \frac{30(x - \frac{9}{5})}{(x - 1)^{1/3}}$$

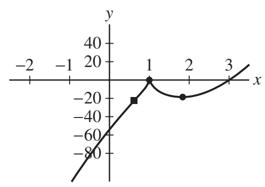
yields critical points at $x = \frac{9}{5}$, $x = 1$.

$$f''(x) = \frac{20(x - \frac{3}{5})}{(x - 1)^{4/3}}$$

yields potential inflection points at $x = \frac{3}{5}$, $x = 1$.

Interval	signs of f' and f''
$(-\infty, \frac{3}{5})$	$+-$
$(\frac{3}{5}, 1)$	$++$
$(1, \frac{9}{5})$	$-+$
$(\frac{9}{5}, \infty)$	$++$

The graph has an inflection point at $x = \frac{3}{5}$, a local maximum at $x = 1$ (at which the graph has a cusp), and a local minimum at $x = \frac{9}{5}$. The sketch looks something like this.



36. Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$ using the formulas

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}, \qquad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

SOLUTION Let $f(x) = \dfrac{x}{x^2+1}$.

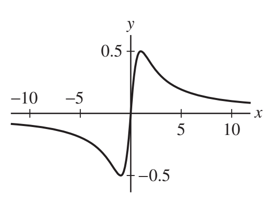
- Because $\lim_{x\to\pm\infty} f(x) = \frac{1}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$, $y = 0$ is a horizontal asymptote for f.
- Now $f'(x) = \dfrac{1-x^2}{\left(x^2+1\right)^2}$ is negative for $x < -1$ and $x > 1$, positive for $-1 < x < 1$, and 0 at $x = \pm 1$. Accordingly, f is decreasing for $x < -1$ and $x > 1$, is increasing for $-1 < x < 1$, has a local minimum value at $x = -1$ and a local maximum value at $x = 1$.
- Moreover,

$$f''(x) = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}.$$

Here is a sign chart for the second derivative, similar to those constructed in various exercises in Section 4.4. (The legend is on page 425.)

x	$\left(-\infty, -\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3}, 0\right)$	0	$\left(0, \sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3}, \infty\right)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	$\frown$	I	$\smile$	I	$\frown$	I	$\smile$

- Here is a graph of $f(x) = \dfrac{x}{x^2+1}$.



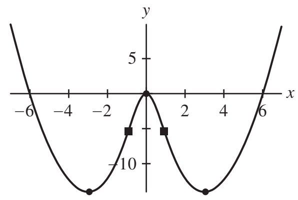
CAS *In Exercises 37–40, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.*

37. $y = x^2 - 10\ln(x^2+1)$

SOLUTION Let $f(x) = x^2 - 10\ln(x^2+1)$. Then $f'(x) = 2x - \dfrac{20x}{x^2+1}$, and

$$f''(x) = 2 - \frac{(x^2+1)(20) - (20x)(2x)}{(x^2+1)^2} = \frac{x^4+12x^2-9}{(x^2+1)^2}.$$

There are critical points at $x = 0$ and $x = \pm 3$, and $x = \pm\sqrt{-6+3\sqrt{5}}$ are candidates for inflection points. Sign analysis shows that $f'(x)$ changes from negative to positive at $x = \pm 3$ and from positive to negative at $x = 0$. Moreover, $f''(x)$ changes from positive to negative at $x = -\sqrt{-6+3\sqrt{5}}$ and from negative to positive at $x = \sqrt{-6+3\sqrt{5}}$. Therefore, f has a local maximum at $x = 0$, local minima at $x = \pm 3$ and points of inflection at $x = \pm\sqrt{-6+3\sqrt{5}}$. Here is a graph of f with the transition points highlighted.



38. $y = e^{-x/2} \ln x$

SOLUTION Let $f(x) = e^{-x/2} \ln x$. Then

$$f'(x) = \frac{e^{-x/2}}{x} - \frac{1}{2}e^{-x/2} \ln x = e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$

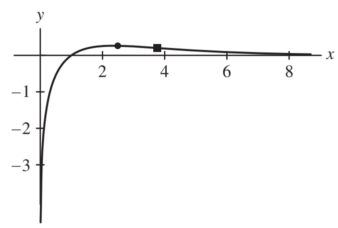
and

$$f''(x) = e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{2x}\right) - \frac{1}{2}e^{-x/2}\left(\frac{1}{x} - \frac{1}{2}\ln x\right)$$
$$= e^{-x/2}\left(-\frac{1}{x^2} - \frac{1}{x} + \frac{1}{4}\ln x\right).$$

There is a critical point at $x = 2.345751$ and a candidate point of inflection at $x = 3.792199$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 2.345751$ and that $f''(x)$ changes from negative to positive at $x = 3.792199$. Therefore, f has a local maximum at $x = 2.345751$ and a point of inflection at $x = 3.792199$. Moreover,

$$\lim_{x\to 0+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0.$$

Here is a graph of f with the transition points highlighted.

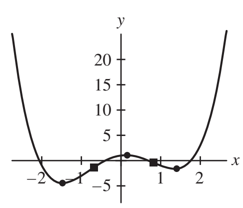


39. $y = x^4 - 4x^2 + x + 1$

SOLUTION Let $f(x) = x^4 - 4x^2 + x + 1$. Then $f'(x) = 4x^3 - 8x + 1$ and $f''(x) = 12x^2 - 8$. The critical points are $x = -1.473$, $x = 0.126$ and $x = 1.347$, while the candidates for points of inflection are $x = \pm\sqrt{\frac{2}{3}}$. Sign analysis reveals that $f'(x)$ changes from negative to positive at $x = -1.473$, from positive to negative at $x = 0.126$ and from negative to positive at $x = 1.347$. For the second derivative, $f''(x)$ changes from positive to negative at $x = -\sqrt{\frac{2}{3}}$ and from negative to positive at $x = \sqrt{\frac{2}{3}}$. Therefore, f has local minima at $x = -1.473$ and $x = 1.347$, a local maximum at $x = 0.126$ and points of inflection at $x = \pm\sqrt{\frac{2}{3}}$. Moreover,

$$\lim_{x\to\pm\infty} f(x) = \infty.$$

Here is a graph of f with the transition points highlighted.

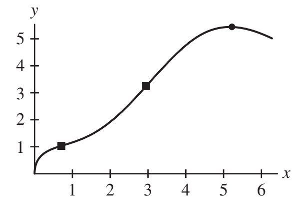


40. $y = 2\sqrt{x} - \sin x, \quad 0 \le x \le 2\pi$

SOLUTION Let $f(x) = 2\sqrt{x} - \sin x$. Then

$$f'(x) = \frac{1}{\sqrt{x}} - \cos x \quad \text{and} \quad f''(x) = -\frac{1}{2}x^{-3/2} + \sin x.$$

On $0 \le x \le 2\pi$, there is a critical point at $x = 5.167866$ and candidate points of inflection at $x = 0.790841$ and $x = 3.047468$. Sign analysis reveals that $f'(x)$ changes from positive to negative at $x = 5.167866$, while $f''(x)$ changes from negative to positive at $x = 0.790841$ and from positive to negative at $x = 3.047468$. Therefore, f has a local maximum at $x = 5.167866$ and points of inflection at $x = 0.790841$ and $x = 3.047468$. Here is a graph of f with the transition points highlighted.



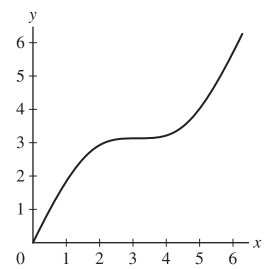
In Exercises 41–46, sketch the graph over the given interval, with all transition points indicated.

41. $y = x + \sin x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = x + \sin x$. Setting $f'(x) = 1 + \cos x = 0$ yields $\cos x = -1$, so that $x = \pi$ is the lone critical point on the interval $[0, 2\pi]$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$ on the interval $[0, 2\pi]$.

Interval	signs of f' and f''
$(0, \pi)$	$+-$
$(\pi, 2\pi)$	$++$

The graph has an inflection point at $x = \pi$, and no local maxima or minima. Here is a sketch of the graph of $f(x)$:

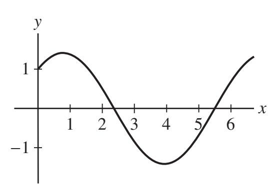


42. $y = \sin x + \cos x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \cos x$. Setting $f'(x) = \cos x - \sin x = 0$ yields $\sin x = \cos x$, so that $\tan x = 1$, and $x = \frac{\pi}{4}, \frac{5\pi}{4}$. Setting $f''(x) = -\sin x - \cos x = 0$ yields $\sin x = -\cos x$, so that $-\tan x = 1$, and $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$.

Interval	signs of f' and f''
$(0, \frac{\pi}{4})$	$+-$
$(\frac{\pi}{4}, \frac{3\pi}{4})$	$--$
$(\frac{3\pi}{4}, \frac{5\pi}{4})$	$-+$
$(\frac{5\pi}{4}, \frac{7\pi}{4})$	$++$
$(\frac{7\pi}{4}, 2\pi)$	$+-$

The graph has a local maximum at $x = \frac{\pi}{4}$, a local minimum at $x = \frac{5\pi}{4}$, and inflection points at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. Here is a sketch of the graph of $f(x)$:

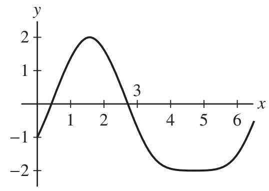


43. $y = 2\sin x - \cos^2 x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = 2\sin x - \cos^2 x$. Then $f'(x) = 2\cos x - 2\cos x\,(-\sin x) = \sin 2x + 2\cos x$ and $f''(x) = 2\cos 2x - 2\sin x$. Setting $f'(x) = 0$ yields $\sin 2x = -2\cos x$, so that $2\sin x\cos x = -2\cos x$. This implies $\cos x = 0$ or $\sin x = -1$, so that $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Setting $f''(x) = 0$ yields $2\cos 2x = 2\sin x$, so that $2\sin(\frac{\pi}{2} - 2x) = 2\sin x$, or $\frac{\pi}{2} - 2x = x \pm 2n\pi$. This yields $3x = \frac{\pi}{2} + 2n\pi$, or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} = \frac{3\pi}{2}$.

Interval	signs of f' and f''
$\left(0, \frac{\pi}{6}\right)$	$++$
$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$+-$
$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$--$
$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$-+$
$\left(\frac{3\pi}{2}, 2\pi\right)$	$++$

The graph has a local maximum at $x = \frac{\pi}{2}$, a local minimum at $x = \frac{3\pi}{2}$, and inflection points at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Here is a graph of f *without* transition points highlighted.

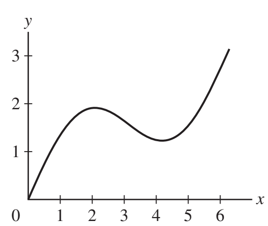


44. $y = \sin x + \frac{1}{2}x, \quad [0, 2\pi]$

SOLUTION Let $f(x) = \sin x + \frac{1}{2}x$. Setting $f'(x) = \cos x + \frac{1}{2} = 0$ yields $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Setting $f''(x) = -\sin x = 0$ yields potential points of inflection at $x = 0, \pi, 2\pi$.

Interval	signs of f' and f''
$\left(0, \frac{2\pi}{3}\right)$	$+-$
$\left(\frac{2\pi}{3}, \pi\right)$	$--$
$\left(\pi, \frac{4\pi}{3}\right)$	$-+$
$\left(\frac{4\pi}{3}, 2\pi\right)$	$++$

The graph has a local maximum at $x = \frac{2\pi}{3}$, a local minimum at $x = \frac{4\pi}{3}$, and an inflection point at $x = \pi$. Here is a graph of f *without* transition points highlighted.

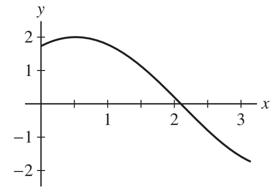


45. $y = \sin x + \sqrt{3}\cos x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x + \sqrt{3}\cos x$. Setting $f'(x) = \cos x - \sqrt{3}\sin x = 0$ yields $\tan x = \frac{1}{\sqrt{3}}$. In the interval $[0, \pi]$, the solution is $x = \frac{\pi}{6}$. Setting $f''(x) = -\sin x - \sqrt{3}\cos x = 0$ yields $\tan x = -\sqrt{3}$. In the interval $[0, \pi]$, the lone solution is $x = \frac{2\pi}{3}$.

Interval	signs of f' and f''
$(0, \pi/6)$	$+-$
$(\pi/6, 2\pi/3)$	$--$
$(2\pi/3, \pi)$	$-+$

The graph has a local maximum at $x = \frac{\pi}{6}$ and a point of inflection at $x = \frac{2\pi}{3}$. A plot without the transition points highlighted is given below:



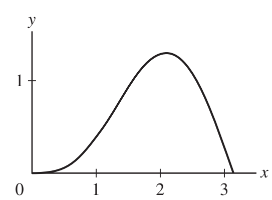
46. $y = \sin x - \frac{1}{2}\sin 2x, \quad [0, \pi]$

SOLUTION Let $f(x) = \sin x - \frac{1}{2}\sin 2x$. Setting $f'(x) = \cos x - \cos 2x = 0$ yields $\cos 2x = \cos x$. Using the double angle formula for cosine, this gives $2\cos^2 x - 1 = \cos x$ or $(2\cos x + 1)(\cos x - 1) = 0$. Solving for $x \in [0, \pi]$, we find $x = 0$ or $\frac{2\pi}{3}$.

Setting $f''(x) = -\sin x + 2\sin 2x = 0$ yields $4\sin x\cos x = \sin x$, so $\sin x = 0$ or $\cos x = \frac{1}{4}$. Hence, there are potential points of inflection at $x = 0$, $x = \pi$ and $x = \cos^{-1}\frac{1}{4} \approx 1.31812$.

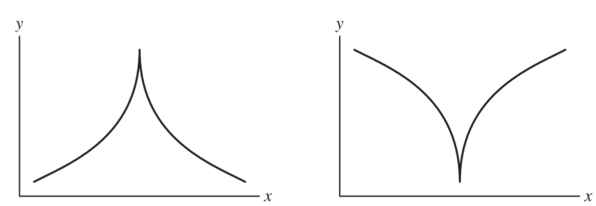
Interval	Sign of f' and f''
$\left(0, \cos^{-1}\frac{1}{4}\right)$	$++$
$\left(\cos^{-1}\frac{1}{4}, \frac{2\pi}{3}\right)$	$+-$
$\left(\frac{2\pi}{3}, \pi\right)$	$--$

The graph of $f(x)$ has a local maximum at $x = \frac{2\pi}{3}$ and a point of inflection at $x = \cos^{-1}\frac{1}{4}$.



47. Are all sign transitions possible? Explain with a sketch why the transitions $++ \rightarrow -+$ and $-- \rightarrow +-$ do not occur if the function is differentiable. (See Exercise 76 for a proof.)

SOLUTION In both cases, there is a point where f is not differentiable at the transition from increasing to decreasing or decreasing to increasing.



48. Suppose that f is twice differentiable satisfying (i) $f(0) = 1$, (ii) $f'(x) > 0$ for all $x \neq 0$, and (iii) $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Let $g(x) = f(x^2)$.

(a) Sketch a possible graph of $f(x)$.

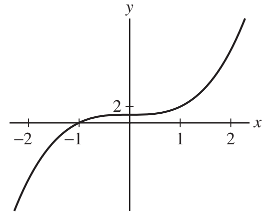
(b) Prove that $g(x)$ has no points of inflection and a unique local extreme value at $x = 0$. Sketch a possible graph of $g(x)$.

SOLUTION

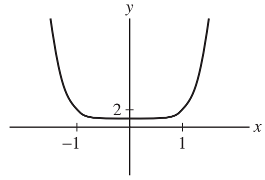
(a) To produce a possible sketch, we give the direction and concavity of the graph over every interval.

Interval	$(-\infty, 0)$	$(0, \infty)$
Direction	↗	↗
Concavity	⌢	⌣

A sketch of one possible such function appears here:



(b) Let $g(x) = f(x^2)$. Then $g'(x) = 2xf'(x^2)$. If $g'(x) = 0$, either $x = 0$ or $f'(x^2) = 0$, which implies that $x = 0$ as well. Since $f'(x^2) > 0$ for all $x \neq 0$, $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$. This gives $g(x)$ a unique local extreme value at $x = 0$, a minimum. $g''(x) = 2f'(x^2) + 4x^2 f''(x^2)$. For all $x \neq 0$, $x^2 > 0$, and so $f''(x^2) > 0$ and $f'(x^2) > 0$. Thus $g''(x) > 0$, and so $g''(x)$ does not change sign, and can have no inflection points. A sketch of $g(x)$ based on the sketch we made for $f(x)$ follows: indeed, this sketch shows a unique local minimum at $x = 0$.



49. Which of the graphs in Figure 18 *cannot* be the graph of a polynomial? Explain.

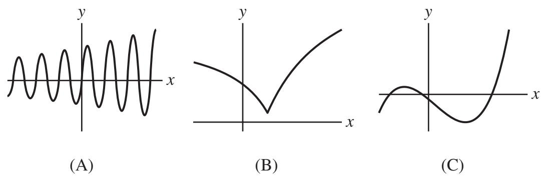


FIGURE 18

SOLUTION Polynomials are everywhere differentiable. Accordingly, graph (B) cannot be the graph of a polynomial, since the function in (B) has a cusp (sharp corner), signifying nondifferentiability at that point.

50. Which curve in Figure 19 is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$? Explain on the basis of horizontal asymptotes.

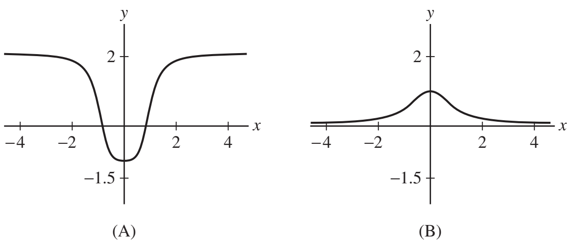


FIGURE 19

SOLUTION Since

$$\lim_{x\to\pm\infty} \frac{2x^4 - 1}{1 + x^4} = \frac{2}{1} \cdot \lim_{x\to\pm\infty} 1 = 2$$

the graph has left and right horizontal asymptotes at $y = 2$, so the left curve is the graph of $f(x) = \dfrac{2x^4 - 1}{1 + x^4}$.

51. Match the graphs in Figure 20 with the two functions $y = \dfrac{3x}{x^2-1}$ and $y = \dfrac{3x^2}{x^2-1}$. Explain.

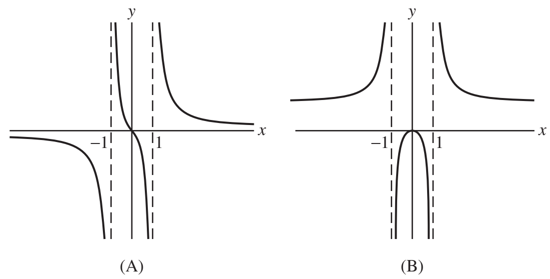


FIGURE 20

SOLUTION Since $\displaystyle\lim_{x\to\pm\infty} \frac{3x^2}{x^2-1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} 1 = 3$, the graph of $y = \dfrac{3x^2}{x^2-1}$ has a horizontal asymptote of $y = 3$; hence, the right curve is the graph of $f(x) = \dfrac{3x^2}{x^2-1}$. Since

$$\lim_{x\to\pm\infty} \frac{3x}{x^2-1} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0,$$

the graph of $y = \dfrac{3x}{x^2-1}$ has a horizontal asymptote of $y = 0$; hence, the left curve is the graph of $f(x) = \dfrac{3x}{x^2-1}$.

52. Match the functions with their graphs in Figure 21.

(a) $y = \dfrac{1}{x^2-1}$ **(b)** $y = \dfrac{x^2}{x^2+1}$

(c) $y = \dfrac{1}{x^2+1}$ **(d)** $y = \dfrac{x}{x^2-1}$

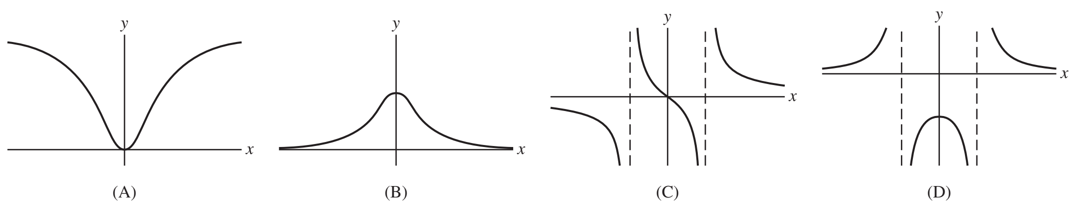


FIGURE 21

SOLUTION

(a) The graph of $\frac{1}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $|x| > 1$ and negative values for $|x| < 1$. Hence, the graph of $\frac{1}{x^2-1}$ is (D).

(b) The graph of $\frac{x^2}{x^2+1}$ should have a horizontal asymptote at $y = 1$ and no vertical asymptotes. Hence, the graph of $\frac{x^2}{x^2+1}$ is (A).

(c) The graph of $\frac{1}{x^2+1}$ should have a horizontal asymptote at $y = 0$ and no vertical asymptotes. Hence, the graph of $\frac{1}{x^2+1}$ is (B).

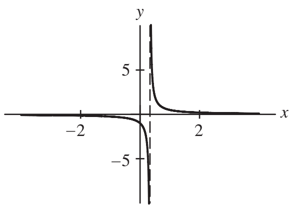
(d) The graph of $\frac{x}{x^2-1}$ should have a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$. Further, the graph should consist of positive values for $-1 < x < 0$ and $x > 1$ and negative values for $x < 1$ and $0 < x < 1$. Hence, the graph of $\frac{x}{x^2-1}$ is (C).

In Exercises 53–70, sketch the graph of the function. Indicate the transition points and asymptotes.

53. $y = \dfrac{1}{3x-1}$

SOLUTION Let $f(x) = \dfrac{1}{3x-1}$. Then $f'(x) = \dfrac{-3}{(3x-1)^2}$, so that f is decreasing for all $x \neq \frac{1}{3}$. Moreover, $f''(x) = \dfrac{18}{(3x-1)^3}$, so that f is concave up for $x > \frac{1}{3}$ and concave down for $x < \frac{1}{3}$. Because $\displaystyle\lim_{x\to\pm\infty} \frac{1}{3x-1} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has a vertical asymptote at $x = \frac{1}{3}$ with

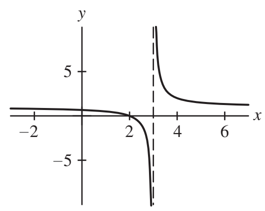
$$\lim_{x\to\frac{1}{3}-} \frac{1}{3x-1} = -\infty \qquad \text{and} \qquad \lim_{x\to\frac{1}{3}+} \frac{1}{3x-1} = \infty.$$



54. $y = \dfrac{x-2}{x-3}$

SOLUTION Let $f(x) = \dfrac{x-2}{x-3}$. Then $f'(x) = \dfrac{-1}{(x-3)^2}$, so that f is decreasing for all $x \neq 3$. Moreover, $f''(x) = \dfrac{2}{(x-3)^3}$, so that f is concave up for $x > 3$ and concave down for $x < 3$. Because $\lim_{x\to\pm\infty} \dfrac{x-2}{x-3} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 3$ with

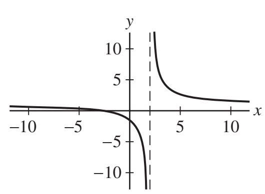
$$\lim_{x\to 3-} \frac{x-2}{x-3} = -\infty \qquad \text{and} \qquad \lim_{x\to 3+} \frac{x-2}{x-3} = \infty.$$



55. $y = \dfrac{x+3}{x-2}$

SOLUTION Let $f(x) = \dfrac{x+3}{x-2}$. Then $f'(x) = \dfrac{-5}{(x-2)^2}$, so that f is decreasing for all $x \neq 2$. Moreover, $f''(x) = \dfrac{10}{(x-2)^3}$, so that f is concave up for $x > 2$ and concave down for $x < 2$. Because $\lim_{x\to\pm\infty} \dfrac{x+3}{x-2} = 1$, f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 2$ with

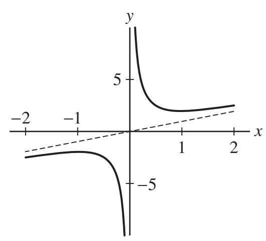
$$\lim_{x\to 2-} \frac{x+3}{x-2} = -\infty \qquad \text{and} \qquad \lim_{x\to 2+} \frac{x+3}{x-2} = \infty.$$



56. $y = x + \dfrac{1}{x}$

SOLUTION Let $f(x) = x + x^{-1}$. Then $f'(x) = 1 - x^{-2}$, so that f is increasing for $x < -1$ and $x > 1$ and decreasing for $-1 < x < 0$ and $0 < x < 1$. Moreover, $f''(x) = 2x^{-3}$, so that f is concave up for $x > 0$ and concave down for $x < 0$. f has no horizontal asymptote and has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} (x + x^{-1}) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} (x + x^{-1}) = \infty.$$



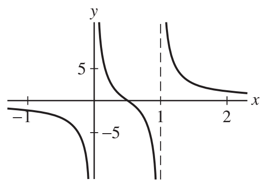
57. $y = \dfrac{1}{x} + \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} + \dfrac{1}{x-1}$. Then $f'(x) = -\dfrac{2x^2 - 2x + 1}{x^2(x-1)^2}$, so that f is decreasing for all $x \neq 0, 1$. Moreover, $f''(x) = \dfrac{2\left(2x^3 - 3x^2 + 3x - 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < \frac{1}{2}$ and $x > 1$ and concave down for $x < 0$ and $\frac{1}{2} < x < 1$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x} + \dfrac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} + \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} + \frac{1}{x-1}\right) = \infty.$$



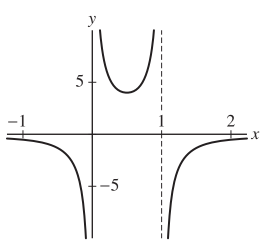
58. $y = \dfrac{1}{x} - \dfrac{1}{x-1}$

SOLUTION Let $f(x) = \dfrac{1}{x} - \dfrac{1}{x-1}$. Then $f'(x) = \dfrac{2x-1}{x^2(x-1)^2}$, so that f is decreasing for $x < 0$ and $0 < x < \frac{1}{2}$ and increasing for $\frac{1}{2} < x < 1$ and $x > 1$. Moreover, $f''(x) = -\dfrac{2\left(3x^2 - 3x + 1\right)}{x^3(x-1)^3}$, so that f is concave up for $0 < x < 1$ and concave down for $x < 0$ and $x > 1$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x} - \dfrac{1}{x-1}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 1$ with

$$\lim_{x\to 0-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty$$

and

$$\lim_{x\to 1-}\left(\frac{1}{x} - \frac{1}{x-1}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 1+}\left(\frac{1}{x} - \frac{1}{x-1}\right) = -\infty.$$



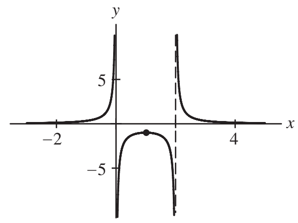
59. $y = \dfrac{1}{x(x-2)}$

SOLUTION Let $f(x) = \dfrac{1}{x(x-2)}$. Then $f'(x) = \dfrac{2(1-x)}{x^2(x-2)^2}$, so that f is increasing for $x < 0$ and $0 < x < 1$ and decreasing for $1 < x < 2$ and $x > 2$. Moreover, $f''(x) = \dfrac{2(3x^2 - 6x + 4)}{x^3(x-2)^3}$, so that f is concave up for $x < 0$ and $x > 2$ and concave down for $0 < x < 2$. Because $\lim_{x\to\pm\infty}\left(\dfrac{1}{x(x-2)}\right) = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x(x-2)}\right) = +\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x(x-2)}\right) = -\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x(x-2)}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x(x-2)}\right) = \infty.$$



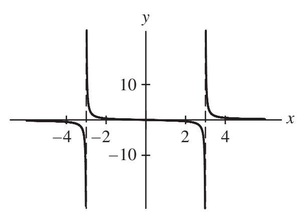
60. $y = \dfrac{x}{x^2-9}$

SOLUTION Let $f(x) = \dfrac{x}{x^2-9}$. Then $f'(x) = -\dfrac{x^2+9}{(x^2-9)^2}$, so that f is decreasing for all $x \neq \pm 3$. Moreover, $f''(x) = \dfrac{6x(x^2+6)}{(x^2-9)^3}$, so that f is concave down for $x < -3$ and for $0 < x < 3$ and is concave up for $-3 < x < 0$ and for $x > 3$. Because $\lim_{x\to\pm\infty} \dfrac{x}{x^2-9} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = \pm 3$, with

$$\lim_{x\to -3-}\left(\frac{x}{x^2-9}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to -3+}\left(\frac{x}{x^2-9}\right) = \infty$$

and

$$\lim_{x\to 3-}\left(\frac{x}{x^2-9}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 3+}\left(\frac{x}{x^2-9}\right) = \infty.$$



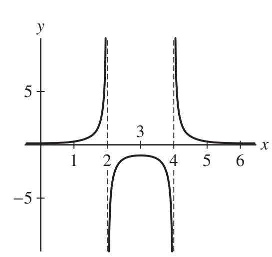
61. $y = \dfrac{1}{x^2-6x+8}$

SOLUTION Let $f(x) = \dfrac{1}{x^2-6x+8} = \dfrac{1}{(x-2)(x-4)}$. Then $f'(x) = \dfrac{6-2x}{\left(x^2-6x+8\right)^2}$, so that f is increasing for $x < 2$ and for $2 < x < 3$, is decreasing for $3 < x < 4$ and for $x > 4$, and has a local maximum at $x = 3$. Moreover, $f''(x) = \dfrac{2\left(3x^2-18x+28\right)}{\left(x^2-6x+8\right)^3}$, so that f is concave up for $x < 2$ and for $x > 4$ and is concave down for $2 < x < 4$. Because $\lim_{x\to\pm\infty} \dfrac{1}{x^2-6x+8} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = 2$ and $x = 4$, with

$$\lim_{x\to 2-}\left(\frac{1}{x^2-6x+8}\right) = \infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2-6x+8}\right) = -\infty$$

and

$$\lim_{x\to 4-}\left(\frac{1}{x^2-6x+8}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 4+}\left(\frac{1}{x^2-6x+8}\right) = \infty.$$



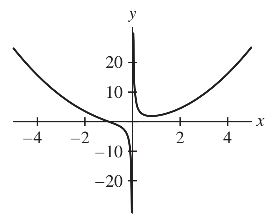
62. $y = \dfrac{x^3+1}{x}$

SOLUTION Let $f(x) = \dfrac{x^3+1}{x} = x^2 + x^{-1}$. Then $f'(x) = 2x - x^{-2}$, so that f is decreasing for $x < 0$ and for $0 < x < \sqrt[3]{1/2}$ and increasing for $x > \sqrt[3]{1/2}$. Moreover, $f''(x) = 2 + 2x^{-3}$, so f is concave up for $x < -1$ and for $x > 0$ and concave down for $-1 < x < 0$. Because

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x} = \infty,$$

f has no horizontal asymptotes. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \frac{x^3+1}{x} = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \frac{x^3+1}{x} = \infty.$$



63. $y = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$

SOLUTION Let $f(x) = 1 - \dfrac{3}{x} + \dfrac{4}{x^3}$. Then

$$f'(x) = \frac{3}{x^2} - \frac{12}{x^4} = \frac{3(x-2)(x+2)}{x^4},$$

so that f is increasing for $|x| > 2$ and decreasing for $-2 < x < 0$ and for $0 < x < 2$. Moreover,

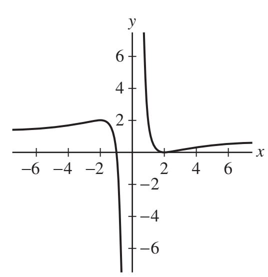
$$f''(x) = -\frac{6}{x^3} + \frac{48}{x^5} = \frac{6(8-x^2)}{x^5},$$

so that f is concave down for $-2\sqrt{2} < x < 0$ and for $x > 2\sqrt{2}$, while f is concave up for $x < -2\sqrt{2}$ and for $0 < x < 2\sqrt{2}$. Because

$$\lim_{x\to\pm\infty} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = 1,$$

f has a horizontal asymptote at $y = 1$. Finally, f has a vertical asymptote at $x = 0$ with

$$\lim_{x\to 0-} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} \left(1 - \frac{3}{x} + \frac{4}{x^3}\right) = \infty.$$



64. $y = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x^2} + \dfrac{1}{(x-2)^2}$. Then

$$f'(x) = -2x^{-3} - 2\,(x-2)^{-3} = -\frac{4(x-1)(x^2-2x+4)}{x^3(x-2)^3},$$

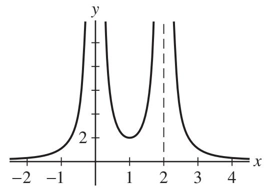
so that f is increasing for $x < 0$ and for $1 < x < 2$, is decreasing for $0 < x < 1$ and for $x > 2$, and has a local minimum at $x = 1$. Moreover, $f''(x) = 6x^{-4} + 6\,(x-2)^{-4}$, so that f is concave up for all $x \neq 0, 2$. Because

$\lim_{x\to\pm\infty}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=0$ and $x=2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2}+\frac{1}{(x-2)^2}\right)=\infty.$$



65. $y=\frac{1}{x^2}-\frac{1}{(x-2)^2}$

SOLUTION Let $f(x)=\frac{1}{x^2}-\frac{1}{(x-2)^2}$. Then $f'(x)=-2x^{-3}+2(x-2)^{-3}$, so that f is increasing for $x<0$ and for $x>2$ and is decreasing for $0<x<2$. Moreover,

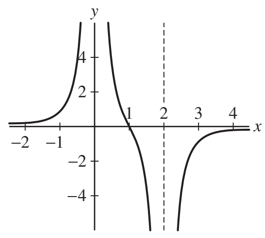
$$f''(x)=6x^{-4}-6(x-2)^{-4}=-\frac{48(x-1)(x^2-2x+2)}{x^4(x-2)^4},$$

so that f is concave up for $x<0$ and for $0<x<1$, is concave down for $1<x<2$ and for $x>2$, and has a point of inflection at $x=1$. Because $\lim_{x\to\pm\infty}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=0$ and $x=2$ with

$$\lim_{x\to 0-}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=\infty \qquad \text{and} \qquad \lim_{x\to 0+}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=\infty$$

and

$$\lim_{x\to 2-}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to 2+}\left(\frac{1}{x^2}-\frac{1}{(x-2)^2}\right)=-\infty.$$



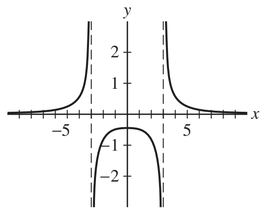
66. $y=\frac{4}{x^2-9}$

SOLUTION Let $f(x)=\frac{4}{x^2-9}$. Then $f'(x)=-\frac{8x}{\left(x^2-9\right)^2}$, so that f is increasing for $x<-3$ and for $-3<x<0$, is decreasing for $0<x<3$ and for $x>3$, and has a local maximum at $x=0$. Moreover, $f''(x)=\frac{24\left(x^2+3\right)}{\left(x^2-9\right)^3}$, so that f is concave up for $x<-3$ and for $x>3$ and is concave down for $-3<x<3$. Because $\lim_{x\to\pm\infty}\frac{4}{x^2-9}=0$, f has a horizontal asymptote at $y=0$. Finally, f has vertical asymptotes at $x=-3$ and $x=3$, with

$$\lim_{x\to -3-}\left(\frac{4}{x^2-9}\right)=\infty \qquad \text{and} \qquad \lim_{x\to -3+}\left(\frac{4}{x^2-9}\right)=-\infty$$

and

$$\lim_{x\to 3-}\left(\frac{4}{x^2-9}\right)=-\infty \qquad \text{and} \qquad \lim_{x\to 3+}\left(\frac{4}{x^2-9}\right)=\infty.$$

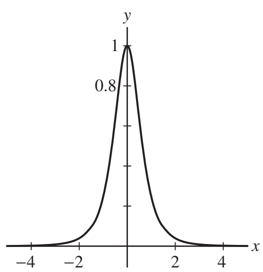


67. $y = \dfrac{1}{(x^2+1)^2}$

SOLUTION Let $f(x) = \dfrac{1}{(x^2+1)^2}$. Then $f'(x) = \dfrac{-4x}{(x^2+1)^3}$, so that f is increasing for $x < 0$, is decreasing for $x > 0$ and has a local maximum at $x = 0$. Moreover,

$$f''(x) = \frac{-4(x^2+1)^3 + 4x \cdot 3(x^2+1)^2 \cdot 2x}{(x^2+1)^6} = \frac{20x^2-4}{(x^2+1)^4},$$

so that f is concave up for $|x| > 1/\sqrt{5}$, is concave down for $|x| < 1/\sqrt{5}$, and has points of inflection at $x = \pm 1/\sqrt{5}$. Because $\lim_{x\to\pm\infty} \dfrac{1}{(x^2+1)^2} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



68. $y = \dfrac{x^2}{(x^2-1)(x^2+1)}$

SOLUTION Let

$$f(x) = \frac{x^2}{(x^2-1)(x^2+1)}.$$

Then

$$f'(x) = -\frac{2x(1+x^4)}{(x-1)^2(x+1)^2(x^2+1)^2},$$

so that f is increasing for $x < -1$ and for $-1 < x < 0$, is decreasing for $0 < x < 1$ and for $x > 1$, and has a local maximum at $x = 0$. Moreover,

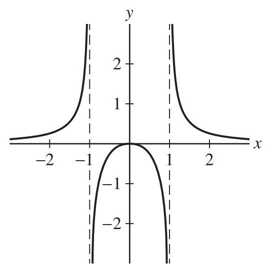
$$f''(x) = \frac{2+24x^4+6x^8}{(x-1)^3(x+1)^3(x^2+1)^3},$$

so that f is concave up for $|x| > 1$ and concave down for $|x| < 1$. Because $\lim_{x\to\pm\infty} \dfrac{x^2}{(x^2-1)(x^2+1)} = 0$, f has a horizontal asymptote at $y = 0$. Finally, f has vertical asymptotes at $x = -1$ and $x = 1$, with

$$\lim_{x\to -1-} \frac{x^2}{(x^2-1)(x^2+1)} = \infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty$$

and

$$\lim_{x\to 1-} \frac{x^2}{(x^2-1)(x^2+1)} = -\infty \qquad \text{and} \qquad \lim_{x\to 1+} \frac{x^2}{(x^2-1)(x^2+1)} = \infty.$$



69. $y = \dfrac{1}{\sqrt{x^2+1}}$

SOLUTION Let $f(x) = \frac{1}{\sqrt{x^2+1}}$. Then

$$f'(x) = -\frac{x}{\sqrt{(x^2+1)^3}} = -x(x^2+1)^{-3/2},$$

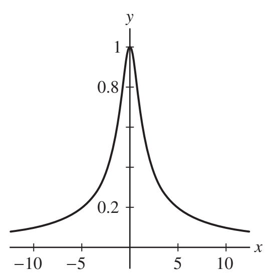
so that f is increasing for $x < 0$ and decreasing for $x > 0$. Moreover,

$$f''(x) = -\frac{3}{2}x(x^2+1)^{-5/2}(-2x) - (x^2+1)^{-3/2} = (2x^2-1)(x^2+1)^{-5/2},$$

so that f is concave down for $|x| < \frac{\sqrt{2}}{2}$ and concave up for $|x| > \frac{\sqrt{2}}{2}$. Because

$$\lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2+1}} = 0,$$

f has a horizontal asymptote at $y = 0$. Finally, f has no vertical asymptotes.



70. $y = \dfrac{x}{\sqrt{x^2+1}}$

SOLUTION Let

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

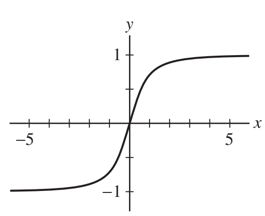
Then

$$f'(x) = (x^2+1)^{-3/2} \quad \text{and} \quad f''(x) = \frac{-3x}{(x^2+1)^{5/2}}.$$

Thus, f is increasing for all x, is concave up for $x < 0$, is concave down for $x > 0$, and has a point of inflection at $x = 0$. Because

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1,$$

f has horizontal asymptotes of $y = -1$ and $y = 1$. There are no vertical asymptotes.



Further Insights and Challenges

*In Exercises 71–75, we explore functions whose graphs approach a nonhorizontal line as $x \to \infty$. A line $y = ax + b$ is called a **slant asymptote** if*

$$\lim_{x\to\infty} (f(x) - (ax + b)) = 0$$

or

$$\lim_{x\to-\infty} (f(x) - (ax + b)) = 0$$

71. Let $f(x) = \dfrac{x^2}{x-1}$ (Figure 22). Verify the following:

(a) $f(0)$ is a local max and $f(2)$ a local min.

(b) f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) $\lim_{x\to1-} f(x) = -\infty$ and $\lim_{x\to1+} f(x) = \infty$.

(d) $y = x + 1$ is a slant asymptote of $f(x)$ as $x \to \pm\infty$.

(e) The slant asymptote lies above the graph of $f(x)$ for $x < 1$ and below the graph for $x > 1$.

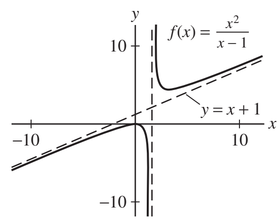


FIGURE 22

SOLUTION Let $f(x) = \dfrac{x^2}{x-1}$. Then $f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$.

(a) Sign analysis of $f''(x)$ reveals that $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$.

(b) Critical points of $f'(x)$ occur at $x = 0$ and $x = 2$. $x = 1$ is not a critical point because it is not in the domain of f. Sign analyses reveal that $x = 2$ is a local minimum of f and $x = 0$ is a local maximum.

(c)

$$\lim_{x\to1-} f(x) = -1 \lim_{x\to1-} \frac{1}{1-x} = -\infty \quad \text{and} \quad \lim_{x\to1+} f(x) = 1 \lim_{x\to1+} \frac{1}{x-1} = \infty.$$

(d) Note that using polynomial division, $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$. Then $\lim_{x\to\pm\infty} (f(x) - (x+1)) = \lim_{x\to\pm\infty} x + 1 + \dfrac{1}{x-1} - (x+1) = \lim_{x\to\pm\infty} \dfrac{1}{x-1} = 0$.

(e) For $x > 1$, $f(x) - (x+1) = \dfrac{1}{x-1} > 0$, so $f(x)$ approaches $x+1$ from above. Similarly, for $x < 1$, $f(x) - (x + 1) = \dfrac{1}{x-1} < 0$, so $f(x)$ approaches $x + 1$ from below.

72. If $f(x) = P(x)/Q(x)$, where P and Q are polynomials of degrees $m + 1$ and m, then by long division, we can write

$$f(x) = (ax + b) + P_1(x)/Q(x)$$

where P_1 is a polynomial of degree $< m$. Show that $y = ax + b$ is the slant asymptote of $f(x)$. Use this procedure to find the slant asymptotes of the following functions:

(a) $y = \dfrac{x^2}{x+2}$

(b) $y = \dfrac{x^3 + x}{x^2 + x + 1}$

SOLUTION Since $\deg(P_1) < \deg(Q)$,

$$\lim_{x\to\pm\infty} \frac{P_1(x)}{Q(x)} = 0.$$

Thus

$$\lim_{x\to\pm\infty} (f(x) - (ax + b)) = 0$$

and $y = ax + b$ is a slant asymptote of f.

(a) $\dfrac{x^2}{x+2} = x - 2 + \dfrac{4}{x+2}$; hence $y = x - 2$ is a slant asymptote of $\dfrac{x^2}{x+2}$.

(b) $\dfrac{x^3+x}{x^2+x+1} = (x-1) + \dfrac{x+1}{x^2-1}$; hence, $y = x - 1$ is a slant asymptote of $\dfrac{x^3+x}{x^2+x+1}$.

73. Sketch the graph of

$$f(x) = \frac{x^2}{x+1}.$$

Proceed as in the previous exercise to find the slant asymptote.

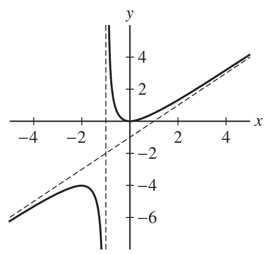
SOLUTION Let $f(x) = \dfrac{x^2}{x+1}$. Then $f'(x) = \dfrac{x(x+2)}{(x+1)^2}$ and $f''(x) = \dfrac{2}{(x+1)^3}$. Thus, f is increasing for $x < -2$ and for $x > 0$, is decreasing for $-2 < x < -1$ and for $-1 < x < 0$, has a local minimum at $x = 0$, has a local maximum at $x = -2$, is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$. Limit analyses give a vertical asymptote at $x = -1$, with

$$\lim_{x\to -1-} \frac{x^2}{x+1} = -\infty \qquad \text{and} \qquad \lim_{x\to -1+} \frac{x^2}{x+1} = \infty.$$

By polynomial division, $f(x) = x - 1 + \dfrac{1}{x+1}$ and

$$\lim_{x\to\pm\infty} \left(x - 1 + \frac{1}{x+1} - (x-1) \right) = 0,$$

which implies that the slant asymptote is $y = x - 1$. Notice that f approaches the slant asymptote as in exercise 71.

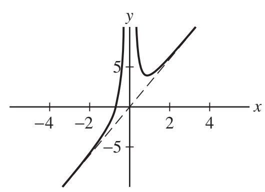


74. Show that $y = 3x$ is a slant asymptote for $f(x) = 3x + x^{-2}$. Determine whether $f(x)$ approaches the slant asymptote from above or below and make a sketch of the graph.

SOLUTION Let $f(x) = 3x + x^{-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - 3x) = \lim_{x\to\pm\infty} (3x + x^{-2} - 3x) = \lim_{x\to\pm\infty} x^{-2} = 0$$

which implies that $3x$ is the slant asymptote of $f(x)$. Since $f(x) - 3x = x^{-2} > 0$ as $x \to \pm\infty$, $f(x)$ approaches the slant asymptote from above in both directions. Moreover, $f'(x) = 3 - 2x^{-3}$ and $f''(x) = 6x^{-4}$. Sign analyses reveal a local minimum at $x = \left(\frac{3}{2}\right)^{-1/3} \approx 0.87358$ and that f is concave up for all $x \neq 0$. Limit analyses give a vertical asymptote at $x = 0$.

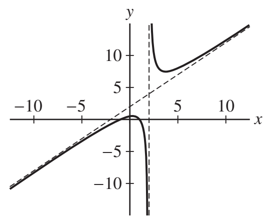


75. Sketch the graph of $f(x) = \dfrac{1-x^2}{2-x}$.

SOLUTION Let $f(x) = \dfrac{1-x^2}{2-x}$. Using polynomial division, $f(x) = x + 2 + \dfrac{3}{x-2}$. Then

$$\lim_{x\to\pm\infty} (f(x) - (x+2)) = \lim_{x\to\pm\infty} \left((x+2) + \frac{3}{x-2} - (x+2) \right) = \lim_{x\to\pm\infty} \frac{3}{x-2} = \frac{3}{1} \cdot \lim_{x\to\pm\infty} x^{-1} = 0$$

which implies that $y = x + 2$ is the slant asymptote of $f(x)$. Since $f(x) - (x + 2) = \dfrac{3}{x-2} > 0$ for $x > 2$, $f(x)$ approaches the slant asymptote from above for $x > 2$; similarly, $\dfrac{3}{x-2} < 0$ for $x < 2$ so $f(x)$ approaches the slant asymptote from below for $x < 2$. Moreover, $f'(x) = \dfrac{x^2 - 4x + 1}{(2-x)^2}$ and $f''(x) = \dfrac{-6}{(2-x)^3}$. Sign analyses reveal a local minimum at $x = 2 + \sqrt{3}$, a local maximum at $x = 2 - \sqrt{3}$ and that f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. Limit analyses give a vertical asymptote at $x = 2$.



76. Assume that $f'(x)$ and $f''(x)$ exist for all x and let c be a critical point of $f(x)$. Show that $f(x)$ cannot make a transition from $++$ to $-+$ at $x = c$. *Hint:* Apply the MVT to $f'(x)$.

SOLUTION Let $f(x)$ be a function such that $f''(x) > 0$ for all x and such that it transitions from $++$ to $-+$ at a critical point c where $f'(c)$ is defined. That is, $f'(c) = 0$, $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Let $g(x) = f'(x)$. The previous statements indicate that $g(c) = 0$, $g(x_0) > 0$ for some $x_0 < c$, and $g(x_1) < 0$ for some $x_1 > c$. By the Mean Value Theorem,

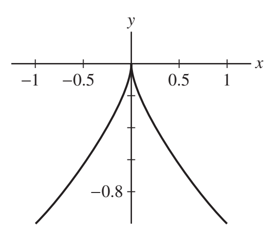
$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} = g'(c_0),$$

for some c_0 between x_0 and x_1. Because $x_1 > c > x_0$ and $g(x_1) < 0 < g(x_0)$,

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} < 0.$$

But, on the other hand $g'(c_0) = f''(c_0) > 0$, so there is a contradiction. This means that our assumption of the existence of such a function $f(x)$ must be in error, so no function can transition from $++$ to $-+$.

If we drop the requirement that $f'(c)$ exist, such a function can be found. The following is a graph of $f(x) = -x^{2/3}$. $f''(x) > 0$ wherever $f''(x)$ is defined, and $f'(x)$ transitions from positive to negative at $x = 0$.



77. Assume that $f''(x)$ exists and $f''(x) > 0$ for all x. Show that $f(x)$ cannot be negative for all x. *Hint:* Show that $f'(b) \neq 0$ for some b and use the result of Exercise 64 in Section 4.4.

SOLUTION Let $f(x)$ be a function such that $f''(x)$ exists and $f''(x) > 0$ for all x. Since $f''(x) > 0$, there is at least one point $x = b$ such that $f'(b) \neq 0$. If not, $f'(x) = 0$ for all x, so $f''(x) = 0$. By the result of Exercise 64 in Section 4.4, $f(x) \geq f(b) + f'(b)(x - b)$. Now, if $f'(b) > 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x > \frac{bf'(b) - f(b)}{f'(b)},$$

a condition that must be met for some x sufficiently large. For such x, $f(x) > f(b) + f'(b)(x - b) > 0$. On the other hand, if $f'(b) < 0$, we find that $f(b) + f'(b)(x - b) > 0$ whenever

$$x < \frac{bf'(b) - f(b)}{f'(b)}.$$

For such an x, $f(x) > f(b) + f'(b)(x - b) > 0$.

4.7 Applied Optimization

Preliminary Questions

1. The problem is to find the right triangle of perimeter 10 whose area is as large as possible. What is the constraint equation relating the base b and height h of the triangle?

SOLUTION The perimeter of a right triangle is the sum of the lengths of the base, the height and the hypotenuse. If the base has length b and the height is h, then the length of the hypotenuse is $\sqrt{b^2+h^2}$ and the perimeter of the triangle is $P = b + h + \sqrt{b^2+h^2}$. The requirement that the perimeter be 10 translates to the constraint equation

$$b + h + \sqrt{b^2+h^2} = 10.$$

2. Describe a way of showing that a continuous function on an open interval (a, b) has a minimum value.

SOLUTION If the function tends to infinity at the endpoints of the interval, then the function must take on a minimum value at a critical point.

3. Is there a rectangle of area 100 of largest perimeter? Explain.

SOLUTION No. Even by fixing the area at 100, we can take one of the dimensions as large as we like thereby allowing the perimeter to become as large as we like.

Exercises

1. Find the dimensions x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

(a) What is the constraint equation relating x and y?

(b) Find a formula for the area in terms of x alone.

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

SOLUTION

(a) The perimeter of the rectangle is 3 meters, so $3 = 2x + 2y$, which is equivalent to $y = \frac{3}{2} - x$.

(b) Using part (a), $A = xy = x(\frac{3}{2} - x) = \frac{3}{2}x - x^2$.

(c) This problem requires optimization over the closed interval $[0, \frac{3}{2}]$, since both x and y must be non-negative.

(d) $A'(x) = \frac{3}{2} - 2x = 0$, which yields $x = \frac{3}{4}$ and consequently, $y = \frac{3}{4}$. Because $A(0) = A(3/2) = 0$ and $A(\frac{3}{4}) = 0.5625$, the maximum area $0.5625\ \text{m}^2$ is achieved with $x = y = \frac{3}{4}$ m.

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?

(a) Express the sum of the areas of the squares in terms of the lengths x and y of the two pieces.

(b) What is the constraint equation relating x and y?

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

SOLUTION Let x and y be the lengths of the pieces.

(a) The perimeter of the first square is x, which implies the length of each side is $\frac{x}{4}$ and the area is $\left(\frac{x}{4}\right)^2$. Similarly, the area of the second square is $\left(\frac{y}{4}\right)^2$. Then the sum of the areas is given by $A = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2$.

(b) $x + y = 12$, so that $y = 12 - x$. Then

$$A(x) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{12-x}{4}\right)^2 = \frac{1}{8}x^2 - \frac{3}{2}x + 9.$$

(c) Since it is possible for the minimum total area to be realized by not cutting the wire at all, optimization over the closed interval [0, 12] suffices.

(d) Solve $A'(x) = \frac{1}{4}x - \frac{3}{2} = 0$ to obtain $x = 6$ m. Now $A(0) = A(12) = 9\ \text{m}^2$, whereas $A(4) = \frac{9}{4}\ \text{m}^2$. Accordingly, the sum of the areas of the squares is minimized if the wire is cut in half.

3. Wire of length 12 m is divided into two pieces and the pieces are bend into a square and a circle. How should this be done in order to minimize the sum of their areas?

SOLUTION Suppose the wire is divided into one piece of length x m that is bent into a circle and a piece of length $12 - x$ m that is bent into a square. Because the circle has circumference x, it follows that the radius of the circle is $x/2\pi$; therefore, the area of the circle is

$$\pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}.$$

As for the square, because the perimeter is $12 - x$, the length of each side is $3 - x/4$ and the area is $(3 - x/4)^2$. Then

$$A(x) = \frac{x^2}{4\pi} + \left(3 - \frac{1}{4}x\right)^2.$$

Now

$$A'(x) = \frac{x}{2\pi} - \frac{1}{2}\left(3 - \frac{1}{4}x\right) = 0$$

when

$$x = \frac{12\pi}{4+\pi} \text{ m} \approx 5.28 \text{ m}.$$

Because $A(0) = 9 \text{ m}^2$, $A(12) = 36/\pi \approx 11.46 \text{ m}^2$, and

$$A\left(\frac{12\pi}{4+\pi}\right) \approx 5.04 \text{ m}^2,$$

we see that the sum of the areas is minimized when approximately 5.28 m of the wire is allotted to the circle.

4. Find the positive number x such that the sum of x and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval?

SOLUTION Let $x > 0$ and $f(x) = x + x^{-1}$. Here we require optimization over the open interval $(0, \infty)$. Solve $f'(x) = 1 - x^{-2} = 0$ for $x > 0$ to obtain $x = 1$. Since $f(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, we conclude that f has an absolute minimum of $f(1) = 2$ at $x = 1$.

5. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

SOLUTION Let $x, y > 0$ be lengths of the side of the L. Since $x + y = 4$ or $y = 4 - x$, the distance between the ends of L is $h(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + (4-x)^2}$. We may equivalently minimize the square of the distance,

$$f(x) = x^2 + y^2 = x^2 + (4-x)^2$$

This is easier computationally (when working by hand). Solve $f'(x) = 4x - 8 = 0$ to obtain $x = 2$ m. Now $f(0) = f(4) = 16$, whereas $f(2) = 8$. Hence the distance between the two ends of the L is minimized when the bend is made at the middle of the wire.

6. Find the dimensions of the box with square base with:

(a) Volume 12 and the minimal surface area.

(b) Surface area 20 and maximal volume.

SOLUTION A box has a square base of side x and height y where $x, y > 0$. Its volume is $V = x^2y$ and its surface area is $S = 2x^2 + 4xy$.

(a) If $V = x^2y = 12$, then $y = 12/x^2$ and $S(x) = 2x^2 + 4x\left(12/x^2\right) = 2x^2 + 48x^{-1}$. Solve $S'(x) = 4x - 48x^{-2} = 0$ to obtain $x = 12^{1/3}$. Since $S(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum surface area is $S(12^{1/3}) = 6\,(12)^{2/3} \approx 31.45$, when $x = 12^{1/3}$ and $y = 12^{1/3}$.

(b) If $S = 2x^2 + 4xy = 20$, then $y = 5x^{-1} - \frac{1}{2}x$ and $V(x) = x^2y = 5x - \frac{1}{2}x^3$. Note that x must lie on the closed interval $[0, \sqrt{10}]$. Solve $V'(x) = 5 - \frac{3}{2}x^2$ for $x > 0$ to obtain $x = \frac{\sqrt{30}}{3}$. Since $V(0) = V(\sqrt{10}) = 0$ and $V\left(\frac{\sqrt{30}}{3}\right) = \frac{10\sqrt{30}}{9}$, the maximum volume is $V(\frac{\sqrt{30}}{3}) = \frac{10}{9}\sqrt{30} \approx 6.086$, when $x = \frac{\sqrt{30}}{3}$ and $y = \frac{\sqrt{30}}{3}$.

7. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 9). Find the dimensions that maximize the area of the corral.

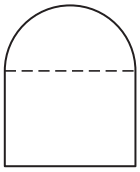


FIGURE 9

SOLUTION Let x be the width of the corral and therefore the diameter of the semicircle, and let y be the height of the rectangular section. Then the perimeter of the corral can be expressed by the equation $2y + x + \frac{\pi}{2}x = 2y + (1 + \frac{\pi}{2})x = 600$ m or equivalently, $y = \frac{1}{2}\left(600 - (1 + \frac{\pi}{2})x\right)$. Since x and y must both be nonnegative, it follows that x must be restricted to the interval $[0, \frac{600}{1+\pi/2}]$. The area of the corral is the sum of the area of the rectangle and semicircle, $A = xy + \frac{\pi}{8}x^2$. Making the substitution for y from the constraint equation,

$$A(x) = \frac{1}{2}x\left(600 - (1 + \frac{\pi}{2})x\right) + \frac{\pi}{8}x^2 = 300x - \frac{1}{2}\left(1 + \frac{\pi}{2}\right)x^2 + \frac{\pi}{8}x^2.$$

Now, $A'(x) = 300 - \left(1 + \frac{\pi}{2}\right)x + \frac{\pi}{4}x = 0$ implies $x = \frac{300}{(1+\frac{\pi}{4})} \approx 168.029746$ m. With $A(0) = 0$ m^2,

$$A\left(\frac{300}{1 + \pi/4}\right) \approx 25204.5 \text{ m}^2 \qquad \text{and} \qquad A\left(\frac{600}{1 + \pi/2}\right) \approx 21390.8 \text{ m}^2,$$

it follows that the corral of maximum area has dimensions

$$x = \frac{300}{1 + \pi/4} \text{ m} \qquad \text{and} \qquad y = \frac{150}{1 + \pi/4} \text{ m}.$$

8. What is the maximum area of a rectangle inscribed in a right triangle with 5 and 8 as in Figure 10. The sides of the rectangle are parallel to the legs of the triangle.

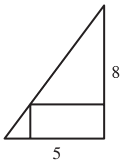


FIGURE 10

SOLUTION Position the triangle with its right angle at the origin, with its side of length 8 along the positive y-axis, and side of length 5 along the positive x-axis. Let $x, y > 0$ be the lengths of sides of the inscribed rectangle along the axes. By similar triangles, we have $\frac{8}{5} = \frac{y}{5-x}$ or $y = 8 - \frac{8}{5}x$. The area of the rectangle is thus $A(x) = xy = 8x - \frac{8}{5}x^2$. To guarantee that both x and y remain nonnegative, we must restrict x to the interval [0, 5]. Solve $A'(x) = 8 - \frac{16}{5}x = 0$ to obtain $x = \frac{5}{2}$. Since $A(0) = A(5) = 0$ and $A(\frac{5}{2}) = 10$, the maximum area is $A\left(\frac{5}{2}\right) = 10$ when $x = \frac{5}{2}$ and $y = 4$.

9. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius $r = 4$ (Figure 11).

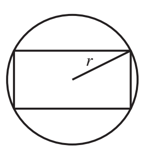


FIGURE 11

SOLUTION Place the center of the circle at the origin with the sides of the rectangle (of lengths $2x > 0$ and $2y > 0$) parallel to the coordinate axes. By the Pythagorean Theorem, $x^2 + y^2 = r^2 = 16$, so that $y = \sqrt{16 - x^2}$. Thus the area of the rectangle is $A(x) = 2x \cdot 2y = 4x\sqrt{16 - x^2}$. To guarantee both x and y are real and nonnegative, we must restrict x to the interval [0, 4]. Solve

$$A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = 0$$

for $x > 0$ to obtain $x = \frac{4}{\sqrt{2}} = 2\sqrt{2}$. Since $A(0) = A(4) = 0$ and $A(2\sqrt{2}) = 32$, the rectangle of maximum area has dimensions $2x = 2y = 4\sqrt{2}$.

10. Find the dimensions x and y of the rectangle inscribed in a circle of radius r that maximizes the quantity xy^2.

SOLUTION Place the center of the circle of radius r at the origin with the sides of the rectangle (of lengths $x > 0$ and $y > 0$) parallel to the coordinate axes. By the Pythagorean Theorem, we have $(\frac{x}{2})^2 + (\frac{y}{2})^2 = r^2$, whence $y^2 = 4r^2 - x^2$. Let $f(x) = xy^2 = 4xr^2 - x^3$. Allowing for degenerate rectangles, we have $0 \le x \le 2r$. Solve $f'(x) = 4r^2 - 3x^2$ for $x \ge 0$ to obtain $x = \frac{2r}{\sqrt{3}}$. Since $f(0) = f(2r) = 0$, the maximal value of f is $f(\frac{2r}{\sqrt{3}}) = \frac{16}{9}\sqrt{3}r^3$ when $x = \frac{2r}{\sqrt{3}}$ and $y = 2\sqrt{\frac{2}{3}}r$.

11. Find the point on the line $y = x$ closest to the point $(1, 0)$. *Hint:* It is equivalent and easier to minimize the *square* of the distance.

SOLUTION With $y = x$, let's equivalently minimize the square of the distance, $f(x) = (x-1)^2 + y^2 = 2x^2 - 2x + 1$, which is computationally easier (when working by hand). Solve $f'(x) = 4x - 2 = 0$ to obtain $x = \frac{1}{2}$. Since $f(x) \to \infty$ as $x \to \pm\infty$, $(\frac{1}{2}, \frac{1}{2})$ is the point on $y = x$ closest to $(1, 0)$.

12. Find the point P on the parabola $y = x^2$ closest to the point $(3, 0)$ (Figure 12).

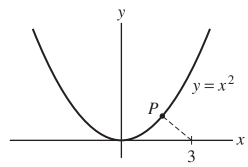


FIGURE 12

SOLUTION With $y = x^2$, let's equivalently minimize the square of the distance,

$$f(x) = (x-3)^2 + y^2 = x^4 + x^2 - 6x + 9.$$

Then

$$f'(x) = 4x^3 + 2x - 6 = 2(x-1)(2x^2 + 2x + 3),$$

so that $f'(x) = 0$ when $x = 1$ (plus two complex solutions, which we discard). Since $f(x) \to \infty$ as $x \to \pm\infty$, $P = (1, 1)$ is the point on $y = x^2$ closest to $(3, 0)$.

13. CAS Find a good numerical approximation to the coordinates of the point on the graph of $y = \ln x - x$ closest to the origin (Figure 13).

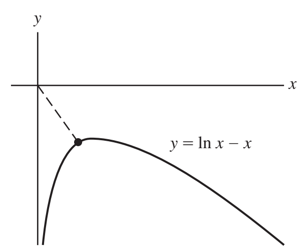


FIGURE 13

SOLUTION The distance from the origin to the point $(x, \ln x - x)$ on the graph of $y = \ln x - x$ is $d = \sqrt{x^2 + (\ln x - x)^2}$. As usual, we will minimize d^2. Let $d^2 = f(x) = x^2 + (\ln x - x)^2$. Then

$$f'(x) = 2x + 2(\ln x - x)\left(\frac{1}{x} - 1\right).$$

To determine x, we need to solve

$$4x + \frac{2\ln x}{x} - 2\ln x - 2 = 0.$$

This yields $x \approx .632784$. Thus, the point on the graph of $y = \ln x - x$ that is closest to the origin is approximately $(0.632784, -1.090410)$.

14. Problem of Tartaglia (1500–1557) Among all positive numbers a, b whose sum is 8, find those for which the product of the two numbers and their difference is largest.

SOLUTION The product of a,b and their difference is $ab(a-b)$. Since $a + b = 8$, $b = 8 - a$ and $a - b = 2a - 8$. Thus, let

$$f(a) = a(8-a)(2a-8) = -2a^3 + 24a^2 - 64a.$$

where $a \in [0, 8]$. Setting $f'(a) = -6a^2 + 48a - 64 = 0$ yields $a = 4 \pm \frac{4}{3}\sqrt{3}$. Now, $f(0) = f(8) = 0$, while

$$f\left(4 - \frac{4}{3}\sqrt{3}\right) < 0 \quad \text{and} \quad f\left(4 + \frac{4}{3}\sqrt{3}\right) > 0.$$

Hence the numbers a, b maximizing the product are

$$a = 4 + \frac{4\sqrt{3}}{3}, \quad \text{and} \quad b = 8 - a = 4 - \frac{4\sqrt{3}}{3}.$$

15. Find the angle θ that maximizes the area of the isosceles triangle whose legs have length ℓ (Figure 14).

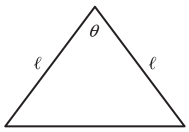


FIGURE 14

SOLUTION The area of the triangle is

$$A(\theta) = \frac{1}{2}\ell^2 \sin\theta,$$

where $0 \le \theta \le \pi$. Setting

$$A'(\theta) = \frac{1}{2}\ell^2 \cos\theta = 0$$

yields $\theta = \frac{\pi}{2}$. Since $A(0) = A(\pi) = 0$ and $A(\frac{\pi}{2}) = \frac{1}{2}\ell^2$, the angle that maximizes the area of the isosceles triangle is $\theta = \frac{\pi}{2}$.

16. A right circular cone (Figure 15) has volume $V = \frac{\pi}{3}r^2h$ and surface area is $S = \pi r\sqrt{r^2+h^2}$. Find the dimensions of the cone with surface area 1 and maximal volume.

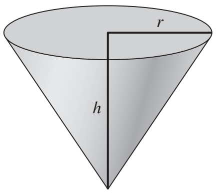


FIGURE 15

SOLUTION We have $\pi r\sqrt{r^2+h^2} = 1$ so $\pi^2 r^2(r^2+h^2) = 1$ and hence $h^2 = \frac{1-\pi^2r^4}{\pi^2r^2}$ and now we must maximize

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(r^2\frac{\sqrt{1-\pi^2r^4}}{\pi r}\right) = \frac{1}{3}r\sqrt{1-\pi^2r^4},$$

where $0 < r \le 1/\sqrt{\pi}$. Because

$$\frac{d}{dr}r\sqrt{1-\pi^2r^4} = \sqrt{1-\pi^2r^4} + \frac{1}{2}r\frac{-4\pi^2r^3}{\sqrt{1-\pi^2r^4}}$$

the relevant critical point is $r = (3\pi^2)^{-1/4}$.

To find h, we back substitute our solution for r in $h^2 = (1-\pi^2r^4)/(\pi^2r^2)$. $r = (3\pi^2)^{-1/4}$, so $r^4 = \frac{1}{3\pi^2}$ and $r^2 = \frac{1}{\sqrt{3}\pi}$; hence, $\pi^2r^4 = \frac{1}{3}$ and $\pi^2r^2 = \frac{\pi}{\sqrt{3}}$, and:

$$h^2 = \left(\frac{2}{3}\right)\Big/\left(\frac{\pi}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}\pi}.$$

From this, $h = \sqrt{2}/\left(3^{1/4}\sqrt{\pi}\right)$. Since

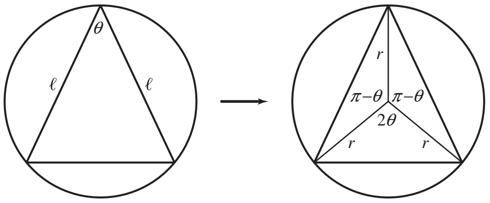
$$\lim_{r\to 0+} V(r) = 0,\ V\left(\frac{1}{\sqrt{\pi}}\right) = 0 \qquad \text{and} \qquad V\left((3\pi^2)^{-1/4}\right) = \frac{1}{3^{7/4}}\sqrt{\frac{2}{\pi}},$$

the cone of surface area 1 with maximal volume has dimensions

$$r = \frac{1}{3^{1/4}\sqrt{\pi}} \qquad \text{and} \qquad h = \frac{\sqrt{2}}{3^{1/4}\sqrt{\pi}}.$$

17. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius r.

SOLUTION Consider the following diagram:



The area of the isosceles triangle is

$$A(\theta) = 2 \cdot \frac{1}{2} r^2 \sin(\pi - \theta) + \frac{1}{2} r^2 \sin(2\theta) = r^2 \sin\theta + \frac{1}{2} r^2 \sin(2\theta),$$

where $0 \le \theta \le \pi$. Solve

$$A'(\theta) = r^2 \cos\theta + r^2 \cos(2\theta) = 0$$

to obtain $\theta = \frac{\pi}{3}, \pi$. Since $A(0) = A(\pi) = 0$ and $A(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4} r^2$, the area of the largest isosceles triangle that can be inscribed in a circle of radius r is $\frac{3\sqrt{3}}{4} r^2$.

18. Find the radius and height of a cylindrical can of total surface area A whose volume is as large as possible. Does there exist a cylinder of surface area A and minimal total volume?

SOLUTION Let a closed cylindrical can be of radius r and height h. Its total surface area is $S = 2\pi r^2 + 2\pi r h = A$, whence $h = \dfrac{A}{2\pi r} - r$. Its volume is thus $V(r) = \pi r^2 h = \frac{1}{2} A r - \pi r^3$, where $0 < r \le \sqrt{\frac{A}{2\pi}}$. Solve $V'(r) = \frac{1}{2} A - 3\pi r^2$ for $r > 0$ to obtain $r = \sqrt{\dfrac{A}{6\pi}}$. Since $V(0) = V(\sqrt{\frac{A}{2\pi}}) = 0$ and

$$V\left(\sqrt{\frac{A}{6\pi}}\right) = \frac{\sqrt{6} A^{3/2}}{18\sqrt{\pi}},$$

the maximum volume is achieved when

$$r = \sqrt{\frac{A}{6\pi}} \quad \text{and} \quad h = \frac{1}{3}\sqrt{\frac{6A}{\pi}}.$$

For a can of total surface area A, there are cans of arbitrarily small volume since $\lim_{r \to 0+} V(r) = 0$.

19. A poster of area 6000 cm^2 has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

SOLUTION Let x be the width of the printed region, and let y be the height. The total printed area is $A = xy$. Because the total area of the poster is 6000 cm^2, we have the constraint $(x + 12)(y + 20) = 6000$, so that $xy + 12y + 20x + 240 = 6000$, or $y = \frac{5760 - 20x}{x + 12}$. Therefore, $A(x) = 20\frac{288x - x^2}{x + 12}$, where $0 \le x \le 288$.

$A(0) = A(288) = 0$, so we are looking for a critical point on the interval $[0, 288]$. Setting $A'(x) = 0$ yields

$$\begin{aligned} 20\frac{(x + 12)(288 - 2x) - (288x - x^2)}{(x + 12)^2} &= 0 \\ \frac{-x^2 - 24x + 3456}{(x + 12)^2} &= 0 \\ x^2 + 24x - 3456 &= 0 \\ (x - 48)(x + 72) &= 0 \end{aligned}$$

Therefore $x = 48$ or $x = -72$. $x = 48$ is the only critical point of $A(x)$ in the interval $[0, 288]$, so $A(48) = 3840$ is the maximum value of $A(x)$ in the interval $[0, 288]$. Now, $y = 20\frac{288 - 48}{48 + 12} = 80$ cm, so the poster with maximum printed area is $48 + 12 = 60$ cm. wide by $80 + 20 = 100$ cm. tall.

20. According to postal regulations, a carton is classified as "oversized" if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with square base that is not oversized and has maximum volume.

SOLUTION Let h denote the height of the carton and s denote the side length of the square base. Clearly the volume will be maximized when the sum of the height and girth equals 108; i.e., $4s + h = 108$, whence $h = 108 - 4s$. Allowing for degenerate cartons, the carton's volume is $V(s) = s^2 h = s^2(108 - 4s)$, where $0 \le s \le 27$. Solve $V'(s) = 216s - 12s^3 = 0$ for s to obtain $s = 0$ or $s = 18$. Since $V(0) = V(27) = 0$, the maximum volume is $V(18) = 11664$ in^3 when $s = 18$ in and $h = 36$ in.

21. Kepler's Wine Barrel Problem In his work *Nova stereometria doliorum vinariorum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius R. *Hint:* Show that an inscribed cylinder has volume $2\pi x(R^2 - x^2)$, where x is one-half the height of the cylinder.

SOLUTION Place the center of the sphere at the origin in three-dimensional space. Let the cylinder be of radius y and half-height x. The Pythagorean Theorem states, $x^2 + y^2 = R^2$, so that $y^2 = R^2 - x^2$. The volume of the cylinder is $V(x) = \pi y^2(2x) = 2\pi\left(R^2 - x^2\right)x = 2\pi R^2 x - 2\pi x^3$. Allowing for degenerate cylinders, we have $0 \le x \le R$. Solve $V'(x) = 2\pi R^2 - 6\pi x^2 = 0$ for $x \ge 0$ to obtain $x = \frac{R}{\sqrt{3}}$. Since $V(0) = V(R) = 0$, the largest volume is $V(\frac{R}{\sqrt{3}}) = \frac{4}{9}\pi\sqrt{3}R^3$ when $x = \frac{R}{\sqrt{3}}$ and $y = \sqrt{\frac{2}{3}}R$.

22. Find the angle θ that maximizes the area of the trapezoid with a base of length 4 and sides of length 2, as in Figure 16.

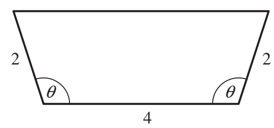


FIGURE 16

SOLUTION Allowing for degenerate trapezoids, we have $0 \le \theta \le \pi$. Via trigonometry and surgery (slice off a right triangle and rearrange the trapezoid into a rectangle), we have that the area of the trapezoid is equivalent to the area of a rectangle of base $4 - 2\cos\theta$ and height $2\sin\theta$; i.e,

$$A(\theta) = (4 - 2\cos\theta)\cdot 2\sin\theta = 8\sin\theta - 4\sin\theta\cos\theta = 8\sin\theta - 2\sin 2\theta,$$

where $0 \le \theta \le \pi$. Solve

$$A'(\theta) = 8\cos\theta - 4\cos 2\theta = 4 + 8\cos\theta - 8\cos^2\theta = 0$$

for $0 \le \theta \le \pi$ to obtain

$$\theta = \theta_0 = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right) \approx 1.94553.$$

Since $A(0) = A(\pi) = 0$ and $A(\theta_0) = 3^{1/4}(3+\sqrt{3})\sqrt{2}$, the area of the trapezoid is maximized when $\theta = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$.

23. A landscape architect wishes to enclose a rectangular garden of area 1,000 m^2 on one side by a brick wall costing \$90/m and on the other three sides by a metal fence costing \$30/m. Which dimensions minimize the total cost?

SOLUTION Let x be the length of the brick wall and y the length of an adjacent side with $x, y > 0$. With $xy = 1000$ or $y = \frac{1000}{x}$, the total cost is

$$C(x) = 90x + 30(x + 2y) = 120x + 60000x^{-1}.$$

Solve $C'(x) = 120 - 60000x^{-2} = 0$ for $x > 0$ to obtain $x = 10\sqrt{5}$. Since $C(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum cost is $C(10\sqrt{5}) = 2400\sqrt{5} \approx \$5366.56$ when $x = 10\sqrt{5} \approx 22.36$ m and $y = 20\sqrt{5} \approx 44.72$ m.

24. The amount of light reaching a point at a distance r from a light source A of intensity I_A is I_A/r^2. Suppose that a second light source B of intensity $I_B = 4I_A$ is located 10 m from A. Find the point on the segment joining A and B where the total amount of light is at a minimum.

SOLUTION Place the segment in the xy-plane with A at the origin and B at $(10, 0)$. Let x be the distance from A. Then $10 - x$ is the distance from B. The total amount of light is

$$f(x) = \frac{I_A}{x^2} + \frac{I_B}{(10-x)^2} = I_A\left(\frac{1}{x^2} + \frac{4}{(10-x)^2}\right).$$

Solve

$$f'(x) = I_A\left(\frac{8}{(10-x)^3} - \frac{2}{x^3}\right) = 0$$

for $0 \le x \le 10$ to obtain

$$4 = \frac{(10-x)^3}{x^3} = \left(\frac{10}{x} - 1\right)^3 \quad \text{or} \quad x = \frac{10}{1+\sqrt[3]{4}} \approx 3.86 \text{ m}.$$

Since $f(x) \to \infty$ as $x \to 0+$ and $x \to 10-$ we conclude that the minimal amount of light occurs 3.86 m from A.

25. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = \dfrac{4-x}{2+x}$ and the axes (Figure 17).

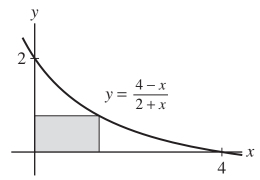


FIGURE 17

SOLUTION Let s be the width of the rectangle. The height of the rectangle is $h = \frac{4-s}{2+s}$, so that the area is

$$A(s) = s\frac{4-s}{2+s} = \frac{4s-s^2}{2+s}.$$

We are maximizing on the closed interval $[0, 4]$. It is obvious from the pictures that $A(0) = A(4) = 0$, so we look for critical points of A.

$$A'(s) = \frac{(2+s)(4-2s)-(4s-s^2)}{(2+s)^2} = -\frac{s^2+4s-8}{(s+2)^2}.$$

The only point where $A'(s)$ doesn't exist is $s = -2$ which isn't under consideration.

Setting $A'(s) = 0$ gives, by the quadratic formula,

$$s = \frac{-4 \pm \sqrt{48}}{2} = -2 \pm 2\sqrt{3}.$$

Of these, only $-2+2\sqrt{3}$ is positive, so this is our lone critical point. $A(-2+2\sqrt{3}) \approx 1.0718 > 0$. Since we are finding the maximum over a closed interval and $-2+2\sqrt{3}$ is the only critical point, the maximum area is $A(-2+2\sqrt{3}) \approx 1.0718$.

26. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of $y = (x+1)^{-2}$ with $x > 0$.

SOLUTION Let $P\left(t, \frac{1}{(t+1)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+1)^2}$ in the first quadrant. The tangent line to the curve at P is

$$L(x) = \frac{1}{(t+1)^2} - \frac{2(x-t)}{(t+1)^3},$$

which has x-intercept $a = \frac{3t+1}{2}$ and y-intercept $b = \frac{3t+1}{(t+1)^3}$. The area of the triangle in question is

$$A(t) = \frac{1}{2}ab = \frac{(3t+1)^2}{4(t+1)^3}.$$

Solve

$$A'(t) = \frac{(3t+1)(3-3t)}{4(t+1)^4} = 0$$

for $0 \le t$ to obtain $t = 1$. Because $A(0) = \frac{1}{4}$, $A(1) = \frac{1}{2}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(1) = \frac{1}{2}$.

27. Find the maximum area of a rectangle circumscribed around a rectangle of sides L and H. *Hint:* Express the area in terms of the angle θ (Figure 18).

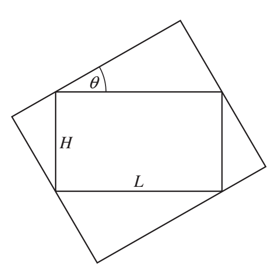


FIGURE 18

SOLUTION Position the $L \times H$ rectangle in the first quadrant of the xy-plane with its "northwest" corner at the origin. Let θ be the angle the base of the circumscribed rectangle makes with the positive x-axis, where $0 \le \theta \le \frac{\pi}{2}$. Then the area of the circumscribed rectangle is $A = LH + 2 \cdot \frac{1}{2}(H\sin\theta)(H\cos\theta) + 2 \cdot \frac{1}{2}(L\sin\theta)(L\cos\theta) = LH + \frac{1}{2}(L^2 + H^2)\sin 2\theta$, which has a maximum value of $LH + \frac{1}{2}(L^2 + H^2)$ when $\theta = \frac{\pi}{4}$ because $\sin 2\theta$ achieves its maximum when $\theta = \frac{\pi}{4}$.

28. A contractor is engaged to build steps up the slope of a hill that has the shape of the graph of $y = x^2(120 - x)/6400$ for $0 \le x \le 80$ with x in meters (Figure 19). What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter.

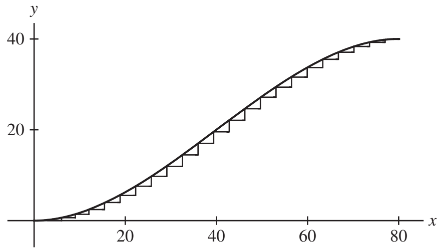


FIGURE 19

SOLUTION Let $f(x) = x^2(120 - x)/6400$. Because the horizontal length of each stair is one-third meter, the vertical rise of each stair is

$$r(x) = f\left(x + \frac{1}{3}\right) - f(x) = \frac{1}{6400}\left(x + \frac{1}{3}\right)^2\left(\frac{359}{3} - x\right) - \frac{1}{6400}x^2(120 - x)$$
$$= \frac{1}{6400}\left(-x^2 + \frac{239}{3}x + \frac{359}{27}\right),$$

where x denotes the location of the beginning of the stair. This is the equation of a downward opening parabola; thus, the maximum occurs when $r'(x) = 0$. Now,

$$r'(x) = \frac{1}{6400}\left(-2x + \frac{239}{3}\right) = 0$$

when $x = 239/6$. Because the stair must start at a location of the form $n/3$ for some integer n, we evaluate $r(x)$ at $x = 119/3$ and $x = 120/3 = 40$. We find

$$r\left(\frac{119}{3}\right) = r(40) = \frac{43199}{172800} \approx 0.249994$$

meters. Thus, the maximum vertical rise of any stair is just below 0.25 meters.

29. Find the equation of the line through $P = (4, 12)$ such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

SOLUTION Let $P = (4, 12)$ be a point in the first quadrant and $y - 12 = m(x - 4)$, $-\infty < m < 0$, be a line through P that cuts the positive x- and y-axes. Then $y = L(x) = m(x - 4) + 12$. The line $L(x)$ intersects the y-axis at $H\,(0, 12 - 4m)$ and the x-axis at $W\left(4 - \frac{12}{m}, 0\right)$. Hence the area of the triangle is

$$A(m) = \frac{1}{2}(12 - 4m)\left(4 - \frac{12}{m}\right) = 48 - 8m - 72m^{-1}.$$

Solve $A'(m) = 72m^{-2} - 8 = 0$ for $m < 0$ to obtain $m = -3$. Since $A \to \infty$ as $m \to -\infty$ or $m \to 0-$, we conclude that the minimal triangular area is obtained when $m = -3$. The equation of the line through $P = (4, 12)$ is $y = -3(x - 4) + 12 = -3x + 24$.

30. Let $P = (a, b)$ lie in the first quadrant. Find the slope of the line through P such that the triangle bounded by this line and the axes in the first quadrant has minimal area. Then show that P is the midpoint of the hypotenuse of this triangle.

SOLUTION Let $P(a, b)$ be a point in the first quadrant (thus $a, b > 0$) and $y - b = m(x - a)$, $-\infty < m < 0$, be a line through P that cuts the positive x- and y-axes. Then $y = L(x) = m(x - a) + b$. The line $L(x)$ intersects the y-axis at $H\,(0, b - am)$ and the x-axis at $W\left(a - \frac{b}{m}, 0\right)$. Hence the area of the triangle is

$$A(m) = \frac{1}{2}(b - am)\left(a - \frac{b}{m}\right) = ab - \frac{1}{2}a^2m - \frac{1}{2}b^2m^{-1}.$$

Solve $A'(m) = \frac{1}{2}b^2m^{-2} - \frac{1}{2}a^2 = 0$ for $m < 0$ to obtain $m = -\frac{b}{a}$. Since $A \to \infty$ as $m \to -\infty$ or $m \to 0-$, we conclude that the minimal triangular area is obtained when $m = -\frac{b}{a}$. For $m = -b/a$, we have $H(0, 2b)$ and $W(2a, 0)$. The midpoint of the line segment connecting H and W is thus $P(a, b)$.

31. Archimedes' Problem A spherical cap (Figure 20) of radius r and height h has volume $V = \pi h^2\left(r - \frac{1}{3}h\right)$ and surface area $S = 2\pi r h$. Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area S.

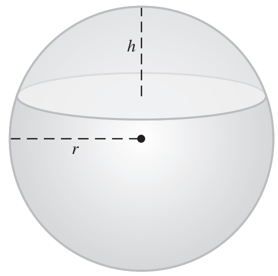


FIGURE 20

SOLUTION Consider all spherical caps of fixed surface area S. Because $S = 2\pi r h$, it follows that

$$r = \frac{S}{2\pi h}$$

and

$$V(h) = \pi h^2\left(\frac{S}{2\pi h} - \frac{1}{3}h\right) = \frac{S}{2}h - \frac{\pi}{3}h^3.$$

Now

$$V'(h) = \frac{S}{2} - \pi h^2 = 0$$

when

$$h^2 = \frac{S}{2\pi} \quad \text{or} \quad h = \frac{S}{2\pi h} = r.$$

Hence, the hemisphere encloses the largest volume among all spherical caps of fixed surface area S.

32. Find the isosceles triangle of smallest area (Figure 21) that circumscribes a circle of radius 1 (from Thomas Simpson's *The Doctrine and Application of Fluxions*, a calculus text that appeared in 1750).

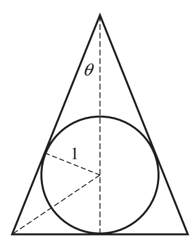


FIGURE 21

SOLUTION From the diagram, we see that the height h and base b of the triangle are $h = 1 + \csc\theta$ and $b = 2h\tan\theta = 2(1 + \csc\theta)\tan\theta$. Thus, the area of the triangle is

$$A(\theta) = \frac{1}{2}hb = (1 + \csc\theta)^2\tan\theta,$$

where $0 < \theta < \pi$. We now set the derivative equal to zero:

$$A'(\theta) = (1 + \csc\theta)(-2\csc\theta + \sec^2\theta(1 + \csc\theta)) = 0.$$

The first factor gives $\theta = 3\pi/2$ which is not in the domain of the problem. To find the roots of the second factor, multiply through by $\cos^2\theta\sin\theta$ to obtain

$$-2\cos^2\theta + \sin\theta + 1 = 0,$$

or

$$2\sin^2\theta + \sin\theta - 1 = 0.$$

This is a quadratic equation in $\sin\theta$ with roots $\sin\theta = -1$ and $\sin\theta = 1/2$. Only the second solution is relevant and gives us $\theta = \pi/6$. Since $A(\theta) \to \infty$ as $\theta \to 0+$ and as $\theta \to \pi-$, we see that the minimum area occurs when the triangle is an equilateral triangle.

33. A box of volume 72 m^3 with square bottom and no top is constructed out of two different materials. The cost of the bottom is \$40/m^2 and the cost of the sides is \$30/m^2. Find the dimensions of the box that minimize total cost.

SOLUTION Let s denote the length of the side of the square bottom of the box and h denote the height of the box. Then

$$V = s^2h = 72 \quad \text{or} \quad h = \frac{72}{s^2}.$$

The cost of the box is

$$C = 40s^2 + 120sh = 40s^2 + \frac{8640}{s},$$

so

$$C'(s) = 80s - \frac{8640}{s^2} = 0$$

when $s = 3\sqrt[3]{4}$ m and $h = 2\sqrt[3]{4}$ m. Because $C \to \infty$ as $s \to 0-$ and as $s \to \infty$, we conclude that the critical point gives the minimum cost.

34. Find the dimensions of a cylinder of volume 1 m^3 of minimal cost if the top and bottom are made of material that costs twice as much as the material for the side.

SOLUTION Let r be the radius in meters of the top and bottom of the cylinder. Let h be the height in meters of the cylinder. Since $V = \pi r^2 h = 1$, we get $h = \frac{1}{\pi r^2}$. Ignoring the actual cost, and using only the proportion, suppose that the sides cost 1 monetary unit per square meter and the top and the bottom 2. The cost of the top and bottom is $2(2\pi r^2)$ and the cost of the sides is $1(2\pi r h) = 2\pi r(\frac{1}{\pi r^2}) = \frac{2}{r}$. Let $C(r) = 4\pi r^2 + \frac{2}{r}$. Because $C(r) \to \infty$ as $r \to 0+$ and as $r \to \infty$, we are looking for critical points of $C(r)$. Setting $C'(r) = 8\pi r - \frac{2}{r^2} = 0$ yields $8\pi r = \frac{2}{r^2}$, so that $r^3 = \frac{1}{4\pi}$. This yields $r = \frac{1}{(4\pi)^{1/3}} \approx 0.430127$. The dimensions that minimize cost are

$$r = \frac{1}{(4\pi)^{1/3}} \text{ m}, \qquad h = \frac{1}{\pi r^2} = 4^{2/3}\pi^{-1/3} \text{ m}.$$

35. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 22. The wall materials cost \$500 per linear meter and your company allocates \$2,400,000 for the project.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

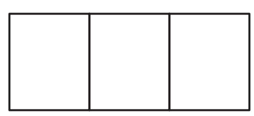


FIGURE 22

SOLUTION Let one compartment have length x and width y. Then total length of the wall of the warehouse is $P = 4x + 6y$ and the constraint equation is cost $= 2{,}400{,}000 = 500(4x + 6y)$, which gives $y = 800 - \frac{2}{3}x$.

(a) Area is given by $A = 3xy = 3x\left(800 - \frac{2}{3}x\right) = 2400x - 2x^2$, where $0 \le x \le 1200$. Then $A'(x) = 2400 - 4x = 0$ yields $x = 600$ and consequently $y = 400$. Since $A(0) = A(1200) = 0$ and $A(600) = 720,000$, the area of the warehouse is maximized when each compartment has length of 600 m and width of 400 m.

(b) The area of one compartment is $600 \cdot 400 = 240,000$ square meters.

36. Suppose, in the previous exercise, that the warehouse consists of n separate spaces of equal size. Find a formula in terms of n for the maximum possible area of the warehouse.

SOLUTION For n compartments, with x and y as before, cost $= 2{,}400{,}000 = 500((n+1)x + 2ny)$ and $y = \dfrac{4800 - (n+1)x}{2n}$. Then

$$A = nxy = x\frac{4800 - (n+1)x}{2} = 2400x - \frac{n+1}{2}x^2$$

and $A'(x) = 2400 - (n+1)x = 0$ yields $x = \dfrac{2400}{n+1}$ and consequently $y = \dfrac{1200}{n}$. Thus the maximum area is given by

$$A = n\left(\frac{2400}{n+1}\right)\left(\frac{1200}{n}\right) = \frac{28{,}800{,}000}{n+1}.$$

37. According to a model developed by economists E. Heady and J. Pesek, if fertilizer made from N pounds of nitrogen and P pounds of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is

$$Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$$

A farmer intends to spend \$30 per acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of N and L produces the highest yield of corn?

SOLUTION The farmer's budget for fertilizer is \$30 per acre, so we have the constraint equation

$$0.25N + 0.2P = 30 \qquad \text{or} \qquad P = 150 - 1.25N$$

Substituting for P in the equation for Y, we find

$$\begin{aligned} Y(N) &= 7.5 + 0.6N + 0.7(150 - 1.25N) - 0.001N^2 - 0.002(150 - 1.25N)^2 + 0.001N(150 - 1.25N) \\ &= 67.5 + 0.625N - 0.005375N^2 \end{aligned}$$

Both N and P must be nonnegative. Since $P = 150 - 1.25N \geq 0$, we require that $0 \leq N \leq 120$. Next,

$$\frac{dY}{dN} = 0.625 - 0.01075N = 0 \quad \Rightarrow \quad N = \frac{0.625}{0.01075} \approx 58.14 \text{ pounds.}$$

Now, $Y(0) = 67.5$, $Y(120) = 65.1$ and $Y(58.14) \approx 85.67$, so the maximum yield of corn occurs for $N \approx 58.14$ pounds and $P \approx 77.33$ pounds.

38. Experiments show that the quantities x of corn and y of soybean required to produce a hog of weight Q satisfy $Q = 0.5x^{1/2}y^{1/4}$. The unit of x, y, and Q is the cwt, an agricultural unit equal to 100 lbs. Find the values of x and y that minimize the cost of a hog of weight $Q = 2.5$ cwt if corn costs \$3/cwt and soy costs \$7/cwt.

SOLUTION With $Q = 2.5$, we find that

$$y = \left(\frac{2.5}{0.5x^{1/2}}\right)^4 = \frac{625}{x^2}.$$

The cost is then

$$C = 3x + 7y = 3x + \frac{4375}{x^2}.$$

Solving

$$\frac{dC}{dx} = 3 - \frac{8750}{x^3} = 0$$

yields $x = \sqrt[3]{8750/3} \approx 14.29$. From this, it follows that $y = 625/14.29^2 \approx 3.06$. The overall cost is $C = 3(14.29) + 7(3.06) \approx \$64.29$.

39. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = \$900$/month. Suppose that one unit becomes vacant with each \$10 increase in rent and that each occupied unit costs \$80/month in maintenance. Which rent r maximizes monthly profit?

SOLUTION Let n denote the number of \$10 increases in rent. Then the monthly profit is given by

$$P(n) = (100 - n)(900 + 10n - 80) = 82000 + 180n - 10n^2,$$

and

$$P'(n) = 180 - 20n = 0$$

when $n = 9$. We know this results in maximum profit because this gives the location of vertex of a downward opening parabola. Thus, monthly profit is maximized with a rent of \$990.

40. An 8-billion-bushel corn crop brings a price of \$2.40/bu. A commodity broker uses the rule of thumb: If the crop is reduced by x percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bu? *Hint:* Revenue is equal to price times crop size.

SOLUTION Let x denote the percentage reduction in crop size. Then the price for corn is $2.40 + 0.10x$, the crop size is $8(1 - 0.01x)$ and the revenue (in billions of dollars) is

$$R(x) = (2.4 + 0.1x)8(1 - 0.01x) = 8(-0.001x^2 + 0.076x + 2.4),$$

where $0 \leq x \leq 100$. Solve

$$R'(x) = -0.002x + 0.076 = 0$$

to obtain $x = 38$ percent. Since $R(0) = 19.2$, $R(38) = 30.752$, and $R(100) = 0$, revenue is maximized when $x = 38$. So we reduce the crop size to

$$8(1 - 0.38) = 4.96 \text{ billion bushels.}$$

The price would be $\$2.40 + 0.10(38) = 2.40 + 3.80 = \$6.20$.

41. The monthly output of a Spanish light bulb factory is $P = 2LK^2$ (in millions), where L is the cost of labor and K is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of L and K would minimize the total cost $L + K$?

SOLUTION Since $P = 1.7$ and $P = 2LK^2$, we have $L = \dfrac{0.85}{K^2}$. Accordingly, the cost of production is

$$C(K) = L + K = K + \frac{0.85}{K^2}.$$

Solve $C'(K) = 1 - \dfrac{1.7}{K^3}$ for $K \geq 0$ to obtain $K = \sqrt[3]{1.7}$. Since $C(K) \to \infty$ as $K \to 0+$ and as $K \to \infty$, the minimum cost of production is achieved for $K = \sqrt[3]{1.7} \approx 1.2$ and $L = 0.6$. The company should invest 1.2 million euros in equipment and 600, 000 euros in labor.

42. The rectangular plot in Figure 23 has size 100 m × 200 m. Pipe is to be laid from A to a point P on side BC and from there to C. The cost of laying pipe along the side of the plot is \$45/m and the cost through the plot is \$80/m (since it is underground).

(a) Let $f(x)$ be the total cost, where x is the distance from P to B. Determine $f(x)$, but note that f is discontinuous at $x = 0$ (when $x = 0$, the cost of the entire pipe is \$45/ft).

(b) What is the most economical way to lay the pipe? What if the cost along the sides is \$65/m?

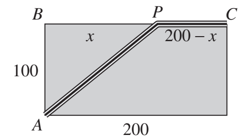


FIGURE 23

SOLUTION

(a) Let x be the distance from P to B. If $x > 0$, then the length of the underground pipe is $\sqrt{100^2 + x^2}$ and the length of the pipe along the side of the plot is $200 - x$. The total cost is

$$f(x) = 80\sqrt{100^2 + x^2} + 45(200 - x).$$

If $x = 0$, all of the pipe is along the side of the plot and $f(0) = 45(200 + 100) = \$13{,}500$.

(b) To locate the critical points of f, solve

$$f'(x) = \frac{80x}{\sqrt{100^2 + x^2}} - 45 = 0.$$

We find $x = \pm 180/\sqrt{7}$. Note that only the positive value is in the domain of the problem. Because $f(0) = \$13{,}500$, $f(180/\sqrt{7}) = \$15{,}614.38$ and $f(200) = \$17{,}888.54$, the most economical way to lay the pipe is to place the pipe along the side of the plot.

If the cost of laying the pipe along the side of the plot is \$65 per meter, then

$$f(x) = 80\sqrt{100^2 + x^2} + 65(200 - x)$$

and

$$f'(x) = \frac{80x}{\sqrt{100^2 + x^2}} - 65.$$

The only critical point in the domain of the problem is $x = 1300/\sqrt{87} \approx 139.37$. Because $f(0) = \$19{,}500$, $f(139.37) = \$17{,}663.69$ and $f(200) = \$17{,}888.54$, the most economical way to lay the pipe is place the underground pipe from A to a point 139.37 meters to the right of B and continuing to C along the side of the plot.

43. Brandon is on one side of a river that is 50 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

SOLUTION Let lengths be in meters, times in seconds, and speeds in m/s. Suppose that Brandon swims diagonally to a point located x meters downstream on the opposite side. Then Brandon then swims a distance $\sqrt{x^2 + 50^2}$ and runs a distance $200 - x$. The total time of the trip is

$$f(x) = \frac{\sqrt{x^2 + 2500}}{1.5} + \frac{200 - x}{4}, \quad 0 \le x \le 200.$$

Solve

$$f'(x) = \frac{2x}{3\sqrt{x^2 + 2500}} - \frac{1}{4} = 0$$

to obtain $x = 30\frac{5}{11} \approx 20.2$ and $f(20.2) \approx 80.9$. Since $f(0) \approx 83.3$ and $f(200) \approx 137.4$, we conclude that the minimal time is 80.9 s. This occurs when Brandon swims diagonally to a point located 20.2 m downstream and then runs the rest of the way.

44. Snell's Law When a light beam travels from a point A above a swimming pool to a point B below the water (Figure 24), it chooses the path that takes the *least time*. Let v_1 be the velocity of light in air and v_2 the velocity in water (it is known that $v_1 > v_2$). Prove Snell's Law of Refraction:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

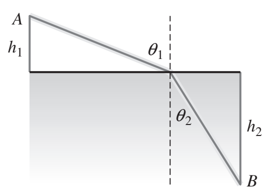


FIGURE 24

SOLUTION The time it takes a beam of light to travel from A to B is

$$f(x) = \frac{a}{v_1} + \frac{b}{v_2} = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + h_2^2}}{v_2}$$

(See diagram below.) Now

$$f'(x) = \frac{x}{v_1\sqrt{x^2 + h_1^2}} - \frac{L - x}{v_2\sqrt{(L-x)^2 + h_2^2}} = 0$$

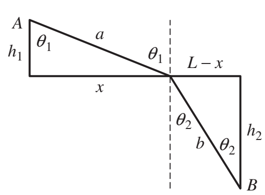
yields

$$\frac{x\Big/\sqrt{x^2 + h_1^2}}{v_1} = \frac{(L-x)\Big/\sqrt{(L-x)^2 + h_2^2}}{v_2} \qquad \text{or} \qquad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

which is Snell's Law. Since

$$f''(x) = \frac{h_1^2}{v_1\left(x^2 + h_1^2\right)^{3/2}} + \frac{h_2^2}{v_2\left((L-x)^2 + h_2^2\right)^{3/2}} > 0$$

for all x, the minimum time is realized when Snell's Law is satisfied.



In Exercises 45–47, a box (with no top) is to be constructed from a piece of cardboard of sides A and B by cutting out squares of length h from the corners and folding up the sides (Figure 26).

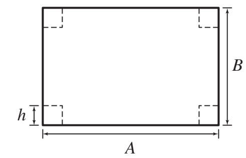


FIGURE 26

45. Find the value of h that maximizes the volume of the box if $A = 15$ and $B = 24$. What are the dimensions of this box?

SOLUTION Once the sides have been folded up, the base of the box will have dimensions $(A - 2h) \times (B - 2h)$ and the height of the box will be h. Thus

$$V(h) = h(A - 2h)(B - 2h) = 4h^3 - 2(A + B)h^2 + ABh.$$

When $A = 15$ and $B = 24$, this gives

$$V(h) = 4h^3 - 78h^2 + 360h,$$

and we need to maximize over $0 \le h \le \frac{15}{2}$. Now,

$$V'(h) = 12h^2 - 156h + 360 = 0$$

yields $h = 3$ and $h = 10$. Because $h = 10$ is not in the domain of the problem and $V(0) = V(15/2) = 0$ and $V(3) = 486$, volume is maximized when $h = 3$. The corresponding dimensions are $9 \times 18 \times 3$.

46. Vascular Branching A small blood vessel of radius r branches off at an angle θ from a larger vessel of radius R to supply blood along a path from A to B. According to Poiseuille's Law, the total resistance to blood flow is proportional to

$$T = \left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4} \right)$$

where a and b are as in Figure 25. Show that the total resistance is minimized when $\cos\theta = (r/R)^4$.

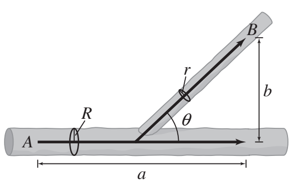


FIGURE 25

SOLUTION With $a, b, r, R > 0$ and $R > r$, let $T(\theta) = \left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4} \right)$. Set

$$T'(\theta) = \left(\frac{b\csc^2\theta}{R^4} - \frac{b\csc\theta\cot\theta}{r^4} \right) = 0.$$

Then

$$\frac{b\left(r^4 - R^4\cos\theta\right)}{R^4 r^4 \sin^2\theta} = 0,$$

so that $\cos\theta = \left(\frac{r}{R}\right)^4$. Since $\lim_{\theta\to 0+} T(\theta) = \infty$ and $\lim_{\theta\to\pi-} T(\theta) = \infty$, the minimum value of $T(\theta)$ occurs when $\cos\theta = \left(\frac{r}{R}\right)^4$.

47. Which values of A and B maximize the volume of the box if $h = 10$ cm and $AB = 900$ cm.

SOLUTION With $h = 10$ and $AB = 900$ (which means that $B = 900/A$), the volume of the box is

$$V(A) = 10(A - 20)\left(\frac{900}{A} - 20\right) = 13{,}000 - 200A - \frac{180{,}000}{A},$$

where $20 \le A \le 45$. Now, solving

$$V'(A) = -200 + \frac{180{,}000}{A^2} = 0$$

yields $A = 30$. Because $V(20) = V(45) = 0$ and $V(30) = 1000\ \text{cm}^3$, maximum volume is achieved with $A = B = 30$ cm.

48. Given n numbers $x_1, \dots, x_n$, find the value of x minimizing the sum of the squares:

$$(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2$$

First solve for $n = 2, 3$ and then try it for arbitrary n.

SOLUTION Note that the sum of squares approaches ∞ as $x \to \pm\infty$, so the minimum must occur at a critical point.

- For $n = 2$: Let $f(x) = (x - x_1)^2 + (x - x_2)^2$. Then setting $f'(x) = 2(x - x_1) + 2(x - x_2) = 0$ yields $x = \frac{1}{2}(x_1 + x_2)$.
- For $n = 3$: Let $f(x) = (x - x_1)^2 + (x - x_2)^2 + (x - x_3)^2$, so that setting $f'(x) = 2(x - x_1) + 2(x - x_2) + 2(x - x_3) = 0$ yields $x = \frac{1}{3}(x_1 + x_2 + x_3)$.
- Let $f(x) = \sum_{k=1}^{n}(x - x_k)^2$. Solve $f'(x) = 2\sum_{k=1}^{n}(x - x_k) = 0$ to obtain $x = \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$.

Note that the optimum value for x is the average of $x_1, \dots, x_n$.

49. A billboard of height b is mounted on the side of a building with its bottom edge at a distance h from the street as in Figure 27. At what distance x should an observer stand from the wall to maximize the angle of observation θ?

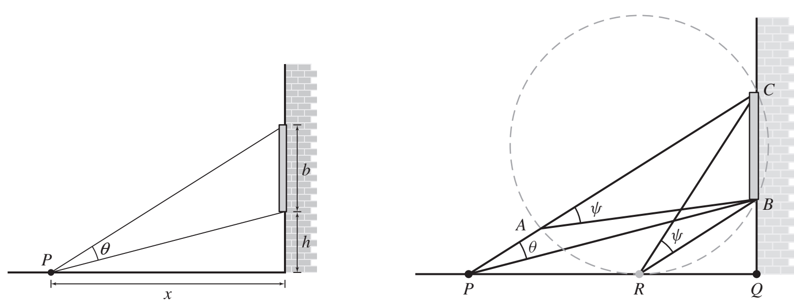


FIGURE 27

SOLUTION From the upper diagram in Figure 27 and the addition formula for the cotangent function, we see that

$$\cot\theta = \frac{1 + \frac{x}{b+h}\frac{x}{h}}{\frac{x}{h} - \frac{x}{b+h}} = \frac{x^2 + h(b+h)}{bx},$$

where b and h are constant. Now, differentiate with respect to x and solve

$$-\csc^2\theta\frac{d\theta}{dx} = \frac{x^2 - h(b+h)}{bx^2} = 0$$

to obtain $x = \sqrt{bh + h^2}$. Since this is the only critical point, and since $\theta \to 0$ as $x \to 0+$ and $\theta \to 0$ as $x \to \infty$, $\theta(x)$ reaches its maximum at $x = \sqrt{bh + h^2}$.

50. Solve Exercise 49 again using geometry rather than calculus. There is a unique circle passing through points B and C which is tangent to the street. Let R be the point of tangency. Note that the two angles labeled ψ in Figure 27 are equal because they subtend equal arcs on the circle.

(a) Show that the maximum value of θ is $\theta = \psi$. *Hint:* Show that $\psi = \theta + \angle PBA$ where A is the intersection of the circle with PC.

(b) Prove that this agrees with the answer to Exercise 49.

(c) Show that $\angle QRB = \angle RCQ$ for the maximal angle ψ.

SOLUTION

(a) We note that $\angle PAB$ is supplementary to both ψ and $\theta + \angle PBA$; hence, $\psi = \theta + \angle PBA$. From here, it is clear that θ is at a maximum when $\angle PBA = 0$; that is, when A coincides with P. This occurs when $P = R$.

(b) To show that the two answers agree, let O be the center of the circle. One observes that if d is the distance from R to the wall, then O has coordinates $(-d, \frac{b}{2} + h)$. This is because the height of the center is equidistant from points B and C and because the center must lie directly above R if the circle is tangent to the floor.

Now we can solve for d. The radius of the circle is clearly $\frac{b}{2} + h$, by the distance formula:

$$\overline{OB}^2 = d^2 + \left(\frac{b}{2} + h - h\right)^2 = \left(\frac{b}{2} + h\right)^2$$

This gives

$$d^2 = \left(\frac{b}{2} + h\right)^2 - \left(\frac{b}{2}\right)^2 = bh + h^2$$

or $d = \sqrt{bh + h^2}$ as claimed.

(c) Observe that the arc RB on the dashed circle is subtended by $\angle QRB$ and also by $\angle RCQ$. Thus, both are equal to one-half the angular measure of the arc.

51. Optimal Delivery Schedule A gas station sells Q gallons of gasoline per year, which is delivered N times per year in equal shipments of Q/N gallons. The cost of each delivery is d dollars and the yearly storage costs are sQT, where T is the length of time (a fraction of a year) between shipments and s is a constant. Show that costs are minimized for $N = \sqrt{sQ/d}$. (*Hint:* $T = 1/N$.) Find the optimal number of deliveries if $Q = 2$ million gal, $d = \$8000$, and $s = 30$ cents/gal-yr. Your answer should be a whole number, so compare costs for the two integer values of N nearest the optimal value.

SOLUTION There are N shipments per year, so the time interval between shipments is $T = 1/N$ years. Hence, the total storage costs per year are sQ/N. The yearly delivery costs are dN and the total costs is $C(N) = dN + sQ/N$. Solving,

$$C'(N) = d - \frac{sQ}{N^2} = 0$$

for N yields $N = \sqrt{sQ/d}$. For the specific case $Q = 2{,}000{,}000$, $d = 8000$ and $s = 0.30$,

$$N = \sqrt{\frac{0.30(2{,}000{,}000)}{8000}} = 8.66.$$

With $C(8) = \$139{,}000$ and $C(9) = \$138{,}667$, the optimal number of deliveries per year is $N = 9$.

52. Victor Klee's Endpoint Maximum Problem Given 40 meters of straight fence, your goal is to build a rectangular enclosure using 80 additional meters of fence that encompasses the greatest area. Let $A(x)$ be the area of the enclosure, with x as in Figure 28.

(a) Find the maximum value of $A(x)$.

(b) Which interval of x values is relevant to our problem? Find the maximum value of $A(x)$ on this interval.

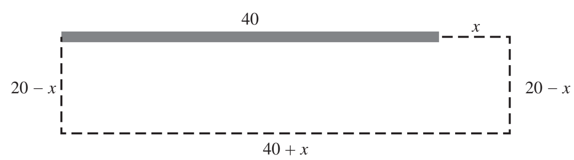


FIGURE 28

SOLUTION

(a) From the diagram, $A(x) = (40 + x)(20 - x) = 800 - 20x - x^2 = 900 - (x + 10)^2$. Thus, the maximum value of $A(x)$ is 900 square meters, occurring when $x = -10$.

(b) For our problem, $x \in [0, 20]$. On this interval, $A(x)$ has no critical points and $A(0) = 800$, while $A(20) = 0$. Thus, on the relevant interval, the maximum enclosed area is 800 square meters.

53. Let (a, b) be a fixed point in the first quadrant and let $S(d)$ be the sum of the distances from $(d, 0)$ to the points $(0, 0)$, (a, b), and $(a, -b)$.

(a) Find the value of d for which $S(d)$ is minimal. The answer depends on whether $b < \sqrt{3}a$ or $b \geq \sqrt{3}a$. *Hint:* Show that $d = 0$ when $b \geq \sqrt{3}a$.

(b) $\boxed{\text{GU}}$ Let $a = 1$. Plot $S(d)$ for $b = 0.5, \sqrt{3}, 3$ and describe the position of the minimum.

SOLUTION

(a) If $d < 0$, then the distance from $(d, 0)$ to the other three points can all be reduced by increasing the value of d. Similarly, if $d > a$, then the distance from $(d, 0)$ to the other three points can all be reduced by decreasing the value of d. It follows that the minimum of $S(d)$ must occur for $0 \leq d \leq a$. Restricting attention to this interval, we find

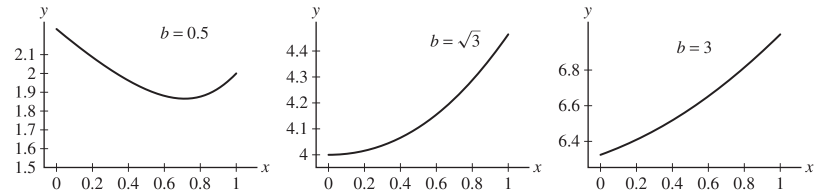
$$S(d) = d + 2\sqrt{(d - a)^2 + b^2}.$$

Solving

$$S'(d) = 1 + \frac{2(d-a)}{\sqrt{(d-a)^2 + b^2}} = 0$$

yields the critical point $d = a - b/\sqrt{3}$. If $b < \sqrt{3}a$, then $d = a - b/\sqrt{3} > 0$ and the minimum occurs at this value of d. On the other hand, if $b \geq \sqrt{3}a$, then the minimum occurs at the endpoint $d = 0$.

(b) Let $a = 1$. Plots of $S(d)$ for $b = 0.5$, $b = \sqrt{3}$ and $b = 3$ are shown below. For $b = 0.5$, the results of (a) indicate the minimum should occur for $d = 1 - 0.5/\sqrt{3} \approx 0.711$, and this is confirmed in the plot. For both $b = \sqrt{3}$ and $b = 3$, the results of (a) indicate that the minimum should occur at $d = 0$, and both of these conclusions are confirmed in the plots.



54. The force F (in Newtons) required to move a box of mass m kg in motion by pulling on an attached rope (Figure 29) is

$$F(\theta) = \frac{fmg}{\cos\theta + f\sin\theta}$$

where θ is the angle between the rope and the horizontal, f is the coefficient of static friction, and $g = 9.8$ m/s^2. Find the angle θ that minimizes the required force F, assuming $f = 0.4$. *Hint:* Find the maximum value of $\cos\theta + f\sin\theta$.

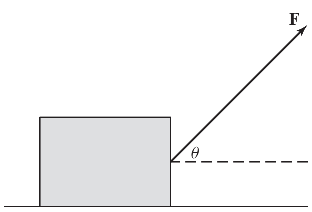


FIGURE 29

SOLUTION Let $F(\alpha) = \dfrac{3.92m}{\sin\alpha + \frac{2}{5}\cos\alpha}$, where $0 \leq \alpha \leq \frac{\pi}{2}$. Solve

$$F'(\alpha) = \frac{3.92m\left(\frac{2}{5}\sin\alpha - \cos\alpha\right)}{\left(\sin\alpha + \frac{2}{5}\cos\alpha\right)^2} = 0$$

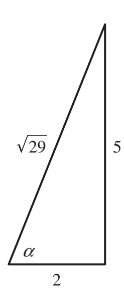
for $0 \leq \alpha \leq \frac{\pi}{2}$ to obtain $\tan\alpha = \frac{5}{2}$. From the diagram below, we note that when $\tan\alpha = \frac{5}{2}$,

$$\sin\alpha = \frac{5}{\sqrt{29}} \qquad \text{and} \qquad \cos\alpha = \frac{2}{\sqrt{29}}.$$

Therefore, at the critical point the force is

$$\frac{3.92m}{\frac{5}{\sqrt{29}} + \frac{2}{5}\frac{2}{\sqrt{29}}} = \frac{5\sqrt{29}}{29}(3.92m) \approx 3.64m.$$

Since $F(0) = \frac{5}{2}(3.92m) = 9.8m$ and $F\left(\frac{\pi}{2}\right) = 3.92m$, we conclude that the minimum force occurs when $\tan\alpha = \frac{5}{2}$.



55. In the setting of Exercise 54, show that for any f the minimal force required is proportional to $1/\sqrt{1+f^2}$.

SOLUTION We minimize $F(\theta)$ by finding the maximum value $g(\theta) = \cos\theta + f\sin\theta$. The angle θ is restricted to the interval $[0, \frac{\pi}{2}]$. We solve for the critical points:

$$g'(\theta) = -\sin\theta + f\cos\theta = 0$$

We obtain

$$f\cos\theta = \sin\theta \Rightarrow \tan\theta = f$$

From the figure below we find that $\cos\theta = 1/\sqrt{1+f^2}$ and $\sin\theta = f/\sqrt{1+f^2}$. Hence

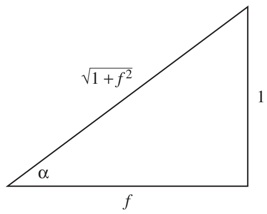
$$g(\theta) = \frac{1}{f} + \frac{f^2}{\sqrt{1+f^2}} = \frac{1+f^2}{\sqrt{1+f^2}} = \sqrt{1+f^2}$$

The values at the endpoints are

$$g(0) = 1, \qquad g\left(\frac{\pi}{2}\right) = f$$

Both of these values are less than $\sqrt{1+f^2}$. Therefore the maximum value of $g(\theta)$ is $\sqrt{1+f^2}$ and the minimum value of $F(\theta)$ is

$$F = \frac{fmg}{g(\theta)} = \frac{fmg}{\sqrt{1+f^2}}$$



56. Bird Migration Ornithologists have found that the power (in joules per second) consumed by a certain pigeon flying at velocity v m/s is described well by the function $P(v) = 17v^{-1} + 10^{-3}v^3$ J/s. Assume that the pigeon can store 5×10^4 J of usable energy as body fat.

(a) Show that at velocity v, a pigeon can fly a total distance of $D(v) = (5 \times 10^4)v/P(v)$ if it uses all of its stored energy.

(b) Find the velocity v_p that *minimizes* $P(v)$.

(c) Migrating birds are smart enough to fly at the velocity that maximizes distance traveled rather than minimizes power consumption. Show that the velocity v_d which maximizes $D(v)$ satisfies $P'(v_d) = P(v_d)/v_d$. Show that v_d is obtained graphically as the velocity coordinate of the point where a line through the origin is tangent to the graph of $P(v)$ (Figure 30).

(d) Find v_d and the maximum distance $D(v_d)$.

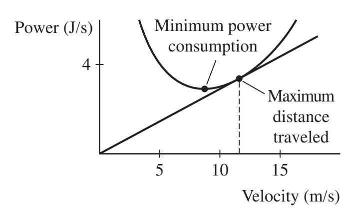


FIGURE 30

SOLUTION

(a) Flying at a velocity v, the birds will exhaust their energy store after $T = \dfrac{5 \cdot 10^4 \text{ joules}}{P(v) \text{ joules/sec}} = \dfrac{5 \cdot 10^4 \text{ sec}}{P(v)}$. The total distance traveled is then $D(v) = vT = \dfrac{5 \cdot 10^4 v}{P(v)}$.

(b) Let $P(v) = 17v^{-1} + 10^{-3}v^3$. Then $P'(v) = -17v^{-2} + 0.003v^2 = 0$ implies $v_p = \left(\frac{17}{0.003}\right)^{1/4} \approx 8.676247$. This critical point is a minimum, because it is the only critical point and $P(v) \to \infty$ both as $v \to 0+$ and as $v \to \infty$.

(c) $D'(v) = \dfrac{P(v) \cdot 5 \cdot 10^4 - 5 \cdot 10^4 v \cdot P'(v)}{(P(v))^2} = 5 \cdot 10^4 \dfrac{P(v) - vP'(v)}{(P(v))^2} = 0$ implies $P(v) - vP'(v) = 0$, or $P'(v) = \dfrac{P(v)}{v}$. Since $D(v) \to 0$ as $v \to 0$ and as $v \to \infty$, the critical point determined by $P'(v) = P(v)/v$ corresponds to a maximum.

Graphically, the expression

$$\frac{P(v)}{v} = \frac{P(v) - 0}{v - 0}$$

is the slope of the line passing through the origin and $(v, P(v))$. The condition $P'(v) = P(v)/v$ which defines v_d therefore indicates that v_d is the velocity component of the point where a line through the origin is tangent to the graph of $P(v)$.

(d) Using $P'(v) = \dfrac{P(v)}{v}$ gives

$$-17v^{-2} + 0.003v^2 = \frac{17v^{-1} + 0.001v^3}{v} = 17v^{-2} + 0.001v^2,$$

which simplifies to $0.002v^4 = 34$ and thus $v_d \approx 11.418583$. The maximum total distance is given by $D(v_d) = \dfrac{5 \cdot 10^4 \cdot v_d}{P(v_d)} = 191.741$ kilometers.

57. The problem is to put a "roof" of side s on an attic room of height h and width b. Find the smallest length s for which this is possible if $b = 27$ and $h = 8$ (Figure 31).

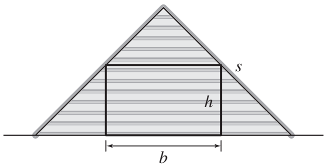
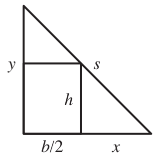


FIGURE 31

SOLUTION Consider the right triangle formed by the right half of the rectangle and its "roof". This triangle has hypotenuse s.



As shown, let y be the height of the roof, and let x be the distance from the right base of the rectangle to the base of the roof. By similar triangles applied to the smaller right triangles at the top and right of the larger triangle, we get:

$$\frac{y-8}{27/2} = \frac{8}{x} \qquad \text{or} \qquad y = \frac{108}{x} + 8.$$

s, y, and x are related by the Pythagorean Theorem:

$$s^2 = \left(\frac{27}{2} + x\right)^2 + y^2 = \left(\frac{27}{2} + x\right)^2 + \left(\frac{108}{x} + 8\right)^2.$$

Since $s > 0$, s^2 is least whenever s is least, so we can minimize s^2 instead of s. Setting the derivative equal to zero yields

$$2\left(\frac{27}{2} + x\right) + 2\left(\frac{108}{x} + 8\right)\left(-\frac{108}{x^2}\right) = 0$$

$$2\left(\frac{27}{2} + x\right) + 2\frac{8}{x}\left(\frac{27}{2} + x\right)\left(-\frac{108}{x^2}\right) = 0$$

$$2\left(\frac{27}{2} + x\right)\left(1 - \frac{864}{x^3}\right) = 0$$

The zeros are $x = -\frac{27}{2}$ (irrelevant) and $x = 6\sqrt[3]{4}$. Since this is the only critical point of s with $x > 0$, and since $s \to \infty$ as $x \to 0$ and $s \to \infty$ as $x \to \infty$, this is the point where s attains its minimum. For this value of x,

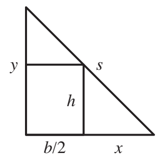
$$s^2 = \left(\frac{27}{2} + 6\sqrt[3]{4}\right)^2 + \left(9\sqrt[3]{2} + 8\right)^2 \approx 904.13,$$

so the smallest roof length is

$$s \approx 30.07.$$

58. Redo Exercise 57 for arbitrary b and h.

SOLUTION Consider the right triangle formed by the right half of the rectangle and its "roof". This triangle has hypotenuse s.



As shown, let y be the height of the roof, and let x be the distance from the right base of the rectangle to the base of the roof. By similar triangles applied to the smaller right triangles at the top and right of the larger triangle, we get:

$$\frac{y-h}{b/2} = \frac{h}{x} \qquad \text{or} \qquad y = \frac{bh}{2x} + h.$$

s, y, and x are related by the Pythagorean Theorem:

$$s^2 = \left(\frac{b}{2} + x\right)^2 + y^2 = \left(\frac{b}{2} + x\right)^2 + \left(\frac{bh}{2x} + h\right)^2.$$

Since $s > 0$, s^2 is least whenever s is least, so we can minimize s^2 instead of s. Setting the derivative equal to zero yields

$$\begin{aligned} 2\left(\frac{b}{2} + x\right) + 2\left(\frac{bh}{2x} + h\right)\left(-\frac{bh}{2x^2}\right) &= 0 \\ 2\left(\frac{b}{2} + x\right) + 2\frac{h}{x}\left(\frac{b}{2} + x\right)\left(-\frac{bh}{2x^2}\right) &= 0 \\ 2\left(\frac{b}{2} + x\right)\left(1 - \frac{bh^2}{2x^3}\right) &= 0 \end{aligned}$$

The zeros are $x = -\frac{b}{2}$ (irrelevant) and

$$x = \frac{b^{1/3}h^{2/3}}{2^{1/3}}.$$

Since this is the only critical point of s with $x > 0$, and since $s \to \infty$ as $x \to 0$ and $s \to \infty$ as $x \to \infty$, this is the point where s attains its minimum. For this value of x,

$$\begin{aligned} s^2 &= \left(\frac{b}{2} + \frac{b^{1/3}h^{2/3}}{2^{1/3}}\right)^2 + \left(\frac{b^{2/3}h^{1/3}}{2^{2/3}} + h\right)^2 \\ &= \frac{b^{2/3}}{2^{2/3}}\left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^2 + h^{2/3}\left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^2 = \left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^3, \end{aligned}$$

so the smallest roof length is

$$s = \left(\frac{b^{2/3}}{2^{2/3}} + h^{2/3}\right)^{3/2}.$$

59. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths $a = 24$ and $b = 3$ (Figure 32).

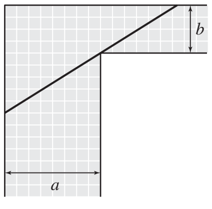


FIGURE 32

SOLUTION In order to find the length of the *longest* pole that can be carried around the corridor, we have to find the *shortest* length from the left wall to the top wall touching the corner of the inside wall. Any pole that does not fit in this shortest space cannot be carried around the corner, so an exact fit represents the longest possible pole.

Let θ be the angle between the pole and a horizontal line to the right. Let c_1 be the length of pole in the corridor of width 24 and let c_2 be the length of pole in the corridor of width 3. By the definitions of sine and cosine,

$$\frac{3}{c_2} = \sin\theta \quad \text{and} \quad \frac{24}{c_1} = \cos\theta,$$

so that $c_1 = \frac{24}{\cos\theta}$, $c_2 = \frac{3}{\sin\theta}$. What must be minimized is the total length, given by

$$f(\theta) = \frac{24}{\cos\theta} + \frac{3}{\sin\theta}.$$

Setting $f'(\theta) = 0$ yields

$$\begin{aligned}\frac{24\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta} &= 0\\ \frac{24\sin\theta}{\cos^2\theta} &= \frac{3\cos\theta}{\sin^2\theta}\\ 24\sin^3\theta &= 3\cos^3\theta\end{aligned}$$

As $\theta < \frac{\pi}{2}$ (the pole is being turned around a corner, after all), we can divide both sides by $\cos^3\theta$, getting $\tan^3\theta = \frac{1}{8}$. This implies that $\tan\theta = \frac{1}{2}$ ($\tan\theta > 0$ as the angle is acute).

Since $f(\theta) \to \infty$ as $\theta \to 0+$ and as $\theta \to \frac{\pi}{2}-$, we can tell that the *minimum* is attained at θ_0 where $\tan\theta_0 = \frac{1}{2}$. Because

$$\tan\theta_0 = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{2},$$

we draw a triangle with opposite side 1 and adjacent side 2. By Pythagoras, $c = \sqrt{5}$, so

$$\sin\theta_0 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\theta_0 = \frac{2}{\sqrt{5}}.$$

From this, we get

$$f(\theta_0) = \frac{24}{\cos\theta_0} + \frac{3}{\sin\theta_0} = \frac{24}{2}\sqrt{5} + 3\sqrt{5} = 15\sqrt{5}.$$

60. Redo Exercise 59 for arbitrary widths a and b.

SOLUTION If the corridors have widths a and b, and if θ is the angle between the beam and the line perpendicular to the corridor of width a, then we have to *minimize*

$$f(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}.$$

Setting the derivative equal to zero,

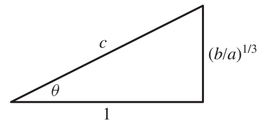
$$a\sec\theta\tan\theta - b\cot\theta\csc\theta = 0,$$

we obtain the critical value θ_0 defined by

$$\tan\theta_0 = \left(\frac{b}{a}\right)^{1/3}$$

and from this we conclude (witness the diagram below) that

$$\cos\theta_0 = \frac{1}{\sqrt{1 + (b/a)^{2/3}}} \quad \text{and} \quad \sin\theta_0 = \frac{(b/a)^{1/3}}{\sqrt{1 + (b/a)^{2/3}}}.$$



This gives the minimum value as

$$\begin{aligned} f(\theta_0) &= a\sqrt{1+(b/a)^{2/3}} + b(b/a)^{-1/3}\sqrt{1+(b/a)^{2/3}} \\ &= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}} \\ &= (a^{2/3}+b^{2/3})^{3/2} \end{aligned}$$

61. Find the minimum length ℓ of a beam that can clear a fence of height h and touch a wall located b ft behind the fence (Figure 33).

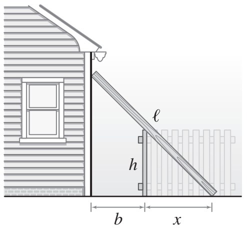


FIGURE 33

SOLUTION Let y be the height of the point where the beam touches the wall in feet. By similar triangles,

$$\frac{y-h}{b} = \frac{h}{x} \quad \text{or} \quad y = \frac{bh}{x} + h$$

and by Pythagoras:

$$\ell^2 = (b+x)^2 + \left(\frac{bh}{x} + h\right)^2.$$

We can minimize ℓ^2 rather than ℓ, so setting the derivative equal to zero gives:

$$2(b+x) + 2\left(\frac{bh}{x}+h\right)\left(-\frac{bh}{x^2}\right) = 2(b+x)\left(1 - \frac{h^2b}{x^3}\right) = 0.$$

The zeroes are $b = -x$ (irrelevant) and $x = \sqrt[3]{h^2b}$. Since $\ell^2 \to \infty$ as $x \to 0+$ and as $x \to \infty$, $x = \sqrt[3]{h^2b}$ corresponds to a minimum for ℓ^2. For this value of x, we have

$$\begin{aligned} \ell^2 &= (b + h^{2/3}b^{1/3})^2 + (h + h^{1/3}b^{2/3})^2 \\ &= b^{2/3}(b^{2/3}+h^{2/3})^2 + h^{2/3}(h^{2/3}+b^{2/3})^2 \\ &= (b^{2/3}+h^{2/3})^3 \end{aligned}$$

and so

$$\ell = (b^{2/3}+h^{2/3})^{3/2}.$$

A beam that clears a fence of height h feet and touches a wall b feet behind the fence must have length at least $\ell = (b^{2/3}+h^{2/3})^{3/2}$ ft.

62. Which value of h maximizes the volume of the box if $A = B$?

SOLUTION When $A = B$, the volume of the box is

$$V(h) = hxy = h(A-2h)^2 = 4h^3 - 4Ah^2 + A^2h,$$

where $0 \le h \le \frac{A}{2}$ (allowing for degenerate boxes). Solve $V'(h) = 12h^2 - 8Ah + A^2 = 0$ for h to obtain $h = \frac{A}{2}$ or $h = \frac{A}{6}$. Because $V(0) = V(\frac{A}{2}) = 0$ and $V(\frac{A}{6}) = \frac{2}{27}A^3$, volume is maximized when $h = \frac{A}{6}$.

63. A basketball player stands d feet from the basket. Let h and α be as in Figure 34. Using physics, one can show that if the player releases the ball at an angle θ, then the initial velocity required to make the ball go through the basket satisfies

$$v^2 = \frac{16d}{\cos^2\theta(\tan\theta - \tan\alpha)}$$

(a) Explain why this formula is meaningful only for $\alpha < \theta < \frac{\pi}{2}$. Why does v approach infinity at the endpoints of this interval?

(b) GU Take $\alpha = \frac{\pi}{6}$ and plot v^2 as a function of θ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Verify that the minimum occurs at $\theta = \frac{\pi}{3}$.

(c) Set $F(\theta) = \cos^2\theta(\tan\theta - \tan\alpha)$. Explain why v is minimized for θ such that $F(\theta)$ is maximized.

(d) Verify that $F'(\theta) = \cos(\alpha - 2\theta)\sec\alpha$ (you will need to use the addition formula for cosine) and show that the maximum value of $F(\theta)$ on $\left[\alpha, \frac{\pi}{2}\right]$ occurs at $\theta_0 = \frac{\alpha}{2} + \frac{\pi}{4}$.

(e) For a given α, the optimal angle for shooting the basket is θ_0 because it minimizes v^2 and therefore minimizes the energy required to make the shot (energy is proportional to v^2). Show that the velocity v_{opt} at the optimal angle θ_0 satisfies

$$v_{\text{opt}}^2 = \frac{32d\cos\alpha}{1 - \sin\alpha} = \frac{32\,d^2}{-h + \sqrt{d^2 + h^2}}$$

(f) GU Show with a graph that for fixed d (say, $d = 15$ ft, the distance of a free throw), v_{opt}^2 is an increasing function of h. Use this to explain why taller players have an advantage and why it can help to jump while shooting.

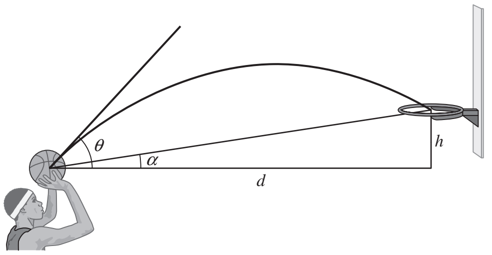


FIGURE 34

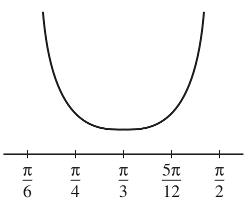
SOLUTION

(a) $\alpha = 0$ corresponds to shooting the ball directly at the basket while $\alpha = \pi/2$ corresponds to shooting the ball directly upward. In neither case is it possible for the ball to go into the basket.

If the angle α is extremely close to 0, the ball is shot almost directly at the basket, so that it must be launched with great speed, as it can only fall an extremely short distance on the way to the basket.

On the other hand, if the angle α is extremely close to $\pi/2$, the ball is launched almost vertically. This requires the ball to travel a great distance upward in order to travel the horizontal distance. In either one of these cases, the ball has to travel at an enormous speed.

(b)



The minimum clearly occurs where $\theta = \pi/3$.

(c) If $F(\theta) = \cos^2\theta\,(\tan\theta - \tan\alpha)$,

$$v^2 = \frac{16d}{\cos^2\theta\,(\tan\theta - \tan\alpha)} = \frac{16d}{F(\theta)}.$$

Since $\alpha \le \theta$, $F(\theta) > 0$, hence v^2 is smallest whenever $F(\theta)$ is greatest.

(d) $F'(\theta) = -2\sin\theta\cos\theta\,(\tan\theta - \tan\alpha) + \cos^2\theta\left(\sec^2\theta\right) = -2\sin\theta\cos\theta\tan\theta + 2\sin\theta\cos\theta\tan\alpha + 1$. We will apply all the double angle formulas:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta;\ \sin 2\theta = 2\sin\theta\cos\theta,$$

getting:

$$\begin{aligned}
F'(\theta) &= 2\sin\theta\cos\theta\tan\alpha - 2\sin\theta\cos\theta\tan\theta + 1\\
&= 2\sin\theta\cos\theta\frac{\sin\alpha}{\cos\alpha} - 2\sin\theta\cos\theta\frac{\sin\theta}{\cos\theta} + 1\\
&= \sec\alpha\left(-2\sin^2\theta\cos\alpha + 2\sin\theta\cos\theta\sin\alpha + \cos\alpha\right)\\
&= \sec\alpha\left(\cos\alpha\left(1 - 2\sin^2\theta\right) + \sin\alpha\,(2\sin\theta\cos\theta)\right)\\
&= \sec\alpha\,(\cos\alpha(\cos 2\theta) + \sin\alpha(\sin 2\theta))\\
&= \sec\alpha\cos(\alpha - 2\theta)
\end{aligned}$$

A critical point of $F(\theta)$ occurs where $\cos(\alpha - 2\theta) = 0$, so that $\alpha - 2\theta = -\frac{\pi}{2}$ (negative because $2\theta > \theta > \alpha$), and this gives us $\theta = \alpha/2 + \pi/4$. The minimum value $F(\theta_0)$ takes place at $\theta_0 = \alpha/2 + \pi/4$.

(e) Plug in $\theta_0 = \alpha/2 + \pi/4$. To find v_{opt}^2 we must simplify

$$\cos^2\theta_0(\tan\theta_0 - \tan\alpha) = \frac{\cos\theta_0(\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha)}{\cos\alpha}$$

By the addition law for sine:

$$\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha = \sin(\theta_0 - \alpha) = \sin(-\alpha/2 + \pi/4)$$

and so

$$\cos\theta_0(\sin\theta_0\cos\alpha - \cos\theta_0\sin\alpha) = \cos(\alpha/2 + \pi/4)\sin(-\alpha/2 + \pi/4)$$

Now use the identity (that follows from the addition law):

$$\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

to get

$$\cos(\alpha/2 + \pi/4)\sin(-\alpha/2 + \pi/4) = (1/2)(1 - \sin\alpha)$$

So we finally get

$$\cos^2\theta_0(\tan\theta_0 - \tan\alpha) = \frac{(1/2)(1 - \sin\alpha)}{\cos\alpha}$$

and therefore

$$v_{\text{opt}}^2 = \frac{32d\cos\alpha}{1 - \sin\alpha}$$

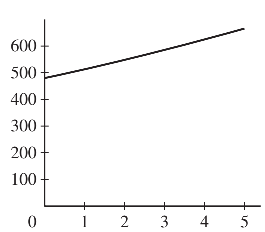
as claimed. From Figure 34 we see that

$$\cos\alpha = \frac{d}{\sqrt{d^2 + h^2}} \qquad \text{and} \qquad \sin\alpha = \frac{h}{\sqrt{d^2 + h^2}}.$$

Substituting these values into the expression for v_{opt}^2 yields

$$v_{\text{opt}}^2 = \frac{32d^2}{-h + \sqrt{d^2 + h^2}}.$$

(f) A sketch of the graph of v_{opt}^2 versus h for $d = 15$ feet is given below: v_{opt}^2 increases with respect to basket height relative to the shooter. This shows that the minimum velocity required to launch the ball to the basket drops as shooter height increases. This shows one of the ways height is an advantage in free throws; a taller shooter need not shoot the ball as hard to reach the basket.



64. Three towns A, B, and C are to be joined by an underground fiber cable as illustrated in Figure 35(A). Assume that C is located directly below the midpoint of $\overline{AB}$. Find the junction point P that minimizes the total amount of cable used.

(a) First show that P must lie directly above C. *Hint:* Use the result of Example 6 to show that if the junction is placed at point Q in Figure 35(B), then we can reduce the cable length by moving Q horizontally over to the point P lying above C.

(b) With x as in Figure 35(A), let $f(x)$ be the total length of cable used. Show that $f(x)$ has a unique critical point c. Compute c and show that $0 \le c \le L$ if and only if $D \le 2\sqrt{3}\,L$.

(c) Find the minimum of $f(x)$ on $[0, L]$ in two cases: $D = 2$, $L = 4$ and $D = 8$, $L = 2$.

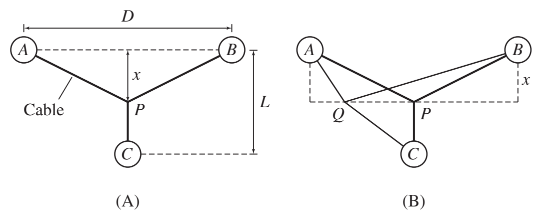


FIGURE 35

SOLUTION

(a) Look at diagram 35(B). Let T be the point directly above Q on $\overline{AB}$. Let $s = AT$ and $D = AB$ so that $TB = D - s$. Let ℓ be the total length of cable from A to Q and B to Q. By the Pythagorean Theorem applied to $\triangle AQT$ and $\triangle BQT$, we get:

$$\ell = \sqrt{s^2 + x^2} + \sqrt{(D-s)^2 + x^2}.$$

From here, it follows that

$$\frac{d\ell}{ds} = \frac{s}{\sqrt{s^2 + x^2}} - \frac{D-s}{\sqrt{(D-s)^2 + x^2}}.$$

Since s and $D - s$ must be non-negative, the only critical point occurs when $s = D/2$. As $\frac{d\ell}{ds}$ changes sign from negative to positive at $s = D/2$, it follows that ℓ is minimized when $s = D/2$, that is, when $Q = P$. Since it is obvious that $PC \le QC$ (QC is the hypotenuse of the triangle $\triangle PQC$), it follows that total cable length is minimized at $Q = P$.

(b) Let $f(x)$ be the total cable length. From diagram 35(A), we get:

$$f(x) = (L - x) + 2\sqrt{x^2 + D^2/4}.$$

Then

$$f'(x) = -1 + \frac{2x}{\sqrt{x^2 + D^2/4}} = 0$$

gives

$$2x = \sqrt{x^2 + D^2/4}$$

or

$$4x^2 = x^2 + D^2/4$$

and the critical point is

$$c = D/2\sqrt{3}.$$

This is the only critical point of f. It lies in the interval $[0, L]$ if and only if $c \le L$, or

$$D \le 2\sqrt{3}L.$$

(c) The minimum of f will depend on whether $D \le 2\sqrt{3}L$.

- $D = 2$, $L = 4$; $2\sqrt{3}L = 8\sqrt{3} > D$, so $c = D/(2\sqrt{3}) = \sqrt{3}/3 \in [0, L]$. $f(0) = L + D = 6$, $f(L) = 2\sqrt{L^2 + D^2/4} = 2\sqrt{17} \approx 8.24621$, and $f(c) = 4 - (\sqrt{3}/3) + 2\sqrt{\frac{1}{3} + 1} = 4 + \sqrt{3} \approx 5.73204$. Therefore, the total length is minimized where $x = c = \sqrt{3}/3$.
- $D = 8$, $L = 2$; $2\sqrt{3}L = 4\sqrt{3} < D$, so c does not lie in the interval $[0, L]$. $f(0) = 2 + 2\sqrt{64/4} = 10$, and $f(L) = 0 + 2\sqrt{4 + 64/4} = 2\sqrt{20} = 4\sqrt{5} \approx 8.94427$. Therefore, the total length is minimized where $x = L$, or where $P = C$.

Further Insights and Challenges

65. Tom and Ali drive along a highway represented by the graph of $f(x)$ in Figure 36. During the trip, Ali views a billboard represented by the segment $\overline{BC}$ along the y-axis. Let Q be the y-intercept of the tangent line to $y = f(x)$. Show that θ is maximized at the value of x for which the angles $\angle QPB$ and $\angle QCP$ are equal. This generalizes Exercise 50 (c) (which corresponds to the case $f(x) = 0$). *Hints:*

(a) Show that $d\theta/dx$ is equal to

$$(b-c)\cdot\frac{(x^2+(xf'(x))^2)-(b-(f(x)-xf'(x)))(c-(f(x)-xf'(x)))}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}$$

(b) Show that the y-coordinate of Q is $f(x) - xf'(x)$.

(c) Show that the condition $d\theta/dx = 0$ is equivalent to

$$PQ^2 = BQ\cdot CQ$$

(d) Conclude that $\triangle QPB$ and $\triangle QCP$ are similar triangles.

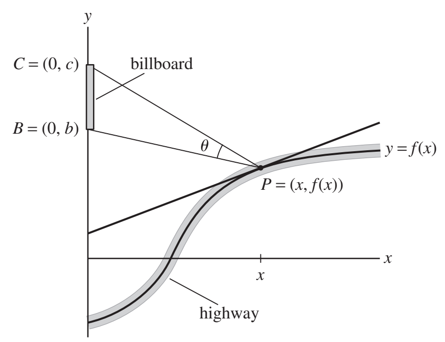


FIGURE 36

SOLUTION

(a) From the figure, we see that

$$\theta(x) = \tan^{-1}\frac{c-f(x)}{x} - \tan^{-1}\frac{b-f(x)}{x}.$$

Then

$$\begin{aligned}\theta'(x) &= \frac{b-(f(x)-xf'(x))}{x^2+(b-f(x))^2} - \frac{c-(f(x)-xf'(x))}{x^2+(c-f(x))^2}\\ &= (b-c)\frac{x^2-bc+(b+c)(f(x)-xf'(x))-(f(x))^2+2xf(x)f'(x)}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}\\ &= (b-c)\frac{(x^2+(xf'(x))^2-(bc-(b+c)(f(x)-xf'(x))+(f(x)-xf'(x))^2)}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}\\ &= (b-c)\frac{(x^2+(xf'(x))^2-(b-(f(x)-xf'(x)))(c-(f(x)-xf'(x)))}{(x^2+(b-f(x))^2)(x^2+(c-f(x))^2)}.\end{aligned}$$

(b) The point Q is the y-intercept of the line tangent to the graph of $f(x)$ at point P. The equation of this tangent line is

$$Y - f(x) = f'(x)(X - x).$$

The y-coordinate of Q is then $f(x) - xf'(x)$.

(c) From the figure, we see that

$$BQ = b-(f(x)-xf'(x)),$$
$$CQ = c-(f(x)-xf'(x))$$

and

$$PQ = \sqrt{x^2+(f(x)-(f(x)-xf'(x)))^2} = \sqrt{x^2+(xf'(x))^2}.$$

Comparing these expressions with the numerator of $d\theta/dx$, it follows that $\dfrac{d\theta}{dx} = 0$ is equivalent to

$$PQ^2 = BQ\cdot CQ.$$

(d) The equation $PQ^2 = BQ \cdot CQ$ is equivalent to

$$\frac{PQ}{BQ} = \frac{CQ}{PQ}.$$

In other words, the sides CQ and PQ from the triangle ΔQCP are proportional in length to the sides PQ and BQ from the triangle ΔQPB. As $\angle PQB = \angle CQP$, it follows that triangles ΔQCP and ΔQPB are similar.

Seismic Prospecting *Exercises 66–68 are concerned with determining the thickness d of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point A to point D separated by a distance s. The first pulse travels directly from A to D along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to D (path ABCD), as in Figure 37. The pulse travels with velocity v_1 in the soil and v_2 in the rock.*

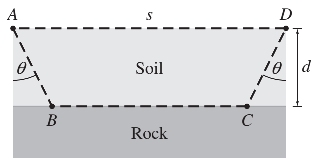


FIGURE 37

66. **(a)** Show that the time required for the first pulse to travel from A to D is $t_1 = s/v_1$.

(b) Show that the time required for the second pulse is

$$t_2 = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

provided that

$$\tan\theta \le \frac{s}{2d} \qquad \boxed{2}$$

(*Note:* If this inequality is not satisfied, then point B does not lie to the left of C.)

(c) Show that t_2 is minimized when $\sin\theta = v_1/v_2$.

SOLUTION

(a) We have time $t_1 = \text{distance/velocity} = s/v_1$.

(b) Let p be the length of the base of the right triangle (opposite the angle θ) and h the length of the hypotenuse of this right triangle. Then $\cos\theta = \frac{d}{h}$ and $h = d\sec\theta$. Moreover, $\tan\theta = \frac{p}{d}$ gives $p = d\tan\theta$. Hence

$$t_2 = t_{AB} + t_{CD} + t_{BC} = \frac{h}{v_1} + \frac{h}{v_1} + \frac{s-2p}{v_2} = \frac{2d}{v_1}\sec\theta + \frac{s-2d\tan\theta}{v_2}$$

(c) Solve $\dfrac{dt_2}{d\theta} = \dfrac{2d\sec\theta\tan\theta}{v_1} - \dfrac{2d\sec^2\theta}{v_2} = 0$ to obtain $\dfrac{\tan\theta}{v_1} = \dfrac{\sec\theta}{v_2}$. Therefore $\dfrac{\sin\theta/\cos\theta}{1/\cos\theta} = \dfrac{v_1}{v_2}$ or $\sin\theta = \dfrac{v_1}{v_2}$.

67. In this exercise, assume that $v_2/v_1 \ge \sqrt{1 + 4(d/s)^2}$.

(a) Show that inequality (2) holds if $\sin\theta = v_1/v_2$.

(b) Show that the minimal time for the second pulse is

$$t_2 = \frac{2d}{v_1}(1-k^2)^{1/2} + \frac{s}{v_2}$$

where $k = v_1/v_2$.

(c) Conclude that $\dfrac{t_2}{t_1} = \dfrac{2d(1-k^2)^{1/2}}{s} + k$.

SOLUTION

(a) If $\sin\theta = \frac{v_1}{v_2}$, then

$$\tan\theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}} = \frac{1}{\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1}}.$$

Because $\frac{v_2}{v_1} \ge \sqrt{1 + 4(\frac{d}{s})^2}$, it follows that

$$\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1} \ge \sqrt{1 + 4\left(\frac{d}{s}\right)^2 - 1} = \frac{2d}{s}.$$

Hence, $\tan\theta \le \frac{s}{2d}$ as required.

(b) For the time-minimizing choice of θ, we have $\sin\theta = \frac{v_1}{v_2}$ from which $\sec\theta = \frac{v_2}{\sqrt{v_2^2 - v_1^2}}$ and $\tan\theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}}$.

Thus

$$\begin{aligned} t_2 &= \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2} = \frac{2d}{v_1}\frac{v_2}{\sqrt{v_2^2 - v_1^2}} + \frac{s - 2d\frac{v_1}{\sqrt{v_2^2 - v_1^2}}}{v_2} \\ &= \frac{2d}{v_1}\left(\frac{v_2}{\sqrt{v_2^2 - v_1^2}} - \frac{v_1^2}{v_2\sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2} \\ &= \frac{2d}{v_1}\left(\frac{v_2^2 - v_1^2}{v_2\sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2} = \frac{2d}{v_1}\left(\frac{\sqrt{v_2^2 - v_1^2}}{\sqrt{v_2^2}}\right) + \frac{s}{v_2} \\ &= \frac{2d}{v_1}\sqrt{1 - \left(\frac{v_1}{v_2}\right)^2} + \frac{s}{v_2} = \frac{2d\left(1 - k^2\right)^{1/2}}{v_1} + \frac{s}{v_2}. \end{aligned}$$

(c) Recall that $t_1 = \frac{s}{v_1}$. We therefore have

$$\begin{aligned} \frac{t_2}{t_1} &= \frac{\frac{2d\left(1-k^2\right)^{1/2}}{v_1} + \frac{s}{v_2}}{\frac{s}{v_1}} \\ &= \frac{2d\left(1 - k^2\right)^{1/2}}{s} + \frac{v_1}{v_2} = \frac{2d\left(1 - k^2\right)^{1/2}}{s} + k. \end{aligned}$$

68. Continue with the assumption of the previous exercise.

(a) Find the thickness of the soil layer, assuming that $v_1 = 0.7v_2$, $t_2/t_1 = 1.3$, and $s = 400$ m.

(b) The times t_1 and t_2 are measured experimentally. The equation in Exercise 67(c) shows that t_2/t_1 is a linear function of $1/s$. What might you conclude if experiments were formed for several values of s and the points $(1/s, t_2/t_1)$ did *not* lie on a straight line?

SOLUTION

(a) Substituting $k = v_1/v_2 = 0.7$, $t_2/t_1 = 1.3$, and $s = 400$ into the equation for t_2/t_1 in Exercise 67(c) gives $1.3 = \frac{2d\sqrt{1 - (0.7)^2}}{400} + 0.7$. Solving for d yields $d \approx 168.03$ m.

(b) If several experiments for different values of s showed that plots of the points $\left(\frac{1}{s}, \frac{t_2}{t_1}\right)$ did *not* lie on a straight line, then we would suspect that $\frac{t_2}{t_1}$ is *not* a linear function of $\frac{1}{s}$ and that a different model is required.

69. In this exercise we use Figure 38 to prove Heron's principle of Example 6 without calculus. By definition, C is the reflection of B across the line $\overline{MN}$ (so that $\overline{BC}$ is perpendicular to $\overline{MN}$ and $BN = CN$. Let P be the intersection of $\overline{AC}$ and $\overline{MN}$. Use geometry to justify:

(a) $\triangle PNB$ and $\triangle PNC$ are congruent and $\theta_1 = \theta_2$.

(b) The paths APB and APC have equal length.

(c) Similarly AQB and AQC have equal length.

(d) The path APC is shorter than AQC for all $Q \neq P$.

Conclude that the shortest path AQB occurs for $Q = P$.

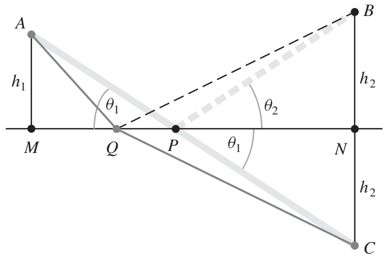


FIGURE 38

SOLUTION

(a) By definition, $\overline{BC}$ is orthogonal to $\overline{QM}$, so triangles $\triangle PNB$ and $\triangle PNC$ are congruent by side–angle–side. Therefore $\theta_1 = \theta_2$

(b) Because $\triangle PNB$ and $\triangle PNC$ are congruent, it follows that $\overline{PB}$ and $\overline{PC}$ are of equal length. Thus, paths APB and APC have equal length.

(c) The same reasoning used in parts (a) and (b) lead us to conclude that $\triangle QNB$ and $\triangle QNC$ are congruent and that $\overline{PB}$ and $\overline{PC}$ are of equal length. Thus, paths AQB and AQC are of equal length.

(d) Consider triangle $\triangle AQC$. By the triangle inequality, the length of side $\overline{AC}$ is less than or equal to the sum of the lengths of the sides $\overline{AQ}$ and $\overline{QC}$. Thus, the path APC is shorter than AQC for all $Q \neq P$.

Finally, the shortest path AQB occurs for $Q = P$.

70. A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius r (Figure 39). The upper radius x can take on any value between 0 and r. Note that $x = 0$ and $x = r$ correspond to a cone and cylinder, respectively. As a function of x, the surface area (not including the top and bottom) is $S(x) = \pi s(r + x)$, where s is the *slant height* as indicated in the figure. Which value of x yields the least expensive design [the minimum value of $S(x)$ for $0 \le x \le r$]?

(a) Show that $S(x) = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) Show that if $r < \sqrt{2}$, then $S(x)$ is an increasing function. Conclude that the cone ($x = 0$) has minimal area in this case.

(c) Assume that $r > \sqrt{2}$. Show that $S(x)$ has two critical points $x_1 < x_2$ in $(0, r)$, and that $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.

(d) Conclude that the minimum occurs at $x = 0$ or x_2.

(e) Find the minimum in the cases $r = 1.5$ and $r = 2$.

(f) Challenge: Let $c = \sqrt{(5 + 3\sqrt{3})/4} \approx 1.597$. Prove that the minimum occurs at $x = 0$ (cone) if $\sqrt{2} < r < c$, but the minimum occurs at $x = x_2$ if $r > c$.

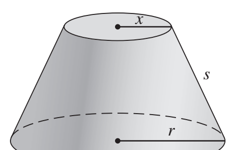


FIGURE 39 Frustrum of height 1 cm.

SOLUTION

(a) Consider a cross-section of the object and notice a triangle can be formed with height 1, hypotenuse s, and base $r - x$. Then, by the Pythagorean Theorem, $s = \sqrt{1 + (r - x)^2}$ and the surface area is $S = \pi(r + x)s = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) $S'(x) = \pi\left(\sqrt{1 + (r - x)^2} - (r + x)(1 + (r - x)^2)^{-1/2}(r - x)\right) = \pi\dfrac{2x^2 - 2rx + 1}{\sqrt{1 + (r - x)^2}} = 0$ yields critical points $x = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - 2}$. If $r < \sqrt{2}$ then there are no real critical points and $S'(x) > 0$ for $x > 0$. Hence, $S(x)$ is increasing everywhere and thus the minimum must occur at the left endpoint, $x = 0$.

(c) For $r > \sqrt{2}$, there are two critical points, $x_1 = \frac{1}{2}r - \frac{1}{2}\sqrt{r^2 - 2}$ and $x_2 = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$. Both values are on the interval $[0, r]$ since $r > \sqrt{r^2 - 2}$. Sign analysis reveals that $S(x)$ is increasing for $0 < x < x_1$, decreasing for $x_1 < x < x_2$ and increasing for $x_2 < x < r$. Hence, $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.

(d) The minimum value of S must occur at an endpoint or a critical point. Since $S(x_1)$ is a local maximum and S increases for $x_2 < x < r$, we conclude that the minimum of S must occur either at $x = 0$ or at $x = x_2$.

(e) If $r = 1.5$ cm, $S(x_2) = 8.8357$ cm^2 and $S(0) = 8.4954$ cm^2, so $S(0) = 8.4954$ cm^2 is the minimum (cone). If $r = 2$ cm, $S(x_2) = 12.852$ cm^2 and $S(0) = 14.0496$ cm^2, so $S(x_2) = 12.852$ cm^2 is the minimum.

(f) *Take a deep breath*. Setting $S(x_2) = S(0)$ produces an equation in r (x_2 is given in r, and so is $S(0)$). By means of a great deal of algebraic labor and a clever substitution, we are going to solve for r. $S(0) = \pi r\sqrt{1 + r^2}$, while, since $x_2 = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$,

$$
\begin{aligned}
S(x_2) &= \pi\left(\frac{3}{2}r + \frac{1}{2}\sqrt{r^2 - 2}\right)\sqrt{1 + (\frac{1}{2}r - \frac{1}{2}\sqrt{r^2 - 2})^2} \\
&= \frac{\pi}{2}\left(3r + \sqrt{r^2 - 2}\right)\sqrt{1 + \frac{1}{4}(r^2 - 2r\sqrt{r^2 - 2} + r^2 - 2)} \\
&= \frac{\pi}{2}\left(3r + \sqrt{r^2 - 2}\right)\sqrt{1 + \frac{1}{2}(r^2 - r\sqrt{r^2 - 2} - 1)}
\end{aligned}
$$

From this, we simplify by squaring and taking out constants:

$$S(x_2)/\pi = \frac{1}{2}\left(3r + \sqrt{r^2-2}\right)\sqrt{1+\frac{1}{2}(r^2 - r\sqrt{r^2-2}-1)}$$
$$(S(x_2)/\pi)^2 = \frac{1}{8}\left(3r + \sqrt{r^2-2}\right)^2\left(2+(r^2 - r\sqrt{r^2-2}-1)\right)$$
$$8(S(x_2)/\pi)^2 = \left(3r + \sqrt{r^2-2}\right)^2\left(r^2 - r\sqrt{r^2-2}+1\right)$$

To solve the equation $S(x_2) = S(0)$, we solve the equivalent equation $8(S(x_2)/\pi)^2 = 8(S(0)/\pi)^2$. $8(S(0)/\pi)^2 = 8r^2(1+r^2) = 8r^2 + 8r^4$. Let $u = r^2 - 2$, so that $\sqrt{r^2-2} = \sqrt{u}$, $r^2 = u+2$, and $r = \sqrt{u+2}$. The expression for $8(S(x_2)/\pi)^2$ is, then:

$$8(S(x_2)/\pi)^2 = \left(3\sqrt{u+2} + \sqrt{u}\right)^2\left((u+2) - \sqrt{u+2}\sqrt{u}+1\right)$$

while

$$8(S(0)/\pi)^2 = 8r^2 + 8r^4 = 8(u+2)(u+3) = 8u^2 + 40u + 48.$$

We compute:

$$\begin{aligned}\left(3\sqrt{u+2}+\sqrt{u}\right)^2 &= 9(u+2) + 6\sqrt{u}\sqrt{u+2} + u \\ &= 10u + 6\sqrt{u}\sqrt{u+2} + 18 \\ \left(10u + 6\sqrt{u}\sqrt{u+2}+18\right)\left(u - \sqrt{u}\sqrt{u+2}+3\right) &= 10u^2 + 6u^{3/2}\sqrt{u+2} + 18u - 10u^{3/2}\sqrt{u+2} - 6u^2 - 12u \\ &\quad - 18\sqrt{u+2}\sqrt{u} + 30u + 18\sqrt{u+2}\sqrt{u} + 54 \\ &= 4u^2 - 4u\left(\sqrt{u}\sqrt{u+2}\right) + 36u + 54\end{aligned}$$

Therefore the equation becomes:

$$\begin{aligned}8(S(0)/\pi)^2 &= 8(S(x_2)/\pi)^2 \\ 8u^2 + 40u + 48 &= 4u^2 - 4u(\sqrt{u}\sqrt{u+2}) + 36u + 54 \\ 4u^2 + 4u - 6 &= -4u(\sqrt{u}\sqrt{u+2}) \\ 16u^4 + 32u^3 - 32u^2 - 48u + 36 &= 16u^2(u)(u+2) \\ 16u^4 + 32u^3 - 32u^2 - 48u + 36 &= 16u^4 + 32u^3 \\ -32u^2 - 48u + 36 &= 0 \\ 8u^2 + 12u - 9 &= 0.\end{aligned}$$

The last quadratic has positive solution:

$$u = \frac{-12 + \sqrt{144 + 4(72)}}{16} = \frac{-12 + 12\sqrt{3}}{16} = \frac{-3+3\sqrt{3}}{4}.$$

Therefore

$$r^2 - 2 = \frac{-3+3\sqrt{3}}{4},$$

so

$$r^2 = \frac{5+3\sqrt{3}}{4}.$$

This gives us that $S(x_2) = S(0)$ when

$$r = c = \sqrt{\frac{5+3\sqrt{3}}{4}}.$$

From part (e) we know that for $r = 1.5 < c$, $S(0)$ is the minimum value for S, but for $r = 2 > c$, $S(x_2)$ is the minimum value. Since $r = c$ is the only solution of $S(0) = S(x_2)$ for $r > \sqrt{2}$, it follows that $S(0)$ provides the minimum value for $\sqrt{2} < r < c$ and $S(x_2)$ provides the minimum when $r > c$.

4.8 Newton's Method

Preliminary Questions

1. How many iterations of Newton's Method are required to compute a root if $f(x)$ is a linear function?

SOLUTION Remember that Newton's Method uses the linear approximation of a function to estimate the location of a root. If the original function is linear, then only one iteration of Newton's Method will be required to compute the root.

2. What happens in Newton's Method if your initial guess happens to be a zero of f?

SOLUTION If x_0 happens to be a zero of f, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - 0 = x_0;$$

in other words, every term in the Newton's Method sequence will remain x_0.

3. What happens in Newton's Method if your initial guess happens to be a local min or max of f?

SOLUTION Assuming that the function is differentiable, then the derivative is zero at a local maximum or a local minimum. If Newton's Method is started with an initial guess such that $f'(x_0) = 0$, then Newton's Method will fail in the sense that x_1 will not be defined. That is, the tangent line will be parallel to the x-axis and will never intersect it.

4. Is the following a reasonable description of Newton's Method: "A root of the equation of the tangent line to $f(x)$ is used as an approximation to a root of $f(x)$ itself"? Explain.

SOLUTION Yes, that is a reasonable description. The iteration formula for Newton's Method was derived by solving the equation of the tangent line to $y = f(x)$ at x_0 for its x-intercept.

Exercises

In this exercise set, all approximations should be carried out using Newton's Method.

In Exercises 1–6, apply Newton's Method to $f(x)$ and initial guess x_0 to calculate x_1, x_2, x_3.

1. $f(x) = x^2 - 6, \quad x_0 = 2$

SOLUTION Let $f(x) = x^2 - 6$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 6}{2x_n}.$$

With $x_0 = 2$, we compute

n	1	2	3
x_n	2.5	2.45	2.44948980

2. $f(x) = x^2 - 3x + 1, \quad x_0 = 3$

SOLUTION Let $f(x) = x^2 - 3x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3}.$$

With $x_0 = 3$, we compute

n	1	2	3
x_n	2.66666667	2.61904762	2.61803445

3. $f(x) = x^3 - 10, \quad x_0 = 2$

SOLUTION Let $f(x) = x^3 - 10$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 10}{3x_n^2}.$$

With $x_0 = 2$ we compute

n	1	2	3
x_n	2.16666667	2.15450362	2.15443469

4. $f(x) = x^3 + x + 1, \quad x_0 = -1$

SOLUTION Let $f(x) = x^3 + x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1}.$$

With $x_0 = -1$ we compute

n	1	2	3
x_n	-0.75	-0.68604651	-0.68233958

5. $f(x) = \cos x - 4x, \quad x_0 = 1$

SOLUTION Let $f(x) = \cos x - 4x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - 4x_n}{-\sin x_n - 4}.$$

With $x_0 = 1$ we compute

n	1	2	3
x_n	0.28540361	0.24288009	0.24267469

6. $f(x) = 1 - x \sin x, \quad x_0 = 7$

SOLUTION Let $f(x) = 1 - x \sin x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1 - x_n \sin x_n}{-x_n \cos x_n - \sin x_n}.$$

With $x_0 = 7$ we compute

n	1	2	3
x_n	6.39354183	6.43930706	6.43911724

7. Use Figure 6 to choose an initial guess x_0 to the unique real root of $x^3 + 2x + 5 = 0$ and compute the first three Newton iterates.

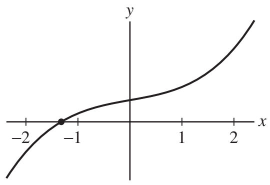


FIGURE 6 Graph of $y = x^3 + 2x + 5$.

SOLUTION Let $f(x) = x^3 + 2x + 5$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 2x_n + 5}{3x_n^2 + 2}.$$

We take $x_0 = -1.4$, based on the figure, and then calculate

n	1	2	3
x_n	-1.330964467	-1.328272820	-1.328268856

8. Approximate a solution of $\sin x = \cos 2x$ in the interval $\left[0, \frac{\pi}{2}\right]$ to three decimal places. Then find the exact solution and compare with your approximation.

SOLUTION Let $f(x) = \sin x - \cos 2x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin x_n - \cos 2x_n}{\cos x_n + 2 \sin 2x_n}.$$

With $x_0 = 0.5$ we find

n	1	2
x_n	0.523775116	0.523598785

The root, to three decimal places, is 0.524. The exact root is $\frac{\pi}{6}$, which is equal to 0.524 to three decimal places.

9. Approximate both solutions of $e^x = 5x$ to three decimal places (Figure 7).

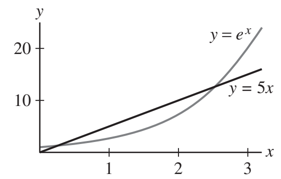


FIGURE 7 Graphs of e^x and $5x$.

SOLUTION We need to solve $e^x - 5x = 0$, so let $f(x) = e^x - 5x$. Then $f'(x) = e^x - 5$. With an initial guess of $x_0 = 0.2$, we calculate

Newton's Method (First root)	$x_0 = 0.2$ (guess)
$x_1 = 0.2 - \dfrac{f(0.2)}{f'(0.2)}$	$x_1 \approx 0.25859$
$x_2 = 0.25859 - \dfrac{f(0.25859)}{f'(0.25859)}$	$x_2 \approx 0.25917$
$x_3 = 0.25917 - \dfrac{f(0.25917)}{f'(0.25917)}$	$x_3 \approx 0.25917$

For the second root, we use an initial guess of $x_0 = 2.5$.

Newton's Method (Second root)	$x_0 = 2.5$ (guess)
$x_1 = 2.5 - \dfrac{f(2.5)}{f'(2.5)}$	$x_1 \approx 2.54421$
$x_2 = 2.54421 - \dfrac{f(2.54421)}{f'(2.54421)}$	$x_2 \approx 2.54264$
$x_3 = 2.54264 - \dfrac{f(2.54264)}{f'(2.54264)}$	$x_3 \approx 2.54264$

Thus the two solutions of $e^x = 5x$ are approximately $r_1 \approx 0.25917$ and $r_2 \approx 2.54264$.

10. The first positive solution of $\sin x = 0$ is $x = \pi$. Use Newton's Method to calculate π to four decimal places.

SOLUTION Let $f(x) = \sin x$. Taking $x_0 = 3$, we have

n	1	2	3
x_n	3.142546543	3.141592653	3.141592654

Hence, $\pi \approx 3.1416$ to four decimal places.

In Exercises 11–14, approximate to three decimal places using Newton's Method and compare with the value from a calculator.

11. $\sqrt{11}$

SOLUTION Let $f(x) = x^2 - 11$, and let $x_0 = 3$. Newton's Method yields:

n	1	2	3
x_n	3.33333333	3.31666667	3.31662479

A calculator yields 3.31662479.

12. $5^{1/3}$

SOLUTION Let $f(x) = x^3 - 5$, and let $x_0 = 2$. Here are approximations to the root of $f(x)$, which is $5^{1/3}$.

n	1	2	3	4
x_n	1.75	1.710884354	1.709976429	1.709975947

A calculator yields 1.709975947.

13. $2^{7/3}$

SOLUTION Note that $2^{7/3} = 4 \cdot 2^{1/3}$. Let $f(x) = x^3 - 2$, and let $x_0 = 1$. Newton's Method yields:

n	1	2	3
x_n	1.33333333	1.26388889	1.25993349

Thus, $2^{7/3} \approx 4 \cdot 1.25993349 = 5.03973397$. A calculator yields 5.0396842.

14. $3^{-1/4}$

SOLUTION Let $f(x) = x^{-4} - 3$, and let $x_0 = 0.8$. Here are approximations to the root of $f(x)$, which is $3^{-1/4}$.

n	1	2	3	4
x_n	0.75424	0.75973342	0.75983565	0.75983569

A calculator yields 0.75983569.

15. Approximate the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ to within an error of at most 10^{-4}. Refer to Figure 5.

SOLUTION Figure 5 from the text suggests the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ is near 2. So let $f(x) = x^4 - 6x^2 + x + 5$ and take $x_0 = 2$.

n	1	2	3	4
x_n	2.111111111	2.093568458	2.093064768	2.093064358

The largest positive root of $x^4 - 6x^2 + x + 5$ is approximately 2.093064358.

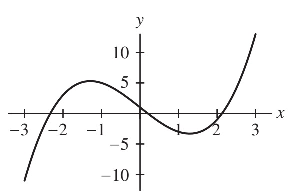
GU *In Exercises 16–19, approximate the root specified to three decimal places using Newton's Method. Use a plot to choose an initial guess.*

16. Largest positive root of $f(x) = x^3 - 5x + 1$.

SOLUTION Let $f(x) = x^3 - 5x + 1$. The graph of $f(x)$ shown below suggests the largest positive root is near $x = 2.2$. Taking $x_0 = 2.2$, Newton's Method gives

n	1	2	3
x_n	2.13193277	2.12842820	2.12841906

The largest positive root of $x^3 - 5x + 1$ is approximately 2.1284.

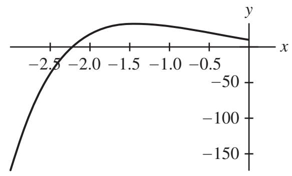


17. Negative root of $f(x) = x^5 - 20x + 10$.

SOLUTION Let $f(x) = x^5 - 20x + 10$. The graph of $f(x)$ shown below suggests taking $x_0 = -2.2$. Starting from $x_0 = -2.2$, the first three iterates of Newton's Method are:

n	1	2	3
x_n	−2.22536529	−2.22468998	−2.22468949

Thus, to three decimal places, the negative root of $f(x) = x^5 - 20x + 10$ is -2.225.



18. Positive solution of $\sin\theta = 0.8\theta$.

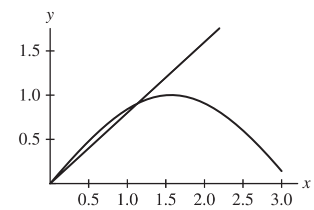
SOLUTION From the graph below, we see that the positive solution to the equation $\sin\theta = 0.8\theta$ is approximately $x = 1.1$. Now, let $f(\theta) = \sin\theta - 0.8\theta$ and define

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} = \theta_n - \frac{\sin\theta_n - 0.8\theta_n}{\cos\theta_n - 0.8}.$$

With $\theta_0 = 1.1$ we find

n	1	2	3
θ_n	1.13235345	1.13110447	1.13110259

Thus, to three decimal places, the positive solution to the equation $\sin\theta = 0.8\theta$ is 1.131.



19. Solution of $\ln(x + 4) = x$.

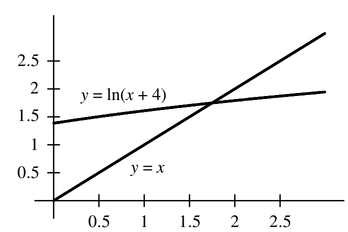
SOLUTION From the graph below, we see that the positive solution to the equation $\ln(x + 4) = x$ is approximately $x = 2$. Now, let $f(x) = \ln(x + 4) - x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n + 4) - x_n}{\frac{1}{x_n+4} - 1}.$$

With $x_0 = 2$ we find

n	1	2	3
x_n	1.750111363	1.749031407	1.749031386

Thus, to three decimal places, the positive solution to the equation $\ln(x + 4) = x$ is 1.749.



20. Let x_1, x_2 be the estimates to a root obtained by applying Newton's Method with $x_0 = 1$ to the function graphed in Figure 8. Estimate the numerical values of x_1 and x_2, and draw the tangent lines used to obtain them.

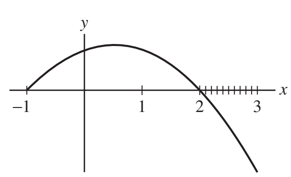
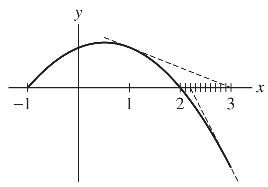


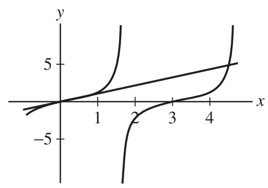
FIGURE 8

SOLUTION The graph with tangent lines drawn on it appears below. The tangent line to the curve at $(x_0, f(x_0))$ has an x-intercept at approximately $x_1 = 3.0$. The tangent line to the curve at $(x_1, f(x_1))$ has an x-intercept at approximately $x_2 = 2.2$.



21. GU Find the smallest positive value of x at which $y = x$ and $y = \tan x$ intersect. *Hint:* Draw a plot.

SOLUTION Here is a plot of $\tan x$ and x on the same axes:



The first intersection with $x > 0$ lies on the second "branch" of $y = \tan x$, between $x = \frac{5\pi}{4}$ and $x = \frac{3\pi}{2}$. Let $f(x) = \tan x - x$. The graph suggests an initial guess $x_0 = \frac{5\pi}{4}$, from which we get the following table:

n	1	2	3	4
x_n	6.85398	21.921	4480.8	7456.27

This is clearly leading nowhere, so we need to try a better initial guess. *Note: This happens with Newton's Method—it is sometimes difficult to choose an initial guess*. We try the point directly between $\frac{5\pi}{4}$ and $\frac{3\pi}{2}$, $x_0 = \frac{11\pi}{8}$:

n	1	2	3	4	5	6	7
x_n	4.64662	4.60091	4.54662	4.50658	4.49422	4.49341	4.49341

The first point where $y = x$ and $y = \tan x$ cross is at approximately $x = 4.49341$, which is approximately 1.4303π.

22. In 1535, the mathematician Antonio Fior challenged his rival Niccolo Tartaglia to solve this problem: A tree stands 12 *braccia* high; it is broken into two parts at such a point that the height of the part left standing is the cube root of the length of the part cut away. What is the height of the part left standing? Show that this is equivalent to solving $x^3 + x = 12$ and find the height to three decimal places. Tartaglia, who had discovered the secret of the cubic equation, was able to determine the exact answer:

$$x = \left(\sqrt[3]{\sqrt{2919} + 54} - \sqrt[3]{\sqrt{2919} - 54}\right) \Big/ \sqrt[3]{9}$$

SOLUTION Suppose that x is the part of the tree left standing, so that x^3 is the part cut away. Since the tree is 12 *braccia* high, this gives that $x + x^3 = 12$. Let $f(x) = x + x^3 - 12$. We are looking for a point where $f(x) = 0$. Using the initial guess $x = 2$ (it seems that most of the tree is cut away), we get the following table:

n	1	2	3	4
x_n	2.15384615385	2.14408201873	2.14404043328	2.14404043253

Hence $x \approx 2.14404043253$. Tartaglia's exact answer is 2.14404043253, so the 4th Newton's Method approximation is accurate to at least 11 decimal places.

23. Find (to two decimal places) the coordinates of the point P in Figure 9 where the tangent line to $y = \cos x$ passes through the origin.

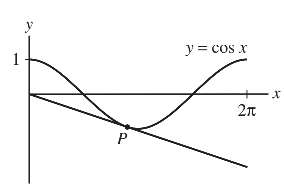


FIGURE 9

SOLUTION Let $(x_r, \cos(x_r))$ be the coordinates of the point P. The slope of the tangent line is $-\sin(x_r)$, so we are looking for a tangent line:

$$y = -\sin(x_r)(x - x_r) + \cos(x_r)$$

such that $y = 0$ when $x = 0$. This gives us the equation:

$$-\sin(x_r)(-x_r) + \cos(x_r) = 0.$$

Let $f(x) = \cos x + x \sin x$. We are looking for the first point $x = r$ where $f(r) = 0$. The sketch given indicates that $x_0 = 3\pi/4$ would be a good initial guess. The following table gives successive Newton's Method approximations:

n	1	2	3	4
x_n	2.931781309	2.803636974	2.798395826	2.798386046

The point P has approximate coordinates $(2.7984, -0.941684)$.

Newton's Method is often used to determine interest rates in financial calculations. In Exercises 24–26, r denotes a yearly interest rate expressed as a decimal (rather than as a percent).

24. If P dollars are deposited every month in an account earning interest at the yearly rate r, then the value S of the account after N years is

$$S = P\left(\frac{b^{12N+1} - b}{b - 1}\right) \qquad \text{where } b = 1 + \frac{r}{12}$$

You have decided to deposit $P = 100$ dollars per month.

(a) Determine S after 5 years if $r = 0.07$ (that is, 7%).

(b) Show that to save \$10,000 after 5 years, you must earn interest at a rate r determined bys the equation $b^{61} - 101b + 100 = 0$. Use Newton's Method to solve for b. Then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

SOLUTION

(a) If $r = 0.07$, $b = 1 + r/12 \approx 1.00583$, and :

$$S = 100\frac{(b^{61} - b)}{b - 1} = 7201.05.$$

(b) If our goal is to get \$10,000 after five years, we need $S = 10{,}000$ when $N = 5$.

$$10{,}000 = 100\left(\frac{b^{61} - b}{b - 1}\right),$$

So that:

$$\begin{aligned} 10{,}000(b - 1) &= 100\left(b^{61} - b\right) \\ 100b - 100 &= b^{61} - b \\ b^{61} - 101b + 100 &= 0 \end{aligned}$$

$b = 1$ is a root, but, since $b - 1$ appears in the denominator of our original equation, it does not satisfy the original equation. Let $f(b) = b^{61} - 101b + 100$. Let's use the initial guess $r = 0.2$, so that $x_0 = 1 + r/12 = 1.016666$.

n	1	2	3
x_n	1.01576	1.01569	1.01569

The solution is approximately $b = 1.01569$. The interest rate r required satisfies $1 + r/12 = 1.01569$, so that $r = 0.01569 \times 12 = 0.18828$. An annual interest rate of 18.828% is required to have \$10,000 after five years.

25. If you borrow L dollars for N years at a yearly interest rate r, your monthly payment of P dollars is calculated using the equation

$$L = P\left(\frac{1-b^{-12N}}{b-1}\right) \qquad \text{where } b = 1 + \frac{r}{12}$$

(a) Find P if $L = \$5000$, $N = 3$, and $r = 0.08$ (8%).
(b) You are offered a loan of $L = \$5000$ to be paid back over 3 years with monthly payments of $P = \$200$. Use Newton's Method to compute b and find the implied interest rate r of this loan. *Hint:* Show that $(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0$.

SOLUTION
(a) $b = (1 + 0.08/12) = 1.00667$

$$P = L\left(\frac{b-1}{1-b^{-12N}}\right) = 5000\left(\frac{1.00667-1}{1-1.00667^{-36}}\right) \approx \$156.69$$

(b) Starting from

$$L = P\left(\frac{1-b^{-12N}}{b-1}\right),$$

divide by P, multiply by $b-1$, multiply by b^{12N} and collect like terms to arrive at

$$(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0.$$

Since $L/P = 5000/200 = 25$, we must solve

$$25b^{37} - 26b^{36} + 1 = 0.$$

Newton's Method gives $b \approx 1.02121$ and

$$r = 12(b-1) = 12(0.02121) \approx 0.25452$$

So the interest rate is around 25.45%.

26. If you deposit P dollars in a retirement fund every year for N years with the intention of then withdrawing Q dollars per year for M years, you must earn interest at a rate r satisfying $P(b^N - 1) = Q(1 - b^{-M})$, where $b = 1 + r$. Assume that \$2,000 is deposited each year for 30 years and the goal is to withdraw \$10,000 per year for 25 years. Use Newton's Method to compute b and then find r. Note that $b = 1$ is a root, but you want the root satisfying $b > 1$.

SOLUTION Substituting $P = 2000$, $Q = 10{,}000$, $N = 30$ and $M = 25$ into the equation $P(b^N - 1) = Q(1 - b^{-M})$ and then rearranging terms, we find that b must satisfy the equation $b^{55} - 6b^{25} + 5 = 0$. Newton's Method with a starting value of $b_0 = 1.1$ yields $b \approx 1.05217$. Thus, $r \approx 0.05217 = 5.217\%$.

27. There is no simple formula for the position at time t of a planet P in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle θ in Figure 10 (note that P determines θ because it is the central angle of point B on the circle). Let $a = OA$ and $e = OS/OA$ (the eccentricity of the orbit).
(a) Show that sector *BSA* has area $(a^2/2)(\theta - e\sin\theta)$.
(b) By Kepler's Second Law, the area of sector *BSA* is proportional to the time t elapsed since the planet passed point A, and because the circle has area πa^2, *BSA* has area $(\pi a^2)(t/T)$, where T is the period of the orbit. Deduce **Kepler's Equation**:

$$\frac{2\pi t}{T} = \theta - e\sin\theta$$

(c) The eccentricity of Mercury's orbit is approximately $e = 0.2$. Use Newton's Method to find θ after a quarter of Mercury's year has elapsed ($t = T/4$). Convert θ to degrees. Has Mercury covered more than a quarter of its orbit at $t = T/4$?

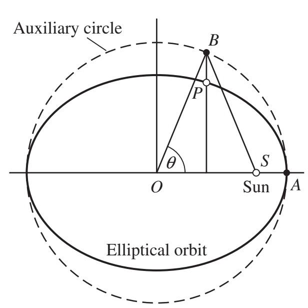


FIGURE 10

SOLUTION

(a) The sector SAB is the slice OAB with the triangle OPS removed. OAB is a central sector with arc θ and radius $\overline{OA} = a$, and therefore has area $\frac{a^2\theta}{2}$. OPS is a triangle with height $a \sin\theta$ and base length $\overline{OS} = ea$. Hence, the area of the sector is

$$\frac{a^2}{2}\theta - \frac{1}{2}ea^2 \sin\theta = \frac{a^2}{2}(\theta - e\sin\theta).$$

(b) Since Kepler's second law indicates that the area of the sector is proportional to the time t since the planet passed point A, we get

$$\pi a^2 (t/T) = a^2/2\,(\theta - e\sin\theta)$$
$$2\pi \frac{t}{T} = \theta - e\sin\theta.$$

(c) If $t = T/4$, the last equation in (b) gives:

$$\frac{\pi}{2} = \theta - e\sin\theta = \theta - .2\sin\theta.$$

Let $f(\theta) = \theta - .2\sin\theta - \frac{\pi}{2}$. We will use Newton's Method to find the point where $f(\theta) = 0$. Since a quarter of the year on Mercury has passed, a good first estimate θ_0 would be $\frac{\pi}{2}$.

n	1	2	3	4
x_n	1.7708	1.76696	1.76696	1.76696

From the point of view of the Sun, Mercury has traversed an angle of approximately 1.76696 radians $= 101.24°$. Mercury has therefore traveled more than one fourth of the way around (from the point of view of central angle) during this time.

28. The roots of $f(x) = \frac{1}{3}x^3 - 4x + 1$ to three decimal places are -3.583, 0.251, and 3.332 (Figure 11). Determine the root to which Newton's Method converges for the initial choices $x_0 = 1.85$, 1.7, and 1.55. The answer shows that a small change in x_0 can have a significant effect on the outcome of Newton's Method.

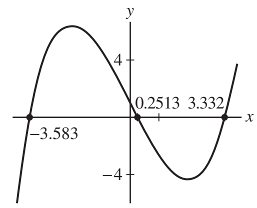


FIGURE 11 Graph of $f(x) = \frac{1}{3}x^3 - 4x + 1$.

SOLUTION Let $f(x) = \frac{1}{3}x^3 - 4x + 1$, and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{3}x_n^3 - 4x_n + 1}{x_n^2 - 4}.$$

- Taking $x_0 = 1.85$, we have

n	1	2	3	4	5	6	7
x_n	−5.58	−4.31	−3.73	−3.59	−3.58294362	−3.582918671	−3.58291867

- Taking $x_0 = 1.7$, we have

n	1	2	3	4	5	6	7	8	9
x_n	−2.05	−33.40	−22.35	−15.02	−10.20	−7.08	−5.15	−4.09	−3.66

n	10	11	12	13
x_n	−3.585312288	−3.582920989	−3.58291867	−3.58291867

- Taking $x_0 = 1.55$, we have

n	1	2	3	4	5	6
x_n	−0.928	0.488	0.245	0.251320515	0.251322863	0.251322863

29. What happens when you apply Newton's Method to find a zero of $f(x) = x^{1/3}$? Note that $x = 0$ is the only zero.

SOLUTION Let $f(x) = x^{1/3}$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

Take $x_0 = 0.5$. Then the sequence of iterates is $-1, 2, -4, 8, -16, 32, -64, \dots$ That is, for any nonzero starting value, the sequence of iterates diverges spectacularly, since $x_n = (-2)^n x_0$. Thus $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 2^n |x_0| = \infty$.

30. What happens when you apply Newton's Method to the equation $x^3 - 20x = 0$ with the unlucky initial guess $x_0 = 2$?

SOLUTION Let $f(x) = x^3 - 20x$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20x_n}{3x_n^2 - 20}.$$

Take $x_0 = 2$. Then the sequence of iterates is $-2, 2, -2, 2, \dots$, which diverges by oscillation.

Further Insights and Challenges

31. Newton's Method can be used to compute reciprocals without performing division. Let $c > 0$ and set $f(x) = x^{-1} - c$.

(a) Show that $x - (f(x)/f'(x)) = 2x - cx^2$.

(b) Calculate the first three iterates of Newton's Method with $c = 10.3$ and the two initial guesses $x_0 = 0.1$ and $x_0 = 0.5$.

(c) Explain graphically why $x_0 = 0.5$ does not yield a sequence converging to $1/10.3$.

SOLUTION

(a) Let $f(x) = \frac{1}{x} - c$. Then

$$x - \frac{f(x)}{f'(x)} = x - \frac{\frac{1}{x} - c}{-x^{-2}} = 2x - cx^2.$$

(b) For $c = 10.3$, we have $f(x) = \frac{1}{x} - 10.3$ and thus $x_{n+1} = 2x_n - 10.3x_n^2$.

- Take $x_0 = 0.1$.

n	1	2	3
x_n	0.097	0.0970873	0.09708738

- Take $x_0 = 0.5$.

n	1	2	3
x_n	-1.575	-28.7004375	-8541.66654

(c) The graph is disconnected. If $x_0 = .5$, $(x_1, f(x_1))$ is on the other portion of the graph, which will never converge to any point under Newton's Method.

In Exercises 32 and 33, consider a metal rod of length L fastened at both ends. If you cut the rod and weld on an additional segment of length m, leaving the ends fixed, the rod will bow up into a circular arc of radius R (unknown), as indicated in Figure 12.

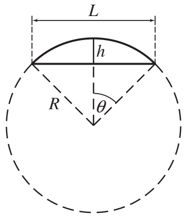


FIGURE 12 The bold circular arc has length $L + m$.

32. Let h be the maximum vertical displacement of the rod.

(a) Show that $L = 2R\sin\theta$ and conclude that

$$h = \frac{L(1 - \cos\theta)}{2\sin\theta}$$

(b) Show that $L + m = 2R\theta$ and then prove

$$\frac{\sin\theta}{\theta} = \frac{L}{L+m} \qquad \boxed{2}$$

SOLUTION

(a) From the figure, we have $\sin\theta = \frac{L/2}{R}$, so that $L = 2R\sin\theta$. Hence

$$h = R - R\cos\theta = R\,(1-\cos\theta) = \frac{\frac{1}{2}L}{\sin\theta}\,(1-\cos\theta) = \frac{L\,(1-\cos\theta)}{2\sin\theta}$$

(b) The arc length $L + m$ is also given by radius $\times$ angle $= R\cdot 2\theta$. Thus, $L + m = 2R\theta$. Dividing $L = 2R\sin\theta$ by $L + m = 2R\theta$ yields

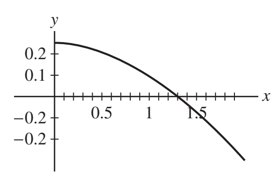
$$\frac{L}{L+m} = \frac{2R\sin\theta}{2R\theta} = \frac{\sin\theta}{\theta}.$$

33. Let $L = 3$ and $m = 1$. Apply Newton's Method to Eq. (2) to estimate θ, and use this to estimate h.

SOLUTION We let $L = 3$ and $m = 1$. We want the solution of:

$$\frac{\sin\theta}{\theta} = \frac{L}{L+m}$$
$$\frac{\sin\theta}{\theta} - \frac{L}{L+m} = 0$$
$$\frac{\sin\theta}{\theta} - \frac{3}{4} = 0.$$

Let $f(\theta) = \frac{\sin\theta}{\theta} - \frac{3}{4}$.



The figure above suggests that $\theta_0 = 1.5$ would be a good initial guess. The Newton's Method approximations for the solution follow:

n	1	2	3	4
θ_n	1.2854388	1.2757223	1.2756981	1.2756981

The angle where $\frac{\sin\theta}{\theta} = \frac{L}{L+m}$ is approximately 1.2757. Hence

$$h = L\frac{1-\cos\theta}{2\sin\theta} \approx 1.11181.$$

34. Quadratic Convergence to Square Roots Let $f(x) = x^2 - c$ and let $e_n = x_n - \sqrt{c}$ be the error in x_n.

(a) Show that $x_{n+1} = \frac{1}{2}(x_n + c/x_n)$ and $e_{n+1} = e_n^2/2x_n$.

(b) Show that if $x_0 > \sqrt{c}$, then $x_n > \sqrt{c}$ for all n. Explain graphically.

(c) Show that if $x_0 > \sqrt{c}$, then $e_{n+1} \le e_n^2/(2\sqrt{c})$.

SOLUTION

(a) Let $f(x) = x^2 - c$. Then

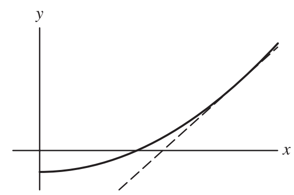
$$xn + 1 = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n} = \frac{x_n^2 + c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right),$$

as long as $x_n \neq 0$. Now

$$\frac{e_n^2}{2x_n} = \frac{\left(x_n - \sqrt{c}\right)^2}{2x_n} = \frac{x_n^2 - 2x_n\sqrt{c} + c}{2x_n} = \frac{1}{2}x_n - \sqrt{c} + \frac{c}{2x_n}$$
$$= \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) - \sqrt{c} = x_{n+1} - \sqrt{c} = e_{n+1}.$$

(b) Since $x_0 > \sqrt{c} \geq 0$, we have $e_0 = x_0 - \sqrt{c} > 0$. Now assume that $e_k > 0$ for $k = n$. Then $0 < e_k = e_n = x_n - \sqrt{c}$, whence $x_n > \sqrt{c} \geq 0$; i.e., $x_n > 0$ and $e_n > 0$. By part (a), we have for $k = n + 1$ that $e_k = e_{n+1} = \dfrac{e_n^2}{2x_n} > 0$ since $x_n > 0$. Thus $e_{n+1} > 0$. Therefore by induction $e_n > 0$ for all $n \geq 0$. Hence $e_n = x_n - \sqrt{c} > 0$ for all $n \geq 0$. Therefore $x_n > \sqrt{c}$ for all $n \geq 0$.

The figure below shows the graph of $f(x) = x^2 - c$. The x-intercept of the graph is, of course, $x = \sqrt{c}$. We see that for any $x_n > \sqrt{c}$, the tangent line to the graph of f intersects the x-axis at a value $x_{n+1} > \sqrt{c}$.



(c) By part (b), if $x_0 > \sqrt{c}$, then $x_n > \sqrt{c}$ for all $n \geq 0$. Accordingly, for all $n \geq 0$ we have $e_{n+1} = \dfrac{e_n^2}{2x_n} < \dfrac{e_n^2}{2\sqrt{c}}$. In other words, $e_{n+1} < \dfrac{e_n^2}{2\sqrt{c}}$ for all $n \geq 0$.

In Exercises 35–37, a flexible chain of length L is suspended between two poles of equal height separated by a distance $2M$ (Figure 13). By Newton's laws, the chain describes a ***catenary*** $y = a\cosh\left(\frac{x}{a}\right)$*, where a is the number such that $L = 2a\sinh\left(\frac{M}{a}\right)$. The sag s is the vertical distance from the highest to the lowest point on the chain.*

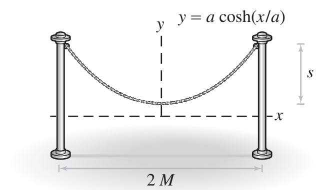


FIGURE 13 Chain hanging between two poles.

35. Suppose that $L = 120$ and $M = 50$.

(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.

(b) Compute the sag s.

SOLUTION

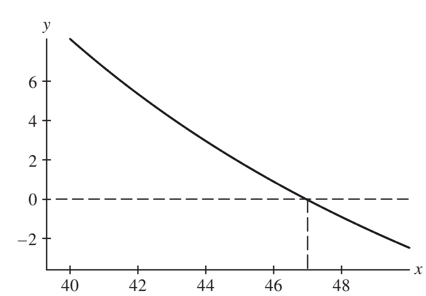
(a) Let

$$f(a) = 2a\sinh\left(\frac{50}{a}\right) - 120.$$

The graph of f shown below suggests $a \approx 47$ is a root of f. Starting with $a_0 = 47$, we find the following approximations using Newton's method:

$$a_1 = 46.95408 \quad \text{and} \quad a_2 = 46.95415$$

Thus, to two decimal places, $a = 46.95$.



(b) The sag is given by

$$s = y(M) - y(0) = \left(a\cosh\frac{M}{a} + C\right) - \left(a\cosh\frac{0}{a} + C\right) = a\cosh\frac{M}{a} - a.$$

Using $M = 50$ and $a = 46.95$, we find $s = 29.24$.

36. Assume that M is fixed.

(a) Calculate $\frac{ds}{da}$. Note that $s = a\cosh\left(\frac{M}{a}\right) - a$.

(b) Calculate $\frac{da}{dL}$ by implicit differentiation using the relation $L = 2a\sinh\left(\frac{M}{a}\right)$.

(c) Use (a) and (b) and the Chain Rule to show that

$$\frac{ds}{dL} = \frac{ds}{da}\frac{da}{dL} = \frac{\cosh(M/a) - (M/a)\sinh(M/a) - 1}{2\sinh(M/a) - (2M/a)\cosh(M/a)} \qquad \boxed{3}$$

SOLUTION The sag in the curve is

$$s = y(M) - y(0) = a\cosh\left(\frac{M}{a}\right) + C - (a\cosh 0 + C) = a\cosh\left(\frac{M}{a}\right) - a.$$

(a) $\dfrac{ds}{da} = \cosh\left(\dfrac{M}{a}\right) - \dfrac{M}{a}\sinh\left(\dfrac{M}{a}\right) - 1$

(b) If we differentiate the relation $L = 2a\sinh\left(\dfrac{M}{a}\right)$ with respect to a, we find

$$0 = 2\frac{da}{dL}\sinh\left(\frac{M}{a}\right) - \frac{2M}{a}\frac{da}{dL}\cosh\left(\frac{M}{a}\right).$$

Solving for da/dL yields

$$\frac{da}{dL} = \left(2\sinh\left(\frac{M}{a}\right) - \frac{2M}{a}\cosh\left(\frac{M}{a}\right)\right)^{-1}.$$

(c) By the Chain Rule,

$$\frac{ds}{dL} = \frac{ds}{da}\cdot\frac{da}{dL}.$$

The formula for ds/dL follows upon substituting the results from parts (a) and (b).

37. Suppose that $L = 160$ and $M = 50$.

(a) Use Newton's Method to find a value of a (to two decimal places) satisfying $L = 2a\sinh(M/a)$.

(b) Use Eq. (3) and the Linear Approximation to estimate the increase in sag Δs for changes in length $\Delta L = 1$ and $\Delta L = 5$.

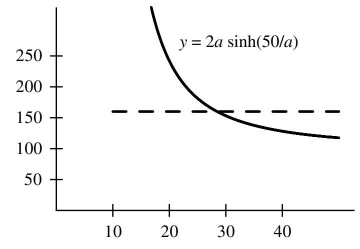
(c) CAS Compute $s(161) - s(160)$ and $s(165) - s(160)$ directly and compare with your estimates in (b).

SOLUTION

(a) Let $f(x) = 2x\sinh(50/x) - 160$. Using the graph below, we select an initial guess of $x_0 = 30$. Newton's Method then yields:

n	1	2	3
x_n	28.30622107	28.45653356	28.45797517

Thus, to two decimal places, $a \approx 28.46$.



(b) With $M = 50$ and $a \approx 28.46$, we find using Eq. (3) that

$$\frac{ds}{dL} = 0.61.$$

By the Linear Approximation,

$$\Delta s \approx \frac{ds}{dL}\cdot\Delta L.$$

If L increases from 160 to 161, then $\Delta L = 1$ and $\Delta s \approx 0.61$; if L increases from 160 to 165, then $\Delta L = 5$ and $\Delta s \approx 3.05$.

(c) When $L = 160$, $a \approx 28.46$ and

$$s(160) = 28.46 \cosh\left(\frac{50}{28.46}\right) - 28.46 \approx 56.45;$$

whereas, when $L = 161$, $a \approx 28.25$ and

$$s(161) = 28.25 \cosh\left(\frac{50}{28.25}\right) - 28.25 \approx 57.07.$$

Therefore, $s(161) - s(160) = 0.62$, very close to the approximation obtained from the Linear Approximation. Moreover, when $L = 165$, $a \approx 27.49$ and

$$s(165) = 27.49 \cosh\left(\frac{50}{27.49}\right) - 27.49 \approx 59.47;$$

thus, $s(165) - s(160) = 3.02$, again very close to the approximation obtained from the Linear Approximation.

4.9 Antiderivatives

Preliminary Questions

1. Find an antiderivative of the function $f(x) = 0$.

SOLUTION Since the derivative of any constant is zero, any constant function is an antiderivative for the function $f(x) = 0$.

2. Is there a difference between finding the general antiderivative of a function $f(x)$ and evaluating $\int f(x)\,dx$?

SOLUTION No difference. The indefinite integral is the symbol for denoting the general antiderivative.

3. Jacques was told that $f(x)$ and $g(x)$ have the same derivative, and he wonders whether $f(x) = g(x)$. Does Jacques have sufficient information to answer his question?

SOLUTION No. Knowing that the two functions have the same derivative is only good enough to tell Jacques that the functions may differ by at most an additive constant. To determine whether the functions are equal for all x, Jacques needs to know the value of each function for a single value of x. If the two functions produce the same output value for a single input value, they must take the same value for all input values.

4. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Which of the following statements are true? Explain.

(a) If $f = g$, then $F = G$.
(b) If F and G differ by a constant, then $f = g$.
(c) If f and g differ by a constant, then $F = G$.

SOLUTION

(a) False. Even if $f(x) = g(x)$, the antiderivatives F and G may differ by an additive constant.
(b) True. This follows from the fact that the derivative of any constant is 0.
(c) False. If the functions f and g are different, then the antiderivatives F and G differ by a linear function: $F(x) - G(x) = ax + b$ for some constants a and b.

5. Is $y = x$ a solution of the following Initial Value Problem?

$$\frac{dy}{dx} = 1, \qquad y(0) = 1$$

SOLUTION Although $\frac{d}{dx}x = 1$, the function $f(x) = x$ takes the value 0 when $x = 0$, so $y = x$ is *not* a solution of the indicated initial value problem.

Exercises

In Exercises 1–8, find the general antiderivative of $f(x)$ and check your answer by differentiating.

1. $f(x) = 18x^2$

SOLUTION

$$\int 18x^2\,dx = 18 \int x^2\,dx = 18 \cdot \frac{1}{3}x^3 + C = 6x^3 + C.$$

As a check, we have

$$\frac{d}{dx}(6x^3 + C) = 18x^2$$

as needed.

2. $f(x) = x^{-3/5}$

SOLUTION

$$\int x^{-3/5}\,dx = \frac{x^{2/5}}{2/5} + C = \frac{5}{2}x^{2/5} + C.$$

As a check, we have

$$\frac{d}{dx}\left(\frac{5}{2}x^{2/5} + C\right) = x^{-3/5}$$

as needed.

3. $f(x) = 2x^4 - 24x^2 + 12x^{-1}$

SOLUTION

$$\begin{aligned}\int (2x^4 - 24x^2 + 12x^{-1})\,dx &= 2\int x^4\,dx - 24\int x^2\,dx + 12\int \frac{1}{x}\,dx\\ &= 2\cdot\frac{1}{5}x^5 - 24\cdot\frac{1}{3}x^3 + 12\ln|x| + C\\ &= \frac{2}{5}x^5 - 8x^3 + 12\ln|x| + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{2}{5}x^5 - 8x^3 + 12\ln|x| + C\right) = 2x^4 - 24x^2 + 12x^{-1}$$

as needed.

4. $f(x) = 9x + 15x^{-2}$

SOLUTION

$$\begin{aligned}\int (9x + 15x^{-2})\,dx &= 9\int x\,dx + 15\int x^{-2}\,dx\\ &= 9\cdot\frac{1}{2}x^2 + 15\cdot\frac{x^{-1}}{-1} + C\\ &= \frac{9}{2}x^2 - 15x^{-1} + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{9}{2}x^2 - 15x^{-1} + C\right) = 9x + 15x^{-2}$$

as needed.

5. $f(x) = 2\cos x - 9\sin x$

SOLUTION

$$\begin{aligned}\int (2\cos x - 9\sin x)\,dx &= 2\int \cos x\,dx - 9\int \sin x\,dx\\ &= 2\sin x - 9(-\cos x) + C = 2\sin x + 9\cos x + C\end{aligned}$$

As a check, we have

$$\frac{d}{dx}(2\sin x + 9\cos x + C) = 2\cos x + 9(-\sin x) = 2\cos x - 9\sin x$$

as needed.

6. $f(x) = 4x^7 - 3\cos x$

SOLUTION

$$\begin{aligned}\int (4x^7 - 3\cos x)\,dx &= 4\int x^7\,dx - 3\int \cos x\,dx\\ &= 4\cdot\frac{1}{8}x^8 - 3\sin x + C = \frac{1}{2}x^8 - 3\sin x + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(\frac{1}{2}x^8 - 3\sin x + C\right) = 4x^7 - 3\cos x,$$

as needed.

7. $f(x) = 12e^x - 5x^{-2}$

SOLUTION

$$\int (12e^x - 5x^{-2})\,dx = 12\int e^x\,dx - 5\int x^{-2}\,dx = 12e^x - 5(-x^{-1}) + C = 12e^x + 5x^{-1} + C.$$

As a check, we have

$$\frac{d}{dx}\left(12e^x + 5x^{-1} + C\right) = 12e^x + 5(-x^{-2}) = 12e^x - 5x^{-2}$$

as needed.

8. $f(x) = e^x - 4\sin x$

SOLUTION

$$\begin{aligned}\int (e^x - 4\sin x)\,dx &= e^x - 4\int \sin x\,dx \\ &= e^x - 4(-\cos x) + C = e^x + 4\cos x + C.\end{aligned}$$

As a check, we have

$$\frac{d}{dx}\left(e^x + 4\cos x + C\right) = e^x - 4\sin x$$

as needed.

9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) $f(x) = \sin x$	(i) $F(x) = \cos(1-x)$
(b) $f(x) = x\sin(x^2)$	(ii) $F(x) = -\cos x$
(c) $f(x) = \sin(1-x)$	(iii) $F(x) = -\frac{1}{2}\cos(x^2)$
(d) $f(x) = x\sin x$	(iv) $F(x) = \sin x - x\cos x$

SOLUTION

(a) An antiderivative of $\sin x$ is $-\cos x$, which is **(ii)**. As a check, we have $\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x$.

(b) An antiderivative of $x\sin(x^2)$ is $-\frac{1}{2}\cos(x^2)$, which is **(iii)**. This is because, by the Chain Rule, we have $\frac{d}{dx}\left(-\frac{1}{2}\cos(x^2)\right) = -\frac{1}{2}\left(-\sin(x^2)\right)\cdot 2x = x\sin(x^2)$.

(c) An antiderivative of $\sin(1-x)$ is $\cos(1-x)$ or **(i)**. As a check, we have $\frac{d}{dx}\cos(1-x) = -\sin(1-x)\cdot(-1) = \sin(1-x)$.

(d) An antiderivative of $x\sin x$ is $\sin x - x\cos x$, which is **(iv)**. This is because

$$\frac{d}{dx}(\sin x - x\cos x) = \cos x - (x(-\sin x) + \cos x\cdot 1) = x\sin x$$

In Exercises 10–39, evaluate the indefinite integral.

10. $\int (9x+2)\,dx$

SOLUTION $\int (9x+2)\,dx = \frac{9}{2}x^2 + 2x + C.$

11. $\int (4 - 18x)\, dx$

SOLUTION $\int (4 - 18x)\, dx = 4x - 9x^2 + C.$

12. $\int x^{-3}\, dx$

SOLUTION $\int x^{-3}\, dx = \dfrac{x^{-2}}{-2} + C = -\dfrac{1}{2}x^{-2} + C.$

13. $\int t^{-6/11}\, dt$

SOLUTION $\int t^{-6/11}\, dt = \dfrac{t^{5/11}}{5/11} + C = \dfrac{11}{5}t^{5/11} + C.$

14. $\int (5t^3 - t^{-3})\, dt$

SOLUTION $\int (5t^3 - t^{-3})\, dt = \dfrac{5}{4}t^4 - \dfrac{t^{-2}}{-2} + C = \dfrac{5}{4}t^4 + \dfrac{1}{2}t^{-2} + C.$

15. $\int (18t^5 - 10t^4 - 28t)\, dt$

SOLUTION $\int (18t^5 - 10t^4 - 28t)\, dt = 3t^6 - 2t^5 - 14t^2 + C.$

16. $\int 14s^{9/5}\, ds$

SOLUTION $\int 14s^{9/5}\, ds = 14 \cdot \dfrac{s^{14/5}}{14/5} + C = 5s^{14/5} + C.$

17. $\int (z^{-4/5} - z^{2/3} + z^{5/4})\, dz$

SOLUTION $\int ((z^{-4/5} - z^{2/3} + z^{5/4})\, dz = \dfrac{z^{1/5}}{1/5} - \dfrac{z^{5/3}}{5/3} + \dfrac{z^{9/4}}{9/4} + C = 5z^{1/5} - \dfrac{3}{5}z^{5/3} + \dfrac{4}{9}z^{9/4} + C.$

18. $\int \dfrac{3}{2}\, dx$

SOLUTION $\int \dfrac{3}{2}\, dx = \dfrac{3}{2}x + C.$

19. $\int \dfrac{1}{\sqrt[3]{x}}\, dx$

SOLUTION $\int \dfrac{1}{\sqrt[3]{x}}\, dx = \int x^{-1/3}\, dx = \dfrac{x^{2/3}}{2/3} + C = \dfrac{3}{2}x^{2/3} + C.$

20. $\int \dfrac{dx}{x^{4/3}}$

SOLUTION $\int \dfrac{dx}{x^{4/3}} = \int x^{-4/3}\, dx = \dfrac{x^{-1/3}}{-1/3} + C = -\dfrac{3}{x^{1/3}} + C.$

21. $\int \dfrac{36\, dt}{t^3}$

SOLUTION $\int \dfrac{36}{t^3}\, dt = \int 36t^{-3}\, dt = 36\dfrac{t^{-2}}{-2} + C = -\dfrac{18}{t^2} + C.$

22. $\int x(x^2 - 4)\, dx$

SOLUTION $\int x(x^2 - 4)\, dx = \int (x^3 - 4x)\, dx = \dfrac{1}{4}x^4 - 2x^2 + C.$

23. $\int (t^{1/2} + 1)(t + 1)\, dt$

SOLUTION

$$\begin{aligned}\int (t^{1/2} + 1)(t + 1)\, dt &= \int (t^{3/2} + t + t^{1/2} + 1)\, dt \\ &= \frac{t^{5/2}}{5/2} + \frac{1}{2}t^2 + \frac{t^{3/2}}{3/2} + t + C \\ &= \frac{2}{5}t^{5/2} + \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + t + C\end{aligned}$$

24. $\int \frac{12 - z}{\sqrt{z}}\, dz$

SOLUTION $\int \frac{12 - z}{\sqrt{z}}\, dz = \int (12z^{-1/2} - z^{1/2})\, dz = 24z^{1/2} - \frac{2}{3}z^{3/2} + C.$

25. $\int \frac{x^3 + 3x - 4}{x^2}\, dx$

SOLUTION

$$\begin{aligned}\int \frac{x^3 + 3x - 4}{x^2}\, dx &= \int (x + 3x^{-1} - 4x^{-2})\, dx \\ &= \frac{1}{2}x^2 + 3\ln|x| + 4x^{-1} + C\end{aligned}$$

26. $\int \left(\frac{1}{3}\sin x - \frac{1}{4}\cos x\right) dx$

SOLUTION $\int \left(\frac{1}{3}\sin x - \frac{1}{4}\cos x\right) dx = -\frac{1}{3}\cos x - \frac{1}{4}\sin x + C.$

27. $\int 12 \sec x \tan x\, dx$

SOLUTION $\int 12 \sec x \tan x\, dx = 12 \sec x + C.$

28. $\int (\theta + \sec^2\theta)\, d\theta$

SOLUTION $\int (\theta + \sec^2\theta)\, d\theta = \frac{1}{2}\theta^2 + \tan\theta + C.$

29. $\int (\csc t \cot t)\, dt$

SOLUTION $\int (\csc t \cot t)\, dt = -\csc t + C.$

30. $\int \sin(7x - 5)\, dx$

SOLUTION $\int \sin(7x - 5)\, dx = -\frac{1}{7}\cos(7x - 5) + C.$

31. $\int \sec^2(7 - 3\theta)\, d\theta$

SOLUTION $\int \sec^2(7 - 3\theta)\, d\theta = -\frac{1}{3}\tan(7 - 3\theta) + C.$

32. $\int (\theta - \cos(1 - \theta))\, d\theta$

SOLUTION $\int (\theta - \cos(1 - \theta))\, d\theta = \frac{1}{2}\theta^2 + \sin(1 - \theta) + C.$

33. $\int 25 \sec^2(3z + 1)\, dz$

SOLUTION $\int 25 \sec^2(3z + 1)\, dz = \frac{25}{3}\tan(3z + 1) + C.$

34. $\int \sec(x + 5)\tan(x + 5)\, dx$

SOLUTION $\int \sec(x + 5)\tan(x + 5)\, dx = \sec(x + 5) + C.$

35. $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right) d\theta$

SOLUTION $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right) d\theta = \frac{1}{3}\sin(3\theta) - 2\tan\left(\frac{\theta}{4}\right) + C.$

36. $\int \left(\frac{4}{x} - e^x\right) dx$

SOLUTION $\int \left(\frac{4}{x} - e^x\right) dx = 4\ln|x| - e^x + C.$

37. $\int (3e^{5x})\, dx$

SOLUTION $\int (3e^{5x})\, dx = \frac{3}{5}e^{5x} + C.$

38. $\int e^{3t-4}\, dt$

SOLUTION $\int e^{3t-4}\, dt = \frac{1}{3}e^{3t-4} + C.$

39. $\int (8x - 4e^{5-2x})\, dx$

SOLUTION $\int (8x - 4e^{5-2x})\, dx = 4x^2 + 2e^{5-2x} + C.$

40. In Figure 3, is graph (A) or graph (B) the graph of an antiderivative of $f(x)$?

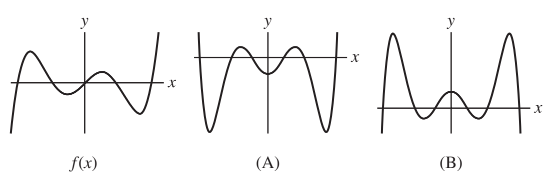


FIGURE 3

SOLUTION Let $F(x)$ be an antiderivative of $f(x)$. By definition, this means $F'(x) = f(x)$. In other words, $f(x)$ provides information as to the increasing/decreasing behavior of $F(x)$. Since, moving left to right, $f(x)$ transitions from $-$ to $+$ to $-$ to $+$ to $-$ to $+$, it follows that $F(x)$ must transition from decreasing to increasing to decreasing to increasing to decreasing to increasing. This describes the graph in (A)!

41. In Figure 4, which of graphs (A), (B), and (C) is *not* the graph of an antiderivative of $f(x)$? Explain.

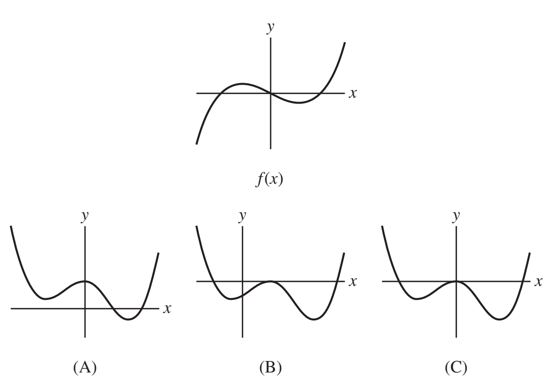


FIGURE 4

SOLUTION Let $F(x)$ be an antiderivative of $f(x)$. Notice that $f(x) = F'(x)$ changes sign from $-$ to $+$ to $-$ to $+$. Hence, $F(x)$ must transition from decreasing to increasing to decreasing to increasing.

- Both graph (A) and graph (C) meet the criteria discussed above and only differ by an additive constant. Thus either could be an antiderivative of $f(x)$.
- Graph (B) does not have the same local extrema as indicated by $f(x)$ and therefore is *not* an antiderivative of $f(x)$.

42. Show that $F(x) = \frac{1}{3}(x+13)^3$ is an antiderivative of $f(x) = (x+13)^2$.

SOLUTION Note that

$$\frac{d}{dx}F(x) = \frac{d}{dx}\frac{1}{3}(x+13)^3 = (x+13)^2.$$

Thus, $F(x) = \frac{1}{3}(x+13)^3$ is an antiderivative of $f(x) = (x+13)^2$.

In Exercises 43–46, verify by differentiation.

43. $\int (x+13)^6\,dx = \frac{1}{7}(x+13)^7 + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{7}(x+13)^7 + C\right) = (x+13)^6$ as required.

44. $\int (x+13)^{-5}\,dx = -\frac{1}{4}(x+13)^{-4} + C$

SOLUTION $\frac{d}{dx}\left(-\frac{1}{4}(x+13)^{-4} + C\right) = (x+13)^{-5}$ as required.

45. $\int (4x+13)^2\,dx = \frac{1}{12}(4x+13)^3 + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{12}(4x+13)^3 + C\right) = \frac{1}{4}(4x+13)^2(4) = (4x+13)^2$ as required.

46. $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$

SOLUTION $\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1} + C\right) = (ax+b)^n$ as required.

In Exercises 47–62, solve the initial value problem.

47. $\frac{dy}{dx} = x^3,\ \ y(0) = 4$

SOLUTION Since $\frac{dy}{dx} = x^3$, we have

$$y = \int x^3\,dx = \frac{1}{4}x^4 + C.$$

Thus,

$$4 = y(0) = \frac{1}{4}0^4 + C = C,$$

so that $C = 4$. Therefore, $y = \frac{1}{4}x^4 + 4$.

48. $\frac{dy}{dt} = 3 - 2t,\ \ y(0) = -5$

SOLUTION Since $\frac{dy}{dt} = 3 - 2t$, we have

$$y = \int (3-2t)\,dt = 3t - t^2 + C.$$

Thus,

$$-5 = y(0) = 3(0) - (0)^2 + C = C,$$

so that $C = -5$. Therefore, $y = 3t - t^2 - 5$.

49. $\frac{dy}{dt} = 2t + 9t^2,\ \ y(1) = 2$

SOLUTION Since $\frac{dy}{dt} = 2t + 9t^2$, we have

$$y = \int (2t + 9t^2)\,dt = t^2 + 3t^3 + C.$$

Thus,

$$2 = y(1) = 1^2 + 3(1)^3 + C,$$

so that $C = -2$. Therefore $y = t^2 + 3t^3 - 2$.

50. $\dfrac{dy}{dx} = 8x^3 + 3x^2, \;\; y(2) = 0$

SOLUTION Since $\frac{dy}{dx} = 8x^3 + 3x^2$, we have

$$y = \int (8x^3 + 3x^2)\,dx = 2x^4 + x^3 + C.$$

Thus

$$0 = y(2) = 2(2)^4 + 2^3 + C,$$

so that $C = -40$. Therefore, $y = 2x^4 + x^3 - 40$.

51. $\dfrac{dy}{dt} = \sqrt{t}, \;\; y(1) = 1$

SOLUTION Since $\frac{dy}{dt} = \sqrt{t} = t^{1/2}$, we have

$$y = \int t^{1/2}\,dt = \frac{2}{3}t^{3/2} + C.$$

Thus

$$1 = y(1) = \frac{2}{3} + C,$$

so that $C = \frac{1}{3}$. Therefore, $y = \frac{2}{3}t^{3/2} + \frac{1}{3}$.

52. $\dfrac{dz}{dt} = t^{-3/2}, \;\; z(4) = -1$

SOLUTION Since $\frac{dz}{dt} = t^{-3/2}$, we have

$$z = \int t^{-3/2}\,dt = -2t^{-1/2} + C.$$

Thus

$$-1 = z(4) = -2(4)^{-1/2} + C,$$

so that $C = 0$. Therefore, $z = -2t^{-1/2}$.

53. $\dfrac{dy}{dx} = (3x + 2)^3, \;\; y(0) = 1$

SOLUTION Since $\frac{dy}{dx} = (3x + 2)^3$, we have

$$y = \int (3x + 2)^3\,dx = \frac{1}{4} \cdot \frac{1}{3}(3x + 2)^4 + C = \frac{1}{12}(3x + 2)^4 + C.$$

Thus,

$$1 = y(0) = \frac{1}{12}(2)^4 + C,$$

so that $C = 1 - \frac{4}{3} = -\frac{1}{3}$. Therefore, $y = \frac{1}{12}(3x + 2)^4 - \frac{1}{3}$.

54. $\dfrac{dy}{dt} = (4t + 3)^{-2}, \;\; y(1) = 0$

SOLUTION Since $\frac{dy}{dt} = (4t + 3)^{-2}$, we have

$$y = \int (4t + 3)^{-2}\,dt = \frac{1}{-1} \cdot \frac{1}{4}(4t + 3)^{-1} + C = -\frac{1}{4}(4t + 3)^{-1} + C.$$

Thus,

$$0 = y(1) = -\frac{1}{4}(7)^{-1} + C,$$

so that $C = \frac{1}{28}$. Therefore, $y = -\frac{1}{4}(4t + 3)^{-1} + \frac{1}{28}$.

55. $\dfrac{dy}{dx} = \sin x, \;\; y\left(\dfrac{\pi}{2}\right) = 1$

SOLUTION Since $\frac{dy}{dx} = \sin x$, we have

$$y = \int \sin x \, dx = -\cos x + C.$$

Thus

$$1 = y\left(\frac{\pi}{2}\right) = 0 + C,$$

so that $C = 1$. Therefore, $y = 1 - \cos x$.

56. $\dfrac{dy}{dz} = \sin 2z, \;\; y\left(\dfrac{\pi}{4}\right) = 4$

SOLUTION Since $\frac{dy}{dz} = \sin 2z$, we have

$$y = \int \sin 2z \, dz = -\frac{1}{2}\cos 2z + C.$$

Thus

$$4 = y\left(\frac{\pi}{4}\right) = 0 + C,$$

so that $C = 4$. Therefore, $y = 4 - \frac{1}{2}\cos 2z$.

57. $\dfrac{dy}{dx} = \cos 5x, \;\; y(\pi) = 3$

SOLUTION Since $\frac{dy}{dx} = \cos 5x$, we have

$$y = \int \cos 5x \, dx = \frac{1}{5}\sin 5x + C.$$

Thus $3 = y(\pi) = 0 + C$, so that $C = 3$. Therefore, $y = 3 + \frac{1}{5}\sin 5x$.

58. $\dfrac{dy}{dx} = \sec^2 3x, \;\; y\left(\dfrac{\pi}{4}\right) = 2$

SOLUTION Since $\frac{dy}{dx} = \sec^2 3x$, we have

$$y = \int \sec^2(3x)\, dx = \frac{1}{3}\tan(3x) + C.$$

Since $y\left(\frac{\pi}{4}\right) = 2$, we get:

$$\begin{aligned} 2 &= \frac{1}{3}\tan\left(3\frac{\pi}{4}\right) + C \\ 2 &= \frac{1}{3}(-1) + C \\ \frac{7}{3} &= C. \end{aligned}$$

Therefore, $y = \frac{1}{3}\tan(3x) + \frac{7}{3}$.

59. $\dfrac{dy}{dx} = e^x, \;\; y(2) = 0$

SOLUTION Since $\frac{dy}{dx} = e^x$, we have

$$y = \int e^x \, dx = e^x + C.$$

Thus,

$$0 = y(2) = e^2 + C,$$

so that $C = -e^2$. Therefore, $y = e^x - e^2$.

60. $\dfrac{dy}{dt} = e^{-t}, \;\; y(0) = 0$

SOLUTION Since $\frac{dy}{dt} = e^{-t}$, we have

$$y = \int e^{-t}\,dt = -e^{-t} + C.$$

Thus,

$$0 = y(0) = -e^0 + C,$$

so that $C = 1$. Therefore, $y = -e^{-t} + 1$.

61. $\dfrac{dy}{dt} = 9e^{12-3t}, \;\; y(4) = 7$

SOLUTION Since $\frac{dy}{dt} = 9e^{12-3t}$, we have

$$y = \int 9e^{12-3t}\,dt = -3e^{12-3t} + C.$$

Thus,

$$7 = y(4) = -3e^0 + C,$$

so that $C = 10$. Therefore, $y = -3e^{12-3t} + 10$.

62. $\dfrac{dy}{dt} = t + 2e^{t-9}, \;\; y(9) = 4$

SOLUTION Since $\frac{dy}{dt} = t + 2e^{t-9}$, we have

$$y = \int (t + 2e^{t-9})\,dt = \frac{1}{2}t^2 + 2e^{t-9} + C.$$

Thus,

$$4 = y(9) = \frac{1}{2}(9)^2 + 2e^0 + C,$$

so that $C = -\frac{77}{2}$. Therefore, $y = \frac{1}{2}t^2 + 2e^{t-9} - \frac{77}{2}$.

In Exercises 63–69, first find f' and then find f.

63. $f''(x) = 12x, \quad f'(0) = 1, \quad f(0) = 2$

SOLUTION Let $f''(x) = 12x$. Then $f'(x) = 6x^2 + C$. Given $f'(0) = 1$, it follows that $1 = 6(0)^2 + C$ and $C = 1$. Thus, $f'(x) = 6x^2 + 1$. Next, $f(x) = 2x^3 + x + C$. Given $f(0) = 2$, it follows that $2 = 2(0)^3 + 0 + C$ and $C = 2$. Finally, $f(x) = 2x^3 + x + 2$.

64. $f''(x) = x^3 - 2x, \quad f'(1) = 0, \quad f(1) = 2$

SOLUTION Let $f''(x) = x^3 - 2x$. Then $f'(x) = \frac{1}{4}x^4 - x^2 + C$. Given $f'(1) = 0$, it follows that $0 = \frac{1}{4}(1)^4 - (1)^2 + C$ and $C = \frac{3}{4}$. Thus, $f'(x) = \frac{1}{4}x^4 - x^2 + \frac{3}{4}$. Next, $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{3}{4}x + C$. Given $f(1) = 2$, it follows that $2 = \frac{1}{20}(1)^5 - \frac{1}{3}(1)^3 + \frac{3}{4} + C$ and $C = \frac{23}{15}$. Finally, $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{3}{4}x + \frac{23}{15}$.

65. $f''(x) = x^3 - 2x + 1, \quad f'(0) = 1, \quad f(0) = 0$

SOLUTION Let $g(x) = f'(x)$. The statement gives us $g'(x) = x^3 - 2x + 1$, $g(0) = 1$. From this, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$. $g(0) = 1$ gives us $1 = C$, so $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1$. $f'(x) = \frac{1}{4}x^4 - x^2 + x + 1$, so $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$. $f(0) = 0$ gives $C = 0$, so

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x.$$

66. $f''(x) = x^3 - 2x + 1, \quad f'(1) = 0, \quad f(1) = 4$

SOLUTION Let $g(x) = f'(x)$. The problem statement gives us $g'(x) = x^3 - 2x + 1$, $g(0) = 0$. From $g'(x)$, we get $g(x) = \frac{1}{4}x^4 - x^2 + x + C$, and from $g(1) = 0$, we get $0 = \frac{1}{4} - 1 + 1 + C$, so that $C = -\frac{1}{4}$. This gives $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x - \frac{1}{4}$. From $f'(x)$, we get $f(x) = \frac{1}{4}(\frac{1}{5}x^5) - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + C = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + C$. From $f(1) = 4$, we get

$$\frac{1}{20} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + C = 4,$$

so that $C = \frac{121}{30}$. Hence,

$$f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + \frac{121}{30}.$$

67. $f''(t) = t^{-3/2}, \quad f'(4) = 1, \quad f(4) = 4$

SOLUTION Let $g(t) = f'(t)$. The problem statement is $g'(t) = t^{-3/2}$, $g(4) = 1$. From $g'(t)$ we get $g(t) = \frac{1}{-1/2}t^{-1/2} + C = -2t^{-1/2} + C$. From $g(4) = 1$ we get $-1 + C = 1$ so that $C = 2$. Hence $f'(t) = g(t) = -2t^{-1/2} + 2$. From $f'(t)$ we get $f(t) = -2\frac{1}{1/2}t^{1/2} + 2t + C = -4t^{1/2} + 2t + C$. From $f(4) = 4$ we get $-8 + 8 + C = 4$, so that $C = 4$. Hence, $f(t) = -4t^{1/2} + 2t + 4$.

68. $f''(\theta) = \cos\theta, \quad f'\left(\frac{\pi}{2}\right) = 1, \quad f\left(\frac{\pi}{2}\right) = 6$

SOLUTION Let $g(\theta) = f'(\theta)$. The problem statement gives

$$g'(\theta) = \cos\theta, \qquad g\left(\frac{\pi}{2}\right) = 1.$$

From $g'(\theta)$ we get $g(\theta) = \sin\theta + C$. From $g(\frac{\pi}{2}) = 1$ we get $1 + C = 1$, so $C = 0$. Hence $f'(\theta) = g(\theta) = \sin\theta$. From $f'(\theta)$ we get $f(\theta) = -\cos\theta + C$. From $f(\frac{\pi}{2}) = 6$ we get $C = 6$, so

$$f(\theta) = -\cos\theta + 6.$$

69. $f''(t) = t - \cos t, \quad f'(0) = 2, \quad f(0) = -2$

SOLUTION Let $g(t) = f'(t)$. The problem statement gives

$$g'(t) = t - \cos t, \qquad g(0) = 2.$$

From $g'(t)$, we get $g(t) = \frac{1}{2}t^2 - \sin t + C$. From $g(0) = 2$, we get $C = 2$. Hence $f'(t) = g(t) = \frac{1}{2}t^2 - \sin t + 2$. From $f'(t)$, we get $f(t) = \frac{1}{2}(\frac{1}{3}t^3) + \cos t + 2t + C$. From $f(0) = -2$, we get $1 + C = -2$, hence $C = -3$, and

$$f(t) = \frac{1}{6}t^3 + \cos t + 2t - 3.$$

70. Show that $F(x) = \tan^2 x$ and $G(x) = \sec^2 x$ have the same derivative. What can you conclude about the relation between F and G? Verify this conclusion directly.

SOLUTION Let $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. Then $f'(x) = 2\tan x \sec^2 x$ and $g'(x) = 2\sec x \cdot \sec x \tan x = 2\tan x \sec^2 x$; hence $f'(x) = g'(x)$. Accordingly, $f(x)$ and $g(x)$ must differ by a constant; i.e., $f(x) - g(x) = \tan^2 x - \sec^2 x = C$ for some constant C. To see that this is true directly, divide the identity $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$. This yields $\tan^2 x + 1 = \sec^2 x$, so that $\tan^2 x - \sec^2 x = -1$.

71. A particle located at the origin at $t = 1$ s moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. State the differential equation with initial condition satisfied by the position $s(t)$ of the particle, and find $s(t)$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = 6t^2 - t,$$

and the associated initial condition is $s(1) = 0$. From the differential equation, we find

$$s(t) = \int (6t^2 - t)\, dt = 2t^3 - \frac{1}{2}t^2 + C.$$

Using the initial condition, it follows that

$$0 = s(1) = 2 - \frac{1}{2} + C \quad \text{so} \quad C = -\frac{3}{2}.$$

Finally,

$$s(t) = 2t^3 - \frac{1}{2}t^2 - \frac{3}{2}.$$

72. A particle moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. Find the particle's position $s(t)$ assuming that $s(2) = 4$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = 6t^2 - t,$$

and the associated initial condition is $s(2) = 4$. From the differential equation, we find

$$s(t) = \int (6t^2 - t)\,dt = 2t^3 - \frac{1}{2}t^2 + C.$$

Using the initial condition, it follows that

$$4 = s(2) = 16 - 2 + C \quad \text{so} \quad C = -10.$$

Finally,

$$s(t) = 2t^3 - \frac{1}{2}t^2 - 10.$$

73. A mass oscillates at the end of a spring. Let $s(t)$ be the displacement of the mass from the equilibrium position at time t. Assuming that the mass is located at the origin at $t = 0$ and has velocity $v(t) = \sin(\pi t/2)$ m/s, state the differential equation with initial condition satisfied by $s(t)$, and find $s(t)$.

SOLUTION The differential equation satisfied by $s(t)$ is

$$\frac{ds}{dt} = v(t) = \sin(\pi t/2),$$

and the associated initial condition is $s(0) = 0$. From the differential equation, we find

$$s(t) = \int \sin(\pi t/2)\,dt = -\frac{2}{\pi}\cos(\pi t/2) + C.$$

Using the initial condition, it follows that

$$0 = s(0) = -\frac{2}{\pi} + C \quad \text{so} \quad C = \frac{2}{\pi}.$$

Finally,

$$s(t) = \frac{2}{\pi}(1 - \cos(\pi t/2)).$$

74. Beginning at $t = 0$ with initial velocity 4 m/s, a particle moves in a straight line with acceleration $a(t) = 3t^{1/2}$ m/s^2. Find the distance traveled after 25 seconds.

SOLUTION Given $a(t) = 3t^{1/2}$ and an initial velocity of 4 m/s, it follows that $v(t)$ satisfies

$$\frac{dv}{dt} = 3t^{1/2}, \quad v(0) = 4.$$

Thus,

$$v(t) = \int 3t^{1/2}\,dt = 2t^{3/2} + C.$$

Using the initial condition, we find

$$4 = v(0) = 2(0)^{3/2} + C \quad \text{so} \quad C = 4$$

and $v(t) = 2t^{3/2} + 4$. Next,

$$s = \int v(t)\,dt = \int (2t^{3/2} + 4)\,dt = \frac{4}{5}t^{5/2} + 4t + C.$$

Finally, the distance traveled after 25 seconds is

$$s(25) - s(0) = \frac{4}{5}(25)^{5/2} + 4(25) = 2600$$

meters.

75. A car traveling 25 m/s begins to decelerate at a constant rate of 4 m/s^2. After how many seconds does the car come to a stop and how far will the car have traveled before stopping?

SOLUTION Since the acceleration of the car is a constant -4m/s^2, v is given by the differential equation:

$$\frac{dv}{dt} = -4, \qquad v(0) = 25.$$

From $\frac{dv}{dt}$, we get $v(t) = \int -4\,dt = -4t + C$. Since $v(0)25$, $C = 25$. From this, $v(t) = -4t + 25\ \frac{\text{m}}{\text{s}}$. To find the time until the car stops, we must solve $v(t) = 0$:

$$\begin{aligned} -4t + 25 &= 0 \\ 4t &= 25 \\ t &= 25/4 = 6.25 \text{ s.} \end{aligned}$$

Now we have a differential equation for $s(t)$. Since we want to know how far the car has traveled from the beginning of its deceleration at time $t = 0$, we have $s(0) = 0$ by definition, so:

$$\frac{ds}{dt} = v(t) = -4t + 25, \qquad s(0) = 0.$$

From this, $s(t) = \int(-4t + 25)\,dt = -2t^2 + 25t + C$. Since $s(0) = 0$, we have $C = 0$, and

$$s(t) = -2t^2 + 25t.$$

At stopping time $t = 0.25$ s, the car has traveled

$$s(6.25) = -2(6.25)^2 + 25(6.25) = 78.125 \text{ m.}$$

76. At time $t = 1$ s, a particle is traveling at 72 m/s and begins to decelerate at the rate $a(t) = -t^{-1/2}$ until it stops. How far does the particle travel before stopping?

SOLUTION With $a(t) = -t^{-1/2}$ and a velocity of 72 m/s at $t = 1$ s, it follows that $v(t)$ satisfies

$$\frac{dv}{dt} = -t^{-1/2}, \quad v(1) = 72.$$

Thus,

$$v(t) = \int -t^{-1/2}\,dt = -2t^{1/2} + C.$$

Using the initial condition, we find

$$72 = v(1) = -2 + C \quad \text{so} \quad C = 74,$$

and $v(t) = 74 - 2t^{1/2}$. The particle comes to rest when

$$74 - 2t^{1/2} = 0 \quad \text{or when} \quad t = 37^2 = 1369$$

seconds. Now,

$$s(t) = \int v(t)\,dt = \int (74 - 2t^{1/2})\,dt = 74t - \frac{4}{3}t^{3/2} + C.$$

The distance traveled by the particle before it comes to rest is then

$$s(1369) - s(1) = 74(1369) - \frac{202612}{3} - 74 + \frac{4}{3} = 33696$$

meters.

77. A 900-kg rocket is released from a space station. As it burns fuel, the rocket's mass decreases and its velocity increases. Let $v(m)$ be the velocity (in meters per second) as a function of mass m. Find the velocity when $m = 729$ if $dv/dm = -50m^{-1/2}$. Assume that $v(900) = 0$.

SOLUTION Since $\frac{dv}{dm} = -50m^{-1/2}$, we have $v(m) = \int -50m^{-1/2}\,dm = -100m^{1/2} + C$. Thus $0 = v(900) = -100\sqrt{900} + C = -3000 + C$, and $C = 3000$. Therefore, $v(m) = 3000 - 100\sqrt{m}$. Accordingly,

$$v(729) = 3000 - 100\sqrt{729} = 3000 - 100(27) = 300 \text{ meters/sec.}$$

78. As water flows through a tube of radius $R = 10$ cm, the velocity v of an individual water particle depends only on its distance r from the center of the tube. The particles at the walls of the tube have zero velocity and $dv/dr = -0.06r$. Determine $v(r)$.

SOLUTION The statement amounts to the differential equation and initial condition:

$$\frac{dv}{dr} = -0.06r, \qquad v(R) = 0.$$

From $\frac{dv}{dr} = -0.06r$, we get

$$v(r) = \int -0.06r\,dr = -0.06\frac{r^2}{2} + C = -0.03r^2 + C.$$

Plugging in $v(R) = 0$, we get $-0.03R^2 + C = 0$, so that $C = 0.03R^2$. Therefore,

$$v(r) = -0.03r^2 + 0.03R^2 = 0.03(R^2 - r^2) \text{ cm/s}.$$

If $R = 10$ centimeters, we get:

$$v(r) = 0.03(10^2 - r^2).$$

79. Verify the linearity properties of the indefinite integral stated in Theorem 4.

SOLUTION To verify the Sum Rule, let $F(x)$ and $G(x)$ be any antiderivatives of $f(x)$ and $g(x)$, respectively. Because

$$\frac{d}{dx}(F(x) + G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x) + g(x),$$

it follows that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$; i.e.,

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

To verify the Multiples Rule, again let $F(x)$ be any antiderivative of $f(x)$ and let c be a constant. Because

$$\frac{d}{dx}(cF(x)) = c\frac{d}{dx}F(x) = cf(x),$$

it follows that $cF(x)$ is and antiderivative of $cf(x)$; i.e.,

$$\int (cf(x))\,dx = c\int f(x)\,dx.$$

Further Insights and Challenges

80. Find constants c_1 and c_2 such that $F(x) = c_1 x\sin x + c_2\cos x$ is an antiderivative of $f(x) = x\cos x$.

SOLUTION Let $F(x) = c_1 x\sin x + c_2\cos x$. If $F(x)$ is to be an antiderivative of $f(x) = x\cos x$, we must have $F'(x) = f(x)$ for all x. Hence $c_1\,(x\cos x + \sin x) - c_2\sin x = x\cos x$ for all x. Equating coefficients on the left- and right-hand sides, we have $c_1 = 1$ (i.e., the coefficients of $x\cos x$ are equal) and $c_1 - c_2 = 0$ (i.e., the coefficients of $\sin x$ are equal). Thus $c_1 = c_2 = 1$ and hence $F(x) = x\sin x + \cos x$. As a check, we have $F'(x) = x\cos x + \sin x - \sin x = x\cos x = f(x)$, as required.

81. Find constants c_1 and c_2 such that $F(x) = c_1 xe^x + c_2 e^x$ is an antiderivative of $f(x) = xe^x$.

SOLUTION Let $F(x) = c_1 xe^x + c_2 e^x$. If $F(x)$ is to be an antiderivative of $f(x) = xe^x$, we must have $F'(x) = f(x)$ for all x. Hence,

$$c_1 xe^x + (c_1 + c_2)e^x = xe^x = 1 \cdot xe^x + 0 \cdot e^x.$$

Equating coefficients of like terms we have $c_1 = 1$ and $c_1 + c_2 = 0$. Thus, $c_1 = 1$ and $c_2 = -1$.

82. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Is it true that $F(x)G(x)$ is an antiderivative of $f(x)g(x)$? Confirm or provide a counterexample.

SOLUTION Let $f(x) = x^2$ and $g(x) = x^3$. Then $F(x) = \frac{1}{3}x^3$ and $G(x) = \frac{1}{4}x^4$ are antiderivatives for $f(x)$ and $g(x)$, respectively. Let $h(x) = f(x)g(x) = x^5$, the general antiderivative of which is $H(x) = \frac{1}{6}x^6 + C$. There is no value of the constant C for which $F(x)G(x) = \frac{1}{12}x^7$ equals $H(x)$. Accordingly, $F(x)G(x)$ is *not* an antiderivative of $h(x) = f(x)g(x)$.

83. Suppose that $F'(x) = f(x)$.
(a) Show that $\frac{1}{2}F(2x)$ is an antiderivative of $f(2x)$.
(b) Find the general antiderivative of $f(kx)$ for $k \neq 0$.

SOLUTION Let $F'(x) = f(x)$.
(a) By the Chain Rule, we have

$$\frac{d}{dx}\left(\frac{1}{2}F(2x)\right) = \frac{1}{2}F'(2x) \cdot 2 = F'(2x) = f(2x).$$

Thus $\frac{1}{2}F(2x)$ is an antiderivative of $f(2x)$.
(b) For nonzero constant k, the Chain Rules gives

$$\frac{d}{dx}\left(\frac{1}{k}F(kx)\right) = \frac{1}{k}F'(kx) \cdot k = F'(kx) = f(kx).$$

Thus $\frac{1}{k}F(kx)$ is an antiderivative of $f(kx)$. Hence the general antiderivative of $f(kx)$ is $\frac{1}{k}F(kx) + C$, where C is a constant.

84. Find an antiderivative for $f(x) = |x|$.

SOLUTION Let $f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$. Then the general antiderivative of $f(x)$ is

$$F(x) = \int f(x)\,dx = \begin{cases} \int x\,dx & \text{for } x \geq 0 \\ \int -x\,dx & \text{for } x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 + C & \text{for } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{for } x < 0 \end{cases}.$$

85. Using Theorem 1, prove that $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n-1$, then $F(x)$ is a polynomial of degree n. Then prove that if $g(x)$ is any function such that $g^{(n)}(x) = 0$, then $g(x)$ is a polynomial of degree at most n.

SOLUTION Suppose $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n-1$. Now, we know that the derivative of a polynomial of degree n is a polynomial of degree $n-1$, and hence an antiderivative of a polynomial of degree $n-1$ is a polynomial of degree n. Thus, by Theorem 1, $F(x)$ can differ from a polynomial of degree n by at most a constant term, which is still a polynomial of degree n. Now, suppose that $g(x)$ is any function such that $g^{(n+1)}(x) = 0$. We know that the $n+1$-st derivative of any polynomial of degree at most n is zero, so by repeated application of Theorem 1, $g(x)$ can differ from a polynomial of degree at most n by at most a constant term. Hence, $g(x)$ is a polynomial of degree at most n.

86. Show that $F(x) = \dfrac{x^{n+1} - 1}{n+1}$ is an antiderivative of $y = x^n$ for $n \neq -1$. Then use L'Hôpital's Rule to prove that

$$\lim_{n \to -1} F(x) = \ln x$$

In this limit, x is fixed and n is the variable. This result shows that, although the Power Rule breaks down for $n = -1$, the antiderivative of $y = x^{-1}$ is a limit of antiderivatives of x^n as $n \to -1$.

SOLUTION If $n \neq -1$, then

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^{n+1} - 1}{n+1}\right) = x^n.$$

Therefore, $F(x)$ is an antiderivative of $y = x^n$. Using L'Hôpital's Rule,

$$\lim_{n \to -1} F(x) = \lim_{n \to -1} \frac{x^{n+1} - 1}{n+1} = \lim_{n \to -1} \frac{x^{n+1}\ln x}{1} = \ln x.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error.

1. $8.1^{1/3} - 2$

SOLUTION Let $f(x) = x^{1/3}$, $a = 8$ and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(a) = \frac{1}{12}$ and, by the Linear Approximation,

$$\Delta f = 8.1^{1/3} - 2 \approx f'(a)\Delta x = \frac{1}{12}(0.1) = 0.00833333.$$

Using a calculator, $8.1^{1/3} - 2 = 0.00829885$. The error in the Linear Approximation is therefore

$$|0.00829885 - 0.00833333| = 3.445 \times 10^{-5}.$$

2. $\dfrac{1}{\sqrt{4.1}} - \dfrac{1}{2}$

SOLUTION Let $f(x) = x^{-1/2}$, $a = 4$ and $\Delta x = 0.1$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(a) = -\frac{1}{16}$ and, by the Linear Approximation,

$$\Delta f = \frac{1}{\sqrt{4.1}} - \frac{1}{2} \approx f'(a)\Delta x = -\frac{1}{16}(0.1) = -0.00625.$$

Using a calculator,

$$\frac{1}{\sqrt{4.1}} - \frac{1}{2} = -0.00613520.$$

The error in the Linear Approximation is therefore

$$|-0.00613520 - (-0.00625)| = 1.148 \times 10^{-4}.$$

3. $625^{1/4} - 624^{1/4}$

SOLUTION Let $f(x) = x^{1/4}$, $a = 625$ and $\Delta x = -1$. Then $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(a) = \frac{1}{500}$ and, by the Linear Approximation,

$$\Delta f = 624^{1/4} - 625^{1/4} \approx f'(a)\Delta x = \frac{1}{500}(-1) = -0.002.$$

Thus $625^{1/4} - 624^{1/4} \approx 0.002$. Using a calculator,

$$625^{1/4} - 624^{1/4} = 0.00200120.$$

The error in the Linear Approximation is therefore

$$|0.00200120 - (0.002)| = 1.201 \times 10^{-6}.$$

4. $\sqrt{101}$

SOLUTION Let $f(x) = \sqrt{x}$ and $a = 100$. Then $f(a) = 10$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = \frac{1}{20}$. The linearization of $f(x)$ at $a = 100$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 10 + \frac{1}{20}(x - 100),$$

and $\sqrt{101} \approx L(101) = 10.05$. Using a calculator, $\sqrt{101} = 10.049876$, so the error in the Linear Approximation is

$$|10.049876 - 10.05| = 1.244 \times 10^{-4}.$$

5. $\dfrac{1}{1.02}$

SOLUTION Let $f(x) = x^{-1}$ and $a = 1$. Then $f(a) = 1$, $f'(x) = -x^{-2}$ and $f'(a) = -1$. The linearization of $f(x)$ at $a = 1$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 1 - (x - 1) = 2 - x,$$

and $\frac{1}{1.02} \approx L(1.02) = 0.98$. Using a calculator, $\frac{1}{1.02} = 0.980392$, so the error in the Linear Approximation is

$$|0.980392 - 0.98| = 3.922 \times 10^{-4}.$$

6. $\sqrt[5]{33}$

SOLUTION Let $f(x) = x^{1/5}$ and $a = 32$. Then $f(a) = 2$, $f'(x) = \frac{1}{5}x^{-4/5}$ and $f'(a) = \frac{1}{80}$. The linearization of $f(x)$ at $a = 32$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{80}(x - 32),$$

and $\sqrt[5]{33} \approx L(33) = 2.0125$. Using a calculator, $\sqrt[5]{33} = 2.012347$, so the error in the Linear Approximation is

$$|2.012347 - 2.0125| = 1.534 \times 10^{-4}.$$

In Exercises 7–12, find the linearization at the point indicated.

7. $y = \sqrt{x}, \quad a = 25$

SOLUTION Let $y = \sqrt{x}$ and $a = 25$. Then $y(a) = 5$, $y' = \frac{1}{2}x^{-1/2}$ and $y'(a) = \frac{1}{10}$. The linearization of y at $a = 25$ is therefore

$$L(x) = y(a) + y'(a)(x - 25) = 5 + \frac{1}{10}(x - 25).$$

8. $v(t) = 32t - 4t^2, \quad a = 2$

SOLUTION Let $v(t) = 32t - 4t^2$ and $a = 2$. Then $v(a) = 48$, $v'(t) = 32 - 8t$ and $v'(a) = 16$. The linearization of $v(t)$ at $a = 2$ is therefore

$$L(t) = v(a) + v'(a)(t - a) = 48 + 16(t - 2) = 16t + 16.$$

9. $A(r) = \frac{4}{3}\pi r^3, \quad a = 3$

SOLUTION Let $A(r) = \frac{4}{3}\pi r^3$ and $a = 3$. Then $A(a) = 36\pi$, $A'(r) = 4\pi r^2$ and $A'(a) = 36\pi$. The linearization of $A(r)$ at $a = 3$ is therefore

$$L(r) = A(a) + A'(a)(r - a) = 36\pi + 36\pi(r - 3) = 36\pi(r - 2).$$

10. $V(h) = 4h(2 - h)(4 - 2h), \quad a = 1$

SOLUTION Let $V(h) = 4h(2 - h)(4 - 2h) = 32h - 32h^2 + 8h^3$ and $a = 1$. Then $V(a) = 8$, $V'(h) = 32 - 64h + 24h^2$ and $V'(a) = -8$. The linearization of $V(h)$ at $a = 1$ is therefore

$$L(h) = V(a) + V'(a)(h - a) = 8 - 8(h - 1) = 16 - 8h.$$

11. $P(x) = e^{-x^2/2}, \quad a = 1$

SOLUTION Let $P(x) = e^{-x^2/2}$ and $a = 1$. Then $P(a) = e^{-1/2}$, $P'(x) = -xe^{-x^2/2}$, and $P'(a) = -e^{-1/2}$. The linearization of $P(x)$ at $a = 1$ is therefore

$$L(x) = P(a) + P'(a)(x - a) = e^{-1/2} - e^{-1/2}(x - 1) = \frac{1}{\sqrt{e}}(2 - x).$$

12. $f(x) = \ln(x + e), \quad a = e$

SOLUTION Let $f(x) = \ln(x + e)$ and $a = e$. Then $f(a) = \ln(2e) = 1 + \ln 2$, $P'(x) = \frac{1}{x+e}$, and $P'(a) = \frac{1}{2e}$. The linearization of $f(x)$ at $a = e$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 1 + \ln 2 + \frac{1}{2e}(x - e).$$

In Exercises 13–18, use the Linear Approximation.

13. The position of an object in linear motion at time t is $s(t) = 0.4t^2 + (t + 1)^{-1}$. Estimate the distance traveled over the time interval $[4, 4.2]$.

SOLUTION Let $s(t) = 0.4t^2 + (t + 1)^{-1}$, $a = 4$ and $\Delta t = 0.2$. Then $s'(t) = 0.8t - (t + 1)^{-2}$ and $s'(a) = 3.16$. Using the Linear Approximation, the distance traveled over the time interval $[4, 4.2]$ is approximately

$$\Delta s = s(4.2) - s(4) \approx s'(a)\Delta t = 3.16(0.2) = 0.632.$$

14. A bond that pays \$10,000 in 6 years is offered for sale at a price P. The percentage yield Y of the bond is

$$Y = 100\left(\left(\frac{10{,}000}{P}\right)^{1/6} - 1\right)$$

Verify that if $P = \$7500$, then $Y = 4.91\%$. Estimate the drop in yield if the price rises to \$7700.

SOLUTION Let $P = \$7500$. Then

$$Y = 100\left(\left(\frac{10{,}000}{7500}\right)^{1/6} - 1\right) = 4.91\%.$$

If the price is raised to \$7700, then $\Delta P = 200$. With

$$\frac{dY}{dP} = -\frac{1}{6}100(10{,}000)^{1/6}P^{-7/6} = -\frac{10^{8/3}}{6}P^{-7/6},$$

we estimate using the Linear Approximation that

$$\Delta Y \approx Y'(7500)\Delta P = -0.46\%.$$

15. When a bus pass from Albuquerque to Los Alamos is priced at p dollars, a bus company takes in a monthly revenue of $R(p) = 1.5p - 0.01p^2$ (in thousands of dollars).

(a) Estimate ΔR if the price rises from \$50 to \$53.

(b) If $p = 80$, how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

SOLUTION

(a) If the price is raised from \$50 to \$53, then $\Delta p = 3$ and

$$\Delta R \approx R'(50)\Delta p = (1.5 - 0.02(50))(3) = 1.5$$

We therefore estimate an increase of \$1500 in revenue.

(b) Because $R'(80) = 1.5 - 0.02(80) = -0.1$, the Linear Approximation gives $\Delta R \approx -0.1\Delta p$. A small increase in price would thus result in a decrease in revenue.

16. A store sells 80 MP4 players per week when the players are priced at $P = \$75$. Estimate the number N sold if P is raised to \$80, assuming that $dN/dP = -4$. Estimate N if the price is lowered to \$69.

SOLUTION If P is raised to \$80, then $\Delta P = 5$. With the assumption that $dN/dP = -4$, we estimate, using the Linear Approximation, that

$$\Delta N \approx \frac{dN}{dP}\Delta P = (-4)(5) = -20;$$

therefore, we estimate that only 60 MP4 players will be sold per week when the price is \$80. On the other hand, if the price is lowered to \$69, then $\Delta P = -6$ and $\Delta N \approx (-4)(-6) = 24$. We therefore estimate that 104 MP4 players will be sold per week when the price is \$69.

17. The circumference of a sphere is measured at $C = 100$ cm. Estimate the maximum percentage error in V if the error in C is at most 3 cm.

SOLUTION The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the circumference is $C = 2\pi r$, where r is the radius of the sphere. Thus, $r = \frac{1}{2\pi}C$ and

$$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{1}{6\pi^2}C^3.$$

Using the Linear Approximation,

$$\Delta V \approx \frac{dV}{dC}\Delta C = \frac{1}{2\pi^2}C^2\Delta C,$$

so

$$\frac{\Delta V}{V} \approx \frac{\frac{1}{2\pi^2}C^2\Delta C}{\frac{1}{6\pi^2}C^3} = 3\frac{\Delta C}{C}.$$

With $C = 100$ cm and ΔC at most 3 cm, we estimate that the maximum percentage error in V is $3\frac{3}{100} = 0.09$, or 9%.

18. Show that $\sqrt{a^2 + b} \approx a + \frac{b}{2a}$ if b is small. Use this to estimate $\sqrt{26}$ and find the error using a calculator.

SOLUTION Let $a > 0$ and let $f(b) = \sqrt{a^2 + b}$. Then

$$f'(b) = \frac{1}{2\sqrt{a^2 + b}}.$$

By the Linear Approximation, $f(b) \approx f(0) + f'(0)b$, so

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}.$$

To estimate $\sqrt{26}$, let $a = 5$ and $b = 1$. Then

$$\sqrt{26} = \sqrt{5^2 + 1} \approx 5 + \frac{1}{10} = 5.1.$$

The error in this estimate is $|\sqrt{26} - 5.1| = 9.80 \times 10^{-4}$.

19. Use the Intermediate Value Theorem to prove that $\sin x - \cos x = 3x$ has a solution, and use Rolle's Theorem to show that this solution is unique.

SOLUTION Let $f(x) = \sin x - \cos x - 3x$, and observe that each root of this function corresponds to a solution of the equation $\sin x - \cos x = 3x$. Now,

$$f\left(-\frac{\pi}{2}\right) = -1 + \frac{3\pi}{2} > 0 \qquad \text{and} \qquad f(0) = -1 < 0.$$

Because f is continuous on $(-\frac{\pi}{2}, 0)$ and $f(-\frac{\pi}{2})$ and $f(0)$ are of opposite sign, the Intermediate Value Theorem guarantees there exists a $c \in (-\frac{\pi}{2}, 0)$ such that $f(c) = 0$. Thus, the equation $\sin x - \cos x = 3x$ has at least one solution.

Next, suppose that the equation $\sin x - \cos x = 3x$ has two solutions, and therefore $f(x)$ has two roots, say a and b. Because f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$, Rolle's Theorem guarantees there exists $c \in (a, b)$ such that $f'(c) = 0$. However,

$$f'(x) = \cos x + \sin x - 3 \leq -1$$

for all x. We have reached a contradiction. Consequently, $f(x)$ has a unique root and the equation $\sin x - \cos x = 3x$ has a unique solution.

20. Show that $f(x) = 2x^3 + 2x + \sin x + 1$ has precisely one real root.

SOLUTION We have $f(0) = 1$ and $f(-1) = -3 + \sin(-1) = -3.84 < 0$. Therefore $f(x)$ has a root in the interval $[-1, 0]$. Now, suppose that $f(x)$ has two real roots, say a and b. Because $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = 0$, Rolle's Theorem guarantees that there exists $c \in (a, b)$ such that $f'(c) = 0$. However

$$f'(x) = 6x^2 + 2 + \cos x > 0$$

for all x (since $2 + \cos x \geq 0$). We have reached a contradiction. Consequently, $f(x)$ must have precisely one real root.

21. Verify the MVT for $f(x) = \ln x$ on $[1, 4]$.

SOLUTION Let $f(x) = \ln x$. On the interval $[1, 4]$, this function is continuous and differentiable, so the MVT applies. Now, $f'(x) = \frac{1}{x}$, so

$$\frac{1}{c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{1}{3}\ln 4,$$

or

$$c = \frac{3}{\ln 4} \approx 2.164 \in (1, 4).$$

22. Suppose that $f(1) = 5$ and $f'(x) \geq 2$ for $x \geq 1$. Use the MVT to show that $f(8) \geq 19$.

SOLUTION Because f is continuous on $[1, 8]$ and differentiable on $(1, 8)$, the Mean Value Theorem guarantees there exists a $c \in (1, 8)$ such that

$$f'(c) = \frac{f(8) - f(1)}{8 - 1} \qquad \text{or} \qquad f(8) = f(1) + 7f'(c).$$

Now, we are given that $f(1) = 5$ and that $f'(x) \geq 2$ for $x \geq 1$. Therefore,

$$f(8) \geq 5 + 7(2) = 19.$$

23. Use the MVT to prove that if $f'(x) \leq 2$ for $x > 0$ and $f(0) = 4$, then $f(x) \leq 2x + 4$ for all $x \geq 0$.

SOLUTION Let $x > 0$. Because f is continuous on $[0, x]$ and differentiable on $(0, x)$, the Mean Value Theorem guarantees there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \qquad \text{or} \qquad f(x) = f(0) + xf'(c).$$

Now, we are given that $f(0) = 4$ and that $f'(x) \leq 2$ for $x > 0$. Therefore, for all $x \geq 0$,

$$f(x) \leq 4 + x(2) = 2x + 4.$$

24. A function $f(x)$ has derivative $f'(x) = \dfrac{1}{x^4+1}$. Where on the interval $[1, 4]$ does $f(x)$ take on its maximum value?

SOLUTION Let

$$f'(x) = \frac{1}{x^4+1}.$$

Because $f'(x)$ is never 0 and exists for all x, the function f has no critical points on the interval $[1, 4]$ and so must take its maximum value at one of the interval endpoints. Moveover, as $f'(x) > 0$ for all x, the function f is increasing for all x. Consequently, on the interval $[1, 4]$, the function f must take its maximum value at $x = 4$.

In Exercises 25–30, find the critical points and determine whether they are minima, maxima, or neither.

25. $f(x) = x^3 - 4x^2 + 4x$

SOLUTION Let $f(x) = x^3 - 4x^2 + 4x$. Then $f'(x) = 3x^2 - 8x + 4 = (3x-2)(x-2)$, so that $x = \frac{2}{3}$ and $x = 2$ are critical points. Next, $f''(x) = 6x - 8$, so $f''(\frac{2}{3}) = -4 < 0$ and $f''(2) = 4 > 0$. Therefore, by the Second Derivative Test, $f(\frac{2}{3})$ is a local maximum while $f(2)$ is a local minimum.

26. $s(t) = t^4 - 8t^2$

SOLUTION Let $s(t) = t^4 - 8t^2$. Then $s'(t) = 4t^3 - 16t = 4t(t-2)(t+2)$, so that $t = 0$, $t = -2$ and $t = 2$ are critical points. Next, $s''(t) = 12t^2 - 16$, so $s''(-2) = 32 > 0$, $s''(0) = -16 < 0$ and $s''(2) = 32 > 0$. Therefore, by the Second Derivative Test, $s(0)$ is a local maximum while $s(-2)$ and $s(2)$ are local minima.

27. $f(x) = x^2(x+2)^3$

SOLUTION Let $f(x) = x^2(x+2)^3$. Then

$$f'(x) = 3x^2(x+2)^2 + 2x(x+2)^3 = x(x+2)^2(3x+2x+4) = x(x+2)^2(5x+4),$$

so that $x = 0$, $x = -2$ and $x = -\frac{4}{5}$ are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, $f(-2)$ is neither a local maximum nor a local minimum, $f(-\frac{4}{5})$ is a local maximum and $f(0)$ is a local minimum.

Interval	$(-\infty, -2)$	$(-2, -\frac{4}{5})$	$(-\frac{4}{5}, 0)$	$(0, \infty)$
Sign of f'	$+$	$+$	$-$	$+$

28. $f(x) = x^{2/3}(1-x)$

SOLUTION Let $f(x) = x^{2/3}(1-x) = x^{2/3} - x^{5/3}$. Then

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2-5x}{3x^{1/3}},$$

so that $x = 0$ and $x = \frac{2}{5}$ are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, $f(0)$ is a local minimum and $f(\frac{2}{5})$ is a local maximum.

Interval	$(-\infty, 0)$	$(0, \frac{2}{5})$	$(\frac{2}{5}, \infty)$
Sign of f'	$-$	$+$	$-$

29. $g(\theta) = \sin^2\theta + \theta$

SOLUTION Let $g(\theta) = \sin^2\theta + \theta$. Then

$$g'(\theta) = 2\sin\theta\cos\theta + 1 = 2\sin 2\theta + 1,$$

so the critical points are

$$\theta = \frac{3\pi}{4} + n\pi$$

for all integers n. Because $g'(\theta) \geq 0$ for all θ, it follows that $g\left(\dfrac{3\pi}{4} + n\pi\right)$ is neither a local maximum nor a local minimum for all integers n.

30. $h(\theta) = 2\cos 2\theta + \cos 4\theta$

SOLUTION Let $h(\theta) = 2\cos 2\theta + \cos 4\theta$. Then

$$h'(\theta) = -4\sin 2\theta - 4\sin 4\theta = -4\sin 2\theta(1 + 2\cos 2\theta),$$

so the critical points are

$$\theta = \frac{n\pi}{2}, \quad \theta = \frac{\pi}{3} + \pi n \quad \text{and} \quad \theta = \frac{2\pi}{3} + \pi n$$

for all integers n. Now,

$$h''(\theta) = -8\cos 2\theta - 16\cos 4\theta,$$

so

$$h''\left(\frac{n\pi}{2}\right) = -8\cos n\pi - 16\cos 2n\pi = -8(-1)^n - 16 < 0;$$

$$h''\left(\frac{\pi}{3} + n\pi\right) = -8\cos\frac{2\pi}{3} - 16\cos\frac{4\pi}{3} = 12 > 0; \text{ and}$$

$$h''\left(\frac{2\pi}{3} + n\pi\right) = -8\cos\frac{4\pi}{3} - 16\cos\frac{8\pi}{3} = 12 > 0,$$

for all integers n. Therefore, by the Second Derivative Test, $h\left(\frac{n\pi}{2}\right)$ is a local maximum, and $h\left(\frac{\pi}{3} + n\pi\right)$ and $h\left(\frac{2\pi}{3} + n\pi\right)$ are local minima for all integers n.

In Exercises 31–38, find the extreme values on the interval.

31. $f(x) = x(10 - x), \quad [-1, 3]$

SOLUTION Let $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, so that $x = 5$ is the only critical point. As this critical point is not in the interval $[-1, 3]$, we only need to check the value of f at the endpoints to determine the extreme values. Because $f(-1) = -11$ and $f(3) = 21$, the maximum value of $f(x) = x(10 - x)$ on the interval $[-1, 3]$ is 21 while the minimum value is -11.

32. $f(x) = 6x^4 - 4x^6, \quad [-2, 2]$

SOLUTION Let $f(x) = 6x^4 - 4x^6$. Then $f'(x) = 24x^3 - 24x^5 = 24x^3(1 - x^2)$, so that the critical points are $x = -1$, $x = 0$ and $x = 1$. The table below lists the value of f at each of the critical points and the endpoints of the interval $[-2, 2]$. Based on this information, the minimum value of $f(x) = 6x^4 - 4x^6$ on the interval $[-2, 2]$ is -170 and the maximum value is 2.

x	-2	-1	0	1	2
$f(x)$	-170	2	0	2	-170

33. $g(\theta) = \sin^2\theta - \cos\theta, \quad [0, 2\pi]$

SOLUTION Let $g(\theta) = \sin^2\theta - \cos\theta$. Then

$$g'(\theta) = 2\sin\theta\cos\theta + \sin\theta = \sin\theta(2\cos\theta + 1) = 0$$

when $\theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$. The table below lists the value of g at each of the critical points and the endpoints of the interval $[0, 2\pi]$. Based on this information, the minimum value of $g(\theta)$ on the interval $[0, 2\pi]$ is -1 and the maximum value is $\frac{5}{4}$.

θ	0	$2\pi/3$	π	$4\pi/3$	2π
$g(\theta)$	-1	5/4	1	5/4	-1

34. $R(t) = \dfrac{t}{t^2 + t + 1}, \quad [0, 3]$

SOLUTION Let $R(t) = \frac{t}{t^2+t+1}$. Then

$$R'(t) = \frac{t^2 + t + 1 - t(2t + 1)}{(t^2 + t + 1)^2} = \frac{1 - t^2}{(t^2 + t + 1)^2},$$

so that the critical points are $t = \pm 1$. Note that only $t = 1$ is on the interval $[0, 3]$. With $R(0) = 0$, $R(1) = \frac{1}{3}$ and $R(3) = \frac{3}{13}$, it follows that the minimum value of $R(t)$ on the interval $[0, 3]$ is 0 and the maximum value is $\frac{1}{3}$.

35. $f(x) = x^{2/3} - 2x^{1/3}$, $[-1, 3]$

SOLUTION Let $f(x) = x^{2/3} - 2x^{1/3}$. Then $f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3}x^{-2/3}(x^{1/3} - 1)$, so that the critical points are $x = 0$ and $x = 1$. With $f(-1) = 3$, $f(0) = 0$, $f(1) = -1$ and $f(3) = \sqrt[3]{9} - 2\sqrt[3]{3} \approx -0.804$, it follows that the minimum value of $f(x)$ on the interval $[-1, 3]$ is -1 and the maximum value is 3.

36. $f(x) = x - \tan x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = x - \tan x$. Then $f'(x) = 1 - \sec^2 x$, so that $x = 0$ is the only critical point on $[-1, 1]$. With $f(-1) = -1 - \tan(-1) > 0$, $f(0) = 0$ and $f(1) = 1 - \tan 1 < 0$, it follows that the minimum value of $f(x)$ on the interval $[-1, 1]$ is $1 - \tan 1 \approx -0.557$ and the maximum value is $-1 - \tan(-1) = -1 + \tan 1 \approx 0.557$.

37. $f(x) = x - 12\ln x$, $[5, 40]$

SOLUTION Let $f(x) = x - 12\ln x$. Then $f'(x) = 1 - \frac{12}{x}$, whence $x = 12$ is the only critical point. The minimum value of f is then $12 - 12\ln 12 \approx -17.818880$, and the maximum value is $40 - 12\ln 40 \approx -4.266553$. Note that $f(5) = 5 - 12\ln 5 \approx -14.313255$.

38. $f(x) = e^x - 20x - 1$, $[0, 5]$

SOLUTION Let $f(x) = e^x - 20x - 1$. Then $f'(x) = e^x - 20$, whence $x = \ln 20$ is the only critical point. The minimum value of f is then $20 - 20\ln 20 - 1 \approx --40.914645$, and the maximum value is $e^5 - 101 \approx 47.413159$. Note that $f(0) = 0$.

39. Find the critical points and extreme values of $f(x) = |x - 1| + |2x - 6|$ in $[0, 8]$.

SOLUTION Let

$$f(x) = |x-1| + |2x-6| = \begin{cases} 7 - 3x, & x < 1 \\ 5 - x, & 1 \le x < 3 \\ 3x - 7, & x \ge 3 \end{cases}.$$

The derivative of $f(x)$ is never zero but does not exist at the transition points $x = 1$ and $x = 3$. Thus, the critical points of f are $x = 1$ and $x = 3$. With $f(0) = 7$, $f(1) = 4$, $f(3) = 2$ and $f(8) = 17$, it follows that the minimum value of $f(x)$ on the interval $[0, 8]$ is 2 and the maximum value is 17.

40. Match the description of $f(x)$ with the graph of its *derivative* $f'(x)$ in Figure 1.

(a) $f(x)$ is increasing and concave up.

(b) $f(x)$ is decreasing and concave up.

(c) $f(x)$ is increasing and concave down.

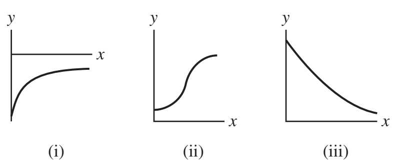


FIGURE 1 Graphs of the derivative.

SOLUTION

(a) If $f(x)$ is increasing and concave up, then $f'(x)$ is positive and increasing. This matches the graph in (ii).

(b) If $f(x)$ is decreasing and concave up, then $f'(x)$ is negative and increasing. This matches the graph in (i).

(c) If $f(x)$ is increasing and concave down, then $f'(x)$ is positive and decreasing. This matches the graph in (iii).

In Exercises 41–46, find the points of inflection.

41. $y = x^3 - 4x^2 + 4x$

SOLUTION Let $y = x^3 - 4x^2 + 4x$. Then $y' = 3x^2 - 8x + 4$ and $y'' = 6x - 8$. Thus, $y'' > 0$ and y is concave up for $x > \frac{4}{3}$, while $y'' < 0$ and y is concave down for $x < \frac{4}{3}$. Hence, there is a point of inflection at $x = \frac{4}{3}$.

42. $y = x - 2\cos x$

SOLUTION Let $y = x - 2\cos x$. Then $y' = 1 + 2\sin x$ and $y'' = 2\cos x$. Thus, $y'' > 0$ and y is concave up on each interval of the form

$$\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right),$$

while $y'' < 0$ and y is concave down on each interval of the form

$$\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right),$$

where n is any integer. Hence, there is a point of inflection at

$$x = \frac{(2n+1)\pi}{2}$$

for each integer n.

43. $y = \dfrac{x^2}{x^2+4}$

SOLUTION Let $y = \dfrac{x^2}{x^2+4} = 1 - \dfrac{4}{x^2+4}$. Then $y' = \dfrac{8x}{(x^2+4)^2}$ and

$$y'' = \frac{(x^2+4)^2(8) - 8x(2)(2x)(x^2+4)}{(x^2+4)^4} = \frac{8(4-3x^2)}{(x^2+4)^3}.$$

Thus, $y'' > 0$ and y is concave up for

$$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}},$$

while $y'' < 0$ and y is concave down for

$$|x| \geq \frac{2}{\sqrt{3}}.$$

Hence, there are points of inflection at

$$x = \pm\frac{2}{\sqrt{3}}.$$

44. $y = \dfrac{x}{(x^2-4)^{1/3}}$

SOLUTION Let $y = \dfrac{x}{(x^2-4)^{1/3}}$. Then

$$y' = \frac{(x^2-4)^{1/3} - \frac{1}{3}x(x^2-4)^{-2/3}(2x)}{(x^2-4)^{2/3}} = \frac{1}{3}\frac{x^2-12}{(x^2-4)^{4/3}}$$

and

$$y'' = \frac{1}{3}\frac{(x^2-4)^{4/3}(2x) - (x^2-12)\frac{4}{3}(x^2-4)^{1/3}(2x)}{(x^2-4)^{8/3}} = \frac{2x(36-x^2)}{9(x^2-4)^{7/3}}.$$

Thus, $y'' > 0$ and y is concave up for $x < -6, -2 < x < 0, 2 < x < 6$, while $y'' < 0$ and y is concave down for $-6 < x < -2, 0 < x < 2, x > 6$. Hence, there are points of inflection at $x = \pm 6$ and $x = 0$. Note that $x = \pm 2$ are not points of inflection because these points are not in the domain of the function.

45. $f(x) = (x^2 - x)e^{-x}$

SOLUTION Let $f(x) = (x^2 - x)e^{-x}$. Then

$$y' = -(x^2 - x)e^{-x} + (2x - 1)e^{-x} = -(x^2 - 3x + 1)e^{-x},$$

and

$$y'' = (x^2 - 3x + 1)e^{-x} - (2x - 3)e^{-x} = e^{-x}(x^2 - 5x + 4) = e^{-x}(x - 1)(x - 4).$$

Thus, $y'' > 0$ and y is concave up for $x < 1$ and for $x > 4$, while $y'' < 0$ and y is concave down for $1 < x < 4$. Hence, there are points of inflection at $x = 1$ and $x = 4$.

46. $f(x) = x(\ln x)^2$

SOLUTION Let $f(x) = x(\ln x)^2$. Then

$$y' = x \cdot 2\ln x \cdot \frac{1}{x} + (\ln x)^2 = 2\ln x + (\ln x)^2,$$

and

$$y'' = \frac{2}{x} + \frac{2}{x} \ln x = \frac{2}{x}(1 + \ln x).$$

Thus, $y'' > 0$ and y is concave up for $x > \frac{1}{e}$, while $y'' < 0$ and y is concave down for $0 < x < \frac{1}{e}$. Hence, there is a point of inflection at $x = \frac{1}{e}$.

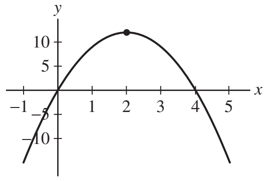
In Exercises 47–56, sketch the graph, noting the transition points and asymptotic behavior.

47. $y = 12x - 3x^2$

SOLUTION Let $y = 12x - 3x^2$. Then $y' = 12 - 6x$ and $y'' = -6$. It follows that the graph of $y = 12x - 3x^2$ is increasing for $x < 2$, decreasing for $x > 2$, has a local maximum at $x = 2$ and is concave down for all x. Because

$$\lim_{x \to \pm\infty} (12x - 3x^2) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

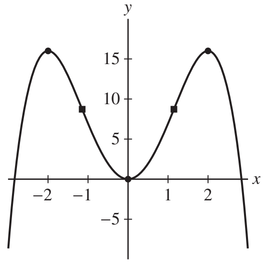


48. $y = 8x^2 - x^4$

SOLUTION Let $y = 8x^2 - x^4$. Then $y' = 16x - 4x^3 = 4x(4 - x^2)$ and $y'' = 16 - 12x^2 = 4(4 - 3x^2)$. It follows that the graph of $y = 8x^2 - x^4$ is increasing for $x < -2$ and $0 < x < 2$, decreasing for $-2 < x < 0$ and $x > 2$, has local maxima at $x = \pm 2$, has a local minimum at $x = 0$, is concave down for $|x| > 2/\sqrt{3}$, is concave up for $|x| < 2/\sqrt{3}$ and has inflection points at $x = \pm 2/\sqrt{3}$. Because

$$\lim_{x \to \pm\infty} (8x^2 - x^4) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

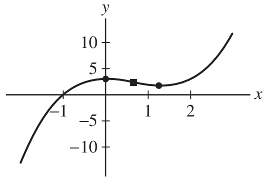


49. $y = x^3 - 2x^2 + 3$

SOLUTION Let $y = x^3 - 2x^2 + 3$. Then $y' = 3x^2 - 4x$ and $y'' = 6x - 4$. It follows that the graph of $y = x^3 - 2x^2 + 3$ is increasing for $x < 0$ and $x > \frac{4}{3}$, is decreasing for $0 < x < \frac{4}{3}$, has a local maximum at $x = 0$, has a local minimum at $x = \frac{4}{3}$, is concave up for $x > \frac{2}{3}$, is concave down for $x < \frac{2}{3}$ and has a point of inflection at $x = \frac{2}{3}$. Because

$$\lim_{x \to -\infty} (x^3 - 2x^2 + 3) = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} (x^3 - 2x^2 + 3) = \infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

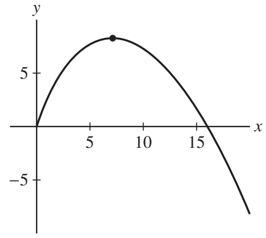


50. $y = 4x - x^{3/2}$

SOLUTION Let $y = 4x - x^{3/2}$. First note that the domain of this function is $x \geq 0$. Now, $y' = 4 - \frac{3}{2}x^{1/2}$ and $y'' = -\frac{3}{4}x^{-1/2}$. It follows that the graph of $y = 4x - x^{3/2}$ is increasing for $0 < x < \frac{64}{9}$, is decreasing for $x > \frac{64}{9}$, has a local maximum at $x = \frac{64}{9}$ and is concave down for all $x > 0$. Because

$$\lim_{x\to\infty}(4x - x^{3/2}) = -\infty,$$

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.



51. $y = \dfrac{x}{x^3+1}$

SOLUTION Let $y = \dfrac{x}{x^3+1}$. Then

$$y' = \frac{x^3 + 1 - x(3x^2)}{(x^3+1)^2} = \frac{1 - 2x^3}{(x^3+1)^2}$$

and

$$y'' = \frac{(x^3+1)^2(-6x^2) - (1-2x^3)(2)(x^3+1)(3x^2)}{(x^3+1)^4} = -\frac{6x^2(2-x^3)}{(x^3+1)^3}.$$

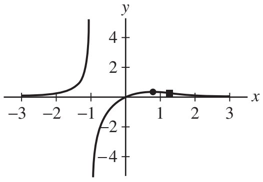
It follows that the graph of $y = \dfrac{x}{x^3+1}$ is increasing for $x < -1$ and $-1 < x < \sqrt[3]{\frac{1}{2}}$, is decreasing for $x > \sqrt[3]{\frac{1}{2}}$, has a local maximum at $x = \sqrt[3]{\frac{1}{2}}$, is concave up for $x < -1$ and $x > \sqrt[3]{2}$, is concave down for $-1 < x < 0$ and $0 < x < \sqrt[3]{2}$ and has a point of inflection at $x = \sqrt[3]{2}$. Note that $x = -1$ is not an inflection point because $x = -1$ is not in the domain of the function. Now,

$$\lim_{x\to\pm\infty}\frac{x}{x^3+1} = 0,$$

so $y = 0$ is a horizontal asymptote. Moreover,

$$\lim_{x\to-1-}\frac{x}{x^3+1} = \infty \qquad \text{and} \qquad \lim_{x\to-1+}\frac{x}{x^3+1} = -\infty,$$

so $x = -1$ is a vertical asymptote. The graph is shown below.



52. $y = \dfrac{x}{(x^2-4)^{2/3}}$

SOLUTION Let $y = \dfrac{x}{(x^2-4)^{2/3}}$. Then

$$y' = \frac{(x^2-4)^{2/3} - \frac{2}{3}x(x^2-4)^{-1/3}(2x)}{(x^2-4)^{4/3}} = -\frac{1}{3}\frac{x^2+12}{(x^2-4)^{5/3}}$$

and

$$y'' = -\frac{1}{3}\frac{(x^2-4)^{5/3}(2x) - (x^2+12)\frac{5}{3}(x^2-4)^{2/3}(2x)}{(x^2-4)^{10/3}} = \frac{4x(x^2+36)}{9(x^2-4)^{8/3}}.$$

It follows that the graph of $y = \dfrac{x}{(x^2-4)^{2/3}}$ is increasing for $-2 < x < 2$, is decreasing for $|x| > 2$, has no local extreme values, is concave up for $0 < x < 2$, $x > 2$, is concave down for $x < -2$, $-2 < x < 0$ and has a point of inflection at $x = 0$. Note that $x = \pm 2$ are neither local extreme values nor inflection points because $x = \pm 2$ are not in the domain of the function. Now,

$$\lim_{x\to\pm\infty} \frac{x}{(x^2-4)^{2/3}} = 0,$$

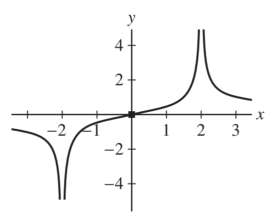
so $y = 0$ is a horizontal asymptote. Moreover,

$$\lim_{x\to -2-} \frac{x}{(x^2-4)^{2/3}} = -\infty \qquad \text{and} \qquad \lim_{x\to -2+} \frac{x}{(x^2-4)^{2/3}} = -\infty$$

while

$$\lim_{x\to 2-} \frac{x}{(x^2-4)^{2/3}} = \infty \qquad \text{and} \qquad \lim_{x\to 2+} \frac{x}{(x^2-4)^{2/3}} = \infty,$$

so $x = \pm 2$ are vertical asymptotes. The graph is shown below.

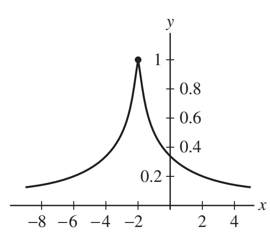


53. $y = \dfrac{1}{|x+2|+1}$

SOLUTION Let $y = \dfrac{1}{|x+2|+1}$. Because

$$\lim_{x\to\pm\infty} \frac{1}{|x+2|+1} = 0,$$

the graph of this function has a horizontal asymptote of $y = 0$. The graph has no vertical asymptotes as $|x+2|+1 \geq 1$ for all x. The graph is shown below. From this graph we see there is a local maximum at $x = -2$.



54. $y = \sqrt{2 - x^3}$

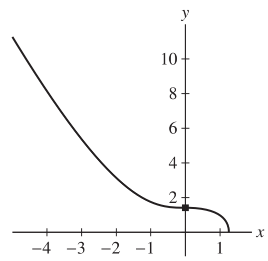
SOLUTION Let $y = \sqrt{2-x^3}$. Note that the domain of this function is $x \leq \sqrt[3]{2}$. Moreover, the graph has no vertical and no horizontal asymptotes. With

$$y' = \frac{1}{2}(2-x^3)^{-1/2}(-3x^2) = -\frac{3x^2}{2\sqrt{2-x^3}}$$

and

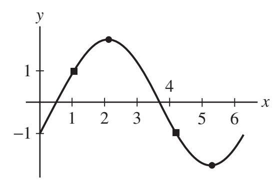
$$y'' = \frac{1}{2}(2-x^3)^{-1/2}(-6x) - \frac{3}{4}x^2(2-x^3)^{-3/2}(3x^2) = \frac{3x(x^3-8)}{4(2-x^3)^{3/2}},$$

it follows that the graph of $y = \sqrt{2-x^3}$ is decreasing over its entire domain, is concave up for $x < 0$, is concave down for $0 < x < \sqrt[3]{2}$ and has a point of inflection at $x = 0$. The graph is shown below.



55. $y = \sqrt{3}\sin x - \cos x$ on $[0, 2\pi]$

SOLUTION Let $y = \sqrt{3}\sin x - \cos x$. Then $y' = \sqrt{3}\cos x + \sin x$ and $y'' = -\sqrt{3}\sin x + \cos x$. It follows that the graph of $y = \sqrt{3}\sin x - \cos x$ is increasing for $0 < x < 5\pi/6$ and $11\pi/6 < x < 2\pi$, is decreasing for $5\pi/6 < x < 11\pi/6$, has a local maximum at $x = 5\pi/6$, has a local minimum at $x = 11\pi/6$, is concave up for $0 < x < \pi/3$ and $4\pi/3 < x < 2\pi$, is concave down for $\pi/3 < x < 4\pi/3$ and has points of inflection at $x = \pi/3$ and $x = 4\pi/3$. The graph is shown below.



56. $y = 2x - \tan x$ on $[0, 2\pi]$

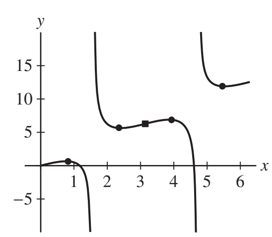
SOLUTION Let $y = 2x - \tan x$. Then $y' = 2 - \sec^2 x$ and $y'' = -2\sec^2 x \tan x$. It follows that the graph of $y = 2x - \tan x$ is increasing for $0 < x < \pi/4, 3\pi/4 < x < 5\pi/4, 7\pi/4 < x < 2\pi$, is decreasing for $\pi/4 < x < \pi/2, \pi/2 < x < 3\pi/4, 5\pi/4 < x < 3\pi/2, 3\pi/2 < x < 7\pi/4$, has local minima at $x = 3\pi/4$ and $x = 7\pi/4$, has local maxima at $x = \pi/4$ and $x = 5\pi/4$, is concave up for $\pi/2 < x < \pi$ and $3\pi/2 < x < 2\pi$, is concave down for $0 < x < \pi/2$ and $\pi < x < 3\pi/2$ and has an inflection point at $x = \pi$. Moreover, because

$$\lim_{x\to\pi/2-} (2x - \tan x) = -\infty \qquad \text{and} \qquad \lim_{x\to\pi/2+} (2x - \tan x) = \infty,$$

while

$$\lim_{x\to 3\pi/2-} (2x - \tan x) = -\infty \qquad \text{and} \qquad \lim_{x\to 3\pi/2+} (2x - \tan x) = \infty,$$

the graph has vertical asymptotes at $x = \pi/2$ and $x = 3\pi/2$. The graph is shown below.



57. Draw a curve $y = f(x)$ for which f' and f'' have signs as indicated in Figure 2.

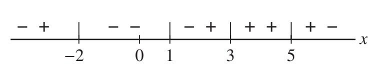
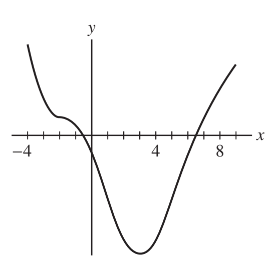


FIGURE 2

SOLUTION The figure below depicts a curve for which $f'(x)$ and $f''(x)$ have the required signs.



58. Find the dimensions of a cylindrical can with a bottom but no top of volume 4 m^3 that uses the least amount of metal.

SOLUTION Let the cylindrical can have height h and radius r. Then

$$V = \pi r^2 h = 4 \quad \text{so} \quad h = \frac{4}{\pi r^2}.$$

The amount of metal needed to make the can is then

$$M = 2\pi r h + \pi r^2 = \frac{8}{r} + \pi r^2.$$

Now,

$$M'(r) = -\frac{8}{r^2} + 2\pi r = 0 \quad \text{when} \quad r = \sqrt[3]{\frac{4}{\pi}}.$$

Because $M \to \infty$ as $r \to 0+$ and as $r \to \infty$, M must achieve its minimum for

$$r = \sqrt[3]{\frac{4}{\pi}} \text{ m}.$$

The height of the can is

$$h = \frac{4}{\pi r^2} = \sqrt[3]{\frac{4}{\pi}} \text{ m}.$$

59. A rectangular box of height h with square base of side b has volume $V = 4$ m^3. Two of the side faces are made of material costing \$40/m^2. The remaining sides cost \$20/m^2. Which values of b and h minimize the cost of the box?

SOLUTION Because the volume of the box is

$$V = b^2 h = 4 \quad \text{it follows that} \quad h = \frac{4}{b^2}.$$

Now, the cost of the box is

$$C = 40(2bh) + 20(2bh) + 20b^2 = 120bh + 20b^2 = \frac{480}{b} + 20b^2.$$

Thus,

$$C'(b) = -\frac{480}{b^2} + 40b = 0$$

when $b = \sqrt[3]{12}$ meters. Because $C(b) \to \infty$ as $b \to 0+$ and as $b \to \infty$, it follows that cost is minimized when $b = \sqrt[3]{12}$ meters and $h = \frac{1}{3}\sqrt[3]{12}$ meters.

60. The corn yield on a certain farm is

$$Y = -0.118x^2 + 8.5x + 12.9 \quad \text{(bushels per acre)}$$

where x is the number of corn plants per acre (in thousands). Assume that corn seed costs \$1.25 (per thousand seeds) and that corn can be sold for \$1.50/bushel. Let $P(x)$ be the profit (revenue minus the cost of seeds) at planting level x.

(a) Compute $P(x_0)$ for the value x_0 that maximizes yield Y.

(b) Find the maximum value of $P(x)$. Does maximum yield lead to maximum profit?

SOLUTION

(a) Let $Y = -0.118x^2 + 8.5x + 12.9$. Then $Y' = -0.236x + 8.5 = 0$ when

$$x_0 = \frac{8.5}{0.236} = 36.017 \text{ thousand corn plants/acre}.$$

Because $Y'' = -0.236 < 0$ for all x, x_0 corresponds to a maximum value for Y. Thus, yield is maximized for a planting level of 36,017 corn plants per acre. At this planting level, the profit is

$$1.5Y(x_0) - 1.25x_0 = 1.5(165.972) - 1.25(36.017) = \$203.94/\text{acre}.$$

(b) As a function of planting level x, the profit is

$$P(x) = 1.5Y(x) - 1.25x = -0.177x^2 + 11.5x + 19.35.$$

Then, $P'(x) = -0.354x + 11.5 = 0$ when

$$x_1 = \frac{11.5}{0.354} = 32.486 \text{ thousand corn plants/acre.}$$

Because $P''(x) = -0.354 < 0$ for all x, x_1 corresponds to a maximum value for P. Thus, profit is maximized for a planting level of 32,486 corn plants per acre.

(c) Note the planting levels obtained in parts (a) and (b) are different. Thus, a maximum yield does not lead to maximum profit.

61. Let $N(t)$ be the size of a tumor (in units of 10^6 cells) at time t (in days). According to the **Gompertz Model**, $dN/dt = N(a - b\ln N)$ where a, b are positive constants. Show that the maximum value of N is $e^{\frac{a}{b}}$ and that the tumor increases most rapidly when $N = e^{\frac{a}{b}-1}$.

SOLUTION Given $dN/dt = N(a - b\ln N)$, the critical points of N occur when $N = 0$ and when $N = e^{a/b}$. The sign of $N'(t)$ changes from positive to negative at $N = e^{a/b}$ so the maximum value of N is $e^{a/b}$. To determine when N changes most rapidly, we calculate

$$N''(t) = N\left(-\frac{b}{N}\right) + a - b\ln N = (a - b) - b\ln N.$$

Thus, $N'(t)$ is increasing for $N < e^{a/b-1}$, is decreasing for $N > e^{a/b-1}$ and is therefore maximum when $N = e^{a/b-1}$. Therefore, the tumor increases most rapidly when $N = e^{\frac{a}{b}-1}$.

62. A truck gets 10 miles per gallon of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by 0.15 mpg for each mile per hour increase above 50 mph.

(a) If the truck driver is paid \$30/hour and diesel fuel costs P = \$3/gal, which speed v between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) GU Plot cost as a function of v (choose the length arbitrarily) and verify your answer to part (a).

(c) GU Do you expect the optimal speed v to increase or decrease if fuel costs go down to P = \$2/gal? Plot the graphs of cost as a function of v for $P = 2$ and $P = 3$ on the same axis and verify your conclusion.

SOLUTION

(a) If the truck travels L miles at a speed of v mph, then the time required is L/v, and the wages paid to the driver are $30L/v$. The cost of the fuel is

$$\frac{3L}{10 - 0.15(v - 50)} = \frac{3L}{17.5 - 0.15v};$$

the total cost is therefore

$$C(v) = \frac{30L}{v} + \frac{3L}{17.5 - 0.15v}.$$

Solving

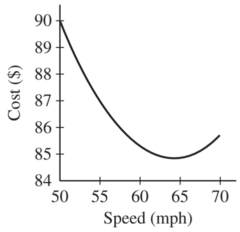
$$C'(v) = L\left(-\frac{30}{v^2} + \frac{0.45}{(17.5 - 0.15v)^2}\right) = 0$$

yields

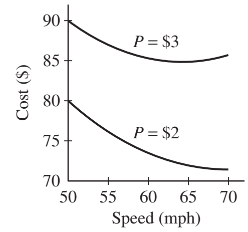
$$v = \frac{175\sqrt{6}}{3 + 1.5\sqrt{6}} \approx 64.2 \text{ mph.}$$

Because $C(50) = 0.9L$, $C(64.2) \approx 0.848L$ and $C(70) \approx 0.857L$, we see that the optimal speed is $v \approx 64.2$ mph.

(b) The cost as a function of speed is shown below for $L = 100$. The optimal speed is clearly around 64 mph.



(c) We expect v to increase if P goes down to \$2 per gallon. When gas is cheaper, it is better to drive faster and thereby save on the driver's wages. The cost as a function of speed for $P = 2$ and $P = 3$ is shown below (with $L = 100$). When $P = 2$, the optimal speed is $v = 70$ mph, which is an increase over the optimal speed when $P = 3$.



63. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius $R = 3$ and height $H = 4$ as in Figure 3. A cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.

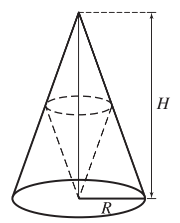


FIGURE 3

SOLUTION Let r denote the radius and h the height of the upside down cone. By similar triangles, we obtain the relation

$$\frac{4-h}{r} = \frac{4}{3} \qquad \text{so} \qquad h = 4\left(1 - \frac{r}{3}\right)$$

and the volume of the upside down cone is

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{4}{3}\pi\left(r^2 - \frac{r^3}{3}\right)$$

for $0 \le r \le 3$. Thus,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(2r - r^2\right),$$

and the critical points are $r = 0$ and $r = 2$. Because $V(0) = V(3) = 0$ and

$$V(2) = \frac{4}{3}\pi\left(4 - \frac{8}{3}\right) = \frac{16}{9}\pi,$$

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius 3 and height 4 is

$$\frac{16}{9}\pi.$$

64. Redo Exercise 63 for arbitrary R and H.

SOLUTION Let r denote the radius and h the height of the upside down cone. By similar triangles, we obtain the relation

$$\frac{H-h}{r} = \frac{H}{R} \qquad \text{so} \qquad h = H\left(1 - \frac{r}{R}\right)$$

and the volume of the upside down cone is

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi H\left(r^2 - \frac{r^3}{R}\right)$$

for $0 \le r \le R$. Thus,

$$\frac{dV}{dr} = \frac{1}{3}\pi H\left(2r - \frac{3r^2}{R}\right),$$

and the critical points are $r = 0$ and $r = 2R/3$. Because $V(0) = V(R) = 0$ and

$$V\left(\frac{2R}{3}\right) = \frac{1}{3}\pi H\left(\frac{4R^2}{9} - \frac{8R^2}{27}\right) = \frac{4}{81}\pi R^2 H,$$

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius R and height H is

$$\frac{4}{81}\pi R^2 H.$$

65. Show that the maximum area of a parallelogram $ADEF$ that is inscribed in a triangle ABC, as in Figure 4, is equal to one-half the area of $\triangle ABC$.

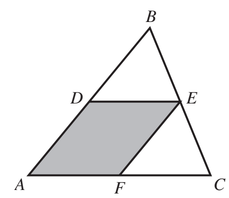


FIGURE 4

SOLUTION Let θ denote the measure of angle BAC. Then the area of the parallelogram is given by $\overline{AD}\cdot\overline{AF}\sin\theta$. Now, suppose that

$$\overline{BE}/\overline{BC} = x.$$

Then, by similar triangles, $\overline{AD} = (1-x)\overline{AB}$, $\overline{AF} = \overline{DE} = x\overline{AC}$, and the area of the parallelogram becomes $\overline{AB}\cdot\overline{AC}x(1-x)\sin\theta$. The function $x(1-x)$ achieves its maximum value of $\frac{1}{4}$ when $x = \frac{1}{2}$. Thus, the maximum area of a parallelogram inscribed in a triangle ΔABC is

$$\frac{1}{4}\overline{AB}\cdot\overline{AC}\sin\theta = \frac{1}{2}\left(\frac{1}{2}\overline{AB}\cdot\overline{AC}\sin\theta\right) = \frac{1}{2}\,(\text{area of }\Delta ABC)\,.$$

66. A box of volume 8 m^3 with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs \$50/m^2 and the metal for the sides costs \$30/m^2. Find the dimensions of the box that minimize total cost.

SOLUTION Let the square base have side length s and the box have height h. Then

$$V = s^2 h = 8 \quad\text{so}\quad h = \frac{8}{s^2}.$$

The cost of the box is then

$$C = 100s^2 + 120sh = 100s^2 + \frac{960}{s}.$$

Now,

$$C'(s) = 200s - \frac{960}{s^2} = 0 \quad\text{when}\quad s = \sqrt[3]{4.8}.$$

Because $C(s)\to\infty$ as $s\to 0+$ and as $s\to\infty$, it follows that total cost is minimized when $s = \sqrt[3]{4.8} \approx 1.69$ meters. The height of the box is

$$h = \frac{8}{s^2} \approx 2.81 \text{ meters}.$$

67. Let $f(x)$ be a function whose graph does not pass through the x-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to Q (Figure 5). Prove that $\overline{PQ}$ is perpendicular to the tangent line to the graph of x_0. *Hint:* Find the minimum value of the *square* of the distance from $(x, f(x))$ to $(a, 0)$.

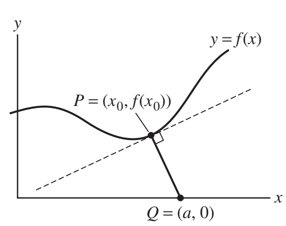


FIGURE 5

SOLUTION Let $P = (a, 0)$ and let $Q = (x_0, f(x_0))$ be the point on the graph of $y = f(x)$ closest to P. The slope of the segment joining P and Q is then

$$\frac{f(x_0)}{x_0 - a}.$$

Now, let

$$q(x) = \sqrt{(x-a)^2 + (f(x))^2},$$

the distance from the arbitrary point $(x, f(x))$ on the graph of $y = f(x)$ to the point P. As $(x_0, f(x_0))$ is the point closest to P, we must have

$$q'(x_0) = \frac{2(x_0 - a) + 2f(x_0)f'(x_0)}{\sqrt{(x_0-a)^2 + (f(x_0))^2}} = 0.$$

Thus,

$$f'(x_0) = -\frac{x_0 - a}{f(x_0)} = -\left(\frac{f(x_0)}{x_0 - a}\right)^{-1}.$$

In other words, the slope of the segment joining P and Q is the negative reciprocal of the slope of the line tangent to the graph of $y = f(x)$ at $x = x_0$; hence; the two lines are perpendicular.

68. Take a circular piece of paper of radius R, remove a sector of angle θ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle θ produces the cup of largest volume?

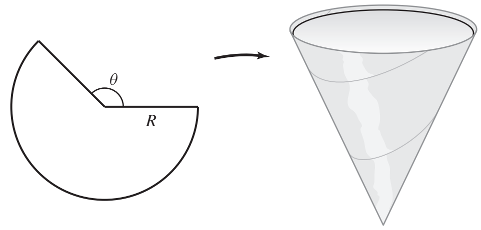


FIGURE 6

SOLUTION Let r denote the radius and h denote the height of the cone-shaped cup. Having removed an angle of θ from the paper, there is an arc of length $(2\pi - \theta)R$ remaining to form the circumference of the cup; hence

$$r = \frac{(2\pi - \theta)R}{2\pi} = \left(1 - \frac{\theta}{2\pi}\right)R.$$

The height of the cup is then

$$h = \sqrt{R^2 - \left(1 - \frac{\theta}{2\pi}\right)^2 R^2} = R\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2},$$

and the volume of the cup is

$$V(\theta) = \frac{1}{3}\pi R^3 \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}$$

for $0 \le \theta \le 2\pi$. Now,

$$\begin{aligned}\frac{dV}{d\theta} &= 2\left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2} + \left(1 - \frac{\theta}{2\pi}\right)^2 \frac{(-2)\left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)}{\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}} \\ &= \left(1 - \frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\frac{2\left(1 - \left(1 - \frac{\theta}{2\pi}\right)^2\right) - \left(1 - \frac{\theta}{2\pi}\right)^2}{\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}},\end{aligned}$$

so that $\theta = 2\pi$ and $\theta = 2\pi \pm \frac{2\pi\sqrt{6}}{3}$ are critical points. With $V(0) = V(2\pi) = 0$ and

$$V\left(2\pi - \frac{2\pi\sqrt{6}}{3}\right) = \frac{2\sqrt{3}}{27}\pi R^3,$$

the volume of the cup is maximized when $\theta = 2\pi - \frac{2\pi\sqrt{6}}{3}$.

69. Use Newton's Method to estimate $\sqrt[3]{25}$ to four decimal places.

SOLUTION Let $f(x) = x^3 - 25$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 25}{3x_n^2}.$$

With $x_0 = 3$, we find

n	1	2	3
x_n	2.925925926	2.924018982	2.924017738

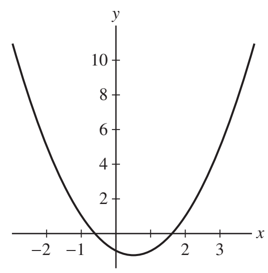
Thus, to four decimal places $\sqrt[3]{25} = 2.9240$.

70. Use Newton's Method to find a root of $f(x) = x^2 - x - 1$ to four decimal places.

SOLUTION Let $f(x) = x^2 - x - 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1}.$$

The graph below suggests the two roots of $f(x)$ are located near $x = -1$ and $x = 2$.



With $x_0 = -1$, we find

n	1	2	3	4
x_n	−0.6666666667	−0.6190476191	−0.6180344477	−0.6180339889

On the other hand, with $x_0 = 2$, we find

n	1	2	3	4
x_n	1.666666667	1.619047619	1.618034448	1.618033989

Thus, to four decimal places, the roots of $f(x) = x^2 - x - 1$ are -0.6180 and 1.6180.

In Exercises 71–84, calculate the indefinite integral.

71. $\int (4x^3 - 2x^2)\,dx$

SOLUTION $\int (4x^3 - 2x^2)\,dx = x^4 - \frac{2}{3}x^3 + C.$

72. $\int x^{9/4}\,dx$

SOLUTION $\int x^{9/4}\,dx = \frac{4}{13}x^{13/4} + C.$

73. $\int \sin(\theta - 8)\,d\theta$

SOLUTION $\int \sin(\theta - 8)\,d\theta = -\cos(\theta - 8) + C.$

74. $\int \cos(5 - 7\theta)\,d\theta$

SOLUTION $\int \cos(5 - 7\theta)\,d\theta = -\frac{1}{7}\sin(5 - 7\theta) + C.$

75. $\int (4t^{-3} - 12t^{-4})\, dt$

SOLUTION $\int (4t^{-3} - 12t^{-4})\, dt = -2t^{-2} + 4t^{-3} + C.$

76. $\int (9t^{-2/3} + 4t^{7/3})\, dt$

SOLUTION $\int (9t^{-2/3} + 4t^{7/3})\, dt = 27t^{1/3} + \frac{6}{5}t^{10/3} + C.$

77. $\int \sec^2 x\, dx$

SOLUTION $\int \sec^2 x\, dx = \tan x + C.$

78. $\int \tan 3\theta \sec 3\theta\, d\theta$

SOLUTION $\int \tan 3\theta \sec 3\theta\, d\theta = \frac{1}{3}\sec 3\theta + C.$

79. $\int (y+2)^4\, dy$

SOLUTION $\int (y+2)^4\, dy = \frac{1}{5}(y+2)^5 + C.$

80. $\int \frac{3x^3 - 9}{x^2}\, dx$

SOLUTION $\int \frac{3x^3 - 9}{x^2}\, dx = \int (3x - 9x^{-2})\, dx = \frac{3}{2}x^2 + 9x^{-1} + C.$

81. $\int (e^x - x)\, dx$

SOLUTION $\int (e^x - x)\, dx = e^x - \frac{1}{2}x^2 + C.$

82. $\int e^{-4x}\, dx$

SOLUTION $\int e^{-4x}\, dx = -\frac{1}{4}e^{-4x} + C.$

83. $\int 4x^{-1}\, dx$

SOLUTION $\int 4x^{-1}\, dx = 4\ln|x| + C.$

84. $\int \sin(4x - 9)\, dx$

SOLUTION $\int \sin(4x - 9)\, dx = -\frac{1}{4}\cos(4x - 9) + C.$

In Exercises 85–90, solve the differential equation with the given initial condition.

85. $\frac{dy}{dx} = 4x^3, \quad y(1) = 4$

SOLUTION Let $\frac{dy}{dx} = 4x^3$. Then

$$y(x) = \int 4x^3\, dx = x^4 + C.$$

Using the initial condition $y(1) = 4$, we find $y(1) = 1^4 + C = 4$, so $C = 3$. Thus, $y(x) = x^4 + 3$.

86. $\dfrac{dy}{dt} = 3t^2 + \cos t, \quad y(0) = 12$

SOLUTION Let $\frac{dy}{dt} = 3t^2 + \cos t$. Then

$$y(t) = \int (3t^2 + \cos t)\,dt = t^3 + \sin t + C.$$

Using the initial condition $y(0) = 12$, we find $y(0) = 0^3 + \sin 0 + C = 12$, so $C = 12$. Thus, $y(t) = t^3 + \sin t + 12$.

87. $\dfrac{dy}{dx} = x^{-1/2}, \quad y(1) = 1$

SOLUTION Let $\frac{dy}{dx} = x^{-1/2}$. Then

$$y(x) = \int x^{-1/2}\,dx = 2x^{1/2} + C.$$

Using the initial condition $y(1) = 1$, we find $y(1) = 2\sqrt{1} + C = 1$, so $C = -1$. Thus, $y(x) = 2x^{1/2} - 1$.

88. $\dfrac{dy}{dx} = \sec^2 x, \quad y\left(\frac{\pi}{4}\right) = 2$

SOLUTION Let $\frac{dy}{dx} = \sec^2 x$. Then

$$y(x) = \int \sec^2 x\,dx = \tan x + C.$$

Using the initial condition $y(\frac{\pi}{4}) = 2$, we find $y(\frac{\pi}{4}) = \tan\frac{\pi}{4} + C = 2$, so $C = 1$. Thus, $y(x) = \tan x + 1$.

89. $\dfrac{dy}{dx} = e^{-x}, \quad y(0) = 3$

SOLUTION Let $\frac{dy}{dx} = e^{-x}$. Then

$$y(x) = \int e^{-x}\,dx = -e^{-x} + C.$$

Using the initial condition $y(0) = 3$, we find $y(0) = -e^0 + C = 3$, so $C = 4$. Thus, $y(x) = 4 - e^{-x}$.

90. $\dfrac{dy}{dx} = e^{4x}, \quad y(1) = 1$

SOLUTION Let $\frac{dy}{dx} = e^{4x}$. Then

$$y(x) = \int e^{4x}\,dx = \frac{1}{4}e^{4x} + C.$$

Using the initial condition $y(1) = 1$, we find $y(1) = \frac{1}{4}e^4 + C = 1$, so $C = 1 - \frac{1}{4}e^4$. Thus, $y(x) = \frac{1}{4}e^{4x} + 1 - \frac{1}{4}e^4$.

91. Find $f(t)$ if $f''(t) = 1 - 2t$, $f(0) = 2$, and $f'(0) = -1$.

SOLUTION Suppose $f''(t) = 1 - 2t$. Then

$$f'(t) = \int f''(t)\,dt = \int (1 - 2t)\,dt = t - t^2 + C.$$

Using the initial condition $f'(0) = -1$, we find $f'(0) = 0 - 0^2 + C = -1$, so $C = -1$. Thus, $f'(t) = t - t^2 - 1$. Now,

$$f(t) = \int f'(t)\,dt = \int (t - t^2 - 1)\,dt = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + C.$$

Using the initial condition $f(0) = 2$, we find $f(0) = \frac{1}{2}0^2 - \frac{1}{3}0^3 - 0 + C = 2$, so $C = 2$. Thus,

$$f(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + 2.$$

92. At time $t = 0$, a driver begins decelerating at a constant rate of -10 m/s^2 and comes to a halt after traveling 500 m. Find the velocity at $t = 0$.

SOLUTION From the constant deceleration of -10 m/s^2, we determine

$$v(t) = \int (-10)\,dt = -10t + v_0,$$

where v_0 is the velocity of the automobile at $t = 0$. Note the automobile comes to a halt when $v(t) = 0$, which occurs at

$$t = \frac{v_0}{10}\text{ s}.$$

The distance traveled during the braking process is

$$s(t) = \int v(t)\,dt = -5t^2 + v_0 t + C,$$

for some arbitrary constant C. We are given that the braking distance is 500 meters, so

$$s\left(\frac{v_0}{10}\right) - s(0) = -5\left(\frac{v_0}{10}\right)^2 + v_0\left(\frac{v_0}{10}\right) + C - C = 500,$$

leading to

$$v_0 = 100\text{ m/s}.$$

93. Find the local extrema of $f(x) = \dfrac{e^{2x}+1}{e^{x+1}}$.

SOLUTION To simplify the differentiation, we first rewrite $f(x) = \frac{e^{2x}+1}{e^{x+1}}$ using the Laws of Exponents:

$$f(x) = \frac{e^{2x}}{e^{x+1}} + \frac{1}{e^{x+1}} = e^{2x-(x+1)} + e^{-(x+1)} = e^{x-1} + e^{-x-1}.$$

Now,

$$f'(x) = e^{x-1} - e^{-x-1}.$$

Setting the derivative equal to zero yields

$$e^{x-1} - e^{-x-1} = 0 \quad\text{or}\quad e^{x-1} = e^{-x-1}.$$

Thus,

$$x - 1 = -x - 1 \quad\text{or}\quad x = 0.$$

Next, we use the Second Derivative Test. With $f''(x) = e^{x-1} + e^{-x-1}$, it follows that

$$f''(0) = e^{-1} + e^{-1} = \frac{2}{e} > 0.$$

Hence, $x = 0$ is a local minimum. Since $f(0) = e^{0-1} + e^{-0-1} = \frac{2}{e}$, we conclude that the point $(0, \frac{2}{e})$ is a local minimum.

94. Find the points of inflection of $f(x) = \ln(x^2+1)$, and at each point, determine whether the concavity changes from up to down or from down to up.

SOLUTION With $f(x) = \ln(x^2+1)$, we find

$$f'(x) = \frac{2x}{x^2+1};\ \text{and}$$

$$f''(x) = \frac{2\left(x^2+1\right) - 2x\cdot 2x}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}$$

Thus, $f''(x) > 0$ for $-1 < x < 1$, whereas $f''(x) < 0$ for $x < -1$ and for $x > 1$. It follows that there are points of inflection at $x = \pm 1$, and that the concavity of f changes from down to up at $x = -1$ and from up to down at $x = 1$.

In Exercises 95–98, find the local extrema and points of inflection, and sketch the graph. Use L'Hôpital's Rule to determine the limits as $x \to 0+$ or $x \to \pm\infty$ if necessary.

95. $y = x\ln x \quad (x > 0)$

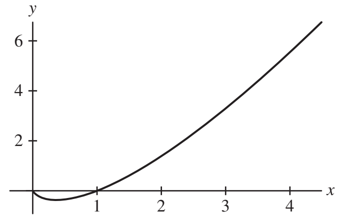
SOLUTION Let $y = x\ln x$. Then

$$y' = \ln x + x\left(\frac{1}{x}\right) = 1 + \ln x,$$

and $y'' = \frac{1}{x}$. Solving $y' = 0$ yields the critical point $x = e^{-1}$. Since $y''(e^{-1}) = e > 0$, the function has a local minimum at $x = e^{-1}$. y'' is positive for $x > 0$, hence the function is concave up for $x > 0$ and there are no points of inflection. As $x \to 0+$ and as $x \to \infty$, we find

$$\lim_{x\to 0+} x \ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0+}(-x) = 0;$$
$$\lim_{x\to\infty} x \ln x = \infty.$$

The graph is shown below:



96. $y = e^{x-x^2}$

SOLUTION Let $y = e^{x-x^2}$. Then $y' = (1-2x)e^{x-x^2}$ and

$$y'' = (1-2x)^2 e^{x-x^2} - 2e^{x-x^2} = (4x^2 - 4x - 1)e^{x-x^2}.$$

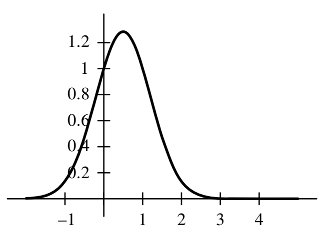
Solving $y' = 0$ yields the critical point $x = \frac{1}{2}$. Since

$$y''\left(\frac{1}{2}\right) = -2e^{1/4} < 0,$$

the function has a local maximum at $x = \frac{1}{2}$. Using the quadratic formula, we find that $y'' = 0$ when $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{2}$. $y'' > 0$ and the function is concave up for $x < \frac{1}{2} - \frac{1}{2}\sqrt{2}$ and for $x > \frac{1}{2} + \frac{1}{2}\sqrt{2}$, whereas $y'' < 0$ and the function is concave down for $\frac{1}{2} - \frac{1}{2}\sqrt{2} < x < \frac{1}{2} + \frac{1}{2}\sqrt{2}$; hence, there are inflection points at $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{2}$. As $x \to \pm\infty$, $x - x^2 \to -\infty$ so

$$\lim_{x\to\pm\infty} e^{x-x^2} = 0.$$

The graph is shown below.



97. $y = x(\ln x)^2 \quad (x > 0)$

SOLUTION Let $y = x(\ln x)^2$. Then

$$y' = x\frac{2\ln x}{x} + (\ln x)^2 = 2\ln x + (\ln x)^2 = \ln x(2 + \ln x),$$

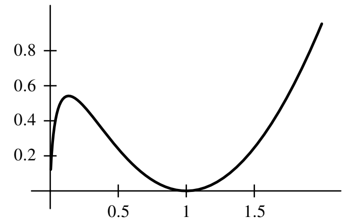
and

$$y'' = \frac{2}{x} + \frac{2\ln x}{x} = \frac{2}{x}(1 + \ln x).$$

Solving $y' = 0$ yields the critical points $x = e^{-2}$ and $x = 1$. Since $y''(e^{-2}) = -2e^2 < 0$ and $y''(1) = 2 > 0$, the function has a local maximum at $x = e^{-2}$ and a local minimum at $x = 1$. $y'' < 0$ and the function is concave down for $x < e^{-1}$, whereas $y'' > 0$ and the function is concave up for $x > e^{-1}$; hence, there is a point of inflection at $x = e^{-1}$. As $x \to 0+$ and as $x \to \infty$, we find

$$\lim_{x\to 0+} x(\ln x)^2 = \lim_{x\to 0+} \frac{(\ln x)^2}{x^{-1}} = \lim_{x\to 0+} \frac{2\ln x \cdot x^{-1}}{-x^{-2}} = \lim_{x\to 0+} \frac{2\ln x}{-x^{-1}} = \lim_{x\to 0+} \frac{2x^{-1}}{x^{-2}} = \lim_{x\to 0+} 2x = 0;$$
$$\lim_{x\to\infty} x(\ln x)^2 = \infty.$$

The graph is shown below:



98. $y = \tan^{-1}\left(\dfrac{x^2}{4}\right)$

SOLUTION Let $y = \tan^{-1}\left(\frac{x^2}{4}\right)$. Then

$$y' = \frac{1}{1+\left(\frac{x^2}{4}\right)^2}\frac{x}{2} = \frac{8x}{x^4+16},$$

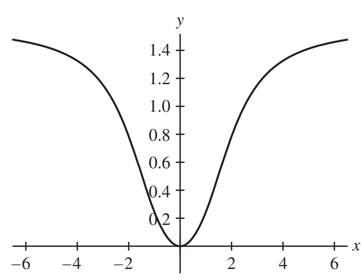
and

$$y'' = \frac{8(x^4+16) - 8x \cdot 4x^3}{(x^4+16)^2} = \frac{128 - 24x^4}{(x^4+16)^2}.$$

Solving $y' = 0$ yields $x = 0$ as the only critical point. Because $y''(0) = \frac{1}{2} > 0$, we conclude the function has a local minimum at $x = 0$. Moreover, $y'' < 0$ for $x < -2 \cdot 3^{-1/4}$ and for $x > 2 \cdot 3^{-1/4}$, whereas $y'' > 0$ for $-2 \cdot 3^{-1/4} < x < 2 \cdot 3^{-1/4}$. Therefore, there are points of inflection at $x = \pm 2 \cdot 3^{-1/4}$. As $x \to \pm\infty$, we find

$$\lim_{x\to\pm\infty} \tan^{-1}\left(\frac{x^2}{4}\right) = \frac{\pi}{2}.$$

The graph is shown below:



99. Explain why L'Hôpital's Rule gives no information about $\lim\limits_{x\to\infty} \dfrac{2x - \sin x}{3x + \cos 2x}$. Evaluate the limit by another method.

SOLUTION As $x \to \infty$, both $2x - \sin x$ and $3x + \cos 2x$ tend toward infinity, so L'Hôpital's Rule applies to $\lim\limits_{x\to\infty} \dfrac{2x - \sin x}{3x + \cos 2x}$; however, the resulting limit, $\lim\limits_{x\to\infty} \dfrac{2 - \cos x}{3 - 2\sin 2x}$, does not exist due to the oscillation of $\sin x$ and $\cos x$ and further applications of L'Hôpital's rule will not change this situation.

To evaluate the limit, we note

$$\lim_{x\to\infty} \frac{2x - \sin x}{3x + \cos 2x} = \lim_{x\to\infty} \frac{2 - \frac{\sin x}{x}}{3 + \frac{\cos 2x}{x}} = \frac{2}{3}.$$

100. Let $f(x)$ be a differentiable function with inverse $g(x)$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = f'(0)^2$$

SOLUTION Since g and f are inverse functions, we have $g(f(x)) = x$ for all x in the domain of f. In particular, for $x = 0$ we have

$$g(0) = g(f(0)) = 0.$$

Therefore, the limit is an indeterminate form of type $\frac{0}{0}$, so we may apply L'Hôpital's Rule. By the Theorem on the derivative of the inverse function, we have

$$g'(x) = \frac{1}{f'(g(x))}.$$

Therefore,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{\frac{1}{f'(g(x))}} = \lim_{x\to 0} f'(x) f'(g(x)) = f'(0) f'(g(0)) = f'(0) \cdot f'(0) = f'(0)^2.$$

In Exercises 101–112, verify that L'Hôpital's Rule applies and evaluate the limit.

101. $\displaystyle\lim_{x\to 3} \frac{4x-12}{x^2-5x+6}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to 3} \frac{4x-12}{x^2-5x+6} = \lim_{x\to 3} \frac{4}{2x-5} = 4.$$

102. $\displaystyle\lim_{x\to -2} \frac{x^3+2x^2-x-2}{x^4+2x^3-4x-8}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to -2} \frac{x^3+2x^2-x-2}{x^4+2x^3-4x-8} = \lim_{x\to -2} \frac{3x^2+4x-1}{4x^3+6x^2-4} = -\frac{3}{12} = -\frac{1}{4}.$$

103. $\displaystyle\lim_{x\to 0+} x^{1/2} \ln x$

SOLUTION First rewrite

$$x^{1/2} \ln x \quad \text{as} \quad \frac{\ln x}{x^{-1/2}}.$$

The rewritten expression is an indeterminate form of type $\frac{\infty}{\infty}$, therefore L'Hôpital's Rule applies. We find

$$\lim_{x\to 0+} x^{1/2} \ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0+} \frac{1/x}{-\frac{1/2}{x}^{-3/2}} = \lim_{x\to 0+} -\frac{x^{1/2}}{2} = 0.$$

104. $\displaystyle\lim_{t\to \infty} \frac{\ln(e^t+1)}{t}$

SOLUTION The given expression is an indeterminate form of type $\frac{\infty}{\infty}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{t\to \infty} \frac{\ln(e^t+1)}{t} = \lim_{t\to \infty} \frac{\frac{e^t}{e^t+1}}{1} = \lim_{t\to \infty} \frac{1}{1+e^{-t}} = 1.$$

105. $\displaystyle\lim_{\theta\to 0} \frac{2\sin\theta - \sin 2\theta}{\sin\theta - \theta\cos\theta}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{\theta\to 0} \frac{2\sin\theta - \sin 2\theta}{\sin\theta - \theta\cos\theta} = \lim_{\theta\to 0} \frac{2\cos\theta - 2\cos 2\theta}{\cos\theta - (\cos\theta - \theta\sin\theta)} = \lim_{\theta\to 0} \frac{2\cos\theta - 2\cos 2\theta}{\theta\sin\theta}$$

$$= \lim_{\theta\to 0} \frac{-2\sin\theta + 4\sin 2\theta}{\sin\theta + \theta\cos\theta} = \lim_{\theta\to 0} \frac{-2\cos\theta + 8\cos 2\theta}{\cos\theta + \cos\theta - \theta\sin\theta} = \frac{-2+8}{1+1-0} = 3.$$

106. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4+x} - 2\sqrt[8]{1+x}}{x^2}$

SOLUTION The given expression is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0} \frac{\sqrt{4+x} - 2\sqrt[8]{1+x}}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}(4+x)^{-1/2} - \frac{1}{4}(1+x)^{-7/8}}{2x}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{4}(4+x)^{-3/2} + \frac{7}{32}(1+x)^{-15/8}}{2} = \frac{-\frac{1}{4}\cdot\frac{1}{8} + \frac{7}{32}}{2} = \frac{3}{32}.$$

107. $\displaystyle\lim_{t\to\infty} \frac{\ln(t+2)}{\log_2 t}$

SOLUTION The limit is an indeterminate form of type $\frac{\infty}{\infty}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{t\to\infty} \frac{\ln(t+2)}{\log_2 t} = \lim_{t\to\infty} \frac{\frac{1}{t+2}}{\frac{1}{t\ln 2}} = \lim_{t\to\infty} \frac{t\ln 2}{t+2} = \lim_{t\to\infty} \frac{\ln 2}{1} = \ln 2.$$

108. $\displaystyle\lim_{x\to 0} \left(\frac{e^x}{e^x-1} - \frac{1}{x}\right)$

SOLUTION First rewrite the function as a quotient:

$$\frac{e^x}{e^x-1} - \frac{1}{x} = \frac{xe^x - e^x + 1}{x(e^x-1)}.$$

The limit is now an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\begin{aligned}\lim_{x\to 0} \left(\frac{e^x}{e^x-1} - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{xe^x + e^x - e^x}{xe^x + e^x - 1} = \lim_{x\to 0} \frac{xe^x}{xe^x + e^x - 1} \\ &= \lim_{x\to 0} \frac{xe^x + e^x}{xe^x + e^x + e^x} = \frac{1}{1+1} = \frac{1}{2}.\end{aligned}$$

109. $\displaystyle\lim_{y\to 0} \frac{\sin^{-1} y - y}{y^3}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{y\to 0} \frac{\sin^{-1} y - y}{y^3} = \lim_{y\to 0} \frac{\frac{1}{\sqrt{1-y^2}} - 1}{3y^2} = \lim_{y\to 0} \frac{y(1-y^2)^{-3/2}}{6y} = \lim_{y\to 0} \frac{(1-y^2)^{-3/2}}{6} = \frac{1}{6}.$$

110. $\displaystyle\lim_{x\to 1} \frac{\sqrt{1-x^2}}{\cos^{-1} x}$

SOLUTION The limit is an indeterminate form $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 1} \frac{\sqrt{1-x^2}}{\cos^{-1} x} = \lim_{x\to 1} \frac{-\frac{x}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 1} x = 1.$$

111. $\displaystyle\lim_{x\to 0} \frac{\sinh(x^2)}{\cosh x - 1}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\lim_{x\to 0} \frac{\sinh(x^2)}{\cosh x - 1} = \lim_{x\to 0} \frac{2x\cosh(x^2)}{\sinh x} = \lim_{x\to 0} \frac{2\cosh(x^2) + 4x^2\sinh(x^2)}{\cosh x} = \frac{2+0}{1} = 2.$$

112. $\displaystyle\lim_{x\to 0} \frac{\tanh x - \sinh x}{\sin x - x}$

SOLUTION The limit is an indeterminate form of type $\frac{0}{0}$; hence, we may apply L'Hôpital's Rule. We find

$$\begin{aligned}\lim_{x\to 0} \frac{\tanh x - \sinh x}{\sin x - x} &= \lim_{x\to 0} \frac{\operatorname{sech}^2 x - \cosh x}{\cos x - 1} = \lim_{x\to 0} \frac{2\operatorname{sech} x(-\operatorname{sech} x \tanh x) - \sinh x}{-\sin x} \\ &= \lim_{x\to 0} \frac{2\operatorname{sech}^2 x \tanh x + \sinh x}{\sin x} = \lim_{x\to 0} \frac{-4\operatorname{sech}^2 x \tanh^2 x + 2\operatorname{sech}^4 x + \cosh x}{\cos x} \\ &= \frac{-4\cdot 1\cdot 0 + 2\cdot 1 + 1}{1} = 3.\end{aligned}$$

113. Let $f(x) = e^{-Ax^2/2}$, where $A > 0$. Given any n numbers $a_1, a_2, \ldots, a_n$, set

$$\Phi(x) = f(x-a_1)f(x-a_2)\cdots f(x-a_n)$$

(a) Assume $n = 2$ and prove that $\Phi(x)$ attains its maximum value at the average $x = \frac{1}{2}(a_1 + a_2)$. *Hint:* Calculate $\Phi'(x)$ using logarithmic differentiation.

(b) Show that for any n, $\Phi(x)$ attains its maximum value at $x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. This fact is related to the role of $f(x)$ (whose graph is a bell-shaped curve) in statistics.

SOLUTION

(a) For $n = 2$ we have,

$$\Phi(x) = f(x - a_1) f(x - a_2) = e^{-\frac{A}{2}(x-a_1)^2} \cdot e^{-\frac{A}{2}(x-a_2)^2} = e^{-\frac{A}{2}\left((x-a_1)^2+(x-a_2)^2\right)}.$$

Since $e^{-\frac{A}{2}y}$ is a decreasing function of y, it attains its maximum value where y is minimum. Therefore, we must find the minimum value of

$$y = (x - a_1)^2 + (x - a_2)^2 = 2x^2 - 2(a_1 + a_2)x + a_1^2 + a_2^2.$$

Now, $y' = 4x - 2(a_1 + a_2) = 0$ when

$$x = \frac{a_1 + a_2}{2}.$$

We conclude that $\Phi(x)$ attains a maximum value at this point.

(b) We have

$$\Phi(x) = e^{-\frac{A}{2}(x-a_1)^2} \cdot e^{-\frac{A}{2}(x-a_2)^2} \cdot \cdots \cdot e^{-\frac{A}{2}(x-a_n)^2} = e^{-\frac{A}{2}\left((x-a_1)^2+\cdots+(x-a_n)^2\right)}.$$

Since the function $e^{-\frac{A}{2}y}$ is a decreasing function of y, it attains a maximum value where y is minimum. Therefore we must minimize the function

$$y = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2.$$

We find the critical points by solving:

$$\begin{aligned} y' &= 2(x - a_1) + 2(x - a_2) + \cdots + 2(x - a_n) = 0 \\ 2nx &= 2(a_1 + a_2 + \cdots + a_n) \\ x &= \frac{a_1 + \cdots + a_n}{n}. \end{aligned}$$

We verify that this point corresponds the minimum value of y by examining the sign of y'' at this point: $y'' = 2n > 0$. We conclude that y attains a minimum value at the point $x = \frac{a_1+\cdots+a_n}{n}$, hence $\Phi(x)$ attains a maximum value at this point.

5 THE INTEGRAL

5.1 Approximating and Computing Area

Preliminary Questions

1. What are the right and left endpoints if [2, 5] is divided into six subintervals?

SOLUTION If the interval [2, 5] is divided into six subintervals, the length of each subinterval is $\frac{5-2}{6} = \frac{1}{2}$. The right endpoints of the subintervals are then $\frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5$, while the left endpoints are $2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}$.

2. The interval [1, 5] is divided into eight subintervals.

(a) What is the left endpoint of the last subinterval?

(b) What are the right endpoints of the first two subintervals?

SOLUTION Note that each of the 8 subintervals has length $\frac{5-1}{8} = \frac{1}{2}$.

(a) The left endpoint of the last subinterval is $5 - \frac{1}{2} = \frac{9}{2}$.

(b) The right endpoints of the first two subintervals are $1 + \frac{1}{2} = \frac{3}{2}$ and $1 + 2\left(\frac{1}{2}\right) = 2$.

3. Which of the following pairs of sums are *not* equal?

(a) $\displaystyle\sum_{i=1}^{4} i, \quad \sum_{\ell=1}^{4} \ell$

(b) $\displaystyle\sum_{j=1}^{4} j^2, \quad \sum_{k=2}^{5} k^2$

(c) $\displaystyle\sum_{j=1}^{4} j, \quad \sum_{i=2}^{5} (i-1)$

(d) $\displaystyle\sum_{i=1}^{4} i(i+1), \quad \sum_{j=2}^{5} (j-1)j$

SOLUTION

(a) Only the name of the index variable has been changed, so these two sums *are* the same.

(b) These two sums are *not* the same; the second squares the numbers two through five while the first squares the numbers one through four.

(c) These two sums *are* the same. Note that when i ranges from two through five, the expression $i - 1$ ranges from one through four.

(d) These two sums *are* the same. Both sums are $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5$.

4. Explain: $\sum_{j=1}^{100} j = \sum_{j=0}^{100} j$ but $\sum_{j=1}^{100} 1$ is not equal to $\sum_{j=0}^{100} 1$.

SOLUTION The first term in the sum $\sum_{j=0}^{100} j$ is equal to zero, so it may be dropped. More specifically,

$$\sum_{j=0}^{100} j = 0 + \sum_{j=1}^{100} j = \sum_{j=1}^{100} j.$$

On the other hand, the first term in $\sum_{j=0}^{100} 1$ is not zero, so this term cannot be dropped. In particular,

$$\sum_{j=0}^{100} 1 = 1 + \sum_{j=1}^{100} 1 \neq \sum_{j=1}^{100} 1.$$

5. Explain why $L_{100} \geq R_{100}$ for $f(x) = x^{-2}$ on [3, 7].

SOLUTION On [3, 7], the function $f(x) = x^{-2}$ is a decreasing function; hence, for any subinterval of [3, 7], the function value at the left endpoint is larger than the function value at the right endpoint. Consequently, L_{100} must be larger than R_{100}.

Exercises

1. Figure 15 shows the velocity of an object over a 3-min interval. Determine the distance traveled over the intervals [0, 3] and [1, 2.5] (remember to convert from km/h to km/min).

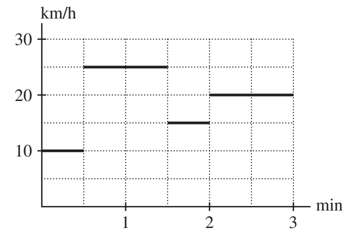


FIGURE 15

SOLUTION The distance traveled by the object can be determined by calculating the area underneath the velocity graph over the specified interval. During the interval [0, 3], the object travels

$$\left(\frac{10}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{25}{60}\right)(1)+\left(\frac{15}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{20}{60}\right)(1)=\frac{23}{24}\approx 0.96 \text{ km}.$$

During the interval [1, 2.5], it travels

$$\left(\frac{25}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{15}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{20}{60}\right)\left(\frac{1}{2}\right)=\frac{1}{2}=0.5 \text{ km}.$$

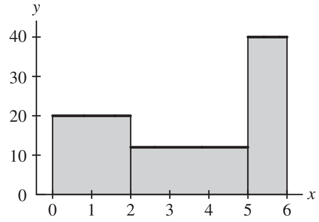
2. An ostrich (Figure 16) runs with velocity 20 km/h for 2 minutes, 12 km/h for 3 minutes, and 40 km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

FIGURE 16 Ostriches can reach speeds as high as 70 km/h.

SOLUTION The total distance traveled by the ostrich is

$$\left(\frac{20}{60}\right)(2)+\left(\frac{12}{60}\right)(3)+\left(\frac{40}{60}\right)(1)=\frac{2}{3}+\frac{3}{5}+\frac{2}{3}=\frac{29}{15}$$

km. This distance is the area under the graph below which shows the ostrich's velocity as a function of time.



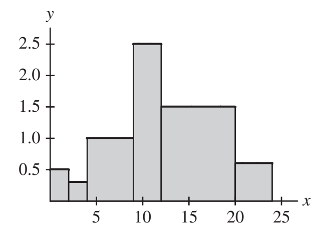
3. A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate $R(t)$ on October 21 is recorded, in centimeters per hour, in the following table, where t is the number of hours since midnight. Compute the total rainfall during this 24-hour period and indicate on a graph how this quantity can be interpreted as an area.

t (h)	0–2	2–4	4–9	9–12	12–20	20–24
$R(t)$ (cm)	0.5	0.3	1.0	2.5	1.5	0.6

SOLUTION Over each interval, the total rainfall is the time interval in hours times the rainfall in centimeters per hour. Thus

$$R=2(0.5)+2(0.3)+5(1.0)+3(2.5)+8(1.5)+4(0.6)=28.5 \text{ cm}.$$

The figure below is a graph of the rainfall as a function of time. The area of the shaded region represents the total rainfall.



4. The velocity of an object is $v(t) = 12t$ m/s. Use Eq. (2) and geometry to find the distance traveled over the time intervals [0, 2] and [2, 5].

SOLUTION By equation Eq. (2), the distance traveled over the time interval $[a, b]$ is

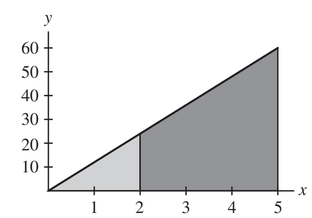
$$\int_a^b v(t)\,dt = \int_a^b 12t\,dt;$$

that is, the distance traveled is the area under the graph of the velocity function over the interval $[a, b]$. The graph below shows the area under the velocity function $v(t) = 12t$ m/s over the intervals [0, 2] and [2, 5]. Over the interval [0, 2], the area is a triangle of base 2 and height 24; therefore, the distance traveled is

$$\frac{1}{2}(2)(24) = 24 \text{ meters.}$$

Over the interval [2, 5], the area is a trapezoid of height 3 and base lengths 24 and 60; therefore, the distance traveled is

$$\frac{1}{2}(3)(24 + 60) = 126 \text{ meters.}$$



5. Compute R_5 and L_5 over [0, 1] using the following values.

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	50	48	46	44	42	40

SOLUTION $\Delta x = \frac{1-0}{5} = 0.2$. Thus,

$$L_5 = 0.2\,(50 + 48 + 46 + 44 + 42) = 0.2(230) = 46,$$

and

$$R_5 = 0.2\,(48 + 46 + 44 + 42 + 40) = 0.2(220) = 44.$$

The average is

$$\frac{46 + 44}{2} = 45.$$

This estimate is frequently referred to as the *Trapezoidal Approximation*.

6. Compute R_6, L_6, and M_3 to estimate the distance traveled over [0, 3] if the velocity at half-second intervals is as follows:

t (s)	0	0.5	1	1.5	2	2.5	3
v (m/s)	0	12	18	25	20	14	20

SOLUTION For R_6 and L_6, $\Delta t = \frac{3-0}{6} = 0.5$. For M_3, $\Delta t = \frac{3-0}{3} = 1$. Then

$$R_6 = 0.5 \text{ s}\,(12 + 18 + 25 + 20 + 14 + 20) \text{ m/s} = 0.5(109) \text{ m} = 54.5 \text{ m},$$

$$L_6 = 0.5\,\text{sec}\,(0 + 12 + 18 + 25 + 20 + 14)\,\text{m/sec} = 0.5(89)\,\text{m} = 44.5\,\text{m},$$

and

$$M_3 = 1\,\text{sec}\,(12 + 25 + 14)\ \text{m/sec} = 51\,\text{m}.$$

7. Let $f(x) = 2x + 3$.
(a) Compute R_6 and L_6 over $[0, 3]$.
(b) Use geometry to find the exact area A and compute the errors $|A - R_6|$ and $|A - L_6|$ in the approximations.

SOLUTION Let $f(x) = 2x + 3$ on $[0, 3]$.

(a) We partition $[0, 3]$ into 6 equally-spaced subintervals. The left endpoints of the subintervals are $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\right\}$ whereas the right endpoints are $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$.

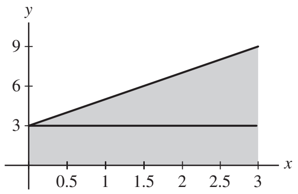
- Let $a = 0$, $b = 3$, $n = 6$, $\Delta x = (b - a)/n = \frac{1}{2}$, and $x_k = a + k\Delta x$, $k = 0, 1, \dots, 5$ (left endpoints). Then

$$L_6 = \sum_{k=0}^{5} f(x_k)\Delta x = \Delta x \sum_{k=0}^{5} f(x_k) = \frac{1}{2}(3 + 4 + 5 + 6 + 7 + 8) = 16.5.$$

- With $x_k = a + k\Delta x$, $k = 1, 2, \dots, 6$ (right endpoints), we have

$$R_6 = \sum_{k=1}^{6} f(x_k)\Delta x = \Delta x \sum_{k=1}^{6} f(x_k) = \frac{1}{2}(4 + 5 + 6 + 7 + 8 + 9) = 19.5.$$

(b) Via geometry (see figure below), the exact area is $A = \frac{1}{2}(3)(6) + 3^2 = 18$. Thus, L_6 underestimates the true area ($L_6 - A = -1.5$), while R_6 overestimates the true area ($R_6 - A = +1.5$).



8. Repeat Exercise 7 for $f(x) = 20 - 3x$ over $[2, 4]$.

SOLUTION Let $f(x) = 20 - 3x$ on $[2, 4]$.

(a) We partition $[2, 4]$ into 6 equally-spaced subintervals. The left endpoints of the subintervals are $\left\{2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}\right\}$ whereas the right endpoints are $\left\{\frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 3\right\}$.

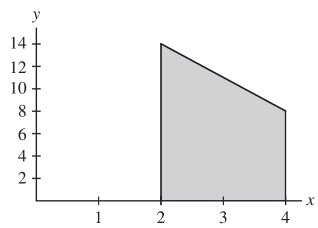
- Let $a = 2$, $b = 4$, $n = 6$, $\Delta x = (b - a)/n = \frac{1}{3}$, and $x_k = a + k\Delta x$, $k = 0, 1, \dots, 5$ (left endpoints). Then

$$L_6 = \sum_{k=0}^{5} f(x_k)\Delta x = \Delta x \sum_{k=0}^{5} f(x_k) = \frac{1}{3}(14 + 13 + 12 + 11 + 10 + 9) = 23.$$

- With $x_k = a + k\Delta x$, $k = 1, 2, \dots, 6$ (right endpoints), we have

$$R_6 = \sum_{k=1}^{6} f(x_k)\Delta x = \Delta x \sum_{k=1}^{6} f(x_k) = \frac{1}{3}(13 + 12 + 11 + 10 + 9 + 8) = 21.$$

(b) Via geometry (see figure below), the exact area is $A = \frac{1}{2}(2)(14 + 8) = 22$. Thus, L_6 overestimates the true area ($L_6 - A = 1$), while R_6 underestimates the true area ($R_6 - A = -1$).



9. Calculate R_3 and L_3

$$\text{for } f(x) = x^2 - x + 4 \quad \text{over } [1, 4]$$

Then sketch the graph of f and the rectangles that make up each approximation. Is the area under the graph larger or smaller than R_3? Is it larger or smaller than L_3?

SOLUTION Let $f(x) = x^2 - x + 4$ and set $a = 1$, $b = 4$, $n = 3$, $\Delta x = (b - a)/n = (4 - 1)/3 = 1$.

(a) Let $x_k = a + k\Delta x$, $k = 0, 1, 2, 3$.

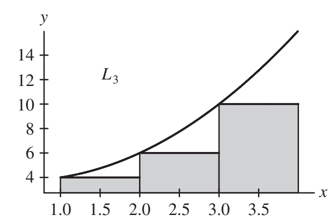
- Selecting the left endpoints of the subintervals, x_k, $k = 0, 1, 2$, or $\{1, 2, 3\}$, we have

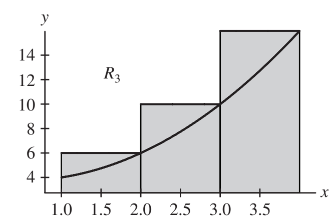
$$L_3 = \sum_{k=0}^{2} f(x_k)\Delta x = \Delta x \sum_{k=0}^{2} f(x_k) = (1)(4 + 6 + 10) = 20.$$

- Selecting the right endpoints of the subintervals, x_k, $k = 1, 2, 3$, or $\{2, 3, 4\}$, we have

$$R_3 = \sum_{k=1}^{3} f(x_k)\Delta x = \Delta x \sum_{k=1}^{3} f(x_k) = (1)(6 + 10 + 16) = 32.$$

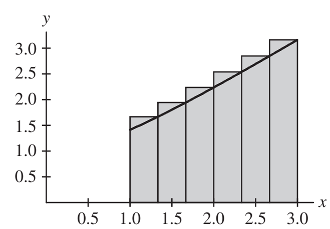
(b) Here are figures of the three rectangles that approximate the area under the curve $f(x)$ over the interval $[1, 4]$. Clearly, the area under the graph is larger than L_3 but smaller than R_3.





10. Let $f(x) = \sqrt{x^2 + 1}$ and $\Delta x = \frac{1}{3}$. Sketch the graph of $f(x)$ and draw the right-endpoint rectangles whose area is represented by the sum $\sum_{i=1}^{6} f(1 + i\Delta x)\Delta x$.

SOLUTION Because $\Delta x = \frac{1}{3}$ and the sum evaluates f at $1 + i\Delta x$ for i from 1 through 6, it follows that the interval over which we are considering f is $[1, 3]$. The sketch of f together with the six rectangles is shown below.



11. Estimate R_3, M_3, and L_6 over $[0, 1.5]$ for the function in Figure 17.

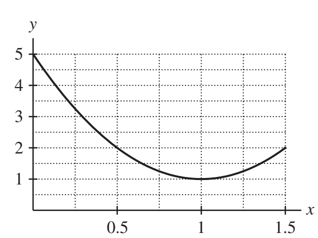


FIGURE 17

SOLUTION Let $f(x)$ on $[0, \frac{3}{2}]$ be given by Figure 17. For $n = 3$, $\Delta x = (\frac{3}{2} - 0)/3 = \frac{1}{2}$, $\{x_k\}_{k=0}^{3} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}\right\}$. Therefore

$$R_3 = \frac{1}{2} \sum_{k=1}^{3} f(x_k) = \frac{1}{2}(2 + 1 + 2) = 2.5,$$

$$M_3 = \frac{1}{2}\sum_{k=1}^{6} f\left(x_k - \frac{1}{2}\Delta x\right) = \frac{1}{2}(3.25 + 1.25 + 1.25) = 2.875.$$

For $n = 6$, $\Delta x = (\frac{3}{2} - 0)/6 = \frac{1}{4}$, $\{x_k\}_{k=0}^{6} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}\right\}$. Therefore

$$L_6 = \frac{1}{4}\sum_{k=0}^{5} f(x_k) = \frac{1}{4}(5 + 3.25 + 2 + 1.25 + 1 + 1.25) = 3.4375.$$

12. Calculate the area of the shaded rectangles in Figure 18. Which approximation do these rectangles represent?

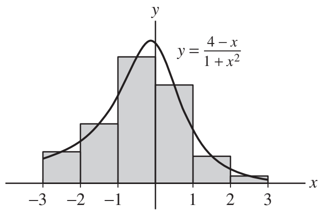


FIGURE 18

SOLUTION Each rectangle in Figure 18 has a width of 1 and the height is taken as the value of the function at the midpoint of the interval. Thus, the area of the shaded rectangles is

$$1\left(\frac{26}{29} + \frac{22}{13} + \frac{18}{5} + \frac{14}{5} + \frac{10}{13} + \frac{6}{29}\right) = \frac{18784}{1885} \approx 9.965.$$

Because there are six rectangles and the height of each rectangle is taken as the value of the function at the midpoint of the interval, the shaded rectangles represent the approximation M_6 to the area under the curve.

In Exercises 13–20, calculate the approximation for the given function and interval.

13. R_3, $f(x) = 7 - x$, $[3, 5]$

SOLUTION Let $f(x) = 7 - x$ on $[3, 5]$. For $n = 3$, $\Delta x = (5 - 3)/3 = \frac{2}{3}$, and $\{x_k\}_{k=0}^{3} = \left\{3, \frac{11}{3}, \frac{13}{3}, 5\right\}$. Therefore

$$R_3 = \frac{2}{3}\sum_{k=1}^{3}(7 - x_k)$$
$$= \frac{2}{3}\left(\frac{10}{3} + \frac{8}{3} + 2\right) = \frac{2}{3}(8) = \frac{16}{3}.$$

14. L_6, $f(x) = \sqrt{6x + 2}$, $[1, 3]$

SOLUTION Let $f(x) = \sqrt{6x + 2}$ on $[1, 3]$. For $n = 6$, $\Delta x = (3 - 1)/6 = \frac{1}{3}$, and $\{x_k\}_{k=0}^{6} = \left\{1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3\right\}$. Therefore

$$L_6 = \frac{1}{3}\sum_{k=0}^{5}\sqrt{6x_k + 2}$$
$$= \frac{1}{3}\left(\sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14} + 4 + \sqrt{18}\right) \approx 7.146368.$$

15. M_6, $f(x) = 4x + 3$, $[5, 8]$

SOLUTION Let $f(x) = 4x + 3$ on $[5, 8]$. For $n = 6$, $\Delta x = (8 - 5)/6 = \frac{1}{2}$, and $\{x_k^*\}_{k=0}^{5} = \{5.25, 5.75, 6.25, 6.75, 7.25, 7.75\}$. Therefore,

$$M_6 = \frac{1}{2}\sum_{k=0}^{5}\left(4x_k^* + 3\right)$$
$$= \frac{1}{2}(24 + 26 + 28 + 30 + 32 + 34)$$
$$= \frac{1}{2}(174) = 87.$$

16. R_5, $f(x) = x^2 + x$, $[-1, 1]$

SOLUTION Let $f(x) = x^2 + x$ on $[-1, 1]$. For $n = 5$, $\Delta x = (1 - (-1))/5 = \frac{2}{5}$, and $\{x_k\}_{k=0}^{5} = \left\{-1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\right\}$. Therefore

$$R_5 = \frac{2}{5}\sum_{k=1}^{5}(x_k^2 + x_k) = \frac{2}{5}\left(\left(\frac{9}{25} - \frac{3}{5}\right) + \left(\frac{1}{25} - \frac{1}{5}\right) + \left(\frac{1}{25} + \frac{1}{5}\right) + \left(\frac{9}{25} + \frac{3}{5}\right) + 2\right)$$
$$= \frac{2}{5}\left(\frac{14}{5}\right) = \frac{28}{25}.$$

17. L_6, $f(x) = x^2 + 3|x|$, $[-2, 1]$

SOLUTION Let $f(x) = x^2 + 3\,|x|$ on $[-2, 1]$. For $n = 6$, $\Delta x = (1 - (-2))/6 = \frac{1}{2}$, and $\{x_k\}_{k=0}^{6} = \{-2, -1.5, -1, -0.5, 0, 0.5, 1\}$. Therefore

$$L_6 = \frac{1}{2}\sum_{k=0}^{5}(x_k^2 + 3\,|x_k|) = \frac{1}{2}(10 + 6.75 + 4 + 1.75 + 0 + 1.75) = 12.125.$$

18. M_4, $f(x) = \sqrt{x}$, $[3, 5]$

SOLUTION Let $f(x) = \sqrt{x}$ on $[3, 5]$. For $n = 4$, $\Delta x = (5 - 3)/4 = \frac{1}{2}$, and $\{x_k^*\}_{k=0}^{3} = \{\frac{13}{4}, \frac{15}{4}, \frac{17}{4}, \frac{19}{4}\}$. Therefore,

$$M_4 = \frac{1}{2}\sum_{k=0}^{3}\sqrt{x_k^*}$$
$$= \frac{1}{2}\left(\frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} + \frac{\sqrt{17}}{2} + \frac{\sqrt{19}}{2}\right) \approx 3.990135.$$

19. L_4, $f(x) = \cos^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = \cos^2 x$ on $[\frac{\pi}{6}, \frac{\pi}{2}]$. For $n = 4$,

$$\Delta x = \frac{(\pi/2 - \pi/6)}{4} = \frac{\pi}{12} \qquad \text{and} \qquad \{x_k\}_{k=0}^{4} = \left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{2}\right\}.$$

Therefore

$$L_4 = \frac{\pi}{12}\sum_{k=0}^{3}\cos^2 x_k \approx 0.410236.$$

20. M_5, $f(x) = \ln x$, $[1, 3]$

SOLUTION Let $f(x) = \ln x$ on $[1, 3]$. For $n = 5$, $\Delta x = (3 - 1)/5 = \frac{2}{5}$, and $\{x_k^*\}_{k=0}^{4} = \{\frac{6}{5}, \frac{8}{5}, 2, \frac{12}{5}, \frac{14}{5}\}$. Therefore,

$$M_5 = \frac{2}{5}\sum_{k=0}^{4}\ln x_k^*$$
$$= \frac{2}{5}\left(\ln\frac{6}{5} + \ln\frac{8}{5} + \ln 2 + \ln\frac{12}{5} + \ln\frac{14}{5}\right) \approx 1.300224.$$

In Exercises 21–26, write the sum in summation notation.

21. $4^7 + 5^7 + 6^7 + 7^7 + 8^7$

SOLUTION The first term is 4^7, and the last term is 8^7, so it seems the kth term is k^7. Therefore, the sum is:

$$\sum_{k=4}^{8} k^7.$$

22. $(2^2 + 2) + (3^2 + 3) + (4^2 + 4) + (5^2 + 5)$

SOLUTION The first term is $2^2 + 2$, and the last term is $5^2 + 5$, so it seems that the sum limits are 2 and 5, and the kth term is $k^2 + k$. Therefore, the sum is:

$$\sum_{k=2}^{5}(k^2 + k).$$

23. $(2^2+2)+(2^3+2)+(2^4+2)+(2^5+2)$

SOLUTION The first term is 2^2+2, and the last term is 2^5+2, so it seems the sum limits are 2 and 5, and the kth term is 2^k+2. Therefore, the sum is:

$$\sum_{k=2}^{5}(2^k+2).$$

24. $\sqrt{1+1^3}+\sqrt{2+2^3}+\cdots+\sqrt{n+n^3}$

SOLUTION The first term is $\sqrt{1+1^3}$ and the last term is $\sqrt{n+n^3}$, so it seems the summation limits are 1 through n, and the k-th term is $\sqrt{k+k^3}$. Therefore, the sum is

$$\sum_{k=1}^{n}\sqrt{k+k^3}.$$

25. $\dfrac{1}{2\cdot 3}+\dfrac{2}{3\cdot 4}+\cdots+\dfrac{n}{(n+1)(n+2)}$

SOLUTION The first summand is $\frac{1}{(1+1)\cdot(1+2)}$. This shows us

$$\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\cdots+\frac{n}{(n+1)(n+2)}=\sum_{i=1}^{n}\frac{i}{(i+1)(i+2)}.$$

26. $e^{\pi}+e^{\pi/2}+e^{\pi/3}+\cdots+e^{\pi/n}$

SOLUTION The first term is $e^{\pi/1}$ and the last term is $e^{\pi/n}$, so it seems the sum limits are 1 and n and the kth term is $e^{\pi/k}$. Therefore, the sum is

$$\sum_{k=1}^{n}e^{\pi/k}.$$

27. Calculate the sums:

(a) $\displaystyle\sum_{i=1}^{5}9$ **(b)** $\displaystyle\sum_{i=0}^{5}4$ **(c)** $\displaystyle\sum_{k=2}^{4}k^3$

SOLUTION

(a) $\displaystyle\sum_{i=1}^{5}9=9+9+9+9+9=45$. Alternatively, $\displaystyle\sum_{i=1}^{5}9=9\sum_{i=1}^{5}1=(9)(5)=45$.

(b) $\displaystyle\sum_{i=0}^{5}4=4+4+4+4+4+4=24$. Alternatively, $\displaystyle\sum_{i=0}^{5}4=4\sum_{i=0}^{5}=(4)(6)=24$.

(c) $\displaystyle\sum_{k=2}^{4}k^3=2^3+3^3+4^3=99$. Alternatively,

$$\sum_{k=2}^{4}k^3=\left(\sum_{k=1}^{4}k^3\right)-\left(\sum_{k=1}^{1}k^3\right)=\left(\frac{4^4}{4}+\frac{4^3}{2}+\frac{4^2}{4}\right)-\left(\frac{1^4}{4}+\frac{1^3}{2}+\frac{1^2}{4}\right)=99.$$

28. Calculate the sums:

(a) $\displaystyle\sum_{j=3}^{4}\sin\left(j\frac{\pi}{2}\right)$ **(b)** $\displaystyle\sum_{k=3}^{5}\frac{1}{k-1}$ **(c)** $\displaystyle\sum_{j=0}^{2}3^{j-1}$

SOLUTION

(a) $\displaystyle\sum_{j=3}^{4}\sin\left(\frac{j\pi}{2}\right)=\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right)=-1+0=-1$.

(b) $\displaystyle\sum_{k=3}^{5}\frac{1}{k-1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$.

(c) $\displaystyle\sum_{j=0}^{2}3^{j-1}=\frac{1}{3}+1+3=\frac{13}{3}$.

29. Let $b_1 = 4$, $b_2 = 1$, $b_3 = 2$, and $b_4 = -4$. Calculate:

(a) $\sum_{i=2}^{4} b_i$ **(b)** $\sum_{j=1}^{2} (2^{b_j} - b_j)$ **(c)** $\sum_{k=1}^{3} kb_k$

SOLUTION

(a) $\sum_{i=2}^{4} b_i = b_2 + b_3 + b_4 = 1 + 2 + (-4) = -1.$

(b) $\sum_{j=1}^{2} \left(2^{b_j} - b_j\right) = (2^4 - 4) + (2^1 - 1) = 13.$

(c) $\sum_{k=1}^{3} kb_k = 1(4) + 2(1) + 3(2) = 12.$

30. Assume that $a_1 = -5$, $\sum_{i=1}^{10} a_i = 20$, and $\sum_{i=1}^{10} b_i = 7$. Calculate:

(a) $\sum_{i=1}^{10} (4a_i + 3)$ **(b)** $\sum_{i=2}^{10} a_i$ **(c)** $\sum_{i=1}^{10} (2a_i - 3b_i)$

SOLUTION

(a) $\sum_{i=1}^{10} (4a_i + 3) = 4\sum_{i=1}^{10} a_i + 3\sum_{i=1}^{10} 1 = 4(20) + 3(10) = 110.$

(b) $\sum_{i=2}^{10} a_i = \sum_{i=1}^{10} a_i - a_1 = 20 - (-5) = 25.$

(c) $\sum_{i=1}^{10} (2a_i - 3b_i) = 2\sum_{i=1}^{10} a_i - 3\sum_{i=1}^{10} b_i = 2(20) - 3(7) = 19.$

31. Calculate $\sum_{j=101}^{200} j$. *Hint:* Write as a difference of two sums and use formula (3).

SOLUTION

$$\sum_{j=101}^{200} j = \sum_{j=1}^{200} j - \sum_{j=1}^{100} j = \left(\frac{200^2}{2} + \frac{200}{2}\right) - \left(\frac{100^2}{2} + \frac{100}{2}\right) = 20100 - 5050 = 15050.$$

32. Calculate $\sum_{j=1}^{30} (2j + 1)^2$. *Hint:* Expand and use formulas (3)–(4).

SOLUTION

$$\begin{aligned}\sum_{j=1}^{30} (2j+1)^2 &= 4\sum_{j=1}^{30} j^2 + 4\sum_{j=1}^{30} j + \sum_{j=1}^{30} 1 \\ &= 4\left(\frac{30^3}{3} + \frac{30^2}{2} + \frac{30}{6}\right) + 4\left(\frac{30^2}{2} + \frac{30}{2}\right) + 30 \\ &= 39{,}710.\end{aligned}$$

In Exercises 33–40, use linearity and formulas (3)–(5) to rewrite and evaluate the sums.

33. $\sum_{j=1}^{20} 8j^3$

SOLUTION $\sum_{j=1}^{20} 8j^3 = 8\sum_{j=1}^{20} j^3 = 8\left(\frac{20^4}{4} + \frac{20^3}{2} + \frac{20^2}{4}\right) = 8(44{,}100) = 352{,}800.$

34. $\sum_{k=1}^{30}(4k-3)$

SOLUTION

$$\begin{aligned}\sum_{k=1}^{30}(4k-3) &= 4\sum_{k=1}^{30}k - 3\sum_{k=1}^{30}1 \\ &= 4\left(\frac{30^2}{2}+\frac{30}{2}\right) - 3(30) = 4(465) - 90 = 1770.\end{aligned}$$

35. $\sum_{n=51}^{150} n^2$

SOLUTION

$$\begin{aligned}\sum_{n=51}^{150} n^2 &= \sum_{n=1}^{150} n^2 - \sum_{n=1}^{50} n^2 \\ &= \left(\frac{150^3}{3}+\frac{150^2}{2}+\frac{150}{6}\right) - \left(\frac{50^3}{3}+\frac{50^2}{2}+\frac{50}{6}\right) \\ &= 1{,}136{,}275 - 42{,}925 = 1{,}093{,}350.\end{aligned}$$

36. $\sum_{k=101}^{200} k^3$

SOLUTION

$$\begin{aligned}\sum_{k=101}^{200} k^3 &= \sum_{k=1}^{200} k^3 - \sum_{k=1}^{100} k^3 \\ &= \left(\frac{200^4}{4}+\frac{200^3}{2}+\frac{200^2}{4}\right) - \left(\frac{100^4}{4}+\frac{100^3}{2}+\frac{100^2}{4}\right) \\ &= 404{,}010{,}000 - 25{,}502{,}500 = 378{,}507{,}500.\end{aligned}$$

37. $\sum_{j=0}^{50} j(j-1)$

SOLUTION

$$\begin{aligned}\sum_{j=0}^{50} j(j-1) &= \sum_{j=0}^{50}(j^2-j) = \sum_{j=0}^{50} j^2 - \sum_{j=0}^{50} j \\ &= \left(\frac{50^3}{3}+\frac{50^2}{2}+\frac{50}{6}\right) - \left(\frac{50^2}{2}+\frac{50}{2}\right) = \frac{50^3}{3} - \frac{50}{3} = \frac{124{,}950}{3} = 41{,}650.\end{aligned}$$

The power sum formula is usable because $\sum_{j=0}^{50} j(j-1) = \sum_{j=1}^{50} j(j-1)$.

38. $\sum_{j=2}^{30}\left(6j+\frac{4j^2}{3}\right)$

SOLUTION

$$\begin{aligned}\sum_{j=2}^{30}\left(6j+\frac{4j^2}{3}\right) &= 6\sum_{j=2}^{30} j + \frac{4}{3}\sum_{j=2}^{30} j^2 = 6\left(\sum_{j=1}^{30} j - \sum_{j=1}^{1} j\right) + \frac{4}{3}\left(\sum_{j=1}^{30} j^2 - \sum_{j=1}^{1} j^2\right) \\ &= 6\left(\frac{30^2}{2}+\frac{30}{2}-1\right) + \frac{4}{3}\left(\frac{30^3}{3}+\frac{30^2}{2}+\frac{30}{6}-1\right) \\ &= 6\,(464) + \frac{4}{3}\,(9454) = 2784 + \frac{37{,}816}{3} = \frac{46{,}168}{3}.\end{aligned}$$

39. $\sum_{m=1}^{30} (4-m)^3$

SOLUTION

$$\begin{aligned}\sum_{m=1}^{30} (4-m)^3 &= \sum_{m=1}^{30} (64 - 48m + 12m^2 - m^3)\\ &= 64\sum_{m=1}^{30} 1 - 48\sum_{m=1}^{30} m + 12\sum_{m=1}^{30} m^2 - \sum_{m=1}^{30} m^3\\ &= 64(30) - 48\frac{(30)(31)}{2} + 12\left(\frac{30^3}{3} + \frac{30^2}{2} + \frac{30}{6}\right) - \left(\frac{30^4}{4} + \frac{30^3}{2} + \frac{30^2}{4}\right)\\ &= 1920 - 22{,}320 + 113{,}460 - 216{,}225 = -123{,}165.\end{aligned}$$

40. $\sum_{m=1}^{20} \left(5 + \frac{3m}{2}\right)^2$

SOLUTION

$$\begin{aligned}\sum_{m=1}^{20} \left(5 + \frac{3m}{2}\right)^2 &= 25\sum_{m=1}^{20} 1 + 15\sum_{m=1}^{20} m + \frac{9}{4}\sum_{m=1}^{20} m^2\\ &= 25(20) + 15\left(\frac{20^2}{2} + \frac{20}{2}\right) + \frac{9}{4}\left(\frac{20^3}{3} + \frac{20^2}{2} + \frac{20}{6}\right)\\ &= 500 + 15(210) + \frac{9}{4}(2870) = 10107.5.\end{aligned}$$

In Exercises 41–44, use formulas (3)–(5) to evaluate the limit.

41. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2}$

SOLUTION Let $s_N = \sum_{i=1}^{N} \frac{i}{N^2}$. Then,

$$s_N = \sum_{i=1}^{N} \frac{i}{N^2} = \frac{1}{N^2}\sum_{i=1}^{N} i = \frac{1}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) = \frac{1}{2} + \frac{1}{2N}.$$

Therefore, $\lim_{N\to\infty} s_N = \frac{1}{2}$.

42. $\lim_{N\to\infty} \sum_{j=1}^{N} \frac{j^3}{N^4}$

SOLUTION Let $s_N = \sum_{j=1}^{N} \frac{j^3}{N^4}$. Then

$$s_N = \frac{1}{N^4}\sum_{j=1}^{N} j^3 = \frac{1}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}\right) = \frac{1}{4} + \frac{1}{2N} + \frac{1}{4N^2}.$$

Therefore, $\lim_{N\to\infty} s_N = \frac{1}{4}$.

43. $\displaystyle\lim_{N\to\infty}\sum_{i=1}^{N}\frac{i^2-i+1}{N^3}$

SOLUTION Let $s_N = \displaystyle\sum_{i=1}^{N}\frac{i^2-i+1}{N^3}$. Then

$$s_N = \sum_{i=1}^{N}\frac{i^2-i+1}{N^3} = \frac{1}{N^3}\left[\left(\sum_{i=1}^{N} i^2\right) - \left(\sum_{i=1}^{N} i\right) + \left(\sum_{i=1}^{N} 1\right)\right]$$
$$= \frac{1}{N^3}\left[\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right) - \left(\frac{N^2}{2}+\frac{N}{2}\right) + N\right] = \frac{1}{3} + \frac{2}{3N^2}.$$

Therefore, $\displaystyle\lim_{N\to\infty} s_N = \frac{1}{3}$.

44. $\displaystyle\lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{i^3}{N^4}-\frac{20}{N}\right)$

SOLUTION Let $s_N = \displaystyle\sum_{i=1}^{N}\left(\frac{i^3}{N^4}-\frac{20}{N}\right)$. Then

$$s_N = \frac{1}{N^4}\sum_{i=1}^{N} i^3 - \frac{20}{N}\sum_{i=1}^{N} 1 = \frac{1}{N^4}\left(\frac{N^4}{4}+\frac{N^3}{2}+\frac{N^2}{4}\right) - 20 = \frac{1}{4} + \frac{1}{2N} + \frac{1}{4N^2} - 20.$$

Therefore, $\displaystyle\lim_{N\to\infty} s_N = \frac{1}{4} - 20 = -\frac{79}{4}$.

In Exercises 45–50, calculate the limit for the given function and interval. Verify your answer by using geometry.

45. $\displaystyle\lim_{N\to\infty} R_N, \quad f(x) = 9x, \quad [0, 2]$

SOLUTION Let $f(x) = 9x$ on $[0, 2]$. Let N be a positive integer and set $a = 0$, $b = 2$, and $\Delta x = (b-a)/N = (2-0)/N = 2/N$. Also, let $x_k = a + k\Delta x = 2k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[0, 2]$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{2}{N}\sum_{k=1}^{N} 9\left(\frac{2k}{N}\right) = \frac{36}{N^2}\sum_{k=1}^{N} k = \frac{36}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right) = 18 + \frac{18}{N}.$$

The area under the graph is

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(18 + \frac{18}{N}\right) = 18.$$

The region under the graph is a triangle with base 2 and height 18. The area of the region is then $\frac{1}{2}(2)(18) = 18$, which agrees with the value obtained from the limit of the right-endpoint approximations.

46. $\displaystyle\lim_{N\to\infty} R_N, \quad f(x) = 3x + 6, \quad [1, 4]$

SOLUTION Let $f(x) = 3x + 6$ on $[1, 4]$. Let N be a positive integer and set $a = 1$, $b = 4$, and $\Delta x = (b-a)/N = (4-1)/N = 3/N$. Also, let $x_k = a + k\Delta x = 1 + 3k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[1, 4]$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{3}{N}\sum_{k=1}^{N}\left(9 + \frac{9k}{N}\right)$$
$$= \frac{27}{N}\sum_{k=1}^{N} 1 + \frac{27}{N^2}\sum_{k=1}^{N} j = \frac{27}{N}(N) + \frac{27}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right)$$
$$= \frac{81}{2} + \frac{27}{2N}.$$

The area under the graph is

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{81}{2} + \frac{27}{2N}\right) = \frac{81}{2}.$$

The region under the graph is a trapezoid with base width 3 and heights 9 and 18. The area of the region is then $\frac{1}{2}(3)(9+18) = \frac{81}{2}$, which agrees with the value obtained from the limit of the right-endpoint approximations.

47. $\lim\limits_{N\to\infty} L_N, \quad f(x) = \frac{1}{2}x + 2, \quad [0, 4]$

SOLUTION Let $f(x) = \frac{1}{2}x + 2$ on $[0, 4]$. Let $N > 0$ be an integer, and set $a = 0$, $b = 4$, and $\Delta x = (4-0)/N = \frac{4}{N}$. Also, let $x_k = 0 + k\Delta x = \frac{4k}{N}$, $k = 0, 1, \ldots, N-1$ be the left endpoints of the N subintervals. Then

$$\begin{aligned} L_N &= \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{4}{N} \sum_{k=0}^{N-1} \left(\frac{1}{2}\left(\frac{4k}{N}\right) + 2 \right) = \frac{8}{N} \sum_{k=0}^{N-1} 1 + \frac{8}{N^2} \sum_{k=0}^{N-1} k \\ &= 8 + \frac{8}{N^2}\left(\frac{(N-1)^2}{2} + \frac{N-1}{2} \right) = 12 - \frac{4}{N}. \end{aligned}$$

The area under the graph is

$$\lim_{N\to\infty} L_N = 12.$$

The region under the curve over $[0, 4]$ is a trapezoid with base width 4 and heights 2 and 4. From this, we get that the area is $\frac{1}{2}(4)(2+4) = 12$, which agrees with the answer obtained from the limit of the left-endpoint approximations.

48. $\lim\limits_{N\to\infty} L_N, \quad f(x) = 4x - 2, \quad [1, 3]$

SOLUTION Let $f(x) = 4x - 2$ on $[1, 3]$. Let $N > 0$ be an integer, and set $a = 1$, $b = 3$, and $\Delta x = (3-1)/N = \frac{2}{N}$. Also, let $x_k = a + k\Delta x = 1 + \frac{2k}{N}$, $k = 0, 1, \ldots, N-1$ be the left endpoints of the N subintervals. Then

$$\begin{aligned} L_N &= \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{2}{N} \sum_{k=0}^{N-1} \left(\frac{8k}{N} + 2 \right) = \frac{16}{N^2} \sum_{k=0}^{N-1} k + \frac{4}{N} \sum_{k=0}^{N-1} 1 \\ &= \frac{16}{N^2}\left(\frac{(N-1)^2}{2} + \frac{N-1}{2} \right) + \frac{4}{N}(N-1) \\ &= 12 - \frac{12}{N} \end{aligned}$$

The area under the graph is

$$\lim_{N\to\infty} L_N = 12.$$

The region under the curve over $[1, 3]$ is a trapezoid with base width 2 and heights 2 and 10. From this, we get that the area is $\frac{1}{2}(2)(2+10) = 12$, which agrees with the answer obtained from the limit of the left-endpoint approximations.

49. $\lim\limits_{N\to\infty} M_N, \quad f(x) = x, \quad [0, 2]$

SOLUTION Let $f(x) = x$ on $[0, 2]$. Let $N > 0$ be an integer and set $a = 0$, $b = 2$, and $\Delta x = (b-a)/N = \frac{2}{N}$. Also, let $x_k^* = 0 + (k - \frac{1}{2})\Delta x = \frac{2k-1}{N}$, $k = 1, 2, \ldots N$, be the midpoints of the N subintervals of $[0, 2]$. Then

$$\begin{aligned} M_N &= \Delta x \sum_{k=1}^{N} f(x_k^*) = \frac{2}{N} \sum_{k=1}^{N} \frac{2k-1}{N} = \frac{2}{N^2} \sum_{k=1}^{N} (2k-1) \\ &= \frac{2}{N^2}\left(2\sum_{k=1}^{N} k - N \right) = \frac{4}{N^2}\left(\frac{N^2}{2} + \frac{N}{2} \right) - \frac{2}{N} = 2. \end{aligned}$$

The area under the curve over $[0, 2]$ is

$$\lim_{N\to\infty} M_N = 2.$$

The region under the curve over $[0, 2]$ is a triangle with base and height 2, and thus area 2, which agrees with the answer obtained from the limit of the midpoint approximations.

50. $\lim_{N\to\infty} M_N, \quad f(x) = 12 - 4x, \quad [2, 6]$

SOLUTION Let $f(x) = 12 - 4x$ on $[2, 6]$. Let $N > 0$ be an integer and set $a = 2$, $b = 6$, and $\Delta x = (b-a)/N = \frac{4}{N}$. Also, let $x_k^* = a + (k - \frac{1}{2})\Delta x = 2 + \frac{4k-2}{N}$, $k = 1, 2, \dots N$, be the midpoints of the N subintervals of $[2, 6]$. Then

$$M_N = \Delta x \sum_{k=1}^{N} f(x_k^*) = \frac{4}{N}\sum_{k=1}^{N}\left(4 - \frac{16k-8}{N}\right)$$

$$= \frac{16}{N}\sum_{k=1}^{N} 1 - \frac{64}{N^2}\sum_{k=1}^{N} k + \frac{32}{N^2}\sum_{k=1}^{N} 1$$

$$= \frac{16}{N}(N) - \frac{64}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{32}{N^2}(N) = -16.$$

The area under the curve over $[2, 6]$ is

$$\lim_{N\to\infty} M_N = -16.$$

The region under the curve over $[2, 6]$ consists of a triangle of base 1 and height 4 above the axis and a triangle of base 3 and height 12 below the axis. The area of this region is therefore

$$\frac{1}{2}(1)(4) - \frac{1}{2}(3)(12) = -16,$$

which agrees with the answer obtained from the limit of the midpoint approximations.

51. Show, for $f(x) = 3x^2 + 4x$ over $[0, 2]$, that

$$R_N = \frac{2}{N}\sum_{j=1}^{N}\left(\frac{24j^2}{N^2} + \frac{16j}{N}\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

SOLUTION Let $f(x) = 3x^2 + 4x$ on $[0, 2]$. Let N be a positive integer and set $a = 0$, $b = 2$, and $\Delta x = (b-a)/N = (2-0)/N = 2/N$. Also, let $x_j = a + j\Delta x = 2j/N$, $j = 1, 2, \dots, N$ be the right endpoints of the N subintervals of $[0, 3]$. Then

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{2}{N}\sum_{j=1}^{N}\left(3\left(\frac{2j}{N}\right)^2 + 4\frac{2j}{N}\right)$$

$$= \frac{2}{N}\sum_{j=1}^{N}\left(\frac{12j^2}{N^2} + \frac{8j}{N}\right)$$

Continuing, we find

$$R_N = \frac{24}{N^3}\sum_{j=1}^{N} j^2 + \frac{16}{N^2}\sum_{j=1}^{N} j$$

$$= \frac{24}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{16}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$

$$= 16 + \frac{20}{N} + \frac{4}{N^2}$$

Thus,

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(16 + \frac{20}{N} + \frac{4}{N^2}\right) = 16.$$

52. Show, for $f(x) = 3x^3 - x^2$ over $[1, 5]$, that

$$R_N = \frac{4}{N}\sum_{j=1}^{N}\left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

SOLUTION Let $f(x) = 3x^3 - x^2$ on $[1, 5]$. Let N be a positive integer and set $a = 1$, $b = 5$, and $\Delta x = (b-a)/N = (5-1)/N = 4/N$. Also, let $x_j = a + j\Delta x = 1 + 4j/N$, $j = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[1, 5]$. Then

$$\begin{aligned} f(x_j) &= 3\left(1 + \frac{4j}{N}\right)^3 - \left(1 + \frac{4j}{N}\right)^2 \\ &= 3\left(1 + \frac{12j}{N} + \frac{48j^2}{N^2} + \frac{64j^3}{N^3}\right) - \left(1 + \frac{8j}{N} + \frac{16j^2}{N^2}\right) \\ &= \frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2. \end{aligned}$$

and

$$R_N = \sum_{j=1}^{N} f(x_j)\Delta x = \frac{4}{N}\sum_{j=1}^{N}\left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right).$$

Continuing, we find

$$\begin{aligned} R_N &= \frac{768}{N^4}\sum_{j=1}^{N} j^3 + \frac{512}{N^3}\sum_{j=1}^{N} j^2 + \frac{112}{N^2}\sum_{j=1}^{N} j + \frac{8}{N}\sum_{j=1}^{N} 1 \\ &= \frac{768}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{2}\right) + \frac{512}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\ &\quad + \frac{112}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{8}{N}(N) \\ &= \frac{1280}{3} + \frac{696}{N} + \frac{832}{3N^2}. \end{aligned}$$

Thus,

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{1280}{3} + \frac{696}{N} + \frac{832}{3N^2}\right) = \frac{1280}{3}.$$

In Exercises 53–60, find a formula for R_N and compute the area under the graph as a limit.

53. $f(x) = x^2$, $[0, 1]$

SOLUTION Let $f(x) = x^2$ on the interval $[0, 1]$. Then $\Delta x = \dfrac{1-0}{N} = \dfrac{1}{N}$ and $a = 0$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{1}{N}\sum_{j=1}^{N} j^2 \frac{1}{N^2} = \frac{1}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}\right) = \frac{1}{3}.$$

54. $f(x) = x^2$, $[-1, 5]$

SOLUTION Let $f(x) = x^2$ on the interval $[-1, 5]$. Then $\Delta x = \dfrac{5-(-1)}{N} = \dfrac{6}{N}$ and $a = -1$. Hence,

$$\begin{aligned} R_N &= \Delta x \sum_{j=1}^{N} f(-1 + j\Delta x) = \frac{6}{N}\sum_{j=1}^{N}\left(-1 + \frac{6j}{N}\right)^2 \\ &= \frac{6}{N}\sum_{j=1}^{N} 1 - \frac{72}{N^2}\sum_{j=1}^{N} j + \frac{216}{N^3}\sum_{j=1}^{N} j^2 \\ &= \frac{6}{N}(N) - \frac{72}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{216}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\ &= 42 + \frac{72}{N} + \frac{36}{N^2} \end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(42 + \frac{72}{N} + \frac{36}{N^2}\right) = 42.$$

55. $f(x) = 6x^2 - 4$, $[2, 5]$

SOLUTION Let $f(x) = 6x^2 - 4$ on the interval $[2, 5]$. Then $\Delta x = \frac{5-2}{N} = \frac{3}{N}$ and $a = 2$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(2 + j\Delta x) = \frac{3}{N}\sum_{j=1}^{N}\left(6\left(2 + \frac{3j}{N}\right)^2 - 4\right) = \frac{3}{N}\sum_{j=1}^{N}\left(20 + \frac{72j}{N} + \frac{54j^2}{N^2}\right)$$

$$= 60 + \frac{216}{N^2}\sum_{j=1}^{N} j + \frac{162}{N^3}\sum_{j=1}^{N} j^2$$

$$= 60 + \frac{216}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{162}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right)$$

$$= 222 + \frac{189}{N} + \frac{27}{N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(222 + \frac{189}{N} + \frac{27}{N^2}\right) = 222.$$

56. $f(x) = x^2 + 7x$, $[6, 11]$

SOLUTION Let $f(x) = x^2 + 7x$ on the interval $[6, 11]$. Then $\Delta x = \frac{11-6}{N} = \frac{5}{N}$ and $a = 6$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(6 + j\Delta x) = \frac{5}{N}\sum_{j=1}^{N}\left[\left(6 + \frac{5j}{N}\right)^2 + 7\left(6 + \frac{5j}{N}\right)\right]$$

$$= \frac{5}{N}\sum_{j=1}^{N}\left(\frac{25j^2}{N^2} + \frac{95j}{N} + 78\right)$$

$$= \frac{125}{N^3}\sum_{j=1}^{N} j^3 + \frac{475}{N^2}\sum_{j=1}^{N} j + \frac{390}{N}\sum_{j=1}^{N} 1$$

$$= \frac{125}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{475}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + 390$$

$$= \frac{4015}{6} + \frac{300}{N} + \frac{125}{6N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{4015}{6} + \frac{300}{N} + \frac{125}{6N^2}\right) = \frac{4015}{6}.$$

57. $f(x) = x^3 - x$, $[0, 2]$

SOLUTION Let $f(x) = x^3 - x$ on the interval $[0, 2]$. Then $\Delta x = \frac{2-0}{N} = \frac{2}{N}$ and $a = 0$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{2}{N}\sum_{j=1}^{N}\left(\left(\frac{2j}{N}\right)^3 - \frac{2j}{N}\right) = \frac{2}{N}\sum_{j=1}^{N}\left(\frac{8j^3}{N^3} - \frac{2j}{N}\right)$$

$$= \frac{16}{N^4}\sum_{j=1}^{N} j^3 - \frac{4}{N^2}\sum_{j=1}^{N} j$$

$$= \frac{16}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{2}\right) - \frac{4}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$

$$= 2 + \frac{6}{N} + \frac{8}{N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(2 + \frac{6}{N} + \frac{8}{N^2}\right) = 2.$$

58. $f(x) = 2x^3 + x^2$, $[-2, 2]$

SOLUTION Let $f(x) = 2x^3 + x^2$ on the interval $[-2, 2]$. Then $\Delta x = \frac{2-(-2)}{N} = \frac{4}{N}$ and $a = -2$. Hence,

$$\begin{aligned} R_N &= \Delta x \sum_{j=1}^{N} f(-2 + j\Delta x) = \frac{4}{N}\sum_{j=1}^{N}\left[2\left(-2+\frac{4j}{N}\right)^3 + \left(-2+\frac{4j}{N}\right)^2\right] \\ &= \frac{4}{N}\sum_{j=1}^{N}\left(\frac{128j^3}{N^3} - \frac{176j^2}{N^2} + \frac{80j}{N} - 12\right) \\ &= \frac{512}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}\right) - \frac{704}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{320}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) - 48 \\ &= \frac{16}{3} + \frac{64}{N} + \frac{32}{3N^2} \end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{16}{3} + \frac{64}{N} + \frac{32}{3N^2}\right) = \frac{16}{3}.$$

59. $f(x) = 2x + 1$, $[a, b]$ (a, b constants with $a < b$)

SOLUTION Let $f(x) = 2x + 1$ on the interval $[a, b]$. Then $\Delta x = \frac{b-a}{N}$. Hence,

$$\begin{aligned} R_N &= \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \frac{(b-a)}{N}\sum_{j=1}^{N}\left(2\left(a + j\frac{(b-a)}{N}\right) + 1\right) \\ &= \frac{(b-a)}{N}(2a+1)\sum_{j=1}^{N} 1 + \frac{2(b-a)^2}{N^2}\sum_{j=1}^{N} j \\ &= \frac{(b-a)}{N}(2a+1)N + \frac{2(b-a)^2}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) \\ &= (b-a)(2a+1) + (b-a)^2 + \frac{(b-a)^2}{N} \end{aligned}$$

and

$$\begin{aligned} \lim_{N\to\infty} R_N &= \lim_{N\to\infty}\left((b-a)(2a+1) + (b-a)^2 + \frac{(b-a)^2}{N}\right) \\ &= (b-a)(2a+1) + (b-a)^2 = (b^2 + b) - (a^2 + a). \end{aligned}$$

60. $f(x) = x^2$, $[a, b]$ (a, b constants with $a < b$)

SOLUTION Let $f(x) = x^2$ on the interval $[a, b]$. Then $\Delta x = \frac{b-a}{N}$. Hence,

$$\begin{aligned} R_N &= \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \frac{(b-a)}{N}\sum_{j=1}^{N}\left(a^2 + 2aj\frac{(b-a)}{N} + j^2\frac{(b-a)^2}{N^2}\right) \\ &= \frac{a^2(b-a)}{N}\sum_{j=1}^{N} 1 + \frac{2a(b-a)^2}{N^2}\sum_{j=1}^{N} j + \frac{(b-a)^3}{N^3}\sum_{j=1}^{N} j^2 \\ &= \frac{a^2(b-a)}{N}N + \frac{2a(b-a)^2}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{(b-a)^3}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\ &= a^2(b-a) + a(b-a)^2 + \frac{a(b-a)^2}{N} + \frac{(b-a)^3}{3} + \frac{(b-a)^3}{2N} + \frac{(b-a)^3}{6N^2} \end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(a^2(b-a)+a(b-a)^2+\frac{a(b-a)^2}{N}+\frac{(b-a)^3}{3}+\frac{(b-a)^3}{2N}+\frac{(b-a)^3}{6N^2}\right)$$

$$= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3} = \frac{1}{3}b^3-\frac{1}{3}a^3.$$

In Exercises 61–64, describe the area represented by the limits.

61. $\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{j=1}^{N}\left(\frac{j}{N}\right)^4$

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^4$$

represents the area between the graph of $f(x)=x^4$ and the x-axis over the interval [0, 1].

62. $\lim\limits_{N\to\infty}\frac{3}{N}\sum\limits_{j=1}^{N}\left(2+\frac{3j}{N}\right)^4$

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{3}{N}\sum_{j=1}^{N}\left(2+j\cdot\frac{3}{N}\right)^4$$

represents the area between the graph of $f(x)=x^4$ and the x-axis over the interval [2, 5].

63. $\lim\limits_{N\to\infty}\frac{5}{N}\sum\limits_{j=0}^{N-1}e^{-2+5j/N}$

SOLUTION The limit

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty}\frac{5}{N}\sum_{j=0}^{N-1}e^{-2+5j/N}$$

represents the area between the graph of $y=e^x$ and the x-axis over the interval $[-2, 3]$.

64. $\lim\limits_{N\to\infty}\frac{\pi}{2N}\sum\limits_{j=1}^{N}\sin\left(\frac{\pi}{3}-\frac{\pi}{4N}+\frac{j\pi}{2N}\right)$

SOLUTION The limit

$$\lim_{N\to\infty}\frac{\pi}{2N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{3}-\frac{\pi}{4N}+\frac{j\pi}{2N}\right)$$

represents the area between the graph of $y=\sin x$ and the x-axis over the interval $[\frac{\pi}{3}, \frac{5\pi}{6}]$.

In Exercises 65–70, express the area under the graph as a limit using the approximation indicated (in summation notation), but do not evaluate.

65. R_N, $f(x)=\sin x$ over $[0,\pi]$

SOLUTION Let $f(x)=\sin x$ over $[0,\pi]$ and set $a=0$, $b=\pi$, and $\Delta x=(b-a)/N=\pi/N$. Then

$$R_N=\Delta x\sum_{k=1}^{N}f(x_k)=\frac{\pi}{N}\sum_{k=1}^{N}\sin\left(\frac{k\pi}{N}\right).$$

Hence

$$\lim_{N\to\infty} R_N=\lim_{N\to\infty}\frac{\pi}{N}\sum_{k=1}^{N}\sin\left(\frac{k\pi}{N}\right)$$

is the area between the graph of $f(x)=\sin x$ and the x-axis over $[0,\pi]$.

66. R_N, $f(x) = x^{-1}$ over $[1, 7]$

SOLUTION Let $f(x) = x^{-1}$ over the interval $[1, 7]$. Then $\Delta x = \dfrac{7-1}{N} = \dfrac{6}{N}$ and $a = 1$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(1 + j\Delta x) = \frac{6}{N} \sum_{j=1}^{N} \left(1 + j\frac{6}{N}\right)^{-1}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} \frac{6}{N} \sum_{j=1}^{N} \left(1 + j\frac{6}{N}\right)^{-1}$$

is the area between the graph of $f(x) = x^{-1}$ and the x-axis over $[1, 7]$.

67. L_N, $f(x) = \sqrt{2x+1}$ over $[7, 11]$

SOLUTION Let $f(x) = \sqrt{2x+1}$ over the interval $[7, 11]$. Then $\Delta x = \dfrac{11-7}{N} = \dfrac{4}{N}$ and $a = 7$. Hence,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(7 + j\Delta x) = \frac{4}{N} \sum_{j=0}^{N-1} \sqrt{2(7 + j\frac{4}{N}) + 1}$$

and

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty} \frac{4}{N} \sum_{j=0}^{N-1} \sqrt{15 + \frac{8j}{N}}$$

is the area between the graph of $f(x) = \sqrt{2x+1}$ and the x-axis over $[7, 11]$.

68. L_N, $f(x) = \cos x$ over $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

SOLUTION Let $f(x) = \cos x$ over the interval $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$. Then $\Delta x = \dfrac{\frac{\pi}{4} - \frac{\pi}{8}}{N} = \dfrac{\frac{\pi}{8}}{N} = \dfrac{\pi}{8N}$ and $a = \frac{\pi}{8}$, Hence:

$$L_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{\pi}{8} + j\Delta x\right) = \frac{\pi}{8N} \sum_{j=0}^{N-1} \cos\left(\frac{\pi}{8} + j\frac{\pi}{8N}\right)$$

and

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty} \frac{\pi}{8N} \sum_{j=0}^{N-1} \cos\left(\frac{\pi}{8} + j\frac{\pi}{8N}\right)$$

is the area between the graph of $f(x) = \cos x$ and the x-axis over $[\frac{\pi}{8}, \frac{\pi}{4}]$.

69. M_N, $f(x) = \tan x$ over $\left[\frac{1}{2}, 1\right]$

SOLUTION Let $f(x) = \tan x$ over the interval $[\frac{1}{2}, 1]$. Then $\Delta x = \frac{1-\frac{1}{2}}{N} = \frac{1}{2N}$ and $a = \frac{1}{2}$. Hence

$$M_N = \Delta x \sum_{j=1}^{N} f\left(\frac{1}{2} + \left(j - \frac{1}{2}\right)\Delta x\right) = \frac{1}{2N} \sum_{j=1}^{N} \tan\left(\frac{1}{2} + \frac{1}{2N}\left(j - \frac{1}{2}\right)\right)$$

and so

$$\lim_{N\to\infty} M_N = \lim_{N\to\infty} \frac{1}{2N} \sum_{j=1}^{N} \tan\left(\frac{1}{2} + \frac{1}{2N}\left(j - \frac{1}{2}\right)\right)$$

is the area between the graph of $f(x) = \tan x$ and the x-axis over $[\frac{1}{2}, 1]$.

70. M_N, $f(x) = x^{-2}$ over $[3, 5]$

SOLUTION Let $f(x) = x^{-2}$ over the interval $[3, 5]$. Then $\Delta x = \frac{5-3}{N} = \frac{2}{N}$ and $a = 3$. Hence

$$M_N = \Delta x \sum_{j=1}^{N} f\left(3 + \left(j - \frac{1}{2}\right)\Delta x\right) = \frac{2}{N} \sum_{j=1}^{N} \left(3 + \frac{2}{N}\left(j - \frac{1}{2}\right)\right)^{-2}$$

and so

$$\lim_{N\to\infty} M_N = \lim_{N\to\infty} \frac{2}{N}\sum_{j=1}^{N}\left(3+\frac{2}{N}\left(j-\frac{1}{2}\right)\right)^{-2}$$

is the area between the graph of $f(x) = x^{-2}$ and the x-axis over [3, 5].

71. Evaluate $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\left(\frac{j}{N}\right)^2}$ by interpreting it as the area of part of a familiar geometric figure.

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\left(\frac{j}{N}\right)^2}$$

represents the area between the graph of $y = f(x) = \sqrt{1-x^2}$ and the x-axis over the interval [0, 1]. This is the portion of the circular disk $x^2 + y^2 \le 1$ that lies in the first quadrant. Accordingly, its area is $\frac{1}{4}\pi\,(1)^2 = \frac{\pi}{4}$.

In Exercises 72–74, let $f(x) = x^2$ and let R_N, L_N, and M_N be the approximations for the interval [0, 1].

72. Show that $R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$. Interpret the quantity $\frac{1}{2N} + \frac{1}{6N^2}$ as the area of a region.

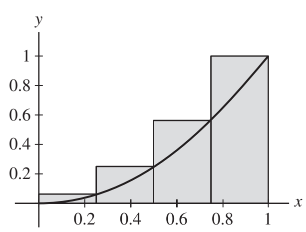
SOLUTION Let $f(x) = x^2$ on [0, 1]. Let $N > 0$ be an integer and set $a = 0$, $b = 1$ and $\Delta x = \frac{1-0}{N} = \frac{1}{N}$. Then

$$R_N = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{1}{N}\sum_{j=1}^{N} j^2 \frac{1}{N^2} = \frac{1}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}.$$

The quantity

$$\frac{1}{2N} + \frac{6}{N^2} \quad \text{in} \quad R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$$

represents the collective area of the parts of the rectangles that lie above the graph of $f(x)$. It is the error between R_N and the true area $A = \frac{1}{3}$.



73. Show that

$$L_N = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}, \qquad M_N = \frac{1}{3} - \frac{1}{12N^2}$$

Then rank the three approximations R_N, L_N, and M_N in order of increasing accuracy (use Exercise 72).

SOLUTION Let $f(x) = x^2$ on [0, 1]. Let N be a positive integer and set $a = 0$, $b = 1$, and $\Delta x = (b-a)/N = 1/N$. Let $x_k = a + k\Delta x = k/N$, $k = 0, 1, \dots, N$ and let $x_k^* = a + (k + \frac{1}{2})\Delta x = (k + \frac{1}{2})/N$, $k = 0, 1, \dots, N-1$. Then

$$L_N = \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{k}{N}\right)^2 = \frac{1}{N^3}\sum_{k=1}^{N-1} k^2$$
$$= \frac{1}{N^3}\left(\frac{(N-1)^3}{3} + \frac{(N-1)^2}{2} + \frac{N-1}{6}\right) = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}$$

$$M_N = \Delta x \sum_{k=0}^{N-1} f(x_k^*) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{k+\frac{1}{2}}{N}\right)^2 = \frac{1}{N^3}\sum_{k=0}^{N-1}\left(k^2+k+\frac{1}{4}\right)$$
$$= \frac{1}{N^3}\left(\left(\sum_{k=1}^{N-1} k^2\right)+\left(\sum_{k=1}^{N-1} k\right)+\frac{1}{4}\left(\sum_{k=0}^{N-1} 1\right)\right)$$
$$= \frac{1}{N^3}\left(\left(\frac{(N-1)^3}{3}+\frac{(N-1)^2}{2}+\frac{N-1}{6}\right)+\left(\frac{(N-1)^2}{2}+\frac{N-1}{2}\right)+\frac{1}{4}N\right) = \frac{1}{3}-\frac{1}{12N^2}$$

The error of R_N is given by $\frac{1}{2N}+\frac{1}{6N^2}$, the error of L_N is given by $-\frac{1}{2N}+\frac{1}{6N^2}$ and the error of M_N is given by $-\frac{1}{12N^2}$. Of the three approximations, R_N is the least accurate, then L_N and finally M_N is the most accurate.

74. For each of R_N, L_N, and M_N, find the smallest integer N for which the error is less than 0.001.

SOLUTION

- For R_N, the error is less than 0.001 when:

$$\frac{1}{2N}+\frac{1}{6N^2} < 0.001.$$

We find an adequate solution in N:

$$\frac{1}{2N}+\frac{1}{6N^2} < 0.001$$
$$3N+1 < 0.006(N^2)$$
$$0 < 0.006N^2-3N-1,$$

in particular, if $N > \frac{3+\sqrt{9.024}}{0.012} = 500.333$. Hence R_{501} is within 0.001 of A.

- For L_N, the error is less than 0.001 if

$$\left|-\frac{1}{2N}+\frac{1}{6N^2}\right| < 0.001.$$

We solve this equation for N:

$$\left|\frac{1}{2N}-\frac{1}{6N^2}\right| < 0.001$$
$$\left|\frac{3N-1}{6N^2}\right| < 0.001$$
$$3N-1 < 0.006N^2$$
$$0 < 0.006N^2-3N+1,$$

which is satisfied if $N > \frac{3+\sqrt{9-0.024}}{0.012} = 499.666$. Therefore, L_{500} is within 0.001 units of A.

- For M_N, the error is given by $-\frac{1}{12N^2}$, so the error is less than 0.001 if

$$\frac{1}{12N^2} < 0.001$$
$$1000 < 12N^2$$
$$9.13 < N$$

Therefore, M_{10} is within 0.001 units of the correct answer.

In Exercises 75–80, use the Graphical Insight on page 291 to obtain bounds on the area.

75. Let A be the area under $f(x)=\sqrt{x}$ over [0, 1]. Prove that $0.51 \le A \le 0.77$ by computing R_4 and L_4. Explain your reasoning.

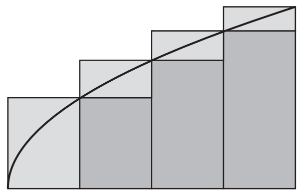
SOLUTION For $n=4$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $\{x_i\}_{i=0}^{4} = \{0 + i\Delta x\} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Therefore,

$$R_4 = \Delta x \sum_{i=1}^{4} f(x_i) = \frac{1}{4}\left(\frac{1}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}+1\right) \approx 0.768$$

$$L_4 = \Delta x \sum_{i=0}^{3} f(x_i) = \frac{1}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right) \approx 0.518.$$

In the plot below, you can see the rectangles whose area is represented by L_4 under the graph and the top of those whose area is represented by R_4 above the graph. The area A under the curve is somewhere between L_4 and R_4, so

$$0.518 \le A \le 0.768.$$



L_4, R_4 and the graph of $f(x)$.

76. Use R_5 and L_5 to show that the area A under $y = x^{-2}$ over [10, 13] satisfies $0.0218 \le A \le 0.0244$.

SOLUTION Let $f(x) = x^{-2}$ over the interval [10, 13]. Because f is a decreasing function over this interval, it follows that $R_N \le A \le L_N$ for all N. Taking $N = 5$, we have $\Delta x = 3/5$ and

$$R_5 = \frac{3}{5}\left(\frac{1}{10.6^2} + \frac{1}{11.2^2} + \frac{1}{11.8^2} + \frac{1}{12.4^2} + \frac{1}{13^2}\right) = 0.021885.$$

Moreover,

$$L_5 = \frac{3}{5}\left(\frac{1}{10^2} + \frac{1}{10.6^2} + \frac{1}{11.2^2} + \frac{1}{11.8^2} + \frac{1}{12.4^2}\right) = 0.0243344.$$

Thus,

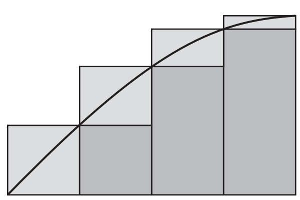
$$0.0218 < R_5 \le A \le L_5 < 0.0244.$$

77. Use R_4 and L_4 to show that the area A under the graph of $y = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ satisfies $0.79 \le A \le 1.19$.

SOLUTION Let $f(x) = \sin x$. $f(x)$ is increasing over the interval $[0, \pi/2]$, so the Insight on page 291 applies, which indicates that $L_4 \le A \le R_4$. For $n = 4$, $\Delta x = \frac{\pi/2-0}{4} = \frac{\pi}{8}$ and $\{x_i\}_{i=0}^{4} = \{0 + i\Delta x\}_{i=0}^{4} = \{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\}$. From this,

$$L_4 = \frac{\pi}{8}\sum_{i=0}^{3} f(x_i) \approx 0.79, \qquad R_4 = \frac{\pi}{8}\sum_{i=1}^{4} f(x_i) \approx 1.18.$$

Hence A is between 0.79 and 1.19.



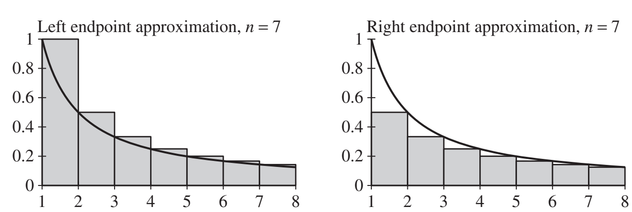
Left and Right endpoint approximations to A.

78. Show that the area A under $f(x) = x^{-1}$ over [1, 8] satisfies

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \le A \le 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}$$

SOLUTION Let $f(x) = x^{-1}$, $1 \le x \le 8$. Since f is decreasing, the left endpoint approximation L_7 overestimates the true area between the graph of f and the x-axis, whereas the right endpoint approximation R_7 underestimates it. Accordingly,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = R_7 < A < L_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$



79. CAS Show that the area A under $y = x^{1/4}$ over [0, 1] satisfies $L_N \le A \le R_N$ for all N. Use a computer algebra system to calculate L_N and R_N for $N = 100$ and 200, and determine A to two decimal places.

SOLUTION On [0, 1], $f(x) = x^{1/4}$ is an increasing function; therefore, $L_N \le A \le R_N$ for all N. We find

$$L_{100} = 0.793988 \quad \text{and} \quad R_{100} = 0.80399,$$

while

$$L_{200} = 0.797074 \quad \text{and} \quad R_{200} = 0.802075.$$

Thus, $A = 0.80$ to two decimal places.

80. CAS Show that the area A under $y = 4/(x^2 + 1)$ over [0, 1] satisfies $R_N \le A \le L_N$ for all N. Determine A to at least three decimal places using a computer algebra system. Can you guess the exact value of A?

SOLUTION On [0, 1], the function $f(x) = 4/(x^2 + 1)$ is decreasing, so $R_N \le A \le L_N$ for all N. From the values in the table below, we find $A = 3.142$ to three decimal places. It appears that the exact value of A is π.

N	R_N	L_N
10	3.03993	3.23992
100	3.13158	3.15158
1000	3.14059	3.14259
10000	3.14149	3.14169
100000	3.14158	3.14160

81. In this exercise, we evaluate the area A under the graph of $y = e^x$ over [0, 1] [Figure 19(A)] using the formula for a geometric sum (valid for $r \ne 1$):

$$1 + r + r^2 + \cdots + r^{N-1} = \sum_{j=0}^{N-1} r^j = \frac{r^N - 1}{r - 1} \tag{8}$$

(a) Show that $L_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} e^{j/N}$.

(b) Apply Eq. (8) with $r = e^{1/N}$ to prove $L_N = \dfrac{e - 1}{N(e^{1/N} - 1)}$.

(c) Compute $A = \displaystyle\lim_{N\to\infty} L_N$ using L'Hôpital's Rule.

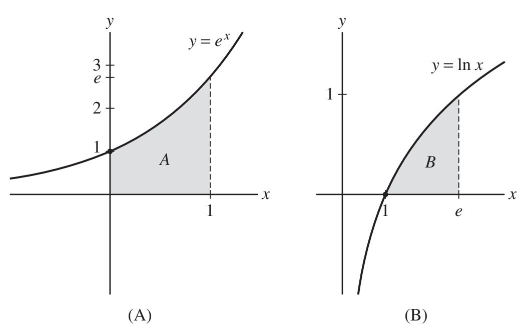


FIGURE 19

SOLUTION

(a) Let $f(x) = e^x$ on [0, 1]. With $n = N$, $\Delta x = (1 - 0)/N = 1/N$ and

$$x_j = a + j\Delta x = \frac{j}{N}$$

for $j = 0, 1, 2, \ldots, N$. Therefore,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \frac{1}{N} \sum_{j=0}^{N-1} e^{j/N}.$$

(b) Applying Eq. (8) with $r = e^{1/N}$, we have

$$L_N = \frac{1}{N} \frac{(e^{1/N})^N - 1}{e^{1/N} - 1} = \frac{e - 1}{N(e^{1/N} - 1)}.$$

Therefore,

$$A = \lim_{N\to\infty} L_N = (e-1)\lim_{N\to\infty} \frac{1}{N(e^{1/N}-1)}.$$

(c) Using L'Hôpital's Rule,

$$A = (e-1)\lim_{N\to\infty} \frac{N^{-1}}{e^{1/N}-1} = (e-1)\lim_{N\to\infty} \frac{-N^{-2}}{-N^{-2}e^{1/N}} = (e-1)\lim_{N\to\infty} e^{-1/N} = e-1.$$

82. Use the result of Exercise 81 to show that the area B under the graph of $f(x) = \ln x$ over $[1, e]$ is equal to 1. *Hint:* Relate B in Figure 19(B) to the area A computed in Exercise 81.

SOLUTION Because $y = \ln x$ and $y = e^x$ are inverse functions, we note that if the area B is reflected across the line $y = x$ and then combined with the area A, we create a rectangle of width 1 and height e. The area of this rectangle is therefore e, and it follows that the area B is equal to e minus the area A. Using the result of Exercise 81, the area B is equal to

$$e - (e-1) = 1.$$

Further Insights and Challenges

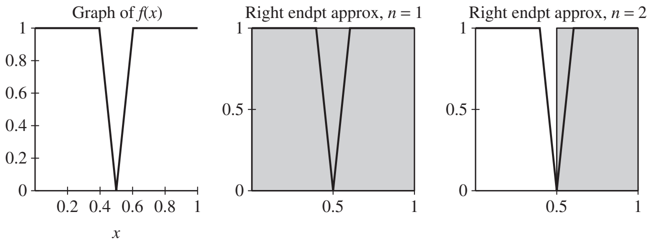
83. Although the accuracy of R_N generally improves as N increases, this need not be true for small values of N. Draw the graph of a positive continuous function $f(x)$ on an interval such that R_1 is closer than R_2 to the exact area under the graph. Can such a function be monotonic?

SOLUTION Let δ be a small positive number less than $\frac{1}{4}$. (In the figures below, $\delta = \frac{1}{10}$. But imagine δ being *very* tiny.) Define $f(x)$ on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} - \delta \\ \frac{1}{2\delta} - \frac{x}{\delta} & \text{if } \frac{1}{2} - \delta \le x < \frac{1}{2} \\ \frac{x}{\delta} - \frac{1}{2\delta} & \text{if } \frac{1}{2} \le x < \frac{1}{2} + \delta \\ 1 & \text{if } \frac{1}{2} + \delta \le x \le 1 \end{cases}$$

Then f is continuous on $[0, 1]$. (Again, just look at the figures.)

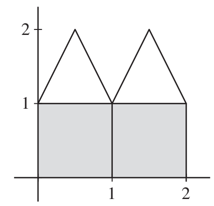
- The exact area between f and the x-axis is $A = 1 - \frac{1}{2}bh = 1 - \frac{1}{2}(2\delta)(1) = 1 - \delta$. (For $\delta = \frac{1}{10}$, we have $A = \frac{9}{10}$.)
- With $R_1 = 1$, the absolute error is $|E_1| = |R_1 - A| = |1 - (1-\delta)| = \delta$. (For $\delta = \frac{1}{10}$, this absolute error is $|E_1| = \frac{1}{10}$.)
- With $R_2 = \frac{1}{2}$, the absolute error is $|E_2| = |R_2 - A| = \left|\frac{1}{2} - (1-\delta)\right| = \left|\delta - \frac{1}{2}\right| = \frac{1}{2} - \delta$. (For $\delta = \frac{1}{10}$, we have $|E_2| = \frac{2}{5}$.)
- Accordingly, R_1 is closer to the exact area A than is R_2. Indeed, the tinier δ is, the more dramatic the effect.
- For a monotonic function, this phenomenon cannot occur. Successive approximations from either side get progressively more accurate.



84. Draw the graph of a positive continuous function on an interval such that R_2 and L_2 are both smaller than the exact area under the graph. Can such a function be monotonic?

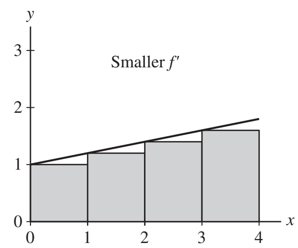
SOLUTION In the plot below, the area under the saw-tooth function $f(x)$ is 3, whereas $L_2 = R_2 = 2$. Thus L_2 and R_2 are both smaller than the exact area. Such a function cannot be monotonic; if $f(x)$ is increasing, then L_N underestimates and R_N overestimates the area for all N, and, if $f(x)$ is decreasing, then L_N overestimates and R_N underestimates the area for all N.

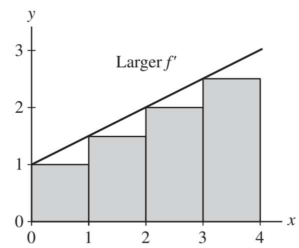
Left/right-endpoint approximation, $n = 2$



85. Explain graphically: *The endpoint approximations are less accurate when $f'(x)$ is large.*

SOLUTION When f' is large, the graph of f is steeper and hence there is more gap between f and L_N or R_N. Recall that the top line segments of the rectangles involved in an endpoint approximation constitute a piecewise constant function. If f' is large, then f is increasing more rapidly and hence is less like a constant function.





86. Prove that for any function $f(x)$ on $[a, b]$,

$$R_N - L_N = \frac{b-a}{N}(f(b) - f(a)) \qquad \boxed{9}$$

SOLUTION For any f (continuous or not) on $I = [a, b]$, partition I into N equal subintervals. Let $\Delta x = (b - a)/N$ and set $x_k = a + k\Delta x$, $k = 0, 1, \dots N$. Then we have the following approximations to the area between the graph of f and the x-axis: the left endpoint approximation $L_N = \Delta x \sum_{k=0}^{N-1} f(x_k)$ and right endpoint approximation $R_N = \Delta x \sum_{k=1}^{N} f(x_k)$. Accordingly,

$$\begin{aligned} R_N - L_N &= \left(\Delta x \sum_{k=1}^{N} f(x_k)\right) - \left(\Delta x \sum_{k=0}^{N-1} f(x_k)\right) \\ &= \Delta x \left(f(x_N) + \left(\sum_{k=1}^{N-1} f(x_k)\right) - f(x_0) - \left(\sum_{k=1}^{N-1} f(x_k)\right)\right) \\ &= \Delta x \left(f(x_N) - f(x_0)\right) = \frac{b-a}{N}\left(f(b) - f(a)\right). \end{aligned}$$

In other words, $R_N - L_N = \dfrac{b-a}{N}(f(b) - f(a))$.

87. In this exercise, we prove that $\lim_{N\to\infty} R_N$ and $\lim_{N\to\infty} L_N$ exist and are equal if $f(x)$ is increasing [the case of $f(x)$ decreasing is similar]. We use the concept of a least upper bound discussed in Appendix B.

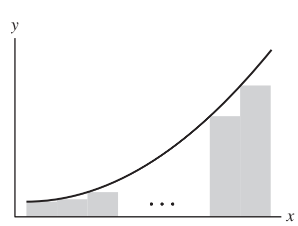
(a) Explain with a graph why $L_N \le R_M$ for all $N, M \ge 1$.

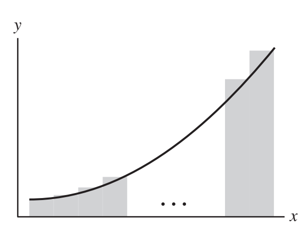
(b) By (a), the sequence $\{L_N\}$ is bounded, so it has a least upper bound L. By definition, L is the smallest number such that $L_N \le L$ for all N. Show that $L \le R_M$ for all M.

(c) According to (b), $L_N \le L \le R_N$ for all N. Use Eq. (9) to show that $\lim_{N\to\infty} L_N = L$ and $\lim_{N\to\infty} R_N = L$.

SOLUTION

(a) Let $f(x)$ be positive and increasing, and let N and M be positive integers. From the figure below at the left, we see that L_N underestimates the area under the graph of $y = f(x)$, while from the figure below at the right, we see that R_M overestimates the area under the graph. Thus, for all $N, M \ge 1$, $L_N \le R_M$.





(b) Because the sequence $\{L_N\}$ is bounded above by R_M for any M, each R_M is an upper bound for the sequence. Furthermore, the sequence $\{L_N\}$ must have a least upper bound, call it L. By definition, the least upper bound must be no greater than any other upper bound; consequently, $L \le R_M$ for all M.

(c) Since $L_N \le L \le R_N$, $R_N - L \le R_N - L_N$, so $|R_N - L| \le |R_N - L_N|$. From this,

$$\lim_{N\to\infty} |R_N - L| \le \lim_{N\to\infty} |R_N - L_N|.$$

By Eq. (9),

$$\lim_{N\to\infty} |R_N - L_N| = \lim_{N\to\infty} \frac{1}{N}|(b-a)(f(b)-f(a))| = 0,$$

so $\lim_{N\to\infty} |R_N - L| \le |R_N - L_N| = 0$, hence $\lim_{N\to\infty} R_N = L$.

Similarly, $|L_N - L| = L - L_N \le R_N - L_N$, so

$$|L_N - L| \le |R_N - L_N| = \frac{(b-a)}{N}(f(b) - f(a)).$$

This gives us that

$$\lim_{N\to\infty} |L_N - L| \le \lim_{N\to\infty} \frac{1}{N}|(b-a)(f(b)-f(a))| = 0,$$

so $\lim_{N\to\infty} L_N = L$.

This proves $\lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = L$.

88. Use Eq. (9) to show that if $f(x)$ is positive and monotonic, then the area A under its graph over $[a, b]$ satisfies

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)| \qquad \boxed{10}$$

SOLUTION Let $f(x)$ be continuous, positive, and monotonic on $[a, b]$. Let A be the area between the graph of f and the x-axis over $[a, b]$. For specificity, say f is increasing. (The case for f decreasing on $[a, b]$ is similar.) As noted in the text, we have $L_N \le A \le R_N$. By Exercise 86 and the fact that A lies between L_N and R_N, we therefore have

$$0 \le R_N - A \le R_N - L_N = \frac{b-a}{N}(f(b) - f(a)).$$

Hence

$$|R_N - A| \le \frac{b-a}{N}(f(b) - f(a)) = \frac{b-a}{N}|f(b) - f(a)|,$$

where $f(b) - f(a) = |f(b) - f(a)|$ because f is increasing on $[a, b]$.

In Exercises 89 and 90, use Eq. (10) to find a value of N such that $|R_N - A| < 10^{-4}$ for the given function and interval.

89. $f(x) = \sqrt{x}$, $[1, 4]$

SOLUTION Let $f(x) = \sqrt{x}$ on $[1, 4]$. Then $b = 4$, $a = 1$, and

$$|R_N - A| \le \frac{4-1}{N}(f(4) - f(1)) = \frac{3}{N}(2-1) = \frac{3}{N}.$$

We need $\frac{3}{N} < 10^{-4}$, which gives $N > 30{,}000$. Thus $|R_{30{,}001} - A| < 10^{-4}$ for $f(x) = \sqrt{x}$ on $[1, 4]$.

90. $f(x) = \sqrt{9 - x^2}$, $[0, 3]$

SOLUTION Let $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$. Then $b = 3$, $a = 0$, and

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)| = \frac{3}{N}(3) = \frac{9}{N}.$$

We need $\frac{9}{N} < 10^{-4}$, which gives $N > 90{,}000$. Thus $|R_{90{,}001} - A| < 10^{-4}$ for $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$.

91. Prove that if $f(x)$ is positive and monotonic, then M_N lies between R_N and L_N and is closer to the actual area under the graph than both R_N and L_N. *Hint:* In the case that $f(x)$ is increasing, Figure 20 shows that the part of the error in R_N due to the ith rectangle is the sum of the areas $A + B + D$, and for M_N it is $|B - E|$. On the other hand, $A \ge E$.

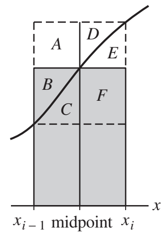


FIGURE 20

SOLUTION Suppose $f(x)$ is monotonic increasing on the interval $[a, b]$, $\Delta x = \dfrac{b-a}{N}$,

$$\{x_k\}_{k=0}^{N} = \{a, a + \Delta x, a + 2\Delta x, \ldots, a + (N-1)\Delta x, b\}$$

and

$$\{x_k^*\}_{k=0}^{N-1} = \left\{\frac{a + (a + \Delta x)}{2}, \frac{(a + \Delta x) + (a + 2\Delta x)}{2}, \ldots, \frac{(a + (N-1)\Delta x) + b}{2}\right\}.$$

Note that $x_i < x_i^* < x_{i+1}$ implies $f(x_i) < f(x_i^*) < f(x_{i+1})$ for all $0 \le i < N$ because $f(x)$ is monotone increasing. Then

$$\left(L_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k)\right) < \left(M_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k^*)\right) < \left(R_N = \frac{b-a}{N}\sum_{k=1}^{N} f(x_k)\right)$$

Similarly, if $f(x)$ is monotone decreasing,

$$\left(L_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k)\right) > \left(M_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k^*)\right) > \left(R_N = \frac{b-a}{N}\sum_{k=1}^{N} f(x_k)\right)$$

Thus, if $f(x)$ is monotonic, then M_N always lies in between R_N and L_N.

Now, as in Figure 20, consider the typical subinterval $[x_{i-1}, x_i]$ and its midpoint x_i^*. We let A, B, C, D, E, and F be the areas as shown in Figure 20. Note that, by the fact that x_i^* is the midpoint of the interval, $A = D + E$ and $F = B + C$. Let E_R represent the right endpoint approximation error $(= A + B + D)$, let E_L represent the left endpoint approximation error $(= C + F + E)$ and let E_M represent the midpoint approximation error $(= |B - E|)$.

- If $B > E$, then $E_M = B - E$. In this case,

  $$E_R - E_M = A + B + D - (B - E) = A + D + E > 0,$$

  so $E_R > E_M$, while

  $$E_L - E_M = C + F + E - (B - E) = C + (B + C) + E - (B - E) = 2C + 2E > 0,$$

  so $E_L > E_M$. Therefore, the midpoint approximation is more accurate than either the left or the right endpoint approximation.
- If $B < E$, then $E_M = E - B$. In this case,

  $$E_R - E_M = A + B + D - (E - B) = D + E + D - (E - B) = 2D + B > 0,$$

  so that $E_R > E_M$ while

  $$E_L - E_M = C + F + E - (E - B) = C + F + B > 0,$$

  so $E_L > E_M$. Therefore, the midpoint approximation is more accurate than either the right or the left endpoint approximation.
- If $B = E$, the midpoint approximation is exactly equal to the area.

Hence, for $B < E$, $B > E$, or $B = E$, the midpoint approximation is more accurate than either the left endpoint or the right endpoint approximation.

5.2 The Definite Integral

Preliminary Questions

1. What is $\int_3^5 dx$ [the function is $f(x) = 1$]?

SOLUTION $\int_3^5 dx = \int_3^5 1 \cdot dx = 1(5-3) = 2.$

2. Let $I = \int_2^7 f(x)\,dx$, where $f(x)$ is continuous. State whether true or false:

(a) I is the area between the graph and the x-axis over $[2, 7]$.
(b) If $f(x) \geq 0$, then I is the area between the graph and the x-axis over $[2, 7]$.
(c) If $f(x) \leq 0$, then $-I$ is the area between the graph of $f(x)$ and the x-axis over $[2, 7]$.

SOLUTION

(a) False. $\int_a^b f(x)\,dx$ is the *signed* area between the graph and the x-axis.
(b) True.
(c) True.

3. Explain graphically: $\int_0^\pi \cos x\,dx = 0$.

SOLUTION Because $\cos(\pi - x) = -\cos x$, the "negative" area between the graph of $y = \cos x$ and the x-axis over $[\frac{\pi}{2}, \pi]$ exactly cancels the "positive" area between the graph and the x-axis over $[0, \frac{\pi}{2}]$.

4. Which is negative, $\int_{-1}^{-5} 8\,dx$ or $\int_{-5}^{-1} 8\,dx$?

SOLUTION Because $-5-(-1) = -4$, $\int_{-1}^{-5} 8\,dx$ is negative.

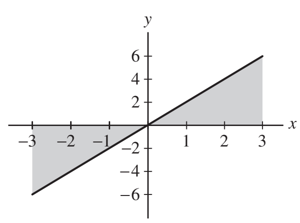
Exercises

In Exercises 1–10, draw a graph of the signed area represented by the integral and compute it using geometry.

1. $\int_{-3}^{3} 2x\,dx$

SOLUTION The region bounded by the graph of $y = 2x$ and the x-axis over the interval $[-3, 3]$ consists of two right triangles. One has area $\frac{1}{2}(3)(6) = 9$ below the axis, and the other has area $\frac{1}{2}(3)(6) = 9$ above the axis. Hence,

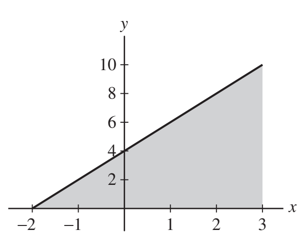
$$\int_{-3}^{3} 2x\,dx = 9 - 9 = 0.$$



2. $\int_{-2}^{3} (2x+4)\,dx$

SOLUTION The region bounded by the graph of $y = 2x + 4$ and the x-axis over the interval $[-2, 3]$ consists of a single right triangle of area $\frac{1}{2}(5)(10) = 25$ above the axis. Hence,

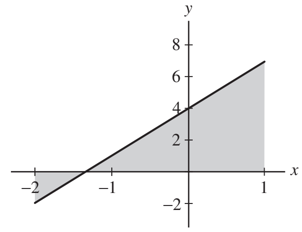
$$\int_{-2}^{3} (2x+4)\,dx = 25.$$



3. $\int_{-2}^{1}(3x+4)\,dx$

SOLUTION The region bounded by the graph of $y = 3x + 4$ and the x-axis over the interval $[-2, 1]$ consists of two right triangles. One has area $\frac{1}{2}(\frac{2}{3})(2) = \frac{2}{3}$ below the axis, and the other has area $\frac{1}{2}(\frac{7}{3})(7) = \frac{49}{6}$ above the axis. Hence,

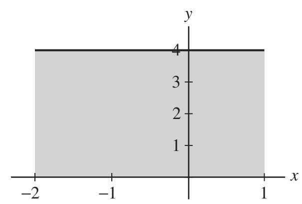
$$\int_{-2}^{1}(3x+4)\,dx = \frac{49}{6} - \frac{2}{3} = \frac{15}{2}.$$



4. $\int_{-2}^{1} 4\,dx$

SOLUTION The region bounded by the graph of $y = 4$ and the x-axis over the interval $[-2, 1]$ is a rectangle of area $(3)(4) = 12$ above the axis. Hence,

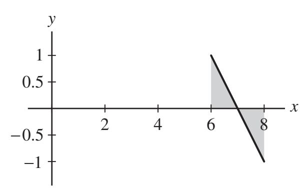
$$\int_{-2}^{1} 4\,dx = 12.$$



5. $\int_{6}^{8}(7-x)\,dx$

SOLUTION The region bounded by the graph of $y = 7 - x$ and the x-axis over the interval $[6, 8]$ consists of two right triangles. One triangle has area $\frac{1}{2}(1)(1) = \frac{1}{2}$ above the axis, and the other has area $\frac{1}{2}(1)(1) = \frac{1}{2}$ below the axis. Hence,

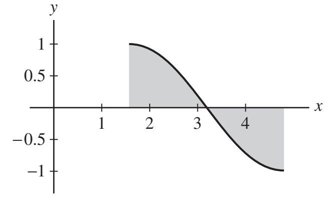
$$\int_{6}^{8}(7-x)\,dx = \frac{1}{2} - \frac{1}{2} = 0.$$



6. $\int_{\pi/2}^{3\pi/2} \sin x\,dx$

SOLUTION The region bounded by the graph of $y = \sin x$ and the x-axis over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ consists of two parts of equal area, one above the axis and the other below the axis. Hence,

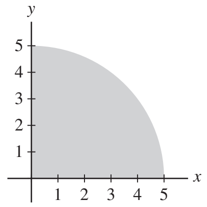
$$\int_{\pi/2}^{3\pi/2} \sin x\,dx = 0.$$



7. $\int_0^5 \sqrt{25 - x^2}\, dx$

SOLUTION The region bounded by the graph of $y = \sqrt{25 - x^2}$ and the x-axis over the interval $[0, 5]$ is one-quarter of a circle of radius 5. Hence,

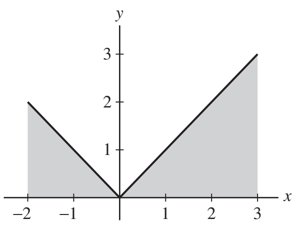
$$\int_0^5 \sqrt{25 - x^2}\, dx = \frac{1}{4}\pi(5)^2 = \frac{25\pi}{4}.$$



8. $\int_{-2}^3 |x|\, dx$

SOLUTION The region bounded by the graph of $y = |x|$ and the x-axis over the interval $[-2, 3]$ consists of two right triangles, both above the axis. One triangle has area $\frac{1}{2}(2)(2) = 2$, and the other has area $\frac{1}{2}(3)(3) = \frac{9}{2}$. Hence,

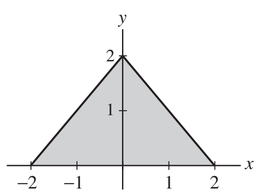
$$\int_{-2}^3 |x|\, dx = \frac{9}{2} + 2 = \frac{13}{2}.$$



9. $\int_{-2}^2 (2 - |x|)\, dx$

SOLUTION The region bounded by the graph of $y = 2 - |x|$ and the x-axis over the interval $[-2, 2]$ is a triangle above the axis with base 4 and height 2. Consequently,

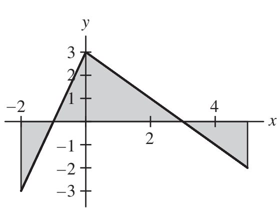
$$\int_{-2}^2 (2 - |x|)\, dx = \frac{1}{2}(2)(4) = 4.$$



10. $\int_{-2}^5 (3 + x - 2|x|)\, dx$

SOLUTION The region bounded by the graph of $y = 3 + x - 2|x|$ and the x-axis over the interval $[-2, 5]$ consists of a triangle below the axis with base 1 and height 3, a triangle above the axis of base 4 and height 3 and a triangle below the axis of base 2 and height 2. Consequently,

$$\int_{-2}^5 (3 + x - 2|x|)\, dx = -\frac{1}{2}(1)(3) + \frac{1}{2}(4)(3) - \frac{1}{2}(2)(2) = \frac{5}{2}.$$



11. Calculate $\int_0^{10} (8-x)\,dx$ in two ways:

(a) As the limit $\lim_{N\to\infty} R_N$

(b) By sketching the relevant signed area and using geometry

SOLUTION Let $f(x) = 8 - x$ over [0, 10]. Consider the integral $\int_0^{10} f(x)\,dx = \int_0^{10} (8-x)\,dx$.

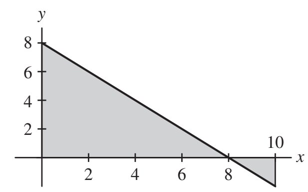
(a) Let N be a positive integer and set $a = 0$, $b = 10$, $\Delta x = (b-a)/N = 10/N$. Also, let $x_k = a + k\Delta x = 10k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of [0, 10]. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{10}{N} \sum_{k=1}^{N} \left(8 - \frac{10k}{N}\right) = \frac{10}{N}\left(8\left(\sum_{k=1}^{N} 1\right) - \frac{10}{N}\left(\sum_{k=1}^{N} k\right)\right)$$
$$= \frac{10}{N}\left(8N - \frac{10}{N}\left(\frac{N^2}{2} + \frac{N}{2}\right)\right) = 30 - \frac{50}{N}.$$

Hence $\lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(30 - \frac{50}{N}\right) = 30.$

(b) The region bounded by the graph of $y = 8 - x$ and the x-axis over the interval [0, 10] consists of two right triangles. One triangle has area $\frac{1}{2}(8)(8) = 32$ above the axis, and the other has area $\frac{1}{2}(2)(2) = 2$ below the axis. Hence,

$$\int_0^{10} (8-x)\,dx = 32 - 2 = 30.$$



12. Calculate $\int_{-1}^{4} (4x-8)\,dx$ in two ways: As the limit $\lim_{N\to\infty} R_N$ and using geometry.

SOLUTION Let $f(x) = 4x - 8$ over $[-1, 4]$. Consider the integral $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{4} (4x-8)\,dx$.

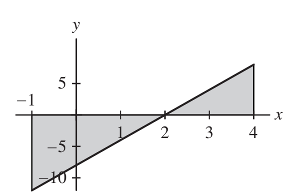
- Let N be a positive integer and set $a = -1$, $b = 4$, $\Delta x = (b-a)/N = 5/N$. Then $x_k = a + k\Delta x = -1 + 5k/N$, $k = 1, 2, \ldots, N$ are the right endpoints of the N subintervals of $[-1, 4]$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{5}{N} \sum_{k=1}^{N} \left(-4 + \frac{20k}{N} - 8\right) = -\frac{60}{N}\left(\sum_{k=1}^{N} 1\right) + \frac{100}{N^2}\left(\sum_{k=1}^{N} k\right)$$
$$= -\frac{60}{N}(N) + \frac{100}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$
$$= -60 + 50 + \frac{50}{N} = -10 + \frac{50}{N}.$$

  Hence $\lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(-10 + \frac{50}{N}\right) = -10.$

- The region bounded by the graph of $y = 4x - 8$ and the x-axis over the interval $[-1, 4]$ consists of a triangle below the axis with base 3 and height 12 and a triangle above the axis with base 2 and height 8. Hence,

$$\int_{-1}^{4} (4x-8)\,dx = -\frac{1}{2}(3)(12) + \frac{1}{2}(2)(8) = -10.$$



In Exercises 13 and 14, refer to Figure 14.

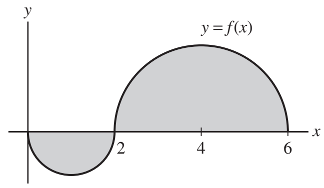


FIGURE 14 The two parts of the graph are semicircles.

13. Evaluate: (a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_0^6 f(x)\,dx$

SOLUTION Let $f(x)$ be given by Figure 14.

(a) The definite integral $\int_0^2 f(x)\,dx$ is the signed area of a semicircle of radius 1 which lies below the x-axis. Therefore,

$$\int_0^2 f(x)\,dx = -\frac{1}{2}\pi\,(1)^2 = -\frac{\pi}{2}.$$

(b) The definite integral $\int_0^6 f(x)\,dx$ is the signed area of a semicircle of radius 1 which lies below the x-axis and a semicircle of radius 2 which lies above the x-axis. Therefore,

$$\int_0^6 f(x)\,dx = \frac{1}{2}\pi\,(2)^2 - \frac{1}{2}\pi\,(1)^2 = \frac{3\pi}{2}.$$

14. Evaluate: (a) $\displaystyle\int_1^4 f(x)\,dx$ (b) $\displaystyle\int_1^6 |f(x)|\,dx$

SOLUTION Let $f(x)$ be given by Figure 14.

(a) The definite integral $\int_1^4 f(x)\,dx$ is the signed area of one-quarter of a circle of radius 1 which lies below the x-axis and one-quarter of a circle of radius 2 which lies above the x-axis. Therefore,

$$\int_1^4 f(x)\,dx = \frac{1}{4}\pi\,(2)^2 - \frac{1}{4}\pi\,(1)^2 = \frac{3}{4}\pi.$$

(b) The definite integral $\int_1^6 |f(x)|\,dx$ is the signed area of one-quarter of a circle of radius 1 and a semicircle of radius 2, both of which lie above the x-axis. Therefore,

$$\int_1^6 |f(x)|\,dx = \frac{1}{2}\pi\,(2)^2 + \frac{1}{4}\pi\,(1)^2 = \frac{9\pi}{4}.$$

In Exercises 15 and 16, refer to Figure 15.

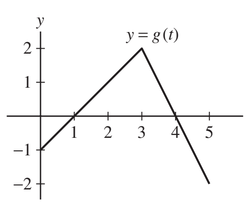


FIGURE 15

15. Evaluate $\displaystyle\int_0^3 g(t)\,dt$ and $\displaystyle\int_3^5 g(t)\,dt$.

SOLUTION

- The region bounded by the curve $y = g(x)$ and the x-axis over the interval $[0, 3]$ is comprised of two right triangles, one with area $\frac{1}{2}$ below the axis, and one with area 2 above the axis. The definite integral is therefore equal to $2 - \frac{1}{2} = \frac{3}{2}$.
- The region bounded by the curve $y = g(x)$ and the x-axis over the interval $[3, 5]$ is comprised of another two right triangles, one with area 1 above the axis and one with area 1 below the axis. The definite integral is therefore equal to 0.

16. Find a, b, and c such that $\int_0^a g(t)\,dt$ and $\int_b^c g(t)\,dt$ are as large as possible.

SOLUTION To make the value of $\int_0^a g(t)\,dt$ as large as possible, we want to include as much positive area as possible. This happens when we take $a = 4$. Now, to make the value of $\int_b^c g(t)\,dt$ as large as possible, we want to make sure to include all of the positive area and only the positive area. This happens when we take $b = 1$ and $c = 4$.

17. Describe the partition P and the set of sample points C for the Riemann sum shown in Figure 16. Compute the value of the Riemann sum.

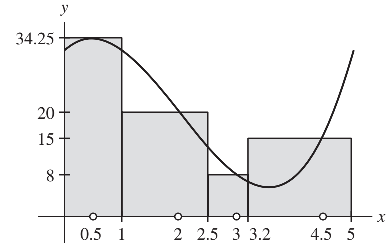


FIGURE 16

SOLUTION The partition P is defined by

$$x_0 = 0 \quad < \quad x_1 = 1 \quad < \quad x_2 = 2.5 \quad < \quad x_3 = 3.2 \quad < \quad x_4 = 5$$

The set of sample points is given by $C = \{c_1 = 0.5, c_2 = 2, c_3 = 3, c_4 = 4.5\}$. Finally, the value of the Riemann sum is

$$34.25(1-0) + 20(2.5-1) + 8(3.2-2.5) + 15(5-3.2) = 96.85.$$

18. Compute $R(f, P, C)$ for $f(x) = x^2 + x$ for the partition P and the set of sample points C in Figure 16.

SOLUTION

$$\begin{aligned} R(f, P, C) &= f(0.5)(1-0) + f(2)(2.5-1) + f(3)(3.2-2.5) + f(4.5)(5-3.2) \\ &= 34.25(1) + 20(1.5) + 8(0.7) + 15(1.8) = 96.85 \end{aligned}$$

In Exercises 19–22, calculate the Riemann sum $R(f, P, C)$ for the given function, partition, and choice of sample points. Also, sketch the graph of f and the rectangles corresponding to $R(f, P, C)$.

19. $f(x) = x, \quad P = \{1, 1.2, 1.5, 2\}, \quad C = \{1.1, 1.4, 1.9\}$

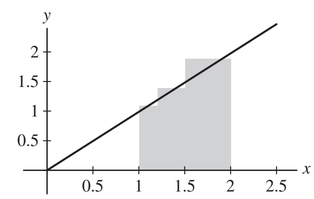
SOLUTION Let $f(x) = x$. With

$$P = \{x_0 = 1, x_1 = 1.2, x_2 = 1.5, x_3 = 2\} \qquad \text{and} \qquad C = \{c_1 = 1.1, c_2 = 1.4, c_3 = 1.9\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= (1.2-1)(1.1) + (1.5-1.2)(1.4) + (2-1.5)(1.9) = 1.59. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



20. $f(x) = 2x + 3, \quad P = \{-4, -1, 1, 4, 8\}, \quad C = \{-3, 0, 2, 5\}$

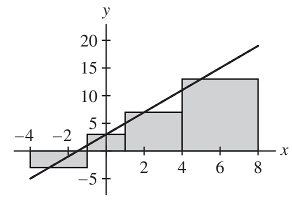
SOLUTION Let $f(x) = 2x + 3$. With

$$P = \{x_0 = -4, x_1 = -1, x_2 = 1, x_3 = 4, x_4 = 8\} \qquad \text{and} \qquad C = \{c_1 = -3, c_2 = 0, c_3 = 2, c_4 = 5\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) + \Delta x_4 f(c_4) \\ &= (-1-(-4))(-3) + (1-(-1))(3) + (4-1)(7) + (8-4)(13) = 70. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



21. $f(x) = x^2 + x, \quad P = \{2, 3, 4.5, 5\}, \quad C = \{2, 3.5, 5\}$

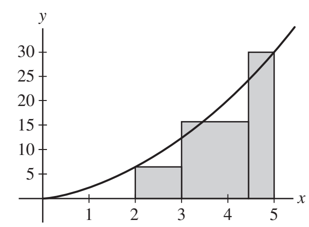
SOLUTION Let $f(x) = x^2 + x$. With

$$P = \{x_0 = 2, x_1 = 3, x_3 = 4.5, x_4 = 5\} \qquad \text{and} \qquad C = \{c_1 = 2, c_2 = 3.5, c_3 = 5\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= (3-2)(6) + (4.5-3)(15.75) + (5-4.5)(30) = 44.625. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



22. $f(x) = \sin x, \quad P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}, \quad C = \{0.4, 0.7, 1.2\}$

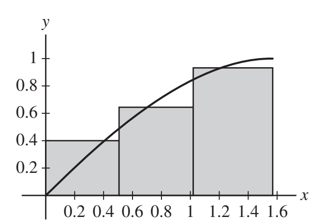
SOLUTION Let $f(x) = \sin x$. With

$$P = \left\{x_0 = 0, x_1 = \frac{\pi}{6}, x_3 = \frac{\pi}{3}, x_4 = \frac{\pi}{2}\right\} \qquad \text{and} \qquad C = \{c_1 = 0.4, c_2 = 0.7, c_3 = 1.2\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= \left(\frac{\pi}{6} - 0\right)(\sin 0.4) + \left(\frac{\pi}{3} - \frac{\pi}{6}\right)(\sin 0.7) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)(\sin 1.2) = 1.029225. \end{aligned}$$

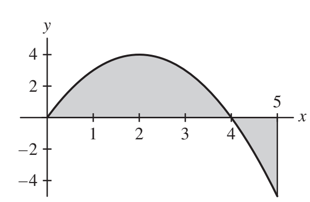
Here is a sketch of the graph of f and the rectangles.



In Exercises 23–28, sketch the signed area represented by the integral. Indicate the regions of positive and negative area.

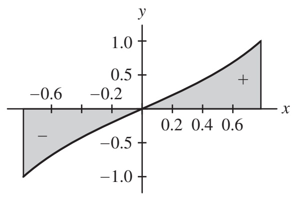
23. $\displaystyle\int_0^5 (4x - x^2)\, dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_0^5 (4x - x^2)\, dx$.



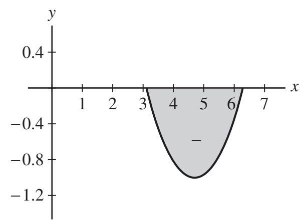
24. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{-\pi/4}^{\pi/4} \tan x\,dx$.



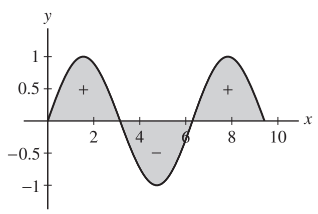
25. $\int_{\pi}^{2\pi} \sin x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{\pi}^{2\pi} \sin x\,dx$.



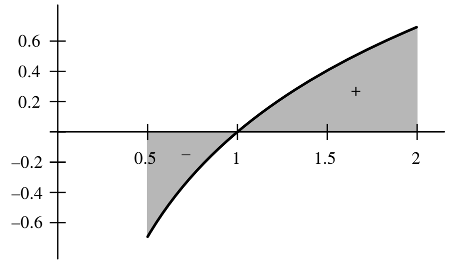
26. $\int_{0}^{3\pi} \sin x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{0}^{3\pi} \sin x\,dx$.



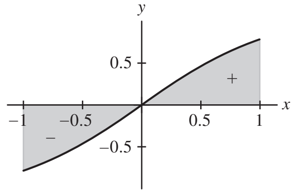
27. $\int_{1/2}^{2} \ln x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{1/2}^{2} \ln x\,dx$.



28. $\int_{-1}^{1} \tan^{-1} x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{-1}^{1} \tan^{-1} x\,dx$.



In Exercises 29–32, determine the sign of the integral without calculating it. Draw a graph if necessary.

29. $\int_{-2}^{1} x^4\,dx$

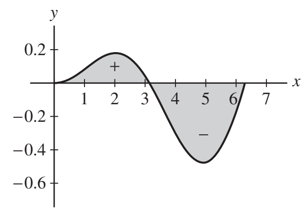
SOLUTION The integrand is always positive. The integral must therefore be positive, since the signed area has only positive part.

30. $\int_{-2}^{1} x^3\,dx$

SOLUTION By symmetry, the positive area from the interval $[0, 1]$ is cancelled by the negative area from $[-1, 0]$. With the interval $[-2, -1]$ contributing more negative area, the definite integral must be negative.

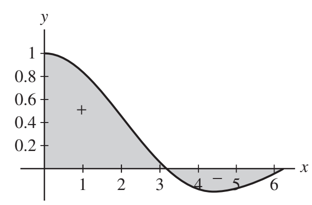
31. GU $\int_{0}^{2\pi} x \sin x\,dx$

SOLUTION As you can see from the graph below, the area below the axis is greater than the area above the axis. Thus, the definite integral is negative.



32. GU $\int_{0}^{2\pi} \frac{\sin x}{x}\,dx$

SOLUTION From the plot below, you can see that the area above the axis is bigger than the area below the axis, hence the integral is positive.



In Exercises 33–42, use properties of the integral and the formulas in the summary to calculate the integrals.

33. $\int_{0}^{4} (6t - 3)\,dt$

SOLUTION $\int_{0}^{4} (6t - 3)\,dt = 6\int_{0}^{4} t\,dt - 3\int_{0}^{4} 1\,dt = 6 \cdot \frac{1}{2}(4)^2 - 3(4 - 0) = 36.$

34. $\int_{-3}^{2} (4x + 7)\,dx$

SOLUTION

$$\begin{aligned}
\int_{-3}^{2} (4x + 7)\,dx &= 4\int_{-3}^{2} x\,dx + 7\int_{-3}^{2} dx \\
&= 4\left(\int_{-3}^{0} x\,dx + \int_{0}^{2} x\,dx\right) + 7(2 - (-3)) \\
&= 4\left(\int_{0}^{2} x\,dx - \int_{0}^{-3} x\,dx\right) + 35 \\
&= 4\left(\frac{1}{2}2^2 - \frac{1}{2}(-3)^2\right) + 35 = 25.
\end{aligned}$$

35. $\int_{0}^{9} x^2\,dx$

SOLUTION By formula (5), $\int_{0}^{9} x^2\,dx = \frac{1}{3}(9)^3 = 243.$

36. $\int_2^5 x^2\,dx$

SOLUTION $\int_2^5 x^2\,dx = \int_0^5 x^2\,dx - \int_0^2 x^2 dx = \frac{1}{3}(5)^3 - \frac{1}{3}(2)^3 = 39.$

37. $\int_0^1 (u^2 - 2u)\,du$

SOLUTION

$$\int_0^1 (u^2 - 2u)\,du = \int_0^1 u^2\,du - 2\int_0^1 u\,du = \frac{1}{3}(1)^3 - 2\left(\frac{1}{2}\right)(1)^2 = \frac{1}{3} - 1 = -\frac{2}{3}.$$

38. $\int_0^{1/2} (12y^2 + 6y)\,dy$

SOLUTION

$$\begin{aligned}\int_0^{1/2} (12y^2 + 6y)\,dy &= 12\int_0^{1/2} y^2\,dy + 6\int_0^{1/2} y\,dy \\ &= 12\cdot\frac{1}{3}\left(\frac{1}{2}\right)^3 + 6\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2 \\ &= \frac{1}{2} + \frac{3}{4} = \frac{5}{4}.\end{aligned}$$

39. $\int_{-3}^1 (7t^2 + t + 1)\,dt$

SOLUTION First, write

$$\begin{aligned}\int_{-3}^1 (7t^2 + t + 1)\,dt &= \int_{-3}^0 (7t^2 + t + 1)\,dt + \int_0^1 (7t^2 + t + 1)\,dt \\ &= -\int_0^{-3} (7t^2 + t + 1)\,dt + \int_0^1 (7t^2 + t + 1)\,dt\end{aligned}$$

Then,

$$\begin{aligned}\int_{-3}^1 (7t^2 + t + 1)\,dt &= -\left(7\cdot\frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 - 3\right) + \left(7\cdot\frac{1}{3}1^3 + \frac{1}{2}1^2 + 1\right) \\ &= -\left(-63 + \frac{9}{2} - 3\right) + \left(\frac{7}{3} + \frac{1}{2} + 1\right) = \frac{196}{3}.\end{aligned}$$

40. $\int_{-3}^3 (9x - 4x^2)\,dx$

SOLUTION First write

$$\begin{aligned}\int_{-3}^3 (9x - 4x^2)\,dx &= \int_{-3}^0 (9x - 4x^2)\,dx + \int_0^3 (9x - 4x^2)\,dx \\ &= -\int_0^{-3} (9x - 4x^2)\,dx + \int_0^3 (9x - 4x^2)\,dx.\end{aligned}$$

Then,

$$\begin{aligned}\int_{-3}^3 (9x - 4x^2)\,dx &= -\left(9\cdot\frac{1}{2}(-3)^2 - 4\cdot\frac{1}{3}(-3)^3\right) + \left(9\cdot\frac{1}{2}(3)^2 - 4\cdot\frac{1}{3}(3)^3\right) \\ &= -\left(\frac{81}{2} + 36\right) + \left(\frac{81}{2} - 36\right) = -72.\end{aligned}$$

41. $\int_{-a}^1 (x^2 + x)\,dx$

SOLUTION First, $\int_0^b (x^2 + x)\,dx = \int_0^b x^2\,dx + \int_0^b x\,dx = \frac{1}{3}b^3 + \frac{1}{2}b^2$. Therefore

$$\begin{aligned}\int_{-a}^1 (x^2 + x)\,dx &= \int_{-a}^0 (x^2 + x)\,dx + \int_0^1 (x^2 + x)\,dx = \int_0^1 (x^2 + x)\,dx - \int_0^{-a} (x^2 + x)\,dx \\ &= \left(\frac{1}{3}\cdot 1^3 + \frac{1}{2}\cdot 1^2\right) - \left(\frac{1}{3}(-a)^3 + \frac{1}{2}(-a)^2\right) = \frac{1}{3}a^3 - \frac{1}{2}a^2 + \frac{5}{6}.\end{aligned}$$

42. $\int_a^{a^2} x^2\,dx$

SOLUTION

$$\int_a^{a^2} x^2\,dx = \int_0^{a^2} x^2\,dx - \int_0^a x^2\,dx = \frac{1}{3}\left(a^2\right)^3 - \frac{1}{3}(a)^3 = \frac{1}{3}a^6 - \frac{1}{3}a^3.$$

In Exercises 43–47, calculate the integral, assuming that

$$\int_0^5 f(x)\,dx = 5, \qquad \int_0^5 g(x)\,dx = 12$$

43. $\int_0^5 (f(x) + g(x))\,dx$

SOLUTION $\int_0^5 (f(x) + g(x))\,dx = \int_0^5 f(x)\,dx + \int_0^5 g(x)\,dx = 5 + 12 = 17.$

44. $\int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right)\,dx$

SOLUTION $\int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right)\,dx = 2\int_0^5 f(x)\,dx - \frac{1}{3}\int_0^5 g(x)\,dx = 2(5) - \frac{1}{3}(12) = 6.$

45. $\int_5^0 g(x)\,dx$

SOLUTION $\int_5^0 g(x)\,dx = -\int_0^5 g(x)\,dx = -12.$

46. $\int_0^5 (f(x) - x)\,dx$

SOLUTION $\int_0^5 (f(x) - x)\,dx = \int_0^5 f(x)\,dx - \int_0^5 x\,dx = 5 - \frac{1}{2}(5)^2 = -\frac{15}{2}.$

47. Is it possible to calculate $\int_0^5 g(x)f(x)\,dx$ from the information given?

SOLUTION It is not possible to calculate $\int_0^5 g(x)f(x)\,dx$ from the information given.

48. Prove by computing the limit of right-endpoint approximations:

$$\boxed{\int_0^b x^3\,dx = \frac{b^4}{4}} \qquad \boxed{9}$$

SOLUTION Let $f(x) = x^3$, $a = 0$ and $\Delta x = (b - a)/N = b/N$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{b}{N}\sum_{k=1}^{N}\left(k^3 \cdot \frac{b^3}{N^3}\right) = \frac{b^4}{N^4}\left(\sum_{k=1}^{N} k^3\right) = \frac{b^4}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}\right) = \frac{b^4}{4} + \frac{b^4}{2N} + \frac{b^4}{4N^2}.$$

Hence $\int_0^b x^3\,dx = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{b^4}{4} + \frac{b^4}{2N} + \frac{b^4}{4N^2}\right) = \frac{b^4}{4}.$

In Exercises 49–54, evaluate the integral using the formulas in the summary and Eq. (9).

49. $\int_0^3 x^3\,dx$

SOLUTION By Eq. (9), $\int_0^3 x^3\,dx = \frac{3^4}{4} = \frac{81}{4}.$

50. $\int_1^3 x^3\,dx$

SOLUTION $\int_1^3 x^3\,dx = \int_0^3 x^3\,dx - \int_0^1 x^3\,dx = \frac{1}{4}(3)^4 - \frac{1}{4}(1)^4 = 20.$

51. $\int_0^3 (x - x^3)\,dx$

SOLUTION $\int_0^3 (x - x^3)\,dx = \int_0^3 x\,dx - \int_0^3 x^3\,dx = \frac{1}{2}3^2 - \frac{1}{4}3^4 = -\frac{63}{4}.$

52. $\int_0^1 (2x^3 - x + 4)\,dx$

SOLUTION Applying the linearity of the definite integral, Eq. (9), the formula from Example 4 and the formula for the definite integral of a constant:

$$\int_0^1 (2x^3 - x + 4)\,dx = 2\int_0^1 x^3\,dx - \int_0^1 x\,dx + \int_0^1 4\,dx = 2 \cdot \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + 4 = 4.$$

53. $\int_0^1 (12x^3 + 24x^2 - 8x)\,dx$

SOLUTION

$$\begin{aligned}\int_0^1 (12x^3 + 24x^2 - 8x)\,dx &= 12\int_0^1 x^3\,dx + 24\int_0^1 x^2 - 8\int_0^1 x\,dx \\ &= 12 \cdot \frac{1}{4}1^4 + 24 \cdot \frac{1}{3}1^3 - 8 \cdot \frac{1}{2}1^2 \\ &= 3 + 8 - 4 = 7\end{aligned}$$

54. $\int_{-2}^2 (2x^3 - 3x^2)\,dx$

SOLUTION

$$\begin{aligned}\int_{-2}^2 (2x^3 - 3x^2)\,dx &= \int_{-2}^0 (2x^3 - 3x^2)\,dx + \int_0^2 (2x^3 - 3x^2)\,dx \\ &= \int_0^2 (2x^3 - 3x^2)\,dx - \int_0^{-2} (2x^3 - 3x^2)\,dx \\ &= 2\int_0^2 x^3\,dx - 3\int_0^2 x^2\,dx - 2\int_0^{-2} x^3\,dx + 3\int_0^{-2} x^2\,dx \\ &= 2 \cdot \frac{1}{4}(2)^4 - 3 \cdot \frac{1}{3}(2)^3 - 2 \cdot \frac{1}{4}(-2)^4 + 3 \cdot \frac{1}{3}(-2)^3 \\ &= 8 - 8 - 8 - 8 = -16.\end{aligned}$$

In Exercises 55–58, calculate the integral, assuming that

$$\int_0^1 f(x)\,dx = 1, \qquad \int_0^2 f(x)\,dx = 4, \qquad \int_1^4 f(x)\,dx = 7$$

55. $\int_0^4 f(x)\,dx$

SOLUTION $\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx = 1 + 7 = 8.$

56. $\int_1^2 f(x)\,dx$

SOLUTION $\int_1^2 f(x)\,dx = \int_0^2 f(x)\,dx - \int_0^1 f(x)\,dx = 4 - 1 = 3.$

57. $\int_4^1 f(x)\,dx$

SOLUTION $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -7.$

58. $\int_2^4 f(x)\,dx$

SOLUTION From Exercise 55, $\int_0^4 f(x)\,dx = 8$. Accordingly,

$$\int_2^4 f(x)\,dx = \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx = 8 - 4 = 4.$$

In Exercises 59–62, express each integral as a single integral.

59. $\int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx$

SOLUTION $\int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx = \int_0^7 f(x)\,dx.$

60. $\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx$

SOLUTION $\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx = \left(\int_2^4 f(x)\,dx + \int_4^9 f(x)\,dx\right) - \int_4^9 f(x)\,dx = \int_2^4 f(x)\,dx.$

61. $\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx$

SOLUTION $\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx = \left(\int_2^5 f(x)\,dx + \int_5^9 f(x)\,dx\right) - \int_2^5 f(x)\,dx = \int_5^9 f(x)\,dx.$

62. $\int_7^3 f(x)\,dx + \int_3^9 f(x)\,dx$

SOLUTION $\int_7^3 f(x)\,dx + \int_3^9 f(x)\,dx = -\int_3^7 f(x)\,dx + \left(\int_3^7 f(x)\,dx + \int_7^9 f(x)\,dx\right) = \int_7^9 f(x)\,dx.$

In Exercises 63–66, calculate the integral, assuming that f is integrable and $\int_1^b f(x)\,dx = 1 - b^{-1}$ for all $b > 0$.

63. $\int_1^5 f(x)\,dx$

SOLUTION $\int_1^5 f(x)\,dx = 1 - 5^{-1} = \frac{4}{5}.$

64. $\int_3^5 f(x)\,dx$

SOLUTION $\int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx = \left(1 - \frac{1}{5}\right) - \left(1 - \frac{1}{3}\right) = \frac{2}{15}.$

65. $\int_1^6 (3f(x) - 4)\,dx$

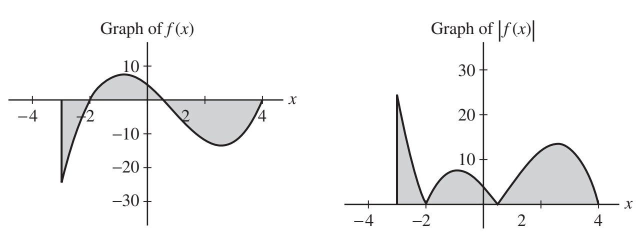
SOLUTION $\int_1^6 (3f(x) - 4)\,dx = 3\int_1^6 f(x)\,dx - 4\int_1^6 1\,dx = 3(1 - 6^{-1}) - 4(6 - 1) = -\frac{35}{2}.$

66. $\int_{1/2}^1 f(x)\,dx$

SOLUTION $\int_{1/2}^1 f(x)\,dx = -\int_1^{1/2} f(x)\,dx = -\left(1 - \left(\frac{1}{2}\right)^{-1}\right) = 1.$

67. Explain the difference in graphical interpretation between $\int_a^b f(x)\,dx$ and $\int_a^b |f(x)|\,dx$.

SOLUTION When $f(x)$ takes on both positive and negative values on $[a, b]$, $\int_a^b f(x)\,dx$ represents the signed area between $f(x)$ and the x-axis, whereas $\int_a^b |f(x)|\,dx$ represents the total (unsigned) area between $f(x)$ and the x-axis. Any negatively signed areas that were part of $\int_a^b f(x)\,dx$ are regarded as positive areas in $\int_a^b |f(x)|\,dx$. Here is a graphical example of this phenomenon.



68. Use the graphical interpretation of the definite integral to explain the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

where $f(x)$ is continuous. Explain also why equality holds if and only if either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x.

SOLUTION Let A_+ denote the unsigned area under the graph of $y = f(x)$ over the interval $[a, b]$ where $f(x) \ge 0$. Similarly, let A_- denote the unsigned area when $f(x) < 0$. Then

$$\int_a^b f(x)\,dx = A_+ - A_-.$$

Moreover,

$$\left|\int_a^b f(x)\,dx\right| \le A_+ + A_- = \int_a^b |f(x)|\,dx.$$

Equality holds if and only if one of the unsigned areas is equal to zero; in other words, if and only if either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x.

69. Let $f(x) = x$. Find an interval $[a, b]$ such that

$$\left|\int_a^b f(x)\,dx\right| = \frac{1}{2} \qquad \text{and} \qquad \int_a^b |f(x)|\,dx = \frac{3}{2}$$

SOLUTION If $a > 0$, then $f(x) \ge 0$ for all $x \in [a, b]$, so

$$\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$$

by the previous exercise. We find a similar result if $b < 0$. Thus, we must have $a < 0$ and $b > 0$. Now,

$$\int_a^b |f(x)|\,dx = \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

Because

$$\int_a^b f(x)\,dx = \frac{1}{2}b^2 - \frac{1}{2}a^2,$$

then

$$\left|\int_a^b f(x)\,dx\right| = \frac{1}{2}|b^2 - a^2|.$$

If $b^2 > a^2$, then

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{3}{2} \quad \text{and} \quad \frac{1}{2}(b^2 - a^2) = \frac{1}{2}$$

yield $a = -1$ and $b = \sqrt{2}$. On the other hand, if $b^2 < a^2$, then

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{3}{2} \quad \text{and} \quad \frac{1}{2}(a^2 - b^2) = \frac{1}{2}$$

yield $a = -\sqrt{2}$ and $b = 1$.

70. Evaluate $I = \int_0^{2\pi} \sin^2 x\,dx$ and $J = \int_0^{2\pi} \cos^2 x\,dx$ as follows. First show with a graph that $I = J$. Then prove that $I + J = 2\pi$.

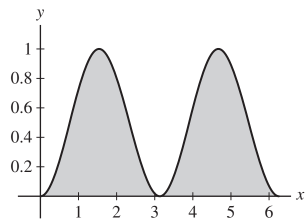
SOLUTION The graphs of $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ are shown below at the left and right, respectively. It is clear that the shaded areas in the two graphs are equal, thus

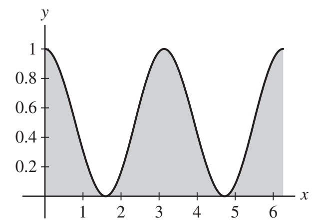
$$I = \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = J.$$

Now, using the fundamental trigonometric identity, we find

$$I + J = \int_0^{2\pi} (\sin^2 x + \cos^2 x)\,dx = \int_0^{2\pi} 1 \cdot dx = 2\pi.$$

Combining this last result with $I = J$ yields $I = J = \pi$.

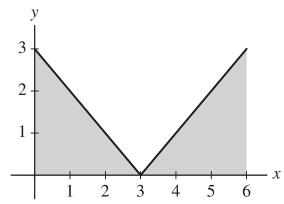




In Exercises 71–74, calculate the integral.

71. $\int_0^6 |3 - x|\, dx$

SOLUTION Over the interval, the region between the curve and the interval [0, 6] consists of two triangles above the x axis, each of which has height 3 and width 3, and so area $\frac{9}{2}$. The total area, hence the definite integral, is 9.

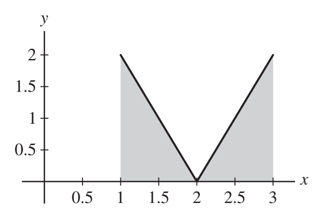


Alternately,

$$\begin{aligned}\int_0^6 |3 - x|\, dx &= \int_0^3 (3 - x)\, dx + \int_3^6 (x - 3)\, dx \\ &= 3\int_0^3 dx - \int_0^3 x\, dx + \left(\int_0^6 x\, dx - \int_0^3 x\, dx\right) - 3\int_3^6 dx \\ &= 9 - \frac{1}{2}3^2 + \frac{1}{2}6^2 - \frac{1}{2}3^2 - 9 = 9.\end{aligned}$$

72. $\int_1^3 |2x - 4|\, dx$

SOLUTION The area between $|2x - 4|$ and the x axis consists of two triangles above the x-axis, each with width 1 and height 2, and hence with area 1. The total area, and hence the definite integral, is 2.



Alternately,

$$\begin{aligned}\int_1^3 |2x - 4|\, dx &= \int_1^2 (4 - 2x)\, dx + \int_2^3 (2x - 4)\, dx \\ &= 4\int_1^2 dx - 2\left(\int_0^2 x\, dx - \int_0^1 x\, dx\right) + 2\left(\int_0^3 x\, dx - \int_0^2 x\, dx\right) - 4\int_2^3 dx \\ &= 4 - 2\left(\frac{1}{2}2^2 - \frac{1}{2}1^2\right) + 2\left(\frac{1}{2}3^2 - \frac{1}{2}2^2\right) - 4 = 2.\end{aligned}$$

73. $\int_{-1}^1 |x^3|\, dx$

SOLUTION

$$|x^3| = \begin{cases} x^3 & x \ge 0 \\ -x^3 & x < 0. \end{cases}$$

Therefore,

$$\int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} -x^3\,dx + \int_0^1 x^3\,dx = \int_0^{-1} x^3\,dx + \int_0^1 x^3\,dx = \frac{1}{4}(-1)^4 + \frac{1}{4}(1)^4 = \frac{1}{2}.$$

74. $\displaystyle\int_0^2 |x^2 - 1|\,dx$

SOLUTION

$$|x^2 - 1| = \begin{cases} x^2 - 1 & 1 \le x \le 2 \\ -(x^2 - 1) & 0 \le x < 1. \end{cases}$$

Therefore,

$$\begin{aligned}\int_0^2 |x^2 - 1|\,dx &= \int_0^1 (1 - x^2)\,dx + \int_1^2 (x^2 - 1)\,dx \\ &= \int_0^1 dx - \int_0^1 x^2\,dx + \left(\int_0^2 x^2\,dx - \int_0^1 x^2\,dx\right) - \int_1^2 1\,dx \\ &= 1 - \frac{1}{3}(1) + \left(\frac{1}{3}(8) - \frac{1}{3}(1)\right) - 1 = 2.\end{aligned}$$

75. Use the Comparison Theorem to show that

$$\int_0^1 x^5\,dx \le \int_0^1 x^4\,dx, \qquad \int_1^2 x^4\,dx \le \int_1^2 x^5\,dx$$

SOLUTION On the interval $[0, 1]$, $x^5 \le x^4$, so, by Theorem 5,

$$\int_0^1 x^5\,dx \le \int_0^1 x^4\,dx.$$

On the other hand, $x^4 \le x^5$ for $x \in [1, 2]$, so, by the same Theorem,

$$\int_1^2 x^4\,dx \le \int_1^2 x^5\,dx.$$

76. Prove that $\displaystyle\frac{1}{3} \le \int_4^6 \frac{1}{x}\,dx \le \frac{1}{2}$.

SOLUTION On the interval $[4, 6]$, $\frac{1}{6} \le \frac{1}{x}$, so, by Theorem 5,

$$\frac{1}{3} = \int_4^6 \frac{1}{6}\,dx \le \int_4^6 \frac{1}{x}\,dx.$$

On the other hand, $\frac{1}{x} \le \frac{1}{4}$ on the interval $[4, 6]$, so

$$\int_4^6 \frac{1}{x}\,dx \le \int_4^6 \frac{1}{4}\,dx = \frac{1}{4}(6 - 4) = \frac{1}{2}.$$

Therefore $\frac{1}{3} \le \int_4^6 \frac{1}{x}\,dx \le \frac{1}{2}$, as desired.

77. Prove that $0.0198 \le \int_{0.2}^{0.3} \sin x\,dx \le 0.0296$. *Hint:* Show that $0.198 \le \sin x \le 0.296$ for x in $[0.2, 0.3]$.

SOLUTION For $0 \le x \le \frac{\pi}{6} \approx 0.52$, we have $\frac{d}{dx}(\sin x) = \cos x > 0$. Hence $\sin x$ is increasing on $[0.2, 0.3]$. Accordingly, for $0.2 \le x \le 0.3$, we have

$$m = 0.198 \le 0.19867 \approx \sin 0.2 \le \sin x \le \sin 0.3 \approx 0.29552 \le 0.296 = M$$

Therefore, by the Comparison Theorem, we have

$$0.0198 = m(0.3 - 0.2) = \int_{0.2}^{0.3} m\,dx \le \int_{0.2}^{0.3} \sin x\,dx \le \int_{0.2}^{0.3} M\,dx = M(0.3 - 0.2) = 0.0296.$$

78. Prove that $0.277 \le \int_{\pi/8}^{\pi/4} \cos x\, dx \le 0.363$.

SOLUTION $\cos x$ is decreasing on the interval $[\pi/8, \pi/4]$. Hence, for $\pi/8 \le x \le \pi/4$,

$$\cos(\pi/4) \le \cos x \le \cos(\pi/8).$$

Since $\cos(\pi/4) = \sqrt{2}/2$,

$$0.277 \le \frac{\pi}{8} \cdot \frac{\sqrt{2}}{2} = \int_{\pi/8}^{\pi/4} \frac{\sqrt{2}}{2}\, dx \le \int_{\pi/8}^{\pi/4} \cos x\, dx.$$

Since $\cos(\pi/8) \le 0.924$,

$$\int_{\pi/8}^{\pi/4} \cos x\, dx \le \int_{\pi/8}^{\pi/4} 0.924\, dx = \frac{\pi}{8}(0.924) \le 0.363.$$

Therefore $0.277 \le \int_{\pi/8}^{\pi/4} \cos x \le 0.363$.

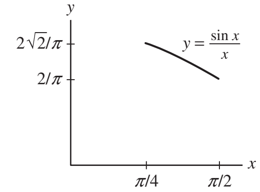
79. Prove that $0 \le \int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx \le \frac{\sqrt{2}}{2}$.

SOLUTION Let

$$f(x) = \frac{\sin x}{x}.$$

As we can see in the sketch below, $f(x)$ is decreasing on the interval $[\pi/4, \pi/2]$. Therefore $f(x) \le f(\pi/4)$ for all x in $[\pi/4, \pi/2]$. $f(\pi/4) = \frac{2\sqrt{2}}{\pi}$, so:

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx \le \int_{\pi/4}^{\pi/2} \frac{2\sqrt{2}}{\pi}\, dx = \frac{\pi}{4}\frac{2\sqrt{2}}{\pi} = \frac{\sqrt{2}}{2}.$$



80. Find upper and lower bounds for $\int_0^1 \frac{dx}{\sqrt{5x^3+4}}$.

SOLUTION Let

$$f(x) = \frac{1}{\sqrt{5x^3+4}}.$$

$f(x)$ is decreasing for x on the interval $[0, 1]$, so $f(1) \le f(x) \le f(0)$ for all x in $[0, 1]$. $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{3}$, so

$$\int_0^1 \frac{1}{3}\, dx \le \int_0^1 f(x)\, dx \le \int_0^1 \frac{1}{2}\, dx$$
$$\frac{1}{3} \le \int_0^1 f(x)\, dx \le \frac{1}{2}.$$

81. Suppose that $f(x) \le g(x)$ on $[a, b]$. By the Comparison Theorem, $\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$. Is it also true that $f'(x) \le g'(x)$ for $x \in [a, b]$? If not, give a counterexample.

SOLUTION The assertion $f'(x) \le g'(x)$ is false. Consider $a = 0$, $b = 1$, $f(x) = x$, $g(x) = 2$. $f(x) \le g(x)$ for all x in the interval $[0, 1]$, but $f'(x) = 1$ while $g'(x) = 0$ for all x.

82. State whether true or false. If false, sketch the graph of a counterexample.

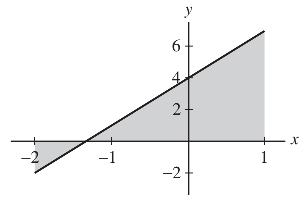
(a) If $f(x) > 0$, then $\int_a^b f(x)\, dx > 0$.

(b) If $\int_a^b f(x)\, dx > 0$, then $f(x) > 0$.

SOLUTION

(a) It is true that if $f(x) > 0$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx > 0$.

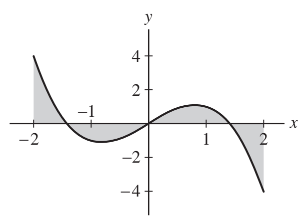
(b) It is *false* that if $\int_a^b f(x)\,dx > 0$, then $f(x) > 0$ for $x \in [a, b]$. Indeed, in Exercise 3, we saw that $\int_{-2}^{1}(3x+4)\,dx = 7.5 > 0$, yet $f(-2) = -2 < 0$. Here is the graph from that exercise.



Further Insights and Challenges

83. Explain graphically: If $f(x)$ is an odd function, then
$$\int_{-a}^{a} f(x)\,dx = 0.$$

SOLUTION If f is an odd function, then $f(-x) = -f(x)$ for all x. Accordingly, for every positively signed area in the right half-plane where f is above the x-axis, there is a corresponding negatively signed area in the left half-plane where f is below the x-axis. Similarly, for every negatively signed area in the right half-plane where f is below the x-axis, there is a corresponding positively signed area in the left half-plane where f is above the x-axis. We conclude that the net area between the graph of f and the x-axis over $[-a, a]$ is 0, since the positively signed areas and negatively signed areas cancel each other out exactly.



84. Compute $\displaystyle\int_{-1}^{1} \sin(\sin(x))(\sin^2(x)+1)\,dx$.

SOLUTION Let $f(x) = \sin(\sin(x))(\sin^2(x)+1))$. $\sin x$ is an odd function, while $\sin^2 x$ is an even function, so:

$$\begin{aligned} f(-x) &= \sin(\sin(-x))(\sin^2(-x)+1) = \sin(-\sin(x))(\sin^2(x)+1) \\ &= -\sin(\sin(x))(\sin^2(x)+1) = -f(x). \end{aligned}$$

Therefore, $f(x)$ is an odd function. The function is odd and the interval is symmetric around the origin so, by the previous exercise, the integral must be zero.

85. Let k and b be positive. Show, by comparing the right-endpoint approximations, that

$$\int_0^b x^k\,dx = b^{k+1}\int_0^1 x^k\,dx$$

SOLUTION Let k and b be any positive numbers. Let $f(x) = x^k$ on $[0, b]$. Since f is continuous, both $\int_0^b f(x)\,dx$ and $\int_0^1 f(x)\,dx$ exist. Let N be a positive integer and set $\Delta x = (b-0)/N = b/N$. Let $x_j = a + j\Delta x = bj/N$, $j = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[0, b]$. Then the right-endpoint approximation to $\int_0^b f(x)\,dx = \int_0^b x^k\,dx$ is

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{b}{N}\sum_{j=1}^{N}\left(\frac{bj}{N}\right)^k = b^{k+1}\left(\frac{1}{N^{k+1}}\sum_{j=1}^{N} j^k\right).$$

In particular, if $b = 1$ above, then the right-endpoint approximation to $\int_0^1 f(x)\,dx = \int_0^1 x^k\,dx$ is

$$S_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^k = \frac{1}{N^{k+1}}\sum_{j=1}^{N} j^k = \frac{1}{b^{k+1}} R_N$$

In other words, $R_N = b^{k+1} S_N$. Therefore,

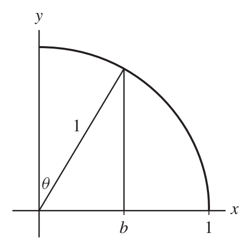
$$\int_0^b x^k\,dx = \lim_{N\to\infty} R_N = \lim_{N\to\infty} b^{k+1} S_N = b^{k+1} \lim_{N\to\infty} S_N = b^{k+1} \int_0^1 x^k\,dx.$$

86. Verify for $0 \le b \le 1$ by interpreting in terms of area:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\sin^{-1} b$$

SOLUTION The function $f(x) = \sqrt{1-x^2}$ is the quarter circle of radius 1 in the first quadrant. For $0 \le b \le 1$, the area represented by the integral $\int_0^b \sqrt{1-x^2}\,dx$ can be divided into two parts. The area of the triangular part is $\frac{1}{2}(b)\sqrt{1-b^2}$ using the Pythagorean Theorem. The area of the sector with angle θ where $\sin\theta = b$, is given by $\frac{1}{2}(1)^2(\theta)$. Thus

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\theta = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\sin^{-1} b.$$



87. Suppose that f and g are continuous functions such that, *for all* a,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} g(x)\,dx$$

Give an *intuitive* argument showing that $f(0) = g(0)$. Explain your idea with a graph.

SOLUTION Let $c = -b$. Since $b < 0$, $c > 0$, so by Eq. (5),

$$\int_0^c x^2\,dx = \frac{1}{3}c^3.$$

Furthermore, x^2 is an even function, so symmetry of the areas gives

$$\int_{-c}^0 x^2\,dx = \int_0^c x^2\,dx.$$

Finally,

$$\int_0^b x^2\,dx = \int_0^{-c} x^2\,dx = -\int_{-c}^0 x^2\,dx = -\int_0^c x^2\,dx = -\frac{1}{3}c^3 = \frac{1}{3}b^3.$$

88. Theorem 4 remains true without the assumption $a \le b \le c$. Verify this for the cases $b < a < c$ and $c < a < b$.

SOLUTION The additivity property of definite integrals states for $a \le b \le c$, we have $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

- Suppose that we have $b < a < c$. By the additivity property, we have $\int_b^c f(x)\,dx = \int_b^a f(x)\,dx + \int_a^c f(x)\,dx$. Therefore, $\int_a^c f(x)\,dx = \int_b^c f(x)\,dx - \int_b^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
- Now suppose that we have $c < a < b$. By the additivity property, we have $\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$. Therefore, $\int_a^c f(x)\,dx = -\int_c^a f(x)\,dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
- Hence the additivity property holds for all real numbers a, b, and c, regardless of their relationship amongst each other.

5.3 The Fundamental Theorem of Calculus, Part I

Preliminary Questions

1. Suppose that $F'(x) = f(x)$ and $F(0) = 3$, $F(2) = 7$.

(a) What is the area under $y = f(x)$ over $[0, 2]$ if $f(x) \geq 0$?

(b) What is the graphical interpretation of $F(2) - F(0)$ if $f(x)$ takes on both positive and negative values?

SOLUTION

(a) If $f(x) \geq 0$ over $[0, 2]$, then the area under $y = f(x)$ is $F(2) - F(0) = 7 - 3 = 4$.

(b) If $f(x)$ takes on both positive and negative values, then $F(2) - F(0)$ gives the signed area between $y = f(x)$ and the x-axis.

2. Suppose that $f(x)$ is a *negative* function with antiderivative F such that $F(1) = 7$ and $F(3) = 4$. What is the area (a positive number) between the x-axis and the graph of $f(x)$ over $[1, 3]$?

SOLUTION $\int_1^3 f(x)\,dx$ represents the *signed* area bounded by the curve and the interval $[1, 3]$. Since $f(x)$ is negative on $[1, 3]$, $\int_1^3 f(x)\,dx$ is the negative of the area. Therefore, if A is the area between the x-axis and the graph of $f(x)$, we have:

$$A = -\int_1^3 f(x)\,dx = -(F(3) - F(1)) = -(4 - 7) = -(-3) = 3.$$

3. Are the following statements true or false? Explain.

(a) FTC I is valid only for positive functions.

(b) To use FTC I, you have to choose the right antiderivative.

(c) If you cannot find an antiderivative of $f(x)$, then the definite integral does not exist.

SOLUTION

(a) False. The FTC I is valid for continuous functions.

(b) False. The FTC I works for any antiderivative of the integrand.

(c) False. If you cannot find an antiderivative of the integrand, you cannot use the FTC I to evaluate the definite integral, but the definite integral may still exist.

4. Evaluate $\int_2^9 f'(x)\,dx$ where $f(x)$ is differentiable and $f(2) = f(9) = 4$.

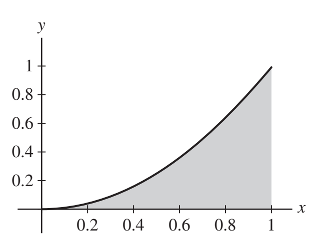
SOLUTION Because f is differentiable, $\int_2^9 f'(x)\,dx = f(9) - f(2) = 4 - 4 = 0$.

Exercises

In Exercises 1–4, sketch the region under the graph of the function and find its area using FTC I.

1. $f(x) = x^2$, $[0, 1]$

SOLUTION

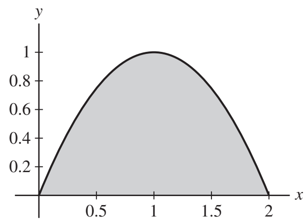


We have the area

$$A = \int_0^1 x^2\,dx = \left.\frac{1}{3}x^3\right|_0^1 = \frac{1}{3}.$$

2. $f(x) = 2x - x^2$, $[0, 2]$

SOLUTION

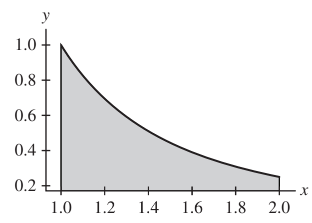


Let A be the area indicated. Then:

$$A = \int_0^2 (2x - x^2)\, dx = \int_0^2 2x\, dx - \int_0^2 x^2\, dx = x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2 = (4-0) - \left(\frac{8}{3} - 0\right) = \frac{4}{3}.$$

3. $f(x) = x^{-2}$, $[1, 2]$

SOLUTION

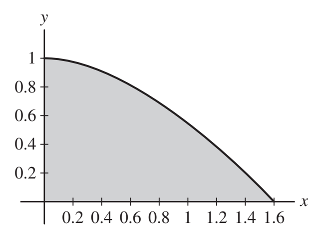


We have the area

$$A = \int_1^2 x^{-2}\, dx = \frac{x^{-1}}{-1}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

4. $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$

SOLUTION



Let A be the shaded area. Then

$$A = \int_0^{\pi/2} \cos x\, dx = \sin x\Big|_0^{\pi/2} = 1 - 0 = 1.$$

In Exercises 5–42, evaluate the integral using FTC I.

5. $\displaystyle\int_3^6 x\, dx$

SOLUTION $\displaystyle\int_3^6 x\, dx = \frac{1}{2}x^2\Big|_3^6 = \frac{1}{2}(6)^2 - \frac{1}{2}(3)^2 = \frac{27}{2}.$

6. $\displaystyle\int_0^9 2\, dx$

SOLUTION $\displaystyle\int_0^9 2\, dx = 2x\Big|_0^9 = 2(9) - 2(0) = 18.$

7. $\displaystyle\int_0^1 (4x - 9x^2)\, dx$

SOLUTION $\displaystyle\int_0^1 (4x - 9x^2)\, dx = (2x^2 - 3x^3)\Big|_0^1 = (2-3) - (0-0) = -1.$

8. $\int_{-3}^{2} u^2\,du$

SOLUTION $\int_{-3}^{2} u^2\,du = \frac{1}{3}u^3\Big|_{-3}^{2} = \frac{1}{3}(2)^3 - \frac{1}{3}(-3)^3 = \frac{35}{3}.$

9. $\int_{0}^{2} (12x^5 + 3x^2 - 4x)\,dx$

SOLUTION $\int_{0}^{2} (12x^5 + 3x^2 - 4x)\,dx = (2x^6 + x^3 - 2x^2)\Big|_{0}^{2} = (128 + 8 - 8) - (0 + 0 - 0) = 128.$

10. $\int_{-2}^{2} (10x^9 + 3x^5)\,dx$

SOLUTION $\int_{-2}^{2} (10x^9 + 3x^5)\,dx = \left(x^{10} + \frac{1}{2}x^6\right)\Big|_{-2}^{2} = \left(2^{10} + \frac{1}{2}2^6\right) - \left(2^{10} + \frac{1}{2}2^6\right) = 0.$

11. $\int_{3}^{0} (2t^3 - 6t^2)\,dt$

SOLUTION $\int_{3}^{0} (2t^3 - 6t^2)\,dt = \left(\frac{1}{2}t^4 - 2t^3\right)\Big|_{3}^{0} = (0 - 0) - \left(\frac{81}{2} - 54\right) = \frac{27}{2}.$

12. $\int_{-1}^{1} (5u^4 + u^2 - u)\,du$

SOLUTION $\int_{-1}^{1} (5u^4 + u^2 - u)\,du = \left(u^5 + \frac{1}{3}u^3 - \frac{1}{2}u^2\right)\Big|_{-1}^{1} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) - \left(-1 - \frac{1}{3} - \frac{1}{2}\right) = \frac{8}{3}.$

13. $\int_{0}^{4} \sqrt{y}\,dy$

SOLUTION $\int_{0}^{4} \sqrt{y}\,dy = \int_{0}^{4} y^{1/2}\,dy = \frac{2}{3}y^{3/2}\Big|_{0}^{4} = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}.$

14. $\int_{1}^{8} x^{4/3}\,dx$

SOLUTION $\int_{1}^{8} x^{4/3}\,dx = \frac{3}{7}x^{7/3}\Big|_{1}^{8} = \frac{3}{7}(128 - 1) = \frac{381}{7}.$

15. $\int_{1/16}^{1} t^{1/4}\,dt$

SOLUTION $\int_{1/16}^{1} t^{1/4}\,dt = \frac{4}{5}t^{5/4}\Big|_{1/16}^{1} = \frac{4}{5} - \frac{1}{40} = \frac{31}{40}.$

16. $\int_{4}^{1} t^{5/2}\,dt$

SOLUTION $\int_{4}^{1} t^{5/2}\,dt = \frac{2}{7}t^{7/2}\Big|_{4}^{1} = \frac{2}{7}(1 - 128) = -\frac{254}{7}.$

17. $\int_{1}^{3} \frac{dt}{t^2}$

SOLUTION $\int_{1}^{3} \frac{dt}{t^2} = \int_{1}^{3} t^{-2}\,dt = -t^{-1}\Big|_{1}^{3} = -\frac{1}{3} + 1 = \frac{2}{3}.$

18. $\int_{1}^{4} x^{-4}\,dx$

SOLUTION $\int_{1}^{4} x^{-4}\,dx = -\frac{1}{3}x^{-3}\Big|_{1}^{4} = -\frac{1}{3}(4)^{-3} + \frac{1}{3} = \frac{21}{64}.$

19. $\int_{1/2}^{1} \frac{8}{x^3}\,dx$

SOLUTION $\int_{1/2}^{1} \frac{8}{x^3}\,dx = \int_{1/2}^{1} 8x^{-3}\,dx = -4x^{-2}\Big|_{1/2}^{1} = -4 + 16 = 12.$

20. $\int_{-2}^{-1} \frac{1}{x^3}\,dx$

SOLUTION $\int_{-2}^{-1} \frac{1}{x^3}\,dx = -\frac{1}{2}x^{-2}\Big|_{-2}^{-1} = -\frac{1}{2}(-1)^{-2} + \frac{1}{2}(-2)^{-2} = -\frac{3}{8}.$

21. $\int_1^2 (x^2 - x^{-2})\,dx$

SOLUTION $\int_1^2 (x^2 - x^{-2})\,dx = \left(\frac{1}{3}x^3 + x^{-1}\right)\Big|_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}.$

22. $\int_1^9 t^{-1/2}\,dt$

SOLUTION $\int_1^9 t^{-1/2}\,dt = 2t^{1/2}\Big|_1^9 = 2(9)^{1/2} - 2(1)^{1/2} = 4.$

23. $\int_1^{27} \frac{t+1}{\sqrt{t}}\,dt$

SOLUTION

$$\int_1^{27} \frac{t+1}{\sqrt{t}}\,dt = \int_1^{27} (t^{1/2} + t^{-1/2})\,dt = \left(\frac{2}{3}t^{3/2} + 2t^{1/2}\right)\Big|_1^{27}$$
$$= \left(\frac{2}{3}(81\sqrt{3}) + 6\sqrt{3}\right) - \left(\frac{2}{3} + 2\right) = 60\sqrt{3} - \frac{8}{3}.$$

24. $\int_{8/27}^{1} \frac{10t^{4/3} - 8t^{1/3}}{t^2}\,dt$

SOLUTION

$$\int_{8/27}^{1} \frac{10t^{4/3} - 8t^{1/3}}{t^2}\,dt = \int_{8/27}^{1} (10t^{-2/3} - 8t^{-5/3})\,dt$$
$$= (30t^{1/3} + 12t^{-2/3})\Big|_{8/27}^{1} = (30 + 12) - (20 + 27) = -5.$$

25. $\int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta$

SOLUTION $\int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta = -\cos\theta\Big|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}.$

26. $\int_{2\pi}^{4\pi} \sin x\,dx$

SOLUTION $\int_{2\pi}^{4\pi} \sin x\,dx = -\cos x\Big|_{2\pi}^{4\pi} = -1 - (-1) = 0.$

27. $\int_0^{\pi/2} \cos\left(\frac{1}{3}\theta\right)d\theta$

SOLUTION $\int_0^{\pi/2} \cos\left(\frac{1}{3}\theta\right)d\theta = 3\sin\left(\frac{1}{3}\theta\right)\Big|_0^{\pi/2} = \frac{3}{2}.$

28. $\int_{\pi/4}^{5\pi/8} \cos 2x\,dx$

SOLUTION $\int_{\pi/4}^{5\pi/8} \cos 2x\,dx = \frac{1}{2}\sin 2x\Big|_{\pi/4}^{5\pi/8} = \frac{1}{2}\sin\frac{5\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} = -\frac{\sqrt{2}}{4} - \frac{1}{2}.$

29. $\int_0^{\pi/6} \sec^2\left(3t - \frac{\pi}{6}\right)dt$

SOLUTION $\int_0^{\pi/6} \sec^2\left(3t - \frac{\pi}{6}\right)dt = \frac{1}{3}\tan\left(3t - \frac{\pi}{6}\right)\Big|_0^{\pi/6} = \frac{1}{3}\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}}.$

30. $\int_0^{\pi/6} \sec\theta \tan\theta \, d\theta$

SOLUTION $\int_0^{\pi/6} \sec\theta \tan\theta \, d\theta = \sec\theta \Big|_0^{\pi/6} = \sec\frac{\pi}{6} - \sec 0 = \frac{2\sqrt{3}}{3} - 1.$

31. $\int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx$

SOLUTION $\int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx = -\frac{1}{5}\csc 5x \Big|_{\pi/20}^{\pi/10} = -\frac{1}{5}\left(1 - \sqrt{2}\right) = \frac{1}{5}(\sqrt{2} - 1).$

32. $\int_{\pi/28}^{\pi/14} \csc^2 7y \, dy$

SOLUTION $\int_{\pi/28}^{\pi/14} \csc^2 7y \, dy = -\frac{1}{7}\cot 7y \Big|_{\pi/28}^{\pi/14} = -\frac{1}{7}\cot\frac{\pi}{2} + \frac{1}{7}\cot\frac{\pi}{4} = \frac{1}{7}.$

33. $\int_0^1 e^x \, dx$

SOLUTION $\int_0^1 e^x \, dx = e^x \Big|_0^1 = e - 1.$

34. $\int_3^5 e^{-4x} \, dx$

SOLUTION $\int_3^5 e^{-4x} \, dx = -\frac{1}{4}e^{-4x} \Big|_3^5 = -\frac{1}{4}e^{-20} + \frac{1}{4}e^{-12}.$

35. $\int_0^3 e^{1-6t} \, dt$

SOLUTION $\int_0^3 e^{1-6t} \, dt = -\frac{1}{6}e^{1-6t} \Big|_0^3 = -\frac{1}{6}e^{-17} + \frac{1}{6}e = \frac{1}{6}(e - e^{-17}).$

36. $\int_2^3 e^{4t-3} \, dt$

SOLUTION $\int_2^3 e^{4t-3} \, dt = \frac{1}{4}e^{4t-3} \Big|_2^3 = \frac{1}{4}e^9 - \frac{1}{4}e^5.$

37. $\int_2^{10} \frac{dx}{x}$

SOLUTION $\int_2^{10} \frac{dx}{x} = \ln|x| \Big|_2^{10} = \ln 10 - \ln 2 = \ln 5.$

38. $\int_{-12}^{-4} \frac{dx}{x}$

SOLUTION $\int_{-12}^{-4} \frac{dx}{x} = \ln|x| \Big|_{-12}^{-4} = \ln|-4| - \ln|-12| = \ln\frac{1}{3} = -\ln 3.$

39. $\int_0^1 \frac{dt}{t+1}$

SOLUTION $\int_0^1 \frac{dt}{t+1} = \ln|t+1| \Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$

40. $\int_1^4 \frac{dt}{5t+4}$

SOLUTION $\int_1^4 \frac{dt}{5t+4} = \frac{1}{5}\ln|5t+4| \Big|_1^4 = \frac{1}{5}\ln 24 - \frac{1}{5}\ln 9 = \frac{1}{5}\ln\frac{24}{9}.$

41. $\int_{-2}^0 (3x - 9e^{3x}) \, dx$

SOLUTION $\int_{-2}^0 (3x - 9e^{3x}) \, dx = \left(\frac{3}{2}x^2 - 3e^{3x}\right)\Big|_{-2}^0 = (0 - 3) - (6 - 3e^{-6}) = 3e^{-6} - 9.$

42. $\int_2^6 \left(x + \frac{1}{x}\right) dx$

SOLUTION $\int_2^6 \left(x + \frac{1}{x}\right) dx = \left(\frac{1}{2}x^2 + \ln|x|\right)\Big|_2^6 = (18 + \ln 6) - (2 + \ln 2) = 16 + \ln 3.$

In Exercises 43–48, write the integral as a sum of integrals without absolute values and evaluate.

43. $\int_{-2}^1 |x|\, dx$

SOLUTION

$$\int_{-2}^1 |x|\, dx = \int_{-2}^0 (-x)\, dx + \int_0^1 x\, dx = -\frac{1}{2}x^2\Big|_{-2}^0 + \frac{1}{2}x^2\Big|_0^1 = 0 - \left(-\frac{1}{2}(4)\right) + \frac{1}{2} = \frac{5}{2}.$$

44. $\int_0^5 |3 - x|\, dx$

SOLUTION

$$\begin{aligned}\int_0^5 |3 - x|\, dx &= \int_0^3 (3 - x)\, dx + \int_3^5 (x - 3)\, dx = \left(3x - \frac{1}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{2}x^2 - 3x\right)\Big|_3^5 \\ &= \left(9 - \frac{9}{2}\right) - 0 + \left(\frac{25}{2} - 15\right) - \left(\frac{9}{2} - 9\right) = \frac{13}{2}.\end{aligned}$$

45. $\int_{-2}^3 |x^3|\, dx$

SOLUTION

$$\begin{aligned}\int_{-2}^3 |x^3|\, dx &= \int_{-2}^0 (-x^3)\, dx + \int_0^3 x^3\, dx = -\frac{1}{4}x^4\Big|_{-2}^0 + \frac{1}{4}x^4\Big|_0^3 \\ &= 0 + \frac{1}{4}(-2)^4 + \frac{1}{4}3^4 - 0 = \frac{97}{4}.\end{aligned}$$

46. $\int_0^3 |x^2 - 1|\, dx$

SOLUTION

$$\begin{aligned}\int_0^3 |x^2 - 1|\, dx &= \int_0^1 (1 - x^2)\, dx + \int_1^3 (x^2 - 1)\, dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 + \left(\frac{1}{3}x^3 - x\right)\Big|_1^3 \\ &= \left(1 - \frac{1}{3}\right) - 0 + (9 - 3) - \left(\frac{1}{3} - 1\right) = \frac{22}{3}.\end{aligned}$$

47. $\int_0^\pi |\cos x|\, dx$

SOLUTION

$$\int_0^\pi |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^\pi (-\cos x)\, dx = \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^\pi = 1 - 0 - (-1 - 0) = 2.$$

48. $\int_0^5 |x^2 - 4x + 3|\, dx$

SOLUTION

$$\begin{aligned}\int_0^5 |x^2 - 4x + 3|\, dx &= \int_0^5 |(x - 3)(x - 1)|\, dx \\ &= \int_0^1 (x^2 - 4x + 3)\, dx + \int_1^3 -(x^2 - 4x + 3)\, dx + \int_3^5 (x^2 - 4x + 3)\, dx\end{aligned}$$

$$= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_1^3 + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_3^5$$
$$= \left(\frac{1}{3} - 2 + 3\right) - 0 - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{125}{3} - 50 + 15\right) - (9 - 18 + 9)$$
$$= \frac{28}{3}.$$

In Exercises 49–54, evaluate the integral in terms of the constants.

49. $\int_1^b x^3\,dx$

SOLUTION $\int_1^b x^3\,dx = \frac{1}{4}x^4\Big|_1^b = \frac{1}{4}b^4 - \frac{1}{4}(1)^4 = \frac{1}{4}\left(b^4 - 1\right)$ for any number b.

50. $\int_b^a x^4\,dx$

SOLUTION $\int_b^a x^4\,dx = \frac{1}{5}x^5\Big|_b^a = \frac{1}{5}a^5 - \frac{1}{5}b^5$ for any numbers a, b.

51. $\int_1^b x^5\,dx$

SOLUTION $\int_1^b x^5\,dx = \frac{1}{6}x^6\Big|_1^b = \frac{1}{6}b^6 - \frac{1}{6}(1)^6 = \frac{1}{6}(b^6 - 1)$ for any number b.

52. $\int_{-x}^x (t^3 + t)\,dt$

SOLUTION

$$\int_{-x}^x (t^3 + t)\,dt = \left(\frac{1}{4}t^4 + \frac{1}{2}t^2\right)\Big|_{-x}^x = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) - \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) = 0.$$

53. $\int_a^{5a} \frac{dx}{x}$

SOLUTION $\int_a^{5a} \frac{dx}{x} = \ln|x|\Big|_a^{5a} = \ln|5a| - \ln|a| = \ln 5.$

54. $\int_b^{b^2} \frac{dx}{x}$

SOLUTION $\int_b^{b^2} \frac{dx}{x} = \ln|x|\Big|_b^{b^2} = \ln|b^2| - \ln|b| = \ln|b|.$

55. Calculate $\int_{-2}^3 f(x)\,dx$, where

$$f(x) = \begin{cases} 12 - x^2 & \text{for } x \le 2 \\ x^3 & \text{for } x > 2 \end{cases}$$

SOLUTION

$$\int_{-2}^3 f(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_{-2}^2 (12 - x^2)\,dx + \int_2^3 x^3\,dx$$
$$= \left(12x - \frac{1}{3}x^3\right)\Big|_{-2}^2 + \frac{1}{4}x^4\Big|_2^3$$
$$= \left(12(2) - \frac{1}{3}2^3\right) - \left(12(-2) - \frac{1}{3}(-2)^3\right) + \frac{1}{4}3^4 - \frac{1}{4}2^4$$
$$= \frac{128}{3} + \frac{65}{4} = \frac{707}{12}.$$

56. Calculate $\int_0^{2\pi} f(x)\,dx$, where

$$f(x) = \begin{cases} \cos x & \text{for } x \le \pi \\ \cos x - \sin 2x & \text{for } x > \pi \end{cases}$$

SOLUTION

$$\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx = \int_0^{\pi} \cos x\,dx + \int_{\pi}^{2\pi} (\cos x - \sin 2x)\,dx$$
$$= \sin x\Big|_0^{\pi} + \left(\sin x + \frac{1}{2}\cos 2x\right)\Big|_{\pi}^{2\pi}$$
$$= (0-0) + \left(\left(0+\frac{1}{2}\right) - \left(0+\frac{1}{2}\right)\right) = 0.$$

57. Use FTC I to show that $\int_{-1}^{1} x^n\,dx = 0$ if n is an odd whole number. Explain graphically.

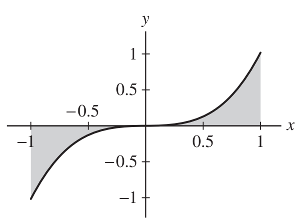
SOLUTION We have

$$\int_{-1}^{1} x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_{-1}^{1} = \frac{(1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1}.$$

Because n is odd, $n+1$ is even, which means that $(-1)^{n+1} = (1)^{n+1} = 1$. Hence

$$\frac{(1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

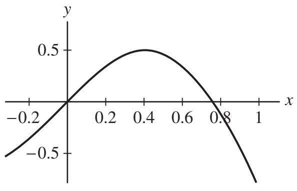
Graphically speaking, for an odd function such as x^3 shown here, the positively signed area from $x=0$ to $x=1$ cancels the negatively signed area from $x=-1$ to $x=0$.

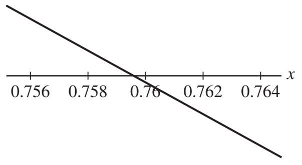


58. CAS Plot the function $f(x) = \sin 3x - x$. Find the positive root of $f(x)$ to three places and use it to find the area under the graph of $f(x)$ in the first quadrant.

SOLUTION The graph of $f(x) = \sin 3x - x$ is shown below at the left. In the figure below at the right, we zoom in on the positive root of $f(x)$ and find that, to three decimal places, this root is approximately $x = 0.760$. The area under the graph of $f(x)$ in the first quadrant is then

$$\int_0^{0.760} (\sin 3x - x)\,dx = \left(-\frac{1}{3}\cos 3x - \frac{1}{2}x^2\right)\Big|_0^{0.760}$$
$$= -\frac{1}{3}\cos(2.28) - \frac{1}{2}(0.760)^2 + \frac{1}{3} \approx 0.262$$





59. Calculate $F(4)$ given that $F(1) = 3$ and $F'(x) = x^2$. *Hint:* Express $F(4) - F(1)$ as a definite integral.

SOLUTION By FTC I,

$$F(4) - F(1) = \int_1^4 x^2\,dx = \frac{4^3 - 1^3}{3} = 21$$

Therefore $F(4) = F(1) + 21 = 3 + 21 = 24$.

60. Calculate $G(16)$, where $dG/dt = t^{-1/2}$ and $G(9) = -5$.

SOLUTION By FTC I,

$$G(16) - G(9) = \int_9^{16} t^{-1/2}\,dt = 2(16^{1/2}) - 2(9^{1/2}) = 2$$

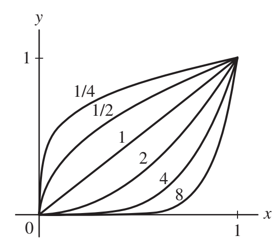
Therefore $G(16) = -5 + 2 = -3$.

61. Does $\int_0^1 x^n\,dx$ get larger or smaller as n increases? Explain graphically.

SOLUTION Let $n \geq 0$ and consider $\int_0^1 x^n\,dx$. (Note: for $n < 0$ the integrand $x^n \to \infty$ as $x \to 0+$, so we exclude this possibility.) Now

$$\int_0^1 x^n\,dx = \left(\frac{1}{n+1}x^{n+1}\right)\bigg|_0^1 = \left(\frac{1}{n+1}(1)^{n+1}\right) - \left(\frac{1}{n+1}(0)^{n+1}\right) = \frac{1}{n+1},$$

which decreases as n increases. Recall that $\int_0^1 x^n\,dx$ represents the area between the positive curve $f(x) = x^n$ and the x-axis over the interval [0, 1]. Accordingly, this area gets smaller as n gets larger. This is readily evident in the following graph, which shows curves for several values of n.



62. Show that the area of the shaded parabolic arch in Figure 8 is equal to four-thirds the area of the triangle shown.

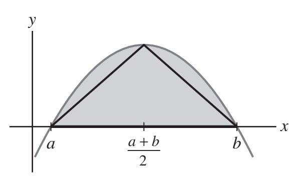


FIGURE 8 Graph of $y = (x - a)(b - x)$.

SOLUTION We first calculate the area of the parabolic arch:

$$\begin{aligned}
\int_a^b (x-a)(b-x)\,dx &= -\int_a^b (x-a)(x-b)\,dx = -\int_a^b (x^2 - ax - bx + ab)\,dx \\
&= -\left(\frac{1}{3}x^3 - \frac{a}{2}x^2 - \frac{b}{2}x^2 + abx\right)\bigg|_a^b \\
&= -\frac{1}{6}\left(2x^3 - 3ax^2 - 3bx^2 + 6abx\right)\bigg|_a^b \\
&= -\frac{1}{6}\left((2b^3 - 3ab^2 - 3b^3 + 6ab^2) - (2a^3 - 3a^3 - 3ba^2 + 6a^2b)\right) \\
&= -\frac{1}{6}\left((-b^3 + 3ab^2) - (-a^3 + 3a^2b)\right) \\
&= -\frac{1}{6}\left(a^3 + 3ab^2 - 3a^2b - b^3\right) = \frac{1}{6}(b-a)^3.
\end{aligned}$$

The indicated triangle has a base of length $b - a$ and a height of

$$\left(\frac{a+b}{2} - a\right)\left(b - \frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^2.$$

Thus, the area of the triangle is

$$\frac{1}{2}(b-a)\left(\frac{b-a}{2}\right)^2 = \frac{1}{8}(b-a)^3.$$

Finally, we note that

$$\frac{1}{6}(b-a)^3 = \frac{4}{3}\cdot\frac{1}{8}(b-a)^3,$$

as required.

Further Insights and Challenges

63. Prove a famous result of Archimedes (generalizing Exercise 62): For $r < s$, the area of the shaded region in Figure 9 is equal to four-thirds the area of triangle $\triangle ACE$, where C is the point on the parabola at which the tangent line is parallel to secant line $\overline{AE}$.

(a) Show that C has x-coordinate $(r+s)/2$.

(b) Show that $ABDE$ has area $(s-r)^3/4$ by viewing it as a parallelogram of height $s-r$ and base of length $\overline{CF}$.

(c) Show that $\triangle ACE$ has area $(s-r)^3/8$ by observing that it has the same base and height as the parallelogram.

(d) Compute the shaded area as the area under the graph minus the area of a trapezoid, and prove Archimedes' result.

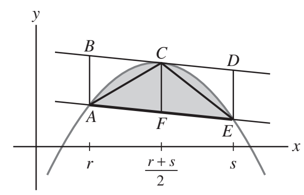


FIGURE 9 Graph of $f(x) = (x-a)(b-x)$.

SOLUTION

(a) The slope of the secant line $\overline{AE}$ is

$$\frac{f(s)-f(r)}{s-r} = \frac{(s-a)(b-s)-(r-a)(b-r)}{s-r} = a+b-(r+s)$$

and the slope of the tangent line along the parabola is

$$f'(x) = a+b-2x.$$

If C is the point on the parabola at which the tangent line is parallel to the secant line $\overline{AE}$, then its x-coordinate must satisfy

$$a+b-2x = a+b-(r+s) \qquad \text{or} \qquad x = \frac{r+s}{2}.$$

(b) Parallelogram $ABDE$ has height $s-r$ and base of length $\overline{CF}$. Since the equation of the secant line $\overline{AE}$ is

$$y = [a+b-(r+s)]\,(x-r) + (r-a)(b-r),$$

the length of the segment $\overline{CF}$ is

$$\left(\frac{r+s}{2}-a\right)\left(b-\frac{r+s}{2}\right) - [a+b-(r+s)]\left(\frac{r+s}{2}-r\right) - (r-a)(b-r) = \frac{(s-r)^2}{4}.$$

Thus, the area of $ABDE$ is $\frac{(s-r)^3}{4}$.

(c) Triangle ACE is comprised of ΔACF and ΔCEF. Each of these smaller triangles has height $\frac{s-r}{2}$ and base of length $\frac{(s-r)^2}{4}$. Thus, the area of ΔACE is

$$\frac{1}{2}\frac{s-r}{2}\cdot\frac{(s-r)^2}{4} + \frac{1}{2}\frac{s-r}{2}\cdot\frac{(s-r)^2}{4} = \frac{(s-r)^3}{8}.$$

(d) The area under the graph of the parabola between $x=r$ and $x=s$ is

$$\begin{aligned}\int_r^s (x-a)(b-x)\,dx &= \left(-abx + \frac{1}{2}(a+b)x^2 - \frac{1}{3}x^3\right)\bigg|_r^s \\ &= -abs + \frac{1}{2}(a+b)s^2 - \frac{1}{3}s^3 + abr - \frac{1}{2}(a+b)r^2 + \frac{1}{3}r^3 \\ &= ab(r-s) + \frac{1}{2}(a+b)(s-r)(s+r) + \frac{1}{3}(r-s)(r^2+rs+s^2),\end{aligned}$$

while the area of the trapezoid under the shaded region is

$$\begin{aligned}&\frac{1}{2}(s-r)\left[(s-a)(b-s)+(r-a)(b-r)\right]\\&\quad=\frac{1}{2}(s-r)\left[-2ab+(a+b)(r+s)-r^2-s^2\right]\\&\quad=ab(r-s)+\frac{1}{2}(a+b)(s-r)(r+s)+\frac{1}{2}(r-s)(r^2+s^2).\end{aligned}$$

Thus, the area of the shaded region is

$$(r-s)\left(\frac{1}{3}r^2+\frac{1}{3}rs+\frac{1}{3}s^2-\frac{1}{2}r^2-\frac{1}{2}s^2\right)=(s-r)\left(\frac{1}{6}r^2-\frac{1}{3}rs+\frac{1}{6}s^2\right)=\frac{1}{6}(s-r)^3,$$

which is four-thirds the area of the triangle ACE.

64. **(a)** Apply the Comparison Theorem (Theorem 5 in Section 5.2) to the inequality $\sin x \le x$ (valid for $x \ge 0$) to prove that

$$1-\frac{x^2}{2}\le \cos x \le 1$$

(b) Apply it again to prove that

$$x-\frac{x^3}{6}\le \sin x \le x \quad (\text{for } x \ge 0)$$

(c) Verify these inequalities for $x = 0.3$.

SOLUTION

(a) We have $\int_0^x \sin t\,dt = -\cos t\Big|_0^x = -\cos x + 1$ and $\int_0^x t\,dt = \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2$. Hence

$$-\cos x + 1 \le \frac{x^2}{2}.$$

Solving, this gives $\cos x \ge 1-\frac{x^2}{2}$. $\cos x \le 1$ follows automatically.

(b) The previous part gives us $1-\frac{t^2}{2}\le \cos t \le 1$, for $t > 0$. Theorem 5 gives us, after integrating over the interval $[0, x]$,

$$x-\frac{x^3}{6}\le \sin x \le x.$$

(c) Substituting $x = 0.3$ into the inequalities obtained in (a) and (b) yields

$$0.955 \le 0.955336489 \le 1 \qquad \text{and} \qquad 0.2955 \le 0.2955202069 \le 0.3,$$

respectively.

65. Use the method of Exercise 64 to prove that

$$\begin{aligned}1-\frac{x^2}{2}&\le \cos x \le 1-\frac{x^2}{2}+\frac{x^4}{24}\\ x-\frac{x^3}{6}&\le \sin x \le x-\frac{x^3}{6}+\frac{x^5}{120} \quad (\text{for } x \ge 0)\end{aligned}$$

Verify these inequalities for $x = 0.1$. Why have we specified $x \ge 0$ for $\sin x$ but not for $\cos x$?

SOLUTION By Exercise 64, $t-\frac{1}{6}t^3 \le \sin t \le t$ for $t > 0$. Integrating this inequality over the interval $[0, x]$, and then solving for $\cos x$, yields:

$$\begin{aligned}\frac{1}{2}x^2-\frac{1}{24}x^4 &\le 1-\cos x \le \frac{1}{2}x^2\\ 1-\frac{1}{2}x^2 &\le \cos x \le 1-\frac{1}{2}x^2+\frac{1}{24}x^4.\end{aligned}$$

These inequalities apply for $x \ge 0$. Since $\cos x$, $1-\frac{x^2}{2}$, and $1-\frac{x^2}{2}+\frac{x^4}{24}$ are all even functions, they also apply for $x \le 0$.

Having established that

$$1-\frac{t^2}{2}\le \cos t\le 1-\frac{t^2}{2}+\frac{t^4}{24},$$

for all $t \ge 0$, we integrate over the interval $[0, x]$, to obtain:

$$x-\frac{x^3}{6}\le \sin x\le x-\frac{x^3}{6}+\frac{x^5}{120}.$$

The functions $\sin x$, $x-\frac{1}{6}x^3$ and $x-\frac{1}{6}x^3+\frac{1}{120}x^5$ are all odd functions, so the inequalities are reversed for $x < 0$.
Evaluating these inequalities at $x = 0.1$ yields

$$0.995000000 \le 0.995004165 \le 0.995004167$$

$$0.0998333333 \le 0.0998334166 \le 0.0998334167,$$

both of which are true.

66. Calculate the next pair of inequalities for $\sin x$ and $\cos x$ by integrating the results of Exercise 65. Can you guess the general pattern?

SOLUTION Integrating

$$t-\frac{t^3}{6}\le \sin t\le t-\frac{t^3}{6}+\frac{t^5}{120} \qquad (\text{for } t\ge 0)$$

over the interval $[0, x]$ yields

$$\frac{x^2}{2}-\frac{x^4}{24}\le 1-\cos x\le \frac{x^2}{2}-\frac{x^4}{24}+\frac{x^6}{720}.$$

Solving for $\cos x$ and yields

$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\le \cos x\le 1-\frac{x^2}{2}+\frac{x^4}{24}.$$

Replacing each x by t and integrating over the interval $[0, x]$ produces

$$x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}\le \sin x\le x-\frac{x^3}{6}+\frac{x^5}{120}.$$

To see the pattern, it is best to compare consecutive inequalities for $\sin x$ and those for $\cos x$:

$$0 \le \sin x \le x$$

$$x-\frac{x^3}{6}\le \sin x\le x$$

$$x-\frac{x^3}{6}\le \sin x\le x-\frac{x^3}{6}+\frac{x^5}{120}.$$

Each iteration adds an additional term. Looking at the highest order terms, we get the following pattern:

$$0$$

$$x$$

$$-\frac{x^3}{6}=-\frac{x^3}{3!}$$

$$\frac{x^5}{5!}$$

We guess that the leading term of the polynomials are of the form

$$(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Similarly, for $\cos x$, the leading terms of the polynomials in the inequality are of the form

$$(-1)^n\frac{x^{2n}}{(2n)!}.$$

67. Use FTC I to prove that if $|f'(x)| \le K$ for $x \in [a, b]$, then $|f(x) - f(a)| \le K|x - a|$ for $x \in [a, b]$.

SOLUTION Let $a > b$ be real numbers, and let $f(x)$ be such that $|f'(x)| \le K$ for $x \in [a, b]$. By FTC,

$$\int_a^x f'(t)\,dt = f(x) - f(a).$$

Since $f'(x) \geq -K$ for all $x \in [a, b]$, we get:

$$f(x) - f(a) = \int_a^x f'(t)\,dt \geq -K(x - a).$$

Since $f'(x) \leq K$ for all $x \in [a, b]$, we get:

$$f(x) - f(a) = \int_a^x f'(t)\,dt \leq K(x - a).$$

Combining these two inequalities yields

$$-K(x - a) \leq f(x) - f(a) \leq K(x - a),$$

so that, by definition,

$$|f(x) - f(a)| \leq K|x - a|.$$

68. **(a)** Use Exercise 67 to prove that $|\sin a - \sin b| \leq |a - b|$ for all a, b.
(b) Let $f(x) = \sin(x + a) - \sin x$. Use part (a) to show that the graph of $f(x)$ lies between the horizontal lines $y = \pm a$.
(c) GU Plot $f(x)$ and the lines $y = \pm a$ to verify (b) for $a = 0.5$ and $a = 0.2$.

SOLUTION

(a) Let $f(x) = \sin x$, so that $f'(x) = \cos x$, and

$$|f'(x)| \leq 1$$

for all x. From Exercise 67, we get:

$$|\sin a - \sin b| \leq |a - b|.$$

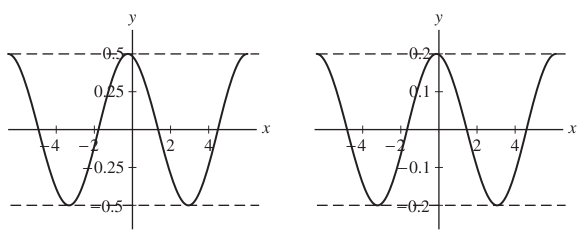
(b) Let $f(x) = \sin(x + a) - \sin(x)$. Applying (a), we get the inequality:

$$|f(x)| = |\sin(x + a) - \sin(x)| \leq |(x + a - x)| = |a|.$$

This is equivalent, by definition, to the two inequalities:

$$-a \leq \sin(x + a) - \sin(x) \leq a.$$

(c) The plots of $y = \sin(x + 0.5) - \sin(x)$ and of $y = \sin(x + 0.2) - \sin(x)$ are shown below. The inequality is satisfied in both plots.



5.4 The Fundamental Theorem of Calculus, Part II

Preliminary Questions

1. Let $G(x) = \int_4^x \sqrt{t^3 + 1}\,dt$.
(a) Is the FTC needed to calculate $G(4)$?
(b) Is the FTC needed to calculate $G'(4)$?

SOLUTION

(a) No. $G(4) = \int_4^4 \sqrt{t^3 + 1}\,dt = 0$.
(b) Yes. By the FTC II, $G'(x) = \sqrt{x^3 + 1}$, so $G'(4) = \sqrt{65}$.

2. Which of the following is an antiderivative $F(x)$ of $f(x) = x^2$ satisfying $F(2) = 0$?

(a) $\int_2^x 2t\,dt$ **(b)** $\int_0^2 t^2\,dt$ **(c)** $\int_2^x t^2\,dt$

SOLUTION The correct answer is **(c)**: $\int_2^x t^2\,dt$.

3. Does every continuous function have an antiderivative? Explain.

SOLUTION Yes. All continuous functions have an antiderivative, namely $\int_a^x f(t)\,dt$.

4. Let $G(x) = \int_4^{x^3} \sin t\,dt$. Which of the following statements are correct?

(a) $G(x)$ is the composite function $\sin(x^3)$.
(b) $G(x)$ is the composite function $A(x^3)$, where

$$A(x) = \int_4^x \sin(t)\,dt$$

(c) $G(x)$ is too complicated to differentiate.
(d) The Product Rule is used to differentiate $G(x)$.
(e) The Chain Rule is used to differentiate $G(x)$.
(f) $G'(x) = 3x^2 \sin(x^3)$.

SOLUTION Statements **(b)**, **(e)**, and **(f)** are correct.

Exercises

1. Write the area function of $f(x) = 2x + 4$ with lower limit $a = -2$ as an integral and find a formula for it.

SOLUTION Let $f(x) = 2x + 4$. The area function with lower limit $a = -2$ is

$$A(x) = \int_a^x f(t)\,dt = \int_{-2}^x (2t + 4)\,dt.$$

Carrying out the integration, we find

$$\int_{-2}^x (2t+4)\,dt = (t^2 + 4t)\Big|_{-2}^x = (x^2 + 4x) - ((-2)^2 + 4(-2)) = x^2 + 4x + 4$$

or $(x+2)^2$. Therefore, $A(x) = (x+2)^2$.

2. Find a formula for the area function of $f(x) = 2x + 4$ with lower limit $a = 0$.

SOLUTION The area function for $f(x) = 2x + 4$ with lower limit $a = 0$ is given by

$$A(x) = \int_0^x (2t+4)\,dt = (t^2 + 4t)\Big|_0^x = x^2 + 4x.$$

3. Let $G(x) = \int_1^x (t^2 - 2)\,dt$. Calculate $G(1)$, $G'(1)$ and $G'(2)$. Then find a formula for $G(x)$.

SOLUTION Let $G(x) = \int_1^x (t^2 - 2)\,dt$. Then $G(1) = \int_1^1 (t^2 - 2)\,dt = 0$. Moreover, $G'(x) = x^2 - 2$, so that $G'(1) = -1$ and $G'(2) = 2$. Finally,

$$G(x) = \int_1^x (t^2 - 2)\,dt = \left(\frac{1}{3}t^3 - 2t\right)\Big|_1^x = \left(\frac{1}{3}x^3 - 2x\right) - \left(\frac{1}{3}(1)^3 - 2(1)\right) = \frac{1}{3}x^3 - 2x + \frac{5}{3}.$$

4. Find $F(0)$, $F'(0)$, and $F'(3)$, where $F(x) = \int_0^x \sqrt{t^2 + t}\,dt$.

SOLUTION By definition, $F(0) = \int_0^0 \sqrt{t^2 + t}\,dt = 0$. By FTC, $F'(x) = \sqrt{x^2 + x}$, so that $F'(0) = \sqrt{0^2 + 0} = 0$ and $F'(3) = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3}$.

5. Find $G(1)$, $G'(0)$, and $G'(\pi/4)$, where $G(x) = \int_1^x \tan t\,dt$.

SOLUTION By definition, $G(1) = \int_1^1 \tan t\,dt = 0$. By FTC, $G'(x) = \tan x$, so that $G'(0) = \tan 0 = 0$ and $G'(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$.

6. Find $H(-2)$ and $H'(-2)$, where $H(x) = \int_{-2}^x \frac{du}{u^2 + 1}$.

SOLUTION By definition, $H(-2) = \int_{-2}^{-2} \frac{du}{u^2+1} = 0$. By FTC, $H'(x) = \frac{1}{x^2+1}$, so $H'(-2) = \frac{1}{5}$.

In Exercises 7–16, find formulas for the functions represented by the integrals.

7. $\int_2^x u^4\,du$

SOLUTION $F(x) = \int_2^x u^4\,du = \frac{1}{5}u^5\Big|_2^x = \frac{1}{5}x^5 - \frac{32}{5}.$

8. $\int_2^x (12t^2 - 8t)\,dt$

SOLUTION $F(x) = \int_2^x (12t^2 - 8t)\,dt = (4t^3 - 4t^2)\Big|_2^x = 4x^3 - 4x^2 - 16.$

9. $\int_0^x \sin u\,du$

SOLUTION $F(x) = \int_0^x \sin u\,du = (-\cos u)\Big|_0^x = 1 - \cos x.$

10. $\int_{-\pi/4}^x \sec^2\theta\,d\theta$

SOLUTION $F(x) = \int_{-\pi/4}^x \sec^2\theta\,d\theta = \tan\theta\Big|_{-\pi/4}^x = \tan x - \tan(-\pi/4) = \tan x + 1.$

11. $\int_4^x e^{3u}\,du$

SOLUTION $F(x) = \int_4^x e^{3u}\,du = \frac{1}{3}e^{3u}\Big|_4^x = \frac{1}{3}e^{3x} - \frac{1}{3}e^{12}.$

12. $\int_x^0 e^{-t}\,dt$

SOLUTION $F(x) = \int_x^0 e^{-t}\,dt = -e^{-t}\Big|_x^0 = -1 + e^{-x}.$

13. $\int_1^{x^2} t\,dt$

SOLUTION $F(x) = \int_1^{x^2} t\,dt = \frac{1}{2}t^2\Big|_1^{x^2} = \frac{1}{2}x^4 - \frac{1}{2}.$

14. $\int_{x/2}^{x/4} \sec^2 u\,du$

SOLUTION $F(x) = \int_{x/2}^{x/4} \sec^2 u\,du = \tan u\Big|_{x/2}^{x/4} = \tan\frac{x}{4} - \tan\frac{x}{2}.$

15. $\int_{3x}^{9x+2} e^{-u}\,du$

SOLUTION $F(x) = \int_{3x}^{9x+2} e^{-u}\,du = -e^{-u}\Big|_{3x}^{9x+2} = -e^{-9x-2} + e^{-3x}.$

16. $\int_2^{\sqrt{x}} \frac{dt}{t}$

SOLUTION $\int_2^{\sqrt{x}} \frac{dt}{t} = \ln|t|\Big|_2^{\sqrt{x}} = \ln\sqrt{x} - \ln 2 = \frac{1}{2}\ln x - \ln 2.$

In Exercises 17–20, express the antiderivative $F(x)$ of $f(x)$ satisfying the given initial condition as an integral.

17. $f(x) = \sqrt{x^3+1}, \quad F(5) = 0$

SOLUTION The antiderivative $F(x)$ of $\sqrt{x^3+1}$ satisfying $F(5) = 0$ is $F(x) = \int_5^x \sqrt{t^3+1}\,dt.$

18. $f(x) = \frac{x+1}{x^2+9}, \quad F(7) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = \frac{x+1}{x^2+9}$ satisfying $F(7) = 0$ is $F(x) = \int_7^x \frac{t+1}{t^2+9}\,dt.$

19. $f(x) = \sec x, \quad F(0) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = \sec x$ satisfying $F(0) = 0$ is $F(x) = \int_0^x \sec t\,dt.$

20. $f(x) = e^{-x^2}, \quad F(-4) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = e^{-x^2}$ satisfying $F(-4) = 0$ is

$$F(x) = \int_{-4}^x e^{-t^2}\,dt.$$

In Exercises 21–24, calculate the derivative.

21. $\dfrac{d}{dx}\displaystyle\int_0^x (t^5 - 9t^3)\,dt$

SOLUTION By FTC II, $\dfrac{d}{dx}\displaystyle\int_0^x (t^5 - 9t^3)\,dt = x^5 - 9x^3$.

22. $\dfrac{d}{d\theta}\displaystyle\int_1^\theta \cot u\,du$

SOLUTION By FTC II, $\dfrac{d}{d\theta}\displaystyle\int_1^\theta \cot u\,du = \cot\theta$.

23. $\dfrac{d}{dt}\displaystyle\int_{100}^t \sec(5x - 9)\,dx$

SOLUTION By FTC II, $\dfrac{d}{dt}\displaystyle\int_{100}^t \sec(5x - 9)\,dx = \sec(5t - 9)$.

24. $\dfrac{d}{ds}\displaystyle\int_{-2}^s \tan\left(\frac{1}{1+u^2}\right)du$

SOLUTION By FTC II, $\dfrac{d}{ds}\displaystyle\int_{-2}^s \tan\left(\frac{1}{1+u^2}\right)du = \tan\left(\frac{1}{1+s^2}\right)$.

25. Let $A(x) = \displaystyle\int_0^x f(t)\,dt$ for $f(x)$ in Figure 8.

(a) Calculate $A(2)$, $A(3)$, $A'(2)$, and $A'(3)$.

(b) Find formulas for $A(x)$ on $[0, 2]$ and $[2, 4]$ and sketch the graph of $A(x)$.

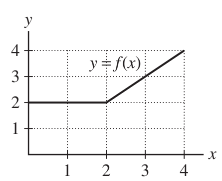


FIGURE 8

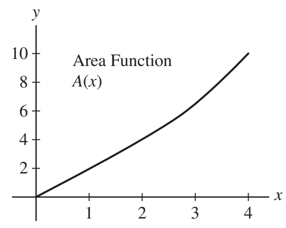
SOLUTION

(a) $A(2) = 2 \cdot 2 = 4$, the area under $f(x)$ from $x = 0$ to $x = 2$, while $A(3) = 2 \cdot 3 + \frac{1}{2} = 6.5$, the area under $f(x)$ from $x = 0$ to $x = 3$. By the FTC, $A'(x) = f(x)$ so $A'(2) = f(2) = 2$ and $A'(3) = f(3) = 3$.

(b) For each $x \in [0, 2]$, the region under the graph of $y = f(x)$ is a rectangle of length x and height 2; for each $x \in [2, 4]$, the region is comprised of a square of side length 2 and a trapezoid of height $x - 2$ and bases 2 and x. Hence,

$$A(x) = \begin{cases} 2x, & 0 \le x < 2 \\ \frac{1}{2}x^2 + 2, & 2 \le x \le 4 \end{cases}$$

A graph of the area function $A(x)$ is shown below.



26. Make a rough sketch of the graph of $A(x) = \displaystyle\int_0^x g(t)\,dt$ for $g(x)$ in Figure 9.

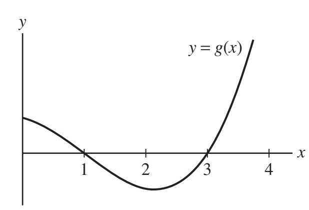
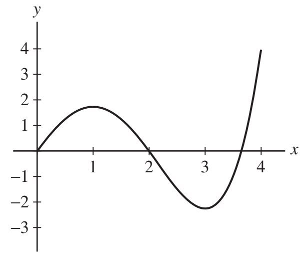


FIGURE 9

SOLUTION The graph of $y = g(x)$ lies above the x-axis over the interval [0, 1], below the x-axis over [1, 3], and above the x-axis over [3, 4]. The corresponding area function should therefore be increasing on (0, 1), decreasing on (1, 3) and increasing on (3, 4). Further, it appears from Figure 9 that the local minimum of the area function at $x = 3$ should be negative. One possible graph of the area function is the following.



27. Verify: $\int_0^x |t|\,dt = \frac{1}{2}x|x|$. *Hint:* Consider $x \geq 0$ and $x \leq 0$ separately.

SOLUTION Let $f(t) = |t| = \begin{cases} t & \text{for } t \geq 0 \\ -t & \text{for } t < 0 \end{cases}$. Then

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \int_0^x t\,dt & \text{for } x \geq 0 \\ \int_0^x -t\,dt & \text{for } x < 0 \end{cases} = \begin{cases} \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2 & \text{for } x \geq 0 \\ \left(-\frac{1}{2}t^2\right)\Big|_0^x = -\frac{1}{2}x^2 & \text{for } x < 0 \end{cases}$$

For $x \geq 0$, we have $F(x) = \frac{1}{2}x^2 = \frac{1}{2}x\,|x|$ since $|x| = x$, while for $x < 0$, we have $F(x) = -\frac{1}{2}x^2 = \frac{1}{2}x\,|x|$ since $|x| = -x$. Therefore, for all real x we have $F(x) = \frac{1}{2}x\,|x|$.

28. Find $G'(1)$, where $G(x) = \int_0^{x^2} \sqrt{t^3 + 3}\,dt$.

SOLUTION By combining the Chain Rule and FTC, $G'(x) = \sqrt{x^6 + 3} \cdot 2x$, so $G'(1) = \sqrt{1 + 3} \cdot 2 = 4$.

In Exercises 29–34, calculate the derivative.

29. $\frac{d}{dx}\int_0^{x^2} \frac{t\,dt}{t+1}$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{dx}\int_0^{x^2} \frac{t\,dt}{t+1} = \frac{x^2}{x^2+1} \cdot 2x = \frac{2x^3}{x^2+1}$.

30. $\frac{d}{dx}\int_1^{1/x} \cos^3 t\,dt$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{dx}\int_1^{1/x} \cos^3 t\,dt = \cos^3\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\cos^3\left(\frac{1}{x}\right)$.

31. $\frac{d}{ds}\int_{-6}^{\cos s} u^4\,du$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{ds}\int_{-6}^{\cos s} u^4\,du = \cos^4 s(-\sin s) = -\cos^4 s \sin s$.

32. $\frac{d}{dx}\int_{x^2}^{x^4} \sqrt{t}\,dt$

Hint for Exercise 32: $F(x) = A(x^4) - A(x^2)$.

SOLUTION Let

$$F(x) = \int_{x^2}^{x^4} \sqrt{t}\,dt = \int_0^{x^4} \sqrt{t}\,dt - \int_0^{x^2} \sqrt{t}\,dt.$$

Applying the Chain Rule combined with FTC, we have

$$F'(x) = \sqrt{x^4} \cdot 4x^3 - \sqrt{x^2} \cdot 2x = 4x^5 - 2x\,|x|.$$

33. $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{x^2} \tan t\,dt$

SOLUTION Let

$$G(x) = \int_{\sqrt{x}}^{x^2} \tan t\,dt = \int_0^{x^2} \tan t\,dt - \int_0^{\sqrt{x}} \tan t\,dt.$$

Applying the Chain Rule combined with FTC twice, we have

$$G'(x) = \tan(x^2)\cdot 2x - \tan(\sqrt{x})\cdot \frac{1}{2}x^{-1/2} = 2x\tan(x^2) - \frac{\tan(\sqrt{x})}{2\sqrt{x}}.$$

34. $\dfrac{d}{du}\displaystyle\int_{-u}^{3u} \sqrt{x^2+1}\,dx$

SOLUTION Let

$$G(x) = \int_{-u}^{3u} \sqrt{x^2+1}\,dx = \int_0^{3u} \sqrt{x^2+1}\,dx - \int_0^{-u} \sqrt{x^2+1}\,dx.$$

Applying the Chain Rule combined with FTC twice, we have

$$G'(x) = 3\sqrt{9u^2+1} + \sqrt{u^2+1}.$$

In Exercises 35–38, with $f(x)$ as in Figure 10 let

$$A(x) = \int_0^x f(t)\,dt \qquad \textit{and} \qquad B(x) = \int_2^x f(t)\,dt.$$

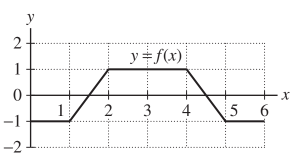


FIGURE 10

35. Find the min and max of $A(x)$ on $[0, 6]$.

SOLUTION The minimum values of $A(x)$ on $[0, 6]$ occur where $A'(x) = f(x)$ goes from negative to positive. This occurs at one place, where $x = 1.5$. The minimum value of $A(x)$ is therefore $A(1.5) = -1.25$. The maximum values of $A(x)$ on $[0, 6]$ occur where $A'(x) = f(x)$ goes from positive to negative. This occurs at one place, where $x = 4.5$. The maximum value of $A(x)$ is therefore $A(4.5) = 1.25$.

36. Find the min and max of $B(x)$ on $[0, 6]$.

SOLUTION The minimum values of $B(x)$ on $[0, 6]$ occur where $B'(x) = f(x)$ goes from negative to positive. This occurs at one place, where $x = 1.5$. The minimum value of $A(x)$ is therefore $B(1.5) = -0.25$. The maximum values of $B(x)$ on $[0, 6]$ occur where $B'(x) = f(x)$ goes from positive to negative. This occurs at one place, where $x = 4.5$. The maximum value of $B(x)$ is therefore $B(4.5) = 2.25$.

37. Find formulas for $A(x)$ and $B(x)$ valid on $[2, 4]$.

SOLUTION On the interval $[2, 4]$, $A'(x) = B'(x) = f(x) = 1$. $A(2) = \displaystyle\int_0^2 f(t)\,dt = -1$ and $B(2) = \displaystyle\int_2^2 f(t)\,dt = 0$. Hence $A(x) = (x-2) - 1$ and $B(x) = (x-2)$.

38. Find formulas for $A(x)$ and $B(x)$ valid on $[4, 5]$.

SOLUTION On the interval $[4, 5]$, $A'(x) = B'(x) = f(x) = -2(x-4.5) = 9-2x$. $A(4) = \displaystyle\int_0^4 f(t)\,dt = 1$ and $B(4) = \displaystyle\int_2^4 f(t)\,dt = 2$. Hence $A(x) = 9x - x^2 - 19$ and $B(x) = 9x - x^2 - 18$.

39. Let $A(x) = \displaystyle\int_0^x f(t)\,dt$, with $f(x)$ as in Figure 11.

(a) Does $A(x)$ have a local maximum at P?

(b) Where does $A(x)$ have a local minimum?

(c) Where does $A(x)$ have a local maximum?
(d) True or false? $A(x) < 0$ for all x in the interval shown.

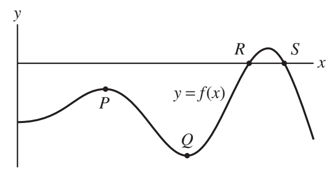


FIGURE 11 Graph of $f(x)$.

SOLUTION

(a) In order for $A(x)$ to have a local maximum, $A'(x) = f(x)$ must transition from positive to negative. As this does not happen at P, $A(x)$ does not have a local maximum at P.
(b) $A(x)$ will have a local minimum when $A'(x) = f(x)$ transitions from negative to positive. This happens at R, so $A(x)$ has a local minimum at R.
(c) $A(x)$ will have a local maximum when $A'(x) = f(x)$ transitions from positive to negative. This happens at S, so $A(x)$ has a local maximum at S.
(d) It is true that $A(x) < 0$ on I since the signed area from 0 to x is clearly always negative from the figure.

40. Determine $f(x)$, assuming that $\int_0^x f(t)\,dt = x^2 + x$.

SOLUTION Let $F(x) = \int_0^x f(t)\,dt = x^2 + x$. Then $F'(x) = f(x) = 2x + 1$.

41. Determine the function $g(x)$ and all values of c such that

$$\int_c^x g(t)\,dt = x^2 + x - 6$$

SOLUTION By the FTC II we have

$$g(x) = \frac{d}{dx}(x^2 + x - 6) = 2x + 1$$

and therefore,

$$\int_c^x g(t)\,dt = x^2 + x - (c^2 + c)$$

We must choose c so that $c^2 + c = 6$. We can take $c = 2$ or $c = -3$.

42. Find $a \le b$ such that $\int_a^b (x^2 - 9)\,dx$ has minimal value.

SOLUTION Let a be given, and let $F_a(x) = \int_a^x (t^2 - 9)\,dt$. Then $F_a'(x) = x^2 - 9$, and the critical points are $x = \pm 3$. Because $F_a''(-3) = -6$ and $F_a''(3) = 6$, we see that $F_a(x)$ has a minimum at $x = 3$. Now, we find a minimizing $\int_a^3 (x^2 - 9)\,dx$. Let $G(x) = \int_x^3 (x^2 - 9)\,dx$. Then $G'(x) = -(x^2 - 9)$, yielding critical points $x = 3$ or $x = -3$. With $x = -3$,

$$G(-3) = \int_{-3}^3 (x^2 - 9)\,dx = \left(\frac{1}{3}x^3 - 9x\right)\Big|_{-3}^3 = -36.$$

With $x = 3$,

$$G(3) = \int_3^3 (x^2 - 9)\,dx = 0.$$

Hence $a = -3$ and $b = 3$ are the values minimizing $\int_a^b (x^2 - 9)\,dx$.

In Exercises 43 and 44, let $A(x) = \int_a^x f(t)\,dt$.

43. **Area Functions and Concavity** Explain why the following statements are true. Assume $f(x)$ is differentiable.
(a) If c is an inflection point of $A(x)$, then $f'(c) = 0$.
(b) $A(x)$ is concave up if $f(x)$ is increasing.
(c) $A(x)$ is concave down if $f(x)$ is decreasing.

SOLUTION

(a) If $x = c$ is an inflection point of $A(x)$, then $A''(c) = f'(c) = 0$.

(b) If $A(x)$ is concave up, then $A''(x) > 0$. Since $A(x)$ is the area function associated with $f(x)$, $A'(x) = f(x)$ by FTC II, so $A''(x) = f'(x)$. Therefore $f'(x) > 0$, so $f(x)$ is increasing.

(c) If $A(x)$ is concave down, then $A''(x) < 0$. Since $A(x)$ is the area function associated with $f(x)$, $A'(x) = f(x)$ by FTC II, so $A''(x) = f'(x)$. Therefore, $f'(x) < 0$ and so $f(x)$ is decreasing.

44. Match the property of $A(x)$ with the corresponding property of the graph of $f(x)$. Assume $f(x)$ is differentiable.

Area function $A(x)$

(a) $A(x)$ is decreasing.

(b) $A(x)$ has a local maximum.

(c) $A(x)$ is concave up.

(d) $A(x)$ goes from concave up to concave down.

Graph of $f(x)$

(i) Lies below the x-axis.

(ii) Crosses the x-axis from positive to negative.

(iii) Has a local maximum.

(iv) $f(x)$ is increasing.

SOLUTION Let $A(x) = \int_a^x f(t)\,dt$ be an area function of $f(x)$. Then $A'(x) = f(x)$ and $A''(x) = f'(x)$.

(a) $A(x)$ is decreasing when $A'(x) = f(x) < 0$, i.e., when $f(x)$ lies below the x-axis. This is choice **(i)**.

(b) $A(x)$ has a local maximum (at x_0) when $A'(x) = f(x)$ changes sign from $+$ to 0 to $-$ as x increases through x_0, i.e., when $f(x)$ crosses the x-axis from positive to negative. This is choice **(ii)**.

(c) $A(x)$ is concave up when $A''(x) = f'(x) > 0$, i.e., when $f(x)$ is increasing. This corresponds to choice **(iv)**.

(d) $A(x)$ goes from concave up to concave down (at x_0) when $A''(x) = f'(x)$ changes sign from $+$ to 0 to $-$ as x increases through x_0, i.e., when $f(x)$ has a local maximum at x_0. This is choice **(iii)**.

45. Let $A(x) = \displaystyle\int_0^x f(t)\,dt$, with $f(x)$ as in Figure 12. Determine:

(a) The intervals on which $A(x)$ is increasing and decreasing

(b) The values x where $A(x)$ has a local min or max

(c) The inflection points of $A(x)$

(d) The intervals where $A(x)$ is concave up or concave down

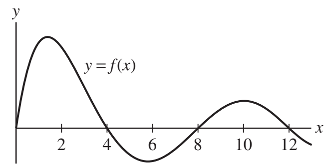


FIGURE 12

SOLUTION

(a) $A(x)$ is increasing when $A'(x) = f(x) > 0$, which corresponds to the intervals $(0, 4)$ and $(8, 12)$. $A(x)$ is decreasing when $A'(x) = f(x) < 0$, which corresponds to the intervals $(4, 8)$ and $(12, \infty)$.

(b) $A(x)$ has a local minimum when $A'(x) = f(x)$ changes from $-$ to $+$, corresponding to $x = 8$. $A(x)$ has a local maximum when $A'(x) = f(x)$ changes from $+$ to $-$, corresponding to $x = 4$ and $x = 12$.

(c) Inflection points of $A(x)$ occur where $A''(x) = f'(x)$ changes sign, or where f changes from increasing to decreasing or vice versa. Consequently, $A(x)$ has inflection points at $x = 2$, $x = 6$, and $x = 10$.

(d) $A(x)$ is concave up when $A''(x) = f'(x)$ is positive or $f(x)$ is increasing, which corresponds to the intervals $(0, 2)$ and $(6, 10)$. Similarly, $A(x)$ is concave down when $f(x)$ is decreasing, which corresponds to the intervals $(2, 6)$ and $(10, \infty)$.

46. Let $f(x) = x^2 - 5x - 6$ and $F(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Find the critical points of $F(x)$ and determine whether they are local minima or local maxima.

(b) Find the points of inflection of $F(x)$ and determine whether the concavity changes from up to down or from down to up.

(c) $\boxed{\text{GU}}$ Plot $f(x)$ and $F(x)$ on the same set of axes and confirm your answers to (a) and (b).

SOLUTION

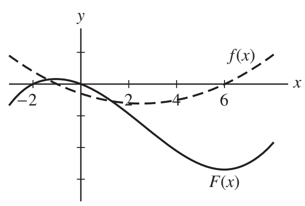
(a) If $F(x) = \int_0^x (t^2 - 5t - 6)\,dt$, then $F'(x) = x^2 - 5x - 6$ and $F''(x) = 2x - 5$. Solving $F'(x) = x^2 - 5x - 6 = 0$ yields critical points $x = -1$ and $x = 6$. Since $F''(-1) = -7 < 0$, there is a local maximum value of F at $x = -1$. Moreover, since $F''(6) = 7 > 0$, there is a local minimum value of F at $x = 6$.

(b) As noted in part (a),

$$F'(x) = x^2 - 5x - 6 \qquad \text{and} \qquad F''(x) = 2x - 5.$$

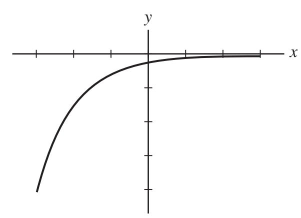
A candidate point of inflection occurs where $F''(x) = 2x - 5 = 0$. Thus $x = \frac{5}{2}$. $F''(x)$ changes from negative to positive at this point, so there is a point of inflection at $x = \frac{5}{2}$ and concavity changes from down to up.

(c) From the graph below, we clearly note that $F(x)$ has a local maximum at $x = -1$, a local minimum at $x = 6$ and a point of inflection at $x = \frac{5}{2}$.



47. Sketch the graph of an increasing function $f(x)$ such that both $f'(x)$ and $A(x) = \int_0^x f(t)\,dt$ are decreasing.

SOLUTION If $f'(x)$ is decreasing, then $f''(x)$ must be negative. Furthermore, if $A(x) = \displaystyle\int_0^x f(t)\,dt$ is decreasing, then $A'(x) = f(x)$ must also be negative. Thus, we need a function which is negative but increasing and concave down. The graph of one such function is shown below.



48. Figure 13 shows the graph of $f(x) = x \sin x$. Let $F(x) = \displaystyle\int_0^x t \sin t\,dt$.

(a) Locate the local max and absolute max of $F(x)$ on $[0, 3\pi]$.

(b) Justify graphically: $F(x)$ has precisely one zero in $[\pi, 2\pi]$.

(c) How many zeros does $F(x)$ have in $[0, 3\pi]$?

(d) Find the inflection points of $F(x)$ on $[0, 3\pi]$. For each one, state whether the concavity changes from up to down or from down to up.

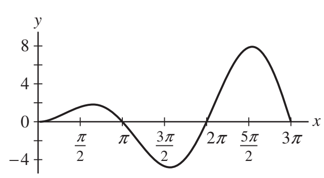


FIGURE 13 Graph of $f(x) = x \sin x$.

SOLUTION Let $F(x) = \int_0^x t \sin t\,dt$. A graph of $f(x) = x \sin x$ is depicted in Figure 13. Note that $F'(x) = f(x)$ and $F''(x) = f'(x)$.

(a) For F to have a local maximum at $x_0 \in (0, 3\pi)$ we must have $F'(x_0) = f(x_0) = 0$ and $F' = f$ must change sign from $+$ to 0 to $-$ as x increases through x_0. This occurs at $x = \pi$. The absolute maximum of $F(x)$ on $[0, 3\pi]$ occurs at $x = 3\pi$ since (from the figure) the signed area between $x = 0$ and $x = c$ is greatest for $x = c = 3\pi$.

(b) At $x = \pi$, the value of F is positive since $f(x) > 0$ on $(0, \pi)$. As x increases along the interval $[\pi, 2\pi]$, we see that F decreases as the negatively signed area accumulates. Eventually the additional negatively signed area "outweighs" the prior positively signed area and F attains the value 0, say at $b \in (\pi, 2\pi)$. Thereafter, on $(b, 2\pi)$, we see that f is negative and thus F becomes and continues to be negative as the negatively signed area accumulates. Therefore, $F(x)$ takes the value 0 exactly once in the interval $[\pi, 2\pi]$.

(c) $F(x)$ has two zeroes in $[0, 3\pi]$. One is described in part (b) and the other must occur in the interval $[2\pi, 3\pi]$ because $F(x) < 0$ at $x = 2\pi$ but clearly the positively signed area over $[2\pi, 3\pi]$ is greater than the previous negatively signed area.

(d) Since f is differentiable, we have that F is twice differentiable on I. Thus $F(x)$ has an inflection point at x_0 provided $F''(x_0) = f'(x_0) = 0$ *and* $F''(x) = f'(x)$ changes sign at x_0. If $F'' = f'$ changes sign from $+$ to 0 to $-$ at x_0, then f has a local maximum at x_0. There is clearly such a value x_0 in the figure in the interval $[\pi/2, \pi]$ and another around $5\pi/2$. Accordingly, F has two inflection points where $F(x)$ changes from concave up to concave down. If $F'' = f'$ changes sign from $-$ to 0 to $+$ at x_0, then f has a local minimum at x_0. From the figure, there is such an x_0 around $3\pi/2$; so F has one inflection point where $F(x)$ changes from concave down to concave up.

49. GU Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2})\,dt$$

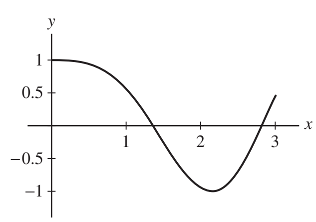
and determine whether it is a local min or max. Then find the smallest positive inflection point of $F(x)$ and use a graph of $y = \cos(x^{3/2})$ to determine whether the concavity changes from up to down or from down to up.

SOLUTION A critical point of $F(x)$ occurs where $F'(x) = \cos(x^{3/2}) = 0$. The smallest positive critical points occurs where $x^{3/2} = \pi/2$, so that $x = (\pi/2)^{2/3}$. $F'(x)$ goes from positive to negative at this point, so $x = (\pi/2)^{2/3}$ corresponds to a local maximum..

Candidate inflection points of $F(x)$ occur where $F''(x) = 0$. By FTC, $F'(x) = \cos(x^{3/2})$, so $F''(x) = -(3/2)x^{1/2}\sin(x^{3/2})$. Finding the smallest positive solution of $F''(x) = 0$, we get:

$$\begin{aligned} -(3/2)x^{1/2}\sin(x^{3/2}) &= 0 \\ \sin(x^{3/2}) &= 0 \qquad \text{(since } x > 0\text{)} \\ x^{3/2} &= \pi \\ x &= \pi^{2/3} \approx 2.14503. \end{aligned}$$

From the plot below, we see that $F'(x) = \cos(x^{3/2})$ changes from decreasing to increasing at $\pi^{2/3}$, so $F(x)$ changes from concave down to concave up at that point.



Further Insights and Challenges

50. Proof of FTC II The proof in the text assumes that $f(x)$ is increasing. To prove it for all continuous functions, let $m(h)$ and $M(h)$ denote the *minimum* and *maximum* of $f(t)$ on $[x, x+h]$ (Figure 14). The continuity of $f(x)$ implies that $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. Show that for $h > 0$,

$$hm(h) \le A(x+h) - A(x) \le hM(h)$$

For $h < 0$, the inequalities are reversed. Prove that $A'(x) = f(x)$.

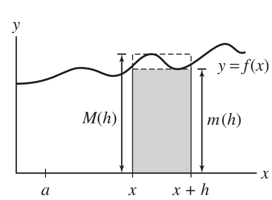


FIGURE 14 Graphical interpretation of $A(x+h) - A(x)$.

SOLUTION Let $f(x)$ be continuous on $[a, b]$. For $h > 0$, let $m(h)$ and $M(h)$ denote the minimum and maximum values of f on $[x, x+h]$. Since f is continuous, we have $\lim_{h\to 0+} m(h) = \lim_{h\to 0+} M(h) = f(x)$. If $h > 0$, then since $m(h) \le f(x) \le M(h)$ on $[x, x+h]$, we have

$$hm(h) = \int_x^{x+h} m(h)\,dt \le \int_x^{x+h} f(t)\,dt = A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M(h)\,dt = hM(h).$$

In other words, $hm(h) \le A(x+h) - A(x) \le hM(h)$. Since $h > 0$, it follows that $m(h) \le \dfrac{A(x+h) - A(x)}{h} \le M(h)$. Letting $h \to 0+$ yields

$$f(x) \le \lim_{h\to 0+} \frac{A(x+h) - A(x)}{h} \le f(x),$$

whence

$$\lim_{h\to 0+} \frac{A(x+h) - A(x)}{h} = f(x)$$

by the Squeeze Theorem. If $h < 0$, then

$$-hm(h) = \int_{x+h}^{x} m(h)\,dt \le \int_{x+h}^{x} f(t)\,dt = A(x) - A(x+h) = \int_{x+h}^{x} f(t)\,dt \le \int_{x+h}^{x} M(h)\,dt = -hM(h).$$

Since $h < 0$, we have $-h > 0$ and thus

$$m(h) \le \frac{A(x) - A(x+h)}{-h} \le M(h)$$

or

$$m(h) \le \frac{A(x+h) - A(x)}{h} \le M(h).$$

Letting $h \to 0-$ gives

$$f(x) \le \lim_{h\to 0-} \frac{A(x+h) - A(x)}{h} \le f(x),$$

so that

$$\lim_{h\to 0-} \frac{A(x+h) - A(x)}{h} = f(x)$$

by the Squeeze Theorem. Since the one-sided limits agree, we therefore have

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

51. Proof of FTC I FTC I asserts that $\int_a^b f(t)\,dt = F(b) - F(a)$ if $F'(x) = f(x)$. Use FTC II to give a new proof of FTC I as follows. Set $A(x) = \int_a^x f(t)\,dt$.

(a) Show that $F(x) = A(x) + C$ for some constant.

(b) Show that $F(b) - F(a) = A(b) - A(a) = \displaystyle\int_a^b f(t)\,dt$.

SOLUTION Let $F'(x) = f(x)$ and $A(x) = \int_a^x f(t)\,dt$.

(a) Then by the FTC, Part II, $A'(x) = f(x)$ and thus $A(x)$ and $F(x)$ are both antiderivatives of $f(x)$. Hence $F(x) = A(x) + C$ for some constant C.

(b)

$$\begin{aligned} F(b) - F(a) &= (A(b) + C) - (A(a) + C) = A(b) - A(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt \end{aligned}$$

which proves the FTC, Part I.

52. Can Every Antiderivative Be Expressed as an Integral? The area function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ for every value of a. However, not all antiderivatives are obtained in this way. The general antiderivative of $f(x) = x$ is $F(x) = \frac{1}{2}x^2 + C$. Show that $F(x)$ is an area function if $C \le 0$ but not if $C > 0$.

SOLUTION Let $f(x) = x$. The general antiderivative of $f(x)$ is $F(x) = \frac{1}{2}x^2 + C$. Let $A(x) = \int_a^x f(t)\,dt = \int_a^x t\,dt = \frac{1}{2}t^2\Big|_a^x = \frac{1}{2}x^2 - \frac{1}{2}a^2$ be an area function of $f(x) = x$. To express $F(x)$ as an area function, we must find a value for a such that $\frac{1}{2}x^2 - \frac{1}{2}a^2 = \frac{1}{2}x^2 + C$, whence $a = \pm\sqrt{-2C}$. If $C \le 0$, then $-2C \ge 0$ and we may choose either $a = \sqrt{-2C}$ or $a = -\sqrt{-2C}$. However, if $C > 0$, then there is no real solution for a and $F(x)$ cannot be expressed as an area function.

53. Prove the formula

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x)$$

SOLUTION Write

$$\int_{u(x)}^{v(x)} f(x)\,dx = \int_{u(x)}^{0} f(x)\,dx + \int_{0}^{v(x)} f(x)\,dx = \int_{0}^{v(x)} f(x)\,dx - \int_{0}^{u(x)} f(x)\,dx.$$

Then, by the Chain Rule and the FTC,

$$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(x)\,dx &= \frac{d}{dx}\int_{0}^{v(x)} f(x)\,dx - \frac{d}{dx}\int_{0}^{u(x)} f(x)\,dx \\ &= f(v(x))v'(x) - f(u(x))u'(x).\end{aligned}$$

54. Use the result of Exercise 53 to calculate

$$\frac{d}{dx}\int_{\ln x}^{e^x} \sin t\,dt$$

SOLUTION By Exercise 53,

$$\frac{d}{dx}\int_{\ln x}^{e^x} \sin t\,dt = e^x \sin e^x - \frac{1}{x}\sin \ln x.$$

5.5 Net Change as the Integral of a Rate

Preliminary Questions

1. A hot metal object is submerged in cold water. The rate at which the object cools (in degrees per minute) is a function $f(t)$ of time. Which quantity is represented by the integral $\int_0^T f(t)\,dt$?

SOLUTION The definite integral $\int_0^T f(t)\,dt$ represents the total drop in temperature of the metal object in the first T minutes after being submerged in the cold water.

2. A plane travels 560 km from Los Angeles to San Francisco in 1 hour. If the plane's velocity at time t is $v(t)$ km/h, what is the value of $\int_0^1 v(t)\,dt$?

SOLUTION The definite integral $\int_0^1 v(t)\,dt$ represents the total distance traveled by the airplane during the one hour flight from Los Angeles to San Francisco. Therefore the value of $\int_0^1 v(t)\,dt$ is 560 km.

3. Which of the following quantities would be naturally represented as derivatives and which as integrals?

(a) Velocity of a train

(b) Rainfall during a 6-month period

(c) Mileage per gallon of an automobile

(d) Increase in the U.S. population from 1990 to 2010

SOLUTION Quantities **(a)** and **(c)** involve rates of change, so these would naturally be represented as derivatives. Quantities **(b)** and **(d)** involve an accumulation, so these would naturally be represented as integrals.

Exercises

1. Water flows into an empty reservoir at a rate of $3000 + 20t$ liters per hour. What is the quantity of water in the reservoir after 5 hours?

SOLUTION The quantity of water in the reservoir after five hours is

$$\int_0^5 (3000 + 20t)\,dt = \left(3000t + 10t^2\right)\Big|_0^5 = 15{,}250 \text{ gallons}.$$

2. A population of insects increases at a rate of $200 + 10t + 0.25t^2$ insects per day. Find the insect population after 3 days, assuming that there are 35 insects at $t = 0$.

SOLUTION The increase in the insect population over three days is

$$\int_0^3 \left(200 + 10t + \frac{1}{4}t^2\right)dt = \left(200t + 5t^2 + \frac{1}{12}t^3\right)\Big|_0^3 = \frac{2589}{4} = 647.25.$$

Accordingly, the population after 3 days is $35 + 647.25 = 682.25$ or 682 insects.

3. A survey shows that a mayoral candidate is gaining votes at a rate of $2000t + 1000$ votes per day, where t is the number of days since she announced her candidacy. How many supporters will the candidate have after 60 days, assuming that she had no supporters at $t = 0$?

SOLUTION The number of supporters the candidate has after 60 days is

$$\int_0^{60} (2000t + 1000)\,dt = (1000t^2 + 1000t)\Big|_0^{60} = 3{,}660{,}000.$$

4. A factory produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week. How many bicycles were produced from the beginning of week 2 to the end of week 3?

SOLUTION The rate of production is $r(t) = 95 + 3t^2 - t$ bicycles per week and the period from the beginning of week 2 to the end of week 3 corresponds to the second and third weeks of production. Accordingly, the number of bikes produced from the beginning of week 2 to the end of week 3 is

$$\int_1^3 r(t)\,dt = \int_1^3 \left(95 + 3t^2 - t\right)\,dt = \left(95t + t^3 - \frac{1}{2}t^2\right)\Big|_1^3 = 212$$

bicycles.

5. Find the displacement of a particle moving in a straight line with velocity $v(t) = 4t - 3$ m/s over the time interval $[2, 5]$.

SOLUTION The displacement is given by

$$\int_2^5 (4t - 3)\,dt = (2t^2 - 3t)\Big|_2^5 = (50 - 15) - (8 - 6) = 33\text{m}.$$

6. Find the displacement over the time interval $[1, 6]$ of a helicopter whose (vertical) velocity at time t is $v(t) = 0.02t^2 + t$ m/s.

SOLUTION Given $v(t) = \frac{1}{50}t^2 + t$ m/s, the change in height over $[1, 6]$ is

$$\int_1^6 v(t)\,dt = \int_1^6 \left(\frac{1}{50}t^2 + t\right)\,dt = \left(\frac{1}{150}t^3 + \frac{1}{2}t^2\right)\Big|_1^6 = \frac{284}{15} \approx 18.93 \text{ m}.$$

7. A cat falls from a tree (with zero initial velocity) at time $t = 0$. How far does the cat fall between $t = 0.5$ and $t = 1$ s? Use Galileo's formula $v(t) = -9.8t$ m/s.

SOLUTION Given $v(t) = -9.8t$ m/s, the total distance the cat falls during the interval $[\frac{1}{2}, 1]$ is

$$\int_{1/2}^1 |v(t)|\,dt = \int_{1/2}^1 9.8t\,dt = 4.9t^2\Big|_{1/2}^1 = 4.9 - 1.225 = 3.675 \text{ m}.$$

8. A projectile is released with an initial (vertical) velocity of 100 m/s. Use the formula $v(t) = 100 - 9.8t$ for velocity to determine the distance traveled during the first 15 seconds.

SOLUTION The distance traveled is given by

$$\begin{aligned}\int_0^{15} |100 - 9.8t|\,dt &= \int_0^{100/9.8} (100 - 9.8t)\,dt + \int_{100/9.8}^{15} (9.8t - 100)\,dt \\ &= \left(100t - 4.9t^2\right)\Big|_0^{100/9.8} + \left(4.9t^2 - 100t\right)\Big|_{100/9.8}^{15} \approx 622.9 \text{ m}.\end{aligned}$$

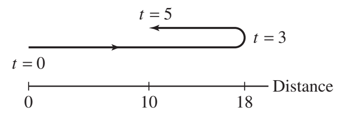
In Exercises 9–12, a particle moves in a straight line with the given velocity (in m/s). Find the displacement and distance traveled over the time interval, and draw a motion diagram like Figure 3 (with distance and time labels).

9. $v(t) = 12 - 4t, \quad [0, 5]$

SOLUTION Displacement is given by $\int_0^5 (12 - 4t)\,dt = (12t - 2t^2)\Big|_0^5 = 10$ ft, while total distance is given by

$$\int_0^5 |12 - 4t|\,dt = \int_0^3 (12 - 4t)\,dt + \int_3^5 (4t - 12)\,dt = (12t - 2t^2)\Big|_0^3 + (2t^2 - 12t)\Big|_3^5 = 26 \text{ ft}.$$

The displacement diagram is given here.



10. $v(t) = 36 - 24t + 3t^2, \quad [0, 10]$

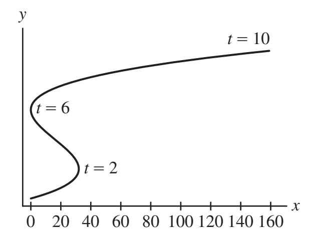
SOLUTION Let $v(t) = 36 - 24t + 3t^2 = 3(t-2)(t-6)$. Displacement is given by

$$\int_0^{10} (36 - 24t + 3t^2)\, dt = (36t - 12t^2 + t^3)\Big|_0^{10} = 160$$

meters. Total distance traveled is given by

$$\begin{aligned}\int_0^{10} |36 - 24t + 3t^2|\, dt &= \int_0^2 (36 - 24t + 3t^2)\, dt + \int_2^6 (24t - 36 - 3t^2)\, dt + \int_6^{10} (36 - 24t + 3t^2)\, dt \\ &= (36t - 12t^2 + t^3)\Big|_0^2 + (12t^2 - 36t - t^3)\Big|_0^{10} + (36t - 12t^2 + t^3)\Big|_6^{10} \\ &= 224 \text{ meters.}\end{aligned}$$

The displacement diagram is given here.

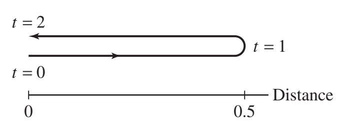


11. $v(t) = t^{-2} - 1, \quad [0.5, 2]$

SOLUTION Displacement is given by $\int_{0.5}^{2} (t^{-2} - 1)\, dt = (-t^{-1} - t)\Big|_{0.5}^{2} = 0$ m, while total distance is given by

$$\int_{0.5}^{2} \left|t^{-2} - 1\right|\, dt = \int_{0.5}^{1} (t^{-2} - 1)\, dt + \int_1^2 (1 - t^{-2})\, dt = (-t^{-1} - t)\Big|_{0.5}^{1} + (t + t^{-1})\Big|_1^2 = 1 \text{ m.}$$

The displacement diagram is given here.

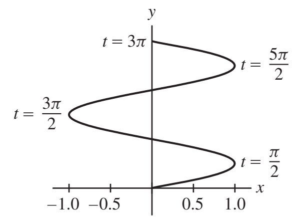


12. $v(t) = \cos t, \quad [0, 3\pi]$

SOLUTION Displacement is given by $\int_0^{3\pi} \cos t\, dt = \sin t\Big|_0^{3\pi} = 0$ meters, while the total distance traveled is given by

$$\begin{aligned}\int_0^{3\pi} |\cos t|\, dt &= \int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^{3\pi/2} \cos t\, dt + \int_{3\pi/2}^{5\pi/2} \cos t\, dt - \int_{5\pi/2}^{3\pi} \cos t\ , dt \\ &= \sin t\Big|_0^{\pi/2} - \sin t\Big|_{\pi/2}^{3\pi/2} + \sin t\Big|_{3\pi/2}^{5\pi/2} - \sin t\Big|_{5\pi/2}^{3\pi} \\ &= 6 \text{ meters.}\end{aligned}$$

The displacement diagram is given here.



13. Find the net change in velocity over [1, 4] of an object with $a(t) = 8t - t^2$ m/s^2.

SOLUTION The net change in velocity is

$$\int_1^4 a(t)\,dt = \int_1^4 (8t - t^2)\,dt = \left(4t^2 - \frac{1}{3}t^3\right)\Big|_1^4 = 39 \text{ m/s}.$$

14. Show that if acceleration is constant, then the change in velocity is proportional to the length of the time interval.

SOLUTION Let $a(t) = a$ be the constant acceleration. Let $v(t)$ be the velocity. Let $[t_1, t_2]$ be the time interval concerned. We know that $v'(t) = a$, and, by FTC,

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a\,dt = a(t_2 - t_1),$$

So the net change in velocity is proportional to the length of the time interval with constant of proportionality a.

15. The traffic flow rate past a certain point on a highway is $q(t) = 3000 + 2000t - 300t^2$ (t in hours), where $t = 0$ is 8 AM. How many cars pass by in the time interval from 8 to 10 AM?

SOLUTION The number of cars is given by

$$\int_0^2 q(t)\,dt = \int_0^2 (3000 + 2000t - 300t^2)\,dt = \left(3000t + 1000t^2 - 100t^3\right)\Big|_0^2$$
$$= 3000(2) + 1000(4) - 100(8) = 9200 \text{ cars}.$$

16. The marginal cost of producing x tablet computers is $C'(x) = 120 - 0.06x + 0.00001x^2$ What is the cost of producing 3000 units if the setup cost is \$90,000? If production is set at 3000 units, what is the cost of producing 200 additional units?

SOLUTION The production coot for producing 3000 units is

$$\int_0^{3000} (120 - 0.06x + 0.00001x^2)\,dx = \left(120x - 0.03x^2 + \frac{1}{3}0.00001x^3\right)\Big|_0^{3000}$$
$$= 360{,}000 - 270{,}000 + 90{,}000 = 180{,}000$$

dollars. Adding in the setup cost, we find the total cost of producing 3000 units is \$270,000. If production is set at 3000 units, the cost of producing an additional 200 units is

$$\int_{3000}^{3200} (120 - 0.06x + 0.00001x^2)\,dx = \left(120x - 0.03x^2 + \frac{1}{3}0.00001x^3\right)\Big|_{3000}^{3200}$$
$$= 384{,}000 - 307{,}200 + 109{,}226.67 - 180{,}000$$

or \$6026.67.

17. A small boutique produces wool sweaters at a marginal cost of $40 - 5[[x/5]]$ for $0 \le x \le 20$, where $[[x]]$ is the greatest integer function. Find the cost of producing 20 sweaters. Then compute the average cost of the first 10 sweaters and the last 10 sweaters.

SOLUTION The total cost of producing 20 sweaters is

$$\int_0^{20} (40 - 5[[x/5]])\,dx = \int_0^5 40\,dx + \int_5^{10} 35\,dx + \int_{10}^{15} 30\,dx + \int_{15}^{20} 25\,dx$$
$$= 40(5) + 35(5) + 30(5) + 25(5) = 650 \text{ dollars}.$$

From this calculation, we see that the cost of the first 10 sweaters is \$375 and the cost of the last ten sweaters is \$275; thus, the average cost of the first ten sweaters is \$37.50 and the average cost of the last ten sweaters is \$27.50.

18. The rate (in liters per minute) at which water drains from a tank is recorded at half-minute intervals. Compute the average of the left- and right-endpoint approximations to estimate the total amount of water drained during the first 3 minutes.

t (min)	0	0.5	1	1.5	2	2.5	3
r (l/min)	50	48	46	44	42	40	38

SOLUTION Let $\Delta t = 0.5$. Then

$$R_N = 0.5(48 + 46 + 44 + 42 + 40 + 38) = 129.0 \text{ liters}$$
$$L_N = 0.5(50 + 48 + 46 + 44 + 42 + 40) = 135.0 \text{ liters}$$

The average of R_N and L_N is $\frac{1}{2}(129 + 135) = 132$ liters.

19. The velocity of a car is recorded at half-second intervals (in feet per second). Use the average of the left- and right-endpoint approximations to estimate the total distance traveled during the first 4 seconds.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	0	12	20	29	38	44	32	35	30

SOLUTION Let $\Delta t = .5$. Then

$$R_N = 0.5 \cdot (12 + 20 + 29 + 38 + 44 + 32 + 35 + 30) = 120 \text{ ft.}$$
$$L_N = 0.5 \cdot (0 + 12 + 20 + 29 + 38 + 44 + 32 + 35) = 105 \text{ ft.}$$

The average of R_N and L_N is 112.5 ft.

20. To model the effects of a **carbon tax** on CO_2 emissions, policymakers study the *marginal cost of abatement* $B(x)$, defined as the cost of increasing CO_2 reduction from x to $x + 1$ tons (in units of ten thousand tons—Figure 4). Which quantity is represented by the area under the curve over [0, 3] in Figure 4?

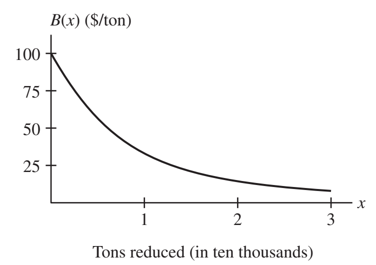


FIGURE 4 Marginal cost of abatement $B(x)$.

SOLUTION The area under the curve over [0, 3] represents the total cost of reducing the amount of CO_2 released into the atmosphere by 3 tons.

21. A megawatt of power is 10^6 W, or 3.6×10^9 J/hour. Which quantity is represented by the area under the graph in Figure 5? Estimate the energy (in joules) consumed during the period 4 PM to 8 PM.

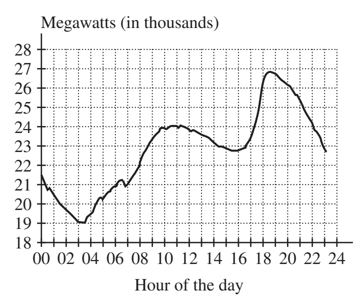


FIGURE 5 Power consumption over 1-day period in California (February 2010).

SOLUTION The area under the graph in Figure 5 represents the total power consumption over one day in California. Assuming $t = 0$ corresponds to midnight, the period 4 PM to 8 PM corresponds to $t = 16$ to $t = 20$. The left and right endpoint approximations are

$$L = 1(22.8 + 23.5 + 26.1 + 26.7) = 99.1 \text{megawatt} \cdot \text{hours}$$
$$R = 1(23.5 + 26.1 + 26.7 + 26.1) = 102.4 \text{megawatt} \cdot \text{hours}$$

The average of these values is

$$100.75 \text{megawatt} \cdot \text{hours} = 3.627 \times 10^{11} \text{ joules.}$$

22. Figure 6 shows the migration rate $M(t)$ of Ireland in the period 1988–1998. This is the rate at which people (in thousands per year) move into or out of the country.

(a) Is the following integral positive or negative? What does this quantity represent?

$$\int_{1988}^{1998} M(t)\,dt$$

(b) Did migration in the period 1988–1998 result in a net influx of people into Ireland or a net outflow of people from Ireland?

(c) During which two years could the Irish prime minister announce, "We've hit an inflection point. We are still losing population, but the trend is now improving."

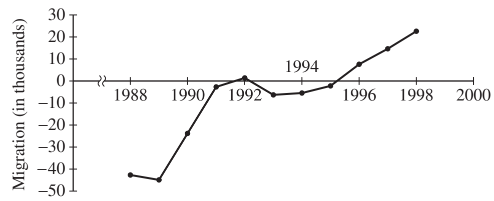


FIGURE 6 Irish migration rate (in thousands per year).

SOLUTION

(a) Because there appears to be more area below the t-axis than above in Figure 6,

$$\int_{1988}^{1998} M(t)\,dt$$

is negative. This quantity represents the net migration from Ireland during the period 1988–1998.

(b) As noted in part (a), there appears to be more area below the t-axis than above in Figure 6, so migration in the period 1988–1998 resulted in a net outflow of people from Ireland.

(c) The prime minister can make this statement when the graph of M is at a local minimum, which appears to be in the years 1989 and 1993.

23. Let $N(d)$ be the number of asteroids of diameter $\le d$ kilometers. Data suggest that the diameters are distributed according to a piecewise power law:

$$N'(d) = \begin{cases} 1.9 \times 10^9 d^{-2.3} & \text{for } d < 70 \\ 2.6 \times 10^{12} d^{-4} & \text{for } d \ge 70 \end{cases}$$

(a) Compute the number of asteroids with diameter between 0.1 and 100 km.

(b) Using the approximation $N(d+1) - N(d) \approx N'(d)$, estimate the number of asteroids of diameter 50 km.

SOLUTION

(a) The number of asteroids with diameter between 0.1 and 100 km is

$$\begin{aligned} \int_{0.1}^{100} N'(d)\,dd &= \int_{0.1}^{70} 1.9 \times 10^9 d^{-2.3}\,dd + \int_{70}^{100} 2.6 \times 10^{12} d^{-4}\,dd \\ &= -\frac{1.9 \times 10^9}{1.3} d^{-1.3}\bigg|_{0.1}^{70} - \frac{2.6 \times 10^{12}}{3} d^{-3}\bigg|_{70}^{100} \\ &= 2.916 \times 10^{10} + 1.66 \times 10^6 \approx 2.916 \times 10^{10}. \end{aligned}$$

(b) Taking $d = 49.5$,

$$N(50.5) - N(49.5) \approx N'(49.5) = 1.9 \times 10^9 49.5^{-2.3} = 240{,}525.79.$$

Thus, there are approximately 240,526 asteroids of diameter 50 km.

24. Heat Capacity The heat capacity $C(T)$ of a substance is the amount of energy (in joules) required to raise the temperature of 1 g by 1°C at temperature T.

(a) Explain why the energy required to raise the temperature from T_1 to T_2 is the area under the graph of $C(T)$ over $[T_1, T_2]$.

(b) How much energy is required to raise the temperature from 50 to 100°C if $C(T) = 6 + 0.2\sqrt{T}$?

SOLUTION

(a) Since $C(T)$ is the energy required to raise the temperature of one gram of a substance by one degree when its temperature is T, the total energy required to raise the temperature from T_1 to T_2 is given by the definite integral $\int_{T_1}^{T_2} C(T)\,dT$. As $C(T) > 0$, the definite integral also represents the area under the graph of $C(T)$.

(b) If $C(T) = 6 + .2\sqrt{T} = 6 + \frac{1}{5}T^{1/2}$, then the energy required to raise the temperature from 50°C to 100°C is $\int_{50}^{100} C(T)\,dT$ or

$$\int_{50}^{100}\left(6+\frac{1}{5}T^{1/2}\right)dT = \left(6T + \frac{2}{15}T^{3/2}\right)\Big|_{50}^{100} = \left(6(100) + \frac{2}{15}(100)^{3/2}\right) - \left(6(50) + \frac{2}{15}(50)^{3/2}\right)$$

$$= \frac{1300 - 100\sqrt{2}}{3} \approx 386.19 \text{ Joules}$$

25. Figure 7 shows the rate $R(t)$ of natural gas consumption (in billions of cubic feet per day) in the mid-Atlantic states (New York, New Jersey, Pennsylvania). Express the total quantity of natural gas consumed in 2009 as an integral (with respect to time t in days). Then estimate this quantity, given the following monthly values of $R(t)$:

3.18, 2.86, 2.39, 1.49, 1.08, 0.80,
1.01, 0.89, 0.89, 1.20, 1.64, 2.52

Keep in mind that the number of days in a month varies with the month.

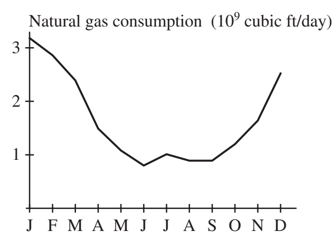


FIGURE 7 Natural gas consumption in 2009 in the mid-Atlantic states

SOLUTION The total quantity of natural gas consumed is given by

$$\int_0^{365} R(t)\,dt.$$

With the given data, we find

$$\int_0^{365} R(t)\,dt \approx 31(3.18) + 28(2.86) + 31(2.39) + 30(1.49) + 31(1.08) + 30(0.80)$$
$$+31(1.01) + 31(0.89) + 30(0.89) + 31(1.20) + 30(1.64) + 31(2.52)$$
$$= 605.05 \text{ billion cubic feet.}$$

26. Cardiac output is the rate R of volume of blood pumped by the heart per unit time (in liters per minute). Doctors measure R by injecting A mg of dye into a vein leading into the heart at $t = 0$ and recording the concentration $c(t)$ of dye (in milligrams per liter) pumped out at short regular time intervals (Figure 8).

(a) Explain: The quantity of dye pumped out in a small time interval $[t, t + \Delta t]$ is approximately $Rc(t)\Delta t$.

(b) Show that $A = R\int_0^T c(t)\,dt$, where T is large enough that all of the dye is pumped through the heart but not so large that the dye returns by recirculation.

(c) Assume $A = 5$ mg. Estimate R using the following values of $c(t)$ recorded at 1-second intervals from $t = 0$ to $t = 10$:

0, 0.4, 2.8, 6.5, 9.8, 8.9,
6.1, 4, 2.3, 1.1, 0

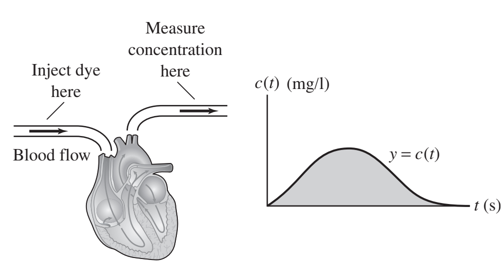


FIGURE 8

SOLUTION

(a) Over a short time interval, $c(t)$ is nearly constant. $Rc(t)$ is the rate of volume of dye (amount of fluid $\times$ concentration of dye in fluid) flowing out of the heart (in mg per minute). Over the short time interval $[t, t + \Delta t]$, the rate of flow of dye is approximately constant at $Rc(t)$ mg/minute. Therefore, the flow of dye over the interval is approximately $Rc(t)\Delta t$ mg.

(b) The rate of flow of dye is $Rc(t)$. Therefore the net flow between time $t = 0$ and time $t = T$ is

$$\int_0^T Rc(t)\,dt = R\int_0^T c(t)\,dt.$$

If T is great enough that all of the dye is pumped through the heart, the net flow is equal to all of the dye, so

$$A = R\int_0^T c(t)\,dt.$$

(c) In the table, $\Delta t = \frac{1}{60}$ minute, and $N = 10$. The right and left hand approximations of $\int_0^T c(t)\,dt$ are:

$$R_{10} = \frac{1}{60}(0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1 + 0) = 0.6983\ \frac{\text{mg}\cdot\text{minute}}{\text{liter}}$$
$$L_{10} = \frac{1}{60}(0 + 0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1) = 0.6983\ \frac{\text{mg}\cdot\text{minute}}{\text{liter}}$$

Both L_N and R_N are the same, so the average of L_N and R_N is 0.6983. Hence,

$$A = R\int_0^T c(t)dt$$
$$5\text{ mg} = R\left(0.6983\ \frac{\text{mg}\cdot\text{minute}}{\text{liter}}\right)$$
$$R = \frac{5}{0.6983}\ \frac{\text{liters}}{\text{minute}} = 7.16\ \frac{\text{liters}}{\text{minute}}.$$

Exercises 27 and 28: A study suggests that the extinction rate $r(t)$ of marine animal families during the Phanerozoic Eon can be modeled by the function $r(t) = 3130/(t + 262)$ for $0 \le t \le 544$, where t is time elapsed (in millions of years) since the beginning of the eon 544 million years ago. Thus, $t = 544$ refers to the present time, $t = 540$ is 4 million years ago, and so on.

27. Compute the average of R_N and L_N with $N = 5$ to estimate the total number of families that became extinct in the periods $100 \le t \le 150$ and $350 \le t \le 400$.

SOLUTION

- $(100 \le t \le 150)$ For $N = 5$,

$$\Delta t = \frac{150 - 100}{5} = 10.$$

  The table of values $\{r(t_i)\}_{i=0\ldots5}$ is given below:

t_i	100	110	120	130	140	150
$r(t_i)$	8.64641	8.41398	8.19372	7.98469	7.78607	7.59709

  The endpoint approximations are:

$$R_N = 10(8.41398 + 8.19372 + 7.98469 + 7.78607 + 7.59709) \approx 399.756 \text{ families}$$
$$L_N = 10(8.64641 + 8.41398 + 8.19372 + 7.98469 + 7.78607) \approx 410.249 \text{ families}$$

  The right endpoint approximation estimates 399.756 families became extinct in the period $100 \le t \le 150$, the left endpoint approximation estimates 410.249 families became extinct during this time. The average of the two is 405.362 families.

- $(350 \le t \le 400)$ For $N = 10$,

$$\Delta t = \frac{400 - 350}{5} = 19.$$

The table of values $\{r(t_i)\}_{i=0\ldots5}$ is given below:

t_i	350	360	370	380	390	400
$r(t_i)$	5.11438	5.03215	4.95253	4.87539	4.80061	4.72810

The endpoint approximations are:

$$R_N = 10(5.03215 + 4.95253 + 4.87539 + 4.80061 + 4.72810) \approx 243.888 \text{ families}$$

$$L_N = 10(5.11438 + 5.03215 + 4.95253 + 4.87539 + 4.80061) \approx 247.751 \text{ families}$$

The right endpoint approximation estimates 243.888 families became extinct in the period $350 \le t \le 400$, the left endpoint approximation estimates 247.751 families became extinct during this time. The average of the two is 245.820 families.

28. CAS Estimate the total number of extinct families from $t = 0$ to the present, using M_N with $N = 544$.

SOLUTION We are estimating

$$\int_0^{544} \frac{3130}{(t+262)}\, dt$$

using M_N with $N = 544$. If $N = 544$, $\Delta t = \dfrac{544-0}{544} = 1$ and $\{t_i^*\}_{i=1,\ldots N} = i\Delta t - (\Delta t/2) = i - \frac{1}{2}$.

$$M_N = \Delta t \sum_{i=1}^{N} r(t_i^*) = 1 \cdot \sum_{i=1}^{544} \frac{3130}{261.5+i} = 3517.3021.$$

Thus, we estimate that 3517 families have become extinct over the past 544 million years.

Further Insights and Challenges

29. Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-2}$, will never pass the point $x = 2$.

SOLUTION The particle's velocity is $v(t) = s'(t) = t^{-2}$, an antiderivative for which is $F(t) = -t^{-1}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)\, du = F(u)\Big|_1^t = F(t) - F(1) = 1 - \frac{1}{t} < 1$$

for all $t \ge 1$. Thus, the particle will never pass $x = 1$, which implies it will never pass $x = 2$ either.

30. Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-1/2}$ moves arbitrarily far from the origin after sufficient time has elapsed.

SOLUTION The particle's velocity is $v(t) = s'(t) = t^{-1/2}$, an antiderivative for which is $F(t) = 2t^{1/2}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)\, du = F(u)\Big|_1^t = F(t) - F(1) = 2\sqrt{t} - 1$$

for all $t \ge 1$. Let $S > 0$ denote an arbitrarily large distance from the origin. We see that for

$$t > \left(\frac{S+1}{2}\right)^2,$$

the particle will be more than S units from the origin. In other words, the particle moves arbitrarily far from the origin after sufficient time has elapsed.

5.6 Substitution Method

Preliminary Questions

1. Which of the following integrals is a candidate for the Substitution Method?

(a) $\int 5x^4 \sin(x^5)\, dx$ **(b)** $\int \sin^5 x \cos x\, dx$ **(c)** $\int x^5 \sin x\, dx$

SOLUTION The function in **(c)**: $x^5 \sin x$ is not of the form $g(u(x))u'(x)$. The function in **(a)** meets the prescribed pattern with $g(u) = \sin u$ and $u(x) = x^5$. Similarly, the function in **(b)** meets the prescribed pattern with $g(u) = u^5$ and $u(x) = \sin x$.

2. Find an appropriate choice of u for evaluating the following integrals by substitution:

(a) $\int x(x^2+9)^4\,dx$ **(b)** $\int x^2 \sin(x^3)\,dx$ **(c)** $\int \sin x \cos^2 x\,dx$

SOLUTION

(a) $x(x^2+9)^4 = \frac{1}{2}(2x)(x^2+9)^4$; hence, $c = \frac{1}{2}$, $f(u) = u^4$, and $u(x) = x^2+9$.

(b) $x^2\sin(x^3) = \frac{1}{3}(3x^2)\sin(x^3)$; hence, $c = \frac{1}{3}$, $f(u) = \sin u$, and $u(x) = x^3$.

(c) $\sin x \cos^2 x = -(-\sin x)\cos^2 x$; hence, $c = -1$, $f(u) = u^2$, and $u(x) = \cos x$.

3. Which of the following is equal to $\int_0^2 x^2(x^3+1)\,dx$ for a suitable substitution?

(a) $\frac{1}{3}\int_0^2 u\,du$ **(b)** $\int_0^9 u\,du$ **(c)** $\frac{1}{3}\int_1^9 u\,du$

SOLUTION With the substitution $u = x^3+1$, the definite integral $\int_0^2 x^2(x^3+1)\,dx$ becomes $\frac{1}{3}\int_1^9 u\,du$. The correct answer is **(c)**.

Exercises

In Exercises 1–6, calculate du.

1. $u = x^3 - x^2$

SOLUTION Let $u = x^3 - x^2$. Then $du = (3x^2 - 2x)\,dx$.

2. $u = 2x^4 + 8x^{-1}$

SOLUTION Let $u = 2x^4 + 8x^{-1}$. Then $du = (8x^3 - 8x^{-2})\,dx$.

3. $u = \cos(x^2)$

SOLUTION Let $u = \cos(x^2)$. Then $du = -\sin(x^2)\cdot 2x\,dx = -2x\sin(x^2)\,dx$.

4. $u = \tan x$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\,dx$.

5. $u = e^{4x+1}$

SOLUTION Let $u = e^{4x+1}$. Then $du = 4e^{4x+1}\,dx$.

6. $u = \ln(x^4+1)$

SOLUTION Let $u = \ln(x^4+1)$. Then $du = \dfrac{4x^3}{x^4+1}\,dx$.

In Exercises 7–22, write the integral in terms of u and du. Then evaluate.

7. $\int (x-7)^3\,dx, \quad u = x-7$

SOLUTION Let $u = x - 7$. Then $du = dx$. Hence

$$\int (x-7)^3\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(x-7)^4 + C.$$

8. $\int (x+25)^{-2}\,dx, \quad u = x+25$

SOLUTION Let $u = x+25$. Then $du = dx$ and

$$\int (x+25)^{-2}\,dx = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x+25} + C.$$

9. $\int t\sqrt{t^2+1}\,dt, \quad u = t^2+1$

SOLUTION Let $u = t^2+1$. Then $du = 2t\,dt$. Hence,

$$\int t\sqrt{t^2+1}\,dt = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(t^2+1)^{3/2} + C.$$

10. $\int (x^3+1)\cos(x^4+4x)\,dx, \quad u = x^4+4x$

SOLUTION Let $u = x^4+4x$. Then $du = (4x^3+4)\,dx = 4(x^3+1)\,dx$ and

$$\int (x^3+1)\cos(x^4+4x)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+4x) + C.$$

11. $\int \frac{t^3}{(4-2t^4)^{11}}\,dt, \quad u = 4-2t^4$

SOLUTION Let $u = 4-2t^4$. Then $du = -8t^3\,dt$. Hence,

$$\int \frac{t^3}{(4-2t^4)^{11}}\,dt = -\frac{1}{8}\int u^{-11}\,du = \frac{1}{80}u^{-10} + C = \frac{1}{80}(4-2t^4)^{-10} + C.$$

12. $\int \sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $du = 4\,dx$ or $\frac{1}{4}du = dx$. Hence

$$\int \sqrt{4u-1}\,dx = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{6}(4x-1)^{3/2} + C.$$

13. $\int x(x+1)^9\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\begin{aligned}\int x(x+1)^9\,dx &= \int (u-1)u^9\,du = \int (u^{10}-u^9)\,du\\ &= \frac{1}{11}u^{11} - \frac{1}{10}u^{10} + C = \frac{1}{11}(x+1)^{11} - \frac{1}{10}(x+1)^{10} + C.\end{aligned}$$

14. $\int x\sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $x = \frac{1}{4}(u+1)$ and $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Hence,

$$\begin{aligned}\int x\sqrt{4x-1}\,dx &= \frac{1}{16}\int (u+1)u^{1/2}\,du = \frac{1}{16}\int (u^{3/2}+u^{1/2})\,du\\ &= \frac{1}{16}\left(\frac{2}{5}u^{5/2}\right) + \frac{1}{16}\left(\frac{2}{3}u^{3/2}\right) + C\\ &= \frac{1}{40}(4x-1)^{5/2} + \frac{1}{24}(4x-1)^{3/2} + C.\end{aligned}$$

15. $\int x^2\sqrt{x+1}\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\begin{aligned}\int x^2\sqrt{x+1}\,dx &= \int (u-1)^2u^{1/2}\,du = \int (u^{5/2}-2u^{3/2}+u^{1/2})\,du\\ &= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C\\ &= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

16. $\int \sin(4\theta-7)\,d\theta, \quad u = 4\theta-7$

SOLUTION Let $u = 4\theta-7$. Then $du = 4\,d\theta$ and

$$\int \sin(4\theta-7)\,d\theta = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(4\theta-7) + C.$$

17. $\int \sin^2\theta\cos\theta\,d\theta, \quad u = \sin\theta$

SOLUTION Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Hence,

$$\int \sin^2\theta\cos\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C.$$

18. $\int \sec^2 x\tan x\,dx, \quad u = \tan x$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\,dx$. Hence

$$\int \sec^2 x\tan x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

19. $\int xe^{-x^2}\,dx, \quad u = -x^2$

SOLUTION Let $u = -x^2$. Then $du = -2x\,dx$ or $-\frac{1}{2}\,du = x\,dx$. Hence,

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

20. $\int (\sec^2 t)e^{\tan t}\,dt, \quad u = \tan t$

SOLUTION Let $u = \tan t$. Then $du = \sec^2 t\,dt$ and

$$\int (\sec^2 t)e^{\tan t}\,dt = \int e^u\,du = e^u + C = e^{\tan t} + C.$$

21. $\int \frac{(\ln x)^2\,dx}{x}, \quad u = \ln x$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

22. $\int \frac{(\tan^{-1}x)^2\,dx}{x^2+1}, \quad u = \tan^{-1}x$

SOLUTION Let $u = \tan^{-1}x$. Then $du = \frac{1}{1+x^2}\,dx$, and

$$\int \frac{(\tan^{-1}x)^2}{x^2+1}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\tan^{-1}x)^3 + C.$$

In Exercises 23–26, evaluate the integral in the form $a\sin(u(x)) + C$ for an appropriate choice of $u(x)$ and constant a.

23. $\int x^3\cos(x^4)\,dx$

SOLUTION Let $u = x^4$. Then $du = 4x^3\,dx$ or $\frac{1}{4}\,du = x^3 dx$. Hence

$$\int x^3\cos(x^4)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4) + C.$$

24. $\int x^2\cos(x^3+1)\,dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ or $\frac{1}{3}\,du = x^2\,dx$. Hence

$$\int x^2\cos(x^3+1)\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C.$$

25. $\int x^{1/2}\cos(x^{3/2})\,dx$

SOLUTION Let $u = x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\,dx$ or $\frac{2}{3}\,du = x^{1/2}\,dx$. Hence

$$\int x^{1/2}\cos(x^{3/2})\,dx = \frac{2}{3}\int \cos u\,du = \frac{2}{3}\sin u + C = \frac{2}{3}\sin(x^{3/2}) + C.$$

26. $\int \cos x \cos(\sin x)\, dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int \cos x \cos(\sin x)\, dx = \int \cos u\, du = \sin u + C.$$

In Exercises 27–72, evaluate the indefinite integral.

27. $\int (4x+5)^9\, dx$

SOLUTION Let $u = 4x+5$. Then $du = 4\, dx$ and

$$\int (4x+5)^9\, dx = \frac{1}{4}\int u^9\, du = \frac{1}{40}u^{10} + C = \frac{1}{40}(4x+5)^{10} + C.$$

28. $\int \frac{dx}{(x-9)^5}$

SOLUTION Let $u = x-9$. Then $du = dx$ and

$$\int \frac{dx}{(x-9)^5} = \int u^{-5}\, du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4(x-9)^4} + C.$$

29. $\int \frac{dt}{\sqrt{t+12}}$

SOLUTION Let $u = t+12$. Then $du = dt$ and

$$\int \frac{dt}{\sqrt{t+12}} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{t+12} + C.$$

30. $\int (9t+2)^{2/3}\, dt$

SOLUTION Let $u = 9t+2$. Then $du = 9\, dt$ and

$$\int (9t+2)^{2/3}\, dt = \frac{1}{9}\int u^{2/3}\, du = \frac{1}{9}\cdot\frac{3}{5}u^{5/3} + C = \frac{1}{15}(9t+2)^{5/3} + C.$$

31. $\int \frac{x+1}{(x^2+2x)^3}\, dx$

SOLUTION Let $u = x^2+2x$. Then $du = (2x+2)\, dx$ or $\frac{1}{2}du = (x+1)\, dx$. Hence

$$\int \frac{x+1}{(x^2+2x)^3}\, dx = \frac{1}{2}\int \frac{1}{u^3}\, du = \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right) + C = -\frac{1}{4}(x^2+2x)^{-2} + C = \frac{-1}{4(x^2+2x)^2} + C.$$

32. $\int (x+1)(x^2+2x)^{3/4}\, dx$

SOLUTION Let $u = x^2+2x$. Then $du = (2x+2)\, dx = 2(x+1)\, dx$ and

$$\begin{aligned}\int (x+1)(x^2+2x)^{3/4}\, dx &= \frac{1}{2}\int u^{3/4}\, du = \frac{1}{2}\cdot\frac{4}{7}u^{7/4} + C\\ &= \frac{2}{7}(x^2+2x)^{7/4} + C.\end{aligned}$$

33. $\int \frac{x}{\sqrt{x^2+9}}\, dx$

SOLUTION Let $u = x^2+9$. Then $du = 2x\, dx$ or $\frac{1}{2}du = x\, dx$. Hence

$$\int \frac{x}{\sqrt{x^2+9}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \frac{1}{2}\frac{\sqrt{u}}{\frac{1}{2}} + C = \sqrt{x^2+9} + C.$$

34. $\int \frac{2x^2+x}{(4x^3+3x^2)^2}\,dx$

SOLUTION Let $u = 4x^3+3x^2$. Then $du = (12x^2+6x)\,dx$ or $\frac{1}{6}du = (2x^2+x)\,dx$. Hence

$$\int (4x^3+3x^2)^{-2}(2x^2+x)\,dx = \frac{1}{6}\int u^{-2}\,du = -\frac{1}{6}u^{-1} + C = -\frac{1}{6}(4x^3+3x^2)^{-1} + C.$$

35. $\int (3x^2+1)(x^3+x)^2\,dx$

SOLUTION Let $u = x^3+x$. Then $du = (3x^2+1)\,dx$. Hence

$$\int (3x^2+1)(x^3+x)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^3+x)^3 + C.$$

36. $\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx$

SOLUTION Let $u = x^5+x^2$. Then $du = (5x^4+2x)\,dx$. Hence

$$\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2}\frac{1}{u^2} + C = -\frac{1}{2}\frac{1}{(x^5+x^2)^2} + C.$$

37. $\int (3x+8)^{11}\,dx$

SOLUTION Let $u = 3x+8$. Then $du = 3\,dx$ and

$$\int (3x+8)^{11}\,dx = \frac{1}{3}\int u^{11}\,du = \frac{1}{36}u^{12} + C = \frac{1}{36}(3x+8)^{12} + C.$$

38. $\int x(3x+8)^{11}\,dx$

SOLUTION Let $u = 3x+8$. Then $du = 3\,dx$, $x = \frac{u-8}{3}$, and

$$\begin{aligned}\int x(3x+8)^{11}\,dx &= \frac{1}{9}\int (u-8)u^{11}\,du = \frac{1}{9}\int (u^{12}-8u^{11})\,du\\ &= \frac{1}{9}\left(\frac{1}{13}u^{13} - \frac{2}{3}u^{12}\right) + C\\ &= \frac{1}{117}(3x+8)^{13} - \frac{2}{27}(3x+8)^{12} + C.\end{aligned}$$

39. $\int x^2\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3+1$. Then $du = 3x^2\,dx$ and

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int u^{1/2}\,du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

40. $\int x^5\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3+1$. Then $du = 3x^2\,dx$, $x^3 = u-1$ and

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= \frac{1}{3}\int (u-1)\sqrt{u}\,du = \frac{1}{3}\int (u^{3/2}-u^{1/2})\,du\\ &= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C.\end{aligned}$$

41. $\int \frac{dx}{(x+5)^3}$

SOLUTION Let $u = x+5$. Then $du = dx$ and

$$\int \frac{dx}{(x+5)^3} = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2}(x+5)^{-2} + C.$$

42. $\int \frac{x^2\,dx}{(x+5)^3}$

SOLUTION Let $u = x + 5$. Then $du = dx$, $x = u - 5$ and

$$\int \frac{x^2\,dx}{(x+5)^3} = \int \frac{(u-5)^2}{u^3}\,du = \int (u^{-1} - 10u^{-2} + 25u^{-3})\,du$$
$$= \ln|u| + 10u^{-1} - \frac{25}{2}u^{-2} + C$$
$$= \ln|x+5| + \frac{10}{x+5} - \frac{25}{2(x+5)^2} + C.$$

43. $\int z^2(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and

$$\int z^2(z^3+1)^{12}\,dz = \frac{1}{3}\int u^{12}\,du = \frac{1}{39}u^{13} + C = \frac{1}{39}(z^3+1)^{13} + C.$$

44. $\int (z^5 + 4z^2)(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$, $z^3 = u - 1$ and

$$\int (z^5+4z^2)(z^3+1)^{12}\,dz = \frac{1}{3}\int (u+3)u^{12}\,du = \frac{1}{3}\int (u^{13} + 3u^{12})\,du$$
$$= \frac{1}{3}\left(\frac{1}{14}u^{14} + \frac{3}{13}u^{13}\right) + C$$
$$= \frac{1}{42}(z^3+1)^{14} + \frac{1}{13}(z^3+1)^{13} + C.$$

45. $\int (x+2)(x+1)^{1/4}\,dx$

SOLUTION Let $u = x + 1$. Then $x = u - 1$, $du = dx$ and

$$\int (x+2)(x+1)^{1/4}\,dx = \int (u+1)u^{1/4}\,du = \int (u^{5/4} + u^{1/4})\,du$$
$$= \frac{4}{9}u^{9/4} + \frac{4}{5}u^{5/4} + C$$
$$= \frac{4}{9}(x+1)^{9/4} + \frac{4}{5}(x+1)^{5/4} + C.$$

46. $\int x^3(x^2-1)^{3/2}\,dx$

SOLUTION Let $u = x^2 - 1$. Then $u + 1 = x^2$ and $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int x^3(x^2-1)^{3/2}\,dx = \int x^2 \cdot x(x^2-1)^{3/2}\,dx$$
$$= \frac{1}{2}\int (u+1)u^{3/2}\,du = \frac{1}{2}\int (u^{5/2} + u^{3/2})\,du$$
$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2}\right) + \frac{1}{2}\left(\frac{2}{5}u^{5/2}\right) + C = \frac{1}{7}(x^2-1)^{7/2} + \frac{1}{5}(x^2-1)^{5/2} + C.$$

47. $\int \sin(8 - 3\theta)\,d\theta$

SOLUTION Let $u = 8 - 3\theta$. Then $du = -3\,d\theta$ and

$$\int \sin(8-3\theta)\,d\theta = -\frac{1}{3}\int \sin u\,du = \frac{1}{3}\cos u + C = \frac{1}{3}\cos(8-3\theta) + C.$$

48. $\int \theta \sin(\theta^2)\, d\theta$

SOLUTION Let $u = \theta^2$. Then $du = 2\theta\, d\theta$ and

$$\int \theta \sin(\theta^2)\, d\theta = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(\theta^2) + C.$$

49. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt$

SOLUTION Let $u = \sqrt{t} = t^{1/2}$. Then $du = \frac{1}{2}t^{-1/2}\, dt$ and

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt = 2\int \cos u\, du = 2\sin u + C = 2\sin\sqrt{t} + C.$$

50. $\int x^2 \sin(x^3 + 1)\, dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\, dx$ or $\frac{1}{3}du = x^2\, dx$. Hence

$$\int x^2 \sin(x^3 + 1)\, dx = \frac{1}{3}\int \sin u\, du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3 + 1) + C.$$

51. $\int \tan(4\theta + 9)\, d\theta$

SOLUTION Let $u = 4\theta + 9$. Then $du = 4\, d\theta$ and

$$\int \tan(4\theta + 9)\, d\theta = \frac{1}{4}\int \tan u\, du = \frac{1}{4}\ln|\sec u| + C = \frac{1}{4}\ln|\sec(4\theta + 9)| + C.$$

52. $\int \sin^8\theta \cos\theta\, d\theta$

SOLUTION Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \sin^8\theta \cos\theta\, d\theta = \int u^8\, du = \frac{1}{9}u^9 + C = \frac{1}{9}\sin^9\theta + C.$$

53. $\int \cot x\, dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$, and

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$$

54. $\int x^{-1/5} \tan x^{4/5}\, dx$

SOLUTION Let $u = x^{4/5}$. Then $du = \frac{4}{5}x^{-1/5}\, dx$ and

$$\int x^{-1/5} \tan x^{4/5}\, dx = \frac{5}{4}\int \tan u\, du = \frac{5}{4}\ln|\sec u| + C = \frac{5}{4}\ln|\sec x^{4/5}| + C.$$

55. $\int \sec^2(4x + 9)\, dx$

SOLUTION Let $u = 4x + 9$. Then $du = 4\, dx$ or $\frac{1}{4}\, du = dx$. Hence

$$\int \sec^2(4x + 9)\, dx = \frac{1}{4}\int \sec^2 u\, du = \frac{1}{4}\tan u + C = \frac{1}{4}\tan(4x + 9) + C.$$

56. $\int \sec^2 x \tan^4 x\, dx$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\, dx$. Hence

$$\int \sec^2 x \tan^4 x\, dx = \int u^4\, du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C.$$

57. $\displaystyle\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$. Hence,

$$\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}} = 2\int \sec^2 u\,dx = 2\tan u + C = 2\tan(\sqrt{x}) + C.$$

58. $\displaystyle\int \frac{\cos 2x}{(1+\sin 2x)^2}\,dx$

SOLUTION Let $u = 1 + \sin 2x$. Then $du = 2\cos 2x$ or $\frac{1}{2}du = \cos 2x\,dx$. Hence

$$\int (1+\sin 2x)^{-2}\cos 2x\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}(1+\sin 2x)^{-1} + C.$$

59. $\displaystyle\int \sin 4x\sqrt{\cos 4x + 1}\,dx$

SOLUTION Let $u = \cos 4x + 1$. Then $du = -4\sin 4x$ or $-\frac{1}{4}du = \sin 4x$. Hence

$$\int \sin 4x\sqrt{\cos 4x + 1}\,dx = -\frac{1}{4}\int u^{1/2}\,du = -\frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{6}(\cos 4x + 1)^{3/2} + C.$$

60. $\displaystyle\int \cos x(3\sin x - 1)\,dx$

SOLUTION Let $u = 3\sin x - 1$. Then $du = 3\cos x\,dx$ or $\frac{1}{3}du = \cos x\,dx$. Hence

$$\int (3\sin x - 1)\cos x\,dx = \frac{1}{3}\int u\,du = \frac{1}{3}\left(\frac{1}{2}u^2\right) + C = \frac{1}{6}(3\sin x - 1)^2 + C.$$

61. $\displaystyle\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta$

SOLUTION Let $u = \sec\theta - 1$. Then $du = \sec\theta\tan\theta\,d\theta$ and

$$\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec\theta - 1)^2 + C.$$

62. $\displaystyle\int \cos t\cos(\sin t)\,dt$

SOLUTION Let $u = \sin t$. Then $du = \cos t\,dt$ and

$$\int \cos t\cos(\sin t)\,dt = \int \cos u\,du = \sin u + C = \sin(\sin t) + C.$$

63. $\displaystyle\int e^{14x-7}\,dx$

SOLUTION Let $u = 14x - 7$. Then $du = 14\,dx$ or $\frac{1}{14}\,du = dx$. Hence,

$$\int e^{14x-7}\,dx = \frac{1}{14}\int e^u\,du = \frac{1}{14}e^u + C = \frac{1}{14}e^{14x-7} + C.$$

64. $\displaystyle\int (x+1)e^{x^2+2x}\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx$ or $\frac{1}{2}\,du = (x+1)\,dx$. Hence,

$$\int (x+1)e^{x^2+2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+2x} + C.$$

65. $\displaystyle\int \frac{e^x\,dx}{(e^x+1)^4}$

SOLUTION Let $u = e^x + 1$. Then $du = e^x\,dx$, and

$$\int \frac{e^x}{(e^x+1)^4}\,dx = \int u^{-4}\,du = -\frac{1}{3u^3} + C = -\frac{1}{3(e^x+1)^3} + C.$$

66. $\int (\sec^2\theta)\, e^{\tan\theta}\, d\theta$

SOLUTION Let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$, and

$$\int (\sec^2\theta)\, e^{\tan\theta}\, d\theta = \int e^u\, du = e^u + C = e^{\tan\theta} + C.$$

67. $\int \frac{e^t\, dt}{e^{2t} + 2e^t + 1}$

SOLUTION Let $u = e^t$. Then $du = e^t\, dt$, and

$$\int \frac{e^t\, dt}{e^{2t} + 2e^t + 1} = \int \frac{du}{u^2 + 2u + 1} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1} + C = -\frac{1}{e^t + 1} + C.$$

68. $\int \frac{dx}{x(\ln x)^2}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{dx}{x(\ln x)^2} = \int u^{-2}\, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

69. $\int \frac{(\ln x)^4\, dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{(\ln x)^4}{x}\, dx = \int u^4\, du = \frac{1}{5}u^5 + C = \frac{1}{5}(\ln x)^5 + C.$$

70. $\int \frac{dx}{x \ln x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

71. $\int \frac{\tan(\ln x)}{x}\, dx$

SOLUTION Let $u = \cos(\ln x)$. Then $du = -\frac{1}{x}\sin(\ln x)\, dx$ or $-du = \frac{1}{x}\sin(\ln x)\, dx$. Hence,

$$\int \frac{\tan(\ln x)}{x}\, dx = \int \frac{\sin(\ln x)}{x\cos(\ln x)}\, dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(\ln x)| + C.$$

72. $\int (\cot x)\ln(\sin x)\, dx$

SOLUTION Let $u = \ln(\sin x)$. Then

$$du = \frac{1}{\sin x}\cos x = \cot x,$$

and

$$\int (\cot x)\ln(\sin x)\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(\sin x))^2 + C.$$

73. Evaluate $\int \frac{dx}{(1+\sqrt{x})^3}$ using $u = 1 + \sqrt{x}$. *Hint:* Show that $dx = 2(u-1)du$.

SOLUTION Let $u = 1 + \sqrt{x}$. Then

$$du = \frac{1}{2\sqrt{x}}\, dx \quad \text{or} \quad dx = 2\sqrt{x}\, du = 2(u-1)\, du.$$

Hence,

$$\int \frac{dx}{(1+\sqrt{x})^3} = 2\int \frac{u-1}{u^3}\,du = 2\int (u^{-2} - u^{-3})\,du$$

$$= -2u^{-1} + u^{-2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C.$$

74. Can They Both Be Right? Hannah uses the substitution $u = \tan x$ and Akiva uses $u = \sec x$ to evaluate $\int \tan x \sec^2 x\,dx$. Show that they obtain different answers, and explain the apparent contradiction.

SOLUTION With the substitution $u = \tan x$, Hannah finds $du = \sec^2 x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\tan^2 x + C_1.$$

On the other hand, with the substitution $u = \sec x$, Akiva finds $du = \sec x \tan x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int \sec x(\tan x \sec x)\,dx = \frac{1}{2}\sec^2 x + C_2$$

Hannah and Akiva have each found a correct antiderivative. To resolve what appears to be a contradiction, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true in their case, note that

$$\left(\frac{1}{2}\sec^2 x + C_2\right) - \left(\frac{1}{2}\tan^2 x + C_1\right) = \frac{1}{2}(\sec^2 x - \tan^2 x) + C_2 - C_1$$

$$= \frac{1}{2}(1) + C_2 - C_1 = \frac{1}{2} + C_2 - C_1, \text{ a constant}$$

Here we used the trigonometric identity $\tan^2 x + 1 = \sec^2 x$.

75. Evaluate $\int \sin x \cos x\,dx$ using substitution in two different ways: first using $u = \sin x$ and then using $u = \cos x$. Reconcile the two different answers.

SOLUTION First, let $u = \sin x$. Then $du = \cos x\,dx$ and

$$\int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1.$$

Next, let $u = \cos x$. Then $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Hence,

$$\int \sin x \cos x\,dx = -\int u\,du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2.$$

To reconcile these two seemingly different answers, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true here, note that $(\frac{1}{2}\sin^2 x + C_1) - (-\frac{1}{2}\cos^2 x + C_2) = \frac{1}{2} + C_1 - C_2$, a constant. Here we used the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

76. Some Choices Are Better Than Others Evaluate

$$\int \sin x\ \cos^2 x\,dx$$

twice. First use $u = \sin x$ to show that

$$\int \sin x\ \cos^2 x\,dx = \int u\sqrt{1-u^2}\,du$$

and evaluate the integral on the right by a further substitution. Then show that $u = \cos x$ is a better choice.

SOLUTION Consider the integral $\int \sin x \cos^2 x\,dx$. If we let $u = \sin x$, then $\cos x = \sqrt{1-u^2}$ and $du = \cos x\,dx$. Hence

$$\int \sin x \cos^2 x\,dx = \int u\sqrt{1-u^2}\,du.$$

Now let $w = 1 - u^2$. Then $dw = -2u\,du$ or $-\frac{1}{2}dw = u\,du$. Therefore,

$$\int u\sqrt{1-u^2}\,du = -\frac{1}{2}\int w^{1/2}\,dw = -\frac{1}{2}\left(\frac{2}{3}w^{3/2}\right) + C$$

$$= -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C$$

$$= -\frac{1}{3}(1-\sin^2 x)^{3/2} + C = -\frac{1}{3}\cos^3 x + C.$$

A better substitution choice is $u = \cos x$. Then $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Hence

$$\int \sin x \cos^2 x\,dx = -\int u^2\,du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C.$$

77. What are the new limits of integration if we apply the substitution $u = 3x + \pi$ to the integral $\int_0^{\pi} \sin(3x+\pi)\,dx$?

SOLUTION The new limits of integration are $u(0) = 3\cdot 0 + \pi = \pi$ and $u(\pi) = 3\pi + \pi = 4\pi$.

78. Which of the following is the result of applying the substitution $u = 4x - 9$ to the integral $\int_2^8 (4x-9)^{20}\,dx$?

(a) $\displaystyle\int_2^8 u^{20}\,du$ **(b)** $\displaystyle\frac{1}{4}\int_2^8 u^{20}\,du$

(c) $\displaystyle 4\int_{-1}^{23} u^{20}\,du$ **(d)** $\displaystyle\frac{1}{4}\int_{-1}^{23} u^{20}\,du$

SOLUTION Let $u = 4x - 9$. Then $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Furthermore, when $x = 2$, $u = -1$, and when $x = 8$, $u = 23$. Hence

$$\int_2^8 (4x-9)^{20}\,dx = \frac{1}{4}\int_{-1}^{23} u^{20}\,du.$$

The answer is therefore **(d)**.

In Exercises 79–90, use the Change-of-Variables Formula to evaluate the definite integral.

79. $\displaystyle\int_1^3 (x+2)^3\,dx$

SOLUTION Let $u = x + 2$. Then $du = dx$. Hence

$$\int_1^3 (x+2)^3\,dx = \int_3^5 u^3\,du = \left.\frac{1}{4}u^4\right|_3^5 = \frac{5^4}{4} - \frac{3^4}{4} = 136.$$

80. $\displaystyle\int_1^6 \sqrt{x+3}\,dx$

SOLUTION Let $u = x + 3$. Then $du = dx$. Hence

$$\int_1^6 \sqrt{x+3}\,dx = \int_4^9 \sqrt{u}\,du = \left.\frac{2}{3}u^{3/2}\right|_4^9 = \frac{2}{3}(27-8) = \frac{38}{3}.$$

81. $\displaystyle\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION Let $u = x^2 + 1$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \left.\frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)\right|_1^2 = -\frac{1}{16} + \frac{1}{4} = \frac{3}{16} = 0.1875.$$

82. $\displaystyle\int_{-1}^2 \sqrt{5x+6}\,dx$

SOLUTION Let $u = 5x + 6$. Then $du = 5\,dx$ or $\frac{1}{5}\,du = dx$. Hence

$$\int_{-1}^2 \sqrt{5x+6}\,dx = \frac{1}{5}\int_1^{16} \sqrt{u}\,du = \left.\frac{1}{5}\left(\frac{2}{3}u^{3/2}\right)\right|_1^{16} = \frac{2}{15}(64-1) = \frac{42}{5}.$$

83. $\displaystyle\int_0^4 x\sqrt{x^2+9}\,dx$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^4 \sqrt{x^2+9}\,dx = \frac{1}{2}\int_9^{25} \sqrt{u}\,du = \left.\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\right|_9^{25} = \frac{1}{3}(125-27) = \frac{98}{3}.$$

84. $\displaystyle\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx$

SOLUTION Let $u = x^2 + 6x + 1$. Then $du = (2x+6)\,dx$ and

$$\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx = 2\int_8^{17} u^{-2}\,du = -\frac{2}{u}\bigg|_8^{17}$$
$$= -\frac{2}{17} + \frac{1}{4} = \frac{9}{68}.$$

85. $\displaystyle\int_0^1 (x+1)(x^2+2x)^5\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$, and

$$\int_0^1 (x+1)(x^2+2x)^5\,dx = \frac{1}{2}\int_0^3 u^5\,du = \frac{1}{12}u^6\bigg|_0^3 = \frac{729}{12} = \frac{243}{4}.$$

86. $\displaystyle\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION Let $u = x - 9$. Then $du = dx$. Hence

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\bigg|_1^8 = 3\,(2-1) = 3.$$

87. $\displaystyle\int_0^1 \theta\tan(\theta^2)\,d\theta$

SOLUTION Let $u = \cos\theta^2$. Then $du = -2\theta\sin\theta^2\,d\theta$ or $-\frac{1}{2}du = \theta\sin\theta^2\,d\theta$. Hence,

$$\int_0^1 \theta\tan(\theta^2)\,d\theta = \int_0^1 \frac{\theta\sin(\theta^2)}{\cos(\theta^2)}\,d\theta = -\frac{1}{2}\int_1^{\cos 1}\frac{du}{u} = -\frac{1}{2}\ln|u|\bigg|_1^{\cos 1} = -\frac{1}{2}\left[\ln(\cos 1) + \ln 1\right] = \frac{1}{2}\ln(\sec 1).$$

88. $\displaystyle\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right)\,dx$

SOLUTION Let $u = 2x - \dfrac{\pi}{6}$. Then $du = 2\,dx$ and

$$\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right)\,dx = \frac{1}{2}\int_{-\pi/6}^{\pi/6}\sec^2 u\,du = \frac{1}{2}\tan u\bigg|_{-\pi/6}^{\pi/6}$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}.$$

89. $\displaystyle\int_0^{\pi/2} \cos^3 x\sin x\,dx$

SOLUTION Let $u = \cos x$. Then $du = -\sin x\,dx$. Hence

$$\int_0^{\pi/2}\cos^3 x\sin x\,dx = -\int_1^0 u^3\,du = \int_0^1 u^3\,du = \frac{1}{4}u^4\bigg|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

90. $\displaystyle\int_{\pi/3}^{\pi/2} \cot^2\frac{x}{2}\csc^2\frac{x}{2}\,dx$

SOLUTION Let $u = \cot\dfrac{x}{2}$. Then $du = -\dfrac{1}{2}\csc^2\dfrac{x}{2}$ and

$$\int_{\pi/3}^{\pi/2}\cot^2\frac{x}{2}\csc^2\frac{x}{2}\,dx = -2\int_{\sqrt{3}}^1 u^2\,du$$
$$= -\frac{2}{3}u^3\bigg|_{\sqrt{3}}^1 = \frac{2}{3}(3\sqrt{3} - 1).$$

91. Evaluate $\displaystyle\int_0^2 r\sqrt{5-\sqrt{4-r^2}}\,dr$.

SOLUTION Let $u = 5 - \sqrt{4-r^2}$. Then

$$du = \frac{r\,dr}{\sqrt{4-r^2}} = \frac{r\,dr}{5-u}$$

so that

$$r\,dr = (5-u)\,du.$$

Hence, the integral becomes:

$$\int_0^2 r\sqrt{5-\sqrt{4-r^2}}\,dr = \int_3^5 \sqrt{u}(5-u)\,du = \int_3^5 \left(5u^{1/2} - u^{3/2}\right)\,du = \left(\frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_3^5$$

$$= \left(\frac{50}{3}\sqrt{5} - 10\sqrt{5}\right) - \left(10\sqrt{3} - \frac{18}{5}\sqrt{3}\right) = \frac{20}{3}\sqrt{5} - \frac{32}{5}\sqrt{3}.$$

92. Find numbers a and b such that

$$\int_a^b (u^2+1)\,du = \int_{-\pi/4}^{\pi/4} \sec^4\theta\,d\theta$$

and evaluate. *Hint:* Use the identity $\sec^2\theta = \tan^2\theta + 1$.

SOLUTION Let $u = \tan\theta$. Then $u^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$ and $du = \sec^2\theta\,d\theta$. Moreover, because

$$\tan\left(-\frac{\pi}{4}\right) = -1 \quad \text{and} \quad \tan\frac{\pi}{4} = 1,$$

it follows that $a = -1$ and $b = 1$. Thus,

$$\int_{-\pi/4}^{\pi/4} \sec^4\theta\,d\theta = \int_{-1}^{1} (u^2+1)\,du = \left(\frac{1}{3}u^3 + u\right)\Big|_{-1}^{1} = \frac{8}{3}.$$

93. Wind engineers have found that wind speed v (in meters/second) at a given location follows a **Rayleigh distribution** of the type

$$W(v) = \frac{1}{32}ve^{-v^2/64}$$

This means that at a given moment in time, the probability that v lies between a and b is equal to the shaded area in Figure 4.

(a) Show that the probability that $v \in [0, b]$ is $1 - e^{-b^2/64}$.

(b) Calculate the probability that $v \in [2, 5]$.

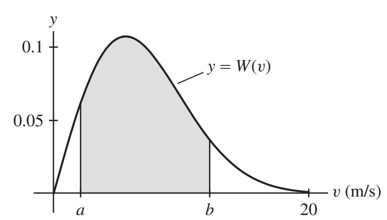


FIGURE 4 The shaded area is the probability that v lies between a and b.

SOLUTION

(a) The probability that $v \in [0, b]$ is

$$\int_0^b \frac{1}{32}ve^{-v^2/64}\,dv.$$

Let $u = -v^2/64$. Then $du = -v/32\,dv$ and

$$\int_0^b \frac{1}{32}ve^{-v^2/64}\,dv = -\int_0^{-b^2/64} e^u\,du = -e^u\Big|_0^{-b^2/64} = -e^{-b^2/64} + 1.$$

(b) The probability that $v \in [2, 5]$ is the probability that $v \in [0, 5]$ minus the probability that $v \in [0, 2]$. By part (a), the probability that $v \in [2, 5]$ is

$$\left(1 - e^{-25/64}\right) - \left(1 - e^{-1/16}\right) = e^{-1/16} - e^{-25/64}.$$

94. Evaluate $\int_0^{\pi/2} \sin^n x \cos x\, dx$ for $n \geq 0$.

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int_0^{\pi/2} \sin^n x \, \cos x\, dx = \int_0^1 u^n\, du = \left.\frac{u^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}.$$

In Exercises 95–96, use substitution to evaluate the integral in terms of $f(x)$.

95. $\int f(x)^3 f'(x)\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int f(x)^3 f'(x)\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}f(x)^4 + C.$$

96. $\int \frac{f'(x)}{f(x)^2}\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int \frac{f'(x)}{f(x)^2}\, dx = \int u^{-2}\, du = -u^{-1} + C = \frac{-1}{f(x)} + C.$$

97. Show that $\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u) \frac{1}{\sqrt{1-u^2}}\, du$.

SOLUTION Let $u = \sin\theta$. Then $u(\pi/6) = 1/2$ and $u(0) = 0$, as required. Furthermore, $du = \cos\theta\, d\theta$, so that

$$d\theta = \frac{du}{\cos\theta}.$$

If $\sin\theta = u$, then $u^2 + \cos^2\theta = 1$, so that $\cos\theta = \sqrt{1-u^2}$. Therefore $d\theta = du/\sqrt{1-u^2}$. This gives

$$\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u)\frac{1}{\sqrt{1-u^2}}\, du.$$

Further Insights and Challenges

98. Use the substitution $u = 1 + x^{1/n}$ to show that

$$\int \sqrt{1 + x^{1/n}}\, dx = n \int u^{1/2}(u-1)^{n-1}\, du$$

Evaluate for $n = 2, 3$.

SOLUTION Let $u = 1 + x^{1/n}$. Then $x = (u-1)^n$ and $dx = n(u-1)^{n-1}\, du$. Accordingly, $\int \sqrt{1 + x^{1/n}}\, dx = n \int u^{1/2}(u-1)^{n-1}\, du$.

For $n = 2$, we have

$$\int \sqrt{1 + x^{1/2}}\, dx = 2\int u^{1/2}(u-1)^1\, du = 2\int (u^{3/2} - u^{1/2})\, du$$
$$= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{4}{5}(1 + x^{1/2})^{5/2} - \frac{4}{3}(1 + x^{1/2})^{3/2} + C.$$

For $n = 3$, we have

$$\int \sqrt{1 + x^{1/3}}\, dx = 3\int u^{1/2}(u-1)^2\, du = 3\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= 3\left(\frac{2}{7}u^{7/2} - (2)\left(\frac{2}{5}\right)u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{6}{7}(1+x^{1/3})^{7/2} - \frac{12}{5}(1+x^{1/3})^{5/2} + 2(1+x^{1/3})^{3/2} + C.$$

99. Evaluate $I = \int_0^{\pi/2} \frac{d\theta}{1+\tan^{6,000}\theta}$. *Hint:* Use substitution to show that I is equal to $J = \int_0^{\pi/2} \frac{d\theta}{1+\cot^{6,000}\theta}$ and then check that $I + J = \int_0^{\pi/2} d\theta$.

SOLUTION To evaluate

$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x},$$

we substitute $t = \pi/2 - x$. Then $dt = -dx$, $x = \pi/2 - t$, $t(0) = \pi/2$, and $t(\pi/2) = 0$. Hence,

$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x} = -\int_{\pi/2}^{0} \frac{dt}{1+\tan^{6000}(\pi/2 - t)} = \int_0^{\pi/2} \frac{dt}{1+\cot^{6000} t}.$$

Let $J = \int_0^{\pi/2} \frac{dt}{1+\cot^{6000}(t)}$. We know $I = J$, so $I + J = 2I$. On the other hand, by the definition of I and J and the linearity of the integral,

$$\begin{aligned} I + J &= \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x} + \frac{dx}{1+\cot^{6000} x} = \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{1+\cot^{6000} x}\right) dx \\ &= \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{1+(1/\tan^{6000} x)}\right) dx \\ &= \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{(\tan^{6000} x + 1)/\tan^{6000} x}\right) dx \\ &= \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{\tan^{6000} x}{1+\tan^{6000} x}\right) dx \\ &= \int_0^{\pi/2} \left(\frac{1+\tan^{6000} x}{1+\tan^{6000} x}\right) dx = \int_0^{\pi/2} 1\, dx = \pi/2. \end{aligned}$$

Hence, $I + J = 2I = \pi/2$, so $I = \pi/4$.

100. Use substitution to prove that $\int_{-a}^{a} f(x)\, dx = 0$ if f is an odd function.

SOLUTION We assume that f is continuous. If $f(x)$ is an odd function, then $f(-x) = -f(x)$. Let $u = -x$. Then $x = -u$ and $du = -dx$ or $-du = dx$. Accordingly,

$$\begin{aligned} \int_{-a}^{a} f(x)\, dx &= \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = -\int_a^0 f(-u)\, du + \int_0^a f(x)\, dx \\ &= \int_0^a f(x)\, dx - \int_0^a f(u)\, du = 0. \end{aligned}$$

101. Prove that $\int_a^b \frac{1}{x}\, dx = \int_1^{b/a} \frac{1}{x}\, dx$ for $a, b > 0$. Then show that the regions under the hyperbola over the intervals [1, 2], [2, 4], [4, 8], ... all have the same area (Figure 5).

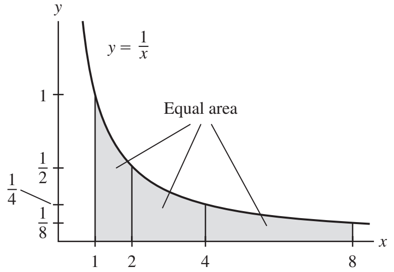


FIGURE 5 The area under $y = \frac{1}{x}$ over $[2^n, 2^{n+1}]$ is the same for all $n = 0, 1, 2, \ldots$.

SOLUTION

(a) Let $u = \frac{x}{a}$. Then $au = x$ and $du = \frac{1}{a}\,dx$ or $a\,du = dx$. Hence

$$\int_a^b \frac{1}{x}\,dx = \int_1^{b/a} \frac{a}{au}\,du = \int_1^{b/a} \frac{1}{u}\,du.$$

Note that $\displaystyle\int_1^{b/a} \frac{1}{u}\,du = \int_1^{b/a} \frac{1}{x}\,dx$ after the substitution $x = u$.

(b) The area under the hyperbola over the interval $[1, 2]$ is given by the definite integral $\int_1^2 \frac{1}{x}\,dx$. Denote this definite integral by A. Using the result from part (a), we find the area under the hyperbola over the interval $[2, 4]$ is

$$\int_2^4 \frac{1}{x}\,dx = \int_1^{4/2} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

Similarly, the area under the hyperbola over the interval $[4, 8]$ is

$$\int_4^8 \frac{1}{x}\,dx = \int_1^{8/4} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

In general, the area under the hyperbola over the interval $[2^n, 2^{n+1}]$ is

$$\int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx = \int_1^{2^{n+1}/2^n} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

102. Show that the two regions in Figure 6 have the same area. Then use the identity $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$ to compute the second area.

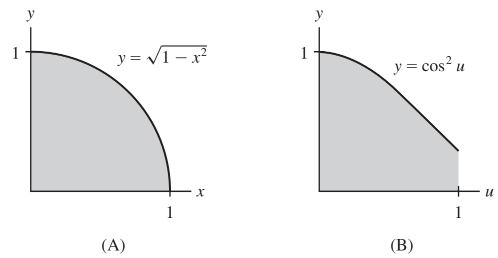


FIGURE 6

SOLUTION The area of the region in Figure 6(A) is given by $\int_0^1 \sqrt{1 - x^2}\,dx$. Let $x = \sin u$. Then $dx = \cos u\,du$ and $\sqrt{1 - x^2} = \sqrt{1 - \sin^2 u} = \cos u$. Hence,

$$\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\pi/2} \cos u \cdot \cos u\,du = \int_0^{\pi/2} \cos^2 u\,du.$$

This last integral represents the area of the region in Figure 6(B). The two regions in Figure 6 therefore have the same area.

Let's now focus on the definite integral $\int_0^{\pi/2} \cos^2 u\,du$. Using the trigonometric identity $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$, we have

$$\int_0^{\pi/2} \cos^2 u\,du = \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2u\,du = \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right)\bigg|_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} - 0 = \frac{\pi}{4}.$$

103. Area of an Ellipse Prove the formula $A = \pi ab$ for the area of the ellipse with equation (Figure 7)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hint: Use a change of variables to show that A is equal to ab times the area of the unit circle.

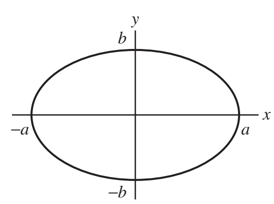


FIGURE 7 Graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; here $a, b > 0$. The area between the part of the ellipse in the upper half-plane, $y = f(x) = \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$, and the x-axis is $\int_{-a}^{a} f(x)\,dx$. By symmetry, the part of the elliptical region in the lower half-plane has the same area. Accordingly, the area enclosed by the ellipse is

$$2\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a}\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\,dx = 2b\int_{-a}^{a}\sqrt{1-(x/a)^2}\,dx$$

Now, let $u = x/a$. Then $x = au$ and $a\,du = dx$. Accordingly,

$$2b\int_{-a}^{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = 2ab\int_{-1}^{1}\sqrt{1-u^2}\,du = 2ab\left(\frac{\pi}{2}\right) = \pi ab$$

Here we recognized that $\int_{-1}^{1}\sqrt{1-u^2}\,du$ represents the area of the upper unit semicircular disk, which by Exercise 102 is $2(\frac{\pi}{4}) = \frac{\pi}{2}$.

5.7 Further Transcendental Functions

Preliminary Questions

1. Find b such that $\int_1^b \frac{dx}{x}$ is equal to

(a) $\ln 3$ **(b)** 3

SOLUTION For $b > 0$,

$$\int_1^b \frac{dx}{x} = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b.$$

(a) For the value of the definite integral to equal $\ln 3$, we must have $b = 3$.
(b) For the value of the definite integral to equal 3, we must have $b = e^3$.

2. Find b such that $\int_0^b \frac{dx}{1+x^2} = \frac{\pi}{3}$.

SOLUTION In general,

$$\int_0^b \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

For the value of the definite integral to equal $\frac{\pi}{3}$, we must have

$$\tan^{-1} b = \frac{\pi}{3} \quad \text{or} \quad b = \tan\frac{\pi}{3} = \sqrt{3}.$$

3. Which integral should be evaluated using substitution?

(a) $\displaystyle\int \frac{9\,dx}{1+x^2}$ **(b)** $\displaystyle\int \frac{dx}{1+9x^2}$

SOLUTION Use the substitution $u = 3x$ on the integral in **(b)**.

4. Which relation between x and u yields $\sqrt{16+x^2} = 4\sqrt{1+u^2}$?

SOLUTION To transform $\sqrt{16+x^2}$ into $4\sqrt{1+u^2}$, make the substitution $x = 4u$.

Exercises

In Exercises 1–10, evaluate the definite integral.

1. $\displaystyle\int_1^9 \frac{dx}{x}$

SOLUTION $\displaystyle\int_1^9 \frac{1}{x}\,dx = \ln|x|\Big|_1^9 = \ln 9 - \ln 1 = \ln 9.$

2. $\displaystyle\int_4^{20} \frac{dx}{x}$

SOLUTION $\displaystyle\int_4^{20} \frac{1}{x}\,dx = \ln|x|\Big|_4^{20} = \ln 20 - \ln 4 = \ln 5.$

3. $\int_1^{e^3} \frac{1}{t}\,dt$

SOLUTION $\int_1^{e^3} \frac{1}{t}\,dt = \ln|t|\Big|_1^{e^3} = \ln e^3 - \ln 1 = 3.$

4. $\int_{-e^2}^{-e} \frac{1}{t}\,dt$

SOLUTION $\int_{-e^2}^{-e} \frac{1}{t}\,dt = \ln|t|\Big|_{-e^2}^{-e} = \ln|-e| - \ln|-e^2| = \ln\frac{e}{e^2} = \ln(1/e) = -1.$

5. $\int_2^{12} \frac{dt}{3t+4}$

SOLUTION Let $u = 3t+4$. Then $du = 3\,dt$ and

$$\int_2^{12} \frac{dt}{3t+4} = \frac{1}{3}\int_{10}^{40} \frac{du}{u} = \frac{1}{3}\ln|u|\Big|_{10}^{40} = \frac{1}{3}(\ln 40 - \ln 10) = \frac{1}{3}\ln 4.$$

6. $\int_e^{e^3} \frac{dt}{t\ln t}$

SOLUTION Let $u = \ln t$. Then $du = (1/t)dt$ and

$$\int_e^{e^3} \frac{1}{t\ln t}\,dt = \int_1^3 \frac{du}{u} = \ln|u|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

7. $\int_{\tan 1}^{\tan 8} \frac{dx}{x^2+1}$

SOLUTION $\int_{\tan 1}^{\tan 8} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{\tan 1}^{\tan 8} = \tan^{-1}(\tan 8) - \tan^{-1}(\tan 1) = 8 - 1 = 7.$

8. $\int_2^7 \frac{x\,dx}{x^2+1}$

SOLUTION Let $u = x^2+1$. Then $du = 2x\,dx$ and

$$\int_2^7 \frac{x\,dx}{x^2+1} = \frac{1}{2}\int_5^{50} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_5^{50} = \frac{1}{2}(\ln 50 - \ln 5) = \frac{1}{2}\ln 10.$$

9. $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

SOLUTION $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$

10. $\int_{-2}^{-2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

SOLUTION $\int_{-2}^{-2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x\Big|_{-2}^{-2/\sqrt{3}} = \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) - \sec^{-1}(-2) = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}.$

11. Use the substitution $u = x/3$ to prove

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

SOLUTION Let $u = x/3$. Then, $x = 3u$, $dx = 3\,du$, $9 + x^2 = 9(1+u^2)$, and

$$\int \frac{dx}{9+x^2} = \int \frac{3\,du}{9(1+u^2)} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$$

12. Use the substitution $u = 2x$ to evaluate $\int \frac{dx}{4x^2+1}$.

SOLUTION Let $u = 2x$. Then, $x = u/2$, $dx = \frac{1}{2}\,du$, $4x^2 + 1 = u^2 + 1$, and

$$\int \frac{dx}{4x^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} 2x + C.$$

In Exercises 13–32, calculate the integral.

13. $\displaystyle\int_0^3 \frac{dx}{x^2+3}$

SOLUTION Let $x = \sqrt{3}u$. Then $dx = \sqrt{3}\,du$ and

$$\int_0^3 \frac{dx}{x^2+3} = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{du}{u^2+1} = \frac{1}{\sqrt{3}}\tan^{-1} u\Big|_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1} 0) = \frac{\pi}{3\sqrt{3}}.$$

14. $\displaystyle\int_0^4 \frac{dt}{4t^2+9}$

SOLUTION Let $t = (3/2)u$. Then $dt = (3/2)\,du$, $4t^2+9 = 9t^2+9 = 9(t^2+1)$, and

$$\int_0^4 \frac{dt}{4t^2+9} = \frac{1}{6}\int_0^{8/3} \frac{du}{u^2+1} = \frac{1}{6}\tan^{-1} u\Big|_0^{8/3} = \frac{1}{6}\tan^{-1}\frac{8}{3}.$$

15. $\displaystyle\int \frac{dt}{\sqrt{1-16t^2}}$

SOLUTION Let $u = 4t$. Then $du = 4\,dt$, and

$$\int \frac{dt}{\sqrt{1-16t^2}} = \int \frac{du}{4\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1} u + C = \frac{1}{4}\sin^{-1}(4t) + C.$$

16. $\displaystyle\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}}$

SOLUTION Let $x = 2u/5$. Then

$$dx = \frac{2}{5}\,du, \quad 4 - 25x^2 = 4(1-u^2),$$

and

$$\begin{aligned}\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}} &= \frac{2}{5}\int_{-1/2}^{1/2} \frac{1}{\sqrt{4(1-u^2)}}\,du\\ &= \frac{1}{5}\sin^{-1} u\Big|_{-1/2}^{1/2}\\ &= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \frac{\pi}{15}.\end{aligned}$$

17. $\displaystyle\int \frac{dt}{\sqrt{5-3t^2}}$

SOLUTION Let $t = \sqrt{5/3}u$. Then $dt = \sqrt{5/3}\,du$ and

$$\int \frac{dt}{\sqrt{5-3t^2}} = \int \frac{\sqrt{5/3}\,du}{\sqrt{5}\sqrt{1-t^2}} = \frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{3}}\sin^{-1} u + C = \frac{1}{\sqrt{3}}\sin^{-1}\sqrt{\frac{3}{5}}t + C.$$

18. $\displaystyle\int_{1/2\sqrt{2}}^{1/2} \frac{dx}{x\sqrt{16x^2-1}}$

SOLUTION Let $x = u/4$. Then $dx = du/4$, $16x^2 - 1 = u^2 - 1$ and

$$\int_{1/2\sqrt{2}}^{1/2} \frac{dx}{x\sqrt{16x^2-1}} = \int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u\Big|_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \frac{\pi}{12}.$$

19. $\displaystyle\int \frac{dx}{x\sqrt{12x^2-3}}$

SOLUTION Let $u = 2x$. Then $du = 2\,dx$ and

$$\int \frac{dx}{x\sqrt{12x^2-3}} = \frac{1}{\sqrt{3}}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{\sqrt{3}}\sec^{-1} u + C = \frac{1}{\sqrt{3}}\sec^{-1}(2x) + C.$$

20. $\displaystyle\int \frac{x\,dx}{x^4+1}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ and

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C.$$

21. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$, and

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1} u + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

22. $\displaystyle\int_{-1/2}^{0} \frac{(x+1)\,dx}{\sqrt{1-x^2}}$

SOLUTION Observe that

$$\int \frac{(x+1)\,dx}{\sqrt{1-x^2}} = \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}}.$$

In the first integral on the right, we let $u = 1 - x^2$, $du = -2x\,dx$. Thus

$$\int \frac{(x+1)\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{u^{1/2}} + \int \frac{1\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

Finally,

$$\int_{-1/2}^{0} \frac{(x+1)\,dx}{\sqrt{1-x^2}} = \left(-\sqrt{1-x^2} + \sin^{-1} x\right)\Big|_{-1/2}^{0} = -1 + \frac{\sqrt{3}}{2} + \frac{\pi}{6}.$$

23. $\displaystyle\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}}$

SOLUTION Let $u = e^x$. Then $du = e^x\,dx$, and

$$\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}} = \int_{1/2}^{1} \frac{du}{1+u^2} = \tan^{-1} u\Big|_{1/2}^{1} = \frac{\pi}{4} - \tan^{-1}(1/2).$$

24. $\displaystyle\int \frac{\ln(\cos^{-1} x)\,dx}{(\cos^{-1} x)\sqrt{1-x^2}}$

SOLUTION Let $u = \ln\cos^{-1} x$. Then $du = \dfrac{1}{\cos^{-1} x}\cdot\dfrac{-1}{\sqrt{1-x^2}}$, and

$$\int \frac{\ln(\cos^{-1} x)\,dx}{(\cos^{-1} x)\sqrt{1-x^2}} = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}(\ln\cos^{-1} x)^2 + C.$$

25. $\displaystyle\int \frac{\tan^{-1} x\,dx}{1+x^2}$

SOLUTION Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$, and

$$\int \frac{\tan^{-1} x\,dx}{1+x^2} = \int u\,du = \frac{1}{2}u^2 + C = \frac{(\tan^{-1} x)^2}{2} + C.$$

26. $\displaystyle\int_{1}^{\sqrt{3}} \frac{dx}{(\tan^{-1} x)(1+x^2)}$

SOLUTION Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$, and

$$\int_{1}^{\sqrt{3}} \frac{dx}{(\tan^{-1} x)(1+x^2)} = \int_{\pi/4}^{\pi/3} \frac{1}{u}\,du = \ln|u|\Big|_{\pi/4}^{\pi/3} = \ln\frac{\pi}{3} - \ln\frac{\pi}{4} = \ln\frac{4}{3}.$$

27. $\int_0^1 3^x\,dx$

SOLUTION $\int_0^1 3^x\,dx = \left.\frac{3^x}{\ln 3}\right|_0^1 = \frac{1}{\ln 3}(3-1) = \frac{2}{\ln 3}.$

28. $\int_0^1 3^{-x}\,dx$

SOLUTION Let $u=-x$. Then $du=-dx$ and

$$\int_0^1 3^{-x}\,dx = -\int_0^{-1} 3^u\,du = -\left.\frac{3^u}{\ln 3}\right|_0^{-1} = \frac{1}{\ln 3}\left(-\frac{1}{3}+1\right) = \frac{2}{3\ln 3}.$$

29. $\int_0^{\log_4(3)} 4^x\,dx$

SOLUTION $\int_0^{\log_4(3)} 4^x\,dx = \left.\frac{4^x}{\ln 4}\right|_0^{\log_4 3} = \frac{1}{\ln 4}(3-1) = \frac{2}{\ln 4} = \frac{1}{\ln 2}.$

30. $\int_0^1 t5^{t^2}\,dt$

SOLUTION Let $u=t^2$. Then $du=2t\,dt$ and

$$\int_0^1 t5^{t^2}\,dt = \frac{1}{2}\int_0^1 5^u\,du = \left.\frac{5^u}{2\ln 5}\right|_0^1 = \frac{5}{2\ln 5} - \frac{1}{2\ln 5} = \frac{2}{\ln 5}.$$

31. $\int 9^x \sin(9^x)\,dx$

SOLUTION Let $u=9^x$. Then $du = 9^x \ln 9\,dx$ and

$$\int 9^x\sin(9^x)\,dx = \frac{1}{\ln 9}\int \sin u\,du = -\frac{1}{\ln 9}\cos u + C = -\frac{1}{\ln 9}\cos(9^x) + C.$$

32. $\int \frac{dx}{\sqrt{5^{2x}-1}}$

SOLUTION First, rewrite

$$\int \frac{dx}{\sqrt{5^{2x}-1}} = \int \frac{dx}{5^x\sqrt{1-5^{-2x}}} = \int \frac{5^{-x}\,dx}{\sqrt{1-5^{-2x}}}.$$

Now, let $u=5^{-x}$. Then $du = -5^{-x}\ln 5\,dx$ and

$$\int \frac{dx}{\sqrt{5^{2x}-1}} = -\frac{1}{\ln 5}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{\ln 5}\sin^{-1} u + C = -\frac{1}{\ln 5}\sin^{-1}(5^{-x}) + C.$$

In Exercises 33–70, evaluate the integral using the methods covered in the text so far.

33. $\int ye^{y^2}\,dy$

SOLUTION Use the substitution $u=y^2$, $du=2y\,dy$. Then

$$\int ye^{y^2}\,dy = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{y^2} + C.$$

34. $\int \frac{dx}{3x+5}$

SOLUTION Let $u=3x+5$. Then $du=3\,dx$ and

$$\int \frac{dx}{3x+5} = \frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+5| + C.$$

35. $\int \frac{x\,dx}{\sqrt{4x^2+9}}$

SOLUTION Let $u=4x^2+9$. Then $du=8x\,dx$ and

$$\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{8}\int u^{-1/2}\,du = \frac{1}{4}u^{1/2} + C = \frac{1}{4}\sqrt{4x^2+9} + C.$$

36. $\int (x - x^{-2})^2\, dx$

SOLUTION $\int (x - x^{-2})^2\, dx = \int (x^2 - 2x^{-1} + x^{-4})\, dx = \frac{1}{3}x^3 - 2\ln|x| - \frac{1}{3}x^{-3} + C.$

37. $\int 7^{-x}\, dx$

SOLUTION Let $u = -x$. Then $du = -dx$ and

$$\int 7^{-x}\, dx = -\int 7^u\, du = -\frac{7^u}{\ln 7} + C = -\frac{7^{-x}}{\ln 7} + C.$$

38. $\int e^{9-12t}\, dt$

SOLUTION Let $u = 9 - 12t$. Then $du = -12\, dt$ and

$$\int e^{9-12t}\, dt = -\frac{1}{12}\int e^u\, du = -\frac{1}{12}e^u + C = -\frac{1}{12}e^{9-12t} + C.$$

39. $\int \sec^2\theta \tan^7\theta\, d\theta$

SOLUTION Let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$ and

$$\int \sec^2\theta \tan^7\theta\, d\theta = \int u^7\, du = \frac{1}{8}u^8 + C = \frac{1}{8}\tan^8\theta + C.$$

40. $\int \frac{\cos(\ln t)\, dt}{t}$

SOLUTION Let $u = \ln t$. Then $du = dt/t$ and

$$\int \frac{\cos(\ln t)\, dt}{t} = \int \cos u\, du = \sin u + C = \sin(\ln t) + C.$$

41. $\int \frac{t\, dt}{\sqrt{7 - t^2}}$

SOLUTION Let $u = 7 - t^2$. Then $du = -2t\, dt$ and

$$\int \frac{t\, dt}{\sqrt{7 - t^2}} = -\frac{1}{2}\int u^{-1/2}\, du = -u^{1/2} + C = -\sqrt{7 - t^2} + C.$$

42. $\int 2^x e^{4x}\, dx$

SOLUTION First, note that

$$2^x = e^{x\ln 2} \quad \text{so} \quad 2^x e^{4x} = e^{(4+\ln 2)x}.$$

Thus,

$$\int 2^x e^{4x}\, dx = \int e^{(4+\ln 2)x}\, dx = \frac{1}{4 + \ln 2}e^{(4+\ln 2)x} + C.$$

43. $\int \frac{(3x + 2)\, dx}{x^2 + 4}$

SOLUTION Write

$$\int \frac{(3x + 2)\, dx}{x^2 + 4} = \int \frac{3x\, dx}{x^2 + 4} + \int \frac{2\, dx}{x^2 + 4}.$$

In the first integral, let $u = x^2 + 4$. Then $du = 2x\, dx$ and

$$\int \frac{3x\, dx}{x^2 + 4} = \frac{3}{2}\int \frac{du}{u} - \frac{3}{2}\ln|u| + C_1 = \frac{3}{2}\ln(x^2 + 4) + C_1.$$

For the second integral, let $x = 2u$. Then $dx = 2\, du$ and

$$\int \frac{2\, dx}{x^2 + 4} = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C_2 = \tan^{-1}(x/2) + C_2.$$

Combining these two results yields

$$\int \frac{(3x+2)\,dx}{x^2+4} = \frac{3}{2}\ln(x^2+4) + \tan^{-1}(x/2) + C.$$

44. $\int \tan(4x+1)\,dx$

SOLUTION First we rewrite $\int \tan(4x+1)\,dx$ as $\int \frac{\sin(4x+1)}{\cos(4x+1)}\,dx$. Let $u = \cos(4x+1)$. Then $du = -4\sin(4x+1)\,dx$, and

$$\int \frac{\sin(4x+1)}{\cos(4x+1)}dx = -\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|\cos(4x+1)| + C.$$

45. $\int \frac{dx}{\sqrt{1-16x^2}}$

SOLUTION Let $u = 4x$. Then $du = 4\,dx$ and

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1}u + C = \frac{1}{4}\sin^{-1}(4x) + C.$$

46. $\int e^t\sqrt{e^t+1}\,dt$

SOLUTION Use the substitution $u = e^t + 1$, $du = e^t\,dt$. Then

$$\int e^t\sqrt{e^t+1}\,dt = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(e^t+1)^{3/2} + C.$$

47. $\int (e^{-x} - 4x)\,dx$

SOLUTION First, observe that

$$\int (e^{-x} - 4x)\,dx = \int e^{-x}\,dx - \int 4x\,dx = \int e^{-x}\,dx - 2x^2.$$

In the remaining integral, use the substitution $u = -x$, $du = -dx$. Then

$$\int e^{-x}\,dx = -\int e^u\,du = -e^u + C = -e^{-x} + C.$$

Finally,

$$\int (e^{-x} - 4x)\,dx = -e^{-x} - 2x^2 + C.$$

48. $\int (7 - e^{10x})\,dx$

SOLUTION First, observe that

$$\int (7 - e^{10x})\,dx = \int 7\,dx - \int e^{10x}\,dx = 7x - \int e^{10x}\,dx.$$

In the remaining integral, use the substitution $u = 10x$, $du = 10\,dx$. Then

$$\int e^{10x}\,dx = \frac{1}{10}\int e^u\,du = \frac{1}{10}e^u + C = \frac{1}{10}e^{10x} + C.$$

Finally,

$$\int (7 - e^{10x})\,dx = 7x - \frac{1}{10}e^{10x} + C.$$

49. $\int \frac{e^{2x} - e^{4x}}{e^x}\,dx$

SOLUTION

$$\int \left(\frac{e^{2x} - e^{4x}}{e^x}\right)dx = \int (e^x - e^{3x})\,dx = e^x - \frac{e^{3x}}{3} + C.$$

50. $\displaystyle\int \frac{dx}{x\sqrt{25x^2-1}}$

SOLUTION Let $u = 5x$. Then $du = 5\,dx$ and

$$\int \frac{dx}{x\sqrt{25x^2-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C = \sec^{-1}(5x) + C.$$

51. $\displaystyle\int \frac{(x+5)\,dx}{\sqrt{4-x^2}}$

SOLUTION Write

$$\int \frac{(x+5)\,dx}{\sqrt{4-x^2}} = \int \frac{x\,dx}{\sqrt{4-x^2}} + \int \frac{5\,dx}{\sqrt{4-x^2}}.$$

In the first integral, let $u = 4 - x^2$. Then $du = -2x\,dx$ and

$$\int \frac{x\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} + C_1 = -\sqrt{4-x^2} + C_1.$$

In the second integral, let $x = 2u$. Then $dx = 2\,du$ and

$$\int \frac{5\,dx}{\sqrt{4-x^2}} = 5\int \frac{du}{\sqrt{1-u^2}} = 5\sin^{-1} u + C_2 = 5\sin^{-1}(x/2) + C_2.$$

Combining these two results yields

$$\int \frac{(x+5)\,dx}{\sqrt{4-x^2}} = -\sqrt{4-x^2} + 5\sin^{-1}(x/2) + C.$$

52. $\displaystyle\int (t+1)\sqrt{t+1}\,dt$

SOLUTION Let $u = t + 1$. Then $du = dt$ and

$$\int (t+1)\sqrt{t+1}\,dt = \int u^{3/2}\,du = \frac{2}{5}u^{5/2} + C = \frac{2}{5}(t+1)^{5/2} + C.$$

53. $\displaystyle\int e^x \cos(e^x)\,dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\,dx$. Then

$$\int e^x \cos(e^x)\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C.$$

54. $\displaystyle\int \frac{e^x}{\sqrt{e^x+1}}\,dx$

SOLUTION Use the substitution $u = e^x + 1$, $du = e^x\,dx$. Then

$$\int \frac{e^x}{\sqrt{e^x+1}}\,dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{e^x+1} + C.$$

55. $\displaystyle\int \frac{dx}{\sqrt{9-16x^2}}$

SOLUTION First rewrite

$$\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{4}{3}x\right)^2}}.$$

Now, let $u = \frac{4}{3}x$. Then $du = \frac{4}{3}\,dx$ and

$$\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1} u + C = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + C.$$

56. $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$

SOLUTION Let $u = \ln(8x-2)$. Then $du = \dfrac{8}{8x-2}dx = \dfrac{4}{4x-1}dx$, and

$$\int \frac{dx}{(4x-1)\ln(8x-2)} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|\ln(8x-2)| + C.$$

57. $\displaystyle\int e^x(e^{2x}+1)^3\,dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\,dx$. Then

$$\begin{aligned}\int e^x(e^{2x}+1)^3\,dx &= \int \left(u^2+1\right)^3\,du = \int \left(u^6+3u^4+3u^2+1\right)du\\ &= \frac{1}{7}u^7 + \frac{3}{5}u^5 + u^3 + u + C = \frac{1}{7}(e^x)^7 + \frac{3}{5}(e^x)^5 + (e^x)^3 + e^x + C\\ &= \frac{e^{7x}}{7} + \frac{3e^{5x}}{5} + e^{3x} + e^x + C.\end{aligned}$$

58. $\displaystyle\int \frac{dx}{x(\ln x)^5}$

SOLUTION Let $u = \ln x$. Then $du = dx/x$ and

$$\int \frac{dx}{x(\ln x)^5} = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4(\ln x)^4} + C.$$

59. $\displaystyle\int \frac{x^2\,dx}{x^3+2}$

SOLUTION Let $u = x^3+2$. Then $du = 3x^2\,dx$, and

$$\int \frac{x^2\,dx}{x^3+2} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|x^3+2| + C.$$

60. $\displaystyle\int \frac{(3x-1)\,dx}{9-2x+3x^2}$

SOLUTION Let $u = 9-2x+3x^2$. Then $du = (-2+6x)\,dx = 2(3x-1)\,dx$, and

$$\int \frac{(3x-1)dx}{9-2x+3x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(9-2x+3x^2) + C.$$

61. $\displaystyle\int \cot x\,dx$

SOLUTION We rewrite $\int \cot x\,dx$ as $\int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, and

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|\sin x| + C.$$

62. $\displaystyle\int \frac{\cos x}{2\sin x+3}\,dx$

SOLUTION Let $u = 2\sin x + 3$. Then $du = 2\cos x\,dx$, and

$$\int \frac{\cos x}{2\sin x+3}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(2\sin x+3) + C,$$

where we have used the fact that $2\sin x + 3 \ge 1$ to drop the absolute value.

63. $\displaystyle\int \frac{4\ln x+5}{x}\,dx$

SOLUTION Let $u = 4\ln x + 5$. Then $du = (4/x)dx$, and

$$\int \frac{4\ln x+5}{x}dx = \frac{1}{4}\int u\,du = \frac{1}{8}u^2 + C = \frac{1}{8}(4\ln x+5)^2 + C.$$

64. $\int (\sec\theta \tan\theta) 5^{\sec\theta}\, d\theta$

SOLUTION Let $u = \sec\theta$. Then $du = \sec\theta \tan\theta\, d\theta$ and

$$\int (\sec\theta \tan\theta) 5^{\sec\theta}\, d\theta = \int 5^u\, du = \frac{5^u}{\ln 5} + C = \frac{5^{\sec\theta}}{\ln 5} + C.$$

65. $\int x 3^{x^2}\, dx$

SOLUTION Let $u = x^2$. Then $du = 2x\, dx$, and

$$\int x 3^{x^2}\, dx = \frac{1}{2}\int 3^u du = \frac{1}{2}\frac{3^u}{\ln 3} + C = \frac{3^{x^2}}{2\ln 3} + C.$$

66. $\int \frac{\ln(\ln x)}{x \ln x}\, dx$

SOLUTION Let $u = \ln(\ln x)$. Then $du = \frac{1}{\ln x} \cdot \frac{1}{x}\, dx$ and

$$\int \frac{\ln(\ln x)}{x \ln x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln(\ln x))^2}{2} + C.$$

67. $\int \cot x \ln(\sin x)\, dx$

SOLUTION Let $u = \ln(\sin x)$. Then

$$du = \frac{1}{\sin x} \cdot \cos x\, dx = \cot x\, dx,$$

and

$$\int \cot x \ln(\sin x)\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln(\sin x))^2}{2} + C.$$

68. $\int \frac{t\, dt}{\sqrt{1 - t^4}}$

SOLUTION Let $u = t^2$. Then $du = 2t\, dt$ and

$$\int \frac{t\, dt}{\sqrt{1 - t^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} t^2 + C.$$

69. $\int t^2 \sqrt{t - 3}\, dt$

SOLUTION Let $u = t - 3$. Then $t = u + 3$, $du = dt$ and

$$\begin{aligned}
\int t^2 \sqrt{t - 3}\, dt &= \int (u + 3)^2 \sqrt{u}\, du \\
&= \int (u^2 + 6u + 9)\sqrt{u}\, du = \int (u^{5/2} + 6u^{3/2} + 9u^{1/2})\, du \\
&= \frac{2}{7}u^{7/2} + \frac{12}{5}u^{5/2} + 6u^{3/2} + C \\
&= \frac{2}{7}(t - 3)^{7/2} + \frac{12}{5}(t - 3)^{5/2} + 6(t - 3)^{3/2} + C.
\end{aligned}$$

70. $\int \cos x 5^{-2\sin x}\, dx$

SOLUTION Let $u = -2\sin x$. Then $du = -2\cos x\, dx$ and

$$\int \cos x 5^{-2\sin x}\, dx = -\frac{1}{2}\int 5^u\, du = -\frac{5^u}{2ln5} + C = -\frac{5^{-2\sin x}}{2\ln 5} + C.$$

71. Use Figure 4 to prove

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$$

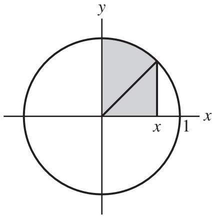


FIGURE 4

SOLUTION The definite integral $\int_0^x \sqrt{1-t^2}\,dt$ represents the area of the region under the upper half of the unit circle from 0 to x. The region consists of a sector of the circle and a right triangle. The sector has a central angle of $\frac{\pi}{2} - \theta$, where $\cos\theta = x$. Hence, the sector has an area of

$$\frac{1}{2}(1)^2\left(\frac{\pi}{2} - \cos^{-1} x\right) = \frac{1}{2}\sin^{-1} x.$$

The right triangle has a base of length x, a height of $\sqrt{1-x^2}$, and hence an area of $\frac{1}{2}x\sqrt{1-x^2}$. Thus,

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x.$$

72. Use the substitution $u = \tan x$ to evaluate

$$\int \frac{dx}{1+\sin^2 x}.$$

Hint: Show that

$$\frac{dx}{1+\sin^2 x} = \frac{du}{1+2u^2}$$

SOLUTION If $u = \tan x$, then $du = \sec^2 x\,dx$ and

$$\frac{du}{1+2u^2} = \frac{\sec^2 x\,dx}{1+2\tan^2 x} = \frac{dx}{\cos^2 x + 2\sin^2 x} = \frac{dx}{\cos^2 x + \sin^2 x + \sin^2 x} = \frac{dx}{1+\sin^2 x}.$$

Thus

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{du}{1+2u^2} = \int \frac{du}{1+(\sqrt{2}u)^2} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u) + C = \frac{1}{\sqrt{2}}\tan^{-1}((\tan x)\sqrt{2}) + C.$$

73. Prove:

$$\int \sin^{-1} t\,dt = \sqrt{1-t^2} + t\sin^{-1} t.$$

SOLUTION Let $G(t) = \sqrt{1-t^2} + t\sin^{-1} t$. Then

$$G'(t) = \frac{d}{dt}\sqrt{1-t^2} + \frac{d}{dt}\left(t\,\sin^{-1} t\right) = \frac{-t}{\sqrt{1-t^2}} + \left(t\cdot\frac{d}{dt}\sin^{-1} t + \sin^{-1} t\right)$$

$$= \frac{-t}{\sqrt{1-t^2}} + \left(\frac{t}{\sqrt{1-t^2}} + \sin^{-1} t\right) = \sin^{-1} t.$$

This proves the formula $\int \sin^{-1} t\,dt = \sqrt{1-t^2} + t\sin^{-1} t$.

74. **(a)** Verify for $r \neq 0$:

$$\int_0^T te^{rt}\,dt = \frac{e^{rT}(rT-1)+1}{r^2} \qquad \boxed{6}$$

Hint: For fixed r, let $F(T)$ be the value of the integral on the left. By FTC II, $F'(t) = te^{rt}$ and $F(0) = 0$. Show that the same is true of the function on the right.

(b) Use L'Hôpital's Rule to show that for fixed T, the limit as $r \to 0$ of the right-hand side of Eq. (6) is equal to the value of the integral for $r = 0$.

SOLUTION
(a) Let

$$f(t) = \frac{e^{rt}}{r^2}(rt - 1) + \frac{1}{r^2}.$$

Then

$$f'(t) = \frac{1}{r^2}\left(e^{rt}r + (rt - 1)(re^{rt})\right) = te^{rt}$$

and

$$f(0) = -\frac{1}{r^2} + \frac{1}{r^2} = 0,$$

as required.
(b) Using L'Hôpital's Rule,

$$\lim_{r\to 0} \frac{e^{rT}(rT - 1) + 1}{r^2} = \lim_{r\to 0} \frac{Te^{rT} + (rT - 1)(Te^{rT})}{2r} = \lim_{r\to 0} \frac{rT^2e^{rT}}{2r} = \lim_{r\to 0} \frac{T^2e^{rT}}{2} = \frac{T^2}{2}.$$

If $r = 0$ then, $\displaystyle\int_0^T te^{rt}\,dt = \int_0^T t\,dt = \left.\frac{t^2}{2}\right|_0^T = \frac{T^2}{2}$.

Further Insights and Challenges

75. Recall that if $f(t) \geq g(t)$ for $t \geq 0$, then for all $x \geq 0$,

$$\int_0^x f(t)\,dt \geq \int_0^x g(t)\,dt \qquad \boxed{7}$$

The inequality $e^t \geq 1$ holds for $t \geq 0$ because $e > 1$. Use Eq. (7) to prove that $e^x \geq 1 + x$ for $x \geq 0$. Then prove, by successive integration, the following inequalities (for $x \geq 0$):

$$e^x \geq 1 + x + \frac{1}{2}x^2, \qquad e^x \geq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

SOLUTION Integrating both sides of the inequality $e^t \geq 1$ yields

$$\int_0^x e^t\,dt = e^x - 1 \geq x \quad \text{or} \quad e^x \geq 1 + x.$$

Integrating both sides of this new inequality then gives

$$\int_0^x e^t\,dt = e^x - 1 \geq x + x^2/2 \quad \text{or} \quad e^x \geq 1 + x + x^2/2.$$

Finally, integrating both sides again gives

$$\int_0^x e^t\,dt = e^x - 1 \geq x + x^2/2 + x^3/6 \quad \text{or} \quad e^x \geq 1 + x + x^2/2 + x^3/6$$

as requested.

76. Generalize Exercise 75; that is, use induction (if you are familiar with this method of proof) to prove that for all $n \geq 0$,

$$e^x \geq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \quad (x \geq 0)$$

SOLUTION For $n = 1$, $e^x \geq 1 + x$ by Exercise 75. Assume the statement is true for $n = k$. We need to prove the statement is true for $n = k + 1$. By the Induction Hypothesis,

$$e^x \geq 1 + x + x^2/2 + \cdots + x^k/k!.$$

Integrating both sides of this inequality yields

$$\int_0^x e^t\,dt = e^x - 1 \geq x + x^2/2 + \cdots + x^{k+1}/(k+1)!$$

or

$$e^x \geq 1 + x + x^2/2 + \cdots + x^{k+1}/(k+1)!$$

as required.

77. Use Exercise 75 to show that $e^x/x^2 \geq x/6$ and conclude that $\lim_{x\to\infty} e^x/x^2 = \infty$. Then use Exercise 76 to prove more generally that $\lim_{x\to\infty} e^x/x^n = \infty$ for all n.

SOLUTION By Exercise 75, $e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Thus

$$\frac{e^x}{x^2} \geq \frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{x}{6} \geq \frac{x}{6}.$$

Since $\lim_{x\to\infty} x/6 = \infty$, $\lim_{x\to\infty} e^x/x^2 = \infty$. More generally, by Exercise 76,

$$e^x \geq 1 + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{(n+1)!}.$$

Thus

$$\frac{e^x}{x^n} \geq \frac{1}{x^n} + \cdots + \frac{x}{(n+1)!} \geq \frac{x}{(n+1)!}.$$

Since $\lim_{x\to\infty} \frac{x}{(n+1)!} = \infty$, $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

Exercises 78–80 develop an elegant approach to the exponential and logarithm functions. Define a function $G(x)$ for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\, dt$$

78. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that $G(x)$ has all the basic properties of the logarithm. Prove the following statements.

(a) $\int_a^{ab} \frac{1}{t}\, dt = \int_1^b \frac{1}{t}\, dt$ for all $a, b > 0$. *Hint:* Use the substitution $u = t/a$.

(b) $G(ab) = G(a) + G(b)$. *Hint:* Break up the integral from 1 to ab into two integrals and use (a).

(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n.

(e) $G(a^{1/n}) = \frac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.

(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.

(g) $G(x)$ is increasing. *Hint:* Use FTC II.

(h) There exists a number a such that $G(a) > 1$. *Hint:* Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$.

(i) $\lim_{x\to\infty} G(x) = \infty$ and $\lim_{x\to 0+} G(x) = -\infty$

(j) There exists a unique number E such that $G(E) = 1$.

(k) $G(E^r) = r$ for every rational number r.

SOLUTION

(a) Let $u = t/a$. Then $du = dt/a$, $u(a) = 1$, $u(ab) = b$ and

$$\int_a^{ab} \frac{1}{t}\, dt = \int_a^{ab} \frac{a}{at}\, dt = \int_1^b \frac{1}{u}\, du = \int_1^b \frac{1}{t}\, dt.$$

(b) Using part (a),

$$G(ab) = \int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^b \frac{1}{t}\, dt = G(a) + G(b).$$

(c) First,

$$G(1) = \int_1^1 \frac{1}{t}\, dt = 0.$$

Next,

$$G(a^{-1}) = G\left(\frac{1}{a}\right) = \int_1^{1/a} \frac{1}{t}\, dt = \int_a^1 \frac{1}{t}\, dt \quad \text{by part (a) with } b = \frac{1}{a}$$
$$= -\int_1^a \frac{1}{t}\, dt = -G(a).$$

(d) Using part (a),

$$G(a^n) = \int_1^{a^n} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{a^2} \frac{1}{t}\,dt + \cdots + \int_{a^{n-1}}^{a^n} \frac{1}{t}\,dt$$
$$= \int_1^{a} \frac{1}{t}\,dt + \int_1^{a} \frac{1}{t}\,dt + \cdots + \int_1^{a} \frac{1}{t}\,dt = nG(a).$$

(e) $G(a) = G((a^{1/n})^n = nG(a^{1/n})$. Thus, $G(a^{1/n}) = \frac{1}{n}G(a)$.

(f) Let $r = m/n$ where m and n are integers. Then

$$\begin{aligned} G(a^r) &= G(a^{m/n}) = G((a^m)^{1/n}) \\ &= \frac{1}{n}G(a^m) \quad \text{by part (e)} \\ &= \frac{m}{n}G(a) \quad \text{by part d} \\ &= rG(a). \end{aligned}$$

(g) By the Fundamental Theorem of Calculus, $G(x)$ is continuous on $(0, \infty)$ and $G'(x) = \frac{1}{x} > 0$ for $x > 0$. Thus, $G(x)$ is increasing and one-to-one for $x > 0$.

(h) First note that

$$G(2) = \int_1^2 \frac{1}{t}\,dt > \frac{1}{2} > 0$$

because $\frac{1}{t} > \frac{1}{2}$ for $t \in (1, 2)$. Now, let $a = 2^m$ for m an integer greater than $1/G(2)$. Then

$$G(a) = G(2^m) = mG(2) > \frac{1}{G(2)} \cdot G(2) = 1.$$

(i) First, let a be the value from part (h) for which $G(a) > 1$ (note that a itself is greater than 1). Now,

$$\lim_{x\to\infty} G(x) = \lim_{m\to\infty} G(a^m) = G(a) \lim_{m\to\infty} m = \infty.$$

For the other limit, let $t = 1/x$ and note

$$\lim_{x\to 0+} G(x) = \lim_{t\to\infty} G\left(\frac{1}{t}\right) = -\lim_{t\to\infty} G(t) = -\infty.$$

(j) By part (c), $G(1) = 0$ and by part (h) there exists an a such that $G(a) > 1$. the Intermediate Value Theorem then guarantees there exists a number E such that $1 < E < a$ and $G(E) = 1$. We know that E is unique because G is one-to-one.

(k) Using part (f) and then part (j),

$$G(E^r) = rG(E) = r \cdot 1 = r.$$

79. Defining e^x Use Exercise 78 to prove the following statements.

(a) $G(x)$ has an inverse with domain $\mathbf{R}$ and range $\{x : x > 0\}$. Denote the inverse by $F(x)$.

(b) $F(x + y) = F(x)F(y)$ for all x, y. *Hint:* It suffices to show that $G(F(x)F(y)) = G(F(x + y))$.

(c) $F(r) = E^r$ for all numbers. In particular, $F(0) = 1$.

(d) $F'(x) = F(x)$. *Hint:* Use the formula for the derivative of an inverse function.

This shows that $E = e$ and $F(x)$ is the function e^x as defined in the text.

SOLUTION

(a) The domain of $G(x)$ is $x > 0$ and, by part (i) of the previous exercise, the range of $G(x)$ is $\mathbf{R}$. Now,

$$G'(x) = \frac{1}{x} > 0$$

for all $x > 0$. Thus, $G(x)$ is increasing on its domain, which implies that $G(x)$ has an inverse. The domain of the inverse is $\mathbf{R}$ and the range is $\{x : x > 0\}$. Let $F(x)$ denote the inverse of $G(x)$.

(b) Let x and y be real numbers and suppose that $x = G(w)$ and $y = G(z)$ for some positive real numbers w and z. Then, using part (b) of the previous exercise

$$F(x + y) = F(G(w) + G(z)) = F(G(wz)) = wz = F(x) + F(y).$$

(c) Let r be any real number. By part (k) of the previous exercise, $G(E^r) = r$. By definition of an inverse function, it then follows that $F(r) = E^r$.

(d) By the formula for the derivative of an inverse function

$$F'(x) = \frac{1}{G'(F(x))} = \frac{1}{1/F(x)} = F(x).$$

80. Defining b^x Let $b > 0$ and let $f(x) = F(xG(b))$ with F as in Exercise 79. Use Exercise 78 (f) to prove that $f(r) = b^r$ for every rational number r. This gives us a way of defining b^x for irrational x, namely $b^x = f(x)$. With this definition, b^x is a differentiable function of x (because F is differentiable).

SOLUTION By Exercise 78 (f),

$$f(r) = F(rG(b)) = F(G(b^r)) = b^r,$$

for every rational number r.

81. The formula $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ is valid for $n \neq -1$. Show that the exceptional case $n = -1$ is a limit of the general case by applying L'Hôpital's Rule to the limit on the left.

$$\lim_{n\to -1} \int_1^x t^n\,dt = \int_1^x t^{-1}\,dt \qquad \text{(for fixed } x > 0)$$

Note that the integral on the left is equal to $\dfrac{x^{n+1}-1}{n+1}$.

SOLUTION

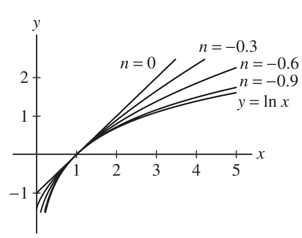
$$\lim_{n\to -1} \int_1^x t^n\,dt = \lim_{n\to -1} \left.\frac{t^{n+1}}{n+1}\right|_1^x = \lim_{n\to -1}\left(\frac{x^{n+1}}{n+1} - \frac{1^{n+1}}{n+1}\right)$$

$$= \lim_{n\to -1} \frac{x^{n+1}-1}{n+1} = \lim_{n\to -1}(x^{n+1})\ln x = \ln x = \int_1^x t^{-1}\,dt$$

Note that when using L'Hôpital's Rule in the second line, we need to differentiate with respect to n.

82. CAS The integral on the left in Exercise 81 is equal to $f_n(x) = \dfrac{x^{n+1}-1}{n+1}$. Investigate the limit graphically by plotting $f_n(x)$ for $n = 0, -0.3, -0.6$, and -0.9 together with $\ln x$ on a single plot.

SOLUTION



83. **(a)** Explain why the shaded region in Figure 5 has area $\int_0^{\ln a} e^y\,dy$.

(b) Prove the formula $\int_1^a \ln x\,dx = a\ln a - \int_0^{\ln a} e^y\,dy$.

(c) Conclude that $\int_1^a \ln x\,dx = a\ln a - a + 1$.

(d) Use the result of (a) to find an antiderivative of $\ln x$.

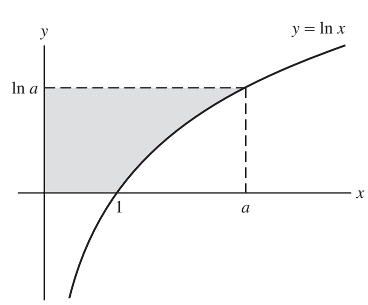


FIGURE 5

SOLUTION

(a) Interpreting the graph with y as the independent variable, we see that the function is $x = e^y$. Integrating in y then gives the area of the shaded region as $\int_0^{\ln a} e^y\,dy$

(b) We can obtain the area under the graph of $y = \ln x$ from $x = 1$ to $x = a$ by computing the area of the rectangle extending from $x = 0$ to $x = a$ horizontally and from $y = 0$ to $y = \ln a$ vertically and then subtracting the area of the shaded region. This yields

$$\int_1^a \ln x\,dx = a\ln a - \int_0^{\ln a} e^y\,dy.$$

(c) By direct calculation

$$\int_0^{\ln a} e^y\,dy = e^y\Big|_0^{\ln a} = a - 1.$$

Thus,

$$\int_1^a \ln x\,dx = a\ln a - (a-1) = a\ln a - a + 1.$$

(d) Based on these results it appears that

$$\int \ln x\,dx = x\ln x - x + C.$$

5.8 Exponential Growth and Decay

Preliminary Questions

1. Two quantities increase exponentially with growth constants $k = 1.2$ and $k = 3.4$, respectively. Which quantity doubles more rapidly?

SOLUTION Doubling time is inversely proportional to the growth constant. Consequently, the quantity with $k = 3.4$ doubles more rapidly.

2. A cell population grows exponentially beginning with one cell. Which takes longer: increasing from one to two cells or increasing from 15 million to 20 million cells?

SOLUTION It takes longer for the population to increase from one cell to two cells, because this requires doubling the population. Increasing from 15 million to 20 million is less than doubling the population.

3. Referring to his popular book *A Brief History of Time*, the renowned physicist Stephen Hawking said, "Someone told me that each equation I included in the book would halve its sales." Find a differential equation satisfied by the function $S(n)$, the number of copies sold if the book has n equations.

SOLUTION Let $S(0)$ denote the sales with no equations in the book. Translating Hawking's observation into an equation yields

$$S(n) = \frac{S(0)}{2^n}.$$

Differentiating with respect to n then yields

$$\frac{dS}{dn} = S(0)\frac{d}{dn}2^{-n} = -\ln 2S(0)2^{-n} = -\ln 2S(n).$$

4. The PV of N dollars received at time T is (choose the correct answer):

(a) The value at time T of N dollars invested today

(b) The amount you would have to invest today in order to receive N dollars at time T

SOLUTION The correct response is **(b)**: the PV of N dollars received at time T is the amount you would have to invest today in order to receive N dollars at time T.

5. In one year, you will be paid \$1. Will the PV increase or decrease if the interest rate goes up?

SOLUTION If the interest rate goes up, the present value of \$1 a year from now will decrease.

Exercises

1. A certain population P of bacteria obeys the exponential growth law $P(t) = 2000e^{1.3t}$ (t in hours).

(a) How many bacteria are present initially?

(b) At what time will there be 10,000 bacteria?

SOLUTION

(a) $P(0) = 2000e^0 = 2000$ bacteria initially.

(b) We solve $2000e^{1.3t} = 10,000$ for t. Thus, $e^{1.3t} = 5$ or

$$t = \frac{1}{1.3}\ln 5 \approx 1.24 \text{ hours.}$$

2. A quantity P obeys the exponential growth law $P(t) = e^{5t}$ (t in years).

(a) At what time t is $P = 10$?

(b) What is the doubling time for P?

SOLUTION

(a) $e^{5t} = 10$ when $t = \frac{1}{5}\ln 10 \approx 0.46$ years.

(b) The doubling time is $\frac{1}{5}\ln 2 \approx 0.14$ years.

3. Write $f(t) = 5(7)^t$ in the form $f(t) = P_0e^{kt}$ for some P_0 and k.

SOLUTION Because $7 = e^{\ln 7}$, it follows that

$$f(t) = 5(7)^t = 5(e^{\ln 7})^t = 5e^{t\ln 7}.$$

Thus, $P_0 = 5$ and $k = \ln 7$.

4. Write $f(t) = 9e^{1.4t}$ in the form $f(t) = P_0b^t$ for some P_0 and b.

SOLUTION Observe that

$$f(t) = 9e^{1.4t} = 9\left(e^{1.4}\right)^t,$$

so $P_0 = 9$ and $b = e^{1.4} \approx 4.0552$.

5. A certain RNA molecule replicates every 3 minutes. Find the differential equation for the number $N(t)$ of molecules present at time t (in minutes). How many molecules will be present after one hour if there is one molecule at $t = 0$?

SOLUTION The doubling time is $\dfrac{\ln 2}{k}$ so $k = \dfrac{\ln 2}{\text{doubling time}}$. Thus, the differential equation is $N'(t) = kN(t) = \dfrac{\ln 2}{3}N(t)$. With one molecule initially,

$$N(t) = e^{(\ln 2/3)t} = 2^{t/3}.$$

Thus, after one hour, there are

$$N(60) = 2^{60/3} = 1{,}048{,}576$$

molecules present.

6. A quantity P obeys the exponential growth law $P(t) = Ce^{kt}$ (t in years). Find the formula for $P(t)$, assuming that the doubling time is 7 years and $P(0) = 100$.

SOLUTION The doubling time is 7 years, so $7 = \ln 2/k$, or $k = \ln 2/7 = 0.099$ years^{-1}. With $P(0) = 100$, it follows that $P(t) = 100e^{0.099t}$.

7. Find all solutions to the differential equation $y' = -5y$. Which solution satisfies the initial condition $y(0) = 3.4$?

SOLUTION $y' = -5y$, so $y(t) = Ce^{-5t}$ for some constant C. The initial condition $y(0) = 3.4$ determines $C = 3.4$. Therefore, $y(t) = 3.4e^{-5t}$.

8. Find the solution to $y' = \sqrt{2}y$ satisfying $y(0) = 20$.

SOLUTION $y' = \sqrt{2}y$, so $y(t) = Ce^{\sqrt{2}t}$ for some constant C. The initial condition $y(0) = 20$ determines $C = 20$. Therefore, $y(t) = 20e^{\sqrt{2}t}$.

9. Find the solution to $y' = 3y$ satisfying $y(2) = 1000$.

SOLUTION $y' = 3y$, so $y(t) = Ce^{3t}$ for some constant C. The initial condition $y(2) = 1000$ determines $C = \dfrac{1000}{e^6}$. Therefore, $y(t) = \dfrac{1000}{e^6}e^{3t} = 1000e^{3(t-2)}$.

10. Find the function $y = f(t)$ that satisfies the differential equation $y' = -0.7y$ and the initial condition $y(0) = 10$.

SOLUTION Given that $y' = -0.7y$ and $y(0) = 10$, then $f(t) = 10e^{-0.7t}$.

11. The decay constant of cobalt-60 is 0.13 year^{-1}. Find its half-life.

SOLUTION Half-life $= \dfrac{\ln 2}{0.13} \approx 5.33$ years.

12. The half-life radium-226 is 1622 years. Find its decay constant.

SOLUTION Half-life $= \dfrac{\ln 2}{k}$ so $k = \dfrac{\ln 2}{\text{half-life}} = \dfrac{\ln 2}{1622} = 4.27 \times 10^{-4}$ years^{-1}.

13. One of the world's smallest flowering plants, *Wolffia globosa* (Figure 13), has a doubling time of approximately 30 hours. Find the growth constant k and determine the initial population if the population grew to 1000 after 48 hours.

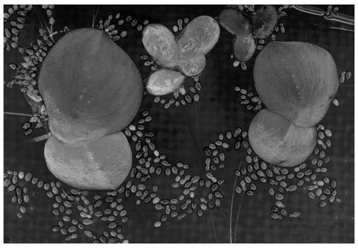


FIGURE 13 The tiny plants are *Wolffia*, with plant bodies smaller than the head of a pin.

SOLUTION By the formula for the doubling time, $30 = \dfrac{\ln 2}{k}$. Therefore,

$$k = \frac{\ln 2}{30} \approx 0.023 \text{ hours}^{-1}.$$

The plant population after t hours is $P(t) = P_0 e^{0.023t}$. If $P(48) = 1000$, then

$$P_0 e^{(0.023)48} = 1000 \quad \Rightarrow \quad P_0 = 1000e^{-(0.023)48} \approx 332$$

14. A 10-kg quantity of a radioactive isotope decays to 3 kg after 17 years. Find the decay constant of the isotope.

SOLUTION $P(t) = 10e^{-kt}$. Thus $P(17) = 3 = 10e^{-17k}$, so $k = \dfrac{\ln(3/10)}{-17} \approx 0.071$ years^{-1}.

15. The population of a city is $P(t) = 2 \cdot e^{0.06t}$ (in millions), where t is measured in years. Calculate the time it takes for the population to double, to triple, and to increase seven-fold.

SOLUTION Since $k = 0.06$, the doubling time is

$$\frac{\ln 2}{k} \approx 11.55 \text{ years.}$$

The tripling time is calculated in the same way as the doubling time. Solve for Δ in the equation

$$\begin{aligned} P(t + \Delta) &= 3P(t) \\ 2 \cdot e^{0.06(t+\Delta)} &= 3(2e^{0.06t}) \\ 2 \cdot e^{0.06t} e^{0.06\Delta} &= 3(2e^{0.06t}) \\ e^{0.06\Delta} &= 3 \\ 0.06\Delta &= \ln 3, \end{aligned}$$

or $\Delta = \ln 3/0.06 \approx 18.31$ years. Working in a similar fashion, we find that the time required for the population to increase seven-fold is

$$\frac{\ln 7}{k} = \frac{\ln 7}{0.06} \approx 32.43 \text{ years.}$$

16. What is the differential equation satisfied by $P(t)$, the number of infected computer hosts in Example 4? Over which time interval would $P(t)$ increase one hundred-fold?

SOLUTION Because the rate constant is $k = 0.0815$ s^{-1}, the differential equation for $P(t)$ is

$$\frac{dP}{dt} = 0.0815P.$$

The time for the number of infected computers to increase one hundred-fold is

$$\frac{\ln 100}{k} = \frac{\ln 100}{0.0815} \approx 56.51 \text{ s.}$$

17. The decay constant for a certain drug is $k = 0.35$ day^{-1}. Calculate the time it takes for the quantity present in the bloodstream to decrease by half, by one-third, and by one-tenth.

SOLUTION The time required for the quantity present in the bloodstream to decrease by half is

$$\frac{\ln 2}{k} = \frac{\ln 2}{0.35} \approx 1.98 \text{ days.}$$

To decay by one-third, the time is

$$\frac{\ln 3}{k} = \frac{\ln 3}{0.35} \approx 3.14 \text{ days.}$$

Finally, to decay by one-tenth, the time is

$$\frac{\ln 10}{k} = \frac{\ln 10}{0.35} \approx 6.58 \text{ days.}$$

18. Light Intensity The intensity of light passing through an absorbing medium decreases exponentially with the distance traveled. Suppose the decay constant for a certain plastic block is $k = 4$ m^{-1}. How thick must the block be to reduce the intensity by a factor of one-third?

SOLUTION Since intensity decreases exponentially, it can be modeled by an exponential decay equation $I(d) = I_0 e^{-kd}$. Assuming $I(0) = 1$, $I(d) = e^{-kd}$. Since the decay constant is $k = 4$, we have $I(d) = e^{-4d}$. Intensity will be reduced by a factor of one-third when $e^{-4d} = \frac{1}{3}$ or when $d = \dfrac{\ln(1/3)}{-4} \approx 0.275$ m.

19. Assuming that population growth is approximately exponential, which of the following two sets of data is most likely to represent the population (in millions) of a city over a 5-year period?

Year	2000	2001	2002	2003	2004
Set I	3.14	3.36	3.60	3.85	4.11
Set II	3.14	3.24	3.54	4.04	4.74

SOLUTION If the population growth is approximately exponential, then the ratio between successive years' data needs to be approximately the same.

Year	2000		2001		2002		2003		2004
Data I	3.14		3.36		3.60		3.85		4.11
Ratios		1.07006		1.07143		1.06944		1.06753	
Data II	3.14		3.24		3.54		4.04		4.74
Ratios		1.03185		1.09259		1.14124		1.17327	

As you can see, the ratio of successive years in the data from "Data I" is very close to 1.07. Therefore, we would expect exponential growth of about $P(t) \approx (3.14)(1.07^t)$.

20. The **atmospheric pressure** $P(h)$ (in kilopascals) at a height h meters above sea level satisfies a differential equation $P' = -kP$ for some positive constant k.

(a) Barometric measurements show that $P(0) = 101.3$ and $P(30,900) = 1.013$. What is the decay constant k?

(b) Determine the atmospheric pressure at $h = 500$.

SOLUTION

(a) Because $P' = -kP$ for some positive constant k, $P(h) = Ce^{-kh}$ where $C = P(0) = 101.3$. Therefore, $P(h) = 101.3e^{-kh}$. We know that $P(30{,}900) = 101.3e^{-30{,}900k} = 1.013$. Solving for k yields

$$k = -\frac{1}{30{,}900} \ln\left(\frac{1.013}{101.3}\right) \approx 0.000149 \text{ meters}^{-1}.$$

(b) $P(500) = 101.3e^{-0.000149(500)} \approx 94.03$ kilopascals.

21. Degrees in Physics One study suggests that from 1955 to 1970, the number of bachelor's degrees in physics awarded per year by U.S. universities grew exponentially, with growth constant $k = 0.1$.
(a) If exponential growth continues, how long will it take for the number of degrees awarded per year to increase 14-fold?
(b) If 2500 degrees were awarded in 1955, in which year were 10,000 degrees awarded?

SOLUTION
(a) The time required for the number of degrees to increase 14-fold is

$$\frac{\ln 14}{k} = \frac{\ln 14}{0.1} \approx 26.39 \text{ years.}$$

(b) The doubling time is $(\ln 2)/0.1 \approx 0.693/0.1 = 6.93$ years. Since degrees are usually awarded once a year, we round off the doubling time to 7 years. The number quadruples after 14 years, so 10, 000 degrees would be awarded in 1969.

22. The **Beer–Lambert Law** is used in spectroscopy to determine the molar absorptivity α or the concentration c of a compound dissolved in a solution at low concentrations (Figure 14). The law states that the intensity I of light as it passes through the solution satisfies $\ln(I/I_0) = \alpha c x$, where I_0 is the initial intensity and x is the distance traveled by the light. Show that I satisfies a differential equation $dI/dx = -kI$ for some constant k.

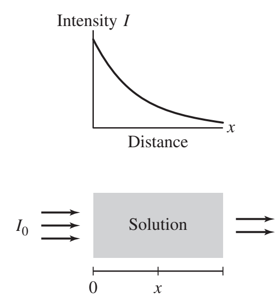


FIGURE 14 Light of intensity passing through a solution.

SOLUTION $\ln\left(\dfrac{I}{I_0}\right) = \alpha c I$ so $\dfrac{I}{I_0} = e^{\alpha c I}$ or $I = I_0 e^{\alpha c I}$. Therefore,

$$\frac{dI}{dx} = I_0 e^{\alpha c I}(\alpha c) = I(\alpha c) = -kI,$$

where $k = -\alpha c$ is a constant.

23. A sample of sheepskin parchment discovered by archaeologists had a C^{14}-to-C^{12} ratio equal to 40% of that found in the atmosphere. Approximately how old is the parchment?

SOLUTION The ratio of C^{14} to C^{12} is $Re^{-0.000121t} = 0.4R$ so $-0.000121t = \ln(0.4)$ or $t = 7572.65 \approx 7600$ years.

24. Chauvet Caves In 1994, three French speleologists (geologists specializing in caves) discovered a cave in southern France containing prehistoric cave paintings. A C^{14} analysis carried out by archeologist Helene Valladas showed the paintings to be between 29,700 and 32,400 years old, much older than any previously known human art. Given that the C^{14}-to-C^{12} ratio of the atmosphere is $R = 10^{-12}$, what range of C^{14}-to-C^{12} ratios did Valladas find in the charcoal specimens?

SOLUTION The C^{14}-C^{12} ratio found in the specimens ranged from

$$10^{-12}e^{-0.000121(32,400)} \approx 1.98 \times 10^{-14}$$

to

$$10^{-12}e^{-0.000121(29,700)} \approx 2.75 \times 10^{-14}.$$

25. A paleontologist discovers remains of animals that appear to have died at the onset of the Holocene ice age, between 10,000 and 12,000 years ago. What range of C^{14}-to-C^{12} ratio would the scientist expect to find in the animal remains?

SOLUTION The scientist would expect to find C^{14}-C^{12} ratios ranging from

$$10^{-12}e^{-0.000121(12,000)} \approx 2.34 \times 10^{-13}$$

to

$$10^{-12}e^{-0.000121(10,000)} \approx 2.98 \times 10^{-13}.$$

26. Inversion of Sugar When cane sugar is dissolved in water, it converts to invert sugar over a period of several hours. The percentage $f(t)$ of unconverted cane sugar at time t (in hours) satisfies $f' = -0.2f$. What percentage of cane sugar remains after 5 hours? After 10 hours?

SOLUTION $f' = -0.2f$, so $f(t) = Ce^{-0.2t}$. Since f is a percentage, at $t = 0$, $C = 100$ percent. Therefore. $f(t) = 100e^{-0.2t}$. Thus $f(5) = 100e^{-0.2(5)} \approx 36.79$ percent and $f(10) = 100e^{-0.2(10)} \approx 13.53$ percent.

27. Continuing with Exercise 26, suppose that 50 grams of sugar are dissolved in a container of water. After how many hours will 20 grams of invert sugar be present?

SOLUTION If there are 20 grams of invert sugar present, then there are 30 grams of unconverted sugar. This means that $f = 60$. Solving

$$100e^{-0.2t} = 60$$

for t yields

$$t = -\frac{1}{0.2}\ln 0.6 \approx 2.55 \text{ hours.}$$

28. Two bacteria colonies are cultivated in a laboratory. The first colony has a doubling time of 2 hours and the second a doubling time of 3 hours. Initially, the first colony contains 1000 bacteria and the second colony 3000 bacteria. At what time t will the sizes of the colonies be equal?

SOLUTION $P_1(t) = 1000e^{k_1 t}$ and $P_2(t) = 3000e^{k_2 t}$. Knowing that $k_1 = \frac{\ln 2}{2}$ hours^{-1} and $k_2 = \frac{\ln 2}{3}$ hours^{-1}, we need to solve $e^{k_1 t} = 3e^{k_2 t}$ for t. Thus

$$k_1 t = \ln(3e^{k_2 t}) = \ln 3 + \ln(e^{k_2 t}) = \ln 3 + k_2 t,$$

so

$$t = \frac{\ln 3}{k_1 - k_2} = \frac{6 \ln 3}{\ln 2} \approx 9.51 \text{ hours.}$$

29. Moore's Law In 1965, Gordon Moore predicted that the number N of transistors on a microchip would increase exponentially.

(a) Does the table of data below confirm Moore's prediction for the period from 1971 to 2000? If so, estimate the growth constant k.

(b) CAS Plot the data in the table.

(c) Let $N(t)$ be the number of transistors t years after 1971. Find an approximate formula $N(t) \approx Ce^{kt}$, where t is the number of years after 1971.

(d) Estimate the doubling time in Moore's Law for the period from 1971 to 2000.

(e) How many transistors will a chip contain in 2015 if Moore's Law continues to hold?

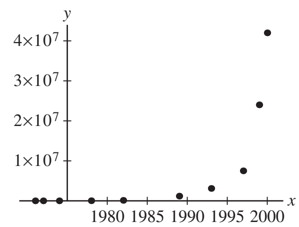
(f) Can Moore have expected his prediction to hold indefinitely?

Processor	Year	No. Transistors
4004	1971	2250
8008	1972	2500
8080	1974	5000
8086	1978	29,000
286	1982	120,000
386 processor	1985	275,000
486 DX processor	1989	1,180,000
Pentium processor	1993	3,100,000
Pentium II processor	1997	7,500,000
Pentium III processor	1999	24,000,000
Pentium 4 processor	2000	42,000,000
Xeon processor	2008	1,900,000,000

SOLUTION

(a) Yes, the graph looks like an exponential graph especially towards the latter years. We estimate the growth constant by setting 1971 as our starting point, so $P_0 = 2250$. Therefore, $P(t) = 2250e^{kt}$. In 2008, $t = 37$. Therefore, $P(37) = 2250e^{37k} = 1{,}900{,}000{,}000$, so $k = \frac{\ln 844{,}444.444}{37} \approx 0.369$. Note: A better estimate can be found by calculating k for each time period and then averaging the k values.

(b)



(c) $N(t) = 2250e^{0.369t}$
(d) The doubling time is $\ln 2/0.369 \approx 1.88$ years.
(e) In 2015, $t = 44$ years. Therefore, $N(44) = 2250e^{0.369(44)} \approx 2.53 \times 10^{10}$.
(f) No, you can't make a microchip smaller than an atom.

30. Assume that in a certain country, the rate at which jobs are created is proportional to the number of people who already have jobs. If there are 15 million jobs at $t = 0$ and 15.1 million jobs 3 months later, how many jobs will there be after 2 years?

SOLUTION Let $J(t)$ denote the number of people, in millions, who have jobs at time t, in months. Because the rate at which jobs are created is proportional to the number of people who already have jobs, $J'(t) = kJ(t)$, for some constant k. Given that $J(0) = 15$, it then follows that $J(t) = 15e^{kt}$. To determine k, we use $J(3) = 15.1$; therefore,

$$k = \frac{1}{3}\ln\left(\frac{15.1}{15}\right) \approx 2.215 \times 10^{-3} \text{ months}^{-1}.$$

Finally, after two years, there are

$$J(24) = 15e^{0.002215(24)} \approx 15.8 \text{ million}$$

jobs.

31. The only functions with a *constant* doubling time are the exponential functions P_0e^{kt} with $k > 0$. Show that the doubling time of linear function $f(t) = at + b$ at time t_0 is $t_0 + b/a$ (which increases with t_0). Compute the doubling times of $f(t) = 3t + 12$ at $t_0 = 10$ and $t_0 = 20$.

SOLUTION Let $f(t) = at + b$ and suppose $f(t_0) = P_0$. The time it takes for the value of f to double is the solution of the equation

$$2P_0 = 2(at_0 + b) = at + b \quad \text{or} \quad t = 2t_0 + b/a.$$

For the function $f(t) = 3t + 12$, $a = 3$, $b = 12$ and $b/a = 4$. With $t_0 = 10$, the doubling time is then 24; with $t_0 = 20$, the doubling time is 44.

32. Verify that the half-life of a quantity that decays exponentially with decay constant k is equal to $(\ln 2)/k$.

SOLUTION Let $y = Ce^{-kt}$ be an exponential decay function. Let t be the half-life of the quantity y, that is, the time t when $y = \frac{C}{2}$. Solving $\frac{C}{2} = Ce^{-kt}$ for t we get $-\ln 2 = -kt$, so $t = \ln 2/k$.

33. Compute the balance after 10 years if \$2000 is deposited in an account paying 9% interest and interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

SOLUTION

(a) $P(10) = 2000(1 + 0.09/4)^{4(10)} = \$4870.38$
(b) $P(10) = 2000(1 + 0.09/12)^{12(10)} = \$4902.71$
(c) $P(10) = 2000e^{0.09(10)} = \$4919.21$

34. Suppose \$500 is deposited into an account paying interest at a rate of 7%, continuously compounded. Find a formula for the value of the account at time t. What is the value of the account after 3 years?

SOLUTION Let $P(t)$ denote the value of the account at time t. Because the initial deposit is \$500 and the account pays interest at a rate of 7%, compounded continuously, it follows that $P(t) = 500e^{0.07t}$. After three years, the value of the account is $P(3) = 500e^{0.07(3)} = \$616.84$.

35. A bank pays interest at a rate of 5%. What is the yearly multiplier if interest is compounded

(a) three times a year? **(b)** continuously?

SOLUTION

(a) $P(t) = P_0\left(1 + \frac{0.05}{3}\right)^{3t}$, so the yearly multiplier is $\left(1 + \frac{0.05}{3}\right)^{3} \approx 1.0508$.
(b) $P(t) = P_0e^{0.05t}$, so the yearly multiplier is $e^{0.05} \approx 1.0513$.

36. How long will it take for \$4000 to double in value if it is deposited in an account bearing 7% interest, continuously compounded?

SOLUTION The doubling time is $\frac{\ln 2}{0.7} \approx 9.9$ years.

37. How much must one invest today in order to receive \$20,000 after 5 years if interest is compounded continuously at the rate $r = 9\%$?

SOLUTION Solving $20{,}000 = P_0e^{0.09(5)}$ for P_0 yields

$$P_0 = \frac{20{,}000}{e^{0.45}} \approx \$12{,}752.56.$$

38. An investment increases in value at a continuously compounded rate of 9%. How large must the initial investment be in order to build up a value of $50,000 over a 7-year period?

SOLUTION Solving $50{,}000 = P_0 e^{0.09(7)}$ for P_0 yields

$$P_0 = \frac{50{,}000}{e^{0.63}} \approx \$26{,}629.59.$$

39. Compute the PV of $5000 received in 3 years if the interest rate is (a) 6% and (b) 11%. What is the PV in these two cases if the sum is instead received in 5 years?

SOLUTION In 3 years:

(a) $PV = 5000e^{-0.06(3)} = \$4176.35$

(b) $PV = 5000e^{-0.11(3)} = \$3594.62$

In 5 years:

(a) $PV = 5000e^{-0.06(5)} = \$3704.09$

(b) $PV = 5000e^{-0.11(5)} = \$2884.75$

40. Is it better to receive $1000 today or $1300 in 4 years? Consider $r = 0.08$ and $r = 0.03$.

SOLUTION Assuming continuous compounding, if $r = 0.08$, then the present value of $1300 four years from now is $1300e^{-0.08(4)} = \$943.99$. It is better to get $1000 now. On the other hand, if $r = 0.03$, the present value of $1300 four years from now is $1300e^{-0.03(4)} = \$1153.00$, so it is better to get the $1,300 in four years.

41. Find the interest rate r if the PV of $8000 to be received in 1 year is $7300.

SOLUTION Solving $7300 = 8000e^{-r(1)}$ for r yields

$$r = -\ln\left(\frac{7300}{8000}\right) = 0.0916,$$

or 9.16%.

42. A company can earn additional profits of $500,000/year for 5 years by investing $2 million to upgrade its factory. Is the investment worthwhile if the interest rate is 6%? (Assume the savings are received as a lump sum at the end of each year.)

SOLUTION The present value of the stream of additional profits is

$$500{,}000(e^{-0.06} + e^{-0.12} + e^{-0.18} + e^{-0.24} + e^{-0.3}) = \$2{,}095{,}700.63.$$

This is more than the $2 million cost of the upgrade, so the upgrade should be made.

43. A new computer system costing $25,000 will reduce labor costs by $7000/year for 5 years.

(a) Is it a good investment if $r = 8\%$?

(b) How much money will the company actually save?

SOLUTION

(a) The present value of the reduced labor costs is

$$7000(e^{-0.08} + e^{-0.16} + e^{-0.24} + e^{-0.32} + e^{-0.4}) = \$27{,}708.50.$$

This is more than the $25,000 cost of the computer system, so the computer system should be purchased.

(b) The present value of the savings is

$$\$27{,}708.50 - \$25{,}000 = \$2708.50.$$

44. After winning $25 million in the state lottery, Jessica learns that she will receive five yearly payments of $5 million beginning immediately.

(a) What is the PV of Jessica's prize if $r = 6\%$?

(b) How much more would the prize be worth if the entire amount were paid today?

SOLUTION

(a) The present value of the prize is

$$5{,}000{,}000(e^{-0.24} + e^{-0.18} + e^{-0.12} + e^{-0.06} + e^{-0.06(0)}) = \$22{,}252{,}915.21.$$

(b) If the entire amount were paid today, the present value would be $25 million, or $2,747,084.79 more than the stream of payments made over five years.

45. Use Eq. (3) to compute the PV of an income stream paying out $R(t) = \$5000$/year continuously for 10 years, assuming $r = 0.05$.

SOLUTION $PV = \int_0^{10} 5000e^{-0.05t}\,dt = -100{,}000e^{-0.05t}\Big|_0^{10} = \$39{,}346.93.$

46. Find the PV of an investment that pays out continuously at a rate of \$800/year for 5 years, assuming $r = 0.08$.

SOLUTION $PV = \int_0^{5} 800e^{-0.08t}\,dt = -10{,}000e^{-0.08t}\Big|_0^{5} = \$3296.80.$

47. Find the PV of an income stream that pays out continuously at a rate $R(t) = \$5000e^{0.1t}$/year for 7 years, assuming $r = 0.05$.

SOLUTION $PV = \int_0^{7} 5000e^{0.1t}e^{-0.05t}\,dt = \int_0^{7} 5000e^{0.05t}\,dt = 100{,}000e^{0.05t}\Big|_0^{7} = \$41{,}906.75.$

48. A commercial property generates income at the rate $R(t)$. Suppose that $R(0) = \$70{,}000$/year and that $R(t)$ increases at a continuously compounded rate of 5%. Find the PV of the income generated in the first 4 years if $r = 6\%$.

SOLUTION $PV = \int_0^{4} 70{,}000e^{0.05t}e^{-0.06t}\,dt = -\frac{70{,}000}{0.01}e^{-0.01t}\Big|_0^{4} = \$274{,}473.93.$

49. Show that an investment that pays out R dollars per year continuously for T years has a PV of $R(1 - e^{-rT})/r$.

SOLUTION The present value of an investment that pays out R dollars/year continuously for T years is

$$PV = \int_0^{T} Re^{-rt}\,dt.$$

Let $u = -rt,\ du = -r\,dt$. Then

$$PV = -\frac{1}{r}\int_0^{-rT} Re^{u}\,du = -\frac{R}{r}e^{u}\Big|_0^{-rT} = -\frac{R}{r}(e^{-rT} - 1) = \frac{R}{r}(1 - e^{-rT}).$$

50. Explain this statement: If T is very large, then the PV of the income stream described in Exercise 49 is approximately R/r.

SOLUTION Because

$$\lim_{T\to\infty} e^{-rT} = \lim_{T\to\infty}\frac{1}{e^{rt}} = 0,$$

it follows that

$$\lim_{T\to\infty}\frac{R}{r}(1 - e^{-rT}) = \frac{R}{r}.$$

51. Suppose that $r = 0.06$. Use the result of Exercise 50 to estimate the payout rate R needed to produce an income stream whose PV is \$20,000, assuming that the stream continues for a large number of years.

SOLUTION From Exercise 50, $PV = \frac{R}{r}$ so $20{,}000 = \frac{R}{0.06}$ or $R = \$1200$.

52. Verify by differentiation:

$$\int te^{-rt}\,dt = -\frac{e^{-rt}(1 + rt)}{r^2} + C \qquad \boxed{5}$$

Use Eq. (5) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)$ dollars per year for 5 years, assuming $r = 0.05$.

SOLUTION

$$\frac{d}{dt}\left(-\frac{e^{-rt}(1 + rt)}{r^2}\right) = \frac{-1}{r^2}\left(e^{-rt}(r) + (1 + rt)(-re^{-rt})\right) = \frac{-1}{r}\left(e^{-rt} - e^{-rt} - rte^{-rt}\right) = te^{-rt}$$

Therefore

$$\begin{aligned} PV &= \int_0^{5}(5000 + 1000t)e^{-0.05t}\,dt = \int_0^{5} 5000e^{-0.05t}\,dt + \int_0^{5} 1000te^{-0.05t}\,dt \\ &= \frac{5000}{-0.05}(e^{-0.05(5)} - 1) - 1000\left(\frac{e^{-0.05(5)}(1 + 0.05(5))}{(0.05)^2}\right) + 1000\frac{1}{(0.05)^2} \\ &= 22{,}119.92 - 389{,}400.39 + 400{,}000 \approx \$32{,}719.53. \end{aligned}$$

53. Use Eq. (5) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)e^{0.02t}$ dollars per year for 10 years, assuming $r = 0.08$.

SOLUTION

$$\begin{aligned} PV &= \int_0^{10} (5000 + 1000t)(e^{0.02t})e^{-0.08t}\,dt = \int_0^{10} 5000e^{-0.06t}\,dt + \int_0^{10} 1000te^{-0.06t}\,dt \\ &= \frac{5000}{-0.06}(e^{-0.06(10)} - 1) - 1000\left(\frac{e^{-0.06(10)}(1 + 0.06(10))}{(0.06)^2}\right) + 1000\frac{1}{(0.06)^2} \\ &= 37{,}599.03 - 243{,}916.28 + 277{,}777.78 \approx \$71{,}460.53. \end{aligned}$$

54. **Banker's Rule of 70** If you earn an interest rate of R percent, continuously compounded, your money doubles after approximately $70/R$ years. For example, at $R = 5\%$, your money doubles after $70/5$ or 14 years. Use the concept of doubling time to justify the Banker's Rule. (*Note:* Sometimes, the rule $72/R$ is used. It is less accurate but easier to apply because 72 is divisible by more numbers than 70.)

SOLUTION The doubling time is

$$t = \frac{\ln 2}{r} = \frac{\ln 2 \cdot 100}{r\%} = \frac{69.93}{r\%} \approx \frac{70}{r\%}.$$

55. **Drug Dosing Interval** Let $y(t)$ be the drug concentration (in mg/kg) in a patient's body at time t. The initial concentration is $y(0) = L$. Additional doses that increase the concentration by an amount d are administered at regular time intervals of length T. In between doses, $y(t)$ decays exponentially—that is, $y' = -ky$. Find the value of T (in terms of k and d) for which the the concentration varies between L and $L - d$ as in Figure 15.

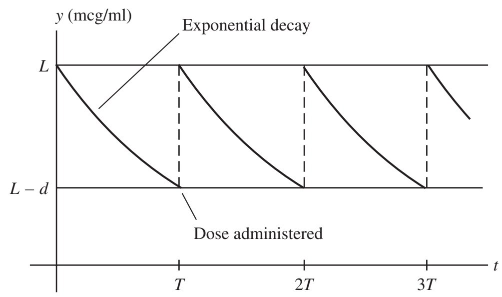


FIGURE 15 Drug concentration with periodic doses.

SOLUTION Because $y' = -ky$ and $y(0) = L$, it follows that $y(t) = Le^{-kt}$. We want $y(T) = L - d$, thus

$$Le^{-kT} = L - d \quad \text{or} \quad T = -\frac{1}{k}\ln\left(1 - \frac{d}{L}\right).$$

Exercises 56 and 57: The ***Gompertz differential equation***

$$\frac{dy}{dt} = ky\ln\left(\frac{y}{M}\right) \qquad \boxed{6}$$

(where M and k are constants) was introduced in 1825 by the English mathematician Benjamin Gompertz and is still used today to model aging and mortality.

56. Show that $y = Me^{ae^{kt}}$ satisfies Eq. (6) for any constant a.

SOLUTION Let $y = Me^{ae^{kt}}$. Then

$$\frac{dy}{dt} = M(kae^{kt})e^{ae^{kt}}$$

and, since

$$\ln(y/M) = ae^{kt},$$

we have

$$ky\ln(y/M) = Mkae^{kt}e^{ae^{kt}} = \frac{dy}{dt}.$$

57. To model mortality in a population of 200 laboratory rats, a scientist assumes that the number $P(t)$ of rats alive at time t (in months) satisfies Eq. (6) with $M = 204$ and $k = 0.15$ month^{-1} (Figure 16). Find $P(t)$ [note that $P(0) = 200$] and determine the population after 20 months.

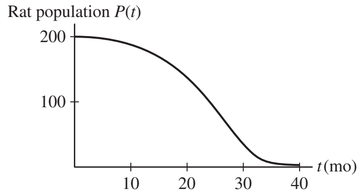


FIGURE 16

SOLUTION The solution to the Gompertz equation with $M = 204$ and $k = 0.15$ is of the form:

$$P(t) = 204e^{ae^{0.15t}}$$

Applying the initial condition allows us to solve for a:

$$200 = 204e^{a}$$
$$\frac{200}{204} = e^{a}$$
$$\ln\left(\frac{200}{204}\right) = a$$

so that $a \approx -0.02$. After $t = 20$ months,

$$P(20) = 204e^{-0.02e^{0.15(20)}} = 136.51,$$

so there are 136 rats.

58. **Isotopes for Dating** Which of the following would be most suitable for dating extremely old rocks: carbon-14 (half-life 5570 years), lead-210 (half-life 22.26 years), or potassium-49 (half-life 1.3 billion years)? Explain why.

SOLUTION For extremely old rocks, you need to have an isotope that decays very slowly. In other words, you want a very large half-life such as Potassium-49; otherwise, the amount of undecayed isotope in the rock sample would be too small to accurately measure.

59. Let $P = P(t)$ be a quantity that obeys an exponential growth law with growth constant k. Show that P increases m-fold after an interval of $(\ln m)/k$ years.

SOLUTION For m-fold growth, $P(t) = mP_0$ for some t. Solving $mP_0 = P_0e^{kt}$ for t, we find $t = \dfrac{\ln m}{k}$

Further Insights and Challenges

60. **Average Time of Decay** Physicists use the radioactive decay law $R = R_0e^{-kt}$ to compute the average or *mean time* M until an atom decays. Let $F(t) = R/R_0 = e^{-kt}$ be the fraction of atoms that have survived to time t without decaying.

(a) Find the inverse function $t(F)$.

(b) By definition of $t(F)$, a fraction $1/N$ of atoms decays in the time interval

$$\left[t\left(\frac{j}{N}\right), t\left(\frac{j-1}{N}\right)\right]$$

Use this to justify the approximation $M \approx \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} t\left(\frac{j}{N}\right)$. Then argue, by passing to the limit as $N \to \infty$, that $M = \int_0^1 t(F)\,dF$. Strictly speaking, this is an *improper integral* because $t(0)$ is infinite (it takes an infinite amount of time for all atoms to decay). Therefore, we define M as a limit

$$M = \lim_{c\to 0}\int_c^1 t(F)\,dF$$

(c) Verify the formula $\int \ln x\,dx = x\ln x - x$ by differentiation and use it to show that for $c > 0$,

$$M = \lim_{c\to 0}\left(\frac{1}{k} + \frac{1}{k}(c\ln c - c)\right)$$

(d) Show that $M = 1/k$ by evaluating the limit (use L'Hôpital's Rule to compute $\lim_{c\to 0} c \ln c$).

(e) What is the mean time to decay for radon (with a half-life of 3.825 days)?

SOLUTION

(a) $F = e^{-kt}$ so $\ln F = -kt$ and $t(F) = \dfrac{\ln F}{-k}$

(b) $M \approx \dfrac{1}{N} \sum_{j=1}^{N} t(j/N)$. For the interval [0, 1], from the approximation given, the subinterval length is $1/N$ and thus the right-hand endpoints have x-coordinate (j/N). Thus we have a Riemann sum and by definition,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} t(j/N) = \int_0^1 t(F)dF.$$

(c) $\dfrac{d}{dx}(x \ln x - x) = x\left(\dfrac{1}{x}\right) + \ln x - 1 = \ln x$. Thus

$$\begin{aligned}\int_c^1 t(F)\,dF &= -\frac{1}{k}(F \ln F - F)\Big|_c^1 = \frac{1}{k}(F - F\ln F)\Big|_c^1 \\ &= \frac{1}{k}(1 - 1\ln 1 - (c - c\ln c)) \\ &= \frac{1}{k} + \frac{1}{k}(c \ln c - c).\end{aligned}$$

(d) By, L'Hôpital's Rule,

$$\lim_{c\to 0+} c \ln c = \lim_{c\to 0+} \frac{\ln c}{c^{-1}} = \lim_{c\to 0+} \frac{c^{-1}}{-c^{-2}} = -\lim_{c\to 0+} c = 0.$$

Thus, $M = \lim_{c\to 0} \displaystyle\int_c^1 t(F)dF = \lim_{c\to 0}\left(\frac{1}{k} + \frac{1}{k}(c\ln c - c)\right) = \frac{1}{k}$.

(e) Since the half-life is 3.825 days, $k = \dfrac{\ln 2}{3.825}$ and $\dfrac{1}{k} = 5.52$. Thus, $M = 5.52$ days.

61. Modify the proof of the relation $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ given in the text to prove $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. *Hint:* Express $\ln(1 + xn^{-1})$ as an integral and estimate above and below by rectangles.

SOLUTION Start by expressing

$$\ln\left(1 + \frac{x}{n}\right) = \int_1^{1+x/n} \frac{dt}{t}.$$

Following the proof in the text, we note that

$$\frac{x}{n+x} \le \ln\left(1 + \frac{x}{n}\right) \le \frac{x}{n}$$

provided $x > 0$, while

$$\frac{x}{n} \le \ln\left(1 + \frac{x}{n}\right) \le \frac{x}{n+x}$$

when $x < 0$. Multiplying both sets of inequalities by n and passing to the limit as $n \to \infty$, the squeeze theorem guarantees that

$$\lim_{n\to\infty}\left(\ln\left(1 + \frac{x}{n}\right)\right)^n = x.$$

Finally,

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

62. Prove that, for $n > 0$,

$$\left(1 + \frac{1}{n}\right)^n \le e \le \left(1 + \frac{1}{n}\right)^{n+1}$$

Hint: Take logarithms and use Eq. (4).

SOLUTION Taking logarithms throughout the desired inequality, we find the equivalent inequality

$$n \ln\left(1 + \frac{1}{n}\right) \le 1 \le (n+1)\ln\left(1 + \frac{1}{n}\right).$$

Multiplying Eq. (4) by n yields

$$\frac{n}{n+1} \le n \ln\left(1 + \frac{1}{n}\right) \le 1,$$

which establishes the left-hand side of the desired inequality. On the other hand, multiplying Eq. (4) by $n+1$ yields

$$1 \le (n+1)\ln\left(1 + \frac{1}{n}\right) \le 1 + \frac{1}{n},$$

which establishes the right-hand side of the desired inequality.

63. A bank pays interest at the rate r, compounded M times yearly. The **effective interest rate** r_e is the rate at which interest, if compounded annually, would have to be paid to produce the same yearly return.

(a) Find r_e if $r = 9\%$ compounded monthly.

(b) Show that $r_e = (1 + r/M)^M - 1$ and that $r_e = e^r - 1$ if interest is compounded continuously.

(c) Find r_e if $r = 11\%$ compounded continuously.

(d) Find the rate r that, compounded weekly, would yield an effective rate of 20%.

SOLUTION

(a) Compounded monthly, $P(t) = P_0(1 + r/12)^{12t}$. By the definition of r_e,

$$P_0(1 + 0.09/12)^{12t} = P_0(1 + r_e)^t$$

so

$$(1 + 0.09/12)^{12t} = (1 + r_e)^t \quad \text{or} \quad r_e = (1 + 0.09/12)^{12} - 1 = 0.0938,$$

or 9.38%

(b) In general,

$$P_0(1 + r/M)^{Mt} = P_0(1 + r_e)^t,$$

so $(1 + r/M)^{Mt} = (1 + r_e)^t$ or $r_e = (1 + r/M)^M - 1$. If interest is compounded continuously, then $P_0 e^{rt} = P_0(1 + r_e)^t$ so $e^{rt} = (1 + r_e)^t$ or $r_e = e^r - 1$.

(c) Using part (b), $r_e = e^{0.11} - 1 \approx 0.1163$ or 11.63%.

(d) Solving

$$0.20 = \left(1 + \frac{r}{52}\right)^{52} - 1$$

for r yields $r = 52(1.2^{1/52} - 1) = 0.1826$ or 18.26%.

CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

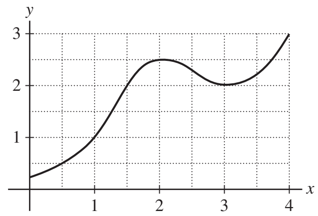


FIGURE 1

1. Estimate L_4 and M_4 on [0, 4].

SOLUTION With $n = 4$ and an interval of [0, 4], $\Delta x = \frac{4-0}{4} = 1$. Then,

$$L_4 = \Delta x(f(0) + f(1) + f(2) + f(3)) = 1\left(\frac{1}{4} + 1 + \frac{5}{2} + 2\right) = \frac{23}{4}$$

and

$$M_4 = \Delta x\left(f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right)\right) = 1\left(\frac{1}{2} + 2 + \frac{9}{4} + \frac{9}{4}\right) = 7.$$

2. Estimate R_4, L_4, and M_4 on [1, 3].

SOLUTION With $n = 4$ and an interval of [1, 3], $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. Then,

$$R_4 = \Delta x\left(f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3)\right) = \frac{1}{2}\left(2 + \frac{5}{2} + \frac{9}{4} + 2\right) = \frac{35}{8};$$

$$L_4 = \Delta x\left(f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right)\right) = \frac{1}{2}\left(1 + 2 + \frac{5}{2} + \frac{9}{4}\right) = \frac{31}{8}; \text{ and}$$

$$M_4 = \Delta x\left(f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{9}{4} + \frac{5}{2} + \frac{17}{8}\right) = \frac{67}{16}.$$

3. Find an interval $[a, b]$ on which R_4 is larger than $\int_a^b f(x)\,dx$. Do the same for L_4.

SOLUTION In general, R_N is larger than $\int_a^b f(x)\,dx$ on any interval $[a, b]$ over which $f(x)$ is increasing. Given the graph of $f(x)$, we may take $[a, b] = [0, 2]$. In order for L_4 to be larger than $\int_a^b f(x)\,dx$, $f(x)$ must be decreasing over the interval $[a, b]$. We may therefore take $[a, b] = [2, 3]$.

4. Justify $\frac{3}{2} \le \int_1^2 f(x)\,dx \le \frac{9}{4}$.

SOLUTION Because $f(x)$ is increasing on [1, 2], we know that

$$L_N \le \int_1^2 f(x)\,dx \le R_N$$

for any N. Now,

$$L_2 = \frac{1}{2}(1+2) = \frac{3}{2} \qquad \text{and} \qquad R_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4},$$

so

$$\frac{3}{2} \le \int_1^2 f(x)\,dx \le \frac{9}{4}.$$

In Exercises 5–8, let $f(x) = x^2 + 3x$.

5. Calculate R_6, M_6, and L_6 for $f(x)$ on the interval [2, 5]. Sketch the graph of $f(x)$ and the corresponding rectangles for each approximation.

SOLUTION Let $f(x) = x^2 + 3x$. A uniform partition of [2, 5] with $N = 6$ subintervals has

$$\Delta x = \frac{5-2}{6} = \frac{1}{2}, \qquad x_j = a + j\Delta x = 2 + \frac{j}{2},$$

and

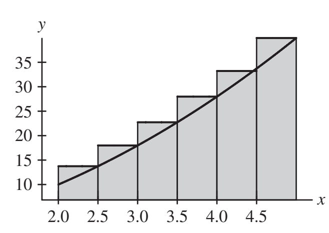
$$x_j^* = a + \left(j - \frac{1}{2}\right)\Delta x = \frac{7}{4} + \frac{j}{2}.$$

Now,

$$R_6 = \Delta x \sum_{j=1}^{6} f(x_j) = \frac{1}{2}\left(f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right) + f(5)\right)$$

$$= \frac{1}{2}\left(\frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4} + 40\right) = \frac{625}{8}.$$

The rectangles corresponding to this approximation are shown below.

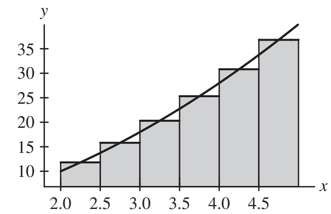


Next,

$$M_6 = \Delta x \sum_{j=1}^{6} f(x_j^*) = \frac{1}{2}\left(f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{189}{16} + \frac{253}{16} + \frac{325}{16} + \frac{405}{16} + \frac{493}{16} + \frac{589}{16}\right) = \frac{2254}{32} = \frac{1127}{16}.$$

The rectangles corresponding to this approximation are shown below.

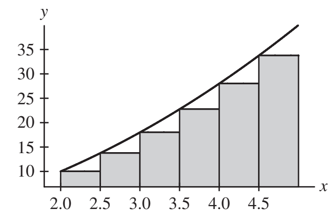


Finally,

$$L_6 = \Delta x \sum_{j=0}^{5} f(x_j) = \frac{1}{2}\left(f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right)\right)$$

$$= \frac{1}{2}\left(10 + \frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4}\right) = \frac{505}{8}.$$

The rectangles corresponding to this approximation are shown below.



6. Use FTC I to evaluate $A(x) = \int_{-2}^{x} f(t)\,dt$.

SOLUTION Let $f(x) = x^2 + 3x$. Then

$$A(x) = \int_{-2}^{x} (t^2 + 3t)\,dt = \left(\frac{1}{3}t^3 + \frac{3}{2}t^2\right)\Big|_{-2}^{x} = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \left(-\frac{8}{3} + 6\right) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{10}{3}.$$

7. Find a formula for R_N for $f(x)$ on [2, 5] and compute $\int_2^5 f(x)\,dx$ by taking the limit.

SOLUTION Let $f(x) = x^2 + 3x$ on the interval [2, 5]. Then $\Delta x = \frac{5-2}{N} = \frac{3}{N}$ and $a = 2$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(2 + j\Delta x) = \frac{3}{N}\sum_{j=1}^{N}\left(\left(2 + \frac{3j}{N}\right)^2 + 3\left(2 + \frac{3j}{N}\right)\right) = \frac{3}{N}\sum_{j=1}^{N}\left(10 + \frac{21j}{N} + \frac{9j^2}{N^2}\right)$$

$$= 30 + \frac{63}{N^2}\sum_{j=1}^{N} j + \frac{27}{N^3}\sum_{j=1}^{N} j^2$$

$$= 30 + \frac{63}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{27}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right)$$

$$= \frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2}\right) = \frac{141}{2}.$$

8. Find a formula for L_N for $f(x)$ on [0, 2] and compute $\int_0^2 f(x)\,dx$ by taking the limit.

SOLUTION Let $f(x) = x^2 + 3x$ and N be a positive integer. Then

$$\Delta x = \frac{2-0}{N} = \frac{2}{N}$$

and

$$x_j = a + j\Delta x = 0 + \frac{2j}{N} = \frac{2j}{N}$$

for $0 \le j \le N$. Thus,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \frac{2}{N}\sum_{j=0}^{N-1}\left(\frac{4j^2}{N^2} + \frac{6j}{N}\right) = \frac{8}{N^3}\sum_{j=0}^{N-1} j^2 + \frac{12}{N^2}\sum_{j=0}^{N-1} j$$
$$= \frac{4(N-1)(2N-1)}{3N^2} + \frac{6(N-1)}{N} = \frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2}.$$

Finally,

$$\int_0^2 f(x)\,dx = \lim_{N\to\infty}\left(\frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2}\right) = \frac{26}{3}.$$

9. Calculate R_5, M_5, and L_5 for $f(x) = (x^2+1)^{-1}$ on the interval [0, 1].

SOLUTION Let $f(x) = (x^2+1)^{-1}$. A uniform partition of [0, 1] with $N = 5$ subintervals has

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}, \qquad x_j = a + j\Delta x = \frac{j}{5},$$

and

$$x_j^* = a + \left(j - \frac{1}{2}\right)\Delta x = \frac{2j-1}{10}.$$

Now,

$$R_5 = \Delta x \sum_{j=1}^{5} f(x_j) = \frac{1}{5}\left(f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1)\right)$$
$$= \frac{1}{5}\left(\frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41} + \frac{1}{2}\right) \approx 0.733732.$$

Next,

$$M_5 = \Delta x \sum_{j=1}^{5} f(x_j^*) = \frac{1}{5}\left(f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right)\right)$$
$$= \frac{1}{5}\left(\frac{100}{101} + \frac{100}{109} + \frac{4}{5} + \frac{100}{149} + \frac{100}{181}\right) \approx 0.786231.$$

Finally,

$$L_5 = \Delta x \sum_{j=0}^{4} f(x_j) = \frac{1}{5}\left(f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right)\right)$$
$$= \frac{1}{5}\left(1 + \frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41}\right) \approx 0.833732.$$

10. Let R_N be the Nth right-endpoint approximation for $f(x) = x^3$ on [0, 4] (Figure 2).

(a) Prove that $R_N = \dfrac{64(N+1)^2}{N^2}$.

(b) Prove that the area of the region within the right-endpoint rectangles above the graph is equal to

$$\frac{64(2N+1)}{N^2}$$

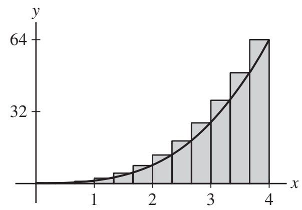


FIGURE 2 Approximation R_N for $f(x) = x^3$ on [0, 4].

SOLUTION

(a) Let $f(x) = x^3$ and N be a positive integer. Then

$$\Delta x = \frac{4-0}{N} = \frac{4}{N} \qquad \text{and} \qquad x_j = a + j\Delta x = 0 + \frac{4j}{N} = \frac{4j}{N}$$

for $0 \le j \le N$. Thus,

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{4}{N}\sum_{j=1}^{N}\frac{64j^3}{N^3} = \frac{256}{N^4}\sum_{j=1}^{N} j^3 = \frac{256}{N^4}\frac{N^2(N+1)^2}{4} = \frac{64(N+1)^2}{N^2}.$$

(b) The area between the graph of $y = x^3$ and the x-axis over [0, 4] is

$$\int_0^4 x^3\,dx = \frac{1}{4}x^4\Big|_0^4 = 64.$$

The area of the region below the right-endpoint rectangles and above the graph is therefore

$$\frac{64(N+1)^2}{N^2} - 64 = \frac{64(2N+1)}{N^2}.$$

11. Which approximation to the area is represented by the shaded rectangles in Figure 3? Compute R_5 and L_5.

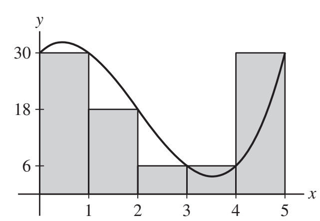


FIGURE 3

SOLUTION There are five rectangles and the height of each is given by the function value at the right endpoint of the subinterval. Thus, the area represented by the shaded rectangles is R_5.

From the figure, we see that $\Delta x = 1$. Then

$$R_5 = 1(30 + 18 + 6 + 6 + 30) = 90 \qquad \text{and} \qquad L_5 = 1(30 + 30 + 18 + 6 + 6) = 90.$$

12. Calculate any two Riemann sums for $f(x) = x^2$ on the interval [2, 5], but choose partitions with at least five subintervals of unequal widths and intermediate points that are neither endpoints nor midpoints.

SOLUTION Let $f(x) = x^2$. Riemann sums will, of course, vary. Here are two possibilities. Take $N = 5$,

$$P = \{x_0 = 2, x_1 = 2.7, x_2 = 3.1, x_3 = 3.6, x_4 = 4.2, x_5 = 5\}$$

and

$$C = \{c_1 = 2.5, c_2 = 3, c_3 = 3.5, c_4 = 4, c_5 = 4.5\}.$$

Then,

$$R(f, P, C) = \sum_{j=1}^{5} \Delta x_j f(c_j) = 0.7(6.25) + 0.4(9) + 0.5(12.25) + 0.6(16) + 0.8(20.25) = 39.9.$$

Alternately, take $N = 6$,

$$P = \{x_0 = 2, x_1 = 2.5, x_2 = 3.5, x_3 = 4, x_4 = 4.25, x_5 = 4.75, x_6 = 5\}$$

and

$$C = \{c_1 = 2.1, c_2 = 3, c_3 = 3.7, c_4 = 4.2, c_5 = 4.5, c_6 = 4.8\}.$$

Then,

$$\begin{aligned} R(f, P, C) &= \sum_{j=1}^{6} \Delta x_j f(c_j) \\ &= 0.5(4.41) + 1(9) + 0.5(13.69) + 0.25(17.64) + 0.5(20.25) + 0.25(23.04) = 38.345. \end{aligned}$$

In Exercises 13–16, express the limit as an integral (or multiple of an integral) and evaluate.

13. $\displaystyle\lim_{N\to\infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right)$

SOLUTION Let $f(x) = \sin x$ and N be a positive integer. A uniform partition of the interval $[\pi/3, \pi/2]$ with N subintervals has

$$\Delta x = \frac{\pi}{6N} \qquad \text{and} \qquad x_j = \frac{\pi}{3} + \frac{\pi j}{6N}$$

for $0 \le j \le N$. Then

$$\frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \int_{\pi/3}^{\pi/2} \sin x\, dx = -\cos x\Big|_{\pi/3}^{\pi/2} = 0 + \frac{1}{2} = \frac{1}{2}.$$

14. $\displaystyle\lim_{N\to\infty} \frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right)$

SOLUTION Let $f(x) = x$ and N be a positive integer. A uniform partition of the interval $[10, 13]$ with N subintervals has

$$\Delta x = \frac{3}{N} \qquad \text{and} \qquad x_j = 10 + \frac{3j}{N}$$

for $0 \le j \le N$. Then

$$\frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) = \Delta x \sum_{j=0}^{N-1} f(x_j) = L_N;$$

consequently,

$$\begin{aligned} \lim_{N\to\infty} \frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) &= \int_{10}^{13} x\, dx = \frac{1}{2}x^2\Big|_{10}^{13} \\ &= \frac{169}{2} - \frac{100}{2} = \frac{69}{2}. \end{aligned}$$

15. $\displaystyle\lim_{N\to\infty} \frac{5}{N}\sum_{j=1}^{N}\sqrt{4+5j/N}$

SOLUTION Let $f(x) = \sqrt{x}$ and N be a positive integer. A uniform partition of the interval [4, 9] with N subintervals has

$$\Delta x = \frac{5}{N} \qquad \text{and} \qquad x_j = 4 + \frac{5j}{N}$$

for $0 \le j \le N$. Then

$$\frac{5}{N}\sum_{j=1}^{N}\sqrt{4+5j/N} = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty}\frac{5}{N}\sum_{j=1}^{N}\sqrt{4+5j/N} = \int_4^9 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg|_4^9 = \frac{54}{3} - \frac{16}{3} = \frac{38}{3}.$$

16. $\displaystyle\lim_{N\to\infty}\frac{1^k + 2^k + \cdots + N^k}{N^{k+1}} \quad (k > 0)$

SOLUTION Observe that

$$\frac{1^k + 2^k + 3^k + \cdots + N^k}{N^{k+1}} = \frac{1}{N}\left[\left(\frac{1}{N}\right)^k + \left(\frac{2}{N}\right)^k + \left(\frac{3}{N}\right)^k + \cdots \left(\frac{N}{N}\right)^k\right] = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^k.$$

Now, let $f(x) = x^k$ and N be a positive integer. A uniform partition of the interval [0, 1] with N subintervals has

$$\Delta x = \frac{1}{N} \qquad \text{and} \qquad x_j = \frac{j}{N}$$

for $0 \le j \le N$. Then

$$\frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^k = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^k = \int_0^1 x^k\,dx = \frac{1}{k+1}x^{k+1}\bigg|_0^1 = \frac{1}{k+1}.$$

In Exercises 17–20, use the given substitution to evaluate the integral.

17. $\displaystyle\int_0^2 \frac{dt}{4t+12}, \quad u = 4t + 12$

SOLUTION Let $u = 4t + 12$. Then $du = 4dt$, and the new limits of integration are $u = 12$ and $u = 20$. Thus,

$$\int_0^2 \frac{dt}{4t+12} = \frac{1}{4}\int_{12}^{20}\frac{du}{u} = \frac{1}{4}\ln u\bigg|_{12}^{20} = \frac{1}{4}(\ln 20 - \ln 12) = \frac{1}{4}\ln\frac{20}{12} = \frac{1}{4}\ln\frac{5}{3}.$$

18. $\displaystyle\int \frac{(x^2+1)\,dx}{(x^3+3x)^4}, \qquad u = x^3 + 3x$

SOLUTION Let $u = x^3 + 3x$. Then $du = (3x^2 + 3)\,dx = 3(x^2 + 1)\,dx$ and

$$\int \frac{(x^2+1)\,dx}{(x^3+3x)^4} = \frac{1}{3}\int u^{-4}\,du = -\frac{1}{9}u^{-3} + C = -\frac{1}{9}(x^3+3x)^{-3} + C.$$

19. $\int_0^{\pi/6} \sin x \cos^4 x\, dx, \qquad u = \cos x$

SOLUTION Let $u = \cos x$. Then $du = -\sin x\, dx$ and the new limits of integration are $u = 1$ and $u = \sqrt{3}/2$. Thus,

$$\begin{aligned}\int_0^{\pi/6} \sin x \cos^4 x\, dx &= -\int_1^{\sqrt{3}/2} u^4\, du \\ &= -\frac{1}{5}u^5\Big|_1^{\sqrt{3}/2} \\ &= \frac{1}{5}\left(1 - \frac{9\sqrt{3}}{32}\right).\end{aligned}$$

20. $\int \sec^2(2\theta)\tan(2\theta)\, d\theta, \quad u = \tan(2\theta)$

SOLUTION Let $u = \tan(2\theta)$. Then $du = 2\sec^2(2\theta)\, d\theta$ and

$$\int \sec^2(2\theta)\tan(2\theta)\, d\theta = \frac{1}{2}\int u\, du = \frac{1}{4}u^2 + C = \frac{1}{4}\tan^2(2\theta) + C.$$

In Exercises 21–70, evaluate the integral.

21. $\int (20x^4 - 9x^3 - 2x)\, dx$

SOLUTION $\int (20x^4 - 9x^3 - 2x)\, dx = 4x^5 - \frac{9}{4}x^4 - x^2 + C.$

22. $\int_0^2 (12x^3 - 3x^2)\, dx$

SOLUTION $\int_0^2 (12x^3 - 3x^2)\, dx = (3x^4 - x^3)\Big|_0^2 = (48 - 8) - 0 = 40.$

23. $\int (2x^2 - 3x)^2\, dx$

SOLUTION $\int (2x^2 - 3x)^2\, dx = \int (4x^4 - 12x^3 + 9x^2)\, dx = \frac{4}{5}x^5 - 3x^4 + 3x^3 + C.$

24. $\int_0^1 (x^{7/3} - 2x^{1/4})\, dx$

SOLUTION $\int_0^1 (x^{7/3} - 2x^{1/4})\, dx = \left(\frac{3}{10}x^{10/3} - \frac{8}{5}x^{5/4}\right)\Big|_0^1 = \frac{3}{10} - \frac{8}{5} = -\frac{13}{10}.$

25. $\int \frac{x^5 + 3x^4}{x^2}\, dx$

SOLUTION $\int \frac{x^5 + 3x^4}{x^2}\, dx = \int (x^3 + 3x^2)\, dx = \frac{1}{4}x^4 + x^3 + C.$

26. $\int_1^3 r^{-4}\, dr$

SOLUTION $\int_1^3 r^{-4}\, dr = -\frac{1}{3}r^{-3}\Big|_1^3 = -\frac{1}{3}\left(\frac{1}{27} - 1\right) = \frac{26}{81}.$

27. $\int_{-3}^{3} |x^2 - 4|\, dx$

SOLUTION

$$\begin{aligned}\int_{-3}^{3} |x^2 - 4|\, dx &= \int_{-3}^{2} (x^2 - 4)\, dx + \int_{-2}^{2} (4 - x^2)\, dx + \int_{2}^{3} (x^2 - 4)\, dx \\ &= \left(\frac{1}{3}x^3 - 4x\right)\Big|_{-3}^{-2} + \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} + \left(\frac{1}{3}x^3 - 4x\right)\Big|_{2}^{3} \\ &= \left(\frac{16}{3} - 3\right) + \left(\frac{16}{3} + \frac{16}{3}\right) + \left(-3 + \frac{16}{3}\right) \\ &= \frac{46}{3}.\end{aligned}$$

28. $\int_{-2}^{4} |(x - 1)(x - 3)|\, dx$

SOLUTION

$$\begin{aligned}\int_{-2}^{4} |(x - 1)(x - 3)|\, dx &= \int_{-2}^{1} (x^2 - 4x + 3)\, dx + \int_{1}^{3} (-x^2 + 4x - 3)\, dx + \int_{3}^{4} (x^2 - 4x + 3)\, dx \\ &= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_{-2}^{1} + \left(-\frac{1}{3}x^3 + 2x^2 - 3x\right)\Big|_{1}^{3} + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_{3}^{4} \\ &= \frac{4}{3} - \left(-\frac{50}{3}\right) + 0 - \left(-\frac{4}{3}\right) + \frac{4}{3} - 0 \\ &= \frac{62}{3}.\end{aligned}$$

29. $\int_{1}^{3} [t]\, dt$

SOLUTION

$$\int_{1}^{3} [t]\, dt = \int_{1}^{2} [t]\, dt + \int_{2}^{3} [t]\, dt = \int_{1}^{2} dt + \int_{2}^{3} 2\, dt = t\Big|_{1}^{2} + 2t\Big|_{2}^{3} = (2 - 1) + (6 - 4) = 3.$$

30. $\int_{0}^{2} (t - [t])^2\, dt$

SOLUTION

$$\begin{aligned}\int_{0}^{2} (t - [t])^2\, dt &= \int_{0}^{1} t^2\, dt + \int_{1}^{2} (t - 1)^2\, dt \\ &= \frac{1}{3}t^3\Big|_{0}^{1} + \frac{1}{3}(t - 1)^3\Big|_{1}^{2} \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

31. $\int (10t - 7)^{14}\, dt$

SOLUTION Let $u = 10t - 7$. Then $du = 10dt$ and

$$\int (10t - 7)^{14}\, dt = \frac{1}{10} \int u^{14}\, du = \frac{1}{150}u^{15} + C = \frac{1}{150}(10t - 7)^{15} + C.$$

32. $\int_{2}^{3} \sqrt{7y - 5}\, dy$

SOLUTION Let $u = 7y - 5$. Then $du = 7dy$ and when $y = 2$, $u = 9$ and when $y = 3$, $u = 16$. Finally,

$$\int_{2}^{3} \sqrt{7y - 5}\, dy = \frac{1}{7} \int_{9}^{16} u^{1/2}\, du = \frac{1}{7} \cdot \frac{2}{3}u^{3/2}\Big|_{9}^{16} = \frac{2}{21}(64 - 27) = \frac{74}{21}.$$

33. $\displaystyle\int \frac{(2x^3+3x)\,dx}{(3x^4+9x^2)^5}$

SOLUTION Let $u = 3x^4 + 9x^2$. Then $du = (12x^3 + 18x)\,dx = 6(2x^3 + 3x)\,dx$ and

$$\int \frac{(2x^3+3x)\,dx}{(3x^4+9x^2)^5} = \frac{1}{6}\int u^{-5}\,du = -\frac{1}{24}u^{-4} + C = -\frac{1}{24}(3x^4+9x^2)^{-4} + C.$$

34. $\displaystyle\int_{-3}^{-1} \frac{x\,dx}{(x^2+5)^2}$

SOLUTION Let $u = x^2 + 5$. Then $du = 2x\,dx$ and

$$\begin{aligned}\int_{-3}^{-1} \frac{x\,dx}{(x^2+5)^2} &= \frac{1}{2}\int_{14}^{6} u^{-2}\,du = -\frac{1}{2}u^{-1}\Big|_{14}^{6}\\ &= -\frac{1}{2}\left(\frac{1}{6}-\frac{1}{14}\right) = -\frac{1}{21}.\end{aligned}$$

35. $\displaystyle\int_0^5 15x\sqrt{x+4}\,dx$

SOLUTION Let $u = x + 4$. Then $x = u - 4$, $du = dx$ and the new limits of integration are $u = 4$ and $u = 9$. Thus,

$$\begin{aligned}\int_0^5 15x\sqrt{x+4}\,dx &= \int_4^9 15(u-4)\sqrt{u}\,du\\ &= 15\int_4^9 (u^{3/2} - 4u^{1/2})\,du\\ &= 15\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right)\Big|_4^9\\ &= 15\left(\left(\frac{486}{5} - 72\right) - \left(\frac{64}{5} - \frac{64}{3}\right)\right)\\ &= 506.\end{aligned}$$

36. $\displaystyle\int t^2\sqrt{t+8}\,dt$

SOLUTION Let $u = t + 8$. Then $du = dt$, $t = u - 8$, and

$$\begin{aligned}\int t^2\sqrt{t+8}\,dt = \int (u-8)^2\sqrt{u}\,du &= \int (u^{5/2} - 16u^{3/2} + 64u^{1/2})\,du\\ &= \frac{2}{7}u^{7/2} - \frac{32}{5}u^{5/2} + \frac{128}{3}u^{3/2} + C\\ &= \frac{2}{7}(t+8)^{7/2} - \frac{32}{5}(t+8)^{5/2} + \frac{128}{3}(t+8)^{3/2} + C.\end{aligned}$$

37. $\displaystyle\int_0^1 \cos\left(\frac{\pi}{3}(t+2)\right)\,dt$

SOLUTION $\displaystyle\int_0^1 \cos\left(\frac{\pi}{3}(t+2)\right)\,dt = \frac{3}{\pi}\sin\left(\frac{\pi}{3}(t+2)\right)\Big|_0^1 = -\frac{3\sqrt{3}}{2\pi}.$

38. $\displaystyle\int_{\pi/2}^{\pi} \sin\left(\frac{5\theta-\pi}{6}\right)\,d\theta$

SOLUTION Let

$$u = \frac{5\theta - \pi}{6} \quad \text{so that} \quad du = \frac{5}{6}d\theta.$$

Then

$$\begin{aligned}\int_{\pi/2}^{\pi} \sin\left(\frac{5\theta-\pi}{6}\right)\,d\theta &= \frac{6}{5}\int_{\pi/4}^{2\pi/3} \sin u\,du\\ &= -\frac{6}{5}\cos u\Big|_{\pi/4}^{2\pi} 3\\ &= -\frac{6}{5}\left(-\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = \frac{3}{5}(1+\sqrt{2}).\end{aligned}$$

39. $\int t^2 \sec^2(9t^3+1)\,dt$

SOLUTION Let $u = 9t^3 + 1$. Then $du = 27t^2\,dt$ and

$$\int t^2 \sec^2(9t^3+1)\,dt = \frac{1}{27}\int \sec^2 u\,du = \frac{1}{27}\tan u + C = \frac{1}{27}\tan(9t^3+1) + C.$$

40. $\int \sin^2(3\theta)\cos(3\theta)\,d\theta$

SOLUTION Let $u = \sin(3\theta)$. Then $du = 3\cos(3\theta)d\theta$ and

$$\int \sin^2(3\theta)\cos(3\theta)\,d\theta = \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}\sin^3(3\theta) + C.$$

41. $\int \csc^2(9-2\theta)\,d\theta$

SOLUTION Let $u = 9 - 2\theta$. Then $du = -2\,d\theta$ and

$$\int \csc^2(9-2\theta)\,d\theta = -\frac{1}{2}\int \csc^2 u\,du = \frac{1}{2}\cot u + C = \frac{1}{2}\cot(9-2\theta) + C.$$

42. $\int \sin\theta\sqrt{4-\cos\theta}\,d\theta$

SOLUTION Let $u = 4 - \cos\theta$. Then $du = \sin\theta\,d\theta$ and

$$\int \sin\theta\sqrt{4-\cos\theta}\,d\theta = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(4-\cos\theta)^{3/2} + C.$$

43. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta$

SOLUTION Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and when $\theta = 0$, $u = 1$ and when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Finally,

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta = -\int_1^{1/2} u^{-2/3}\,du = -3u^{1/3}\Big|_1^{1/2} = -3(2^{-1/3}-1) = 3 - \frac{3\sqrt[3]{4}}{2}.$$

44. $\int \frac{\sec^2 t\,dt}{(\tan t - 1)^2}$

SOLUTION Let $u = \tan t - 1$. Then $du = \sec^2 t\,dt$ and

$$\int \frac{\sec^2 t\,dt}{(\tan t - 1)^2} = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\tan t - 1} + C.$$

45. $\int e^{9-2x}\,dx$

SOLUTION Let $u = 9 - 2x$. Then $du = -2\,dx$, and

$$\int e^{9-2x}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{9-2x} + C.$$

46. $\int_1^3 e^{4x-3}\,dx$

SOLUTION $\int_1^3 e^{4x-3}\,dx = \frac{1}{4}e^{4x-3}\Big|_1^3 = \frac{1}{4}(e^9 - e).$

47. $\int x^2 e^{x^3}\,dx$

SOLUTION Let $u = x^3$. Then $du = 3x^2\,dx$, and

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

48. $\int_0^{\ln 3} e^{x-e^x}\,dx$

SOLUTION Note $e^{x-e^x} = e^x e^{-e^x}$. Now, let $u = e^x$. Then $du = e^x\,dx$, and the new limits of integration are $u = e^0 = 1$ and $u = e^{\ln 3} = 3$. Thus,

$$\int_0^{\ln 3} e^{x-e^x}\,dx = \int_0^{\ln 3} e^x e^{-e^x}\,dx = \int_1^3 e^{-u}\,du = -e^{-t}\Big|_1^3 = -(e^{-3} - e^{-1}) = e^{-1} - e^{-3}.$$

49. $\int e^x 10^x\,dx$

SOLUTION $\int e^x 10^x\,dx = \int (10e)^x\,dx = \dfrac{(10e)^x}{\ln(10e)} + C = \dfrac{(10e)^x}{\ln 10 + \ln e} + C = \dfrac{10^x e^x}{\ln 10 + 1} + C.$

50. $\int e^{-2x}\sin(e^{-2x})\,dx$

SOLUTION Let $u = e^{-2x}$. Then $du = -2e^{-2x}dx$, and

$$\int e^{-2x}\sin\left(e^{-2x}\right)\,dx = -\frac{1}{2}\int \sin u\,du = \frac{\cos u}{2} + C = \frac{1}{2}\cos\left(e^{-2x}\right) + C.$$

51. $\int \dfrac{e^{-x}\,dx}{(e^{-x}+2)^3}$

SOLUTION Let $u = e^{-x} + 2$. Then $du = -e^{-x}\,dx$ and

$$\int \frac{e^{-x}\,dx}{(e^{-x}+2)^3} = -\int u^{-3}\,du = \frac{1}{2u^2} + C = \frac{1}{2(e^{-x}+2)^2} + C.$$

52. $\int \sin\theta\cos\theta e^{\cos^2\theta+1}\,d\theta$

SOLUTION Let $u = \cos^2\theta + 1$. Then $du = -2\sin\theta\cos\theta\,d\theta$ and

$$\int \sin\theta\cos\theta e^{\cos^2\theta+1}\,d\theta = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos^2\theta+1} + C.$$

53. $\int_0^{\pi/6}\tan 2\theta\,d\theta$

SOLUTION $\int_0^{\pi/6}\tan 2\theta\,d\theta = \dfrac{1}{2}\ln|\sec 2\theta|\Big|_0^{\pi/6} = \dfrac{1}{2}\ln 2.$

54. $\int_{\pi/3}^{2\pi/3}\cot\left(\frac{1}{2}\theta\right)\,d\theta$

SOLUTION

$$\begin{aligned}\int_{\pi/3}^{2\pi/3}\cot\left(\frac{1}{2}\theta\right)\,d\theta &= 2\ln\left|\sin\frac{\theta}{2}\right|\Big|_{\pi/3}^{2\pi}3\\ &= 2\left(\ln\sin\frac{\pi}{3} - \ln\sin\frac{\pi}{6}\right)\\ &= 2\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = \ln 3.\end{aligned}$$

55. $\int \dfrac{dt}{t(1+(\ln t)^2)}$

SOLUTION Let $u = \ln t$. Then, $du = \frac{1}{t}\,dt$ and

$$\int \frac{dt}{t(1+(\ln t)^2)} = \int \frac{du}{1+u^2} = \tan^{-1}u + C = \tan^{-1}(\ln t) + C.$$

56. $\displaystyle\int \frac{\cos(\ln x)\,dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{dx}{x}$, and

$$\int \frac{\cos(\ln x)}{x}dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C.$$

57. $\displaystyle\int_1^e \frac{\ln x\,dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{dx}{x}$ and the new limits of integration are $u = \ln 1 = 0$ and $u = \ln e = 1$. Thus,

$$\int_1^e \frac{\ln x\,dx}{x} = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}.$$

58. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{dx}{x\sqrt{\ln x}} = \int u^{-1/2}\,du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C.$$

59. $\displaystyle\int \frac{dx}{4x^2+9}$

SOLUTION Let $u = \frac{2x}{3}$. Then $x = \frac{3}{2}u$, $dx = \frac{3}{2}\,du$, and

$$\int \frac{dx}{4x^2+9} = \int \frac{\frac{3}{2}\,du}{4\cdot\frac{9}{4}u^2+9} = \frac{1}{6}\int \frac{du}{u^2+1} = \frac{1}{6}\tan^{-1}u + C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C.$$

60. $\displaystyle\int_0^{0.8} \frac{dx}{\sqrt{1-x^2}}$

SOLUTION $\displaystyle\int_0^{0.8} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_0^{0.8} = \sin^{-1}0.8 - \sin^{-1}0 = \sin^{-1}0.8.$

61. $\displaystyle\int_4^{12} \frac{dx}{x\sqrt{x^2-1}}$

SOLUTION $\displaystyle\int_4^{12} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x\Big|_4^{12} = \sec^{-1}12 - \sec^{-1}4.$

62. $\displaystyle\int_0^3 \frac{x\,dx}{x^2+9}$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\,dx$, and the new limits of integration are $u = 9$ and $u = 18$. Thus,

$$\int_0^3 \frac{x\,dx}{x^2+9} = \frac{1}{2}\int_9^{18} \frac{du}{u} = \frac{1}{2}\ln u\Big|_9^{18} = \frac{1}{2}(\ln 18 - \ln 9) = \frac{1}{2}\ln\frac{18}{9} = \frac{1}{2}\ln 2.$$

63. $\displaystyle\int_0^3 \frac{dx}{x^2+9}$

SOLUTION Let $u = \frac{x}{3}$. Then $du = \frac{dx}{3}$, and the new limits of integration are $u = 0$ and $u = 1$. Thus,

$$\int_0^3 \frac{dx}{x^2+9} = \frac{1}{3}\int_0^1 \frac{dt}{t^2+1} = \frac{1}{3}\tan^{-1}t\Big|_0^1 = \frac{1}{3}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}.$$

64. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$

SOLUTION Let $u = e^x$. Then

$$du = e^x\,dx \quad\Rightarrow\quad du = u\,dx \quad\Rightarrow\quad u^{-1}du = dx$$

By substitution, we obtain

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}}$$
$$= \sec^{-1} u + C = \sec^{-1}(e^x) + C$$

65. $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^4}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$, and $\sqrt{1-x^4} = \sqrt{1-u^2}$. Thus,

$$\int \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(x^2) + C.$$

66. $\displaystyle\int_0^1 \frac{dx}{25-x^2}$

SOLUTION Let $x = 5u$. Then $dx = 5\,du$, and the new limits of integration are $u = 0$ and $u = \frac{1}{5}$. Thus,

$$\int_0^1 \frac{dx}{25-x^2} = \frac{1}{25}\int_0^{1/5} \frac{5\,du}{1-u^2} = \frac{5}{25}\int_0^{1/5} \frac{du}{1-u^2}$$
$$= \frac{1}{5}\tanh^{-1}u\Big|_0^{1/5} = \frac{1}{5}\left(\tanh^{-1}\frac{1}{5} - \tanh^{-1}0\right) = \frac{1}{5}\tanh^{-1}\frac{1}{5}.$$

67. $\displaystyle\int_0^4 \frac{dx}{2x^2+1}$

SOLUTION Let $u = \sqrt{2}x$. Then $du = \sqrt{2}\,dx$, and the new limits of integration are $u = 0$ and $u = 4\sqrt{2}$. Thus,

$$\int_0^4 \frac{dx}{2x^2+1} = \int_0^{4\sqrt{2}} \frac{\frac{1}{\sqrt{2}}\,du}{u^2+1} = \frac{1}{\sqrt{2}}\int_0^{4\sqrt{2}} \frac{du}{u^2+1}$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}u\Big|_0^{4\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\tan^{-1}(4\sqrt{2}) - \tan^{-1}0\right) = \frac{1}{\sqrt{2}}\tan^{-1}(4\sqrt{2}).$$

68. $\displaystyle\int_5^8 \frac{dx}{x\sqrt{x^2-16}}$

SOLUTION Let $x = 4u$. Then $dx = 4\,du$, and the new limits of integration are $u = \frac{5}{4}$ and $u = 2$. Thus,

$$\int_5^8 \frac{dx}{x\sqrt{x^2-16}} = \frac{1}{4}\int_{5/4}^2 \frac{du}{u\sqrt{u^2-1}} = \frac{1}{4}\left(\sec^{-1}u\right)\Big|_{5/4}^2 = \frac{1}{4}\left(\sec^{-1}2 - \sec^{-1}\frac{5}{4}\right) = \frac{1}{4}\left(\frac{\pi}{3} - \sec^{-1}\frac{5}{4}\right).$$

69. $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^3\,dx}{1+x^2}$

SOLUTION Let $u = \tan^{-1}x$. Then

$$du = \frac{1}{1+x^2}\,dx$$

and

$$\int_0^1 \frac{(\tan^{-1}x)^3\,dx}{1+x^2} = \int_0^{\pi/4} u^3\,du = \frac{1}{4}u^4\Big|_0^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4}\right)^4 = \frac{\pi^4}{1024}.$$

70. $\displaystyle\int \frac{\cos^{-1}t\,dt}{\sqrt{1-t^2}}$

SOLUTION Let $u = \cos^{-1}t$. Then $du = -\frac{1}{\sqrt{1-t^2}}\,dt$, and

$$\int \frac{\cos^{-1}t}{\sqrt{1-t^2}}\,dt = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\left(\cos^{-1}t\right)^2 + C.$$

71. Combine to write as a single integral:

$$\int_0^8 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_8^6 f(x)\,dx$$

SOLUTION First, rewrite

$$\int_0^8 f(x)\,dx = \int_0^6 f(x)\,dx + \int_6^8 f(x)\,dx$$

and observe that

$$\int_8^6 f(x)\,dx = -\int_6^8 f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx + \int_8^6 f(x)\,dx = \int_0^6 f(x)\,dx.$$

Finally,

$$\int_0^8 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_8^6 f(x)\,dx = \int_0^6 f(x)\,dx + \int_{-2}^0 f(x)\,dx = \int_{-2}^6 f(x)\,dx.$$

72. Let $A(x) = \int_0^x f(x)\,dx$, where $f(x)$ is the function shown in Figure 4. Identify the location of the local minima, the local maxima, and points of inflection of $A(x)$ on the interval $[0, E]$, as well as the intervals where $A(x)$ is increasing, decreasing, concave up, or concave down. Where does the absolute max of $A(x)$ occur?

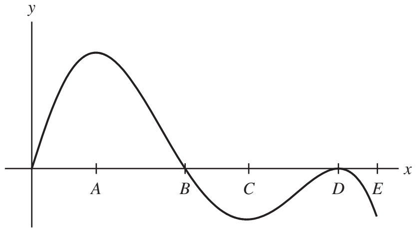


FIGURE 4

SOLUTION Let $f(x)$ be the function shown in Figure 4 and define

$$A(x) = \int_0^x f(x)\,dx.$$

Then $A'(x) = f(x)$ and $A''(x) = f'(x)$. Hence, $A(x)$ is increasing when $f(x)$ is positive, is decreasing when $f(x)$ is negative, is concave up when $f(x)$ is increasing and is concave down when $f(x)$ is decreasing. Thus, $A(x)$ is increasing for $0 < x < B$, is decreasing for $B < x < D$ and for $D < x < E$, has a local maximum at $x = B$ and no local minima. Moreover, $A(x)$ is concave up for $0 < x < A$ and for $C < x < D$, is concave down for $A < x < C$ and for $D < x < E$, and has a point of inflection at $x = A$, $x = C$ and $x = D$. The absolute maximum value for $A(x)$ occurs at $x = B$.

73. Find the local minima, the local maxima, and the inflection points of $A(x) = \displaystyle\int_3^x \frac{t\,dt}{t^2+1}$.

SOLUTION Let

$$A(x) = \int_3^x \frac{t\,dt}{t^2+1}.$$

Then

$$A'(x) = \frac{x}{x^2+1}$$

and

$$A''(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Now, $x = 0$ is the only critical point of A; because $A''(0) > 0$, it follows that A has a local minimum at $x = 0$. There are no local maxima. Moreover, $A(x)$ is concave down for $|x| > 1$ and concave up for $|x| < 1$. $A(x)$ therefore has inflection points at $x = \pm 1$.

74. A particle starts at the origin at time $t = 0$ and moves with velocity $v(t)$ as shown in Figure 5.

(a) How many times does the particle return to the origin in the first 12 seconds?

(b) What is the particle's maximum distance from the origin?

(c) What is particle's maximum distance to the left of the origin?

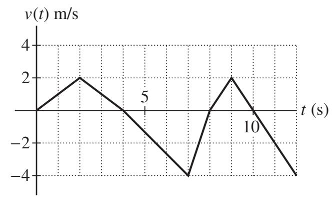


FIGURE 5

SOLUTION Because the particle starts at the origin, the position of the particle is given by

$$s(t) = \int_0^t v(\tau)\,d\tau;$$

that is by the signed area between the graph of the velocity and the t-axis over the interval $[0, t]$. Using the geometry in Figure 5, we see that $s(t)$ is increasing for $0 < t < 4$ and for $8 < t < 10$ and is decreasing for $4 < t < 8$ and for $10 < t < 12$. Furthermore,

$$s(0) = 0 \text{ m}, \quad s(4) = 4 \text{ m}, \quad s(8) = -4 \text{ m}, \quad s(10) = -2 \text{ m}, \quad \text{and} \quad s(12) = -6 \text{ m}.$$

(a) In the first 12 seconds, the particle returns to the origin once, sometime between $t = 4$ and $t = 8$ seconds.

(b) The particle's maximum distance from the origin is 6 meters (to the left at $t = 12$ seconds).

(c) The particle's distance to the left of the origin is 6 meters.

75. On a typical day, a city consumes water at the rate of $r(t) = 100 + 72t - 3t^2$ (in thousands of gallons per hour), where t is the number of hours past midnight. What is the daily water consumption? How much water is consumed between 6 PM and midnight?

SOLUTION With a consumption rate of $r(t) = 100 + 72t - 3t^2$ thousand gallons per hour, the daily consumption of water is

$$\int_0^{24} (100 + 72t - 3t^2)\,dt = \left(100t + 36t^2 - t^3\right)\Big|_0^{24} = 100(24) + 36(24)^2 - (24)^3 = 9312,$$

or 9.312 million gallons. From 6 PM to midnight, the water consumption is

$$\begin{aligned}\int_{18}^{24} (100 + 72t - 3t^2)\,dt &= \left(100t + 36t^2 - t^3\right)\Big|_{18}^{24} \\ &= 100(24) + 36(24)^2 - (24)^3 - \left(100(18) + 36(18)^2 - (18)^3\right) \\ &= 9312 - 7632 = 1680,\end{aligned}$$

or 1.68 million gallons.

76. The learning curve in a certain bicycle factory is $L(x) = 12x^{-1/5}$ (in hours per bicycle), which means that it takes a bike mechanic $L(n)$ hours to assemble the nth bicycle. If a mechanic has produced 24 bicycles, how long does it take her or him to produce the second batch of 12?

SOLUTION The second batch of 12 bicycles consists of bicycles 13 through 24. The time it takes to produce these bicycles is

$$\int_{13}^{24} 12x^{-1/5}\,dx = 15x^{4/5}\Big|_{13}^{24} = 15\left(24^{4/5} - 13^{4/5}\right) \approx 73.91 \text{ hours}.$$

77. Cost engineers at NASA have the task of projecting the cost P of major space projects. It has been found that the cost C of developing a projection increases with P at the rate $dC/dP \approx 21P^{-0.65}$, where C is in thousands of dollars and P in millions of dollars. What is the cost of developing a projection for a project whose cost turns out to be $P = \$35$ million?

SOLUTION Assuming it costs nothing to develop a projection for a project with a cost of \$0, the cost of developing a projection for a project whose cost turns out to be \$35 million is

$$\int_0^{35} 21P^{-0.65}\,dP = 60P^{0.35}\Big|_0^{35} = 60(35)^{0.35} \approx 208.245,$$

or \$208,245.

78. An astronomer estimates that in a certain constellation, the number of stars per magnitude m, per degree-squared of sky, is equal to $A(m) = 2.4 \times 10^{-6} m^{7.4}$ (fainter stars have higher magnitudes). Determine the total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky.

SOLUTION The total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky is

$$\int_6^{15} A(m)\,dm = \int_6^{15} 2.4 \times 10^{-6} m^{7.4}\,dm$$
$$= \frac{2}{7} \times 10^{-6} m^{8.4} \Big|_6^{15}$$
$$\approx 2162$$

79. Evaluate $\displaystyle\int_{-8}^{8} \frac{x^{15}\,dx}{3+\cos^2 x}$, using the properties of odd functions.

SOLUTION Let $f(x) = \frac{x^{15}}{3+\cos^2 x}$ and note that

$$f(-x) = \frac{(-x)^{15}}{3+\cos^2(-x)} = -\frac{x^{15}}{\cos^2 x} = -f(x).$$

Because $f(x)$ is an odd function and the interval $-8 \le x \le 8$ is symmetric about $x = 0$, it follows that

$$\int_{-8}^{8} \frac{x^{15}\,dx}{3+\cos^2 x} = 0.$$

80. Evaluate $\int_0^1 f(x)\,dx$, assuming that $f(x)$ is an even continuous function such that

$$\int_1^2 f(x)\,dx = 5, \qquad \int_{-2}^{1} f(x)\,dx = 8$$

SOLUTION Using the given information

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_1^2 f(x)\,dx = 13.$$

Because $f(x)$ is an even function, it follows that

$$\int_{-2}^{0} f(x)\,dx = \int_0^2 f(x)\,dx,$$

so

$$\int_0^2 f(x)\,dx = \frac{13}{2}.$$

Finally,

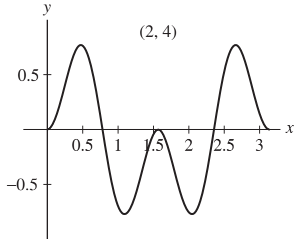
$$\int_0^1 f(x)\,dx = \int_0^2 f(x)\,dx - \int_1^2 f(x)\,dx = \frac{13}{2} - 5 = \frac{3}{2}.$$

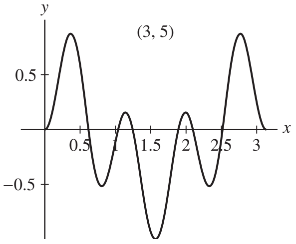
81. GU Plot the graph of $f(x) = \sin mx \sin nx$ on $[0, \pi]$ for the pairs $(m, n) = (2, 4), (3, 5)$ and in each case guess the value of $I = \int_0^\pi f(x)\,dx$. Experiment with a few more values (including two cases with $m = n$) and formulate a conjecture for when I is zero.

SOLUTION The graphs of $f(x) = \sin mx \sin nx$ with $(m, n) = (2, 4)$ and $(m, n) = (3, 5)$ are shown below. It appears as if the positive areas balance the negative areas, so we expect that

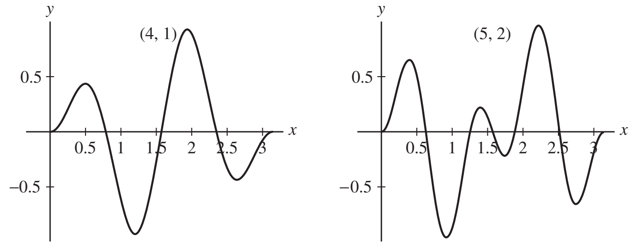
$$I = \int_0^\pi f(x)\,dx = 0$$

in these cases.





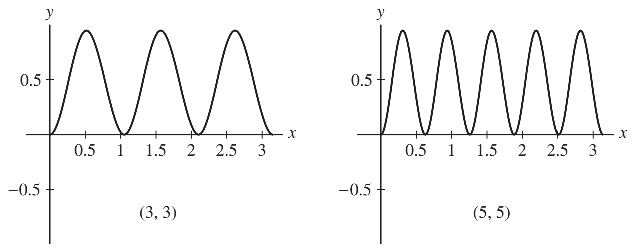
We arrive at the same conclusion for the cases $(m, n) = (4, 1)$ and $(m, n) = (5, 2)$.



However, when $(m, n) = (3, 3)$ and when $(m, n) = (5, 5)$, the value of

$$I = \int_0^{\pi} f(x)\, dx$$

is clearly not zero as there is no negative area.



We therefore conjecture that I is zero whenever $m \neq n$.

82. Show that

$$\int x\, f(x)\, dx = xF(x) - G(x)$$

where $F'(x) = f(x)$ and $G'(x) = F(x)$. Use this to evaluate $\int x \cos x\, dx$.

SOLUTION Suppose $F'(x) = f(x)$ and $G'(x) = F(x)$. Then

$$\frac{d}{dx}(xF(x) - G(x)) = xF'(x) + F(x) - G'(x) = xf(x) + F(x) - F(x) = xf(x).$$

Therefore, $xF(x) - G(x)$ is an antiderivative of $xf(x)$ and

$$\int xf(x)\, dx = xF(x) - G(x) + C.$$

To evaluate $\int x \cos x\, dx$, note that $f(x) = \cos x$. Thus, we may take $F(x) = \sin x$ and $G(x) = -\cos x$. Finally,

$$\int x \cos x\, dx = x \sin x + \cos x + C.$$

83. Prove

$$2 \le \int_1^2 2^x\, dx \le 4 \qquad \text{and} \qquad \frac{1}{9} \le \int_1^2 3^{-x}\, dx \le \frac{1}{3}$$

SOLUTION The function $f(x) = 2^x$ is increasing, so $1 \le x \le 2$ implies that $2 = 2^1 \le 2^x \le 2^2 = 4$. Consequently,

$$2 = \int_1^2 2\, dx \le \int_1^2 2^x\, dx \le \int_1^2 4\, dx = 4.$$

On the other hand, the function $f(x) = 3^{-x}$ is decreasing, so $1 \le x \le 2$ implies that

$$\frac{1}{9} = 3^{-2} \le 3^{-x} \le 3^{-1} = \frac{1}{3}.$$

It then follows that

$$\frac{1}{9} = \int_1^2 \frac{1}{9}\, dx \le \int_1^2 3^{-x}\, dx \le \int_1^2 \frac{1}{3}\, dx = \frac{1}{3}.$$

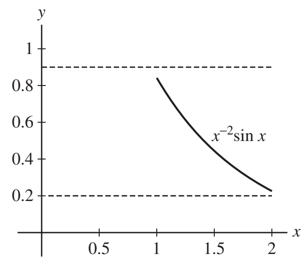
84. GU Plot the graph of $f(x) = x^{-2}\sin x$, and show that $0.2 \le \int_1^2 f(x)\,dx \le 0.9$.

SOLUTION Let $f(x) = x^{-2}\sin x$. From the figure below, we see that

$$0.2 \le f(x) \le 0.9$$

for $1 \le x \le 2$. Therefore,

$$0.2 = \int_0^1 0.2\,dx \le \int_0^1 f(x)\,dx \le \int_0^1 0.9\,dx = 0.9.$$



85. Find upper and lower bounds for $\int_0^1 f(x)\,dx$, for $f(x)$ in Figure 6.

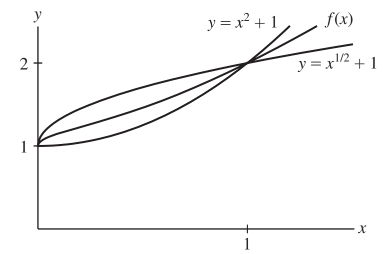


FIGURE 6

SOLUTION From the figure, we see that the inequalities $x^2 + 1 \le f(x) \le \sqrt{x} + 1$ hold for $0 \le x \le 1$. Because

$$\int_0^1 (x^2 + 1)\,dx = \left(\frac{1}{3}x^3 + x\right)\Big|_0^1 = \frac{4}{3}$$

and

$$\int_0^1 (\sqrt{x} + 1)\,dx = \left(\frac{2}{3}x^{3/2} + x\right)\Big|_0^1 = \frac{5}{3},$$

it follows that

$$\frac{4}{3} \le \int_0^1 f(x)\,dx \le \frac{5}{3}.$$

In Exercises 86–91, find the derivative.

86. $A'(x)$, where $A(x) = \int_3^x \sin(t^3)\,dt$

SOLUTION Let $A(x) = \int_3^x \sin(t^3)\,dt$. Then $A'(x) = \sin(x^3)$.

87. $A'(\pi)$, where $A(x) = \int_2^x \frac{\cos t}{1+t}\,dt$

SOLUTION Let $A(x) = \int_2^x \frac{\cos t}{1+t}\,dt$. Then $A'(x) = \frac{\cos x}{1+x}$ and

$$A'(\pi) = \frac{\cos \pi}{1+\pi} = -\frac{1}{1+\pi}.$$

88. $\frac{d}{dy}\int_{-2}^{y} 3^x\,dx$

SOLUTION $\frac{d}{dy}\int_{-2}^{y} 3^x\,dx = 3^y.$

89. $G'(x)$, where $G(x) = \int_{-2}^{\sin x} t^3\,dt$

SOLUTION Let $G(x) = \int_{-2}^{\sin x} t^3\,dt$. Then

$$G'(x) = \sin^3 x \frac{d}{dx}\sin x = \sin^3 x \cos x.$$

90. $G'(2)$, where $G(x) = \int_0^{x^3} \sqrt{t+1}\,dt$

SOLUTION Let $G(x) = \int_0^{x^3} \sqrt{t+1}\,dt$. Then

$$G'(x) = \sqrt{x^3+1}\frac{d}{dx}x^3 = 3x^2\sqrt{x^3+1}$$

and $G'(2) = 3(2)^2\sqrt{8+1} = 36$.

91. $H'(1)$, where $H(x) = \int_{4x^2}^{9} \frac{1}{t}\,dt$

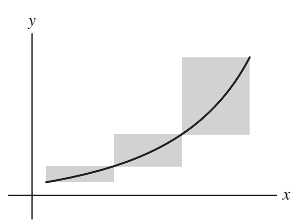
SOLUTION Let $H(x) = \int_{4x^2}^{9} \frac{1}{t}\,dt = -\int_9^{4x^2} \frac{1}{t}\,dt$. Then

$$H'(x) = -\frac{1}{4x^2}\frac{d}{dx}4x^2 = -\frac{8x}{4x^2} = -\frac{2}{x}$$

and $H'(1) = -2$.

92. Explain with a graph: If $f(x)$ is increasing and concave up on $[a, b]$, then L_N is more accurate than R_N. Which is more accurate if $f(x)$ is increasing and concave down?

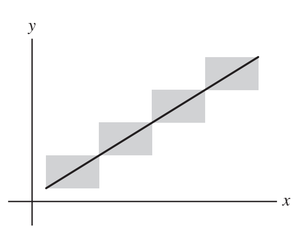
SOLUTION Consider the figure below, which displays a portion of the graph of an increasing, concave up function.



The shaded rectangles represent the differences between the right-endpoint approximation R_N and the left-endpoint approximation L_N. In particular, the portion of each rectangle that lies below the graph of $y = f(x)$ is the amount by which L_N underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of $y = f(x)$ is the amount by which R_N overestimates the area. Because the graph of $y = f(x)$ is increasing and concave up, the lower portion of each shaded rectangle is smaller than the upper portion. Therefore, L_N is more accurate (introduces less error) than R_N. By similar reasoning, if $f(x)$ is increasing and concave down, then R_N is more accurate than L_N.

93. Explain with a graph: If $f(x)$ is linear on $[a, b]$, then the $\int_a^b f(x)\,dx = \frac{1}{2}(R_N + L_N)$ for all N.

SOLUTION Consider the figure below, which displays a portion of the graph of a linear function.



The shaded rectangles represent the differences between the right-endpoint approximation R_N and the left-endpoint approximation L_N. In particular, the portion of each rectangle that lies below the graph of $y = f(x)$ is the amount by which L_N underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of $y = f(x)$ is the amount by which R_N overestimates the area. Because the graph of $y = f(x)$ is a line, the lower portion of each shaded rectangle is exactly the same size as the upper portion. Therefore, if we average L_N and R_N, the error in the two approximations will exactly cancel, leaving

$$\frac{1}{2}(R_N + L_N) = \int_a^b f(x)\,dx.$$

94. In this exercise, we prove

$$x - \frac{x^2}{2} \le \ln(1+x) \le x \qquad (\text{for } x > 0) \qquad \boxed{1}$$

(a) Show that $\ln(1+x) = \int_0^x \frac{dt}{1+t}$ for $x > 0$.
(b) Verify that $1 - t \le \frac{1}{1+t} \le 1$ for all $t > 0$.
(c) Use (b) to prove Eq. (1).
(d) Verify Eq. (1) for $x = 0.5$, 0.1, and 0.01.

SOLUTION
(a) Let $x > 0$. Then

$$\int_0^x \frac{dt}{1+t} = \ln(1+t)\Big|_0^x = \ln(1+x) - \ln 1 = \ln(1+x).$$

(b) For $t > 0$, $1 + t > 1$, so $\frac{1}{1+t} < 1$. Moreover, $(1-t)(1+t) = 1 - t^2 < 1$. Because $1 + t > 0$, it follows that $1 - t < \frac{1}{1+t}$. Hence,

$$1 - t \le \frac{1}{1+t} \le 1.$$

(c) Integrating each expression in the result from part (b) from $t = 0$ to $t = x$ yields

$$x - \frac{x^2}{2} \le \ln(1+x) \le x.$$

(d) For $x = 0.5$, $x = 0.1$ and $x = 0.01$, we obtain the string of inequalities

$$\begin{aligned} 0.375 \le\ & 0.405465 \le 0.5 \\ 0.095 \le\ & 0.095310 \le 0.1 \\ 0.00995 \le\ & 0.00995033 \le 0.01, \end{aligned}$$

respectively.

95. Let

$$F(x) = x\sqrt{x^2-1} - 2\int_1^x \sqrt{t^2-1}\,dt$$

Prove that $F(x)$ and $\cosh^{-1} x$ differ by a constant by showing that they have the same derivative. Then prove they are equal by evaluating both at $x = 1$.

SOLUTION Let

$$F(x) = x\sqrt{x^2-1} - 2\int_1^x \sqrt{t^2-1}\,dt.$$

Then

$$\frac{dF}{dx} = \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} - 2\sqrt{x^2-1} = \frac{x^2}{\sqrt{x^2-1}} - \sqrt{x^2-1} = \frac{1}{\sqrt{x^2-1}}.$$

Also, $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$; therefore, $F(x)$ and $\cosh^{-1}x$ have the same derivative. We conclude that $F(x)$ and $\cosh^{-1}x$ differ by a constant:

$$F(x) = \cosh^{-1}x + C.$$

Now, let $x = 1$. Because $F(1) = 0$ and $\cosh^{-1} 1 = 0$, it follows that $C = 0$. Therefore,

$$F(x) = \cosh^{-1}x.$$

96. Let $f(x)$ be a positive increasing continuous function on $[a, b]$, where $0 \le a < b$ as in Figure 7. Show that the shaded region has area

$$I = bf(b) - af(a) - \int_a^b f(x)\,dx \qquad \boxed{2}$$

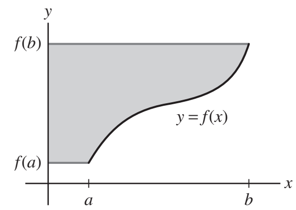


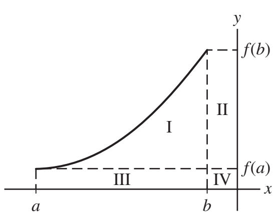
FIGURE 7

SOLUTION We can construct the shaded region in Figure 7 by taking a rectangle of length b and height $f(b)$ and removing a rectangle of length a and height $f(a)$ as well as the region between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$. The area of the resulting region is then the area of the large rectangle minus the area of the small rectangle and minus the area under the curve $y = f(x)$; that is,

$$I = bf(b) - af(a) - \int_a^b f(x)\,dx.$$

97. How can we interpret the quantity I in Eq. (2) if $a < b \le 0$? Explain with a graph.

SOLUTION We will consider each term on the right-hand side of (2) separately. For convenience, let **I**, **II**, **III** and **IV** denote the area of the similarly labeled region in the diagram below.



Because $b < 0$, the expression $bf(b)$ is the opposite of the area of the rectangle along the right; that is,

$$bf(b) = -\mathbf{II} - \mathbf{IV}.$$

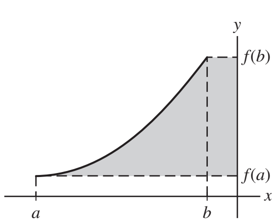
Similarly,

$$-af(a) = \mathbf{III} + \mathbf{IV} \qquad \text{and} \qquad -\int_a^b f(x)\,dx = -\mathbf{I} - \mathbf{III}.$$

Therefore,

$$bf(b) - af(a) - \int_a^b f(x)\,dx = -\mathbf{I} - \mathbf{II};$$

that is, the opposite of the area of the shaded region shown below.



98. The isotope thorium-234 has a half-life of 24.5 days.

(a) What is the differential equation satisfied by $y(t)$, the amount of thorium-234 in a sample at time t?

(b) At $t = 0$, a sample contains 2 kg of thorium-234. How much remains after 40 days?

SOLUTION

(a) By the equation for half-life,

$$24.5 = \frac{\ln 2}{k}, \quad \text{so} \quad k = \frac{\ln 2}{24.5} \approx 0.028 \text{ days}^{-1}.$$

Therefore, the differential equation for $y(t)$ is

$$y' = -0.028y.$$

(b) If there are 2 kg of thorium-234 at $t = 0$, then $y(t) = 2e^{-0.028t}$. After 40 days, the amount of thorium-234 is

$$y(40) = 2e^{-0.028(40)} = 0.653 \text{ kg}.$$

99. The Oldest Snack Food? In Bat Cave, New Mexico, archaeologists found ancient human remains, including cobs of popping corn whose C^{14}-to-C^{12} ratio was approximately 48% of that found in living matter. Estimate the age of the corn cobs.

SOLUTION Let t be the age of the corn cobs. The C^{14} to C^{12} ratio decreased by a factor of $e^{-0.000121t}$ which is equal to 0.48. That is:

$$e^{-0.000121t} = 0.48,$$

so

$$-0.000121t = \ln 0.48,$$

and

$$t = -\frac{1}{0.000121} \ln 0.48 \approx 6065.9.$$

We conclude that the age of the corn cobs is approximately 6065.9 years.

100. The C^{14}-to-C^{12} ratio of a sample is proportional to the disintegration rate (number of beta particles emitted per minute) that is measured directly with a Geiger counter. The disintegration rate of carbon in a living organism is 15.3 beta particles per minute per gram. Find the age of a sample that emits 9.5 beta particles per minute per gram.

SOLUTION Let t be the age of the sample in years. Because the disintegration rate for the sample has dropped from 15.3 beta particles/min per gram to 9.5 beta particles/min per gram and the C^{14} to C^{12} ratio is proportional to the disintegration rate, it follows that

$$e^{-0.000121t} = \frac{9.5}{15.3},$$

so

$$t = -\frac{1}{0.000121} \ln \frac{9.5}{15.3} \approx 3938.5.$$

We conclude that the sample is approximately 3938.5 years old.

101. What is the interest rate if the PV of \$50,000 to be delivered in 3 years is \$43,000?

SOLUTION Let r denote the interest rate. The present value of \$50,000 received in 3 years with an interest rate of r is $50{,}000e^{-3r}$. Thus, we need to solve

$$43{,}000 = 50{,}000e^{-3r}$$

for r. This yields

$$r = -\frac{1}{3} \ln \frac{43}{50} = 0.0503.$$

Thus, the interest rate is 5.03%.

102. An equipment upgrade costing \$1 million will save a company \$320,000 per year for 4 years. Is this a good investment if the interest rate is $r = 5\%$? What is the largest interest rate that would make the investment worthwhile? Assume that the savings are received as a lump sum at the end of each year.

SOLUTION With an interest rate of $r = 5\%$, the present value of the four payments is

$$\$320{,}000\left(e^{-0.05} + e^{-0.1} + e^{-0.15} + e^{-0.2}\right) = \$1{,}131{,}361.78.$$

As this is greater than the \$1 million cost of the upgrade, this is a good investment. To determine the largest interest rate that would make the investment worthwhile, we must solve the equation

$$320{,}000\left(e^{-r} + e^{-2r} + e^{-3r} + e^{-4r}\right) = 1{,}000{,}000$$

for r. Using a computer algebra system, we find $r = 10.13\%$.

103. Find the PV of an income stream paying out continuously at a rate of $5000e^{-0.1t}$ dollars per year for 5 years, assuming an interest rate of $r = 4\%$.

SOLUTION $PV = \int_0^5 5000e^{-0.1t}e^{-0.04t}\,dt = \int_0^5 5000e^{-0.14t}\,dt = \frac{5000}{-0.14}e^{-0.14t}\Big|_0^5 = \$17,979.10.$

104. Calculate the limit:

(a) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$ **(b)** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{4n}$ **(c)** $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{3n}$

SOLUTION

(a) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{n/4}\right)^{n/4}\right]^4 = e^4.$

(b) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{4n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^4 = e^4.$

(c) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n/4}\right)^{n/4}\right]^{12} = e^{12}.$

6 APPLICATIONS OF THE INTEGRAL

6.1 Area Between Two Curves

Preliminary Questions

1. What is the area interpretation of $\displaystyle\int_a^b \big(f(x) - g(x)\big)\,dx$ if $f(x) \ge g(x)$?

SOLUTION Because $f(x) \ge g(x)$, $\int_a^b (f(x) - g(x))\,dx$ represents the area of the region bounded between the graphs of $y = f(x)$ and $y = g(x)$, bounded on the left by the vertical line $x = a$ and on the right by the vertical line $x = b$.

2. Is $\displaystyle\int_a^b \big(f(x) - g(x)\big)\,dx$ still equal to the area between the graphs of f and g if $f(x) \ge 0$ but $g(x) \le 0$?

SOLUTION Yes. Since $f(x) \ge 0$ and $g(x) \le 0$, it follows that $f(x) - g(x) \ge 0$.

3. Suppose that $f(x) \ge g(x)$ on [0, 3] and $g(x) \ge f(x)$ on [3, 5]. Express the area between the graphs over [0, 5] as a sum of integrals.

SOLUTION Remember that to calculate an area between two curves, one must subtract the equation for the lower curve from the equation for the upper curve. Over the interval [0, 3], $y = f(x)$ is the upper curve. On the other hand, over the interval [3, 5], $y = g(x)$ is the upper curve. The area between the graphs over the interval [0, 5] is therefore given by

$$\int_0^3 (f(x) - g(x))\,dx + \int_3^5 (g(x) - f(x))\,dx.$$

4. Suppose that the graph of $x = f(y)$ lies to the left of the y-axis. Is $\int_a^b f(y)\,dy$ positive or negative?

SOLUTION If the graph of $x = f(y)$ lies to the left of the y-axis, then for each value of y, the corresponding value of x is less than zero. Hence, the value of $\int_a^b f(y)\,dy$ is negative.

Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 9).

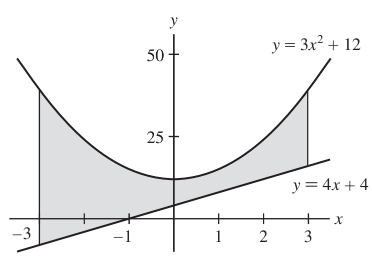


FIGURE 9

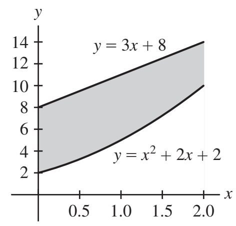
SOLUTION As the graph of $y = 3x^2 + 12$ lies above the graph of $y = 4x + 4$ over the interval $[-3, 3]$, the area between the graphs is

$$\int_{-3}^{3} \Big((3x^2 + 12) - (4x + 4)\Big)\,dx = \int_{-3}^{3} (3x^2 - 4x + 8)\,dx = \Big(x^3 - 2x^2 + 8x\Big)\Big|_{-3}^{3} = 102.$$

2. Find the area of the region between the graphs of $f(x) = 3x + 8$ and $g(x) = x^2 + 2x + 2$ over [0, 2].

SOLUTION From the diagram below, we see that the graph of $f(x) = 3x + 8$ lies above the graph of $g(x) = x^2 + 2x + 2$ over the interval [0, 2]. Thus, the area between the graphs is

$$\int_0^2 \Big[(3x + 8) - \Big(x^2 + 2x + 2\Big)\Big]\,dx = \int_0^2 \Big(-x^2 + x + 6\Big)\,dx = \Big(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\Big)\Big|_0^2 = \frac{34}{3}.$$



3. Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$ (Figure 10).

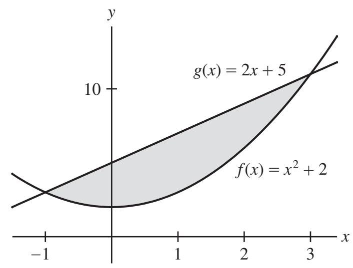


FIGURE 10

SOLUTION From the figure, we see that the graph of $g(x) = 2x + 5$ lies above the graph of $f(x) = x^2 + 2$ over the interval $[-1, 3]$. Thus, the area between the graphs is

$$\int_{-1}^{3} \left[(2x+5) - \left(x^2+2\right)\right] dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx$$

$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3}$$

$$= 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}.$$

4. Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$ (Figure 11).

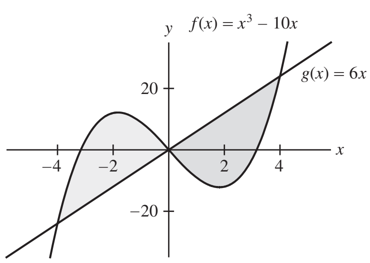


FIGURE 11

SOLUTION From the figure, we see that the graph of $f(x) = x^3 - 10x$ lies above the graph of $g(x) = 6x$ over the interval $[-4, 0]$, while the graph of $g(x) = 6x$ lies above the graph of $f(x) = x^3 - 10x$ over the interval $[0, 4]$. Thus, the area enclosed by the two graphs is

$$A = \int_{-4}^{0} \left(x^3 - 10x - 6x\right) dx + \int_{0}^{4} \left(6x - (x^3 - 10x)\right) dx$$

$$= \int_{-4}^{0} (x^3 - 16x)\, dx + \int_{0}^{4} (16x - x^3)\, dx$$

$$= \left(\frac{1}{4}x^4 - 8x^2\right)\Big|_{-4}^{0} + \left(8x^2 - \frac{1}{4}x^4\right)\Big|_{0}^{4}$$

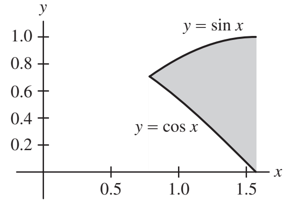
$$= 64 + 64 = 128.$$

In Exercises 5 and 6, sketch the region between $y = \sin x$ and $y = \cos x$ over the interval and find its area.

5. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

SOLUTION Over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, the graph of $y = \cos x$ lies below that of $y = \sin x$ (see the sketch below). Hence, the area between the two curves is

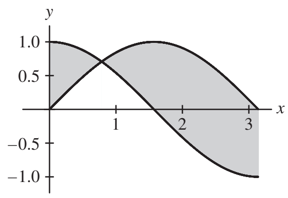
$$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2} = (0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2} - 1.$$



6. $[0, \pi]$

SOLUTION Over the interval $[0, \frac{\pi}{4}]$, the graph of $y = \sin x$ lies below that of $y = \cos x$, while over the interval $[\frac{\pi}{4}, \pi]$, the orientation of the graphs is reversed (see the sketch below). The area between the graphs over $[0, \pi]$ is then

$$\begin{aligned} &\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx \\ &= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi} \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) + (1 - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}. \end{aligned}$$



In Exercises 7 and 8, let $f(x) = 20 + x - x^2$ and $g(x) = x^2 - 5x$.

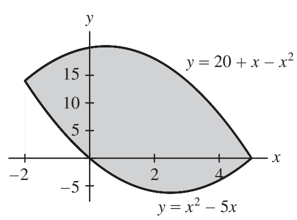
7. Sketch the region enclosed by the graphs of $f(x)$ and $g(x)$ and compute its area.

SOLUTION Setting $f(x) = g(x)$ gives $20 + x - x^2 = x^2 - 5x$, which simplifies to

$$0 = 2x^2 - 6x - 20 = 2(x - 5)(x + 2).$$

Thus, the curves intersect at $x = -2$ and $x = 5$. With $y = 20 + x - x^2$ being the upper curve (see the sketch below), the area between the two curves is

$$\int_{-2}^{5} \left((20 + x - x^2) - (x^2 - 5x)\right) dx = \int_{-2}^{5} \left(20 + 6x - 2x^2\right) dx = \left(20x + 3x^2 - \frac{2}{3}x^3\right)\Big|_{-2}^{5} = \frac{343}{3}.$$



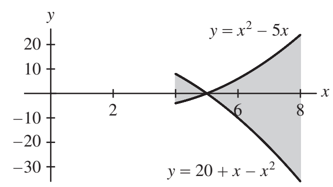
8. Sketch the region between the graphs of $f(x)$ and $g(x)$ over [4, 8] and compute its area as a sum of two integrals.

SOLUTION Setting $f(x) = g(x)$ gives $20 + x - x^2 = x^2 - 5x$, which simplifies to

$$0 = 2x^2 - 6x - 20 = 2(x-5)(x+2).$$

Thus, the curves intersect at $x = -2$ and $x = 5$. Over the interval [4, 5], $y = 20 + x - x^2$ is the upper curve but over the interval [5, 8], $y = x^2 - 5x$ is the upper curve (see the sketch below). The area between the two curves over the interval [4, 8] is then

$$\int_4^5 \Big((20 + x - x^2) - (x^2 - 5x)\Big)\,dx + \int_5^8 \Big((x^2 - 5x) - (20 + x - x^2)\Big)\,dx$$
$$= \int_4^5 \Big(-2x^2 + 6x + 20\Big)\,dx + \int_5^8 \Big(2x^2 - 6x - 20\Big)\,dx$$
$$= \left(-\frac{2}{3}x^2 + 3x^2 + 20x\right)\Big|_4^5 + \left(\frac{2}{3}x^3 - 3x^2 - 20x\right)\Big|_5^8 = \frac{19}{3} + 81 = \frac{262}{3}.$$



9. Find the area between $y = e^x$ and $y = e^{2x}$ over [0, 1].

SOLUTION As the graph of $y = e^{2x}$ lies above the graph of $y = e^x$ over the interval [0, 1], the area between the graphs is

$$\int_0^1 (e^{2x} - e^x)\,dx = \left(\frac{1}{2}e^{2x} - e^x\right)\Big|_0^1 = \frac{1}{2}e^2 - e - \left(\frac{1}{2} - 1\right) = \frac{1}{2}e^2 - e + \frac{1}{2}.$$

10. Find the area of the region bounded by $y = e^x$ and $y = 12 - e^x$ and the y-axis.

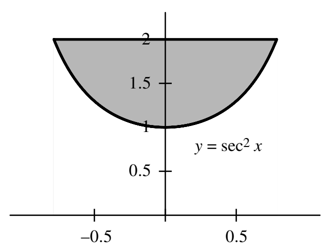
SOLUTION The two graphs intersect when $e^x = 12 - e^x$, or when $x = \ln 6$. As the graph of $y = 12 - e^x$ lies above the graph of $y = e^x$ over the interval $[0, \ln 6]$, the area between the graphs is

$$\int_0^{ln 6} \left(12 - e^x - e^x\right)\,dx = (12x - 2e^x)\Big|_0^{\ln 6} = 12\ln 6 - 12 - (0 - 2) = 12\ln 6 - 10.$$

11. Sketch the region bounded by the line $y = 2$ and the graph of $y = \sec^2 x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and find its area.

SOLUTION A sketch of the region bounded by $y = \sec^2 x$ and $y = 2$ is shown below. Note the region extends from $x = -\frac{\pi}{4}$ on the left to $x = \frac{\pi}{4}$ on the right. As the graph of $y = 2$ lies above the graph of $y = \sec^2 x$, the area between the graphs is

$$\int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\,dx = (2x - \tan x)\Big|_{-\pi/4}^{\pi/4} = \left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right) = \pi - 2.$$



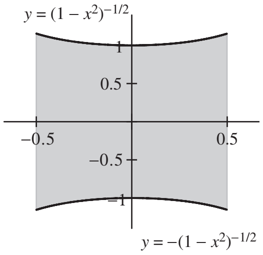
12. Sketch the region bounded by

$$y = \frac{1}{\sqrt{1 - x^2}} \quad \text{and} \quad y = -\frac{1}{\sqrt{1 - x^2}}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$ and find its area.

SOLUTION A sketch of the region bounded by $y = \frac{1}{\sqrt{1-x^2}}$ and $y = -\frac{1}{\sqrt{1-x^2}}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$ is shown below. As the graph of $y = \frac{1}{\sqrt{1-x^2}}$ lies above the graph of $y = -\frac{1}{\sqrt{1-x^2}}$, the area between the graphs is

$$\int_{-1/2}^{1/2} \left[\frac{1}{\sqrt{1-x^2}} - \left(-\frac{1}{\sqrt{1-x^2}}\right)\right] dx = 2\sin^{-1} x \Big|_{-1/2}^{1/2} = 2\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{2\pi}{3}.$$



In Exercises 13–16, find the area of the shaded region in Figures 12–15.

13.

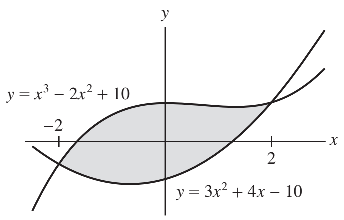


FIGURE 12

SOLUTION As the graph of $y = x^3 - 2x^2 + 10$ lies above the graph of $y = 3x^2 + 4x - 10$, the area of the shaded region is

$$\int_{-2}^{2} \left((x^3 - 2x^2 + 10) - (3x^2 + 4x - 10)\right) dx = \int_{-2}^{2} \left(x^3 - 5x^2 - 4x + 20\right) dx$$
$$= \left(\frac{1}{4}x^4 - \frac{5}{3}x^3 - 2x^2 + 20x\right)\Big|_{-2}^{2} = \frac{160}{3}.$$

14.

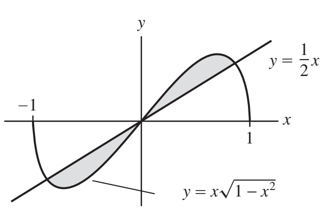


FIGURE 13

SOLUTION Setting $\frac{1}{2}x = x\sqrt{1-x^2}$ yields $x = 0$ or $\frac{1}{2} = \sqrt{1-x^2}$, so that $x = \pm\frac{\sqrt{3}}{2}$. Over the interval $[-\frac{\sqrt{3}}{2}, 0]$, $y = \frac{1}{2}x$ is the upper curve but over the interval $[0, \frac{\sqrt{3}}{2}]$, $y = x\sqrt{1-x^2}$ is the upper curve. The area of the shaded region is then

$$\int_{-\sqrt{3}/2}^{0} \left(\frac{1}{2}x - x\sqrt{1-x^2}\right) dx + \int_{0}^{\sqrt{3}/2} \left(x\sqrt{1-x^2} - \frac{1}{2}x\right) dx$$
$$= \left(\frac{1}{4}x^2 + \frac{1}{3}(1-x^2)^{3/2}\right)\Big|_{-\sqrt{3}/2}^{0} + \left(-\frac{1}{3}(1-x^2)^{3/2} - \frac{1}{4}x^2\right)\Big|_{0}^{\sqrt{3}/2} = \frac{5}{48} + \frac{5}{48} = \frac{5}{24}.$$

15.

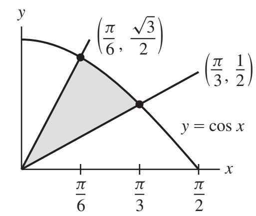


FIGURE 14

SOLUTION The line on the top-left has equation $y = \frac{3\sqrt{3}}{\pi}x$, and the line on the bottom-right has equation $y = \frac{3}{2\pi}x$. Thus, the area to the left of $x = \frac{\pi}{6}$ is

$$\int_0^{\pi/6} \left(\frac{3\sqrt{3}}{\pi}x - \frac{3}{2\pi}x\right) dx = \left(\frac{3\sqrt{3}}{2\pi}x^2 - \frac{3}{4\pi}x^2\right)\Bigg|_0^{\pi/6} = \frac{3\sqrt{3}}{2\pi}\frac{\pi^2}{36} - \frac{3}{4\pi}\frac{\pi^2}{36} = \frac{(2\sqrt{3}-1)\pi}{48}.$$

The area to the right of $x = \frac{\pi}{6}$ is

$$\int_{\pi/6}^{\pi/3} \left(\cos x - \frac{3}{2\pi}x\right) dx = \left(\sin x - \frac{3}{4\pi}x^2\right)\Bigg|_{\pi/6}^{\pi/3} = \frac{8\sqrt{3} - 8 - \pi}{16}.$$

The entire area is then

$$\frac{(2\sqrt{3}-1)\pi}{48} + \frac{8\sqrt{3}-8-\pi}{16} = \frac{12\sqrt{3} - 12 + (\sqrt{3}-2)\pi}{24}.$$

16.

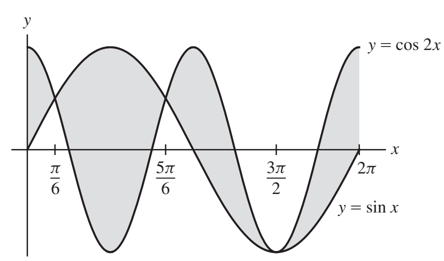


FIGURE 15

SOLUTION Over the interval $[0, \pi/6]$, the graph of $y = \cos 2x$ lies above the graph of $y = \sin x$. The orientation of the two graphs reverses over $[\pi/6, 5\pi/6]$ and reverses again over $[5\pi/6, 2\pi]$. Thus, the area between the two graphs is given by

$$A = \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{5\pi/6} (\sin x - \cos 2x)\, dx + \int_{5\pi/6}^{2\pi} (\cos 2x - \sin x)\, dx.$$

Carrying out the integration and evaluation, we find

$$\begin{aligned} A &= \left(\frac{1}{2}\sin 2x + \cos x\right)\Bigg|_0^{\pi/6} + \left(-\cos x - \frac{1}{2}\sin 2x\right)\Bigg|_{\pi/6}^{5\pi/6} + \left(\frac{1}{2}\sin 2x + \cos x\right)\Bigg|_{5\pi/6}^{2\pi} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right) + 1 - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right) \\ &= 3\sqrt{3}. \end{aligned}$$

In Exercises 17 and 18, find the area between the graphs of $x = \sin y$ and $x = 1 - \cos y$ over the given interval (Figure 16).

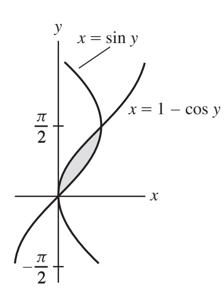


FIGURE 16

17. $0 \le y \le \frac{\pi}{2}$

SOLUTION As shown in the figure, the graph on the right is $x = \sin y$ and the graph on the left is $x = 1 - \cos y$. Therefore, the area between the two curves is given by

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy = \left(-\cos y - y + \sin y\right)\Big|_0^{\pi/2} = \left(-\frac{\pi}{2} + 1\right) - (-1) = 2 - \frac{\pi}{2}.$$

18. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

SOLUTION The shaded region in the figure shows the area between the graphs from $y = 0$ to $y = \frac{\pi}{2}$. It is bounded on the right by $x = \sin y$ and on the left by $x = 1 - \cos y$. Therefore, the area between the graphs from $y = 0$ to $y = \frac{\pi}{2}$ is

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy = \left(-\cos y - y + \sin y\right)\Big|_0^{\pi/2} = \left(-\frac{\pi}{2} + 1\right) - (-1) = 2 - \frac{\pi}{2}.$$

The graphs cross at $y = 0$. Since $x = 1 - \cos y$ lies to the right of $x = \sin y$ on the interval $[-\frac{\pi}{2}, 0]$ along the y-axis, the area between the graphs from $y = -\frac{\pi}{2}$ to $y = 0$ is

$$\int_{-\pi/2}^{0} ((1 - \cos y) - \sin y)\, dy = \left(y - \sin y + \cos y\right)\Big|_{-\pi/2}^{0} = 1 - \left(-\frac{\pi}{2} + 1\right) = \frac{\pi}{2}.$$

The total area between the graphs from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$ is the sum

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy + \int_{-\pi/2}^{0} ((1 - \cos y) - \sin y)\, dy = 2 - \frac{\pi}{2} + \frac{\pi}{2} = 2.$$

19. Find the area of the region lying to the right of $x = y^2 + 4y - 22$ and to the left of $x = 3y + 8$.

SOLUTION Setting $y^2 + 4y - 22 = 3y + 8$ yields

$$0 = y^2 + y - 30 = (y + 6)(y - 5),$$

so the two curves intersect at $y = -6$ and $y = 5$. The area in question is then given by

$$\int_{-6}^{5} \left((3y + 8) - (y^2 + 4y - 22)\right) dy = \int_{-6}^{5} \left(-y^2 - y + 30\right) dy = \left(-\frac{y^3}{3} - \frac{y^2}{2} + 30y\right)\Bigg|_{-6}^{5} = \frac{1331}{6}.$$

20. Find the area of the region lying to the right of $x = y^2 - 5$ and to the left of $x = 3 - y^2$.

SOLUTION Setting $y^2 + 5 = 3 - y^2$ yields $2y^2 = 8$ or $y = \pm 2$. The area of the region enclosed by the two graphs is then

$$\int_{-2}^{2} \left((3 - y^2) - (y^2 + 5)\right) dy = \int_{-2}^{2} \left(8 - 2y^2\right) dy = \left(8y - \frac{2}{3}y^3\right)\Big|_{-2}^{2} = \frac{64}{3}.$$

21. Figure 17 shows the region enclosed by $x = y^3 - 26y + 10$ and $x = 40 - 6y^2 - y^3$. Match the equations with the curves and compute the area of the region.

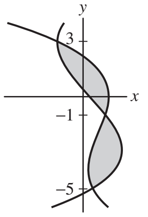


FIGURE 17

SOLUTION Substituting $y = 0$ into the equations for both curves indicates that the graph of $x = y^3 - 26y + 10$ passes through the point $(10, 0)$ while the graph of $x = 40 - 6y^2 - y^3$ passes through the point $(40, 0)$. Therefore, over the

y-interval $[-1, 3]$, the graph of $x = 40 - 6y^2 - y^3$ lies to the right of the graph of $x = y^3 - 26y + 10$. The orientation of the two graphs is reversed over the y-interval $[-5, -1]$. Hence, the area of the shaded region is

$$\int_{-5}^{-1} \left((y^3 - 26y + 10) - (40 - 6y^2 - y^3)\right) dy + \int_{-1}^{3} \left((40 - 6y^2 - y^3) - (y^3 - 26y + 10)\right) dy$$
$$= \int_{-5}^{-1} \left(2y^3 + 6y^2 - 26y - 30\right) dy + \int_{-1}^{3} \left(-2y^3 - 6y^2 + 26y + 30\right) dy$$
$$= \left(\frac{1}{2}y^4 + 2y^3 - 13y^2 - 30y\right)\Big|_{-5}^{-1} + \left(-\frac{1}{2}y^4 - 2y^3 + 13y^2 + 30y\right)\Big|_{-1}^{3} = 256.$$

22. Figure 18 shows the region enclosed by $y = x^3 - 6x$ and $y = 8 - 3x^2$. Match the equations with the curves and compute the area of the region.

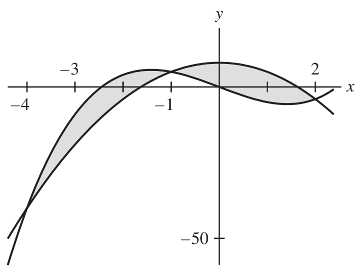


FIGURE 18 Region between $y = x^3 - 6x$ and $y = 8 - 3x^2$.

SOLUTION Setting $x^3 - 6x = 8 - 3x^2$ yields $(x + 1)(x + 4)(x - 2) = 0$, so the two curves intersect at $x = -4$, $x = -1$ and $x = 2$. Over the interval $[-4, -1]$, $y = x^3 - 6x$ is the upper curve, while $y = 8 - 3x^2$ is the upper curve over the interval $[-1, 2]$. The area of the region enclosed by the two curves is then

$$\int_{-4}^{-1} \left((x^3 - 6x) - (8 - 3x^2)\right) dx + \int_{-1}^{2} \left((8 - 3x^2) - (x^3 - 6x)\right) dx$$
$$= \left(\frac{1}{4}x^4 - 3x^2 - 8x + x^3\right)\Big|_{-4}^{-1} + \left(8x - x^3 - \frac{1}{4}x^4 + 3x^2\right)\Big|_{-1}^{2} = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}.$$

In Exercises 23 and 24, find the area enclosed by the graphs in two ways: by integrating along the x-axis and by integrating along the y-axis.

23. $x = 9 - y^2, \quad x = 5$

SOLUTION Along the y-axis, we have points of intersection at $y = \pm 2$. Therefore, the area enclosed by the two curves is

$$\int_{-2}^{2} \left(9 - y^2 - 5\right) dy = \int_{-2}^{2} \left(4 - y^2\right) dy = \left(4y - \frac{1}{3}y^3\right)\Big|_{-2}^{2} = \frac{32}{3}.$$

Along the x-axis, we have integration limits of $x = 5$ and $x = 9$. Therefore, the area enclosed by the two curves is

$$\int_{5}^{9} 2\sqrt{9 - x}\, dx = -\frac{4}{3}(9 - x)^{3/2}\Big|_{5}^{9} = 0 - \left(-\frac{32}{3}\right) = \frac{32}{3}.$$

24. The *semicubical parabola* $y^2 = x^3$ and the line $x = 1$.

SOLUTION Since $y^2 = x^3$, it follows that $x \geq 0$ since $y^2 \geq 0$. Therefore, $y = \pm x^{3/2}$, and the area of the region enclosed by the semicubical parabola and $x = 1$ is

$$\int_{0}^{1} \left(x^{3/2} - (-x^{3/2})\right) dx = \int_{0}^{1} 2x^{3/2}\, dx = \frac{4}{5}x^{5/2}\Big|_{0}^{1} = \frac{4}{5}.$$

Along the y-axis, we have integration limits of $y = \pm 1$. Therefore, the area enclosed by the two curves is

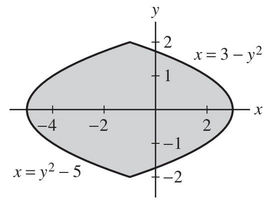
$$\int_{-1}^{1} \left(1 - y^{2/3}\right) dy = \left(y - \frac{3}{5}y^{5/3}\right)\Big|_{-1}^{1} = \left(1 - \frac{3}{5}\right) - \left(-1 + \frac{3}{5}\right) = \frac{4}{5}.$$

In Exercises 25 and 26, find the area of the region using the method (integration along either the x- or the y-axis) that requires you to evaluate just one integral.

25. Region between $y^2 = x + 5$ and $y^2 = 3 - x$

SOLUTION From the figure below, we see that integration along the x-axis would require two integrals, but integration along the y-axis requires only one integral. Setting $y^2 - 5 = 3 - y^2$ yields points of intersection at $y = \pm 2$. Thus, the area is given by

$$\int_{-2}^{2} \left((3 - y^2) - (y^2 + 5) \right) dy = \int_{-2}^{2} \left(8 - 2y^2 \right) dy = \left(8y - \frac{2}{3}y^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



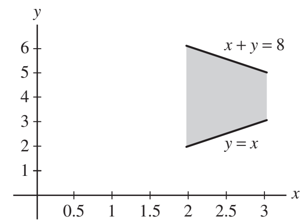
26. Region between $y = x$ and $x + y = 8$ over $[2, 3]$

SOLUTION From the figure below, we see that integration along the y-axis would require three integrals, but integration along the x-axis requires only one integral. The area of the region is then

$$\int_{2}^{3} \left((8 - x) - x \right) dx = (8x - x^2) \Big|_{2}^{3} = (24 - 9) - (16 - 4) = 3.$$

As a check, the area of a trapezoid is given by

$$\frac{h}{2}(b_1 + b_2) = \frac{1}{2}(4 + 2) = 3.$$

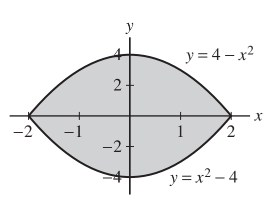


In Exercises 27–44, sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis.

27. $y = 4 - x^2, \quad y = x^2 - 4$

SOLUTION Setting $4 - x^2 = x^2 - 4$ yields $2x^2 = 8$ or $x^2 = 4$. Thus, the curves $y = 4 - x^2$ and $y = x^2 - 4$ intersect at $x = \pm 2$. From the figure below, we see that $y = 4 - x^2$ lies above $y = x^2 - 4$ over the interval $[-2, 2]$; hence, the area of the region enclosed by the curves is

$$\int_{-2}^{2} \left((4 - x^2) - (x^2 - 4) \right) dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left(8x - \frac{2}{3}x^3 \right) \Big|_{-2}^{2} = \frac{64}{3}.$$



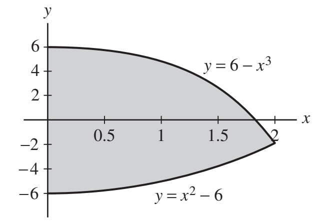
28. $y = x^2 - 6, \quad y = 6 - x^3, \quad y$-axis

SOLUTION Setting $x^2 - 6 = 6 - x^3$ yields

$$0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),$$

so the curves $y = x^2 - 6$ and $y = 6 - x^3$ intersect at $x = 2$. Using the graph shown below, we see that $y = 6 - x^3$ lies above $y = x^2 - 6$ over the interval [0, 2]; hence, the area of the region enclosed by these curves and the y-axis is

$$\int_0^2 \left((6 - x^3) - (x^2 - 6)\right) dx = \int_0^2 (-x^3 - x^2 + 12)\, dx = \left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x\right)\Big|_0^2 = \frac{52}{3}.$$

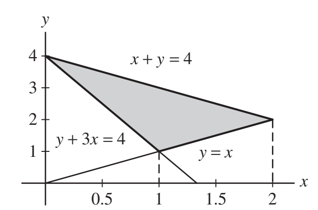


29. $x + y = 4, \quad x - y = 0, \quad y + 3x = 4$

SOLUTION From the graph below, we see that the top of the region enclosed by the three lines is always bounded by $x + y = 4$. On the other hand, the bottom of the region is bounded by $y + 3x = 4$ for $0 \le x \le 1$ and by $x - y = 0$ for $1 \le x \le 2$. The total area of the region is then

$$\int_0^1 ((4 - x) - (4 - 3x))\ dx + \int_1^2 ((4 - x) - x)\ dx = \int_0^1 2x\, dx + \int_1^2 (4 - 2x)\, dx$$

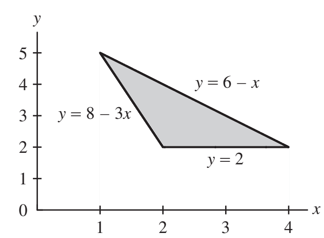
$$= x^2\Big|_0^1 + (4x - x^2)\Big|_1^2 = 1 + (8 - 4) - (4 - 1) = 2.$$



30. $y = 8 - 3x, \quad y = 6 - x, \quad y = 2$

SOLUTION From the figure below, we see that the graph of $y = 6 - x$ lies to the right of the graph of $y = 8 - 3x$, so integration in y is most appropriate for this problem. Setting $8 - 3x = 6 - x$ yields $x = 1$, so the y-coordinate of the point of intersection between $y = 8 - 3x$ and $y = 6 - x$ is 5. The area bounded by the three given curves is thus

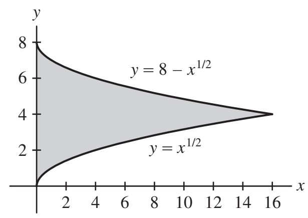
$$\begin{aligned} A &= \int_2^5 \left((6 - y) - \frac{1}{3}(8 - y)\right) dy \\ &= \int_2^5 \left(\frac{10}{3} - \frac{2}{3}y\right) dy \\ &= \left(\frac{10}{3}y - \frac{1}{3}y^2\right)\Big|_2^5 \\ &= \left(\frac{50}{3} - \frac{25}{3}\right) - \left(\frac{20}{3} - \frac{4}{3}\right) \\ &= 3. \end{aligned}$$



31. $y = 8 - \sqrt{x}, \quad y = \sqrt{x}, \quad x = 0$

SOLUTION Setting $8 - \sqrt{x} = \sqrt{x}$ yields $\sqrt{x} = 4$ or $x = 16$. Using the graph shown below, we see that $y = 8 - \sqrt{x}$ lies above $y = \sqrt{x}$ over the interval $[0, 16]$. The area of the region enclosed by these two curves and the y-axis is then

$$\int_0^{16} \left(8 - \sqrt{x} - \sqrt{x}\right) dx = \int_0^{16} \left(8 - 2\sqrt{x}\right) dx = \left(8x - \frac{4}{3}x^{3/2}\right)\bigg|_0^{16} = \frac{128}{3}.$$



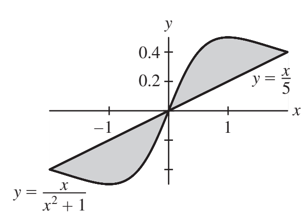
32. $y = \dfrac{x}{x^2+1}, \quad y = \dfrac{x}{5}$

SOLUTION Setting

$$\frac{x}{x^2+1} = \frac{x}{5} \quad \text{yields} \quad x = -2, 0, 2.$$

From the figure below, we see that the graph of $y = x/5$ lies above the graph of $y = x/(x^2+1)$ over $[-2, 0]$ and that the orientation is reversed over $[0, 2]$. Thus,

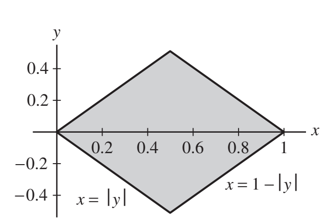
$$\begin{aligned} A &= \int_{-2}^{0} \left(\frac{x}{5} - \frac{x}{x^2+1}\right) dx + \int_0^2 \left(\frac{x}{x^2+1} - \frac{x}{5}\right) dx \\ &= \left(\frac{x^2}{10} - \frac{1}{2}\ln(x^2+1)\right)\bigg|_{-2}^{0} + \left(\frac{1}{2}\ln(x^2+1) - \frac{x^2}{10}\right)\bigg|_0^2 \\ &= \left(0 - \frac{2}{5} + \frac{1}{2}\ln 5\right) + \left(\frac{1}{2}\ln 5 - \frac{2}{5} - 0\right) \\ &= \ln 5 - \frac{4}{5}. \end{aligned}$$



33. $x = |y|, \quad x = 1 - |y|$

SOLUTION From the graph below, we see that the region enclosed by the curves $x = |y|$ and $x = 1 - |y|$ is symmetric with respect to the x-axis. We can therefore determine the total area by doubling the area in the first quadrant. For $y > 0$, setting $y = 1 - y$ yields $y = \frac{1}{2}$ as the point of intersection. Moreover, $x = 1 - |y| = 1 - y$ lies to the right of $x = |y| = y$, so the total area of the region is

$$2\int_0^{1/2} \left((1-y) - y\right) dy = 2\left(y - y^2\right)\bigg|_0^{1/2} = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}.$$



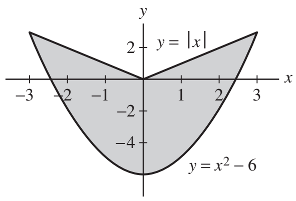
34. $y = |x|, \quad y = x^2 - 6$

SOLUTION From the graph below, we see that the region enclosed by the curves $y = |x|$ and $y = x^2 - 6$ is symmetric with respect to the y-axis. We can therefore determine the total area of the region by doubling the area of the portion of the region to the right of the y-axis. For $x > 0$, setting $x = x^2 - 6$ yields

$$0 = x^2 - x - 6 = (x-3)(x+2),$$

so the curves intersect at $x = 3$. Moreover, on the interval $[0, 3]$, $y = |x| = x$ lies above $y = x^2 - 6$. Therefore, the area of the region enclosed by the two curves is

$$2\int_0^3 \left(x - (x^2 - 6)\right) dx = 2\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right)\Bigg|_0^3 = 2\left(\frac{9}{2} - 9 + 18\right) = 27.$$



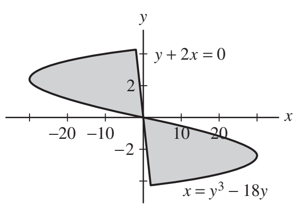
35. $x = y^3 - 18y, \quad y + 2x = 0$

SOLUTION Setting $y^3 - 18y = -\frac{y}{2}$ yields

$$0 = y^3 - \frac{35}{2}y = y\left(y^2 - \frac{35}{2}\right),$$

so the points of intersection occur at $y = 0$ and $y = \pm\frac{\sqrt{70}}{2}$. From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the second quadrant is identical to the region enclosed in the fourth quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the second quadrant. In the second quadrant, $y + 2x = 0$ lies to the right of $x = y^3 - 18y$, so the total area enclosed by the two curves is

$$2\int_0^{\sqrt{70}/2} \left(-\frac{y}{2} - (y^3 - 18y)\right) dy = 2\left(\frac{35}{4}y^2 - \frac{1}{4}y^4\right)\Bigg|_0^{\sqrt{70}/2} = 2\left(\frac{1225}{8} - \frac{1225}{16}\right) = \frac{1225}{8}.$$



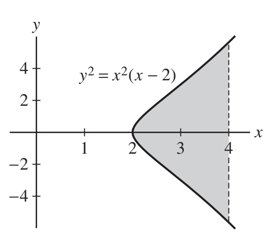
36. $y = x\sqrt{x-2}, \quad y = -x\sqrt{x-2}, \quad x = 4$

SOLUTION Note that $y = x\sqrt{x-2}$ and $y = -x\sqrt{x-2}$ are the upper and lower branches, respectively, of the curve $y^2 = x^2(x-2)$. The area enclosed by this curve and the vertical line $x = 4$ is

$$\int_2^4 \left(x\sqrt{x-2} - (-x\sqrt{x-2})\right) dx = \int_2^4 2x\sqrt{x-2}\,dx.$$

Substitute $u = x - 2$. Then $du = dx$, $x = u + 2$ and

$$\int_2^4 2x\sqrt{x-2}\,dx = \int_0^2 2(u+2)\sqrt{u}\,du = \int_0^2 \left(2u^{3/2} + 4u^{1/2}\right) du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right)\Bigg|_0^2 = \frac{128\sqrt{2}}{15}.$$



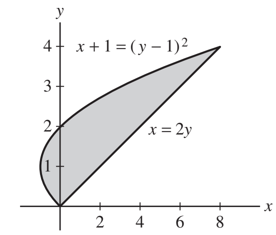
37. $x = 2y, \quad x + 1 = (y - 1)^2$

SOLUTION Setting $2y = (y-1)^2 - 1$ yields

$$0 = y^2 - 4y = y(y - 4),$$

so the two curves intersect at $y = 0$ and at $y = 4$. From the graph below, we see that $x = 2y$ lies to the right of $x + 1 = (y-1)^2$ over the interval $[0, 4]$ along the y-axis. Thus, the area of the region enclosed by the two curves is

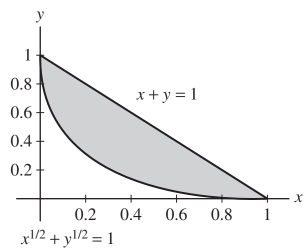
$$\int_0^4 \left(2y - ((y-1)^2 - 1)\right)\, dy = \int_0^4 \left(4y - y^2\right)\, dy = \left(2y^2 - \frac{1}{3}y^3\right)\Big|_0^4 = \frac{32}{3}.$$



38. $x + y = 1, \quad x^{1/2} + y^{1/2} = 1$

SOLUTION From the graph below, we see that the two curves intersect at $x = 0$ and at $x = 1$ and that $x + y = 1$ lies above $x^{1/2} + y^{1/2} = 1$. The area of the region enclosed by the two curves is then

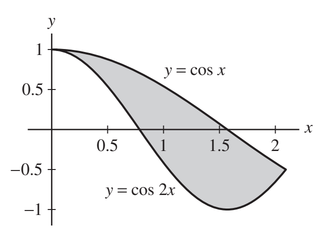
$$\int_0^1 \left((1 - x) - (1 - \sqrt{x})^2\right)\, dx = \int_0^1 \left(-2x + 2\sqrt{x}\right)\, dx = \left(-x^2 + \frac{4}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{3}.$$



39. $y = \cos x, \quad y = \cos 2x, \quad x = 0, \quad x = \dfrac{2\pi}{3}$

SOLUTION From the graph below, we see that $y = \cos x$ lies above $y = \cos 2x$ over the interval $[0, \frac{2\pi}{3}]$. The area of the region enclosed by the two curves is therefore

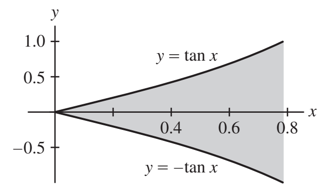
$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \frac{1}{2}\sin 2x\right)\Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.$$



40. $y = \tan x, \quad y = -\tan x, \quad x = \dfrac{\pi}{4}$

SOLUTION Because the graph of $y = \tan x$ lies above the graph of $y = -\tan x$ over the interval $[0, \pi/4]$, the area bounded by the two curves is

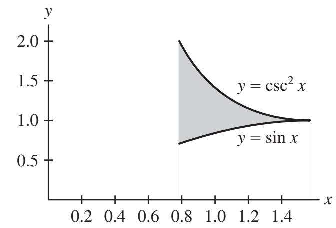
$$\begin{aligned} A &= \int_0^{\pi/4} (\tan x - (-\tan x))\, dx = 2\int_0^{\pi/4} \tan x\, dx \\ &= 2\ln|\sec x|\Big|_0^{\pi/4} \\ &= 2\ln 2 - 2\ln 1 = 2\ln 2. \end{aligned}$$



41. $y = \sin x, \quad y = \csc^2 x, \quad x = \dfrac{\pi}{4}$

SOLUTION Over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, $y = \csc^2 x$ lies above $y = \sin x$. The area of the region enclosed by the two curves is then

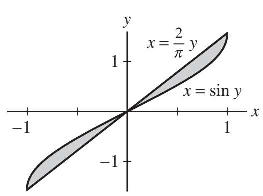
$$\int_{\pi/4}^{\pi/2} \left(\csc^2 x - \sin x\right) dx = \left(-\cot x + \cos x\right)\Big|_{\pi/4}^{\pi/2} = (0 - 0) - \left(-1 + \frac{\sqrt{2}}{2}\right) = 1 - \frac{\sqrt{2}}{2}.$$



42. $x = \sin y, \quad x = \dfrac{2}{\pi} y$

SOLUTION Here, integration along the y-axis will require less work than integration along the x-axis. The curves intersect when $\frac{2y}{\pi} = \sin y$ or when $y = 0, \pm\frac{\pi}{2}$. From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the first quadrant is identical to the region enclosed in the third quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the first quadrant. In the first quadrant, $x = \sin y$ lies to the right of $x = \frac{2y}{\pi}$, so the total area enclosed by the two curves is

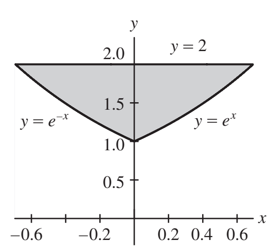
$$2\int_0^{\pi/2} \left(\sin y - \frac{2}{\pi} y\right) dy = 2\left(-\cos y - \frac{1}{\pi} y^2\right)\Big|_0^{\pi/2} = 2\left[\left(0 - \frac{\pi}{4}\right) - (-1 - 0)\right] = 2 - \frac{\pi}{2}.$$



43. $y = e^x, \quad y = e^{-x}, \quad y = 2$

SOLUTION From the figure below, we see that integration in y would be most appropriate - unfortunately, we have not yet learned how to integrate $\ln y$. Consequently, we will calculate the area using two integrals in x:

$$\begin{aligned} A &= \int_{-\ln 2}^{0} (2 - e^{-x})\, dx + \int_0^{\ln 2} (2 - e^x)\, dx \\ &= \left(2x + e^{-x}\right)\Big|_{-\ln 2}^{0} + \left(2x - e^x\right)\Big|_0^{\ln 2} \\ &= 1 - (-2\ln 2 + 2) + (2\ln 2 - 2) - (-1) = 4\ln 2 - 2. \end{aligned}$$



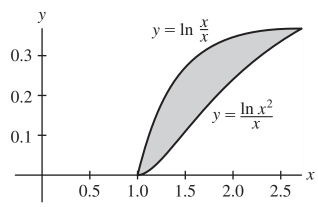
44. $y = \dfrac{\ln x}{x}, \quad y = \dfrac{(\ln x)^2}{x}$

SOLUTION Setting

$$\frac{\ln x}{x} = \frac{(\ln x)^2}{2} \quad \text{yields} \quad x = 1, e.$$

From the figure below, we see that the graph of $y = \ln x/x$ lies above the graph of $y = (\ln x)^2/x$ over the interval $[1, e]$. Thus, the area between the two curves is

$$\begin{aligned} A &= \int_1^e \left(\frac{\ln x}{x} - \frac{(\ln x)^2}{x} \right) dx \\ &= \left(\frac{1}{2}(\ln x)^2 - \frac{1}{3}(\ln x)^3 \right)\Bigg|_1^e \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$



45. CAS Plot

$$y = \frac{x}{\sqrt{x^2+1}} \quad \text{and} \quad y = (x-1)^2$$

on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

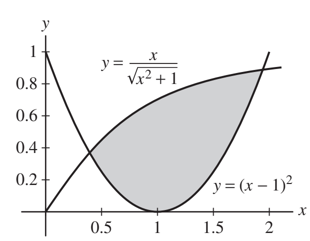
SOLUTION Using a computer algebra system, we find that the curves

$$y = \frac{x}{\sqrt{x^2+1}} \qquad \text{and} \qquad y = (x-1)^2$$

intersect at $x = 0.3943285581$ and at $x = 1.942944418$. From the graph below, we see that $y = \frac{x}{\sqrt{x^2+1}}$ lies above $y = (x-1)^2$, so the area of the region enclosed by the two curves is

$$\int_{0.3943285581}^{1.942944418} \left(\frac{x}{\sqrt{x^2+1}} - (x-1)^2 \right) dx = 0.7567130951$$

The value of the definite integral was also obtained using a computer algebra system.



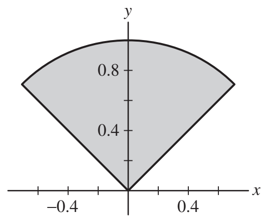
46. Sketch a region whose area is represented by

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x| \right) dx$$

and evaluate using geometry.

SOLUTION Matching the integrand $\sqrt{1-x^2} - |x|$ with the $y_{\text{TOP}} - y_{\text{BOT}}$ template for calculating area, we see that the region in question is bounded along the top by the curve $y = \sqrt{1-x^2}$ (the upper half of the unit circle) and is bounded along the bottom by the curve $y = |x|$. Hence, the region is $\frac{1}{4}$ of the unit circle (see the figure below). The area of the region must then be

$$\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}.$$



47. Athletes 1 and 2 run along a straight track with velocities $v_1(t)$ and $v_2(t)$ (in m/s) as shown in Figure 19.

(a) Which of the following is represented by the area of the shaded region over [0, 10]?

i. The distance between athletes 1 and 2 at time $t = 10$ s.

ii. The difference in the distance traveled by the athletes over the time interval [0, 10].

(b) Does Figure 19 give us enough information to determine who is ahead at time $t = 10$ s?

(c) If the athletes begin at the same time and place, who is ahead at $t = 10$ s? At $t = 25$ s?

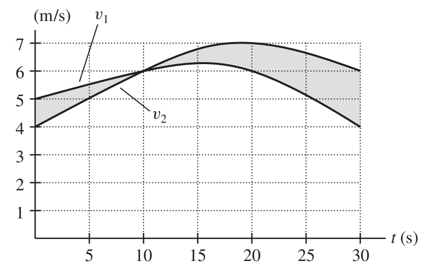


FIGURE 19

SOLUTION

(a) The area of the shaded region over [0, 10] represents **(ii)**: the difference in the distance traveled by the athletes over the time interval [0, 10].

(b) No, Figure 19 does not give us enough information to determine who is ahead at time $t = 10$ s. We would additionally need to know the relative position of the runners at $t = 0$ s.

(c) If the athletes begin at the same time and place, then athlete 1 is ahead at $t = 10$ s because the velocity graph for athlete 1 lies above the velocity graph for athlete 2 over the interval [0, 10]. Over the interval [10, 25], the velocity graph for athlete 2 lies above the velocity graph for athlete 1 and appears to have a larger area than the area between the graphs over [0, 10]. Thus, it appears that athlete 2 is ahead at $t = 25$ s.

48. Express the area (not signed) of the shaded region in Figure 20 as a sum of three integrals involving $f(x)$ and $g(x)$.

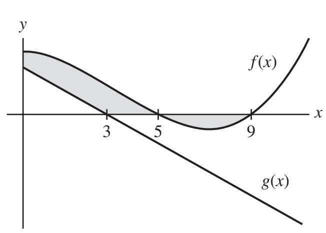


FIGURE 20

SOLUTION Because either the curve bounding the top of the region or the curve bounding the bottom of the region or both change at $x = 3$ and at $x = 5$, the area is calculated using three integrals. Specifically, the area is

$$\int_0^3 (f(x) - g(x))\,dx + \int_3^5 (f(x) - 0)\,dx + \int_5^9 (0 - f(x))\,dx$$

$$= \int_0^3 (f(x) - g(x))\,dx + \int_3^5 f(x)\,dx - \int_5^9 f(x)\,dx.$$

49. Find the area enclosed by the curves $y = c - x^2$ and $y = x^2 - c$ as a function of c. Find the value of c for which this area is equal to 1.

SOLUTION The curves intersect at $x = \pm\sqrt{c}$, with $y = c - x^2$ above $y = x^2 - c$ over the interval $[-\sqrt{c}, \sqrt{c}]$. The area of the region enclosed by the two curves is then

$$\int_{-\sqrt{c}}^{\sqrt{c}} \left((c - x^2) - (x^2 - c)\right) dx = \int_{-\sqrt{c}}^{\sqrt{c}} \left(2c - 2x^2\right) dx = \left(2cx - \frac{2}{3}x^3\right)\Big|_{-\sqrt{c}}^{\sqrt{c}} = \frac{8}{3}c^{3/2}.$$

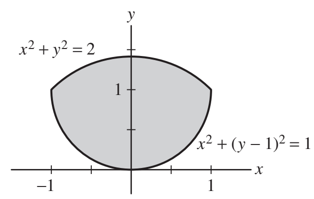
In order for the area to equal 1, we must have $\frac{8}{3}c^{3/2} = 1$, which gives

$$c = \frac{9^{1/3}}{4} \approx 0.520021.$$

50. Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y - 1)^2 = 1$.

SOLUTION Setting $2 - y^2 = 1 - (y - 1)^2$ yields $y = 1$. The two circles therefore intersect at the points $(1, 1)$ and $(-1, 1)$. From the graph below, we see that over the interval $[-1, 1]$, the upper half of the circle $x^2 + y^2 = 2$ lies above the lower half of the circle $x^2 + (y - 1)^2 = 1$. The area enclosed by the two circles is therefore given by the integral

$$\int_{-1}^{1} \left(\sqrt{2 - x^2} - (1 - \sqrt{1 - x^2})\right) dx.$$



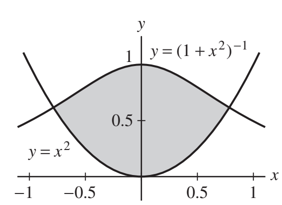
51. Set up (but do not evaluate) an integral that expresses the area between the graphs of $y = (1 + x^2)^{-1}$ and $y = x^2$.

SOLUTION Setting $(1 + x^2)^{-1} = x^2$ yields $x^4 + x^2 - 1 = 0$. This is a quadratic equation in the variable x^2. By the quadratic formula,

$$x^2 = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

As x^2 must be nonnegative, we discard $\frac{-1-\sqrt{5}}{2}$. Finally, we find the two curves intersect at $x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$. From the graph below, we see that $y = (1 + x^2)^{-1}$ lies above $y = x^2$. The area enclosed by the two curves is then

$$\int_{-\sqrt{\frac{-1+\sqrt{5}}{2}}}^{\sqrt{\frac{-1+\sqrt{5}}{2}}} \left((1 + x^2)^{-1} - x^2\right) dx.$$

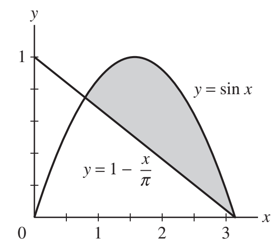


52. CAS Find a numerical approximation to the area above $y = 1 - (x/\pi)$ and below $y = \sin x$ (find the points of intersection numerically).

SOLUTION The region in question is shown in the figure below. Using a computer algebra system, we find that $y = 1 - x/\pi$ and $y = \sin x$ intersect on the left at $x = 0.8278585215$. Analytically, we determine the two curves intersect on the right at $x = \pi$. The area above $y = 1 - x/\pi$ and below $y = \sin x$ is then

$$\int_{0.8278585215}^{\pi} \left(\sin x - \left(1 - \frac{x}{\pi}\right)\right) dx = 0.8244398727,$$

where the definite integral was evaluated using a computer algebra system.

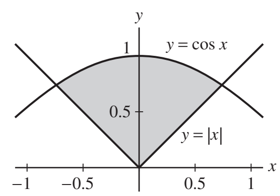


53. CAS Find a numerical approximation to the area above $y = |x|$ and below $y = \cos x$.

SOLUTION The region in question is shown in the figure below. We see that the region is symmetric with respect to the y-axis, so we can determine the total area of the region by doubling the area of the portion in the first quadrant. Using a computer algebra system, we find that $y = \cos x$ and $y = |x|$ intersect at $x = 0.7390851332$. The area of the region between the two curves is then

$$2\int_0^{0.7390851332} (\cos x - x)\, dx = 0.8009772242,$$

where the definite integral was evaluated using a computer algebra system.

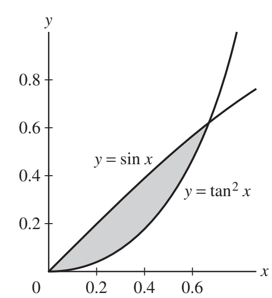


54. CAS Use a computer algebra system to find a numerical approximation to the number c (besides zero) in $\left[0, \frac{\pi}{2}\right]$, where the curves $y = \sin x$ and $y = \tan^2 x$ intersect. Then find the area enclosed by the graphs over $[0, c]$.

SOLUTION The region in question is shown in the figure below. Using a computer algebra system, we find that $y = \sin x$ and $y = \tan^2 x$ intersect at $x = 0.6662394325$. The area of the region enclosed by the two curves is then

$$\int_0^{0.6662394325} \left(\sin x - \tan^2 x\right) dx = 0.09393667698,$$

where the definite integral was evaluated using a computer algebra system.



55. The back of Jon's guitar (Figure 21) is 19 inches long. Jon measured the width at 1-in. intervals, beginning and ending $\frac{1}{2}$ in. from the ends, obtaining the results

$$6,\ 9,\ 10.25,\ 10.75,\ 10.75,\ 10.25,\ 9.75,\ 9.5,\ 10,\ 11.25,$$
$$12.75,\ 13.75,\ 14.25,\ 14.5,\ 14.5,\ 14,\ 13.25,\ 11.25,\ 9$$

Use the midpoint rule to estimate the area of the back.

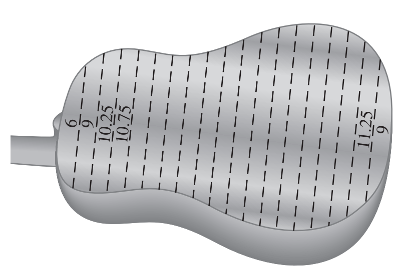


FIGURE 21 Back of guitar.

SOLUTION Note that the measurements were taken at the midpoint of each one-inch section of the guitar. For example, in the 0 to 1 inch section, the midpoint would be at $\frac{1}{2}$ inch, and thus the approximate area of the first rectangle would be $1 \cdot 6$ inches^2. An approximation for the entire area is then

$$\begin{aligned} A &= 1(6 + 9 + 10.25 + 10.75 + 10.75 + 10.25 + 9.75 + 9.5 + 10 + 11.25 \\ &\quad + 12.75 + 13.75 + 14.25 + 14.5 + 14.5 + 14 + 13.25 + 11.25 + 9) \\ &= 214.75 \text{ in}^2. \end{aligned}$$

56. Referring to Figure 1 at the beginning of this section, estimate the projected number of additional joules produced in the years 2009–2030 as a result of government stimulus spending in 2009–2010. *Note:* One watt is equal to one joule per second, and one gigawatt is 10^9 watts.

SOLUTION We make some rough estimates of the areas depicted in Figure 1. From 2009 through 2012, the area between the curves is roughly a right triangle with a base of 3 and a height of 40; from 2012 through 2020, the area is roughly an 8 by 40 rectangle. Finally, from 2020 through 2030, the area is roughly a trapezoid with height 10 and bases 40 and 27. Thus, additional energy produced is approximately

$$\frac{1}{2}(3)(40) + 8(40) + \frac{1}{2}(10)(40 + 27) = 715 \text{ gigawatt-years.}$$

Because 1 gigawatt is equal to 10^9 joules per second and 1 year (assuming 365 days) is equal to 31536000 seconds, the additional joules produced in the years 2009–2030 as a result of government stimulus spending in 2009–2010 is approximately 2.25×10^{19}.

Exercises 57 and 58 use the notation and results of Exercises 49–51 of Section 3.4. For a given country, $F(r)$ is the fraction of total income that goes to the bottom rth fraction of households. The graph of $y = F(r)$ is called the Lorenz curve.

57. Let A be the area between $y = r$ and $y = F(r)$ over the interval [0, 1] (Figure 22). The **Gini index** is the ratio $G = A/B$, where B is the area under $y = r$ over [0, 1].

(a) Show that $G = 2\int_0^1 (r - F(r))\, dr$.

(b) Calculate G if

$$F(r) = \begin{cases} \frac{1}{3}r & \text{for } 0 \le r \le \frac{1}{2} \\ \frac{5}{3}r - \frac{2}{3} & \text{for } \frac{1}{2} \le r \le 1 \end{cases}$$

(c) The Gini index is a measure of income distribution, with a lower value indicating a more equal distribution. Calculate G if $F(r) = r$ (in this case, all households have the same income by Exercise 51(b) of Section 3.4).

(d) What is G if all of the income goes to one household? *Hint:* In this extreme case, $F(r) = 0$ for $0 \le r < 1$.

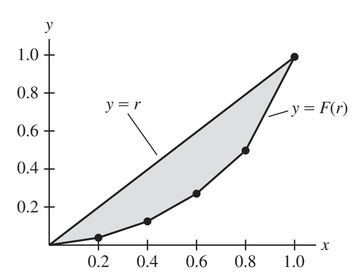


FIGURE 22 Lorenz Curve for U.S. in 2001.

SOLUTION

(a) Because the graph of $y = r$ lies above the graph of $y = F^{\cdot\cdot}$ in Figure 22,

$$A = \int_0^1 (r - F(r))\, dr.$$

Moreover,

$$B = \int_0^1 r\, dr = \frac{1}{2}r^2\bigg|_0^1 = \frac{1}{2}.$$

Thus,

$$G = \frac{A}{B} = 2\int_0^1 (r - F(r))\, dr.$$

(b) With the given $F(r)$,

$$\begin{aligned} G &= 2\int_0^{1/2}\left(r-\frac{1}{3}r\right)\,dr + 2\int_{1/2}^{1}\left(r-\left(\frac{5}{3}r-\frac{2}{3}\right)\right)\,dr \\ &= \frac{4}{3}\int_0^{1/2} r\,dr - \frac{4}{3}\int_{1/2}^{1}(r-1)\,dr \\ &= \frac{2}{3}r^2\Big|_0^{1/2} - \frac{4}{3}\left(\frac{1}{2}r^2 - r\right)\Big|_{1/2}^{1} \\ &= \frac{1}{6} - \frac{4}{3}\left(-\frac{1}{2}\right) + \frac{4}{3}\left(-\frac{3}{8}\right) = \frac{1}{3}. \end{aligned}$$

(c) If $F(r) = r$, then

$$G = 2\int_0^1 (r-r)\,dr = 0.$$

(d) If $F(r) = 0$ for $0 \le r < 1$, then

$$G = 2\int_0^1 (r-0)\,dr = 2\left(\frac{1}{2}r^2\right)\Big|_0^1 = 2\left(\frac{1}{2}\right) = 1.$$

58. Calculate the Gini index of the United States in the year 2001 from the Lorenz curve in Figure 22, which consists of segments joining the data points in the following table.

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.035	0.123	0.269	0.499	1

SOLUTION From part (a) of the previous exercise,

$$G = 2\int_0^1 (r - F(r))\,dr = 1 - 2\int_0^1 F(r)\,dr.$$

Because $F(r)$ consists of segments joining the data points in the given table, the area under the graph of $y = F(r)$ consists of a triangle and four trapezoids. The area is

$$\frac{1}{2}(0.2)(0.035) + \frac{1}{2}(0.2)(0.035+0.123) + \frac{1}{2}(0.2)(0.123+0.269) + \frac{1}{2}(0.2)(0.269+0.499) + \frac{1}{2}(0.2)(0.499+1)$$

or 0.2852. Finally,

$$G = 1 - 2(0.2852) = 0.4296.$$

Further Insights and Challenges

59. Find the line $y = mx$ that divides the area under the curve $y = x(1-x)$ over [0, 1] into two regions of equal area.

SOLUTION First note that

$$\int_0^1 x(1-x)\,dx = \int_0^1 \left(x - x^2\right)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{6}.$$

Now, the line $y = mx$ and the curve $y = x(1-x)$ intersect when $mx = x(1-x)$, or at $x = 0$ and at $x = 1 - m$. The area of the region enclosed by the two curves is then

$$\int_0^{1-m} (x(1-x) - mx)\,dx = \int_0^{1-m} \left((1-m)x - x^2\right)\,dx = \left((1-m)\frac{x^2}{2} - \frac{1}{3}x^3\right)\Big|_0^{1-m} = \frac{1}{6}(1-m)^3.$$

To have $\frac{1}{6}(1-m)^3 = \frac{1}{2}\cdot\frac{1}{6}$ requires

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.206299.$$

60. CAS Let c be the number such that the area under $y = \sin x$ over $[0, \pi]$ is divided in half by the line $y = cx$ (Figure 23). Find an equation for c and solve this equation *numerically* using a computer algebra system.

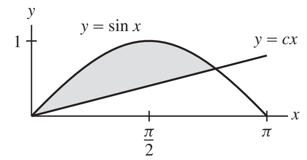


FIGURE 23

SOLUTION First note that

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 2.$$

Now, let $y = cx$ and $y = \sin x$ intersect at $x = a$. Then $ca = \sin a$, which gives $c = \frac{\sin a}{a}$ and $y = cx = \frac{\sin a}{a}x$. Then

$$\int_0^a \left(\sin x - \frac{\sin a}{a}x\right) dx = \left(-\cos x - \frac{\sin a}{2a}x^2\right)\Big|_0^a = 1 - \cos a - \frac{a\sin a}{2}.$$

We need

$$1 - \cos a - \frac{a\sin a}{2} = \frac{1}{2}(2) = 1,$$

which gives $a = 2.458714176$ and finally

$$c = \frac{\sin a}{a} = 0.2566498570.$$

61. Explain geometrically (without calculation):

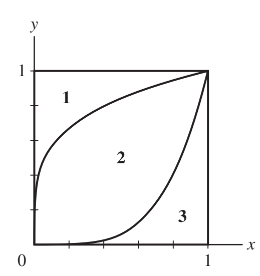
$$\int_0^1 x^n\,dx + \int_0^1 x^{1/n}\,dx = 1 \qquad (\text{for } n > 0)$$

SOLUTION Let A_1 denote the area of region 1 in the figure below. Define A_2 and A_3 similarly. It is clear from the figure that

$$A_1 + A_2 + A_3 = 1.$$

Now, note that x^n and $x^{1/n}$ are inverses of each other. Therefore, the graphs of $y = x^n$ and $y = x^{1/n}$ are symmetric about the line $y = x$, so regions 1 and 3 are also symmetric about $y = x$. This guarantees that $A_1 = A_3$. Finally,

$$\int_0^1 x^n\,dx + \int_0^1 x^{1/n}\,dx = A_3 + (A_2 + A_3) = A_1 + A_2 + A_3 = 1.$$



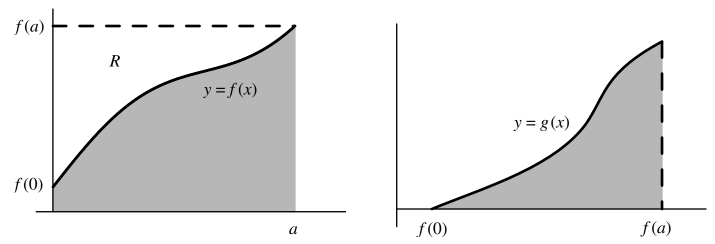
62. Let $f(x)$ be an increasing function with inverse $g(x)$. Explain geometrically:

$$\int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(x)\,dx = af(a)$$

SOLUTION The region whose area is represented by $\int_0^a f(x)\,dx$ is shown as the shaded portion of the graph below on the left, and the region whose area is represented by $\int_{f(0)}^{f(a)} g(x)\,dx$ is shown as the shaded portion of the graph below on the right. Because f and g are inverse functions, the graph of $y = f(x)$ is obtained by reflecting the graph of $y = g(x)$

through the line $y = x$. It then follows that if we were to reflect the shaded region in the graph below on the right through the line $y = x$, the reflected region would coincide exactly with the region R in the graph below on the left. Thus

$$\int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(x)\,dx = \text{ area of a rectangle with width } a \text{ and height } f(a) = af(a).$$



6.2 Setting Up Integrals: Volume, Density, Average Value

Preliminary Questions

1. What is the average value of $f(x)$ on $[0, 4]$ if the area between the graph of $f(x)$ and the x-axis is equal to 12?

SOLUTION Assuming that $f(x) \geq 0$ over the interval $[1, 4]$, the fact that the area between the graph of f and the x-axis is equal to 9 indicates that $\int_1^4 f(x)\,dx = 9$. The average value of f over the interval $[1, 4]$ is then

$$\frac{\int_1^4 f(x)\,dx}{4-1} = \frac{9}{3} = 3.$$

2. Find the volume of a solid extending from $y = 2$ to $y = 5$ if every cross section has area $A(y) = 5$.

SOLUTION Because the cross-sectional area of the solid is constant, the volume is simply the cross-sectional area times the length, or $5 \times 3 = 15$.

3. What is the definition of flow rate?

SOLUTION The flow rate of a fluid is the volume of fluid that passes through a cross-sectional area at a given point per unit time.

4. Which assumption about fluid velocity did we use to compute the flow rate as an integral?

SOLUTION To express flow rate as an integral, we assumed that the fluid velocity depended only on the radial distance from the center of the tube.

5. The average value of $f(x)$ on $[1, 4]$ is 5. Find $\displaystyle\int_1^4 f(x)\,dx$.

SOLUTION

$$\begin{aligned}\int_1^4 f(x)\,dx &= \text{average value on } [1,4] \times \text{ length of } [1,4] \\ &= 5 \times 3 = 15.\end{aligned}$$

Exercises

1. Let V be the volume of a pyramid of height 20 whose base is a square of side 8.

(a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height y.

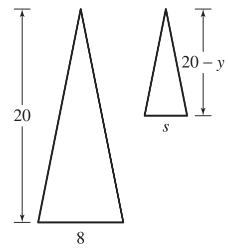
(b) Calculate V by integrating the cross-sectional area.

SOLUTION

(a) We can use similar triangles to determine the side length, s, of the square cross section at height y. Using the diagram below, we find

$$\frac{8}{20} = \frac{s}{20-y} \qquad \text{or} \qquad s = \frac{2}{5}(20-y).$$

The area of the cross section at height y is then given by $\frac{4}{25}(20-y)^2$.



(b) The volume of the pyramid is

$$\int_0^{20} \frac{4}{25}(20-y)^2\,dy = -\frac{4}{75}(20-y)^3\Big|_0^{20} = \frac{1280}{3}.$$

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4 [Figure 17(A)].

(a) Use similar triangles to find the area of a horizontal cross section at a height y.

(b) Calculate V by integrating the cross-sectional area.

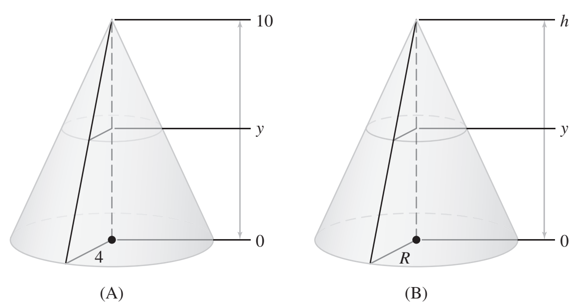


FIGURE 17 Right circular cones.

SOLUTION

(a) If r is the radius at height y (see Figure 17), then

$$\frac{10}{4} = \frac{10-y}{r}$$

from similar triangles, which implies that $r = 4 - \frac{2}{5}y$. The area of the cross-section at height y is then

$$A = \pi\left(4 - \frac{2}{5}y\right)^2.$$

(b) The volume of the cone is

$$V = \int_0^{10} \pi\left(4-\frac{2}{5}y\right)^2 dy = -\frac{5\pi}{6}\left(4-\frac{2}{5}y\right)^3\Big|_0^{10} = \frac{160\pi}{3}.$$

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height h whose base is a circle of radius R [Figure 17(B)].

SOLUTION

(a) From similar triangles (see Figure 17),

$$\frac{h}{h-y} = \frac{R}{r_0},$$

where r_0 is the radius of the cone at a height of y. Thus, $r_0 = R - \frac{Ry}{h}$.

(b) The volume of the cone is

$$\pi\int_0^h \left(R - \frac{Ry}{h}\right)^2 dy = \frac{-h\pi}{R}\,\frac{\left(R-\frac{Ry}{h}\right)^3}{3}\Bigg|_0^h = \frac{h\pi}{R}\,\frac{R^3}{3} = \frac{\pi R^2 h}{3}.$$

4. Calculate the volume of the ramp in Figure 18 in three ways by integrating the area of the cross sections:

(a) Perpendicular to the x-axis (rectangles).

(b) Perpendicular to the y-axis (triangles).

(c) Perpendicular to the z-axis (rectangles).

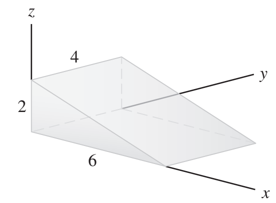


FIGURE 18 Ramp of length 6, width 4, and height 2.

SOLUTION

(a) Cross sections perpendicular to the x-axis are rectangles of width 4 and height $2 - \frac{1}{3}x$. The volume of the ramp is then

$$\int_0^6 4\left(-\frac{1}{3}x + 2\right)\,dx = \left(-\frac{2}{3}x^2 + 8x\right)\bigg|_0^6 = 24.$$

(b) Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2}\cdot 2\cdot 6\right)\,dy = (6y)\bigg|_0^4 = 24.$$

(c) Cross sections perpendicular to the z-axis are rectangles of length $6 - 3z$ and width 4. The volume of the ramp is then

$$\int_0^2 4\,(-3(z-2))\,dz = (-6z^2 + 24z)\bigg|_0^2 = 24.$$

5. Find the volume of liquid needed to fill a sphere of radius R to height h (Figure 19).

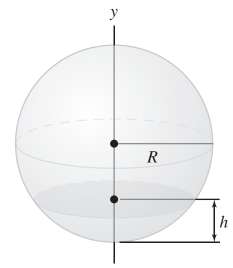


FIGURE 19 Sphere filled with liquid to height h.

SOLUTION The radius r at any height y is given by $r = \sqrt{R^2 - (R-y)^2}$. Thus, the volume of the filled portion of the sphere is

$$\pi\int_0^h r^2\,dy = \pi\int_0^h \left(R^2 - (R-y)^2\right)\,dy = \pi\int_0^h (2Ry - y^2)\,dy = \pi\left(Ry^2 - \frac{y^3}{3}\right)\bigg|_0^h = \pi\left(Rh^2 - \frac{h^3}{3}\right).$$

6. Find the volume of the wedge in Figure 20(A) by integrating the area of vertical cross sections.

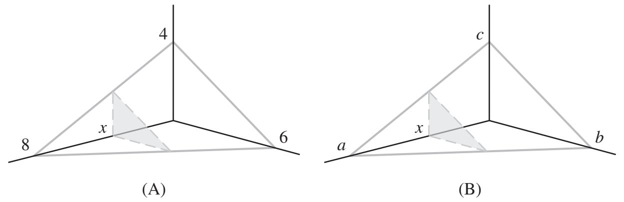


FIGURE 20

SOLUTION Cross sections of the wedge taken perpendicular to the x-axis are right triangles. Using similar triangles, we find the base and the height of the cross sections to be $\frac{3}{4}(8-x)$ and $\frac{1}{2}(8-x)$, respectively. The volume of the wedge is then

$$\frac{3}{16}\int_0^8 (8-x)^2\,dx = \frac{3}{16}\int_0^8 \left(64 - 16x + x^2\right)\,dx = \frac{3}{16}\left(64x - 8x^2 + \frac{1}{3}x^3\right)\bigg|_0^8 = 32.$$

7. Derive a formula for the volume of the wedge in Figure 20(B) in terms of the constants a, b, and c.

SOLUTION The line from c to a is given by the equation $(z/c) + (x/a) = 1$ and the line from b to a is given by $(y/b) + (x/a) = 1$. The cross sections perpendicular to the x-axis are right triangles with height $c(1 - x/a)$ and base $b(1 - x/a)$. Thus we have

$$\int_0^a \frac{1}{2}bc\,(1 - x/a)^2\,dx = -\frac{1}{6}abc\left(1 - \frac{x}{a}\right)^3\Big|_0^a = \frac{1}{6}abc.$$

8. Let B be the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose vertical cross sections perpendicular to the x-axis are equilateral triangles. Show that the vertical cross sections have area $A(x) = \sqrt{3}(1 - x^2)$ and compute the volume of B.

SOLUTION At the arbitrary location x, the side of the equilateral triangle cross section that lies in the base of the solid extends from the top half of the unit circle (with $y = \sqrt{1 - x^2}$) to the bottom half (with $y = -\sqrt{1 - x^2}$). The equilateral triangle therefore has sides of length $s = 2\sqrt{1 - x^2}$ and an area of

$$A(x) = \frac{s^2\sqrt{3}}{4} = \sqrt{3}(1 - x^2).$$

Finally, the volume of the solid is

$$\sqrt{3}\int_{-1}^1 \left(1 - x^2\right)\,dx = \sqrt{3}\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^1 = \frac{4\sqrt{3}}{3}.$$

In Exercises 9–14, find the volume of the solid with the given base and cross sections.

9. The base is the unit circle $x^2 + y^2 = 1$, and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

SOLUTION At each location x, the side of the triangular cross section that lies in the base of the solid extends from the top half of the unit circle (with $y = \sqrt{1 - x^2}$) to the bottom half (with $y = -\sqrt{1 - x^2}$). The triangle therefore has base and height equal to $2\sqrt{1 - x^2}$ and area $2(1 - x^2)$. The volume of the solid is then

$$\int_{-1}^1 2(1 - x^2)\,dx = 2\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^1 = \frac{8}{3}.$$

10. The base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis. The cross sections perpendicular to the y-axis are semicircles.

SOLUTION The diameter of the semicircle lies in the base of the solid and thus has length $1 - y$ for each y. The area of the semicircle is then

$$\frac{1}{2}\pi\left(\frac{1 - y}{2}\right)^2 = \frac{1}{8}\pi(1 - y)^2.$$

Finally, the volume of the solid is

$$\frac{\pi}{8}\int_0^1 (1 - y)^2\,dy = \frac{\pi}{8}\int_0^1 (1 - 2y + y^2)\,dy = \frac{\pi}{8}\left(y - y^2 + \frac{1}{3}y^3\right)\Big|_0^1 = \frac{\pi}{24}.$$

11. The base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \le x \le 3$. The cross sections perpendicular to the x-axis are squares.

SOLUTION For each x, the base of the square cross section extends from the semicircle $y = \sqrt{9 - x^2}$ to the x-axis. The square therefore has a base with length $\sqrt{9 - x^2}$ and an area of $\left(\sqrt{9 - x^2}\right)^2 = 9 - x^2$. The volume of the solid is then

$$\int_{-3}^3 \left(9 - x^2\right)\,dx = \left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^3 = 36.$$

12. The base is a square, one of whose sides is the interval $[0, \ell]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

SOLUTION For each x, the rectangular cross section has base ℓ and height x^2. The cross-sectional area is then ℓx^2, and the volume of the solid is

$$\int_0^\ell \left(\ell x^2\right)\,dx = \left(\frac{1}{3}\ell x^3\right)\Big|_0^\ell = \frac{1}{3}\ell^4.$$

13. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

SOLUTION At any location y, the distance to the parabola from the y-axis is $\sqrt{y}$. Thus the base of the square will have length $2\sqrt{y}$. Therefore the volume is

$$\int_0^3 \left(2\sqrt{y}\right)\left(2\sqrt{y}\right)\,dy = \int_0^3 4y\,dy = 2y^2\Big|_0^3 = 18.$$

14. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are rectangles of height y^3.

SOLUTION As in previous exercise, for each y, the width of the rectangle will be $2\sqrt{y}$. Because the height is y^3, the volume of the solid is given by

$$2\int_0^3 y^{7/2}\,dy = \frac{4}{9}y^{9/2}\Big|_0^3 = 36\sqrt{3}.$$

15. Find the volume of the solid whose base is the region $|x| + |y| \le 1$ and whose vertical cross sections perpendicular to the y-axis are semicircles (with diameter along the base).

SOLUTION The region R in question is a diamond shape connecting the points $(1, 0)$, $(0, -1)$, $(-1, 0)$, and $(0, 1)$. Thus, in the lower half of the xy-plane, the radius of the circles is $y + 1$ and in the upper half, the radius is $1 - y$. Therefore, the volume is

$$\frac{\pi}{2}\int_{-1}^0 (y+1)^2\,dy + \frac{\pi}{2}\int_0^1 (1-y)^2\,dy = \frac{\pi}{2}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{\pi}{3}.$$

16. Show that a pyramid of height h whose base is an equilateral triangle of side s has volume $\frac{\sqrt{3}}{12}hs^2$.

SOLUTION Using similar triangles, the side length of the equilateral triangle at height x above the base is

$$\frac{s(h-x)}{h};$$

the area of the cross section is therefore given by

$$\frac{\sqrt{3}}{4}\left(\frac{s(h-x)}{h}\right)^2.$$

Thus, the volume of the pyramid is

$$\frac{s^2\sqrt{3}}{4h^2}\int_0^h (h-x)^2\,dx = \left(-\frac{s^2\sqrt{3}}{12h^2}(h-x)^3\right)\Big|_0^h = \frac{\sqrt{3}}{12}s^2h.$$

17. The area of an ellipse is πab, where a and b are the lengths of the semimajor and semiminor axes (Figure 21). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.

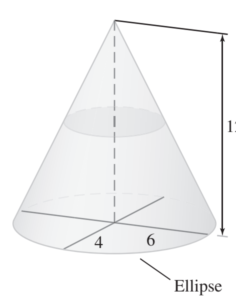


FIGURE 21

SOLUTION At each height y, the elliptical cross section has major axis $\frac{1}{2}(12 - y)$ and minor axis $\frac{1}{3}(12 - y)$. The cross-sectional area is then $\frac{\pi}{6}(12 - y)^2$, and the volume is

$$\int_0^{12} \frac{\pi}{6}(12-y)^2\,dy = -\frac{\pi}{18}(12-y)^3\Big|_0^{12} = 96\pi.$$

18. Find the volume V of a *regular* tetrahedron (Figure 22) whose face is an equilateral triangle of side s. The tetrahedron has height $h = \sqrt{2/3}s$.

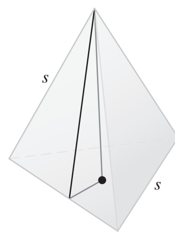


FIGURE 22

SOLUTION Our first task is to determine the relationship between the height of the tetrahedron, h, and the side length of the equilateral triangles, s. Let B be the orthocenter of the tetrahedron (the point directly below the apex), and let b denote the distance from B to each corner of the base triangle. By the Law of Cosines, we have

$$s^2 = b^2 + b^2 - 2b^2 \cos 120° = 3b^2,$$

so $b^2 = \frac{1}{3}s^2$. Thus

$$h^2 = s^2 - b^2 = \frac{2}{3}s^2 \quad \text{or} \quad h = s\sqrt{\frac{2}{3}}.$$

Therefore, using similar triangles, the side length of the equilateral triangle at height z above the base is

$$s\left(\frac{h-z}{h}\right) = s - \frac{z}{\sqrt{2/3}}.$$

The volume of the tetrahedron is then given by

$$\int_0^{s\sqrt{2/3}} \frac{\sqrt{3}}{4}\left(s - \frac{z}{\sqrt{2/3}}\right)^2 dz = -\frac{\sqrt{2}}{12}\left(s - \frac{z}{\sqrt{2/3}}\right)^3\Bigg|_0^{s\sqrt{2/3}} = \frac{s^3\sqrt{2}}{12}.$$

19. A frustum of a pyramid is a pyramid with its top cut off [Figure 23(A)]. Let V be the volume of a frustum of height h whose base is a square of side a and whose top is a square of side b with $a > b \geq 0$.

(a) Show that if the frustum were continued to a full pyramid, it would have height $ha/(a-b)$ [Figure 23(B)].

(b) Show that the cross section at height x is a square of side $(1/h)(a(h-x)+bx)$.

(c) Show that $V = \frac{1}{3}h(a^2 + ab + b^2)$. A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4000 years ago.

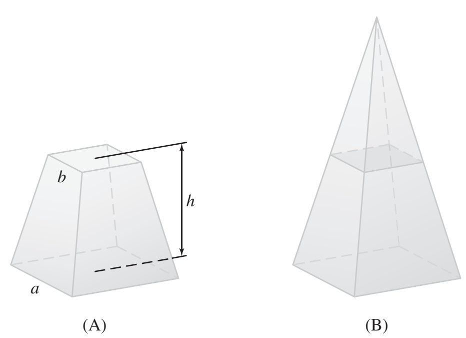


FIGURE 23

SOLUTION

(a) Let H be the height of the full pyramid. Using similar triangles, we have the proportion

$$\frac{H}{a} = \frac{H-h}{b}$$

which gives

$$H = \frac{ha}{a-b}.$$

(b) Let w denote the side length of the square cross section at height x. By similar triangles, we have

$$\frac{a}{H} = \frac{w}{H-x}.$$

Substituting the value for H from part (a) gives

$$w = \frac{a(h-x)+bx}{h}.$$

(c) The volume of the frustrum is

$$\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2 dx = \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right) dx$$
$$= \frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\bigg|_0^h = \frac{h}{3}\left(a^2+ab+b^2\right).$$

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane (Figure 24).

FIGURE 24

SOLUTION Place the center of the base at the origin. Then, for each x, the vertical cross section taken perpendicular to the x-axis is a rectangle of base $2\sqrt{r^2-x^2}$ and height x. The volume of the solid enclosed by the plane and the cylinder is therefore

$$\int_0^r 2x\sqrt{r^2-x^2}\,dx = \int_0^{r^2} \sqrt{u}\,du = \left(\frac{2}{3}u^{3/2}\right)\bigg|_0^{r^2} = \frac{2}{3}r^3.$$

21. The solid S in Figure 25 is the intersection of two cylinders of radius r whose axes are perpendicular.

(a) The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. Find the strip's width.

(b) Find the area of the horizontal cross section of S at distance y.

(c) Find the volume of S as a function of r.

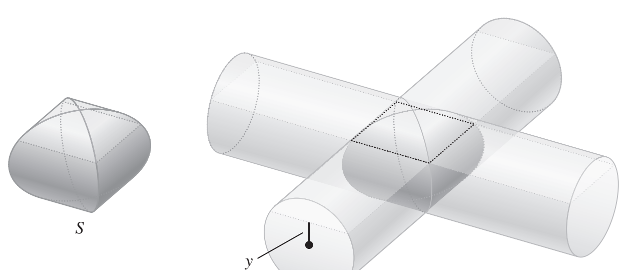


FIGURE 25 Two cylinders intersecting at right angles.

SOLUTION

(a) The horizontal cross section at distance y from the central axis (for $-r \le y \le r$) is a square of width $w = 2\sqrt{r^2-y^2}$.

(b) The area of the horizontal cross section of S at distance y from the central axis is $w^2 = 4(r^2-y^2)$.

(c) The volume of the solid S is then

$$4\int_{-r}^r \left(r^2-y^2\right) dy = 4\left(r^2y - \frac{1}{3}y^3\right)\bigg|_{-r}^r = \frac{16}{3}r^3.$$

22. Let S be the intersection of two cylinders of radius r whose axes intersect at an angle θ. Find the volume of S as a function of r and θ.

SOLUTION Each cross section at distance y from the central axis (for $-r \le y \le r$) is a rhombus with side length $\dfrac{2\sqrt{r^2-y^2}}{\sin\theta}$. The area of each rhombus is $\dfrac{4(r^2-y^2)}{\sin\theta}$, and thus the volume of the solid will be

$$\frac{4}{\sin\theta}\int_{-r}^{r}\left(r^2-y^2\right)dy = \frac{16r^3}{3\sin\theta}.$$

23. Calculate the volume of a cylinder inclined at an angle $\theta = 30^\circ$ with height 10 and base of radius 4 (Figure 26).

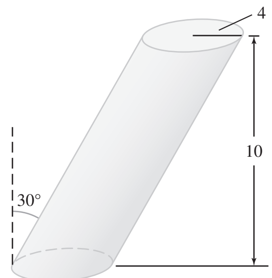


FIGURE 26 Cylinder inclined at an angle $\theta = 30^\circ$.

SOLUTION The area of each circular cross section is $\pi(4)^2 = 16\pi$, hence the volume of the cylinder is

$$\int_0^{10} 16\pi\,dx = (16\pi x)\Big|_0^{10} = 160\pi$$

24. The areas of cross sections of Lake Nogebow at 5-meter intervals are given in the table below. Figure 27 shows a contour map of the lake. Estimate the volume V of the lake by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

Depth (m)	0	5	10	15	20
Area (million m^2)	2.1	1.5	1.1	0.835	0.217

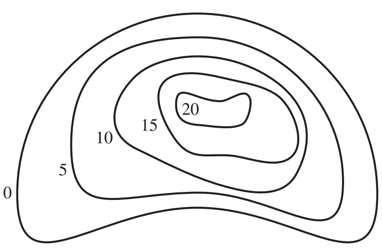


FIGURE 27 Depth contour map of Lake Nogebow.

SOLUTION The volume of the lake is

$$\int_0^{20} A(z)\,dz,$$

where $A(z)$ denotes the cross-sectional area of the lake at depth z. The right- and left-endpoint approximations to this integral, with $\Delta z = 5$, are

$$R = 5\,(1.5 + 1.1 + 0.835 + 0.217) = 18.26$$

$$L = 5\,(2.1 + 1.5 + 1.1 + 0.835) = 27.675$$

Thus

$$V \approx \frac{1}{2}(18.26 + 27.675) = 22.97 \text{ million m}^3.$$

25. Find the total mass of a 1-m rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 1$.

SOLUTION The total mass of the rod is

$$\int_0^1 \rho(x)\,dx = \int_0^1 \left(10(x+1)^{-2}\right)dx = \left(-10(x+1)^{-1}\right)\Big|_0^1 = 5\text{ kg}.$$

26. Find the total mass of a 2-m rod whose linear density function is $\rho(x) = 1 + 0.5\sin(\pi x)$ kg/m for $0 \le x \le 2$.

SOLUTION The total mass of the rod is

$$\int_0^2 \rho(x)\,dx = \int_0^2 (1 + 0.5\sin\pi x)\,dx = \left(x - 0.5\frac{\cos\pi x}{\pi}\right)\Big|_0^2 = 2 \text{ kg},$$

27. A mineral deposit along a strip of length 6 cm has density $s(x) = 0.01x(6-x)$ g/cm for $0 \le x \le 6$. Calculate the total mass of the deposit.

SOLUTION The total mass of the deposit is

$$\int_0^6 s(x)\,dx = \int_0^6 0.01x(6-x)\,dx = \left(0.03x^2 - \frac{0.01}{3}x^3\right)\Big|_0^6 = 0.36 \text{ g}.$$

28. Charge is distributed along a glass tube of length 10 cm with linear charge density $\rho(x) = x(x^2+1)^{-2} \times 10^{-4}$ coulombs per centimeter for $0 \le x \le 10$. Calculate the total charge.

SOLUTION The total charge along the tube is

$$\int_0^{10} \rho(x)\,dx = 10^{-4}\int_0^{10} \frac{x}{(x^2+1)^2}\,dx = 10^{-4}\left(-\frac{1}{2}(x^2+1)^{-1}\right)\Big|_0^{10} = 5 \times 10^{-5}\left(1 - \frac{1}{101}\right) = 4.95 \times 10^{-5}$$

coulombs.

29. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 4(1 + r^2)^{1/3}$ (in thousands per square mile).

SOLUTION The total population is

$$\begin{aligned} 2\pi\int_0^{10} r\cdot\rho(r)\,dr &= 2\pi\int_0^{10} 4r(1+r^2)^{1/3}\,dr = 3\pi(1+r^2)^{4/3}\Big|_0^{10} \\ &\approx 4423.59 \text{ thousand} \approx 4.4 \text{ million}. \end{aligned}$$

30. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1+r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5-km radius of the clearing.

SOLUTION The number of gorillas within a 5-km radius of the clearing is

$$2\pi\int_0^5 r\cdot\rho(r)\,dr = \int_0^5 \frac{104\pi r}{(1+r^2)^2} = -\frac{52\pi}{1+r^2}\Big|_0^5 = 50\pi \approx 157.$$

31. Table 1 lists the population density (in people per square kilometer) as a function of distance r (in kilometers) from the center of a rural town. Estimate the total population within a 1.2-km radius of the center by taking the average of the left- and right-endpoint approximations.

TABLE 1 **Population Density**

r	$\rho(r)$	r	$\rho(r)$
0.0	125.0	0.8	56.2
0.2	102.3	1.0	46.0
0.4	83.8	1.2	37.6
0.6	68.6		

SOLUTION The total population is given by

$$2\pi\int_0^{1.2} r\cdot\rho(r)\,dr.$$

With $\Delta r = 0.2$, the left- and right-endpoint approximations to the required definite integral are

$$\begin{aligned} L_6 &= 0.2(2\pi)[0(125) + (0.2)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46)] \\ &= 233.86; \\ R_{10} &= 0.2(2\pi)[(0.2)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46) + (1.2)(37.6)] \\ &= 290.56. \end{aligned}$$

This gives an average of 262.21. Thus, there are roughly 262 people within a 1.2-km radius of the town center.

32. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

SOLUTION The total mass of the plate is

$$2\pi \int_0^{20} r \cdot \rho(r)\, dr = 2\pi \int_0^{20} \left(0.03r + 0.01r\cos(\pi r^2)\right) dr = 2\pi \left(0.015r^2 + \frac{0.01}{2\pi}\sin(\pi r^2)\right)\Big|_0^{20} = 12\pi \text{ grams.}$$

33. The density of deer in a forest is the radial function $\rho(r) = 150(r^2+2)^{-2}$ deer per square kilometer, where r is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region $2 \le r \le 5$ km.

SOLUTION The number of deer in the region $2 \le r \le 5$ km is

$$2\pi \int_2^5 r\,(150)\left(r^2+2\right)^{-2} dr = -150\pi\left(\frac{1}{r^2+2}\right)\Big|_2^5 = -150\pi\left(\frac{1}{27} - \frac{1}{6}\right) \approx 61 \text{ deer.}$$

34. Show that a circular plate of radius 2 cm with radial mass density $\rho(r) = \frac{4}{r}$ g/cm^2 has finite total mass, even though the density becomes infinite at the origin.

SOLUTION The total mass of the plate is

$$2\pi \int_0^2 r\left(\frac{4}{r}\right) dr = 2\pi \int_0^2 4\, dr = 16\pi \text{ g.}$$

35. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance r cm from the center is $v(r) = (16 - r^2)$ cm/s.

SOLUTION The flow rate is

$$2\pi \int_0^R r v(r)\, dr = 2\pi \int_0^4 r\left(16 - r^2\right) dr = 2\pi\left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4 = 128\pi \,\frac{\text{cm}^3}{\text{s}}.$$

36. The velocity of fluid particles flowing through a tube of radius 5 cm is $v(r) = (10 - 0.3r - 0.34r^2)$ cm/s, where r cm is the distance from the center. What quantity per second of fluid flows through the portion of the tube where $0 \le r \le 2$?

SOLUTION The flow rate through the portion of the tube where $0 \le r \le 2$ is

$$\begin{aligned} 2\pi \int_0^2 r v(r)\, dr &= 2\pi \int_0^2 r\left(10 - 0.3r - 0.34r^2\right) dr = 2\pi \int_0^2 \left(10r - 0.3r^2 - 0.34r^3\right) dr \\ &= 2\pi\left(5r^2 - 0.1r^3 - 0.085r^4\right)\Big|_0^2 \\ &= 112.09\,\frac{\text{cm}^3}{\text{s}} \end{aligned}$$

37. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function $v(r) = 0.5(r-1)(3-r)$ cm/s.

SOLUTION The flow rate is

$$2\pi \int_1^3 r(0.5)(r-1)(3-r)\, dr = \pi \int_1^3 \left(-r^3 + 4r^2 - 3r\right) dr = \pi\left(-\frac{1}{4}r^4 + \frac{4}{3}r^3 - \frac{3}{2}r^2\right)\Big|_1^3 = \frac{8\pi}{3}\,\frac{\text{cm}^3}{\text{s}}.$$

38. Let $v(r)$ be the velocity of blood in an arterial capillary of radius $R = 4 \times 10^{-5}$ m. Use Poiseuille's Law (Example 6) with $k = 10^6$ (m-s)$^{-1}$ to determine the velocity at the center of the capillary and the flow rate (use correct units).

SOLUTION According to Poiseuille's Law, $v(r) = k(R^2 - r^2)$. With $R = 4 \times 10^{-5}$ m and $k = 10^6$ (m-s)$^{-1}$,

$$v(0) = 0.0016 \text{ m/s}.$$

The flow rate through the capillary is

$$2\pi \int_0^R kr(R^2 - r^2)\, dr = 2\pi k\left(\frac{R^2r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R = 2\pi k\frac{R^4}{4} \approx 4.02 \times 10^{-12}\,\frac{\text{m}^3}{\text{s}}.$$

In Exercises 39–48, calculate the average over the given interval.

39. $f(x) = x^3$, $[0, 4]$

SOLUTION The average is

$$\frac{1}{4-0}\int_0^4 x^3\,dx = \frac{1}{4}\int_0^4 x^3\,dx = \left.\frac{1}{16}x^4\right|_0^4 = 16.$$

40. $f(x) = x^3$, $[-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)}\int_{-1}^1 x^3\,dx = \frac{1}{2}\int_{-1}^1 x^3\,dx = \left.\frac{1}{8}x^4\right|_{-1}^1 = 0.$$

41. $f(x) = \cos x$, $\left[0, \frac{\pi}{6}\right]$

SOLUTION The average is

$$\frac{1}{\pi/6-0}\int_0^{\pi/6}\cos x\,dx = \frac{6}{\pi}\int_0^{\pi/6}\cos x\,dx = \left.\frac{6}{\pi}\sin x\right|_0^{\pi/6} = \frac{3}{\pi}.$$

42. $f(x) = \sec^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

SOLUTION The average is

$$\frac{1}{\pi/3-\pi/6}\int_{\pi/6}^{\pi/3}\sec^2 x\,dx = \frac{6}{\pi}\int_{\pi/6}^{\pi/3}\sec^2 x\,dx = \left.\frac{6}{\pi}\tan x\right|_{\pi/6}^{\pi/3} = \frac{6}{\pi}\left(\sqrt{3}-\frac{\sqrt{3}}{3}\right) = \frac{4\sqrt{3}}{\pi}.$$

43. $f(s) = s^{-2}$, $[2, 5]$

SOLUTION The average is

$$\frac{1}{5-2}\int_2^5 s^{-2}\,ds = \left.-\frac{1}{3}s^{-1}\right|_2^5 = \frac{1}{10}.$$

44. $f(x) = \dfrac{\sin(\pi/x)}{x^2}$, $[1, 2]$

SOLUTION The average is

$$\frac{1}{2-1}\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin u\,du = \left.-\frac{1}{\pi}\cos u\right|_{\pi/2}^{\pi} = \frac{1}{\pi}.$$

45. $f(x) = 2x^3 - 6x^2$, $[-1, 3]$

SOLUTION The average is

$$\frac{1}{3-(-1)}\int_{-1}^3 (2x^3-6x^2)\,dx = \frac{1}{4}\int_{-1}^3 (2x^3-6x^2)\,dx = \left.\frac{1}{4}\left(\frac{1}{2}x^4 - 2x^3\right)\right|_{-1}^3 = \frac{1}{4}\left(-\frac{27}{2}-\frac{5}{2}\right) = -4.$$

46. $f(x) = \dfrac{1}{x^2+1}$, $[-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)}\int_{-1}^1 \frac{1}{x^2+1}\,dx = \left.\frac{1}{2}\tan^{-1} x\right|_{-1}^1 = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}.$$

47. $f(x) = x^n$ for $n \ge 0$, $[0, 1]$

SOLUTION For $n > -1$, the average is

$$\frac{1}{1-0}\int_0^1 x^n\,dx = \int_0^1 x^n\,dx = \left.\frac{1}{n+1}x^{n+1}\right|_0^1 = \frac{1}{n+1}.$$

48. $f(x) = e^{-nx}, \quad [-1, 1]$

SOLUTION The average is

$$\frac{1}{1-(-1)} \int_{-1}^{1} e^{-nx}\, dx = \frac{1}{2}\left(-\frac{1}{n}e^{-nx}\right)\Big|_{-1}^{1} = \frac{1}{2}\left(-\frac{1}{n}e^{-n} + \frac{1}{n}e^{n}\right) = \frac{1}{n}\sinh n.$$

49. The temperature (in °C) at time t (in hours) in an art museum varies according to $T(t) = 20 + 5\cos\left(\frac{\pi}{12}t\right)$. Find the average over the time periods [0, 24] and [2, 6].

SOLUTION

- The average temperature over the 24-hour period is

$$\frac{1}{24-0}\int_0^{24}\left(20 + 5\cos\left(\frac{\pi}{12}t\right)\right) dt = \frac{1}{24}\left(20t + \frac{60}{\pi}\sin\left(\frac{\pi}{12}t\right)\right)\Big|_0^{24} = 20°\text{C}.$$

- The average temperature over the 4-hour period is

$$\frac{1}{6-2}\int_2^{6}\left(20 + 5\cos\left(\frac{\pi}{12}t\right)\right) dt = \frac{1}{4}\left(20t + \frac{60}{\pi}\sin\left(\frac{\pi}{12}t\right)\right)\Big|_2^{6} = 22.4°\text{C}.$$

50. A ball thrown in the air vertically from ground level with initial velocity 18 m/s has height $h(t) = 18t - 9.8t^2$ at time t (in seconds). Find the average height and the average speed over the time interval extending from the ball's release to its return to ground level.

SOLUTION Let $h(t) = 18t - 9.8t^2$. The ball is at ground level when $t = 0$ s and when

$$t = \frac{18}{9.8} = \frac{9}{4.9}\text{ s.}$$

The average height of the ball is then

$$\begin{aligned}\frac{1}{\frac{9}{4.9}-0}\int_0^{9/4.9}(18t - 9.8t^2)\, dt &= \frac{4.9}{9}\left(9t^2 - \frac{9.8}{3}t^3\right)\Big|_0^{9/4.9}\\ &= \frac{4.9}{9}\left[9\left(\frac{9}{4.9}\right)^2 - \frac{9.8}{3}\left(\frac{9}{4.9}\right)^3\right]\\ &= 5.51\text{ m.}\end{aligned}$$

The average speed is given by

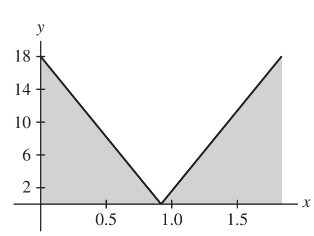
$$\frac{1}{\frac{9}{4.9}-0}\int_0^{9/4.9}|v(t)|\, dt.$$

Now, $v(t) = h'(t) = 18 - 19.6t$. From the figure below, which shows the graph of $|v(t)|$ over the interval $[0, 9/4.9]$, we see that

$$\int_0^{9/4.9}|v(t)|\, dt = \left(\frac{9}{9.8}\right)18.$$

Thus, the average speed is

$$\frac{4.9}{9}\left(\frac{9}{9.8}\right)18 = 9\text{ m/s.}$$



51. Find the average speed over the time interval $[1, 5]$ of a particle whose position at time t is $s(t) = t^3 - 6t^2$ m/s.

SOLUTION The average speed over the time interval $[1, 5]$ is

$$\frac{1}{5-1}\int_1^5 |s'(t)|\,dt.$$

Because $s'(t) = 3t^2 - 12t = 3t(t-4)$, it follows that

$$\begin{aligned}\int_1^5 |s'(t)|\,dt &= \int_1^4 (12t - 3t^2)\,dt + \int_4^5 (3t^2 - 12t)\,dt\\ &= (6t^2 - t^3)\Big|_1^4 + (t^3 - 6t^2)\Big|_4^5\\ &= (96-64) - (6-1) + (125-150) - (64-96)\\ &= 34.\end{aligned}$$

Thus, the average speed is

$$\frac{34}{4} = \frac{17}{2}\ \text{m/s}.$$

52. An object with zero initial velocity accelerates at a constant rate of 10 m/s^2. Find its average velocity during the first 15 seconds.

SOLUTION An acceleration $a(t) = 10$ gives $v(t) = 10t + c$ for some constant c and zero initial velocity implies $c = 0$. Thus the average velocity is given by

$$\frac{1}{15-0}\int_0^{15} 10t\,dt = \frac{1}{3}t^2\Big|_0^{15} = 75\ \text{m/s}.$$

53. The acceleration of a particle is $a(t) = 60t - 4t^3$ m/s^2. Compute the average acceleration and the average speed over the time interval $[2, 6]$, assuming that the particle's initial velocity is zero.

SOLUTION The average acceleration over the time interval $[2, 6]$ is

$$\begin{aligned}\frac{1}{6-2}\int_2^6 (60t - 4t^3)\,dt &= \frac{1}{4}(30t^2 - t^4)\Big|_2^6\\ &= \frac{1}{4}\left[(1080-1296) - (120-16)\right]\\ &= -\frac{320}{4} = -80\ \text{m/s}^2.\end{aligned}$$

Given $a(t) = 60t - 4t^3$ and $v(0) = 0$, it follows that $v(t) = 30t^2 - t^4$. Now, average speed is given by

$$\frac{1}{6-2}\int_2^6 |v(t)|\,dt.$$

Based on the formula for $v(t)$,

$$\begin{aligned}\int_2^6 |v(t)|\,dt &= \int_2^{\sqrt{30}} (30t^2 - t^4)\,dt + \int_{\sqrt{30}}^6 (t^4 - 30t^2)\,dt\\ &= \left(10t^3 - \frac{1}{5}t^5\right)\Big|_2^{\sqrt{30}} + \left(\frac{1}{5}t^5 - 10t^3\right)\Big|_{\sqrt{30}}^6\\ &= 120\sqrt{30} - \frac{368}{5} - \frac{3024}{5} + 120\sqrt{30}\\ &= 240\sqrt{30} - \frac{3392}{5}.\end{aligned}$$

Finally, the average speed is

$$\frac{1}{4}\left(240\sqrt{30} - \frac{3392}{5}\right) = 60\sqrt{30} - \frac{848}{5} \approx 159.03\ \text{m/s}.$$

54. What is the average area of the circles whose radii vary from 0 to R?

SOLUTION The average area is

$$\frac{1}{R-0}\int_0^R \pi r^2\,dr = \frac{\pi}{3R}r^3\Big|_0^R = \frac{1}{3}\pi R^2.$$

55. Let M be the average value of $f(x) = x^4$ on [0, 3]. Find a value of c in [0, 3] such that $f(c) = M$.

SOLUTION We have

$$M = \frac{1}{3-0}\int_0^3 x^4\,dx = \frac{1}{3}\int_0^3 x^4\,dx = \frac{1}{15}x^5\Big|_0^3 = \frac{81}{5}.$$

Then $M = f(c) = c^4 = \frac{81}{5}$ implies $c = \frac{3}{5^{1/4}} = 2.006221$.

56. Let $f(x) = \sqrt{x}$. Find a value of c in [4, 9] such that $f(c)$ is equal to the average of f on [4, 9].

SOLUTION The average value is

$$\frac{1}{9-4}\int_4^9 \sqrt{x}\,dx = \frac{1}{5}\int_4^9 \sqrt{x}\,dx = \frac{2}{15}x^{3/2}\Big|_4^9 = \frac{38}{15}.$$

Then $f(c) = \sqrt{c} = \frac{38}{15}$ implies

$$c = \left(\frac{38}{15}\right)^2 = \frac{1444}{225} \approx 6.417778.$$

57. Let M be the average value of $f(x) = x^3$ on $[0, A]$, where $A > 0$. Which theorem guarantees that $f(c) = M$ has a solution c in $[0, A]$? Find c.

SOLUTION The Mean Value Theorem for Integrals guarantees that $f(c) = M$ has a solution c in $[0, A]$. With $f(x) = x^3$ on $[0, A]$,

$$M = \frac{1}{A-0}\int_0^A x^3\,dx = \frac{1}{A}\,\frac{1}{4}x^4\Big|_0^A = \frac{A^3}{4}.$$

Solving $f(c) = c^3 = \frac{A^3}{4}$ for c yields

$$c = \frac{A}{\sqrt[3]{4}}.$$

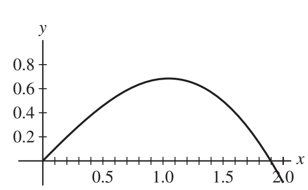
58. CAS Let $f(x) = 2\sin x - x$. Use a computer algebra system to plot $f(x)$ and estimate:

(a) The positive root α of $f(x)$.

(b) The average value M of $f(x)$ on $[0, \alpha]$.

(c) A value $c \in [0, \alpha]$ such that $f(c) = M$.

SOLUTION Let $f(x) = 2\sin x - x$. A graph of $y = f(x)$ is shown below. From this graph, the positive root of $f(x)$ appears to be roughly $x = 1.9$.



(a) Using a computer algebra system, solving the equation

$$2\sin\alpha - \alpha = 0$$

yields $\alpha = 1.895494267$.

(b) The average value of $f(x)$ on $[0, \alpha]$ is

$$M = \frac{1}{\alpha - 0}\int_0^\alpha f(x)\,dx = 0.4439980667.$$

(c) Solving

$$f(c) = 2\sin c - c = 0.4439980667$$

yields either $c = 0.4805683082$ or $c = 1.555776337$.

59. Which of $f(x) = x\sin^2 x$ and $g(x) = x^2\sin^2 x$ has a larger average value over $[0, 1]$? Over $[1, 2]$?

SOLUTION The functions f and g differ only in the power of x multiplying $\sin^2 x$. It is also important to note that $\sin^2 x \geq 0$ for all x. Now, for each $x \in (0, 1)$, $x > x^2$ so

$$f(x) = x\sin^2 x > x^2\sin^2 x = g(x).$$

Thus, over $[0, 1]$, $f(x)$ will have a larger average value than $g(x)$. On the other hand, for each $x \in (1, 2)$, $x^2 > x$, so

$$g(x) = x^2\sin^2 x > x\sin^2 x = f(x).$$

Thus, over $[1, 2]$, $g(x)$ will have the larger average value.

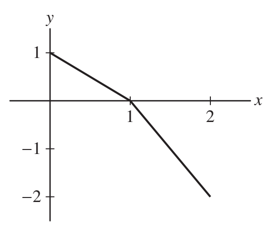
60. Find the average of $f(x) = ax + b$ over the interval $[-M, M]$, where a, b, and M are arbitrary constants.

SOLUTION The average is

$$\frac{1}{M-(-M)}\int_{-M}^{M}(ax+b)\,dx = \frac{1}{2M}\int_{-M}^{M}(ax+b)\,dx = \frac{1}{2M}\left(\frac{a}{2}x^2 + bx\right)\Bigg|_{-M}^{M} = b.$$

61. Sketch the graph of a function $f(x)$ such that $f(x) \geq 0$ on $[0, 1]$ and $f(x) \leq 0$ on $[1, 2]$, whose average on $[0, 2]$ is negative.

SOLUTION Many solutions will exist. One could be



62. Give an example of a function (necessarily discontinuous) that does not satisfy the conclusion of the MVT for Integrals.

SOLUTION There are an infinite number of discontinuous functions that do not satisfy the conclusion of the Mean Value Theorem for Integrals. Consider the function on $[-1, 1]$ such that for $x < 0$, $f(x) = -1$ and for $x \geq 0$, $f(x) = 1$. Clearly the average value is 0 but $f(c) \neq 0$ for all c in $[-1, 1]$.

Further Insights and Challenges

63. An object is tossed into the air vertically from ground level with initial velocity v_0 ft/s at time $t = 0$. Find the average speed of the object over the time interval $[0, T]$, where T is the time the object returns to earth.

SOLUTION The height is given by $h(t) = v_0 t - 16t^2$. The ball is at ground level at time $t = 0$ and $T = v_0/16$. The velocity is given by $v(t) = v_0 - 32t$ and thus the speed is given by $s(t) = |v_0 - 32t|$. The average speed is

$$\begin{aligned}\frac{1}{v_0/16 - 0}\int_0^{v_0/16}|v_0 - 32t|\,dt &= \frac{16}{v_0}\int_0^{v_0/32}(v_0 - 32t)\,dt + \frac{16}{v_0}\int_{v_0/32}^{v_0/16}(32t - v_0)\,dt\\ &= \frac{16}{v_0}\left(v_0 t - 16t^2\right)\Big|_0^{v_0/32} + \frac{16}{v_0}\left(16t^2 - v_0 t\right)\Big|_{v_0/32}^{v_0/16} = v_0/2.\end{aligned}$$

64. Review the MVT stated in Section 4.3 (Theorem 1, p. 266) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for Integrals.

SOLUTION The Mean Value Theorem essentially states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$. Let F be any antiderivative of f. Then

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b-a}(F(b) - F(a)) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

6.3 Volumes of Revolution

Preliminary Questions

1. Which of the following is a solid of revolution?

(a) Sphere **(b)** Pyramid **(c)** Cylinder **(d)** Cube

SOLUTION The sphere and the cylinder have circular cross sections; hence, these are solids of revolution. The pyramid and cube do not have circular cross sections, so these are not solids of revolution.

2. True or false? When the region under a single graph is rotated about the x-axis, the cross sections of the solid perpendicular to the x-axis are circular disks.

SOLUTION True. The cross sections will be disks with radius equal to the value of the function.

3. True or false? When the region between two graphs is rotated about the x-axis, the cross sections to the solid perpendicular to the x-axis are circular disks.

SOLUTION False. The cross sections may be washers.

4. Which of the following integrals expresses the volume obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the x-axis? [Assume $f(x) \geq g(x) \geq 0$.]

(a) $\pi \int_a^b \big(f(x) - g(x)\big)^2\, dx$

(b) $\pi \int_a^b \big(f(x)^2 - g(x)^2\big)\, dx$

SOLUTION The correct answer is **(b)**. Cross sections of the solid will be washers with outer radius $f(x)$ and inner radius $g(x)$. The area of the washer is then $\pi f(x)^2 - \pi g(x)^2 = \pi(f(x)^2 - g(x)^2)$.

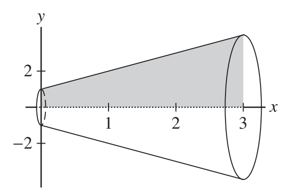
Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the region under the graph of $f(x)$ about the x-axis over the given interval, (b) describe the cross section perpendicular to the x-axis located at x, and (c) calculate the volume of the solid.

1. $f(x) = x + 1, \quad [0, 3]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius $x + 1$.

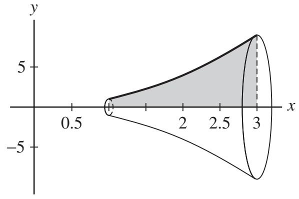
(c) The volume of the solid of revolution is

$$\pi \int_0^3 (x+1)^2\, dx = \pi \int_0^3 (x^2 + 2x + 1)\, dx = \pi \left(\frac{1}{3}x^3 + x^2 + x\right)\Big|_0^3 = 21\pi.$$

2. $f(x) = x^2, \quad [1, 3]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk of radius x^2.

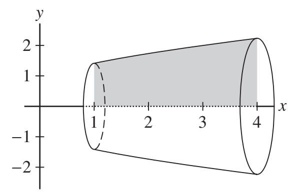
(c) The volume of the solid of revolution is

$$\pi \int_1^3 \left(x^2\right)^2 dx = \pi \left(\frac{x^5}{5}\right)\Bigg|_1^3 = \frac{242\pi}{5}.$$

3. $f(x) = \sqrt{x+1}, \quad [1, 4]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius $\sqrt{x+1}$.

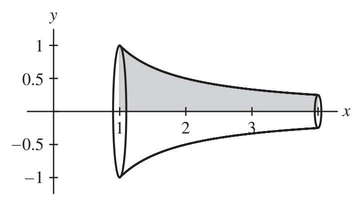
(c) The volume of the solid of revolution is

$$\pi \int_1^4 (\sqrt{x+1})^2\, dx = \pi \int_1^4 (x+1)\, dx = \pi \left(\frac{1}{2}x^2 + x\right)\Bigg|_1^4 = \frac{21\pi}{2}.$$

4. $f(x) = x^{-1}, \quad [1, 4]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius x^{-1}.

(c) The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-1})^2\, dx = \pi \int_1^4 x^{-2}\, dx = \pi\, (-x)^{-1}\Big|_1^4 = \frac{3\pi}{4}.$$

In Exercises 5–12, find the volume of revolution about the x-axis for the given function and interval.

5. $f(x) = x^2 - 3x, \quad [0, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^3 (x^2 - 3x)^2\, dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\, dx = \pi \left(\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3\right)\Bigg|_0^3 = \frac{81\pi}{10}.$$

6. $f(x) = \dfrac{1}{x^2}, \quad [1, 4]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-2})^2\, dx = \pi \int_1^4 x^{-4}\, dx = \pi \left(-\frac{1}{3}x^{-3}\right)\Bigg|_1^4 = \frac{21\pi}{64}.$$

7. $f(x) = x^{5/3}, \quad [1, 8]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^8 (x^{5/3})^2\, dx = \pi \int_1^8 x^{10/3}\, dx = \frac{3\pi}{13} x^{13/3}\Bigg|_1^8 = \frac{3\pi}{13}(2^{13} - 1) = \frac{24573\pi}{13}.$$

8. $f(x) = 4 - x^2$, $[0, 2]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^2 (4 - x^2)^2\, dx = \pi \int_0^2 (16 - 8x^2 + x^4)\, dx = \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right)\Big|_0^2 = \frac{256\pi}{15}.$$

9. $f(x) = \dfrac{2}{x+1}$, $[1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 \left(\frac{2}{x+1} \right)^2 dx = 4\pi \int_1^3 (x+1)^{-2}\, dx = -4\pi\, (x+1)^{-1} \Big|_1^3 = \pi.$$

10. $f(x) = \sqrt{x^4 + 1}$, $[1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 (\sqrt{x^4+1})^2\, dx = \pi \int_1^3 (x^4 + 1)\, dx = \pi \left(\frac{1}{5}x^5 + x \right)\Big|_1^3 = \frac{252\pi}{5}.$$

11. $f(x) = e^x$, $[0, 1]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^1 (e^x)^2\, dx = \frac{1}{2}\pi e^{2x} \Big|_0^1 = \frac{1}{2}\pi(e^2 - 1).$$

12. $f(x) = \sqrt{\cos x \sin x}$, $\left[0, \frac{\pi}{2}\right]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^{\pi/2} (\sqrt{\cos x \sin x})^2\, dx = \pi \int_0^{\pi/2} (\cos x \sin x)\, dx = \frac{\pi}{2} \int_0^{\pi/2} \sin 2x\, dx = \frac{\pi}{4}\,(-\cos 2x)\Big|_0^{\pi/2} = \frac{\pi}{2}.$$

In Exercises 13 and 14, R is the shaded region in Figure 11.

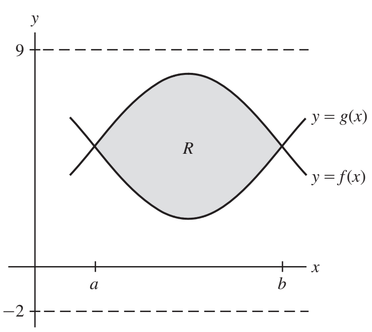


FIGURE 11

13. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating region R about $y = -2$?
- (i) $(f(x)^2 + 2^2) - (g(x)^2 + 2^2)$
- (ii) $(f(x) + 2)^2 - (g(x) + 2)^2$
- (iii) $(f(x)^2 - 2^2) - (g(x)^2 - 2^2)$
- (iv) $(f(x) - 2)^2 - (g(x) - 2)^2$

SOLUTION when the region R is rotated about $y = -2$, the outer radius is $f(x) - (-2) = f(x) + 2$ and the inner radius is $g(x) - (-2) = g(x) + 2$. Thus, the appropriate integrand is **(ii)**: $(f(x) + 2)^2 - (g(x) + 2)^2$.

14. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating R about $y = 9$?
- (i) $(9 + f(x))^2 - (9 + g(x))^2$
- (ii) $(9 + g(x))^2 - (9 + f(x))^2$
- (iii) $(9 - f(x))^2 - (9 - g(x))^2$
- (iv) $(9 - g(x))^2 - (9 - f(x))^2$

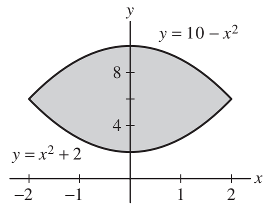
SOLUTION when the region R is rotated about $y = 9$, the outer radius is $9 - g(x)$ and the inner radius is $9 - f(x)$. Thus, the appropriate integrand is **(iv)**: $(9 - g(x))^2 - (9 - f(x))^2$.

In Exercises 15–20, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the x-axis located at x, and (c) find the volume of the solid obtained by rotating the region about the x-axis.

15. $y = x^2 + 2, \quad y = 10 - x^2$

SOLUTION

(a) Setting $x^2 + 2 = 10 - x^2$ yields $2x^2 = 8$, or $x^2 = 4$. The two curves therefore intersect at $x = \pm 2$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$.

(c) The volume of the solid of revolution is

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2\right) dx = \pi \int_{-2}^{2} (96 - 24x^2)\, dx = \pi \left(96x - 8x^3\right)\Big|_{-2}^{2} = 256\pi.$$

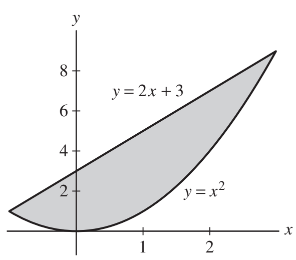
16. $y = x^2, \quad y = 2x + 3$

SOLUTION

(a) Setting $x^2 = 2x + 3$ yields

$$0 = x^2 - 2x - 3 = (x - 3)(x + 1).$$

The two curves therefore intersect at $x = -1$ and $x = 3$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 2x + 3$ and inner radius $r = x^2$.

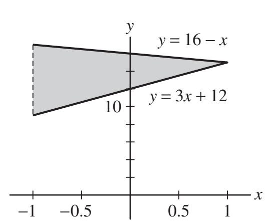
(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{3} \left((2x + 3)^2 - (x^2)^2\right) dx = \pi \int_{-1}^{3} (4x^2 + 12x + 9 - x^4)\, dx = \pi \left(\frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5\right)\Big|_{-1}^{3} = \frac{1088\pi}{15}.$$

17. $y = 16 - x, \quad y = 3x + 12, \quad x = -1$

SOLUTION

(a) Setting $16 - x = 3x + 12$, we find that the two lines intersect at $x = 1$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$.

(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{1} \left((16 - x)^2 - (3x + 12)^2\right) dx = \pi \int_{-1}^{1} (112 - 104x - 8x^2)\, dx = \pi \left(112x - 52x^2 - \frac{8}{3}x^3\right)\Big|_{-1}^{1} = \frac{656\pi}{3}.$$

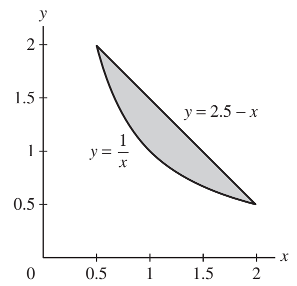
18. $y = \dfrac{1}{x}, \quad y = \dfrac{5}{2} - x$

SOLUTION

(a) Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x - 2)\left(x - \frac{1}{2}\right).$$

The two curves therefore intersect at $x = 2$ and $x = \frac{1}{2}$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = \frac{5}{2} - x$ and inner radius $r = x^{-1}$.

(c) The volume of the solid of revolution is

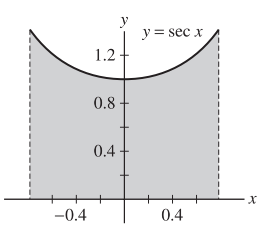
$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x\right)^2 - \left(\frac{1}{x}\right)^2\right) dx = \pi \int_{1/2}^{2} \left(\frac{25}{4} - 5x + x^2 - x^{-2}\right) dx$$

$$= \pi \left(\frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + x^{-1}\right)\Big|_{1/2}^{2} = \frac{9\pi}{8}.$$

19. $y = \sec x, \quad y = 0, \quad x = -\dfrac{\pi}{4}, \quad x = \dfrac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

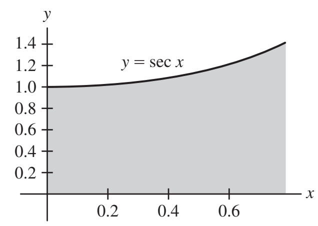
(c) The volume of the solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2\, dx = \pi\,(\tan x)\Big|_{-\pi/4}^{\pi/4} = 2\pi.$$

20. $y = \sec x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

(c) The volume of the solid of revolution is

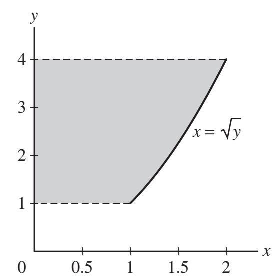
$$\pi \int_0^{\pi/4} (\sec x)^2 \, dx = \pi \left(\tan x\right) \Big|_0^{\pi/4} = \pi.$$

In Exercises 21–24, find the volume of the solid obtained by rotating the region enclosed by the graphs about the y-axis over the given interval.

21. $x = \sqrt{y}, \quad x = 0; \quad 1 \le y \le 4$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{y}$. The volume of the solid of revolution is

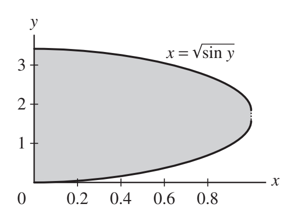
$$\pi \int_1^4 \left(\sqrt{y}\right)^2 \, dy = \frac{\pi y^2}{2}\Big|_1^4 = \frac{15\pi}{2}.$$



22. $x = \sqrt{\sin y}, \quad x = 0; \quad 0 \le y \le \pi$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{\sin y}$. The volume of the solid of revolution is

$$\pi \int_0^{\pi} \left(\sqrt{\sin y}\right)^2 \, dy = \pi \left(-\cos y\right) \Big|_0^{\pi} = 2\pi.$$



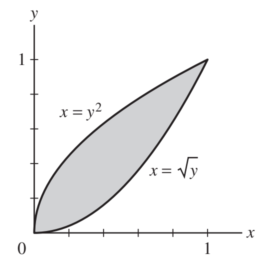
23. $x = y^2, \quad x = \sqrt{y}$

SOLUTION Setting $y^2 = \sqrt{y}$ and then squaring both sides yields

$$y^4 = y \quad \text{or} \quad y^4 - y = y(y^3 - 1) = 0,$$

so the two curves intersect at $y = 0$ and $y = 1$. When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = y^2$. The volume of the solid of revolution is

$$\pi \int_0^1 \left((\sqrt{y})^2 - (y^2)^2\right) \, dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{3\pi}{10}.$$



24. $x = 4 - y, \quad x = 16 - y^2$

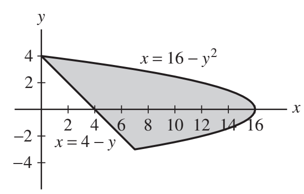
SOLUTION Setting $4 - y = 16 - y^2$ yields

$$0 = y^2 - y - 12 = (y - 4)(y + 3),$$

so the two curves intersect at $y = -3$ and $y = 4$. When the region enclosed by the two curves (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = 16 - y^2$ and inner radius $r = 4 - y$. The volume of the solid of revolution is

$$\pi \int_{-3}^{4} \left((16 - y^2)^2 - (4 - y)^2\right) dy = \pi \int_{-3}^{4} \left(y^4 - 33y^2 + 8y + 240\right) dy$$

$$= \pi \left(\frac{1}{5}y^5 - 11y^3 + 4y^2 + 240y\right)\bigg|_{-3}^{4} = \frac{4802\pi}{5}.$$



25. Rotation of the region in Figure 12 about the y-axis produces a solid with two types of different cross sections. Compute the volume as a sum of two integrals, one for $-12 \le y \le 4$ and one for $4 \le y \le 12$.

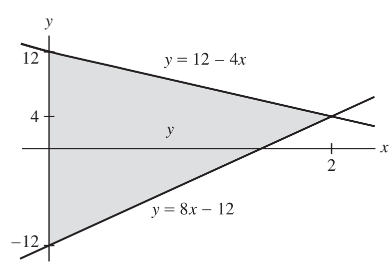


FIGURE 12

SOLUTION For $-12 \le y \le 4$, the cross section is a disk with radius $\frac{1}{8}(y + 12)$; for $4 \le y \le 12$, the cross section is a disk with radius $\frac{1}{4}(12 - y)$. Therefore, the volume of the solid of revolution is

$$V = \frac{\pi}{8} \int_{-12}^{4} (y + 12)^2 \, dy + \frac{\pi}{4} \int_{4}^{12} (12 - y)^2 \, dy$$

$$= \frac{\pi}{24}(y + 12)^3\bigg|_{-12}^{4} - \frac{\pi}{12}(12 - y)^3\bigg|_{4}^{12}$$

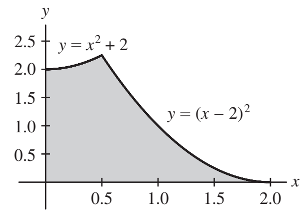
$$= \frac{512\pi}{3} + \frac{128\pi}{3} = \frac{640\pi}{3}.$$

26. Let R be the region enclosed by $y = x^2 + 2$, $y = (x - 2)^2$ and the axes $x = 0$ and $y = 0$. Compute the volume V obtained by rotating R about the x-axis. *Hint:* Express V as a sum of two integrals.

SOLUTION Setting $x^2 + 2 = (x - 2)^2$ yields $4x = 2$ or $x = 1/2$. When the region enclosed by the two curves and the coordinate axes (shown in the figure below) is rotated about the x-axis, there are two different cross sections. For

$0 \le x \le 1/2$, the cross section is a disk of radius x^2+2; for $1/2 \le x \le 2$, the cross section is a disk of radius $(x-2)^2$. The volume of the solid of revolution is therefore

$$V = \pi \int_0^{1/2} (x^2+2)\,dx + \pi \int_{1/2}^{2} (x-2)^2\,dx$$
$$= \pi\left(\frac{1}{3}x^3 + 2x\right)\Big|_0^{1/2} + \frac{\pi}{3}(x-2)^3\Big|_{1/2}^{2}$$
$$= \frac{25\pi}{24} + \frac{9\pi}{8} = \frac{13\pi}{6}.$$



In Exercises 27–32, find the volume of the solid obtained by rotating region A in Figure 13 about the given axis.

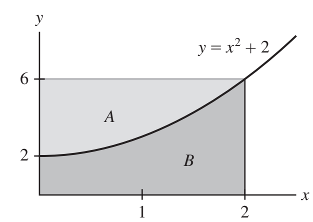


FIGURE 13

27. x-axis

SOLUTION Rotating region A about the x-axis produces a solid whose cross sections are washers with outer radius $R = 6$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((6)^2 - (x^2+2)^2\right) dx = \pi \int_0^2 (32 - 4x^2 - x^4)\,dx = \pi\left(32x - \frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{704\pi}{15}.$$

28. $y = -2$

SOLUTION Rotating region A about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 6 - (-2) = 8$ and inner radius $r = x^2 + 2 - (-2) = x^2 + 4$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((8)^2 - (x^2+4)^2\right) dx = \pi \int_0^2 (48 - 8x^2 - x^4)\,dx = \pi\left(48x - \frac{8}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{1024\pi}{15}.$$

29. $y = 2$

SOLUTION Rotating the region A about $y = 2$ produces a solid whose cross sections are washers with outer radius $R = 6 - 2 = 4$ and inner radius $r = x^2 + 2 - 2 = x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(4^2 - (x^2)^2\right) dx = \pi\left(16x - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{128\pi}{5}.$$

30. y-axis

SOLUTION Rotating region A about the y-axis produces a solid whose cross sections are disks with radius $R = \sqrt{y-2}$. Note that here we need to integrate along the y-axis. The volume of the solid of revolution is

$$\pi \int_2^6 (\sqrt{y-2})^2\,dy = \pi \int_2^6 (y-2)\,dy = \pi\left(\frac{1}{2}y^2 - 2y\right)\Big|_2^6 = 8\pi.$$

31. $x = -3$

SOLUTION Rotating region A about $x = -3$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$ and inner radius $r = 0 - (-3) = 3$. The volume of the solid of revolution is

$$\pi \int_2^6 \left((3 + \sqrt{y-2})^2 - (3)^2\right) dy = \pi \int_2^6 (6\sqrt{y-2} + y - 2)\, dy = \pi \left(4(y-2)^{3/2} + \frac{1}{2}y^2 - 2y\right)\Big|_2^6 = 40\pi.$$

32. $x = 2$

SOLUTION Rotating region A about $x = 2$ produces a solid whose cross sections are washers with outer radius $R = 2 - 0 = 2$ and inner radius $r = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_2^6 \left(2^2 - (2 - \sqrt{y-2})^2\right) dy = \pi \int_2^6 \left(4\sqrt{y-2} - y + 2\right) dy = \pi \left(\frac{8}{3}(y-2)^{3/2} - \frac{1}{2}y^2 + 2y\right)\Big|_2^6 = \frac{40\pi}{3}.$$

In Exercises 33–38, find the volume of the solid obtained by rotating region B in Figure 13 about the given axis.

33. x-axis

SOLUTION Rotating region B about the x-axis produces a solid whose cross sections are disks with radius $R = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^2 (x^2+2)^2\, dx = \pi \int_0^2 (x^4 + 4x^2 + 4)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right)\Big|_0^2 = \frac{376\pi}{15}.$$

34. $y = -2$

SOLUTION Rotating region B about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = x^2 + 2 - (-2) = x^2 + 4$ and inner radius $r = 0 - (-2) = 2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((x^2+4)^2 - (2)^2\right) dx = \pi \int_0^2 (x^4 + 8x^2 + 12)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Big|_0^2 = \frac{776\pi}{15}.$$

35. $y = 6$

SOLUTION Rotating region B about $y = 6$ produces a solid whose cross sections are washers with outer radius $R = 6 - 0 = 6$ and inner radius $r = 6 - (x^2 + 2) = 4 - x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(6^2 - (4 - x^2)^2\right) dy = \pi \int_0^2 \left(20 + 8x^2 - x^4\right) dy = \pi \left(20x + \frac{8}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{824\pi}{15}.$$

36. y-axis

Hint for Exercise 36: Express the volume as a sum of two integrals along the y-axis or use Exercise 30.

SOLUTION Rotating region B about the y-axis produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2$ and inner radius $r = \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 \left((2)^2 - (\sqrt{y-2})^2\right) dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (6 - y)\, dy$$
$$= \pi\,(4y)\Big|_0^2 + \pi \left(6y - \frac{1}{2}y^2\right)\Big|_2^6 = 16\pi.$$

Alternately, we recognize that rotating both region A and region B about the y-axis produces a cylinder of radius $R = 2$ and height $h = 6$. The volume of this cylinder is $\pi(2)^2 \cdot 6 = 24\pi$. In Exercise 30, we found that the volume of the solid generated by rotating region A about the y-axis to be 8π. Therefore, the volume of the solid generated by rotating region B about the y-axis is $24\pi - 8\pi = 16\pi$.

37. $x = 2$

SOLUTION Rotating region B about $x = 2$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a disk with radius $R = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 (2 - \sqrt{y-2})^2\, dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (2 + y - 4\sqrt{y-2})\, dy$$
$$= \pi\,(4y)\Big|_0^2 + \pi \left(2y + \frac{1}{2}y^2 - \frac{8}{3}(y-2)^{3/2}\right)\Big|_2^6 = \frac{32\pi}{3}.$$

38. $x = -3$

SOLUTION Rotating region B about $x = -3$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = 0 - (-3) = 3$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$. The volume of the solid of revolution is

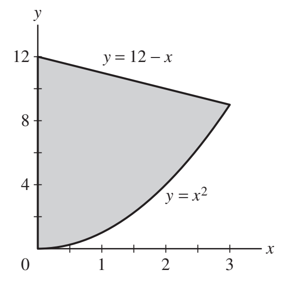
$$\begin{aligned}\pi \int_0^2 \left((5)^2 - (3)^2\right) dy + \pi \int_2^6 \left((5)^2 - (\sqrt{y-2} + 3)^2\right) dy \\ = \pi \int_0^2 16\, dy + \pi \int_2^6 (18 - y - 6\sqrt{y-2})\, dy \\ = \pi\,(16y)\Big|_0^2 + \pi \left(18y - \frac{1}{2}y^2 - 4(y-2)^{3/2}\right)\Big|_2^6 = 56\pi.\end{aligned}$$

In Exercises 39–52, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

39. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = -2$

SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (shown in the figure below) about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 12 - x - (-2) = 14 - x$ and inner radius $r = x^2 - (-2) = x^2 + 2$. The volume of the solid of revolution is

$$\begin{aligned}\pi \int_0^3 \left((14 - x)^2 - (x^2 + 2)^2\right) dx &= \pi \int_0^3 (192 - 28x - 3x^2 - x^4)\, dx \\ &= \pi \left(192x - 14x^2 - x^3 - \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1872\pi}{5}.\end{aligned}$$



40. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = 15$

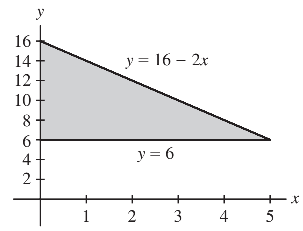
SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (see the figure in the previous exercise) about $y = 15$ produces a solid whose cross sections are washers with outer radius $R = 15 - x^2$ and inner radius $r = 15 - (12 - x) = 3 + x$. The volume of the solid of revolution is

$$\begin{aligned}\pi \int_0^3 \left((15 - x^2)^2 - (3 + x)^2\right) dx &= \pi \int_0^3 (216 - 6x - 31x^2 + x^4)\, dx \\ &= \pi \left(216x - 3x^2 - \frac{31}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1953\pi}{5}.\end{aligned}$$

41. $y = 16 - 2x, \quad y = 6, \quad x = 0, \quad$ about x-axis

SOLUTION Rotating the region enclosed by $y = 16 - 2x$, $y = 6$ and the y-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are washers with outer radius $R = 16 - 2x$ and inner radius $r = 6$. The volume of the solid of revolution is

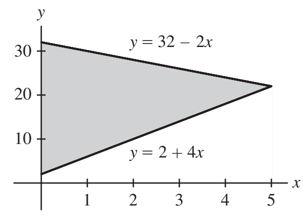
$$\begin{aligned}\pi \int_0^5 \left((16 - 2x)^2 - 6^2\right) dx &= \pi \int_0^5 (220 - 64x + 4x^2)\, dx \\ &= \pi \left(220x - 32x^2 + \frac{4}{3}x^3\right)\Big|_0^5 = \frac{1400\pi}{3}.\end{aligned}$$



42. $y = 32 - 2x, \quad y = 2 + 4x, \quad x = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $y = 32 - 2x$, $y = 2 + 4x$ and the y-axis (shown in the figure below) about the y-axis produces a solid with two different cross sections. For $2 \le y \le 22$, the cross section is a disk of radius $\frac{1}{4}(y-2)$; for $22 \le y \le 32$, the cross section is a disk of radius $\frac{1}{2}(32-y)$. The volume of the solid of revolution is

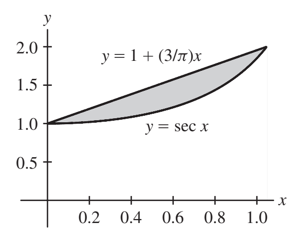
$$\begin{aligned} V &= \frac{\pi}{4}\int_2^{22} (y-2)^2\,dy + \frac{\pi}{2}\int_{22}^{32} (32-y)^2\,dy \\ &= \frac{\pi}{12}(y-2)^3\Big|_2^{22} - \frac{\pi}{6}(32-y)^3\Big|_{22}^{32} \\ &= \frac{2000\pi}{3} + \frac{500\pi}{3} = \frac{2500\pi}{3}. \end{aligned}$$



43. $y = \sec x, \quad y = 1 + \dfrac{3}{\pi}x,$ about x-axis

SOLUTION We first note that $y = \sec x$ and $y = 1 + (3/\pi)x$ intersect at $x = 0$ and $x = \pi/3$. Rotating the region enclosed by $y = \sec x$ and $y = 1 + (3/\pi)x$ (shown in the figure below) about the x-axis produces a cross section that is a washer with outer radius $R = 1 + (3/\pi)x$ and inner radius $r = \sec x$. The volume of the solid of revolution is

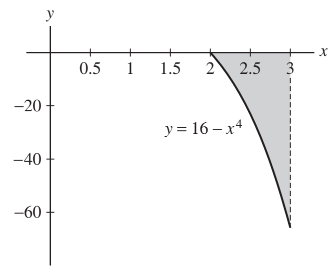
$$\begin{aligned} V &= \pi\int_0^{\pi/3} \left(\left(1 + \frac{3}{\pi}x\right)^2 - \sec^2 x\right) dx \\ &= \pi\int_0^{\pi/3} \left(1 + \frac{6}{\pi}x + \frac{9}{\pi^2}x^2 - \sec^2 x\right) dx \\ &= \pi\left(x + \frac{3}{\pi}x^2 + \frac{3}{\pi^2}x^3 - \tan x\right)\Big|_0^{\pi/3} \\ &= \pi\left(\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{9} - \sqrt{3}\right) = \frac{7\pi^2}{9} - \sqrt{3}\pi. \end{aligned}$$



44. $x = 2, \quad x = 3, \quad y = 16 - x^4, \quad y = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $x = 2$, $x = 3$, $y = 16 - x^4$ and the x-axis (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = 3$ and inner radius $r = \sqrt[4]{16-y}$. The volume of the solid of revolution is

$$\pi\int_{-65}^{0}\left(9 - \sqrt{16-y}\right) dy = \left(9y + \frac{2}{3}(16-y)^{3/2}\right)\Big|_{-65}^{0} = \frac{425\pi}{3}.$$



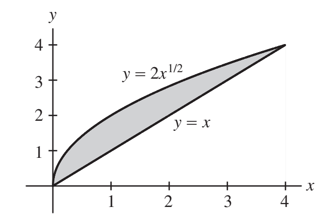
45. $y = 2\sqrt{x}, \quad y = x, \quad$ about $x = -2$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = y - (-2) = y + 2$ and inner radius $r = \frac{1}{4}y^2 - (-2) = \frac{1}{4}y^2 + 2$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((y+2)^2 - \left(\frac{1}{4}y^2 + 2 \right)^2 \right) dy \\ &= \pi \int_0^4 \left(4y - \frac{1}{16}y^4 \right) dy \\ &= \pi \left(2y^2 - \frac{1}{80}y^5 \right) \Big|_0^4 \\ &= \pi \left(32 - \frac{64}{5} \right) = \frac{96\pi}{5}. \end{aligned}$$



46. $y = 2\sqrt{x}, \quad y = x, \quad$ about $y = 4$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

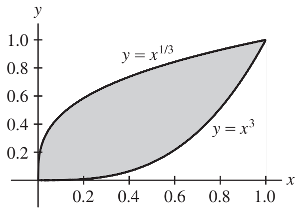
so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure from the previous exercise) about $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - x$ and inner radius $r = 4 - 2\sqrt{x}$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((4-x)^2 - (4 - 2\sqrt{x})^2 \right) dy \\ &= \pi \int_0^4 \left(x^2 - 12x + 16\sqrt{x} \right) dy \\ &= \pi \left(\frac{1}{3}x^3 - 6x^2 + \frac{32}{3}x^{3/2} \right) \Big|_0^4 \\ &= \pi \left(\frac{64}{3} - 96 + \frac{256}{3} \right) = \frac{32\pi}{3}. \end{aligned}$$

47. $y = x^3, \quad y = x^{1/3}, \quad$ for $x \geq 0, \quad$ about y-axis

SOLUTION Rotating the region enclosed by $y = x^3$ and $y = x^{1/3}$ (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = y^{1/3}$ and inner radius $r = y^3$. The volume of the solid of revolution is

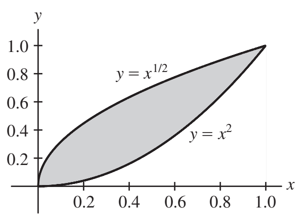
$$\pi \int_0^1 \left((y^{1/3})^2 - (y^3)^2 \right) dy = \pi \int_0^1 (y^{2/3} - y^6)\, dy = \pi \left(\frac{3}{5}y^{5/3} - \frac{1}{7}y^7 \right) \Big|_0^1 = \frac{16\pi}{35}.$$



48. $y = x^2$, $y = x^{1/2}$, about $x = -2$

SOLUTION Rotating the region enclosed by $y = x^2$ and $y = x^{1/2}$ (shown in the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y} - (-2) = \sqrt{y} + 2$ and inner radius $r = y^2 - (-2) = y^2 + 2$. The volume of the solid of revolution is

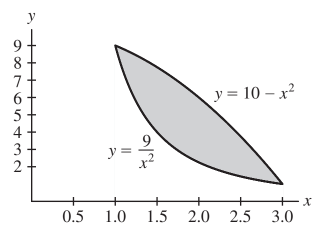
$$\pi \int_0^1 \left((\sqrt{y}+2)^2 - (y^2+2)^2\right) dy = \pi \int_0^1 \left(y + 4\sqrt{y} - y^4 - 4y^2\right) dy$$
$$= \pi \left(\frac{1}{2}y^2 + \frac{8}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{4}{3}y^3\right)\Big|_0^1$$
$$= \pi \left(\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3}\right) = \frac{49\pi}{30}.$$



49. $y = \dfrac{9}{x^2}$, $y = 10 - x^2$, $x \ge 0$, about $y = 12$

SOLUTION The region enclosed by the two curves is shown in the figure below. Rotating this region about $y = 12$ produces a solid whose cross sections are washers with outer radius $R = 12 - 9x^{-2}$ and inner radius $r = 12 - (10 - x^2) = 2 + x^2$. The volume of the solid of revolution is

$$\pi \int_1^3 \left((12 - 9x^{-2})^2 - (x^2+2)^2\right) dx = \pi \int_1^3 \left(140 - 4x^2 - x^4 - 216x^{-2} + 81x^{-4}\right) dx$$
$$= \pi \left(140x - \frac{4}{3}x^3 - \frac{1}{5}x^5 + 216x^{-1} - 27x^{-3}\right)\Big|_1^3 = \frac{1184\pi}{15}.$$



50. $y = \dfrac{9}{x^2}$, $y = 10 - x^2$, $x \ge 0$, about $x = -1$

SOLUTION The region enclosed by the two curves is shown in the figure from the previous exercise. Rotating this region about $x = -1$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{10-y} - (-1) = \sqrt{10-y} + 1$ and inner radius $r = 3y^{-1/2} - (-1) = 3y^{-1/2} + 1$. The volume of the solid of revolution is

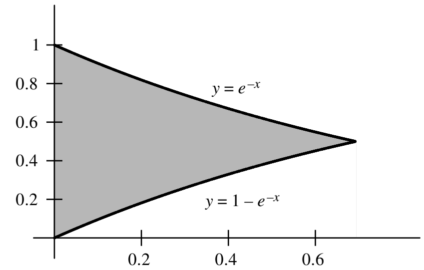
$$V = \pi \int_1^9 \left((\sqrt{10-y}+1)^2 - (3y^{-1/2}+1)^2\right) dy$$
$$= \pi \int_1^9 \left(10 - y + 2\sqrt{10-y} - 9y^{-1} - 6y^{-1/2}\right) dy$$
$$= \pi \left(10y - \frac{1}{2}y^2 - \frac{4}{3}(10-y)^{3/2} - 9\ln y - 12\sqrt{y}\right)\Big|_1^9$$

$$= \pi\left(\left(90 - \frac{81}{2} - \frac{4}{3} - 9\ln 9 - 36\right) - \left(10 - \frac{1}{2} - 36 - 12\right)\right)$$
$$= \pi\left(\frac{73}{6} - 9\ln 9 + \frac{77}{2}\right) = \left(\frac{152}{3} - 9\ln 9\right)\pi.$$

51. $y = e^{-x}, \quad y = 1 - e^{-x}, \quad x = 0, \quad \text{about } y = 4$

SOLUTION Rotating the region enclosed by $y = 1 - e^{-x}$, $y = e^{-x}$ and the y-axis (shown in the figure below) about the line $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - (1 - e^{-x}) = 3 + e^{-x}$ and inner radius $r = 4 - e^{-x}$. The volume of the solid of revolution is

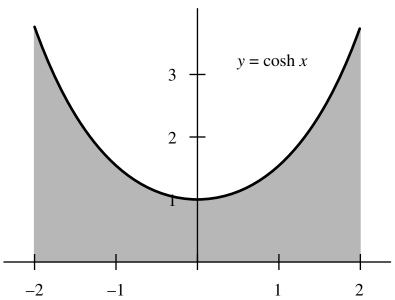
$$\pi\int_0^{\ln 2}\left((3 + e^{-x})^2 - (4 - e^{-x})^2\right)dx = \pi\int_0^{\ln 2}(14e^{-x} - 7)\,dx = \pi(-14e^{-x} - 7x)\Big|_0^{\ln 2}$$
$$= \pi(-7 - 7\ln 2 + 14) = 7\pi(1 - \ln 2).$$



52. $y = \cosh x, \quad x = \pm 2, \quad \text{about } x\text{-axis}$

SOLUTION Rotating the region enclosed by $y = \cosh x$, $x = \pm 2$ and the x-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are disks with radius $R = \cosh x$. The volume of the solid of revolution is

$$\pi\int_{-2}^{2}\cosh^2 x\,dx = \frac{1}{2}\pi\int_{-2}^{2}(1 + \cosh 2x)\,dx = \frac{1}{2}\pi\left(x + \frac{1}{2}\sinh 2x\right)\Big|_{-2}^{2}$$
$$= \frac{1}{2}\pi\left[\left(2 + \frac{1}{2}\sinh 4\right) - \left(-2 + \frac{1}{2}\sinh(-4)\right)\right] = \frac{1}{2}\pi(4 + \sinh 4).$$



53. The bowl in Figure 14(A) is 21 cm high, obtained by rotating the curve in Figure 14(B) as indicated. Estimate the volume capacity of the bowl shown by taking the average of right- and left-endpoint approximations to the integral with $N = 7$.

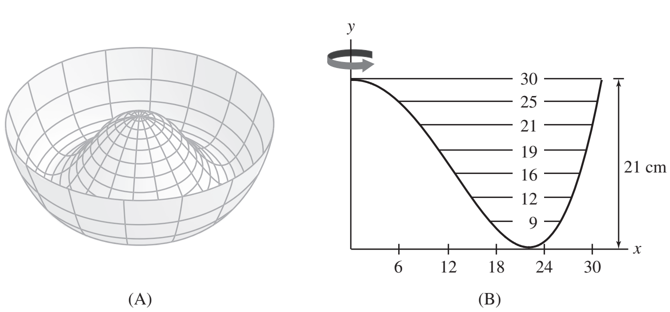


FIGURE 14

SOLUTION Using the given values for the inner radii and the values in Figure 14(B), which indicate the difference between the inner and outer radii, we find

$$R_7 = 3\pi\left((23^2 - 14^2) + (25^2 - 13^2) + (26^2 - 10^2) + (27^2 - 8^2) + (28^2 - 7^2) + (29^2 - 4^2) + (30^2 - 0^2)\right)$$
$$= 3\pi(4490) = 13470\pi$$

and

$$L_7 = 3\pi\left((20^2 - 20^2) + (23^2 - 14^2) + (25^2 - 13^2) + (26^2 - 10^2) + (27^2 - 8^2) + (28^2 - 7^2) + (29^2 - 4^2)\right)$$
$$= 3\pi(3590) = 10770\pi$$

Averaging these two values, we estimate that the volume capacity of the bowl is

$$V = 12120\pi \approx 38076.1 \text{ cm}^3.$$

54. The region between the graphs of $f(x)$ and $g(x)$ over [0, 1] is revolved about the line $y = -3$. Use the midpoint approximation with values from the following table to estimate the volume V of the resulting solid.

x	0.1	0.3	0.5	0.7	0.9
$f(x)$	8	7	6	7	8
$g(x)$	2	3.5	4	3.5	2

SOLUTION The volume of the resulting solid is

$$V = \pi\int_0^1 \left((f(x)+3)^2 - (g(x)+3)^2\right)\,dx$$
$$\approx 0.2\pi\left((11^2 - 5^2) + (10^2 - 6.5^2) + (9^2 - 7^2) + (10^2 - 6.5^2) + (11^2 - 5^2)\right)$$
$$= 0.2\pi(96 + 57.75 + 32 + 57.75 + 96) = 67.9\pi.$$

55. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

SOLUTION The segment joining $(0, h)$ and $(r, 0)$ has the equation

$$y = -\frac{h}{r}x + h \quad \text{or} \quad x = \frac{r}{h}(h - y).$$

Rotating the region under this segment about the y-axis produces a cone with volume

$$\frac{\pi r^2}{h^2}\int_0^h (h-y)^2\,dx = -\frac{\pi r^2}{3h^2}(h-y)^3\bigg|_0^h$$
$$= \frac{1}{3}\pi r^2 h.$$

56. The **torus** (doughnut-shaped solid) in Figure 15 is obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$. *Hint:* Evaluate the integral by interpreting it as the area of a circle.

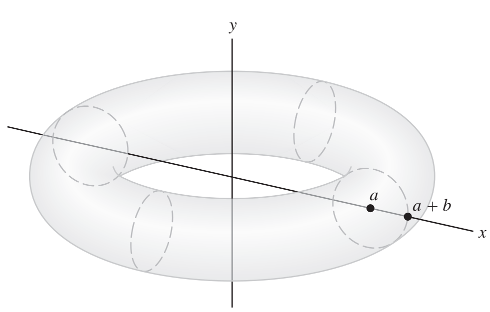


FIGURE 15 Torus obtained by rotating a circle about the y-axis.

SOLUTION Rotating the region enclosed by the circle $(x-a)^2+y^2=b^2$ about the y-axis produces a torus whose cross sections are washers with outer radius $R=a+\sqrt{b^2-y^2}$ and inner radius $r=a-\sqrt{b^2-y^2}$. The volume of the torus is then

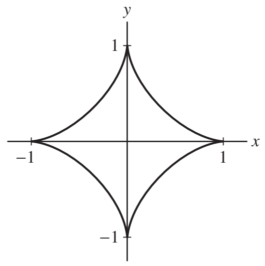
$$\pi\int_{-b}^{b}\left(\left(a+\sqrt{b^2-y^2}\right)^2-\left(a-\sqrt{b^2-y^2}\right)^2\right)dy=4a\pi\int_{-b}^{b}\sqrt{b^2-y^2}\,dy.$$

Now, the remaining definite integral is one-half the area of a circle of radius b; therefore, the volume of the torus is

$$4a\pi\cdot\frac{1}{2}\pi b^2=2\pi^2ab^2.$$

57. $\boxed{\text{GU}}$ Sketch the hypocycloid $x^{2/3}+y^{2/3}=1$ and find the volume of the solid obtained by revolving it about the x-axis.

SOLUTION A sketch of the hypocycloid is shown below.



For the hypocycloid, $y=\pm\left(1-x^{2/3}\right)^{3/2}$. Rotating this region about the x-axis will produce a solid whose cross sections are disks with radius $R=\left(1-x^{2/3}\right)^{3/2}$. Thus the volume of the solid of revolution will be

$$\pi\int_{-1}^{1}\left((1-x^{2/3})^{3/2}\right)^2dx=\pi\left(\frac{-x^3}{3}+\frac{9}{7}x^{7/3}-\frac{9}{5}x^{5/3}+x\right)\Bigg|_{-1}^{1}=\frac{32\pi}{105}.$$

58. The solid generated by rotating the region between the branches of the hyperbola $y^2-x^2=1$ about the x-axis is called a **hyperboloid** (Figure 16). Find the volume of the hyperboloid for $-a\le x\le a$.

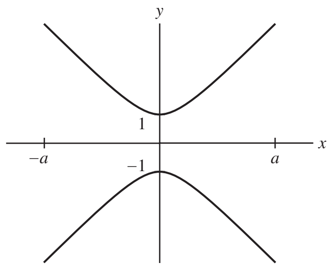


FIGURE 16 The hyperbola with equation $y^2-x^2=1$.

SOLUTION Each cross section is a disk of radius $R=\sqrt{1+x^2}$, so the volume of the hyperboloid is

$$\pi\int_{-a}^{a}\left(\sqrt{1+x^2}\right)^2dx=\pi\int_{-a}^{a}(1+x^2)\,dx=\pi\left(x+\frac{1}{3}x^3\right)\Bigg|_{-a}^{a}=\pi\left(\frac{2a^3+6a}{3}\right)$$

59. A "bead" is formed by removing a cylinder of radius r from the center of a sphere of radius R (Figure 17). Find the volume of the bead with $r=1$ and $R=2$.

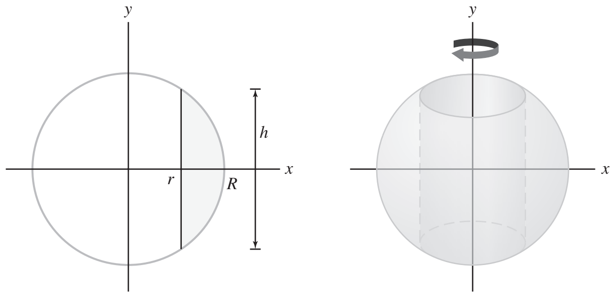


FIGURE 17 A bead is a sphere with a cylinder removed.

SOLUTION The equation of the outer circle is $x^2 + y^2 = 2^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{3}$. Each cross section of the bead is a washer with outer radius $\sqrt{4 - y^2}$ and inner radius 1, so the volume is given by

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(\left(\sqrt{4 - y^2} \right)^2 - 1^2 \right) dy = \pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(3 - y^2 \right) dy = 4\pi\sqrt{3}.$$

Further Insights and Challenges

60. Find the volume V of the bead (Figure 17) in terms of r and R. Then show that $V = \frac{\pi}{6}h^3$, where h is the height of the bead. This formula has a surprising consequence: Since V can be expressed in terms of h alone, it follows that two beads of height 1 cm, one formed from a sphere the size of an orange and the other from a sphere the size of the earth, would have the same volume! Can you explain intuitively how this is possible?

SOLUTION The equation for the outer circle of the bead is $x^2 + y^2 = R^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{R^2 - r^2}$. Each cross section of the bead is a washer with outer radius $\sqrt{R^2 - y^2}$ and inner radius r, so the volume is

$$\pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\left(\sqrt{R^2 - y^2} \right)^2 - r^2 \right) dy = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} (R^2 - r^2 - y^2)\, dy$$

$$= \pi \left((R^2 - r^2)y - \frac{1}{3}y^3 \right) \Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = \frac{4}{3}(R^2 - r^2)^{3/2}\pi.$$

Now, $h = 2\sqrt{R^2 - r^2} = 2(R^2 - r^2)^{1/2}$, which gives $h^3 = 8(R^2 - r^2)^{3/2}$ and finally $(R^2 - r^2)^{3/2} = \frac{1}{8}h^3$. Substituting into the expression for the volume gives $V = \frac{\pi}{6}h^3$. The beads may have the same volume but clearly the wall of the earth-sized bead must be extremely thin while the orange-sized bead would be thicker.

61. The solid generated by rotating the region inside the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ around the x-axis is called an **ellipsoid**. Show that the ellipsoid has volume $\frac{4}{3}\pi ab^2$. What is the volume if the ellipse is rotated around the y-axis?

SOLUTION

- Rotating the ellipse about the x-axis produces an ellipsoid whose cross sections are disks with radius $R = b\sqrt{1 - (x/a)^2}$. The volume of the ellipsoid is then

$$\pi \int_{-a}^{a} \left(b\sqrt{1 - (x/a)^2} \right)^2 dx = b^2\pi \int_{-a}^{a} \left(1 - \frac{1}{a^2}x^2 \right) dx = b^2\pi \left(x - \frac{1}{3a^2}x^3 \right) \Big|_{-a}^{a} = \frac{4}{3}\pi ab^2.$$

- Rotating the ellipse about the y-axis produces an ellipsoid whose cross sections are disks with radius $R = a\sqrt{1 - (y/b)^2}$. The volume of the ellipsoid is then

$$\int_{-b}^{b} \left(a\sqrt{1 - (y/b)^2} \right)^2 dy = a^2\pi \int_{-b}^{b} \left(1 - \frac{1}{b^2}y^2 \right) dy = a^2\pi \left(y - \frac{1}{3b^2}y^3 \right) \Big|_{-b}^{b} = \frac{4}{3}\pi a^2 b.$$

62. The curve $y = f(x)$ in Figure 18, called a **tractrix**, has the following property: the tangent line at each point (x, y) on the curve has slope

$$\frac{dy}{dx} = \frac{-y}{\sqrt{1 - y^2}}$$

Let R be the shaded region under the graph of $0 \le x \le a$ in Figure 18. Compute the volume V of the solid obtained by revolving R around the x-axis in terms of the constant $c = f(a)$. *Hint:* Use the substitution $u = f(x)$ to show that

$$V = \pi \int_c^1 u\sqrt{1 - u^2}\, du$$

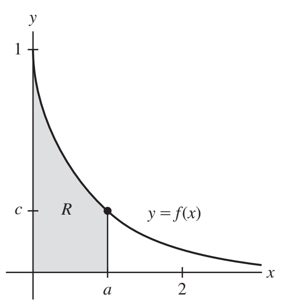


FIGURE 18 The tractrix.

SOLUTION Let $y = f(x)$ be the tractrix depicted in Figure 18. Rotating the region R about the x-axis produces a solid whose cross sections are disks with radius $f(x)$. The volume of the resulting solid is then

$$V = \pi \int_0^a [f(x)]^2\, dx.$$

Now, let $u = f(x)$. Then

$$du = f'(x)\,dx = \frac{-f(x)}{\sqrt{1-[f(x)]^2}}\,dx = \frac{-u}{\sqrt{1-u^2}}\,dx;$$

hence,

$$dx = -\frac{\sqrt{1-u^2}}{u}\,du,$$

and

$$V = \pi \int_1^c u^2 \left(-\frac{\sqrt{1-u^2}}{u}\,du\right) = \pi \int_c^1 u\sqrt{1-u^2}\,du.$$

Carrying out the integration, we find

$$V = -\frac{\pi}{3}(1-u^2)^{3/2}\Big|_c^1 = \frac{\pi}{3}(1-c^2)^{3/2}.$$

63. Verify the formula

$$\int_{x_1}^{x_2} (x-x_1)(x-x_2)\,dx = \frac{1}{6}(x_1-x_2)^3 \qquad \boxed{3}$$

Then prove that the solid obtained by rotating the shaded region in Figure 19 about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B and H as in the figure. *Hint:* Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx+c)^2$, where $x_1 < x_2$. Show that

$$V = \pi \int_{x_1}^{x_2} f(x)\,dx$$

and use Eq. (3).

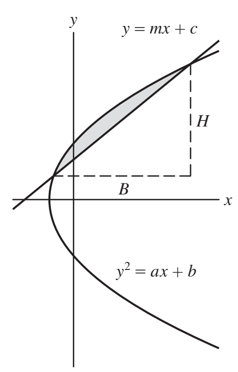


FIGURE 19 The line $y = mx + c$ intersects the parabola $y^2 = ax + b$ at two points above the x-axis.

SOLUTION First, we calculate

$$\int_{x_1}^{x_2} (x-x_1)(x-x_2)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{2}(x_1+x_2)x^2 + x_1x_2x\right)\Big|_{x_1}^{x_2} = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1x_2^2 - \frac{1}{6}x_2^3$$
$$= \frac{1}{6}\left(x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 - x_2^3\right) = \frac{1}{6}(x_1-x_2)^3.$$

Now, consider the region enclosed by the parabola $y^2 = ax + b$ and the line $y = mx + c$, and let x_1 and x_2 denote the x-coordinates of the points of intersection between the two curves with $x_1 < x_2$. Rotating the region about the y-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{ax+b}$ and inner radius $r = mx + c$. The volume of the solid of revolution is then

$$V = \pi \int_{x_1}^{x_2} \left(ax + b - (mx+c)^2\right)\,dx$$

Because x_1 and x_2 are roots of the equation $ax + b - (mx + c)^2 = 0$ and $ax + b - (mx + c)^2$ is a quadratic polynomial in x with leading coefficient $-m^2$, it follows that $ax + b - (mx + c)^2 = -m^2(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi m^2 \int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{\pi}{6} m^2 (x_2 - x_1)^3,$$

where we have used Eq. (3). From the diagram, we see that

$$B = x_2 - x_1 \qquad \text{and} \qquad H = mB,$$

so

$$V = \frac{\pi}{6} m^2 B^3 = \frac{\pi}{6} B\,(mB)^2 = \frac{\pi}{6} BH^2.$$

64. Let R be the region in the unit circle lying above the cut with the line $y = mx + b$ (Figure 20). Assume the points where the line intersects the circle lie above the x-axis. Use the method of Exercise 63 to show that the solid obtained by rotating R about the x-axis has volume $V = \frac{\pi}{6} hd^2$, with h and d as in the figure.

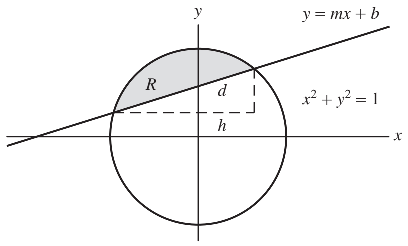


FIGURE 20

SOLUTION Let x_1 and x_2 denote the x-coordinates of the points of intersection between the circle $x^2 + y^2 = 1$ and the line $y = mx + b$ with $x_1 < x_2$. Rotating the region enclosed by the two curves about the x-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{1 - x^2}$ and inner radius $r = mx + b$. The volume of the resulting solid is then

$$V = \pi \int_{x_1}^{x_2} \left((1 - x^2) - (mx + b)^2 \right) dx$$

Because x_1 and x_2 are roots of the equation $(1 - x^2) - (mx + b)^2 = 0$ and $(1 - x^2) - (mx + b)^2$ is a quadratic polynomial in x with leading coefficient $-(1 + m^2)$, it follows that $(1 - x^2) - (mx + b)^2 = -(1 + m^2)(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi(1 + m^2) \int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{\pi}{6}(1 + m^2)(x_2 - x_1)^3.$$

From the diagram, we see that $h = x_2 - x_1$. Moreover, by the Pythagorean theorem, $d^2 = h^2 + (mh)^2 = (1 + m^2)h^2$. Thus,

$$V = \frac{\pi}{6}(1 + m^2)h^3 = \frac{\pi}{6} h \left[(1 + m^2)h^2 \right] = \frac{\pi}{6} hd^2.$$

6.4 The Method of Cylindrical Shells

Preliminary Questions

1. Consider the region $\mathcal{R}$ under the graph of the constant function $f(x) = h$ over the interval $[0, r]$. Give the height and the radius of the cylinder generated when $\mathcal{R}$ is rotated about:

(a) the x-axis **(b)** the y-axis

SOLUTION

(a) When the region is rotated about the x-axis, each shell will have radius h and height r.
(b) When the region is rotated about the y-axis, each shell will have radius r and height h.

2. Let V be the volume of a solid of revolution about the y-axis.

(a) Does the Shell Method for computing V lead to an integral with respect to x or y?
(b) Does the Disk or Washer Method for computing V lead to an integral with respect to x or y?

SOLUTION

(a) The Shell method requires slicing the solid parallel to the axis of rotation. In this case, that will mean slicing the solid in the vertical direction, so integration will be with respect to x.
(b) The Disk or Washer method requires slicing the solid perpendicular to the axis of rotation. In this case, that means slicing the solid in the horizontal direction, so integration will be with respect to y.

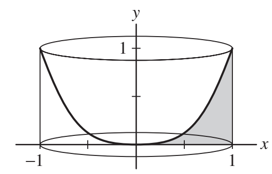
Exercises

In Exercises 1–6, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y-axis, and find its volume.

1. $f(x) = x^3$, $[0, 1]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height x^3, so the volume of the solid is

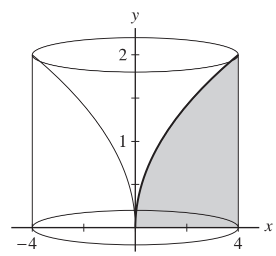
$$2\pi \int_0^1 x \cdot x^3\, dx = 2\pi \int_0^1 x^4\, dx = 2\pi \left(\frac{1}{5}x^5\right)\Big|_0^1 = \frac{2}{5}\pi.$$



2. $f(x) = \sqrt{x}$, $[0, 4]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\sqrt{x}$, so the volume of the solid is

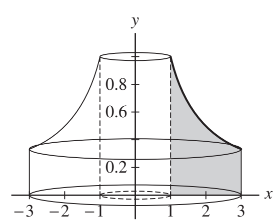
$$2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left(\frac{2}{5}x^{5/2}\right)\Big|_0^4 = \frac{128}{5}\pi.$$



3. $f(x) = x^{-1}$, $[1, 3]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height x^{-1}, so the volume of the solid is

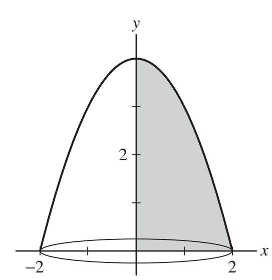
$$2\pi \int_1^3 x(x^{-1})\, dx = 2\pi \int_1^3 1\, dx = 2\pi\ (x)\Big|_1^3 = 4\pi.$$



4. $f(x) = 4 - x^2$, $[0, 2]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $4 - x^2$, so the volume of the solid is

$$2\pi \int_0^2 x(4 - x^2)\, dx = 2\pi \int_0^2 (4x - x^3)\, dx = 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 8\pi.$$



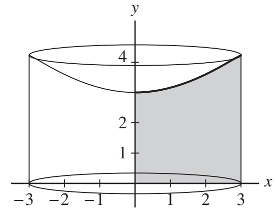
5. $f(x) = \sqrt{x^2 + 9}$, $[0, 3]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\sqrt{x^2 + 9}$, so the volume of the solid is

$$2\pi \int_0^3 x\sqrt{x^2 + 9}\,dx.$$

Let $u = x^2 + 9$. Then $du = 2x\,dx$ and

$$2\pi \int_0^3 x\sqrt{x^2 + 9}\,dx = \pi \int_9^{18} \sqrt{u}\,du = \pi \left(\frac{2}{3}u^{3/2}\right)\Big|_9^{18} = 18\pi(2\sqrt{2} - 1).$$



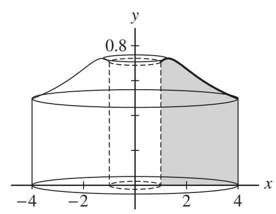
6. $f(x) = \dfrac{x}{\sqrt{1 + x^3}}$, $[1, 4]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\dfrac{x}{\sqrt{1 + x^3}}$, so the volume of the solid is

$$2\pi \int_1^4 x\left(\frac{x}{\sqrt{1 + x^3}}\right) dx = 2\pi \int_1^4 \frac{x^2}{\sqrt{1 + x^3}}\,dx.$$

Let $u = 1 + x^3$. Then $du = 3x^2\,dx$ and

$$2\pi \int_1^4 \frac{x^2}{\sqrt{1 + x^3}}\,dx = \frac{2}{3}\pi \int_2^{65} u^{-1/2}\,du = \frac{2}{3}\pi \left(2u^{1/2}\right)\Big|_2^{65} = \frac{4\pi}{3}\left(\sqrt{65} - \sqrt{2}\right).$$

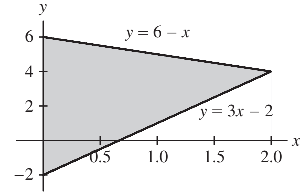


In Exercises 7–12, use the Shell Method to compute the volume obtained by rotating the region enclosed by the graphs as indicated, about the y-axis.

7. $y = 3x - 2$, $y = 6 - x$, $x = 0$

SOLUTION The region enclosed by $y = 3x - 2$, $y = 6 - x$ and $x = 0$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $6 - x - (3x - 2) = 8 - 4x$. The volume of the resulting solid is

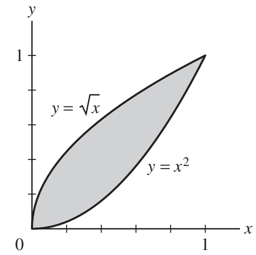
$$2\pi \int_0^2 x(8 - 4x)\,dx = 2\pi \int_0^2 (8x - 4x^2)\,dx = 2\pi \left(4x^2 - \frac{4}{3}x^3\right)\Big|_0^2 = \frac{32}{3}\pi.$$



8. $y = \sqrt{x}, \quad y = x^2$

SOLUTION The region enclosed by $y = \sqrt{x}$ and $y = x^2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $\sqrt{x} - x^2$. The volume of the resulting solid is

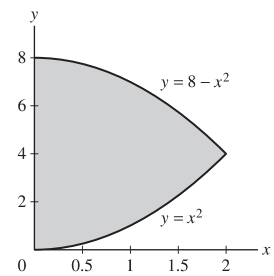
$$2\pi \int_0^1 x(\sqrt{x} - x^2)\, dx = 2\pi \int_0^1 (x^{3/2} - x^3)\, dx = 2\pi \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \right)\Big|_0^1 = \frac{3}{10}\pi.$$



9. $y = x^2, \quad y = 8 - x^2, \quad x = 0, \quad$ for $x \geq 0$

SOLUTION The region enclosed by $y = x^2$, $y = 8 - x^2$ and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius x and height $8 - x^2 - x^2 = 8 - 2x^2$. The volume of the resulting solid is

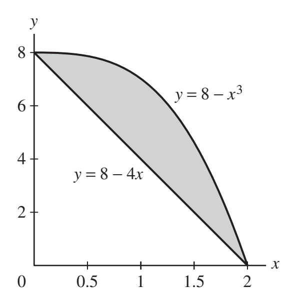
$$2\pi \int_0^2 x(8 - 2x^2)\, dx = 2\pi \int_0^2 (8x - 2x^3)\, dx = 2\pi \left(4x^2 - \frac{1}{2}x^4 \right)\Big|_0^2 = 16\pi.$$



10. $y = 8 - x^3, \quad y = 8 - 4x, \quad$ for $x \geq 0$

SOLUTION The region enclosed by $y = 8 - x^3$ and $y = 8 - 4x$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $(8 - x^3) - (8 - 4x) = 4x - x^3$. The volume of the resulting solid is

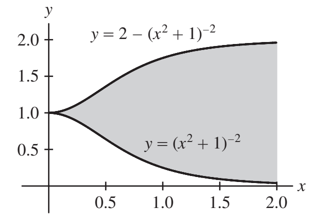
$$2\pi \int_0^2 x(4x - x^3)\, dx = 2\pi \int_0^2 (4x^2 - x^4)\, dx = 2\pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \right)\Big|_0^2 = \frac{128\pi}{15}.$$



11. $y = (x^2 + 1)^{-2}, \quad y = 2 - (x^2 + 1)^{-2}, \quad x = 2$

SOLUTION The region enclosed by $y = (x^2 + 1)^{-2}$, $y = 2 - (x^2 + 1)^{-2}$ and $x = 2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $2 - (x^2 + 1)^{-2} - (x^2 + 1)^{-2} = 2 - 2(x^2 + 1)^{-2}$. The volume of the resulting solid is

$$2\pi \int_0^2 x(2 - 2(x^2 + 1)^{-2})\, dx = 2\pi \int_0^2 \left(2x - \frac{2x}{(x^2 + 1)^{-2}} \right) dx = 2\pi \left(x^2 + \frac{1}{x^2 + 1} \right)\Big|_0^2 = \frac{32}{5}\pi.$$

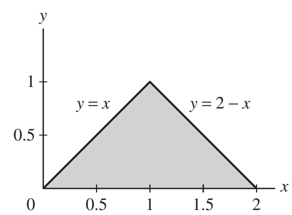


12. $y = 1 - |x - 1|, \quad y = 0$

SOLUTION The region enclosed by $y = 1 - |x - 1|$ and the x-axis is shown below. When rotating this region about the y-axis, two different shells are generated. For each $x \in [0, 1]$, the shell has radius x and height x; for each $x \in [1, 2]$, the shell has radius x and height $2 - x$. The volume of the resulting solid is

$$2\pi \int_0^1 x(x)\,dx + 2\pi \int_1^2 x(2-x)\,dx = 2\pi \int_0^1 (x^2)\,dx + 2\pi \int_1^2 (2x - x^2)\,dx$$

$$= 2\pi \left(\frac{1}{3}x^3\right)\Big|_0^1 + 2\pi \left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2 = 2\pi.$$

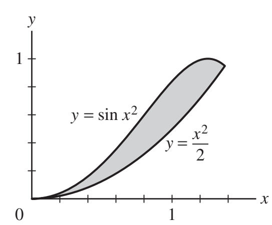


In Exercises 13 and 14, use a graphing utility to find the points of intersection of the curves numerically and then compute the volume of rotation of the enclosed region about the y-axis.

13. GU $y = \frac{1}{2}x^2, \quad y = \sin(x^2)$

SOLUTION The region enclosed by $y = \frac{1}{2}x^2$ and $y = \sin x^2$ is shown below. When rotating this region about the y-axis, each shell has radius x and height $\sin x^2 - \frac{1}{2}x^2$. Using a computer algebra system, we find that the x-coordinate of the point of intersection on the right is $x = 1.376769504$. Thus, the volume of the resulting solid of revolution is

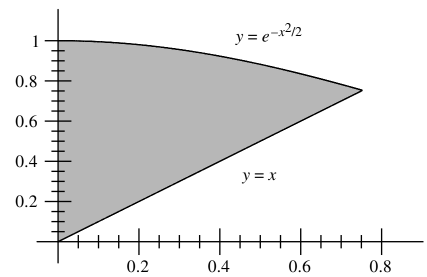
$$2\pi \int_0^{1.376769504} x\left(\sin x^2 - \frac{1}{2}x^2\right)\,dx = 1.321975576.$$



14. GU $y = e^{-x^2/2}, \quad y = x, \quad x = 0$

SOLUTION The region enclosed by $y = e^{-x^2/2}$, $y = x$ and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius x and height $e^{-x^2/2} - x$. Using a computer algebra system, we find that the x-coordinate of the point of intersection on the right is $x = 0.7530891650$. Thus, volume of the resulting solid of revolution is

$$2\pi \int_0^{0.7530891650} x(e^{-x^2/2} - x)\,dx = 0.6568505551$$

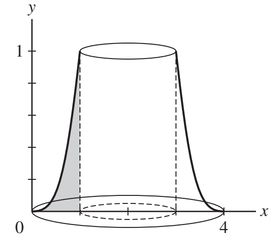


In Exercises 15–20, sketch the solid obtained by rotating the region underneath the graph of $f(x)$ over the interval about the given axis, and calculate its volume using the Shell Method.

15. $f(x) = x^3$, $[0, 1]$, about $x = 2$

SOLUTION A sketch of the solid is shown below. Each shell has radius $2 - x$ and height x^3, so the volume of the solid is

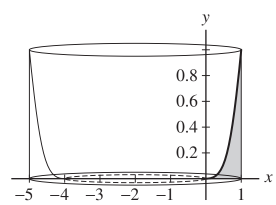
$$2\pi \int_0^1 (2-x)\left(x^3\right) dx = 2\pi \int_0^1 (2x^3 - x^4)\, dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{3\pi}{5}.$$



16. $f(x) = x^3$, $[0, 1]$, about $x = -2$

SOLUTION A sketch of the solid is shown below. Each shell has radius $x - (-2) = x + 2$ and height x^3, so the volume of the solid is

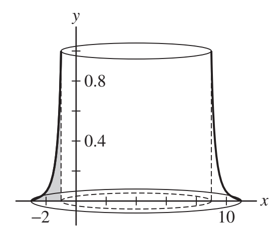
$$2\pi \int_0^1 (2+x)\left(x^3\right) dx = 2\pi \int_0^1 (2x^3 + x^4)\, dx = 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{7\pi}{5}.$$



17. $f(x) = x^{-4}$, $[-3, -1]$, about $x = 4$

SOLUTION A sketch of the solid is shown below. Each shell has radius $4 - x$ and height x^{-4}, so the volume of the solid is

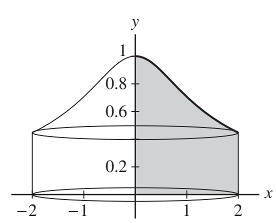
$$2\pi \int_{-3}^{-1} (4-x)\left(x^{-4}\right) dx = 2\pi \int_{-3}^{-1} (4x^{-4} - x^{-3})\, dx = 2\pi \left(\frac{1}{2}x^{-2} - \frac{4}{3}x^{-3}\right)\Big|_{-3}^{-1} = \frac{280\pi}{81}.$$



18. $f(x) = \dfrac{1}{\sqrt{x^2+1}}$, $[0, 2]$, about $x = 0$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\dfrac{1}{\sqrt{x^2+1}}$, so the volume of the solid is

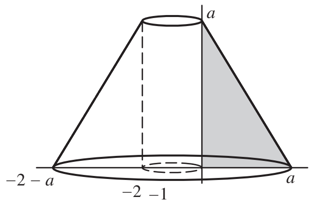
$$2\pi \int_0^2 x\left(\frac{1}{\sqrt{x^2+1}}\right) dx = 2\pi\left(\sqrt{x^2+1}\right)\Big|_0^2 = 2\pi(\sqrt{5} - 1).$$



19. $f(x) = a - x$ with $a > 0$, $[0, a]$, about $x = -1$

SOLUTION A sketch of the solid is shown below. Each shell has radius $x - (-1) = x + 1$ and height $a - x$, so the volume of the solid is

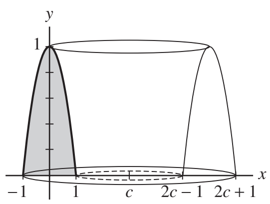
$$\begin{aligned} 2\pi \int_0^a (x+1)(a-x)\,dx &= 2\pi \int_0^a \left(a + (a-1)x - x^2\right) dx \\ &= 2\pi \left(ax + \frac{a-1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^a \\ &= 2\pi \left(a^2 + \frac{a^2(a-1)}{2} - \frac{a^3}{3}\right) = \frac{a^2(a+3)}{3}\pi. \end{aligned}$$



20. $f(x) = 1 - x^2$, $[-1, 1]$, $x = c$ with $c > 1$

SOLUTION A sketch of the solid is shown below. Each shell has radius $c - x$ and height $1 - x^2$, so the volume of the solid is

$$2\pi \int_{-1}^1 (c-x)\left(1-x^2\right) dx = 2\pi \int_{-1}^1 \left(x^3 - cx^2 - x + c\right) dx = 2\pi \left(\frac{1}{4}x^4 - \frac{c}{3}x^3 - \frac{1}{2}x^2 + cx\right)\Big|_{-1}^1 = \frac{8c\pi}{3}.$$

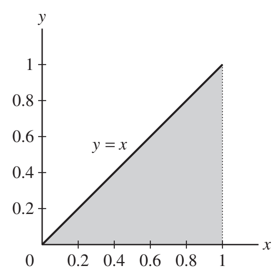


In Exercises 21–26, sketch the enclosed region and use the Shell Method to calculate the volume of rotation about the x-axis.

21. $x = y$, $y = 0$, $x = 1$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $1 - y$. The volume of the resulting solid is

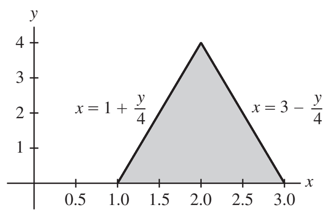
$$2\pi \int_0^1 y(1-y)\,dy = 2\pi \int_0^1 (y - y^2)\,dy = 2\pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1 = \frac{\pi}{3}.$$



22. $x = \frac{1}{4}y + 1$, $x = 3 - \frac{1}{4}y$, $y = 0$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $2 - \frac{1}{2}y$. The volume of the resulting solid is

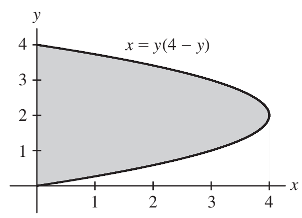
$$2\pi \int_0^4 y\left(2 - \frac{1}{2}y\right) dy = 2\pi \int_0^4 \left(2y - \frac{1}{2}y^2\right) dy = 2\pi \left(y^2 - \frac{1}{6}y^3\right)\Big|_0^4 = \frac{32\pi}{3}.$$



23. $x = y(4 - y), \quad y = 0$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $y(4 - y)$. The volume of the resulting solid is

$$2\pi \int_0^4 y^2(4 - y)\,dy = 2\pi \int_0^4 (4y^2 - y^3)\,dy = 2\pi \left(\frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^4 = \frac{128\pi}{3}.$$



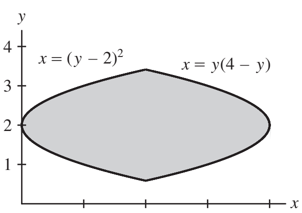
24. $x = y(4 - y), \quad x = (y - 2)^2$

SOLUTION Setting $y(4 - y) = (y - 2)^2$ yields

$$y^2 - 4y + 2 = 0 \quad \text{or} \quad y = 2 \pm \sqrt{2}.$$

When the region shown below is rotated about the x-axis, each shell has radius y and height $-2y^2 + 8y - 4$. The volume of the resulting solid is

$$2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} y(-2y^2 + 8y - 4)\,dy = 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (-2y^3 + 8y^2 - 4y)\,dy = 2\pi \left(-\frac{1}{2}y^4 + \frac{8}{3}y^3 - 2y^2\right)\Big|_{2-\sqrt{2}}^{2+\sqrt{2}} = \frac{64\pi\sqrt{2}}{3}.$$



25. $y = 4 - x^2, \quad x = 0, \quad y = 0$

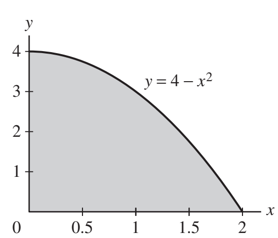
SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $\sqrt{4 - y}$. The volume of the resulting solid is

$$2\pi \int_0^4 y\sqrt{4 - y}\,dy.$$

Let $u = 4 - y$. Then $du = -dy$, $y = 4 - u$, and

$$2\pi \int_0^4 y\sqrt{4 - y}\,dy = -2\pi \int_4^0 (4 - u)\sqrt{u}\,du = 2\pi \int_0^4 \left(4\sqrt{u} - u^{3/2}\right)\,du$$

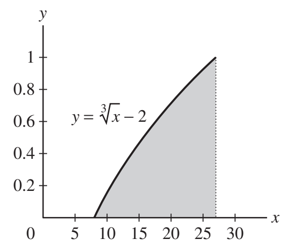
$$= 2\pi \left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^4 = \frac{256\pi}{15}.$$



26. $y = x^{1/3} - 2, \quad y = 0, \quad x = 27$

SOLUTION When the region shown below is rotated about the x-axis, each shell has radius y and height $27 - (y+2)^3$. The volume of the resulting solid is

$$2\pi \int_0^1 y \cdot \left(27 - (y+2)^3\right) dy = 2\pi \int_0^1 \left(19y - 12y^2 - 6y^3 - y^4\right) dy$$
$$= 2\pi \left(\frac{19}{2}y^2 - 4y^3 - \frac{3}{2}y^4 - \frac{1}{5}y^5\right)\Big|_0^1 = \frac{38\pi}{5}.$$



27. Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \le x \le 2$ about:

(a) the x-axis

(b) the y-axis

SOLUTION

(a) x-axis: Using the disk method, the cross sections are disks with radius $R = 8 - x^3$; hence the volume of the solid is

$$\pi \int_0^2 (8 - x^3)^2\, dx = \pi \left(64x - 4x^4 + \frac{1}{7}x^7\right)\Big|_0^2 = \frac{576\pi}{7}.$$

With the shell method, each shell has radius y and height $(8-y)^{1/3}$. The volume of the solid is

$$2\pi \int_0^8 y\,(8-y)^{1/3}\, dy$$

Let $u = 8 - y$. Then $dy = -du$, $y = 8 - u$ and

$$2\pi \int_0^8 y\,(8-y)^{1/3}\, dy = 2\pi \int_0^8 (8-u) \cdot u^{1/3}\, du = 2\pi \int_0^8 (8u^{1/3} - u^{4/3})\, du$$
$$= 2\pi \left(6u^{4/3} - \frac{3}{7}u^{7/3}\right)\Big|_0^8 = \frac{576\pi}{7}.$$

(b) y-axis: With the shell method, each shell has radius x and height $8 - x^3$. The volume of the solid is

$$2\pi \int_0^2 x(8 - x^3)\, dx = 2\pi \left(4x^2 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{96\pi}{5}.$$

Using the disk method, the cross sections are disks with radius $R = (8-y)^{1/3}$. The volume is then given by

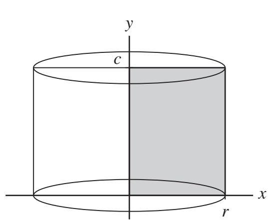
$$\pi \int_0^8 (8-y)^{2/3}\, dy = -\frac{3\pi}{5}(8-y)^{5/3}\Big|_0^8 = \frac{96\pi}{5}.$$

28. Sketch the solid of rotation about the y-axis for the region under the graph of the constant function $f(x) = c$ (where $c > 0$) for $0 \le x \le r$.

(a) Find the volume without using integration.

(b) Use the Shell Method to compute the volume.

SOLUTION



(a) The solid is simply a cylinder with height c and radius r. The volume is given by $\pi r^2 c$.
(b) Each shell has radius x and height c, so the volume is

$$2\pi \int_0^r cx\,dx = 2\pi \left(c\frac{1}{2}x^2 \right)\Big|_0^r = \pi r^2 c.$$

29. The graph in Figure 11(A) can be described by both $y = f(x)$ and $x = h(y)$, where h is the inverse of f. Let V be the volume obtained by rotating the region under the graph about the y-axis.
(a) Describe the figures generated by rotating segments $\overline{AB}$ and $\overline{CB}$ about the y-axis.
(b) Set up integrals that compute V by the Shell and Disk Methods.

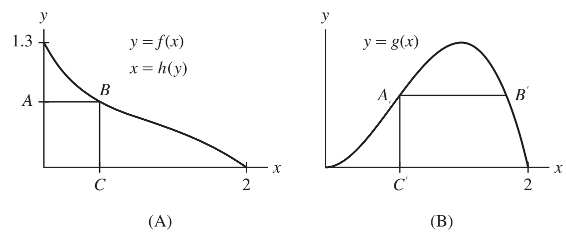


FIGURE 11

SOLUTION

(a) When rotated about the y-axis, the segment $\overline{AB}$ generates a disk with radius $R = h(y)$ and the segment $\overline{CB}$ generates a shell with radius x and height $f(x)$.
(b) Based on Figure 11(A) and the information from part (a), when using the Shell Method,

$$V = 2\pi \int_0^2 xf(x)\,dx;$$

when using the Disk Method,

$$V = \pi \int_0^{1.3} (h(y))^2\,dy.$$

30. Let W be the volume of the solid obtained by rotating the region under the graph in Figure 11(B) about the y-axis.
(a) Describe the figures generated by rotating segments $\overline{A'B'}$ and $\overline{A'C'}$ about the y-axis.
(b) Set up an integral that computes W by the Shell Method.
(c) Explain the difficulty in computing W by the Washer Method.

SOLUTION

(a) When rotated about the y-axis, the segment $\overline{A'B'}$ generates a washer and the segment $\overline{C'A'}$ generates a shell with radius x and height $g(x)$.
(b) Using Figure 11(B) and the information from part (a),

$$W = 2\pi \int_0^2 xg(x)\,dx.$$

(c) The function $g(x)$ is not one-to-one, which makes it difficult to determine the inner and outer radius of each washer.

31. Let R be the region under the graph of $y = 9 - x^2$ for $0 \le x \le 2$. Use the Shell Method to compute the volume of rotation of R about the x-axis as a sum of two integrals along the y-axis. *Hint:* The shells generated depend on whether $y \in [0, 5]$ or $y \in [5, 9]$.

SOLUTION The region R is sketched below. When rotating this region about the x-axis, we produce a solid with two different shell structures. For $0 \le y \le 5$, the shell has radius y and height 2; for $5 \le y \le 9$, the shell has radius y and height $\sqrt{9 - y}$. The volume of the solid is therefore

$$V = 2\pi \int_0^5 2y\,dy + 2\pi \int_5^9 y\sqrt{9 - y}\,dy$$

For the first integral, we calculate

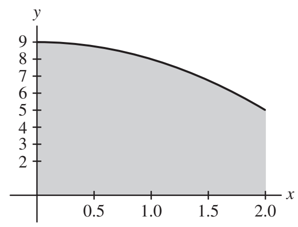
$$2\pi \int_0^5 2y\,dy = 2\pi y^2 \Big|_0^5 = 50\pi.$$

For the second integral, we make the substitution $u = 9 - y$, $du = -dy$ and find

$$\begin{aligned} 2\pi \int_5^9 y\sqrt{9-y}\,dy &= -2\pi \int_4^0 (9-u)\sqrt{u}\,du \\ &= 2\pi \int_0^4 (9u^{1/2} - u^{3/2})\,du \\ &= 2\pi \left(6u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^4 \\ &= 2\pi \left(48 - \frac{64}{5}\right) = \frac{352\pi}{5}. \end{aligned}$$

Thus, the total volume is

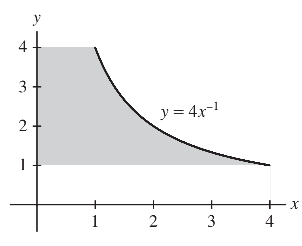
$$V = 50\pi + \frac{352\pi}{5} = \frac{602\pi}{5}.$$



32. Let R be the region under the graph of $y = 4x^{-1}$ for $1 \le y \le 4$. Use the Shell Method to compute the volume of rotation of R about the y-axis as a sum of two integrals along the x-axis.

SOLUTION The region R is sketched below. When rotating this region about the y-axis, we produce a solid with two different shell structures. For $0 \le x \le 1$, the shell has radius x and height 3; for $1 \le x \le 4$, the shell has radius x and height $4x^{-1} - 1$. The volume of the solid is therefore

$$\begin{aligned} V &= 2\pi \int_0^1 3x\,dx + 2\pi \int_1^4 x(4x^{-1} - 1)\,dx \\ &= 2\pi \int_0^1 3x\,dx + 2\pi \int_1^4 (4 - x)\,dx \\ &= 2\pi\, \frac{3}{2}x^2\Big|_0^1 + 2\pi \left(4x - \frac{1}{2}x^2\right)\Big|_1^4 \\ &= 3\pi + 2\pi\left(8 - \frac{7}{2}\right) = 12\pi. \end{aligned}$$



In Exercises 33–38, use the Shell Method to find the volume obtained by rotating region A in Figure 12 about the given axis.

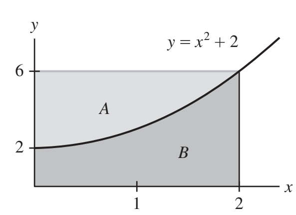


FIGURE 12

33. y-axis

SOLUTION When rotating region A about the y-axis, each shell has radius x and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 x(4-x^2)\,dx = 2\pi\int_0^2 (4x-x^3)\,dx = 2\pi\left(2x^2-\frac{1}{4}x^4\right)\Big|_0^2 = 8\pi.$$

34. $x=-3$

SOLUTION When rotating region A about $x=-3$, each shell has radius $x-(-3)=x+3$ and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 (x+3)(4-x^2)\,dx = 2\pi\int_0^2 (4x-x^3+12-3x^2)\,dx = 2\pi\left(2x^2-\frac{1}{4}x^4+12x-x^3\right)\Big|_0^2 = 40\pi.$$

35. $x=2$

SOLUTION When rotating region A about $x=2$, each shell has radius $2-x$ and height $6-(x^2+2)=4-x^2$. The volume of the resulting solid is

$$2\pi\int_0^2 (2-x)\left(4-x^2\right)\,dx = 2\pi\int_0^2 \left(8-2x^2-4x+x^3\right)\,dx = 2\pi\left(8x-\frac{2}{3}x^3-2x^2+\frac{1}{4}x^4\right)\Big|_0^2 = \frac{40\pi}{3}.$$

36. x-axis

SOLUTION When rotating region A about the x-axis, each shell has radius y and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 y\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $y=u+2$ and

$$2\pi\int_2^6 y\sqrt{y-2}\,dy = 2\pi\int_0^4 (u+2)\sqrt{u}\,du = 2\pi\left(\frac{2}{5}u^{5/2}+\frac{4}{3}u^{3/2}\right)\Big|_0^4 = \frac{704\pi}{15}.$$

37. $y=-2$

SOLUTION When rotating region A about $y=-2$, each shell has radius $y-(-2)=y+2$ and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 (y+2)\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $y+2=u+4$ and

$$2\pi\int_2^6 (y+2)\sqrt{y-2}\,dy = 2\pi\int_0^4 (u+4)\sqrt{u}\,du = 2\pi\left(\frac{2}{5}u^{5/2}+\frac{8}{3}u^{3/2}\right)\Big|_0^4 = \frac{1024\pi}{15}.$$

38. $y=6$

SOLUTION When rotating region A about $y=6$, each shell has radius $6-y$ and height $\sqrt{y-2}$. The volume of the resulting solid is

$$2\pi\int_2^6 (6-y)\sqrt{y-2}\,dy$$

Let $u=y-2$. Then $du=dy$, $6-y=4-u$ and

$$2\pi\int_2^6 (6-y)\sqrt{y-2}\,dy = 2\pi\int_0^4 (4-u)\sqrt{u}\,du = 2\pi\left(\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)\Big|_0^4 = \frac{256\pi}{15}.$$

In Exercises 39–44, use the most convenient method (Disk or Shell Method) to find the volume obtained by rotating region B in Figure 12 about the given axis.

39. y-axis

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius x and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 x(x^2+2)\,dx = 2\pi \int_0^2 (x^3+2x)\,dx = 2\pi \left(\frac{1}{4}x^4 + x^2\right)\Big|_0^2 = 16\pi.$$

40. $x = -3$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius $x - (-3) = x + 3$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (x+3)(x^2+2)\,dx = 2\pi \int_0^2 (x^3+3x^2+2x+6)\,dx = 2\pi \left(\frac{1}{4}x^4 + x^3 + x^2 + 6x\right)\Big|_0^2 = 56\pi.$$

41. $x = 2$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is parallel to the axis of rotation, we will use the Shell Method. Each shell has radius $2 - x$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (2-x)\left(x^2+2\right)\,dx = 2\pi \int_0^2 \left(2x^2 - x^3 + 4 - 2x\right)\,dx = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 + 4x - x^2\right)\Big|_0^2 = \frac{32\pi}{3}.$$

42. x-axis

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = x^2 + 2$ and inner radius $r = 0$. The volume of the solid is then

$$\begin{aligned}\pi \int_0^2 (x^2+2)^2\,dx &= \pi \int_0^2 (x^4+4x^2+4)\,dx \\ &= \pi \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right)\Big|_0^2 \\ &= \pi\left(\frac{32}{5} + \frac{32}{3} + 8\right) = \frac{376\pi}{15}.\end{aligned}$$

43. $y = -2$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = x^2 + 2 - (-2) = x^2 + 4$ and inner radius $r = 0 - (-2) = 2$. The volume of the solid is then

$$\begin{aligned}\pi \int_0^2 \left((x^2+4)^2 - 2^2\right)\,dx &= \pi \int_0^2 (x^4+8x^2+12)\,dx \\ &= \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Big|_0^2 \\ &= \pi\left(\frac{32}{5} + \frac{64}{3} + 24\right) = \frac{776\pi}{15}.\end{aligned}$$

44. $y = 8$

SOLUTION Because a vertical slice of region B will produce a solid with a single cross section while a horizontal slice will produce a solid with two different cross sections, we will use a vertical slice. Now, because a vertical slice is perpendicular to the axis of rotation, we will use the Disk Method. Each disk has outer radius $R = 8 - 0 = 8$ and inner radius $r = 8 - (x^2 + 2) = 6 - x^2$. The volume of the solid is then

$$\pi \int_0^2 \left(8^2 - (6-x^2)^2\right) dx = \pi \int_0^2 (28 + 12x^2 - x^4)\, dx$$
$$= \pi \left(28x + 4x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$
$$= \pi \left(56 + 32 - \frac{32}{5}\right) = \frac{408\pi}{5}.$$

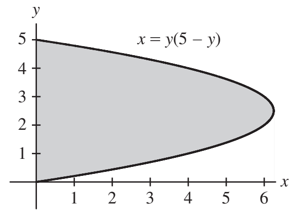
In Exercises 45–50, use the most convenient method (Disk or Shell Method) to find the given volume of rotation.

45. Region between $x = y(5-y)$ and $x = 0$, rotated about the y-axis

SOLUTION Examine the figure below, which shows the region bounded by $x = y(5-y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

Now, suppose the region is rotated about the y-axis. Because a horizontal slice is perpendicular to the y-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = y(5-y)$, so the volume is

$$\pi \int_0^5 y^2(5-y)^2\, dy = \pi \int_0^5 (25y^2 - 10y^3 + y^4)\, dy = \pi \left(\frac{25}{3}y^3 - \frac{5}{2}y^4 + \frac{1}{5}y^5\right)\Big|_0^5 = \frac{625\pi}{6}.$$



46. Region between $x = y(5-y)$ and $x = 0$, rotated about the x-axis

SOLUTION Examine the figure from the previous exercise, which shows the region bounded by $x = y(5-y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

Now, suppose the region is rotated about the x-axis. Because a horizontal slice is parallel to the x-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of y and a height of $y(5-y)$, so the volume is

$$2\pi \int_0^5 y^2(5-y)\, dy = 2\pi \int_0^5 (5y^2 - y^3)\, dy = 2\pi \left(\frac{5}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^5 = \frac{625\pi}{6}.$$

47. Region in Figure 13, rotated about the x-axis

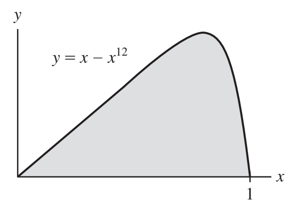


FIGURE 13

SOLUTION Examine Figure 13. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x - x^{12}$ and the bottom lies along the curve $y = 0$ (the x-axis). On the other hand, if the region is sliced horizontally, the equation $y = x - x^{12}$ must be solved for x in order to determine the endpoint locations. Clearly, it will be easier to slice the region vertically.

Now, suppose the region in Figure 13 is rotated about the x-axis. Because a vertical slice is perpendicular to the x-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = x - x^{12}$, so the volume is

$$\pi \int_0^1 \left(x - x^{12}\right)^2 dx = \pi \left(\frac{1}{3}x^3 - \frac{1}{7}x^{14} + \frac{1}{25}x^{25}\right)\Big|_0^1 = \frac{121\pi}{525}.$$

48. Region in Figure 13, rotated about the y-axis

SOLUTION Examine Figure 13. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x - x^{12}$ and the bottom lies along the curve $y = 0$ (the x-axis). On the other hand, if the region is sliced horizontally, the equation $y = x - x^{12}$ must be solved for x in order to determine the endpoint locations. Clearly, it will be easier to slice the region vertically.

Now suppose the region is rotated about the y-axis. Because a vertical slice is parallel to the y-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has radius x and height $x - x^{12}$, so the volume is

$$2\pi \int_0^1 x(x - x^{12})\,dx = 2\pi \left(\frac{1}{3}x^3 - \frac{1}{14}x^{14}\right)\Big|_0^1 = \frac{11\pi}{21}.$$

49. Region in Figure 14, rotated about $x = 4$

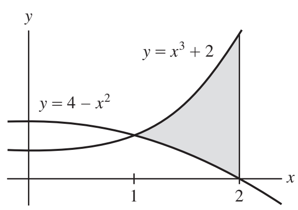


FIGURE 14

SOLUTION Examine Figure 14. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x^3 + 2$ and the bottom lies along the curve $y = 4 - x^2$. On the other hand, the left end of a horizontal slice switches from $y = 4 - x^2$ to $y = x^3 + 2$ at $y = 3$. Here, vertical slices will be more convenient.

Now, suppose the region in Figure 14 is rotated about $x = 4$. Because a vertical slice is parallel to $x = 4$, we will calculate the volume of the resulting solid using the shell method. Each shell has radius $4 - x$ and height $x^3 + 2 - (4 - x^2) = x^3 + x^2 - 2$, so the volume is

$$2\pi \int_1^2 (4 - x)(x^3 + x^2 - 2)\,dx = 2\pi \left(-\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + x^2 - 8x\right)\Big|_1^2 = \frac{563\pi}{30}.$$

50. Region in Figure 14, rotated about $y = -2$

SOLUTION Examine Figure 14. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x^3 + 2$ and the bottom lies along the curve $y = 4 - x^2$. On the other hand, the left end of a horizontal slice switches from $y = 4 - x^2$ to $y = x^3 + 2$ at $y = 3$. Here, vertical slices will be more convenient.

Now suppose the region is rotated about $y = -2$. Because a vertical slice is perpendicular to $y = -2$, we will calculate the volume of the resulting solid using the disk method. Each cross section is a washer with outer radius $R = x^3 + 2 - (-2) = x^3 + 4$ and inner radius $r = 4 - x^2 - (-2) = 6 - x^2$, so the volume is

$$\pi \int_1^2 \left((x^3 + 4)^2 - (6 - x^2)^2\right)\,dx = \pi \left(\frac{1}{7}x^7 - \frac{1}{5}x^5 + 2x^4 + 4x^3 - 20x\right)\Big|_1^2 = \frac{1748\pi}{35}.$$

In Exercises 51–54, use the Shell Method to find the given volume of rotation.

51. A sphere of radius r

SOLUTION A sphere of radius r can be generated by rotating the region under the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis. Each shell has radius y and height

$$\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.$$

Thus, the volume of the sphere is

$$2\pi \int_0^r 2y\sqrt{r^2 - y^2}\,dy.$$

Let $u = r^2 - y^2$. Then $du = -2y\,dy$ and

$$2\pi \int_0^r 2y\sqrt{r^2 - y^2}\,dy = 2\pi \int_0^{r^2} \sqrt{u}\,du = 2\pi \left(\frac{2}{3}u^{3/2}\right)\Big|_0^{r^2} = \frac{4}{3}\pi r^3.$$

52. The "bead" formed by removing a cylinder of radius r from the center of a sphere of radius R (compare with Exercise 59 in Section 6.3)

SOLUTION Each shell has radius x and height $2\sqrt{R^2-x^2}$. The volume of the bead is then

$$2\pi \int_r^R 2x\sqrt{R^2-x^2}\,dx.$$

Let $u = R^2 - x^2$. Then $du = -2x\,dx$ and

$$2\pi \int_r^R 2x\sqrt{R^2-x^2}\,dx = 2\pi \int_0^{R^2-r^2} \sqrt{u}\,du = 2\pi \left(\frac{2}{3}u^{3/2}\right)\Big|_0^{R^2-r^2} = \frac{4}{3}\pi(R^2-r^2)^{3/2}.$$

53. The torus obtained by rotating the circle $(x-a)^2+y^2=b^2$ about the y-axis, where $a>b$ (compare with Exercise 53 in Section 5.3). *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

SOLUTION When rotating the region enclosed by the circle $(x-a)^2+y^2=b^2$ about the y-axis each shell has radius x and height

$$\sqrt{b^2-(x-a)^2} - \left(-\sqrt{b^2-(x-a)^2}\right) = 2\sqrt{b^2-(x-a)^2}.$$

The volume of the resulting torus is then

$$2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2-(x-a)^2}\,dx.$$

Let $u = x - a$. Then $du = dx$, $x = u + a$ and

$$\begin{aligned} 2\pi \int_{a-b}^{a+b} 2x\sqrt{b^2-(x-a)^2}\,dx &= 2\pi \int_{-b}^{b} 2(u+a)\sqrt{b^2-u^2}\,du \\ &= 4\pi \int_{-b}^{b} u\sqrt{b^2-u^2}\,du + 4a\pi \int_{-b}^{b} \sqrt{b^2-u^2}\,du. \end{aligned}$$

Now,

$$\int_{-b}^{b} u\sqrt{b^2-u^2}\,du = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius b; thus,

$$\int_{-b}^{b} \sqrt{b^2-u^2}\,du = \frac{1}{2}\pi b^2.$$

Finally, the volume of the torus is

$$4\pi(0) + 4a\pi\left(\frac{1}{2}\pi b^2\right) = 2\pi^2 ab^2.$$

54. The "paraboloid" obtained by rotating the region between $y = x^2$ and $y = c$ ($c > 0$) about the y-axis

SOLUTION When we rotate the region in the first quadrant bounded by $y = x^2$ and $y = c$ about the y-axis, each shell has a radius of x and a height of $c - x^2$. The volume of the paraboloid is then

$$2\pi \int_0^{\sqrt{c}} x(c-x^2)\,dx = 2\pi \int_0^{\sqrt{c}} (cx - x^3)\,dx = 2\pi \left(\frac{1}{2}cx^2 - \frac{1}{4}x^4\right)\Big|_0^{\sqrt{c}} = \frac{1}{2}\pi c^2.$$

Further Insights and Challenges

55. The surface area of a sphere of radius r is $4\pi r^2$. Use this to derive the formula for the volume V of a sphere of radius R in a new way.

(a) Show that the volume of a thin spherical shell of inner radius r and thickness Δr is approximately $4\pi r^2 \Delta r$.

(b) Approximate V by decomposing the sphere of radius R into N thin spherical shells of thickness $\Delta r = R/N$.

(c) Show that the approximation is a Riemann sum that converges to an integral. Evaluate the integral.

SOLUTION

(a) The volume of a thin spherical shell of inner radius r and thickness Δx is given by the product of the surface area of the shell, $4\pi r^2$ and the thickness. Thus, we have $4\pi r^2 \Delta x$.

(b) The volume of the sphere is approximated by

$$R_N = 4\pi \left(\frac{R}{N}\right) \sum_{k=1}^{N} (x_k)^2$$

where $x_k = k\frac{R}{N}$.

(c) $V = 4\pi \lim_{N\to\infty} \left(\frac{R}{N}\right) \sum_{k=1}^{N} (x_k)^2 = 4\pi \int_0^R x^2\,dx = 4\pi \left(\frac{1}{3}x^3\right)\Big|_0^R = \frac{4}{3}\pi R^3.$

56. Show that the solid (an **ellipsoid**) obtained by rotating the region R in Figure 15 about the y-axis has volume $\frac{4}{3}\pi a^2 b$.

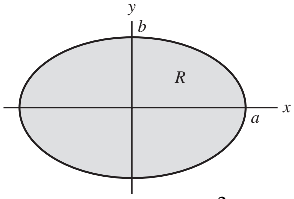


FIGURE 15 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION Let's slice the portion of the ellipse in the first and fourth quadrants horizontally and rotate the slices about the y-axis. The resulting ellipsoid has cross sections that are disks with radius

$$R = \sqrt{a^2 - \frac{a^2 y^2}{b^2}}.$$

Thus, the volume of the ellipsoid is

$$\pi \int_{-b}^{b} \left(a^2 - \frac{a^2 y^2}{b^2}\right) dy = \pi \left(a^2 y - \frac{a^2 y^3}{3b^2}\right)\Big|_{-b}^{b} = \pi \left[\left(a^2 b - \frac{a^2 b}{3}\right) - \left(-a^2 b + \frac{a^2 b}{3}\right)\right] = \frac{4}{3}\pi a^2 b.$$

57. The bell-shaped curve $y = f(x)$ in Figure 16 satisfies $dy/dx = -xy$. Use the Shell Method and the substitution $u = f(x)$ to show that the solid obtained by rotating the region R about the y-axis has volume $V = 2\pi(1 - c)$, where $c = f(a)$. Observe that as $c \to 0$, the region R becomes infinite but the volume V approaches 2π.

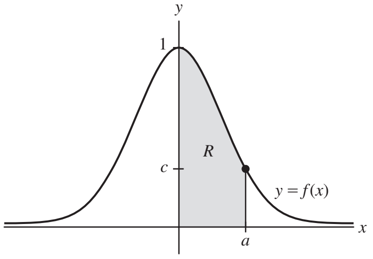


FIGURE 16 The bell-shaped curve.

SOLUTION Let $y = f(x)$ be the exponential function depicted in Figure 16. When rotating the region R about the y-axis, each shell in the resulting solid has radius x and height $f(x)$. The volume of the solid is then

$$V = 2\pi \int_0^a x f(x)\,dx.$$

Now, let $u = f(x)$. Then $du = f'(x)\,dx = -xf(x)\,dx$; hence, $xf(x)dx = -du$, and

$$V = 2\pi \int_1^c (-du) = 2\pi \int_c^1 du = 2\pi(1 - c).$$

6.5 Work and Energy

Preliminary Questions

1. Why is integration needed to compute the work performed in stretching a spring?

SOLUTION Recall that the force needed to extend or compress a spring depends on the amount by which the spring has already been extended or compressed from its equilibrium position. In other words, the force needed to move a spring is variable. Whenever the force is variable, work needs to be computed with an integral.

2. Why is integration needed to compute the work performed in pumping water out of a tank but not to compute the work performed in lifting up the tank?

SOLUTION To lift a tank through a vertical distance d, the force needed to move the tank remains constant; hence, no integral is needed to calculate the work done in lifting the tank. On the other hand, pumping water from a tank requires that different layers of the water be lifted through different distances, and, depending on the shape of the tank, may require different forces. Thus, pumping water from a tank requires that an integral be evaluated.

3. Which of the following represents the work required to stretch a spring (with spring constant k) a distance x beyond its equilibrium position: kx, $-kx$, $\frac{1}{2}mk^2$, $\frac{1}{2}kx^2$, or $\frac{1}{2}mx^2$?

SOLUTION The work required to stretch a spring with spring constant k a distance x beyond its equilibrium position is

$$\int_0^x ky\,dy = \frac{1}{2}ky^2\Big|_0^x = \frac{1}{2}kx^2.$$

Exercises

1. How much work is done raising a 4-kg mass to a height of 16 m above ground?

SOLUTION The force needed to lift a 4-kg object is a constant

$$(4\text{ kg})(9.8\text{ m/s}^2) = 39.2\text{ N}.$$

The work done in lifting the object to a height of 16 m is then

$$(39.2\text{ N})(16\text{ m}) = 627.2\text{ J}.$$

2. How much work is done raising a 4-lb mass to a height of 16 ft above ground?

SOLUTION The force needed to lift a 4-lb object is a constant 4 lb. The work done in lifting the object to a height of 16 ft is then

$$(4\text{ lb})(16\text{ ft}) = 64\text{ ft-lb}.$$

In Exercises 3–6, compute the work (in joules) required to stretch or compress a spring as indicated, assuming a spring constant of $k = 800$ N/m.

3. Stretching from equilibrium to 12 cm past equilibrium

SOLUTION The work required to stretch the spring 12 cm past equilibrium is

$$\int_0^{0.12} 800x\,dx = 400x^2\Big|_0^{0.12} = 5.76\text{ J}.$$

4. Compressing from equilibrium to 4 cm past equilibrium

SOLUTION The work required to compress the spring 4 cm past equilibrium is

$$\int_0^{-0.04} 800x\,dx = 400x^2\Big|_0^{-0.04} = 0.64\text{ J}.$$

5. Stretching from 5 cm to 15 cm past equilibrium

SOLUTION The work required to stretch the spring from 5 cm to 15 cm past equilibrium is

$$\int_{0.05}^{0.15} 800x\,dx = 400x^2\Big|_{0.05}^{0.15} = 8\text{ J}.$$

6. Compressing 4 cm more when it is already compressed 5 cm

SOLUTION The work required to compress the spring from 5 cm to 9 cm past equilibrium is

$$\int_{-0.05}^{-0.09} 800x\,dx = 400x^2\Big|_{-0.05}^{-0.09} = 2.24\text{ J}.$$

7. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant as follows:

$$\int_0^{0.1} kx\,dx = \frac{1}{2}kx^2\Big|_0^{0.1} = 0.005k = 5\text{ J}.$$

Thus, $k = 1000$ N/m. Next, we calculate the work required to stretch the spring 15 cm beyond equilibrium:

$$\int_0^{0.15} 1000x\,dx = 500x^2\Big|_0^{0.15} = 11.25\text{ J}.$$

8. To create images of samples at the molecular level, atomic force microscopes use silicon micro-cantilevers that obey Hooke's Law $F(x) = -kx$, where x is the distance through which the tip is deflected (Figure 6). Suppose that 10^{-17} J of work are required to deflect the tip a distance 10^{-8} m. Find the deflection if a force of 10^{-9} N is applied to the tip.

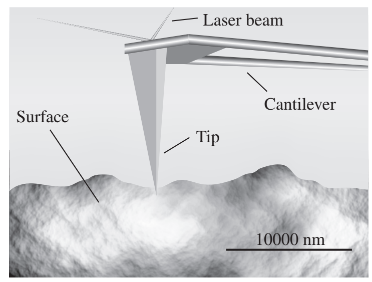


FIGURE 6

SOLUTION First, we determine the value of the constant k. Knowing it takes 10^{-17} J of work to deflect the tip a distance 10^{-8} m, it follows that

$$\frac{1}{2}k(10^{-8})^2 = 10^{-17} \quad \text{or} \quad k = \frac{1}{5}\text{ N/m}.$$

Now, the deflection produced by a force of 10^{-9} N can be determined as

$$x = \frac{F}{k} = \frac{10^{-9}}{1/5} = 5 \times 10^{-9}\text{ m}.$$

9. A spring obeys a force law $F(x) = -kx^{1.1}$ with $k = 100$ N/m. Find the work required to stretch a spring 0.3 m past equilibrium.

SOLUTION The work required to stretch this spring 0.3 m past equilibrium is

$$\int_0^{0.3} 100x^{1.1}\,dx = \frac{100}{1.1}x^{2.1}\Big|_0^{0.3} \approx 7.25\text{ J}.$$

10. Show that the work required to stretch a spring from position a to position b is $\frac{1}{2}k(b^2 - a^2)$, where k is the spring constant. How do you interpret the negative work obtained when $|b| < |a|$?

SOLUTION The work required to stretch a spring from position a to position b is

$$\int_a^b kx\,dx = \frac{1}{2}kx^2\Big|_a^b = \frac{1}{2}k(b^2 - a^2).$$

When $|b| < |a|$, the "negative work" is the work done by the spring to return to its equilibrium position.

In Exercises 11–14, use the method of Examples 2 and 3 to calculate the work against gravity required to build the structure out of a lightweight material of density 600 kg/m^3.

11. Box of height 3 m and square base of side 2 m

SOLUTION The volume of one layer is $4\Delta y$ m^3 and so the weight of one layer is $23520\Delta y$ N. Thus, the work done against gravity to build the tower is

$$W = \int_0^3 23520y\,dy = 11760y^2\Big|_0^3 = 105840\text{ J}.$$

12. Cylindrical column of height 4 m and radius 0.8 m

SOLUTION The area of the base is 0.64π m^2, so the volume of each small layer is $0.64\pi\,\Delta y$ m^3. The weight of one layer is then $3763.2\pi\,\Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^4 3763.2\pi y\,dy = 30105.6\pi\text{ J} \approx 94579.5\text{ J}.$$

13. Right circular cone of height 4 m and base of radius 1.2 m

SOLUTION By similar triangles, the layer of the cone at a height y above the base has radius $r = 0.3(4 - y)$ meters. Thus, the volume of the small layer at this height is $0.09\pi(4 - y)^2 \Delta y$ m^3, and the weight is $529.2\pi(4 - y)^2 \Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^4 529.2\pi(4 - y)^2 y\, dy = 11289.6\pi \text{ J} \approx 35467.3 \text{ J}.$$

14. Hemisphere of radius 0.8 m

SOLUTION The area of one layer is $\pi(0.64 - y^2)$ m^2, so the volume of each small layer is $\pi(0.64 - y^2)\Delta y$ m^3. The weight of one layer is then $5880\pi(0.64 - y^2)\Delta y$ N. Finally, the total work done against gravity to build the tower is

$$\int_0^{0.8} 5880\pi(0.64 - y^2) y\, dy = 602.112\pi \text{ J} \approx 1891.6 \text{ J}.$$

15. Built around 2600 BCE, the Great Pyramid of Giza in Egypt (Figure 7) is 146 m high and has a square base of side 230 m. Find the work (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m^3.

FIGURE 7 The Great Pyramid in Giza, Egypt.

SOLUTION From similar triangles, the area of one layer is

$$\left(230 - \frac{230}{146} y\right)^2 \text{ m}^2,$$

so the volume of each small layer is

$$\left(230 - \frac{230}{146} y\right)^2 \Delta y \text{ m}^3.$$

The weight of one layer is then

$$19600\left(230 - \frac{230}{146} y\right)^2 \Delta y \text{ N}.$$

Finally, the total work needed to build the pyramid was

$$\int_0^{146} 19600\left(230 - \frac{230}{146} y\right)^2 y\, dy \approx 1.84 \times 10^{12} \text{ J}.$$

16. Calculate the work (against gravity) required to build a box of height 3 m and square base of side 2 m out of material of variable density, assuming that the density at height y is $f(y) = 1000 - 100y$ kg/m^3.

SOLUTION The volume of one layer is $4\Delta y$ m^3 and so the weight of one layer is $(4000 - 400y)\Delta y$ N. Thus, the work done against gravity to build the tower is

$$W = \int_0^3 (4000 - 400y) y\, dy = \left(2000y^2 - \frac{400}{3} y^3\right)\Big|_0^3 = 14400 \text{ J}.$$

In Exercises 17–22, calculate the work (in joules) required to pump all of the water out of a full tank. Distances are in meters, and the density of water is 1000 kg/m^3.

17. Rectangular tank in Figure 8; water exits from a small hole at the top.

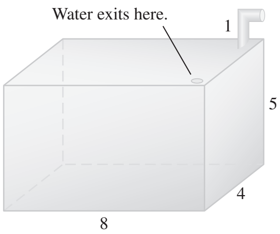


FIGURE 8

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313600\Delta y \text{ N}.$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_0^5 313600y\,dy = 156800y^2\Big|_0^5 = 3.92 \times 10^6 \text{ J}.$$

18. Rectangular tank in Figure 8; water exits through the spout.

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313600\Delta y \text{ N}.$$

Each layer must be lifted $y + 1$ meters, so the total work needed to empty the tank is

$$\int_0^5 313600(y+1)\,dy = 156800(y+1)^2\Big|_0^5 = 5.488 \times 10^6 \text{ J}.$$

19. Hemisphere in Figure 9; water exits through the spout.

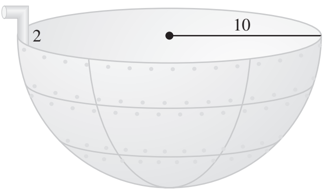


FIGURE 9

SOLUTION Place the origin at the center of the hemisphere, and let the positive y-axis point downward. The radius of a layer of water at depth y is $\sqrt{100 - y^2}$ m, so the volume of the layer is $\pi(100 - y^2)\Delta y$ m^3, and the force needed to lift the layer is $9800\pi(100 - y^2)\Delta y$ N. The layer must be lifted $y + 2$ meters, so the total work needed to empty the tank is

$$\int_0^{10} 9800\pi(100 - y^2)(y+2)\,dy = \frac{112700000\pi}{3} \text{ J} \approx 1.18 \times 10^8 \text{ J}.$$

20. Conical tank in Figure 10; water exits through the spout.

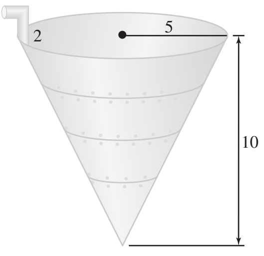


FIGURE 10

SOLUTION Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2 \text{ m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the tank is

$$\int_0^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12 - y)\, dy = \pi(3.675 \times 10^6) \text{ J} \approx 1.155 \times 10^7 \text{ J}.$$

21. Horizontal cylinder in Figure 11; water exits from a small hole at the top. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

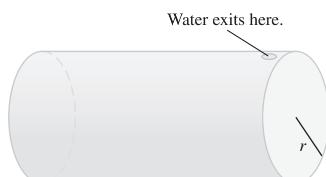


FIGURE 11

SOLUTION Place the origin along the axis of the cylinder. At location y, the layer of water is a rectangular slab of length ℓ, width $2\sqrt{r^2 - y^2}$ and thickness Δy. Thus, the volume of the layer is $2\ell\sqrt{r^2 - y^2}\Delta y$, and the force needed to lift the layer is $19{,}600\ell\sqrt{r^2 - y^2}\Delta y$. The layer must be lifted a distance $r - y$, so the total work needed to empty the tank is given by

$$\int_{-r}^{r} 19{,}600\ell\sqrt{r^2 - y^2}(r - y)\, dy = 19{,}600\ell r\int_{-r}^{r}\sqrt{r^2 - y^2}\, dy - 19{,}600\ell\int_{-r}^{r} y\sqrt{r^2 - y^2}\, dy.$$

Now,

$$\int_{-r}^{r} y\sqrt{r^2 - y^2}\, du = 0$$

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius r; thus,

$$\int_{-r}^{r}\sqrt{r^2 - y^2}\, dy = \frac{1}{2}\pi r^2.$$

Finally, the total work needed to empty the tank is

$$19{,}600\ell r\left(\frac{1}{2}\pi r^2\right) - 19{,}600\ell(0) = 9800\ell\pi r^3 \text{ J}.$$

22. Trough in Figure 12; water exits by pouring over the sides.

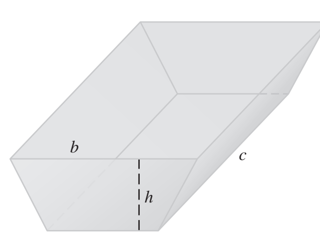


FIGURE 12

SOLUTION Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at a height of y meters is

$$w = a + \frac{y\,(b-a)}{h}\ \text{m}^2,$$

so the volume of each layer is

$$c\left(a + \frac{y\,(b-a)}{h}\right)\Delta y\ \text{m}^3.$$

Thus, the force needed to lift the layer is

$$9800c\left(a + \frac{y\,(b-a)}{h}\right)\Delta y\ \text{N}.$$

Each layer must be lifted $h - y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y\,(b-a)}{h}\right)dy = 9800\,c\left(\frac{ah^2}{3} + \frac{bh^2}{6}\right)\ \text{J}.$$

23. Find the work W required to empty the tank in Figure 8 through the hole at the top if the tank is half full of water.

SOLUTION Place the origin on the top of the box, and let the positive y-axis point downward. Note that with this coordinate system, the bottom half of the box corresponds to y values from 2.5 to 5. The volume of one layer of water is $32\Delta y$ m^3, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313{,}600\Delta y\ \text{N}.$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_{2.5}^5 313{,}600y\,dy = 156{,}800y^2\Big|_{2.5}^5 = 2.94 \times 10^6\ \text{J}.$$

24. Assume the tank in Figure 8 is full of water and let W be the work required to pump out half of the water through the hole at the top. Do you expect W to equal the work computed in Exercise 23? Explain and then compute W.

SOLUTION Recall that the origin was placed at the top of the box with the positive y-axis pointing downward. Pumping out half the water from a full tank would involve y values ranging from $y = 0$ to $y = 2.5$, whereas pumping out a half-full tank would involve y values ranging from $y = 2.5$ to $y = 5$. Because pumping out half the water from a full tank requires moving the layers of water a shorter distance than pumping out a half-full tank, we do not expect that W would be equal to the work computed in Exercise 23.

To compute W, we proceed as in Exercise 17 and Exercise 23, to find

$$W = \int_0^{2.5} 313{,}600y\,dy = 980{,}000\ \text{J}.$$

It is reassuring to note that

$$\text{Work(Exercise 23)} + \text{Work(Exercise 24)} = \text{Work(Exercise 17)}.$$

25. Assume the tank in Figure 10 is full. Find the work required to pump out half of the water. *Hint:* First, determine the level H at which the water remaining in the tank is equal to one-half the total capacity of the tank.

SOLUTION Our first step is to determine the level H at which the water remaining in the tank is equal to one-half the total capacity of the tank. From Figure 10 and similar triangles, we see that the radius of the cone at level H is $H/2$ so the volume of water is

$$V = \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi\left(\frac{H}{2}\right)^2 H = \frac{1}{12}\pi H^3.$$

The total capacity of the tank is $250\pi/3$ m^3, so the water level when the water remaining in the tank is equal to one-half the total capacity of the tank satisfies

$$\frac{1}{12}\pi H^3 = \frac{125}{3}\pi \quad \text{or} \quad H = \frac{10}{2^{1/3}}\ \text{m}.$$

Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Now, consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2\ \text{m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the half-full tank is

$$\int_{10/2^{1/3}}^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12 - y)\, dy \approx 3.79 \times 10^6 \text{ J}.$$

26. Assume that the tank in Figure 10 is full.

(a) Calculate the work $F(y)$ required to pump out water until the water level has reached level y.

(b) CAS Plot $F(y)$.

(c) What is the significance of $F'(y)$ as a rate of change?

(d) CAS If your goal is to pump out all of the water, at which water level y_0 will half of the work be done?

SOLUTION

(a) Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2 \text{ m}^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ m}^3.$$

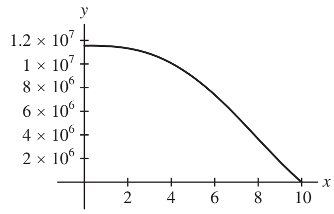
Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \Delta y \text{ N}.$$

The layer must be lifted $12 - y$ meters, so the total work needed to pump out water until the water level has reached level y is

$$\int_y^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12 - y)\, dy = 3{,}675{,}000\pi - 9800\pi y^3 + \frac{1225\pi}{2} y^4 \text{ J}.$$

(b) A plot of $F(y)$ is shown below.



(c) First, note that $F'(y) < 0$; as y increases, less water is being pumped from the tank, so $F(y)$ decreases. Therefore, when the water level in the tank has reached level y, we can interpret $-F'(y)$ as the amount of work per meter needed to remove the next layer of water from the tank. In other words, $-F'(y)$ is a "marginal work" function.

(d) The amount of work needed to empty the tank is $3{,}675{,}000\pi$ J. Half of this work will be done when the water level reaches height y_0 satisfying

$$3{,}675{,}000\pi - 9800\pi y_0^3 + \frac{1225\pi}{2} y_0^4 = 1{,}837{,}500\pi.$$

Using a computer algebra system, we find $y_0 = 6.91$ m.

27. Calculate the work required to lift a 10-m chain over the side of a building (Figure 13) Assume that the chain has a density of 8 kg/m. *Hint:* Break up the chain into N segments, estimate the work performed on a segment, and compute the limit as $N \to \infty$ as an integral.

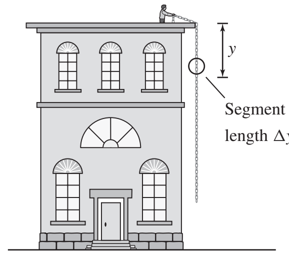


FIGURE 13 The small segment of the chain of length Δy located y meters from the top is lifted through a vertical distance y.

SOLUTION In this example, each part of the chain is lifted a different distance. Therefore, we divide the chain into N small segments of length $\Delta y = 10/N$. Suppose that the ith segment is located a distance y_i from the top of the building. This segment weighs $8(9.8)\Delta y$ kilograms and it must be lifted approximately y_i meters (not exactly y_i meters, because each point along the segment is a slightly different distance from the top). The work W_i done on this segment is approximately $W_i \approx 78.4 y_i \Delta y$ N. The total work W is the sum of the W_i and we have

$$W = \sum_{j=1}^{N} W_i \approx \sum_{j=1}^{N} 78.4 y_j \, \Delta y.$$

Passing to the limit as $N \to \infty$, we obtain

$$W = \int_0^{10} 78.4 \, y \, dy = 39.2 y^2 \Big|_0^{10} = 3920 \text{ J}.$$

28. How much work is done lifting a 3-m chain over the side of a building if the chain has mass density 4 kg/m?

SOLUTION Consider a segment of the chain of length Δy located a distance y_j meters from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (4\Delta y)(9.8) y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^3 4 \cdot 9.8 y \, dy = 19.6 y^2 \Big|_0^3 = 176.4 \text{ J}.$$

29. A 6-m chain has mass 18 kg. Find the work required to lift the chain over the side of a building.

SOLUTION First, note that the chain has a mass density of 3 kg/m. Now, consider a segment of the chain of length Δy located a distance y_j feet from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (3\Delta y) 9.8 y_j \text{ ft-lb}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^6 29.4 y \, dy = 14.7 y^2 \Big|_0^6 = 529.2 \text{ J}.$$

30. A 10-m chain with mass density 4 kg/m is initially coiled on the ground. How much work is performed in lifting the chain so that it is fully extended (and one end touches the ground)?

SOLUTION Consider a segment of the chain of length Δy that must be lifted y_j feet off the ground. The work needed to lift this segment of the chain is approximately

$$W_j \approx (4\Delta y) 9.8 y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^{10} 39.2 y \, dy = 19.6 y^2 \Big|_0^{10} = 1960 \text{ J}.$$

31. How much work is done lifting a 12-m chain that has mass density 3 kg/m (initially coiled on the ground) so that its top end is 10 m above the ground?

SOLUTION Consider a segment of the chain of length Δy that must be lifted y_j feet off the ground. The work needed to lift this segment of the chain is approximately

$$W_j \approx (3\Delta y)9.8y_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta y \to 0$, it follows that the total work is

$$\int_0^{10} 29.4y\,dy = 14.7y^2\Big|_0^{10} = 1470 \text{ J}.$$

32. A 500-kg wrecking ball hangs from a 12-m cable of density 15 kg/m attached to a crane. Calculate the work done if the crane lifts the ball from ground level to 12 m in the air by drawing in the cable.

SOLUTION We will treat the cable and the wrecking ball separately. Consider a segment of the cable of length Δy that must be lifted y_j feet. The work needed to lift the cable segment is approximately

$$W_j \approx (15\Delta y)9.8y_j \text{ J}.$$

Summing over all of the segments of the cable and passing to the limit as $\Delta y \to 0$, it follows that lifting the cable requires

$$\int_0^{12} 147y\,dy = 73.5y^2\Big|_0^{12} = 10{,}584 \text{ J}.$$

Lifting the 500 kg wrecking ball 12 meters requires an additional 58,800 J. Thus, the total work is 69,384 J.

33. Calculate the work required to lift a 3-m chain over the side of a building if the chain has variable density of $\rho(x) = x^2 - 3x + 10$ kg/m for $0 \le x \le 3$.

SOLUTION Consider a segment of the chain of length Δx that must be lifted x_j feet. The work needed to lift this segment is approximately

$$W_j \approx \left(\rho(x_j)\Delta x\right) 9.8x_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work is

$$\begin{aligned}\int_0^3 9.8\rho(x)x\,dx &= 9.8\int_0^3 \left(x^3 - 3x^2 + 10x\right)dx\\ &= 9.8\left(\frac{1}{4}x^4 - x^3 + 5x^2\right)\Big|_0^3 = 374.85 \text{ J}.\end{aligned}$$

34. A 3-m chain with linear mass density $\rho(x) = 2x(4-x)$ kg/m lies on the ground. Calculate the work required to lift the chain so that its bottom is 2 m above ground.

SOLUTION Consider a segment of the chain of length Δx that must be lifted x_j feet. The work needed to lift this segment is approximately

$$W_j \approx \left(\rho(x_j)\Delta x\right) 9.8x_j \text{ J}.$$

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work needed to fully extend the chain is

$$\begin{aligned}\int_0^3 9.8\rho(x)x\,dx &= 9.8\int_0^3 \left(8x^2 - 2x^3\right)dx\\ &= 9.8\left(\frac{8}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^3 = 308.7 \text{ J}.\end{aligned}$$

Lifting the entire chain, which weighs

$$\int_0^3 9.8\rho(x)\,dx = 9.8\int_0^3 \left(8x - 2x^2\right)dx = 9.8\left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 176.4 \text{ N}$$

another two meters requires an additional 352.8 J of work. The total work is therefore 661.5 J.

Exercises 35–37: The gravitational force between two objects of mass m and M, separated by a distance r, has magnitude GMm/r^2*, where* $G = 6.67 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-1}}$.

35. Show that if two objects of mass M and m are separated by a distance r_1, then the work required to increase the separation to a distance r_2 is equal to $W = GMm(r_1^{-1} - r_2^{-1})$.

SOLUTION The work required to increase the separation from a distance r_1 to a distance r_2 is

$$\int_{r_1}^{r_2} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r}\bigg|_{r_1}^{r_2} = GMm(r_1^{-1} - r_2^{-1}).$$

36. Use the result of Exercise 35 to calculate the work required to place a 2000-kg satellite in an orbit 1200 km above the surface of the earth. Assume that the earth is a sphere of radius $R_e = 6.37 \times 10^6$ m and mass $M_e = 5.98 \times 10^{24}$ kg. Treat the satellite as a point mass.

SOLUTION The satellite will move from a distance $r_1 = R_e$ to a distance $r_2 = R_e + 1{,}200{,}000$. Thus, from Exercise 35,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(2000)\left(\frac{1}{6.37 \times 10^6} - \frac{1}{6.37 \times 10^6 + 1{,}200{,}000}\right) \approx 1.99 \times 10^{10}\ \mathrm{J}.$$

37. Use the result of Exercise 35 to compute the work required to move a 1500-kg satellite from an orbit 1000 to an orbit 1500 km above the surface of the earth.

SOLUTION The satellite will move from a distance $r_1 = R_e + 1{,}000{,}000$ to a distance $r_2 = R_e + 1{,}500{,}000$. Thus, from Exercise 35,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1500) \times \left(\frac{1}{6.37 \times 10^6 + 1{,}000{,}000} - \frac{1}{6.37 \times 10^6 + 1{,}500{,}000}\right)$$
$$\approx 5.16 \times 10^9\ \mathrm{J}.$$

38. The pressure P and volume V of the gas in a cylinder of length 0.8 meters and radius 0.2 meters, with a movable piston, are related by $PV^{1.4} = k$, where k is a constant (Figure 14). When the piston is fully extended, the gas pressure is 2000 kilopascals (one kilopascal is 10^3 newtons per square meter).

(a) Calculate k.

(b) The force on the piston is PA, where A is the piston's area. Calculate the force as a function of the length x of the column of gas.

(c) Calculate the work required to compress the gas column from 0.8 m to 0.5 m.

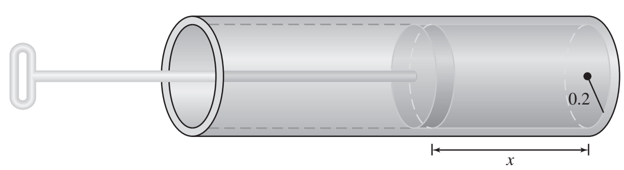


FIGURE 14 Gas in a cylinder with a piston.

SOLUTION

(a) We have $P = 2 \times 10^6$ and $V = 0.032\pi$. Thus

$$k = 2 \times 10^6 (0.032\pi)^{1.4} = 80{,}213.9.$$

(b) The area of the piston is $A = 0.04\pi$ and the volume of the cylinder as a function of x is $V = 0.04\pi x$, which gives $P = k/V^{1.4} = k/(0.04\pi x)^{1.4}$. Thus

$$F = PA = \frac{k}{(0.04\pi x)^{1.4}} 0.04\pi = k(0.04\pi)^{-0.4} x^{-1.4}.$$

(c) Since the force is pushing against the piston, in order to calculate work, we must calculate the integral of the opposite force, i.e., we have

$$W = -k(0.04\pi)^{-0.4} \int_{0.8}^{0.5} x^{-1.4}\,dx = -k(0.04\pi)^{-0.4} \frac{1}{-0.4} x^{-0.4}\bigg|_{0.8}^{0.5} = 103{,}966.7\ \mathrm{J}.$$

Further Insights and Challenges

39. Work-Energy Theorem An object of mass m moves from x_1 to x_2 during the time interval $[t_1, t_2]$ due to a force $F(x)$ acting in the direction of motion. Let $x(t)$, $v(t)$, and $a(t)$ be the position, velocity, and acceleration at time t. The object's kinetic energy is $\text{KE} = \frac{1}{2}mv^2$.

(a) Use the change-of-variables formula to show that the work performed is equal to

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))v(t)\,dt$$

(b) Use Newton's Second Law, $F(x(t)) = ma(t)$, to show that

$$\frac{d}{dt}\left(\frac{1}{2}mv(t)^2\right) = F(x(t))v(t)$$

(c) Use the FTC to prove the Work-Energy Theorem: The change in kinetic energy during the time interval $[t_1, t_2]$ is equal to the work performed.

SOLUTION

(a) Let $x_1 = x(t_1)$ and $x_2 = x(t_2)$, then $x = x(t)$ gives $dx = v(t)\,dt$. By substitution we have

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))v(t)\,dt.$$

(b) Knowing $F(x(t)) = m \cdot a(t)$, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}m \cdot v(t)^2\right) &= m \cdot v(t)\,v'(t) && \text{(Chain Rule)} \\ &= m \cdot v(t)\,a(t) \\ &= v(t) \cdot F(x(t)) && \text{(Newton's 2nd law)} \end{aligned}$$

(c) From the FTC,

$$\frac{1}{2}m \cdot v(t)^2 = \int F(x(t))\,v(t)\,dt.$$

Since $KE = \frac{1}{2}\,m\,v^2$,

$$\Delta KE = KE(t_2) - KE(t_1) = \frac{1}{2}m\,v(t_2)^2 - \frac{1}{2}m\,v(t_1)^2 = \int_{t_1}^{t_2} F(x(t))\,v(t)\,dt.$$

(d)

$$\begin{aligned} W &= \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))\,v(t)\,dt && \text{(Part (a))} \\ &= KE(t_2) - KE(t_1) \\ &= \Delta KE && \text{(as required)} \end{aligned}$$

40. A model train of mass 0.5 kg is placed at one end of a straight 3-m electric track. Assume that a force $F(x) = (3x - x^2)$ N acts on the train at distance x along the track. Use the Work-Energy Theorem (Exercise 39) to determine the velocity of the train when it reaches the end of the track.

SOLUTION We have

$$W = \int_0^3 F(x)\,dx = \int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3 = 4.5 \text{ J}.$$

Then the change in KE must be equal to W, which gives

$$4.5 = \frac{1}{2}m(v(t_2)^2 - v(t_1)^2)$$

Note that $v(t_1) = 0$ as the train was placed on the track with no initial velocity and $m = 0.5$. Thus

$$v(t_2) = \sqrt{18} = 4.242641 \text{ m/sec.}$$

41. With what initial velocity v_0 must we fire a rocket so it attains a maximum height r above the earth? *Hint:* Use the results of Exercises 35 and 39. As the rocket reaches its maximum height, its KE decreases from $\frac{1}{2}mv_0^2$ to zero.

SOLUTION The work required to move the rocket a distance r from the surface of the earth is

$$W(r) = GM_e m\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right).$$

As the rocket climbs to a height r, its kinetic energy is reduced by the amount $W(r)$. The rocket reaches its maximum height when its kinetic energy is reduced to zero, that is, when

$$\frac{1}{2}mv_0^2 = GM_e m\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right).$$

Therefore, its initial velocity must be

$$v_0 = \sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)}.$$

42. With what initial velocity must we fire a rocket so it attains a maximum height of $r = 20$ km above the surface of the earth?

SOLUTION Using the result of the previous exercise with $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, $M_e = 5.98 \times 10^{24}$ kg, $R_e = 6.37 \times 10^6$ m and $r = 20{,}000$ m,

$$v_0 = \sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)} = 626 \text{ m/sec.}$$

43. Calculate **escape velocity,** the minimum initial velocity of an object to ensure that it will continue traveling into space and never fall back to earth (assuming that no force is applied after takeoff). *Hint:* Take the limit as $r \to \infty$ in Exercise 41.

SOLUTION The result of Exercise 41 leads to an interesting conclusion. The initial velocity v_0 required to reach a height r does not increase beyond all bounds as r tends to infinity; rather, it approaches a finite limit, called the escape velocity:

$$v_{\text{esc}} = \lim_{r\to\infty}\sqrt{2GM_e\left(\frac{1}{R_e} - \frac{1}{r + R_e}\right)} = \sqrt{\frac{2GM_e}{R_e}}$$

In other words, v_{esc} is large enough to insure that the rocket reaches a height r for every value of r! Therefore, a rocket fired with initial velocity v_{esc} never returns to earth. It continues traveling indefinitely into outer space.

Now, let's see how large escape velocity actually is:

$$v_{\text{esc}} = \left(\frac{2 \cdot 6.67 \times 10^{-11} \cdot 5.989 \times 10^{24}}{6.37 \times 10^6}\right)^{1/2} \approx 11{,}190 \text{ m/sec.}$$

Since one meter per second is equal to 2.236 miles per hour, escape velocity is approximately $11{,}190(2.236) = 25{,}020$ miles per hour.

CHAPTER REVIEW EXERCISES

1. Compute the area of the region in Figure 1(A) enclosed by $y = 2 - x^2$ and $y = -2$.

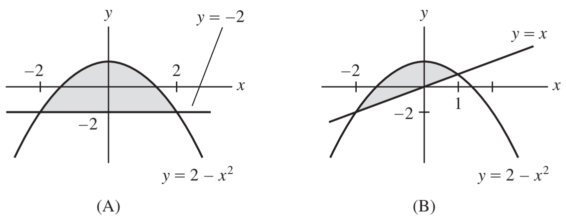


FIGURE 1

SOLUTION The graphs of $y = 2 - x^2$ and $y = -2$ intersect where $2 - x^2 = -2$, or $x = \pm 2$. Therefore, the enclosed area lies over the interval $[-2, 2]$. The region enclosed by the graphs lies below $y = 2 - x^2$ and above $y = -2$, so the area is

$$\int_{-2}^{2}\Big((2 - x^2) - (-2)\Big)\,dx = \int_{-2}^{2}(4 - x^2)\,dx = \left(4x - \frac{1}{3}x^3\right)\Bigg|_{-2}^{2} = \frac{32}{3}.$$

2. Compute the area of the region in Figure 1(B) enclosed by $y = 2 - x^2$ and $y = x$.

SOLUTION The graphs of $y = 2 - x^2$ and $y = x$ intersect where $2 - x^2 = x$, which simplifies to

$$0 = x^2 + x - 2 = (x + 2)(x - 1).$$

Thus, the graphs intersect at $x = -2$ and $x = 1$. As the graph of $y = x$ lies below the graph of $y = 2 - x^2$ over the interval $[-2, 1]$, the area between the graphs is

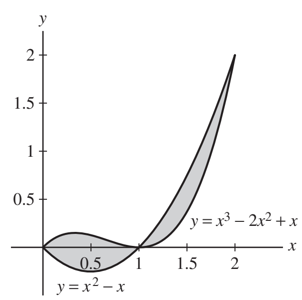
$$\int_{-2}^{1} \left((2 - x^2) - x\right)\, dx = \left(2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{-2}^{1} = \frac{9}{2}.$$

In Exercises 3–12, find the area of the region enclosed by the graphs of the functions.

3. $y = x^3 - 2x^2 + x, \quad y = x^2 - x$

SOLUTION The region bounded by the graphs of $y = x^3 - 2x^2 + x$ and $y = x^2 - x$ over the interval $[0, 2]$ is shown below. For $x \in [0, 1]$, the graph of $y = x^3 - 2x^2 + x$ lies above the graph of $y = x^2 - x$, whereas, for $x \in [1, 2]$, the graph of $y = x^2 - x$ lies above the graph of $y = x^3 - 2x^2 + x$. The area of the region is therefore given by

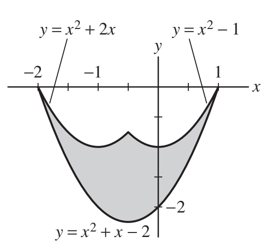
$$\begin{aligned}
\int_0^1 &\left((x^3 - 2x^2 + x) - (x^2 - x)\right)\, dx + \int_1^2 \left((x^2 - x) - (x^3 - 2x^2 + x)\right)\, dx \\
&= \left(\frac{1}{4}x^4 - x^3 + x^2\right)\Big|_0^1 + \left(x^3 - x^2 - \frac{1}{4}x^4\right)\Big|_1^2 \\
&= \frac{1}{4} - 1 + 1 + (8 - 4 - 4) - \left(1 - 1 - \frac{1}{4}\right) = \frac{1}{2}.
\end{aligned}$$



4. $y = x^2 + 2x, \quad y = x^2 - 1, \quad h(x) = x^2 + x - 2$

SOLUTION The region bounded by the graphs of $y = x^2 + 2x$, $y = x^2 - 1$ and $y = x^2 + x - 2$ is shown below. For each $x \in [-2, -\frac{1}{2}]$, the graph of $y = x^2 + 2x$ lies above the graph of $y = x^2 + x - 2$, whereas, for each $x \in [-\frac{1}{2}, 1]$, the graph of $y = x^2 - 1$ lies above the graph of $y = x^2 + x - 2$. The area of the region is therefore given by

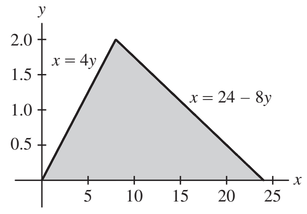
$$\begin{aligned}
\int_{-2}^{-1/2} &\left((x^2 + 2x) - (x^2 + x - 2)\right)\, dx + \int_{-1/2}^{1} \left((x^2 - 1) - (x^2 + x - 2)\right)\, dx \\
&= \left(\frac{1}{2}x^2 + 2x\right)\Big|_{-2}^{-1/2} + \left(-\frac{1}{2}x^2 + x\right)\Big|_{-1/2}^{1} \\
&= \left(\frac{1}{8} - 1\right) - (2 - 4) + \left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{8} - \frac{1}{2}\right) = \frac{9}{4}.
\end{aligned}$$



5. $x = 4y, \quad x = 24 - 8y, \quad y = 0$

SOLUTION The region bounded by the graphs $x = 4y$, $x = 24 - 8y$ and $y = 0$ is shown below. For each $0 \le y \le 2$, the graph of $x = 24 - 8y$ lies to the right of $x = 4y$. The area of the region is therefore

$$A = \int_0^2 (24 - 8y - 4y)\, dy = \int_0^2 (24 - 12y)\, dy$$
$$= (24y - 6y^2)\Big|_0^2 = 24.$$



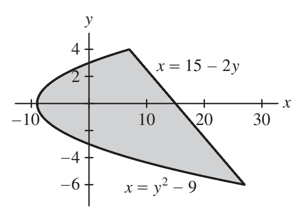
6. $x = y^2 - 9, \quad x = 15 - 2y$

SOLUTION Setting $y^2 - 9 = 15 - 2y$ yields

$$y^2 + 2y - 24 = (y + 6)(y - 4) = 0,$$

so the two curves intersect at $y = -6$ and $y = 4$. The region bounded by the graphs $x = y^2 - 9$ and $x = 15 - 2y$ is shown below. For each $-6 \le y \le 4$, the graph of $x = 15 - 2y$ lies to the right of $x = y^2 - 9$. The area of the region is therefore

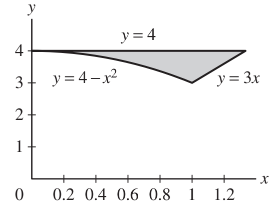
$$A = \int_{-6}^4 \left(15 - 2y - (y^2 - 9)\right) dy = \int_{-6}^4 (24 - 2y - y^2)\, dy$$
$$= \left(24y - y^2 - \frac{1}{3}y^3\right)\Big|_{-6}^4$$
$$= \left(\frac{176}{3} - (-108)\right) = \frac{500}{3}.$$



7. $y = 4 - x^2, \quad y = 3x, \quad y = 4$

SOLUTION The region bounded by the graphs of $y = 4 - x^2$, $y = 3x$ and $y = 4$ is shown below. For $x \in [0, 1]$, the graph of $y = 4$ lies above the graph of $y = 4 - x^2$, whereas, for $x \in [1, \frac{4}{3}]$, the graph of $y = 4$ lies above the graph of $y = 3x$. The area of the region is therefore given by

$$\int_0^1 (4 - (4 - x^2))\, dx + \int_1^{4/3} (4 - 3x)\, dx = \frac{1}{3}x^3\Big|_0^1 + \left(4x - \frac{3}{2}x^2\right)\Big|_1^{4/3} = \frac{1}{3} + \left(\frac{16}{3} - \frac{8}{3}\right) - \left(4 - \frac{3}{2}\right) = \frac{1}{2}.$$



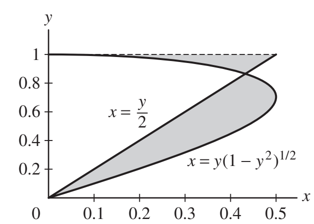
8. $\boxed{\text{GU}}$ $x = \frac{1}{2}y, \quad x = y\sqrt{1-y^2}, \quad 0 \le y \le 1$

SOLUTION The region bounded by the graphs of $x = y/2$ and $x = y\sqrt{1-y^2}$ over the interval [0, 1] is shown below. For $y \in [0, \frac{\sqrt{3}}{2}]$, the graph of $x = y\sqrt{1-y^2}$ lies to the right of the graph of $x = y/2$, whereas, for $y \in [\frac{\sqrt{3}}{2}, 1]$, the graph of $x = y/2$ lies to the right of the graph of $x = y\sqrt{1-y^2}$. The area of the region is therefore given by

$$\int_0^{\sqrt{3}/2} \left(y\sqrt{1-y^2} - \frac{y}{2} \right) dy + \int_{\sqrt{3}/2}^1 \left(\frac{y}{2} - y\sqrt{1-y^2} \right) dy$$

$$= \left(-\frac{1}{3}(1-y^2)^{3/2} - \frac{y^2}{4} \right) \Bigg|_0^{\sqrt{3}/2} + \left(\frac{y^2}{4} + \frac{1}{3}(1-y^2)^{3/2} \right) \Bigg|_{\sqrt{3}/2}^1$$

$$= -\frac{1}{24} - \frac{3}{16} + \frac{1}{3} + \frac{1}{4} - \frac{3}{16} - \frac{1}{24} = \frac{1}{8}.$$



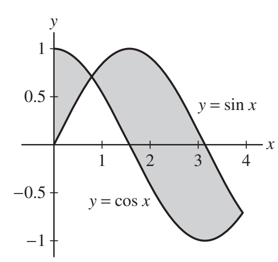
9. $y = \sin x, \quad y = \cos x, \quad 0 \le x \le \frac{5\pi}{4}$

SOLUTION The region bounded by the graphs of $y = \sin x$ and $y = \cos x$ over the interval $[0, \frac{5\pi}{4}]$ is shown below. For $x \in [0, \frac{\pi}{4}]$, the graph of $y = \cos x$ lies above the graph of $y = \sin x$, whereas, for $x \in [\frac{\pi}{4}, \frac{5\pi}{4}]$, the graph of $y = \sin x$ lies above the graph of $y = \cos x$. The area of the region is therefore given by

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4}$$

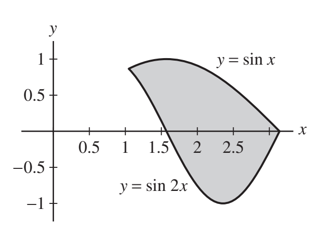
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 3\sqrt{2} - 1.$$



10. $f(x) = \sin x, \quad g(x) = \sin 2x, \quad \frac{\pi}{3} \le x \le \pi$

SOLUTION The region bounded by the graphs of $y = \sin x$ and $y = \sin 2x$ over the interval $[\frac{\pi}{3}, \pi]$ is shown below. As the graph of $y = \sin x$ lies above the graph of $y = \sin 2x$, the area of the region is given by

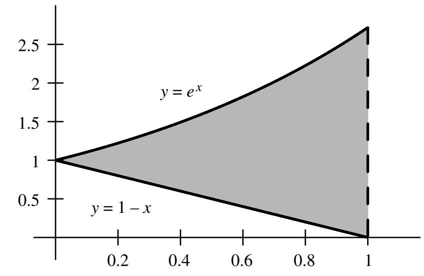
$$\int_{\pi/3}^{\pi} (\sin x - \sin 2x)\, dx = \left(-\cos x + \frac{1}{2}\cos 2x \right)\Bigg|_{\pi/3}^{\pi} = \left(1 + \frac{1}{2} \right) - \left(-\frac{1}{2} - \frac{1}{4} \right) = \frac{9}{4}.$$



11. $y = e^x, \quad y = 1 - x, \quad x = 1$

SOLUTION The region bounded by the graphs of $y = e^x$, $y = 1 - x$ and $x = 1$ is shown below. As the graph of $y = e^x$ lies above the graph of $y = 1 - x$, the area of the region is given by

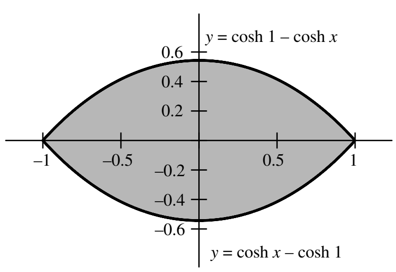
$$\int_0^1 \left(e^x - (1-x)\right)\, dx = \left(e^x - x + \frac{1}{2}x^2\right)\Big|_0^1 = \left(e - 1 + \frac{1}{2}\right) - 1 = e - \frac{3}{2}.$$



12. $y = \cosh 1 - \cosh x, \quad y = \cosh x - \cosh 1$

SOLUTION The region bounded by the graphs of $y = \cosh 1 - \cosh x$, $y = \cosh x - \cosh 1$ is shown below. As the graph of $y = \cosh 1 - \cosh x$ lies above the graph of $y = \cosh x - \cosh 1$, the area of the region is given by

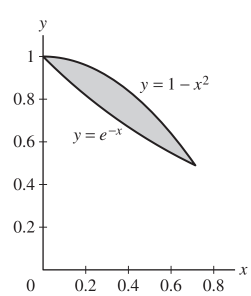
$$\begin{aligned}\int_{-1}^1 ((\cosh 1 - \cosh x) - (\cosh x - \cosh 1))\, dx &= (2x \cosh 1 - 2\sinh x)\Big|_{-1}^1 \\ &= (2\cosh 1 - 2\sinh 1) - (-2\cosh 1 + 2\sinh 1) \\ &= 4\cosh 1 - 4\sinh 1 = 4e^{-1}.\end{aligned}$$



13. GU Use a graphing utility to locate the points of intersection of $y = e^{-x}$ and $y = 1 - x^2$ and find the area between the two curves (approximately).

SOLUTION The region bounded by the graphs of $y = e^{-x}$ and $y = 1 - x^2$ is shown below. One point of intersection clearly occurs at $x = 0$. Using a computer algebra system, we find that the other point of intersection occurs at $x = 0.7145563847$. As the graph of $y = 1 - x^2$ lies above the graph of $y = e^{-x}$, the area of the region is given by

$$\int_0^{0.7145563847} \left(1 - x^2 - e^{-x}\right)\, dx = 0.08235024596$$



14. Figure 2 shows a solid whose horizontal cross section at height y is a circle of radius $(1+y)^{-2}$ for $0 \le y \le H$. Find the volume of the solid.

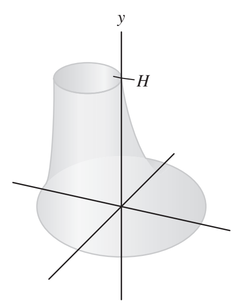


FIGURE 2

SOLUTION The area of each horizontal cross section is $A(y) = \pi(1+y)^{-4}$. Therefore, the volume of the solid is

$$\int_0^H \pi(1+y)^{-4}\,dy = \pi\left.\frac{(1+y)^{-3}}{-3}\right|_0^H = \pi\left(\frac{(1+H)^{-3}}{-3} + \frac{1}{3}\right) = \frac{\pi}{3}\left(1 - \frac{1}{(1+H)^3}\right).$$

15. The base of a solid is the unit circle $x^2 + y^2 = 1$, and its cross sections perpendicular to the x-axis are rectangles of height 4. Find its volume.

SOLUTION Because the cross sections are rectangles of constant height 4, the figure is a cylinder of radius 1 and height 4. The volume is therefore $\pi r^2 h = 4\pi$.

16. The base of a solid is the triangle bounded by the axes and the line $2x + 3y = 12$, and its cross sections perpendicular to the y-axis have area $A(y) = (y+2)$. Find its volume.

SOLUTION The volume of this solid is

$$V = \int_0^4 A(y)\,dy = \int_0^4 (y+2)\,dy = \left.\left(\frac{1}{2}y^2 + 2y\right)\right|_0^4 = 16.$$

17. Find the total mass of a rod of length 1.2 m with linear density $\rho(x) = (1 + 2x + \frac{2}{9}x^3)$ kg/m.

SOLUTION The total weight of the rod is

$$\int_0^{1.2} \rho(x)\,dx = \left.\left(x + x^2 + \frac{1}{18}x^4\right)\right|_0^{1.2} = 2.7552\text{ kg}.$$

18. Find the flow rate (in the correct units) through a pipe of diameter 6 cm if the velocity of fluid particles at a distance r from the center of the pipe is $v(r) = (3 - r)$ cm/s.

SOLUTION The flow rate through the pipe is

$$2\pi\int_0^3 r\,v(r)\,dr = 2\pi\int_0^3 (3r - r^2)\,dr = 2\pi\left.\left(\frac{3}{2}r^2 - \frac{1}{3}r^3\right)\right|_0^3 = 2\pi\left(\frac{27}{2} - 9\right) = 9\pi\,\frac{\text{cm}^3}{\text{s}}.$$

In Exercises 19–24, find the average value of the function over the interval.

19. $f(x) = x^3 - 2x + 2, \quad [-1, 2]$

SOLUTION The average value is

$$\frac{1}{2-(-1)}\int_{-1}^2 \left(x^3 - 2x + 2\right)\,dx = \frac{1}{3}\left.\left(\frac{1}{4}x^4 - x^2 + 2x\right)\right|_{-1}^2 = \frac{1}{3}\left[(4 - 4 + 4) - \left(\frac{1}{4} - 1 - 2\right)\right] = \frac{9}{4}.$$

20. $f(x) = |x|, \quad [-4, 4]$

SOLUTION The average value is

$$\frac{1}{4-(-4)}\int_{-4}^4 |x|\,dx = \frac{1}{8}\left(\int_{-4}^0 (-x)\,dx + \int_0^4 x\,dx\right) = \frac{1}{8}\left(\left.-\frac{1}{2}x^2\right|_{-4}^0 + \left.\frac{1}{2}x^2\right|_0^4\right) = \frac{1}{8}\left[(0+8) + (8-0)\right] = 2.$$

21. $f(x) = x\cosh(x^2)$, $[0, 1]$

SOLUTION The average value is

$$\frac{1}{1-0}\int_0^1 x\cosh(x^2)\,dx.$$

To evaluate the integral, let $u = x^2$. Then $du = 2x\,dx$ and

$$\frac{1}{1-0}\int_0^1 x\cosh(x^2)\,dx = \frac{1}{2}\int_0^1 \cosh u\,du = \frac{1}{2}\sinh u\Big|_0^1 = \frac{1}{2}\sinh 1.$$

22. $f(x) = \dfrac{e^x}{1+e^{2x}}$, $\left[0, \dfrac{1}{2}\right]$

SOLUTION The average value is

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2} \frac{e^x}{1+e^{2x}}\,dx.$$

To evaluate the integral, let $u = e^x$. Then $du = e^x\,dx$ and

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2} \frac{e^x}{1+e^{2x}}\,dx = 2\int_1^{\sqrt{e}} \frac{du}{1+u^2} = 2\tan^{-1}u\Big|_1^{\sqrt{e}} = 2\left(\tan^{-1}\sqrt{e} - \frac{\pi}{4}\right).$$

23. $f(x) = \sqrt{9-x^2}$, $[0, 3]$ *Hint:* Use geometry to evaluate the integral.

SOLUTION The region below the graph of $y = \sqrt{9-x^2}$ but above the x-axis over the interval $[0, 3]$ is one-quarter of a circle of radius 3; consequently,

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}.$$

The average value is then

$$\frac{1}{3-0}\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{3}\left(\frac{9\pi}{4}\right) = \frac{3\pi}{4}.$$

24. $f(x) = x[x]$, $[0, 3]$, where $[x]$ is the greatest integer function.

SOLUTION The average value is

$$\begin{aligned}\frac{1}{3-0}\int_0^3 x[x]\,dx &= \frac{1}{3}\left(\int_0^1 x\cdot 0\,dx + \int_1^2 x\cdot 1\,dx + \int_2^3 x\cdot 2\,dx\right)\\ &= \frac{1}{3}\left(\frac{1}{2}x^2\Big|_1^2 + x^2\Big|_2^3\right) = \frac{1}{3}\left(2 - \frac{1}{2} + 9 - 4\right) = \frac{13}{6}.\end{aligned}$$

25. Find $\displaystyle\int_2^5 g(t)\,dt$ if the average value of $g(t)$ on $[2, 5]$ is 9.

SOLUTION The average value of the function $g(t)$ on $[2, 5]$ is given by

$$\frac{1}{5-2}\int_2^5 g(t)\,dt = \frac{1}{3}\int_2^5 g(t)\,dt.$$

Therefore,

$$\int_2^5 g(t)\,dt = 3(\text{average value}) = 3(9) = 27.$$

26. The average value of $R(x)$ over $[0, x]$ is equal to x for all x. Use the FTC to determine $R(x)$.

SOLUTION The average value of the function $R(x)$ over $[0, x]$ is

$$\frac{1}{x-0}\int_0^x R(t)\,dt = \frac{1}{x}\int_0^x R(t)\,dt.$$

Given that the average value is equal to x, it follows that

$$\int_0^x R(t)\,dt = x^2.$$

Differentiating both sides of this equation and using the Fundamental Theorem of Calculus on the left-hand side yields

$$R(x) = 2x.$$

27. Use the Washer Method to find the volume obtained by rotating the region in Figure 3 about the x-axis.

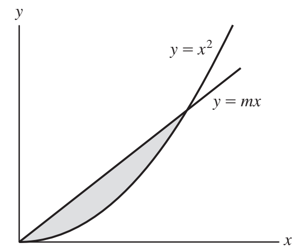


FIGURE 3

SOLUTION Setting $x^2 = mx$ yields $x(x-m) = 0$, so the two curves intersect at $(0, 0)$ and (m, m^2). To use the washer method, we must slice the solid perpendicular to the axis of rotation; as we are revolving about the y-axis, this implies a horizontal slice and integration in y. For each $y \in [0, m^2]$, the cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = \frac{y}{m}$. The volume of the solid is therefore given by

$$\pi\int_0^{m^2}\left((\sqrt{y})^2 - \left(\frac{y}{m}\right)^2\right)dy = \pi\left(\frac{1}{2}y^2 - \frac{y^3}{3m^2}\right)\Bigg|_0^{m^2} = \pi\left(\frac{m^4}{2} - \frac{m^4}{3}\right) = \frac{\pi}{6}m^4.$$

28. Use the Shell Method to find the volume obtained by rotating the region in Figure 3 about the x-axis.

SOLUTION Setting $x^2 = mx$ yields $x(x-m) = 0$, so the two curves intersect at $(0, 0)$ and (m, m^2). To use the shell method, we must slice the solid parallel to the axis of rotation; as we are revolving about the x-axis, this implies a horizontal slice and integration in y. For each $y \in [0, m^2]$, the shell has radius y and height $\sqrt{y} - \frac{y}{m}$. The volume of the solid is therefore given by

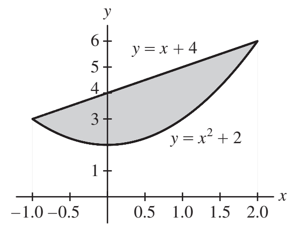
$$2\pi\int_0^{m^2} y\left(\sqrt{y} - \frac{y}{m}\right)dy = 2\pi\left(\frac{2}{5}y^{5/2} - \frac{y^3}{3m}\right)\Bigg|_0^{m^2} = 2\pi\left(\frac{2m^5}{5} - \frac{m^5}{3}\right) = \frac{2\pi}{15}m^5.$$

In Exercises 29–40, use any method to find the volume of the solid obtained by rotating the region enclosed by the curves about the given axis.

29. $y = x^2 + 2, \quad y = x + 4, \quad x$-axis

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 + 2$ and $y = x + 4$ (see the figure below) vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [-1, 2]$, the washer has outer radius $x + 4$ and inner radius $x^2 + 2$. The volume of the solid is therefore given by

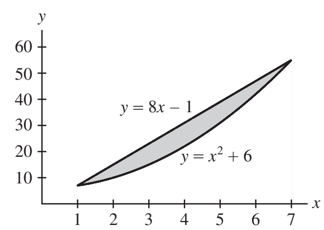
$$\begin{aligned}\pi\int_{-1}^{2}((x+4)^2 - (x^2+2)^2)\,dx &= \pi\int_{-1}^{2}(-x^4 - 3x^2 + 8x + 12)\,dx\\ &= \pi\left(-\frac{1}{5}x^5 - x^3 + 4x^2 + 12x\right)\Bigg|_{-1}^{2}\\ &= \pi\left(\frac{128}{5} + \frac{34}{5}\right) = \frac{162\pi}{5}.\end{aligned}$$



30. $y = x^2 + 6, \quad y = 8x - 1, \quad y\text{-axis}$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 + 6$ and $y = 8x - 1$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [1, 7]$, the shell has radius x and height $8x - 1 - (x^2 + 6) = -x^2 + 8x - 7$. The volume of the solid is therefore given by

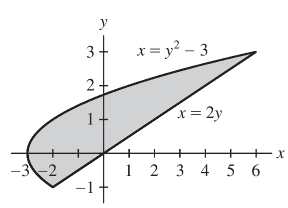
$$\begin{aligned} 2\pi \int_1^7 x(-x^2 + 8x - 7)\,dx &= 2\pi \int_1^7 (-x^3 + 8x^2 - 7x)\,dx \\ &= 2\pi \left(-\frac{1}{4}x^4 + \frac{8}{3}x^3 - \frac{7}{2}x^2 \right)\Big|_1^7 \\ &= 2\pi \left(\frac{1715}{12} + \frac{13}{12} \right) = 288\pi. \end{aligned}$$



31. $x = y^2 - 3, \quad x = 2y, \quad \text{axis } y = 4$

SOLUTION Let's choose to slice the region bounded by the graphs of $x = y^2 - 3$ and $x = 2y$ (see the figure below) horizontally. Because a horizontal slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $y \in [-1, 3]$, the shell has radius $4 - y$ and height $2y - (y^2 - 3) = 3 + 2y - y^2$. The volume of the solid is therefore given by

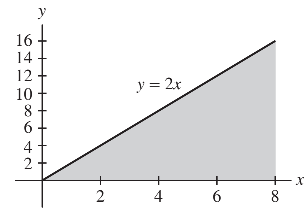
$$\begin{aligned} 2\pi \int_{-1}^3 (4 - y)(3 + 2y - y^2)\,dy &= 2\pi \int_{-1}^3 (12 + 5y - 6y^2 + y^3)\,dy \\ &= 2\pi \left(12y + \frac{5}{2}y^2 - 2y^3 + \frac{1}{4}y^4 \right)\Big|_{-1}^3 \\ &= 2\pi \left(\frac{99}{4} + \frac{29}{4} \right) = 64\pi. \end{aligned}$$



32. $y = 2x, \quad y = 0, \quad x = 8, \quad \text{axis } x = -3$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = 2x$, $y = 0$ and $x = 8$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 8]$, the shell has radius $x - (-3) = x + 3$ and height $2x$. The volume of the solid is therefore given by

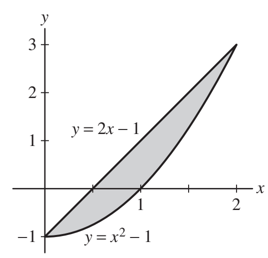
$$2\pi \int_0^8 (x + 3)(2x)\,dx = 4\pi \left(\frac{1}{3}x^3 + \frac{3}{2}x^2 \right)\Big|_0^8 = 4\pi \left(\frac{512}{3} + 96 \right) = \frac{3200\pi}{3}.$$



33. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } x = -2$

SOLUTION The region bounded by the graphs of $y = x^2 - 1$ and $y = 2x - 1$ is shown below. Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 2]$, the shell has radius $x - (-2) = x + 2$ and height $(2x - 1) - (x^2 - 1) = 2x - x^2$. The volume of the solid is therefore given by

$$2\pi \int_0^2 (x + 2)(2x - x^2)\, dx = 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2 = 2\pi(8 - 4) = 8\pi.$$



34. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } y = 4$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = x^2 - 1$ and $y = 2x - 1$ (see the figure in the previous exercise) vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [0, 2]$, the cross section is a washer with outer radius $R = 4 - (x^2 - 1) = 5 - x^2$ and inner radius $r = 4 - (2x - 1) = 5 - 2x$. The volume of the solid is therefore given by

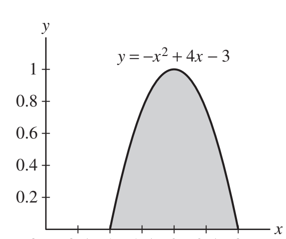
$$\pi \int_0^2 \left((5 - x^2)^2 - (5 - 2x)^2\right) dx = \pi \left(10x^2 - \frac{14}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = \pi \left(40 - \frac{112}{3} + \frac{32}{5}\right) = \frac{136\pi}{15}.$$

35. $y = -x^2 + 4x - 3, \quad y = 0, \quad \text{axis } y = -1$

SOLUTION The region bounded by the graph of $y = -x^2 + 4x - 3$ and the x-axis is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = -x^2 + 4x - 3 - (-1) = -x^2 + 4x - 2$ and inner radius $r = 0 - (-1) = 1$. The volume of the solid is therefore given by

$$\pi \int_1^3 \left((-x^2 + 4x - 2)^2 - 1\right) dx = \pi \left(\frac{1}{5}x^5 - 2x^4 + \frac{20}{3}x^3 - 8x^2 + 3x\right)\Big|_1^3$$

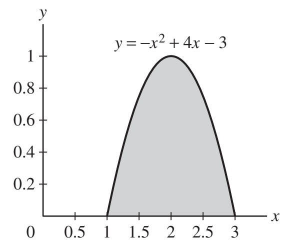
$$= \pi \left[\left(\frac{243}{5} - 162 + 180 - 72 + 9\right) - \left(\frac{1}{5} - 2 + \frac{20}{3} - 8 + 3\right)\right] = \frac{56\pi}{15}.$$



36. $y = -x^2 + 4x - 3, \quad y = 0, \quad \text{axis } x = 4$

SOLUTION The region bounded by the graph of $y = -x^2 + 4x - 3$ and the x-axis is shown in the previous exercise. Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the shell has radius $4 - x$ and height $-x^2 + 4x - 3$. The volume of the solid is therefore given by

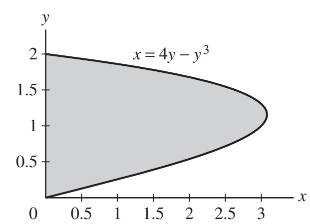
$$\begin{aligned} 2\pi \int_1^3 (4-x)(-x^2+4x-3)\,dx &= 2\pi \int_1^3 (x^3 - 8x^2 + 19x - 12)\,dx \\ &= 2\pi \left(\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x \right)\Bigg|_1^3 \\ &= 2\pi \left(-\frac{9}{4} + \frac{59}{12} \right) = \frac{16\pi}{3}. \end{aligned}$$



37. $x = 4y - y^3, \quad x = 0, \quad y \ge 0, \quad x\text{-axis}$

SOLUTION The region bounded by the graphs of $x = 4y - y^3$ and $x = 0$ for $y \ge 0$ is shown below. Let's choose to slice this region horizontally. Because a horizontal slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $y \in [0, 2]$, the shell has radius y and height $4y - y^3$. The volume of the solid is therefore given by

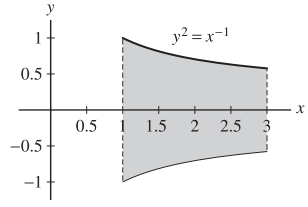
$$\begin{aligned} 2\pi \int_0^2 y(4y - y^3)\,dy &= 2\pi \int_0^2 (4y^2 - y^4)\,dy \\ &= 2\pi \left(\frac{4}{3}y^3 - \frac{1}{5}y^5 \right)\Bigg|_0^2 \\ &= 2\pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{128\pi}{15}. \end{aligned}$$



38. $y^2 = x^{-1}, \quad x = 1, \quad x = 3, \quad \text{axis } y = -3$

SOLUTION The region bounded by the graphs of $y^2 = x^{-1}$, $x = 1$ and $x = 3$ is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = \frac{1}{\sqrt{x}} - (-3) = 3 + \frac{1}{\sqrt{x}}$ and inner radius $r = -\frac{1}{\sqrt{x}} - (-3) = 3 - \frac{1}{\sqrt{x}}$. The volume of the solid is therefore given by

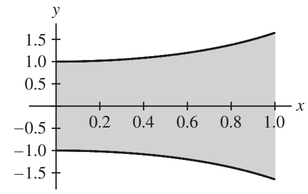
$$\pi \int_1^3 \left(\left(3 + \frac{1}{\sqrt{x}}\right)^2 - \left(3 - \frac{1}{\sqrt{x}}\right)^2 \right) dx = 12\pi \int_1^3 x^{-1/2}\,dx = 24\pi\sqrt{x}\Big|_1^3 = 24\pi(\sqrt{3} - 1).$$



39. $y = e^{-x^2/2}, \quad y = -e^{-x^2/2}, \quad x = 0, \quad x = 1, \quad y\text{-axis}$

SOLUTION Let's choose to slice the region bounded by the graphs of $y = e^{-x^2/2}$ and $y = -e^{-x^2/2}$ (see the figure below) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, 1]$, the shell has radius x and height $e^{-x^2/2} - (-e^{-x^2/2}) = 2e^{-x^2/2}$. The volume of the solid is therefore given by

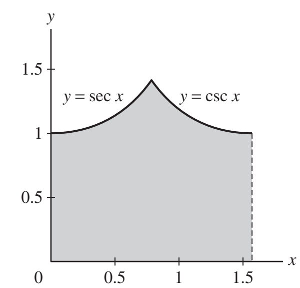
$$2\pi \int_0^1 2xe^{-x^2/2}\, dx = -4\pi e^{-x^2/2}\Big|_0^1$$
$$= -4\pi(e^{-1/2} - 1) = 4\pi(1 - e^{-1/2}).$$



40. $y = \sec x, \quad y = \csc x, \quad y = 0, \quad x = 0, \quad x = \dfrac{\pi}{2}, \quad x\text{-axis}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, cross sections for $x \in [0, \pi/4]$ are circular disks with radius $R = \sec x$, whereas cross sections for $x \in [\pi/4, \pi/2]$ are circular disks with radius $R = \csc x$.

(c) The volume of the solid of revolution is

$$\pi \int_0^{\pi/4} \sec^2 x\, dx + \pi \int_{\pi/4}^{\pi/2} \csc^2 x\, dx = \pi\,(\tan x)\Big|_0^{\pi/4} + \pi\,(-\cot x)\Big|_{\pi/4}^{\pi/2} = \pi\,(1) + \pi\,(1) = 2\pi.$$

In Exercises 41–44, find the volume obtained by rotating the region about the given axis. The regions refer to the graph of the hyperbola $y^2 - x^2 = 1$ *in Figure 4.*

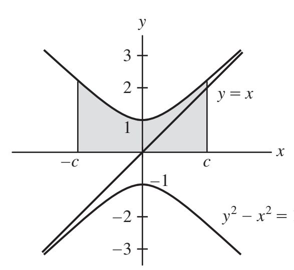


FIGURE 4

41. The shaded region between the upper branch of the hyperbola and the x-axis for $-c \le x \le c$, about the x-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [-c, c]$, cross sections are circular disks with radius $R = \sqrt{1 + x^2}$. The volume of the solid is therefore given by

$$\pi \int_{-c}^{c} (1 + x^2)\, dx = \pi\,\left(x + \frac{1}{3}x^3\right)\Big|_{-c}^{c} = \pi\left[\left(c + \frac{c^3}{3}\right) - \left(-c - \frac{c^3}{3}\right)\right] = 2\pi\left(c + \frac{c^3}{3}\right).$$

42. The region between the upper branch of the hyperbola and the x-axis for $0 \le x \le c$, about the y-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each $x \in [0, c]$, the shell has radius x and height $\sqrt{1+x^2}$. The volume of the solid is therefore given by

$$2\pi \int_0^c x\sqrt{1+x^2}\,dx = \frac{2\pi}{3}(1+x^2)^{3/2}\Big|_0^c = \frac{2\pi}{3}\left((1+c^2)^{3/2}-1\right).$$

43. The region between the upper branch of the hyperbola and the line $y = x$ for $0 \le x \le c$, about the x-axis.

SOLUTION Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [0, c]$, cross sections are washers with outer radius $R = \sqrt{1+x^2}$ and inner radius $r = x$. The volume of the solid is therefore given by

$$\pi \int_0^c \left((1+x^2) - x^2\right)\,dx = \pi x\Big|_0^c = c\pi.$$

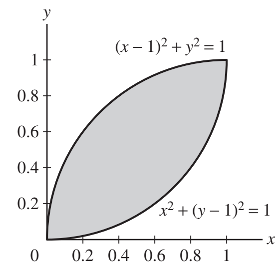
44. The region between the upper branch of the hyperbola and $y = 2$, about the y-axis.

SOLUTION The upper branch of the hyperbola and the horizontal line $y = 2$ intersect when $x = \pm\sqrt{3}$. Using the shell method, each shell has radius x and height $2 - \sqrt{1+x^2}$. The volume of the solid is therefore given by

$$2\pi \int_0^{\sqrt{3}} x\left(2 - \sqrt{1+x^2}\right)\,dx = 2\pi\left(x^2 - \frac{1}{3}(1+x^2)^{3/2}\right)\Big|_0^{\sqrt{3}} = 2\pi\left(3 - \frac{8}{3} + \frac{1}{3}\right) = \frac{4\pi}{3}.$$

45. Let R be the intersection of the circles of radius 1 centered at $(1, 0)$ and $(0, 1)$. Express as an integral (but do not evaluate): **(a)** the area of R and **(b)** the volume of revolution of R about the x-axis.

SOLUTION The region R is shown below.



(a) A vertical slice of R has its top along the upper left arc of the circle $(x-1)^2 + y^2 = 1$ and its bottom along the lower right arc of the circle $x^2 + (y-1)^2 = 1$. The area of R is therefore given by

$$\int_0^1 \left(\sqrt{1-(x-1)^2} - (1 - \sqrt{1-x^2})\right)\,dx.$$

(b) If we revolve R about the x-axis and use the washer method, each cross section is a washer with outer radius $\sqrt{1-(x-1)^2}$ and inner radius $1 - \sqrt{1-x^2}$. The volume of the solid is therefore given by

$$\pi \int_0^1 \left[(1-(x-1)^2) - (1-\sqrt{1-x^2})^2\right]\,dx.$$

46. Let $a > 0$. Show that the volume obtained when the region between $y = a\sqrt{x - ax^2}$ and the x-axis is rotated about the x-axis is independent of the constant a.

SOLUTION Setting $a\sqrt{x-ax^2} = 0$ yields $x = 0$ and $x = 1/a$. Using the washer method, cross sections are circular disks with radius $R = a\sqrt{x-ax^2}$. The volume of the solid is therefore given by

$$\pi \int_0^{1/a} a^2(x - ax^2)\,dx = \pi\left(\frac{1}{2}a^2x^2 - \frac{1}{3}a^3x^3\right)\Big|_0^{1/a} = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6},$$

which is independent of the constant a.

47. If 12 J of work are needed to stretch a spring 20 cm beyond equilibrium, how much work is required to compress it 6 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant k as follows:

$$\frac{1}{2}k(0.2)^2 = 12 \quad \text{so} \quad k = 600 \text{ N/m}.$$

Now, the work needed to compress the spring 6 cm beyond equilibrium is

$$W = \int_0^{0.06} 600x\,dx = 300x^2\Big|_0^{0.06} = 1.08 \text{ J}.$$

48. A spring whose equilibrium length is 15 cm exerts a force of 50 N when it is stretched to 20 cm. Find the work required to stretch the spring from 22 to 24 cm.

SOLUTION A force of 50 N is exerted when the spring is stretched 5 cm = 0.05 m from its equilibrium length; therefore, the value of the spring constant is $k = 1000$ N/m. The work required to stretch the spring from a length of 22 cm to a length of 24 cm is then

$$\int_{0.07}^{0.09} 1000x\,dx = 500x^2\Big|_{0.07}^{0.09} = 500(0.09^2 - 0.07^2) = 1.6 \text{ J}.$$

49. If 18 ft-lb of work are needed to stretch a spring 1.5 ft beyond equilibrium, how far will the spring stretch if a 12-lb weight is attached to its end?

SOLUTION First, we determine the value of the spring constant as follows:

$$\frac{1}{2}k(1.5)^2 = 18 \quad \text{so} \quad k = 16 \text{ lb/ft}.$$

Now, if a 12-lb weight is attached to the end of the spring, balancing the forces acting on the weight, we have $12 = 16d$, which implies $d = 0.75$ ft. A 12-lb weight will therefore stretch the spring 9 inches.

50. Let W be the work (against the sun's gravitational force) required to transport an 80-kg person from Earth to Mars when the two planets are aligned with the sun at their minimal distance of 55.7×10^6 km. Use Newton's Universal Law of Gravity (see Exercises 35–37 in Section 6.5) to express W as an integral and evaluate it. The sun has mass $M_s = 1.99 \times 10^{30}$ kg, and the distance from the sun to the earth is 149.6×10^6 km.

SOLUTION According to Newton's Universal Law of Gravity, the gravitational force between the person and the sun is

$$\frac{GM_s m}{r^2},$$

where $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-1} is a constant, $M_s = 1.99 \times 10^{30}$ kg is the mass of the sun, $m = 80$ kg is the mass of the person, and r is the distance between the sun and the person. The work against the sun's gravitational force required to transport the person from Earth to Mars when the two planets are aligned with the sun is therefore given by

$$W = \int_{r_{se}}^{r_{se}+r_{em}} \frac{GM_s m}{r^2}\,dr = GM_s m\left(\frac{1}{r_{se}} - \frac{1}{r_{se}+r_{em}}\right),$$

where $r_{se} = 149.6 \times 10^6$ km is the distance from the sun to Earth and $r_{em} = 55.7 \times 10^6$ km is the distance from Earth to Mars. Converting the distances to meters and substituting the known values into the formula for W yields

$$W = (6.67 \times 10^{-11})(1.99 \times 10^{30})(80)\left(\frac{1}{149.6 \times 10^9} - \frac{1}{205.3 \times 10^9}\right) \approx 1.93 \times 10^{10} \text{ J}.$$

In Exercises 51 and 52, water is pumped into a spherical tank of radius 2 m *from a source located* 1 m *below a hole at the bottom (Figure 5). The density of water is* 1000 kg/m^3.

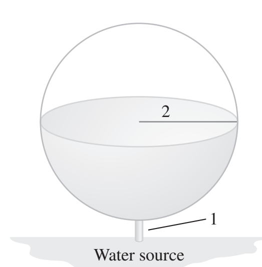


FIGURE 5

51. Calculate the work required to fill the tank.

SOLUTION Place the origin at the base of the sphere with the positive y-axis pointing upward. The equation for the great circle of the sphere is then $x^2 + (y-2)^2 = 4$. At location y, the horizontal cross section is a circle of radius $\sqrt{4-(y-2)^2} = \sqrt{4y-y^2}$; the volume of the layer is then $\pi(4y-y^2)\Delta y$ m^3, and the force needed to lift the layer is $1000(9.8)\pi(4y-y^2)\Delta y$ N. The layer of water must be lifted $y+1$ meters, so the work required to fill the tank is given by

$$\begin{aligned} 9800\pi \int_0^4 (y+1)(4y-y^2)\,dy &= 9800\pi \int_0^4 (3y^2+4y-y^3)\,dy \\ &= 9800\pi \left(y^3 + 2y^2 - \frac{1}{4}y^4 \right)\Big|_0^4 \\ &= 313{,}600\pi \approx 985{,}203.5 \text{ J}. \end{aligned}$$

52. Calculate the work $F(h)$ required to fill the tank to level h meters in the sphere.

SOLUTION Place the origin at the base of the sphere with the positive y-axis pointing upward. The equation for the great circle of the sphere is then $x^2 + (y-2)^2 = 4$. At location y, the horizontal cross section is a circle of radius $\sqrt{4-(y-2)^2} = \sqrt{4y-y^2}$; the volume of the layer is then $\pi(4y-y^2)\Delta y$ m^3, and the force needed to lift the layer is $1000(9.8)\pi(4y-y^2)\Delta y$ N. The layer of water must be lifted $y+1$ meters, so the work required to fill the tank is given by

$$\begin{aligned} 9800\pi \int_0^h (y+1)(4y-y^2)\,dy &= 9800\pi \int_0^h (3y^2+4y-y^3)\,dy \\ &= 9800\pi \left(y^3 + 2y^2 - \frac{1}{4}y^4 \right)\Big|_0^h \\ &= 9800\pi \left(h^3 + 2h^2 - \frac{1}{4}h^4 \right) \text{ J}. \end{aligned}$$

53. A tank of mass 20 kg containing 100 kg of water (density 1000 kg/m^3) is raised vertically at a constant speed of 100 m/min for one minute, during which time it leaks water at a rate of 40 kg/min. Calculate the total work performed in raising the container.

SOLUTION Let t denote the elapsed time in minutes and let y denote the height of the container. Given that the speed of ascent is 100 m/min, $y = 100t$; moreover, the mass of water in the container is

$$100 - 40t = 100 - 0.4y\text{kg}.$$

The force needed to lift the container and its contents is then

$$9.8\,(20 + (100 - 0.4y)) = 1176 - 3.92y \text{ N},$$

and the work required to lift the container and its contents is

$$\int_0^{100} (1176 - 3.92y)\,dy = (1176y - 1.96y^2)\Big|_0^{100} = 98{,}000\text{J}.$$

7 TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

Preliminary Questions

1. Which derivative rule is used to derive the Integration by Parts formula?

SOLUTION The Integration by Parts formula is derived from the Product Rule.

2. For each of the following integrals, state whether substitution or Integration by Parts should be used:

$$\int x\cos(x^2)\,dx,\qquad \int x\cos x\,dx,\qquad \int x^2e^x\,dx,\qquad \int xe^{x^2}\,dx$$

SOLUTION

(a) $\int x\cos(x^2)\,dx$: use the substitution $u = x^2$.

(b) $\int x\cos x\,dx$: use Integration by Parts.

(c) $\int x^2e^x\,dx$; use Integration by Parts.

(d) $\int xe^{x^2}\,dx$; use the substitution $u = x^2$.

3. Why is $u = \cos x$, $v' = x$ a poor choice for evaluating $\int x\cos x\,dx$?

SOLUTION Transforming $v' = x$ into $v = \frac{1}{2}x^2$ increases the power of x and makes the new integral harder than the original.

Exercises

In Exercises 1–6, evaluate the integral using the Integration by Parts formula with the given choice of u and v'.

1. $\int x\sin x\,dx;\quad u = x,\ v' = \sin x$

SOLUTION Using the given choice of u and v' results in

$$\begin{aligned} u &= x & v &= -\cos x\\ u' &= 1 & v' &= \sin x \end{aligned}$$

Using Integration by Parts,

$$\int x\sin x\,dx = x(-\cos x) - \int (1)(-\cos x)\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

2. $\int xe^{2x}\,dx;\quad u = x,\ v' = e^{2x}$

SOLUTION Using $u = x$ and $v' = e^{2x}$ gives us

$$\begin{aligned} u &= x & v &= \tfrac{1}{2}e^{2x}\\ u' &= 1 & v' &= e^{2x} \end{aligned}$$

Integration by Parts gives us

$$\int xe^{2x}\,dx = x\left(\frac{1}{2}e^{2x}\right) - \int (1)\frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}\right)e^{2x} + C = \frac{1}{4}e^{2x}\,(2x-1) + C.$$

3. $\int (2x+9)e^x\,dx$; $u = 2x+9$, $v' = e^x$

SOLUTION Using $u = 2x+9$ and $v' = e^x$ gives us

$$\begin{aligned} u &= 2x+9 & v &= e^x \\ u' &= 2 & v' &= e^x \end{aligned}$$

Integration by Parts gives us

$$\int (2x+9)e^x\,dx = (2x+9)e^x - \int 2e^x\,dx = (2x+9)e^x - 2e^x + C = e^x(2x+7) + C.$$

4. $\int x\cos 4x\,dx$; $u = x$, $v' = \cos 4x$

SOLUTION Using $u = x$ and $v' = \cos 4x$ gives us

$$\begin{aligned} u &= x & v &= \tfrac{1}{4}\sin 4x \\ u' &= 1 & v' &= \cos 4x \end{aligned}$$

Integration by Parts gives us

$$\begin{aligned} \int x\cos 4x\,dx &= \frac{1}{4}x\sin 4x - \int (1)\frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + C \\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C. \end{aligned}$$

5. $\int x^3\ln x\,dx$; $u = \ln x$, $v' = x^3$

SOLUTION Using $u = \ln x$ and $v' = x^3$ gives us

$$\begin{aligned} u &= \ln x & v &= \tfrac{1}{4}x^4 \\ u' &= \tfrac{1}{x} & v' &= x^3 \end{aligned}$$

Integration by Parts gives us

$$\begin{aligned} \int x^3\ \ln x\,dx &= (\ln x)\left(\frac{1}{4}x^4\right) - \int \left(\frac{1}{x}\right)\left(\frac{1}{4}x^4\right)dx \\ &= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C = \frac{x^4}{16}(4\ln x - 1) + C. \end{aligned}$$

6. $\int \tan^{-1}x\,dx$; $u = \tan^{-1}x$, $v' = 1$

SOLUTION Using $u = \tan^{-1}x$ and $v' = 1$ gives us

$$\begin{aligned} u &= \tan^{-1}x & v &= x \\ u' &= \frac{1}{x^2+1} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \left(\frac{1}{x^2+1}\right)x\,dx.$$

For the integral on the right we'll use the substitution $w = x^2+1$, $dw = 2x\,dx$. Then we have

$$\begin{aligned} \int \tan^{-1}x\,dx &= x\tan^{-1}x - \frac{1}{2}\int \left(\frac{1}{x^2+1}\right)2x\,dx = x\tan^{-1}x - \frac{1}{2}\int \frac{dw}{w} \\ &= x\tan^{-1}x - \frac{1}{2}\ln|w| + C = x\tan^{-1}x - \frac{1}{2}\ln|x^2+1| + C. \end{aligned}$$

In Exercises 7–36, evaluate using Integration by Parts.

7. $\int (4x-3)e^{-x}\,dx$

SOLUTION Let $u = 4x - 3$ and $v' = e^{-x}$. Then we have

$$u = 4x - 3 \qquad v = -e^{-x}$$
$$u' = 4 \qquad v' = e^{-x}$$

Using Integration by Parts, we get

$$\int (4x-3)e^{-x}\,dx = (4x-3)(-e^{-x}) - \int (4)(-e^{-x})\,dx$$
$$= -e^{-x}(4x-3) + 4\int e^{-x}\,dx = -e^{-x}(4x-3) - 4e^{-x} + C = -e^{-x}(4x+1) + C.$$

8. $\int (2x+1)e^{x}\,dx$

SOLUTION Let $u = 2x + 1$ and $v' = e^{-x}$. Then we have

$$u = 2x + 1 \qquad v = -e^{-x}$$
$$u' = 2 \qquad v' = e^{-x}$$

Using Integration by Parts, we get

$$\int (2x+1)\,e^{-x}\,dx = (2x+1)(-e^{-x}) - \int (2)(-e^{-x})\,dx$$
$$= -(2x+1)e^{-x} + 2\int e^{-x}\,dx = -(2x+1)e^{-x} - 2e^{-x} + C = -e^{-x}(2x+3) + C.$$

9. $\int x\,e^{5x+2}\,dx$

SOLUTION Let $u = x$ and $v' = e^{5x+2}$. Then we have

$$u = x \qquad v = \frac{1}{5}e^{5x+2}$$
$$u' = 1 \qquad v' = e^{5x+2}$$

Using Integration by Parts, we get

$$\int xe^{5x+2}\,dx = x\left(\frac{1}{5}e^{5x+2}\right) - \int (1)\left(\frac{1}{5}e^{5x+2}\right)dx = \frac{1}{5}xe^{5x+2} - \frac{1}{5}\int e^{5x+2}\,dx$$
$$= \frac{1}{5}xe^{5x+2} - \frac{1}{25}e^{5x+2} + C = \left(\frac{x}{5} - \frac{1}{25}\right)e^{5x+2} + C$$

10. $\int x^2e^x\,dx$

SOLUTION Let $u = x^2$ and $v' = e^x$. Then we have

$$u = x^2 \qquad v = e^x$$
$$u' = 2x \qquad v' = e^x$$

Using Integration by Parts, we get

$$\int x^2\,e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

We must apply Integration by Parts again to evaluate $\int xe^x\,dx$. Taking $u = x$ and $v' = e^x$, we get

$$\int xe^x\,dx = xe^x - \int (1)e^x\,dx = xe^x - e^x + C.$$

Plugging this into the original equation gives us

$$\int x^2\,e^x\,dx = x^2e^x - 2\left(xe^x - e^x\right) + C = e^x(x^2 - 2x + 2) + C.$$

11. $\int x \cos 2x\, dx$

SOLUTION Let $u = x$ and $v' = \cos 2x$. Then we have

$$u = x \quad v = \tfrac{1}{2}\sin 2x$$
$$u' = 1 \quad v' = \cos 2x$$

Using Integration by Parts, we get

$$\int x \cos 2x\, dx = x\left(\frac{1}{2}\sin 2x\right) - \int (1)\left(\frac{1}{2}\sin 2x\right) dx$$
$$= \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

12. $\int x \sin(3 - x)\, dx$

SOLUTION Let $u = x$ and $v' = \sin(3 - x)$. Then we have

$$u = x \quad v = \cos(3 - x)$$
$$u' = 1 \quad v' = \sin(3 - x)$$

Using Integration by Parts, we get

$$\int x\sin(3 - x)\, dx = x\cos(3 - x) - \int (1)\cos(3 - x)\, dx = x\cos(3 - x) + \sin(3 - x) + C$$

13. $\int x^2 \sin x\, dx$

SOLUTION Let $u = x^2$ and $v' = \sin x$. Then we have

$$u = x^2 \quad v = -\cos x$$
$$u' = 2x \quad v' = \sin x$$

Using Integration by Parts, we get

$$\int x^2\sin x\, dx = x^2(-\cos x) - \int 2x(-\cos x)\, dx = -x^2\cos x + 2\int x\cos x\, dx.$$

We must apply Integration by Parts again to evaluate $\int x\cos x\, dx$. Taking $u = x$ and $v' = \cos x$, we get

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C.$$

Plugging this into the original equation gives us

$$\int x^2\sin x\, dx = -x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

14. $\int x^2 \cos 3x\, dx$

SOLUTION Let $u = x^2$ and $v' = \cos 3x$. Then we have

$$u = x^2 \quad v = \frac{1}{3}\sin 3x$$
$$u' = 2x \quad v' = \cos 3x$$

Using Integration by Parts, we get

$$\int x^2\cos 3x\, dx = \frac{1}{3}x^2\sin 3x - \int (2x)\frac{1}{3}\sin 3x\, dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\int x\sin 3x\, dx$$

Use Integration by Parts again on this integral, with $u = x$ and $v' = \sin 3x$ to get

$$\int x^2\cos 3x\, dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\, dx\right)$$
$$= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$$

15. $\int e^{-x} \sin x \, dx$

SOLUTION Let $u = e^{-x}$ and $v' = \sin x$. Then we have

$$u = e^{-x} \qquad v = -\cos x$$
$$u' = -e^{-x} \qquad v' = \sin x$$

Using Integration by Parts, we get

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \int (-e^{-x})(-\cos x)\, dx = -e^{-x} \cos x - \int e^{-x} \cos x \, dx.$$

We must apply Integration by Parts again to evaluate $\int e^{-x} \cos x \, dx$. Using $u = e^{-x}$ and $v' = \cos x$, we get

$$\int e^{-x} \cos x \, dx = e^{-x} \sin x - \int (-e^{-x})(\sin x)\, dx = e^{-x} \sin x + \int e^{-x} \sin x \, dx.$$

Plugging this into the original equation, we get

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \left[e^{-x} \sin x + \int e^{-x} \sin x \, dx \right].$$

Solving this equation for $\int e^{-x} \sin x \, dx$ gives us

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C.$$

16. $\int e^{x} \sin 2x \, dx$

SOLUTION Let $u = \sin 2x$ and $v' = e^{x}$. Then we have

$$u = \sin 2x \qquad v = e^{x}$$
$$u' = 2\cos 2x \qquad v' = e^{x}$$

Using Integration by Parts, we get

$$\int e^{x} \sin 2x \, dx = e^{x} \sin 2x - 2 \int e^{x} \cos 2x \, dx.$$

We must apply Integration by Parts again to evaluate $\int e^{x} \cos 2x \, dx$. Using $u = \cos 2x$ and $v' = e^{x}$, we get

$$\int e^{x} \cos 2x \, dx = e^{x} \cos 2x - \int (-2 \sin 2x) e^{x} \, dx = e^{x} \cos 2x + 2 \int e^{x} \sin 2x \, dx.$$

Plugging this into the original equation, we get

$$\int e^{x} \sin 2x \, dx = e^{x} \sin 2x - 2 \left[e^{x} \cos 2x + 2 \int e^{x} \sin 2x \, dx \right] = e^{x} \sin 2x - 2e^{x} \cos 2x - 4 \int e^{x} \sin 2x \, dx.$$

Solving this equation for $\int e^{x} \sin 2x \, dx$ gives us

$$\int e^{x} \sin 2x \, dx = \frac{1}{5} e^{x} (\sin 2x - 2\cos 2x) + C.$$

17. $\int e^{-5x} \sin x \, dx$

SOLUTION Let $u = \sin x$ and $v' = e^{-5x}$. Then we have

$$u = \sin x \qquad v = -\frac{1}{5} e^{-5x}$$
$$u' = \cos x \qquad v' = e^{-5x}$$

Using Integration by Parts, we get

$$\int e^{-5x} \sin x \, dx = -\frac{1}{5} e^{-5x} \sin x - \int \cos x \left(-\frac{1}{5} e^{-5x} \right) dx = -\frac{1}{5} e^{-5x} \sin x + \frac{1}{5} \int e^{-5x} \cos x \, dx$$

Apply Integration by Parts again to this integral, with $u = \cos x$ and $v' = e^{-5x}$ to get

$$\int e^{-5x} \cos x \, dx = -\frac{1}{5} e^{-5x} \cos x - \frac{1}{5} \int e^{-5x} \sin x \, dx$$

Plugging this into the original equation, we get

$$\begin{aligned} \int e^{-5x} \sin x \, dx &= -\frac{1}{5} e^{-5x} \sin x + \frac{1}{5} \left(-\frac{1}{5} e^{-5x} \cos x - \frac{1}{5} \int e^{-5x} \sin x \, dx \right) \\ &= -\frac{1}{5} e^{-5x} \sin x - \frac{1}{25} e^{-5x} \cos x - \frac{1}{25} \int e^{-5x} \sin x \, dx \end{aligned}$$

Solving this equation for $\int e^{-5x} \sin x \, dx$ gives us

$$\int e^{-5x} \sin x \, dx = -\frac{5}{26} e^{-5x} \sin x - \frac{1}{26} e^{-5x} \cos x + C = -\frac{1}{26} e^{-5x} (5 \sin x + \cos x) + C$$

18. $\int e^{3x} \cos 4x \, dx$

SOLUTION Let $u = \cos 4x$ and $v' = e^{3x}$. Then we have

$$\begin{aligned} u &= \cos 4x & v &= \frac{1}{3} e^{3x} \\ u' &= -4 \sin 4x & v' &= e^{3x} \end{aligned}$$

Using Integration by Parts, we get

$$\int e^{3x} \cos 4x \, dx = \frac{1}{3} e^{3x} \cos 4x - \int \frac{1}{3} e^{3x} (-4 \sin 4x) \, dx = \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \int e^{3x} \sin 4x \, dx$$

Apply Integration by Parts again to this integral, with $u = \sin 4x$ and $v' = e^{3x}$, to get

$$\int e^{3x} \sin 4x \, dx = \frac{1}{3} e^{3x} \sin 4x - \int \frac{1}{3} e^{3x} \cdot 4 \cos 4x \, dx = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx$$

Plugging this into the original equation, we get

$$\begin{aligned} \int e^{3x} \cos 4x \, dx &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{3} \left(\frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int e^{3x} \cos 4x \, dx \right) \\ &= \frac{1}{3} e^{3x} \cos 4x + \frac{4}{9} e^{3x} \sin 4x - \frac{16}{9} \int e^{3x} \cos 4x \, dx \end{aligned}$$

Solving this equation for $\int e^{3x} \cos 4x \, dx$ gives us

$$\int e^{3x} \cos 4x \, dx = \frac{3}{25} e^{3x} \cos 4x + \frac{4}{25} e^{3x} \sin 4x = \frac{1}{25} e^{3x} (3 \cos 4x + 4 \sin 4x) + C$$

19. $\int x \ln x \, dx$

SOLUTION Let $u = \ln x$ and $v' = x$. Then we have

$$\begin{aligned} u &= \ln x & v &= \tfrac{1}{2} x^2 \\ u' &= \tfrac{1}{x} & v' &= x \end{aligned}$$

Using Integration by Parts, we get

$$\begin{aligned} \int x \ln x \, dx &= \frac{1}{2} x^2 \ln x - \int \left(\frac{1}{x} \right) \left(\frac{1}{2} x^2 \right) dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \left(\frac{x^2}{2} \right) + C = \frac{1}{4} x^2 (2 \ln x - 1) + C. \end{aligned}$$

20. $\displaystyle\int \frac{\ln x}{x^2}\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-2}$. Then we have

$$u = \ln x \qquad v = -x^{-1}$$
$$u' = \tfrac{1}{x} \qquad v' = x^{-2}$$

Using Integration by Parts, we get

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx = -\frac{1}{x}\ln x + \int x^{-2}\,dx$$
$$= -\frac{1}{x}\ln x - \frac{1}{x} + C = -\frac{1}{x}(\ln x + 1) + C.$$

21. $\displaystyle\int x^2 \ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^2$. Then we have

$$u = \ln x \qquad v = \tfrac{1}{3}x^3$$
$$u' = \tfrac{1}{x} \qquad v' = x^2$$

Using Integration by Parts, we get

$$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{x}\left(\frac{1}{3}x^3\right)dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + C = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C.$$

22. $\displaystyle\int x^{-5}\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-5}$. Then we have

$$u = \ln x \qquad v = -\frac{1}{4}x^{-4}$$
$$u' = \frac{1}{x} \qquad v = x^{-5}$$

Using Integration by Parts, we get

$$\int x^{-5}\ln x\,dx = -\frac{1}{4}x^{-4}\ln x + \int \frac{1}{4}x^{-4}\frac{1}{x}\,dx = -\frac{1}{4}x^{-4}\ln x + \frac{1}{4}\int x^{-5}\,dx$$
$$= -\frac{1}{4}x^{-4}\ln x - \frac{1}{16}x^{-4} + C = -\frac{1}{4x^4}\left(\ln x + \frac{1}{4}\right) + C$$

23. $\displaystyle\int (\ln x)^2\,dx$

SOLUTION Let $u = (\ln x)^2$ and $v' = 1$. Then we have

$$u = (\ln x)^2 \qquad v = x$$
$$u' = \frac{2}{x}\ln x \qquad v' = 1$$

Using Integration by Parts, we get

$$\int (\ln x)^2\,dx = (\ln x)^2(x) - \int \left(\frac{2}{x}\ln x\right)x\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

We must apply Integration by Parts again to evaluate $\int \ln x\,dx$. Using $u = \ln x$ and $v' = 1$, we have

$$\int \ln x\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Plugging this into the original equation, we get

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\,(x\ln x - x) + C = x\left[(\ln x)^2 - 2\ln x + 2\right] + C.$$

24. $\int x(\ln x)^2\,dx$

SOLUTION Let $u = (\ln x)^2$, $v' = x$. Then we have

$$u = (\ln x)^2 \qquad v = \frac{1}{2}x^2$$
$$u' = \frac{2\ln x}{x} \qquad v' = x$$

Using Integration by Parts, we get

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x^2\frac{\ln x}{x}\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$$

Apply Integration by Parts again to this integral, with $u = \ln x$, $v' = x$, to get

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Plug this back into the first formula to get

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C = \frac{1}{2}x^2\left((\ln x)^2 - \ln x + \frac{1}{2}\right) + C$$

25. $\int x\sec^2 x\,dx$

SOLUTION Let $u = x$ and $v' = \sec^2 x$. Then we have

$$u = x \qquad v = \tan x$$
$$u' = 1 \qquad v' = \sec^2 x$$

Using Integration by Parts, we get

$$\int x\sec^2 x\,dx = x\tan x - \int (1)\tan x\,dx = x\tan x - \ln|\sec x| + C.$$

26. $\int x\tan x\sec x\,dx$

SOLUTION Let $u = x$ and $v' = \tan x\sec x$. Then we have

$$u = x \qquad v = \sec x$$
$$u' = 1 \qquad v' = \tan x\sec x$$

Using Integration by Parts, we get

$$\int x\tan x\sec x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$$

27. $\int \cos^{-1} x\,dx$

SOLUTION Let $u = \cos^{-1} x$ and $v' = 1$. Then we have

$$u = \cos^{-1} x \qquad v = x$$
$$u' = \frac{-1}{\sqrt{1-x^2}} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\,dx.$$

We can evaluate $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\,dx$, and we have

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = x\cos^{-1} x - \frac{1}{2}\int w^{-1/2}\,dw$$
$$= x\cos^{-1} x - \frac{1}{2}(2w^{1/2}) + C = x\cos^{-1} x - \sqrt{1-x^2} + C.$$

28. $\int \sin^{-1} x\, dx$

SOLUTION Let $u = \sin^{-1} x$ and $v' = 1$. Then we have

$$\begin{aligned} u &= \sin^{-1} x & v &= x \\ u' &= \frac{1}{\sqrt{1-x^2}} & v' &= 1 \end{aligned}$$

Using Integration by Parts, we get

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

We can evaluate $\int \frac{x}{\sqrt{1-x^2}}\, dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\, dx$, and we have

$$\begin{aligned} \int \sin^{-1} x\, dx &= x \sin^{-1} x + \frac{1}{2} \int \frac{-2x\, dx}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int w^{-1/2}\, dw \\ &= x \sin^{-1} x + \frac{1}{2}(2w^{1/2}) + C = x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

29. $\int \sec^{-1} x\, dx$

SOLUTION We are forced to choose $u = \sec^{-1} x$, $v' = 1$, so that $u' = \frac{1}{x\sqrt{x^2-1}}$ and $v = x$. Using Integration by parts, we get:

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \frac{x\, dx}{x\sqrt{x^2-1}} = x \sec^{-1} x - \int \frac{dx}{\sqrt{x^2-1}}.$$

Via the substitution $\sqrt{x^2-1} = \tan\theta$ (so that $x = \sec\theta$ and $dx = \sec\theta \tan\theta d\theta$), we get:

$$\begin{aligned} \int \sec^{-1} x\, dx &= x \sec^{-1} x - \int \frac{\sec\theta \tan\theta d\theta}{\tan\theta} = x \sec^{-1} x - \int \sec\theta d\theta \\ &= x \sec^{-1} x - \ln|\sec\theta + \tan\theta| + C = x \sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C. \end{aligned}$$

30. $\int x 5^x\, dx$

SOLUTION Let $u = x$ and $v' = 5^x$. Then we have

$$\begin{aligned} u &= x & v &= \frac{5^x}{\ln 5} \\ u' &= 1 & v' &= 5^x \end{aligned}$$

Using Integration by Parts, we get

$$\begin{aligned} \int x\, 5^x\, dx &= x\left(\frac{5^x}{\ln 5}\right) - \int (1)\frac{5^x}{\ln 5}\, dx = \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5}\int 5^x\, dx \\ &= \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5}\left(\frac{5^x}{\ln 5}\right) + C = \frac{5^x}{\ln 5}\left(x - \frac{1}{\ln 5}\right) + C. \end{aligned}$$

31. $\int 3^x \cos x\, dx$

SOLUTION Let $u = \cos x$ and $v' = 3^x$. Then we have

$$\begin{aligned} u &= \cos x & v &= \frac{3^x}{\ln 3} \\ u' &= -\sin x & v' &= 3^x \end{aligned}$$

Using Integration by Parts, we get

$$\int 3^x \cos x\, dx = \frac{3^x}{\ln 3} \cos x + \frac{1}{\ln 3} \int 3^x \sin x\, dx$$

Apply Integration by Parts to the remaining integral, with $u = \sin x$ and $v' = 3^x$; then

$$\int 3^x \sin x\,dx = \frac{3^x}{\ln 3}\sin x - \frac{1}{\ln 3}\int 3^x \cos x\,dx$$

Plug this into the first equation to get

$$\begin{aligned}\int 3^x \cos x\,dx &= \frac{3^x}{\ln 3}\cos x + \frac{1}{\ln 3}\left(\frac{3^x}{\ln 3}\sin x - \frac{1}{\ln 3}\int 3^x \cos x\,dx\right)\\ &= \frac{3^x}{\ln 3}\cos x + \frac{3^x}{(\ln 3)^2}\sin x - \frac{1}{(\ln 3)^2}\int 3^x \cos x\,dx\end{aligned}$$

Solving for $\int 3^x \cos x\,dx$ gives

$$\int 3^x \cos x\,dx = \frac{3^x \ln 3 \cos x}{1 + (\ln 3)^2} + \frac{3^x \sin x}{1 + (\ln 3)^2} + C = \frac{3^x}{1 + (\ln 3)^2}(\ln 3 \cos x + \sin x) + C$$

32. $\int x \sinh x\,dx$

SOLUTION Let $u = x$, $v' = \sinh x$. Then

$$\begin{aligned} u &= x & v &= \cosh x\\ u' &= 1 & v' &= \sinh x\end{aligned}$$

Integration by Parts gives us

$$\int x \sinh x\,dx = x \cosh x - \int \cosh x\,dx = x \cosh x - \sinh x + C$$

33. $\int x^2 \cosh x\,dx$

SOLUTION Let $u = x^2$, $v' = \cosh x$. Then

$$\begin{aligned} u &= x^2 & v &= \sinh x\\ u' &= 2x & v' &= \cosh x\end{aligned}$$

Integration by Parts gives us (along with Exercise 32)

$$\int x^2 \cosh x\,dx = x^2 \sinh x - 2\int x \sinh x, dx = x^2 \sinh x - 2x \cosh x + 2 \sinh x + C$$

34. $\int \cos x \cosh x\,dx$

SOLUTION Let $u = \cos x$ and $v' = \cosh x$. Then

$$\begin{aligned} u &= \cos x & v &= \sinh x\\ u' &= -\sin x & v' &= \cosh x\end{aligned}$$

Integration by Parts gives us

$$\int \cos x \cosh x\,dx = \cos x \sinh x - \int (-\sin x) \sinh x\,dx = \cos x \sinh x + \int \sin x \sinh x\,dx.$$

We must apply Integration by Parts again to evaluate $\int \sin x \sinh x\,dx$. Using $u = \sin x$ and $v' = \sinh x$, we find

$$\int \sin x \sinh x\,dx = \sin x \cosh x - \int \cos x \cosh x\,dx.$$

Plugging this into the original equation, we have

$$\int \cos x \cosh x\,dx = \cos x \sinh x + \sin x \cosh x - \int \cos x \cosh x\,dx.$$

Solving this equation for $\int \cos x \cosh x\, dx$ yields

$$\int \cos x \cosh x\, dx = \frac{1}{2}(\cos x \sinh x + \sin x \cosh x) + C.$$

35. $\int \tanh^{-1} 4x\, dx$

SOLUTION Using $u = \tanh^{-1} 4x$ and $v' = 1$ gives us

$$u = \tanh^{-1} 4x \qquad v = x$$
$$u' = \frac{4}{1-16x^2} \qquad v' = 1$$

Integration by Parts gives us

$$\int \tanh^{-1} 4x\, dx = x \tanh^{-1} 4x - \int \left(\frac{4}{1-16x^2}\right) x\, dx.$$

For the integral on the right we'll use the substitution $w = 1 - 16x^2$, $dw = -32x\, dx$. Then we have

$$\begin{aligned}\int \tanh^{-1} 4x\, dx &= x \tanh^{-1} 4x + \frac{1}{8}\int \frac{dw}{w} = x \tanh^{-1} 4x + \frac{1}{8} \ln |w| + C\\ &= x \tanh^{-1} 4x + \frac{1}{8} \ln |1 - 16x^2| + C.\end{aligned}$$

36. $\int \sinh^{-1} x\, dx$

SOLUTION Using $u = \sinh^{-1} x$ and $v' = 1$ gives us

$$u = \sinh^{-1} x \qquad v = x$$
$$u' = \frac{1}{\sqrt{1+x^2}} \qquad v' = 1$$

Integration by Parts gives us

$$\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \left(\frac{1}{\sqrt{1+x^2}}\right) x\, dx.$$

For the integral on the right we'll use the substitution $w = 1 + x^2$, $dw = 2x\, dx$. Then we have

$$\begin{aligned}\int \sinh^{-1} x\, dx &= x \sinh^{-1} x - \frac{1}{2}\int \frac{dw}{\sqrt{w}} = x \sinh^{-1} x - \sqrt{w} + C\\ &= x \sinh^{-1} x - \sqrt{1+x^2} + C.\end{aligned}$$

In Exercises 37 and 38, evaluate using substitution and then Integration by Parts.

37. $\int e^{\sqrt{x}}\, dx$ *Hint:* Let $u = x^{1/2}$

SOLUTION Let $w = x^{1/2}$. Then $dw = \frac{1}{2}x^{-1/2}dx$, or $dx = 2\,x^{1/2}\,dw = 2w\,dw$. Now,

$$\int e^{\sqrt{x}}\, dx = 2\int we^w\, dw.$$

Using Integration by Parts with $u = w$ and $v' = e^w$, we get

$$2\int we^w\, dw = 2(we^w - e^w) + C.$$

Substituting back, we find

$$\int e^{\sqrt{x}}\, dx = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

38. $\int x^3 e^{x^2}\,dx$

SOLUTION Let $w = x^2$. Then $dw = 2x\,dx$, and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^w\,dw.$$

Using Integration by Parts, we let $u = w$ and $v' = e^w$. Then we have

$$\int we^w\,dw = we^w - \int (1)e^w\,dw = we^w - e^w + C.$$

Substituting back in terms of x, we get

$$\int x^3\,e^{x^2}\,dx = \frac{1}{2}\left(x^2e^{x^2} - e^{x^2}\right) + C.$$

In Exercises 39–48, evaluate using Integration by Parts, substitution, or both if necessary.

39. $\int x\cos 4x\,dx$

SOLUTION Let $u = x$ and $v' = \cos 4x$. Then we have

$$\begin{array}{ll} u = x & v = \frac{1}{4}\sin 4x \\ u' = 1 & v' = \cos 4x \end{array}$$

Using Integration by Parts, we get

$$\begin{aligned}\int x\cos 4x\,dx &= \frac{1}{4}x\sin 4x - \int (1)\frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + C \\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.\end{aligned}$$

40. $\int \frac{\ln(\ln x)\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and we have

$$\int \frac{\ln(\ln x)\,dx}{x} = \int \ln w\,dw$$

Now we can use Integration by Parts, letting $u = \ln w$ and $v' = 1$. Then $u' = 1/w$, $v = w$, and

$$\int \ln w\,dw = w\ln w - \int \frac{1}{w}(w)\,dw = w\ln w - w + C.$$

Substituting back in terms of x, we get

$$\int \frac{\ln(\ln x)\,dx}{x} = (\ln x)\ln(\ln x) - \ln x + C.$$

41. $\int \frac{x\,dx}{\sqrt{x+1}}$

SOLUTION Let $u = x + 1$. Then $du = dx$, $x = u - 1$, and

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x+1}} &= \int \frac{(u-1)\,du}{\sqrt{u}} = \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du = \int (u^{1/2} - u^{-1/2})\,du \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.\end{aligned}$$

42. $\int x^2(x^3+9)^{15}\,dx$

SOLUTION Note that $(x^3 + 0)' = 3x^2$, so use substitution with $u = x^3 + 9$, $du = 3x^2\,dx$. Then

$$\int x^2(x^3+9)^{15}\,dx = \frac{1}{3}\int u^{15}\,du = \frac{1}{48}u^{16} + C = \frac{1}{48}(x^3+9)^{16} + C$$

43. $\int \cos x \ln(\sin x)\,dx$

SOLUTION Let $w = \sin x$. Then $dw = \cos x\,dx$, and

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw.$$

Now use Integration by Parts with $u = \ln w$ and $v' = 1$. Then $u' = 1/w$ and $v = w$, which gives us

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw = w \ln w - w + C = \sin x \ln(\sin x) - \sin x + C.$$

44. $\int \sin \sqrt{x}\,dx$

SOLUTION First use substitution, with $w = \sqrt{x}$ and $dw = dx/(2\sqrt{x})$. This gives us

$$\int \sin \sqrt{x}\,dx = \int \frac{(2\sqrt{x}) \sin \sqrt{x}\,dx}{(2\sqrt{x})} = 2 \int w \sin w\,dw.$$

Now use Integration by Parts, with $u = w$ and $v' = \sin w$. Then we have

$$\begin{aligned}\int \sin \sqrt{x}\,dx &= 2 \int w \sin w\,dw = 2\left(-w \cos w - \int -\cos w\,dw\right)\\ &= 2(-w \cos w + \sin w) + C = 2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C.\end{aligned}$$

45. $\int \sqrt{x} e^{\sqrt{x}}\,dx$

SOLUTION Let $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}}\,dx$ and

$$\int \sqrt{x} e^{\sqrt{x}}\,dx = 2 \int w^2 e^w\,dw.$$

Now, use Integration by Parts with $u = w^2$ and $v' = e^w$. This gives

$$\int \sqrt{x} e^{\sqrt{x}}\,dx = 2 \int w^2 e^w\,dw = 2w^2 e^w - 4 \int w e^w\,dw.$$

We need to use Integration by Parts again, this time with $u = w$ and $v' = e^w$. We find

$$\int w e^w\,dw = w e^w - \int e^w\,dw = w e^w - e^w + C;$$

finally,

$$\int \sqrt{x} e^{\sqrt{x}}\,dx = 2w^2 e^w - 4 w e^w + 4 e^w + C = 2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4 e^{\sqrt{x}} + C.$$

46. $\int \frac{\tan \sqrt{x}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$ and $du = \frac{1}{2} x^{-1/2}$. Then

$$\int \frac{\tan \sqrt{x}\,dx}{\sqrt{x}} = 2 \int \tan u\,du = -2 \ln|\cos u| + C = -2 \ln|\cos \sqrt{x}| + C$$

47. $\int \frac{\ln(\ln x) \ln x\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and

$$\int \frac{\ln(\ln x) \ln x\,dx}{x} = \int w \ln w\,dw.$$

Now use Integration by Parts, with $u = \ln w$ and $v' = w$. Then,

$$\begin{aligned} u &= \ln w & v &= \frac{1}{2} w^2\\ u' &= w^{-1} & v' &= w \end{aligned}$$

and

$$\int \frac{\ln(\ln x)\ln x\,dx}{x} = \frac{1}{2}w^2\ln w - \frac{1}{2}\int w\,dw = \frac{1}{2}w^2\ln w - \frac{1}{2}\left(\frac{w^2}{2}\right) + C$$
$$= \frac{1}{2}(\ln x)^2\ln(\ln x) - \frac{1}{4}(\ln x)^2 + C = \frac{1}{4}(\ln x)^2[2\ln(\ln x) - 1] + C.$$

48. $\int \sin(\ln x)\,dx$

SOLUTION Let $u = \sin(\ln x)$ and $v' = 1$. Then we have

$$u = \sin(\ln x) \qquad v = x$$
$$u' = \frac{\cos(\ln x)}{x} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int (x)\frac{\cos(\ln x)}{x}\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

We must use Integration by Parts again to evaluate $\int \cos(\ln x)\,dx$. Let $u = \cos(\ln x)$ and $v' = 1$. Then

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int(-\sin(\ln x))\,dx\right]$$
$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

Solving this equation for $\int \sin(\ln x)\,dx$, we get

$$\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

In Exercises 49–54, compute the definite integral.

49. $\int_0^3 xe^{4x}\,dx$

SOLUTION Let $u = x$, $v' = e^{4x}$. Then $u' = 1$ and $v = \frac{1}{4}e^{4x}$. Using Integration by Parts,

$$\int_0^3 xe^{4x}\,dx = \left(\frac{1}{4}xe^{4x}\right)\Big|_0^3 - \frac{1}{4}\int_0^3 e^{4x}\,dx = \frac{3}{4}e^{12} - \frac{1}{16}e^{12} + \frac{1}{16} = \frac{11}{16}e^{12} + \frac{1}{16}$$

50. $\int_0^{\pi/4} x\sin 2x\,dx$

SOLUTION Let $u = x$ and $v' = \sin 2x$. Then $u' = 1$ and $v = -\frac{1}{2}\cos 2x$. Using Integration by Parts,

$$\int_0^{\pi/4} x\sin(2x)\,dx = -\frac{1}{2}x\cos 2x\Big|_0^{\pi/4} - \int_0^{\pi/4}\left(-\frac{1}{2}\cos 2x\right)dx = \left(-\frac{1}{2}x\cos 2x + \left(\frac{1}{2}\right)\frac{\sin 2x}{2}\right)\Big|_0^{\pi/4}$$
$$= \left(-\frac{1}{2}\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right) - (0+0) = \frac{1}{4}.$$

51. $\int_1^2 x\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. Using Integration by Parts gives

$$\int_1^2 x\ln x\,dx = \left(\frac{1}{2}x^2\ln x\right)\Big|_1^2 - \frac{1}{2}\int_1^2 x\,dx = 2\ln 2 - \frac{1}{4}x^2\Big|_1^2 = 2\ln 2 - \frac{3}{4}$$

52. $\int_1^e \frac{\ln x\,dx}{x^2}$

SOLUTION Let $u = \ln x$ and $v' = x^{-2}$. Then $u' = x^{-1}$ and $v = -x^{-1}$. Using Integration by Parts gives

$$\int_1^e \frac{\ln x\,dx}{x^2} = -\frac{\ln x}{x}\Big|_1^e + \int_1^e x^{-2}\,dx = -e^{-1} - x^{-1}\Big|_1^e = 1 - \frac{2}{e}$$

53. $\int_0^{\pi} e^x \sin x\, dx$

SOLUTION Let $u = \sin x$ and $v' = e^x$; then $u' = \cos x$ and $v = e^x$. Integration by Parts gives

$$\int_0^{\pi} e^x \sin x\, dx = e^x \sin x \Big|_0^{\pi} - \int_0^{\pi} e^x \cos x\, dx = -\int_0^{\pi} e^x \cos x\, dx$$

Apply integration by parts again to this integral, with $u = \cos x$ and $v' = e^x$; then $u' = -\sin x$ and $v = e^x$, so we get

$$\int_0^{\pi} e^x \sin x\, dx = -\left(\left(e^x \cos x\right)\Big|_0^{\pi} + \int_0^{\pi} e^x \sin x\, dx \right) = e^{\pi} + 1 - \int_0^{\pi} e^x \sin x\, dx$$

Solving for $\int_0^{\pi} e^x \sin x\, dx$ gives

$$\int_0^{\pi} e^x \sin x\, dx = \frac{e^{\pi} + 1}{2}$$

54. $\int_0^{1} \tan^{-1} x\, dx$

SOLUTION Let $u = \tan^{-1} x$ and $v' = 1$. Then we have

$$\begin{aligned} u &= \tan^{-1} x & v &= x \\ u' &= \frac{1}{x^2+1} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \left(\frac{1}{x^2+1} \right) x\, dx.$$

For the integral on the right we'll use the substitution $w = x^2 + 1$, $dw = 2x\, dx$. Then we have

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{dw}{w} = x \tan^{-1} x - \frac{1}{2} \ln |w| + C = x \tan^{-1} x - \frac{1}{2} \ln |x^2 + 1| + C.$$

Now we can compute the definite integral:

$$\int_0^{1} \tan^{-1} x\, dx = \left(x \tan^{-1} x - \frac{1}{2} \ln |x^2 + 1| \right) \Big|_0^1 = \left((1) \tan^{-1}(1) - \frac{1}{2} \ln 2 \right) - (0) = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

55. Use Eq. (5) to evaluate $\int x^4 e^x\, dx$.

SOLUTION

$$\begin{aligned} \int x^4 e^x\, dx &= x^4 e^x - 4 \int x^3 e^x\, dx = x^4 e^x - 4 \left[x^3 e^x - 3 \int x^2 e^x\, dx \right] \\ &= x^4 e^x - 4x^3 e^x + 12 \int x^2 e^x\, dx = x^4 e^x - 4x^3 e^x + 12 \left[x^2 e^x - 2 \int x e^x\, dx \right] \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \int x e^x\, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \left[x e^x - \int e^x\, dx \right] \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \left[x e^x - e^x \right] + C. \end{aligned}$$

Thus,

$$\int x^4 e^x\, dx = e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) + C.$$

56. Use substitution and then Eq. (5) to evaluate $\int x^4 e^{7x}\,dx$.

SOLUTION Let $u = 7x$. Then $du = 7dx$, and

$$\int x^4 e^{7x}\,dx = \frac{1}{7^5}\int (7x)^4 e^{7x}(7dx) = \frac{1}{7^5}\int u^4 e^u\,du.$$

Now use the result from Exercise 55:

$$\begin{aligned}\int x^4 e^{7x}\,dx &= \frac{1}{7^5}e^u[u^4 - 4u^3 + 12u^2 - 24u + 24] + C\\ &= \frac{1}{7^5}e^{7x}[(7x)^4 - 4(7x)^3 + 12(7x)^2 - 24(7x) + 24] + C\\ &= \frac{1}{7^5}e^{7x}[2401x^4 - 1372x^3 + 588x^2 - 168x + 24] + C.\end{aligned}$$

57. Find a reduction formula for $\int x^n e^{-x}\,dx$ similar to Eq. (5).

SOLUTION Let $u = x^n$ and $v' = e^{-x}$. Then

$$\begin{array}{ll} u = x^n & v = -e^{-x} \\ u' = nx^{n-1} & v' = e^{-x} \end{array}$$

Using Integration by Parts, we get

$$\int x^n e^{-x}\,dx = -x^n e^{-x} - \int nx^{n-1}(-e^{-x})\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx.$$

58. Evaluate $\int x^n \ln x\,dx$ for $n \neq -1$. Which method should be used to evaluate $\int x^{-1}\ln x\,dx$?

SOLUTION Let $u = \ln x$ and $v' = x^n$. Then we have

$$\begin{array}{ll} u = \ln x & v = \dfrac{x^{n+1}}{n+1} \\ u' = \dfrac{1}{x} & v' = x^n \end{array}$$

and

$$\begin{aligned}\int x^n \ln x\,dx &= \frac{x^{n+1}}{n+1}\ln x - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx\\ &= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C.\end{aligned}$$

For $n = -1$, $\int x^{-1}\ln x\,dx$, use the substitution $u = \ln x$, $du = dx/x$. Then

$$\int x^{-1}\ln x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

In Exercises 59–66, indicate a good method for evaluating the integral (but do not evaluate). Your choices are algebraic manipulation, substitution (specify u and du), and Integration by Parts (specify u and v′). If it appears that the techniques you have learned thus far are not sufficient, state this.

59. $\int \sqrt{x}\ln x\,dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = \sqrt{x}$.

60. $\int \frac{x^2 - \sqrt{x}}{2x}\,dx$

SOLUTION Use algebraic manipulation:

$$\frac{x^2 - \sqrt{x}}{2x} = \frac{x}{2} - \frac{1}{2\sqrt{x}}.$$

61. $\int \frac{x^3\,dx}{\sqrt{4-x^2}}$

SOLUTION Use substitution, followed by algebraic manipulation: Let $u = 4 - x^2$. Then $du = -2x\,dx$, $x^2 = 4 - u$, and

$$\int \frac{x^3}{\sqrt{4-x^2}}dx = -\frac{1}{2}\int \frac{(x^2)(-2x\,dx)}{\sqrt{u}} = -\frac{1}{2}\int \frac{(4-u)(du)}{\sqrt{u}} = -\frac{1}{2}\int \left(\frac{4}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right) du.$$

62. $\int \frac{dx}{\sqrt{4-x^2}}$

SOLUTION The techniques learned so far are insufficient. This problem requires the technique of trigonometric substitution.

63. $\int \frac{x+2}{x^2+4x+3}\,dx$

SOLUTION Use substitution. Let $u = x^2 + 4x + 3$; then $du = 2x + 4\,dx = 2(x+2)\,dx$, and

$$\int \frac{x+2}{x^2+4x+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

64. $\int \frac{dx}{(x+2)(x^2+4x+3)}$

SOLUTION The techniques learned so far are insufficient. This problem requires the technique of trigonometric substitution.

65. $\int x\sin(3x+4)\,dx$

SOLUTION Use Integration by Parts, with $u = x$ and $v' = \sin(3x+4)$.

66. $\int x\cos(9x^2)\,dx$

SOLUTION Use substitution, with $u = 9x^2$ and $du = 18x\,dx$.

67. Evaluate $\int (\sin^{-1} x)^2\,dx$. *Hint:* Use Integration by Parts first and then substitution.

SOLUTION First use integration by parts with $v' = 1$ to get

$$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2\int \frac{x\sin^{-1} x\,dx}{\sqrt{1-x^2}}.$$

Now use substitution on the integral on the right, with $u = \sin^{-1} x$. Then $du = dx/\sqrt{1-x^2}$ and $x = \sin u$, and we get (using Integration by Parts again)

$$\int \frac{x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = \int u\sin u\,du = -u\cos u + \sin u + C = -\sqrt{1-x^2}\,\sin^{-1} x + x + C.$$

where $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}$. So the final answer is

$$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + C.$$

68. Evaluate $\int \frac{(\ln x)^2\,dx}{x^2}$. *Hint:* Use substitution first and then Integration by Parts.

SOLUTION Let $w = \ln x$. Then $dw = dx/x,\ e^w = x$, and

$$\int \frac{(\ln x)^2\,dx}{x^2} = \int \frac{w^2 dw}{e^w}.$$

Now use Integration by Parts, with $u = w^2$ and $v' = e^{-w}$:

$$\begin{aligned}\int \frac{w^2 dw}{e^w} &= -w^2 e^{-w} - \int 2w(-e^{-w})\,dw = -w^2 e^{-w} + 2(-we^{-w} - e^{-w}) + C\\ &= -e^{-w}(w^2 + 2w + 2) + C = -e^{-\ln x}((\ln x)^2 + 2\ln x + 2) + C.\end{aligned}$$

The final answer is

$$\int \frac{(\ln x)^2\,dx}{x^2} = \frac{-[(\ln x)^2 + 2\ln x + 2]}{x} + C.$$

69. Evaluate $\int x^7 \cos(x^4)\,dx$.

SOLUTION First, let $w = x^4$. Then $dw = 4x^3\,dx$ and

$$\int x^7 \cos(x^4)\,dx = \frac{1}{4}\int w \cos x\,dw.$$

Now, use Integration by Parts with $u = w$ and $v' = \cos w$. Then

$$\int x^7 \cos(x^4)\,dx = \frac{1}{4}\left(w \sin w - \int \sin w\,dw\right) = \frac{1}{4} w \sin w + \frac{1}{4}\cos w + C = \frac{1}{4}x^4 \sin(x^4) + \frac{1}{4}\cos(x^4) + C.$$

70. Find $f(x)$, assuming that

$$\int f(x)e^x\,dx = f(x)e^x - \int x^{-1}e^x\,dx$$

SOLUTION We see that Integration by Parts was applied to $\int f(x)\,e^x\,dx$ with $u = f(x)$ and $v' = e^x$, and that therefore $f'(x) = u' = x^{-1}$. Thus $f(x) = \ln x + C$ for any constant C.

71. Find the volume of the solid obtained by revolving the region under $y = e^x$ for $0 \le x \le 2$ about the y-axis.

SOLUTION By the Method of Cylindrical Shells, the volume V of the solid is

$$V = \int_a^b (2\pi r)h\,dx = 2\pi \int_0^2 xe^x\,dx.$$

Using Integration by Parts with $u = x$ and $v' = e^x$, we find

$$V = 2\pi \quad (xe^x - e^x)\Big|_0^2 = 2\pi\big[(2e^2 - e^2) - (0 - 1)\big] = 2\pi(e^2 + 1).$$

72. Find the area enclosed by $y = \ln x$ and $y = (\ln x)^2$.

SOLUTION The two graphs intersect at $x = 1$ and at $x = e$, and $\ln x$ is above $(\ln x)^2$, so the area is

$$\int_1^e \ln x - (\ln x)^2\,dx = \int_1^e \ln x\,dx - \int_1^e (\ln x)^2\,dx$$

Using integration by parts for the second integral, let $u = (\ln x)^2$, $v' = 1$; then $u' = \frac{2\ln x}{x}$ and $v = x$, so that

$$\int_1^e (\ln x)^2\,dx = \left(x(\ln x)^2\right)\Big|_1^e - 2\int_1^e \ln x = e - 2\int_1^e \ln x$$

Substituting this back into the original equation gives

$$\int_1^e \ln x - (\ln x)^2\,dx = 3\int_1^e \ln x\,dx - e$$

We use integration by parts to evaluate the remaining integral, with $u = \ln x$ and $v' = 1$; then $u' = \frac{1}{x}$ and $v = x$, so that

$$\int_1^e \ln x\,dx = x\ln x\Big|_1^e - \int_1^e 1\,dx = e - (e - 1) = 1$$

and thus, substituting back in, the value of the original integral is

$$\int_1^e \ln x - (\ln x)^2\,dx = 3\int_1^e \ln x\,dx - e = 3 - e$$

73. Recall that the *present value* (PV) of an investment that pays out income continuously at a rate $R(t)$ for T years is $\int_0^T R(t)e^{-rt}\,dt$, where r is the interest rate. Find the PV if $R(t) = 5000 + 100t$ \$/year, $r = 0.05$ and $T = 10$ years.

SOLUTION The present value is given by

$$PV = \int_0^T R(t)e^{-rt}\,dt = \int_0^{10} (5000 + 100t)e^{-rt}\,dt = 5000\int_0^{10} e^{-rt}\,dt + 100\int_0^{10} te^{-rt}\,dt.$$

Using Integration by Parts for the integral on the right, with $u = t$ and $v' = e^{-rt}$, we find

$$\begin{aligned} PV &= 5000\left(-\frac{1}{r}e^{-rt}\right)\Big|_0^{10} + 100\left[\left(-\frac{t}{r}e^{-rt}\right)\Big|_0^{10} - \int_0^{10} \frac{-1}{r}e^{-rt}\,dt\right] \\ &= -\frac{5000}{r}e^{-rt}\Big|_0^{10} - \frac{100}{r}\left(te^{-rt} + \frac{1}{r}e^{-rt}\right)\Big|_0^{10} \\ &= -\frac{5000}{r}(e^{-10r} - 1) - \frac{100}{r}\left[\left(10e^{-10r} + \frac{1}{r}e^{-10r}\right) - \left(0 + \frac{1}{r}\right)\right] \\ &= e^{-10r}\left[-\frac{5000}{r} - \frac{1000}{r} - \frac{100}{r^2}\right] + \frac{5000}{r} + \frac{100}{r^2} \\ &= \frac{5000r + 100 - e^{-10r}(6000r + 100)}{r^2}. \end{aligned}$$

74. Derive the reduction formula

$$\int (\ln x)^k\,dx = x(\ln x)^k - k\int (\ln x)^{k-1}\,dx \qquad \boxed{6}$$

SOLUTION Use Integration by Parts with $u = (\ln x)^k$ and $v' = 1$. Then $u' = k(\ln x)^{k-1}/x$, $v = x$, and we get

$$\int (\ln x)^k\,dx = x(\ln x)^k - k\int \frac{(\ln x)^{k-1}x\,dx}{x} = x(\ln x)^k - k\int (\ln x)^{k-1}\,dx.$$

75. Use Eq. (6) to calculate $\int (\ln x)^k\,dx$ for $k = 2, 3$.

SOLUTION

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2(x\ln x - x) + C = x(\ln x)^2 - 2x\ln x + 2x + C;$$

$$\begin{aligned} \int (\ln x)^3\,dx &= x(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2x\ln x + 2x\right] + C \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C. \end{aligned}$$

76. Derive the reduction formulas

$$\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$$

$$\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

SOLUTION For $\int x^n\cos x\,dx$, let $u = x^n$ and $v' = \cos x$. Then we have

$$\begin{array}{ll} u = x^n & v = \sin x \\ u' = nx^{n-1} & v' = \cos x \end{array}$$

Using Integration by Parts, we get

$$\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx.$$

For $\int x^n\sin x\,dx$, let $u = x^n$ and $v' = \sin x$. Then we have

$$\begin{array}{ll} u = x^n & v = -\cos x \\ u' = nx^{n-1} & v' = \sin x \end{array}$$

Using Integration by Parts, we get

$$\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx.$$

77. Prove that $\int xb^x\,dx = b^x\left(\frac{x}{\ln b} - \frac{1}{\ln^2 b}\right) + C$.

SOLUTION Let $u = x$ and $v' = b^x$. Then $u' = 1$ and $v = b^x/\ln b$. Using Integration by Parts, we get

$$\int x\,b^x\,dx = \frac{xb^x}{\ln b} - \frac{1}{\ln b}\int b^x\,dx = \frac{xb^x}{\ln b} - \frac{1}{\ln b}\cdot\frac{b^x}{\ln b} + C = b^x\left(\frac{x}{\ln b} - \frac{1}{(\ln b)^2}\right) + C.$$

78. Define $P_n(x)$ by

$$\int x^n e^x\,dx = P_n(x)\,e^x + C$$

Use Eq. (5) to prove that $P_n(x) = x^n - nP_{n-1}(x)$. Use this recursion relation to find $P_n(x)$ for $n = 1, 2, 3, 4$. Note that $P_0(x) = 1$.

SOLUTION Use induction on n. Clearly for $n = 0$, we have

$$\int x^0 e^x\,dx = \int e^x\,dx = e^x + C = (1)e^x + C$$

so we may take $P_0(x) = 1 = x^0 - 0$. Now assume that

$$\int x^n e^x\,dx = P_n(x)e^x + C$$

where $P_n(x) = x^n - nP_{n-1}(x)$. Then using Eq. (5) with $n + 1$ in place of n gives

$$\int x^{n+1}e^x\,dx = x^{n+1}e^x - (n+1)\int x^n e^x\,dx = x^{n+1}e^x - (n+1)(P_n(x)e^x + C_1)$$
$$= (x^{n+1} - (n+1)P_n(x))e^x + C$$

Thus we may define $P_{n+1}(x) = x^{n+1} - (n+1)P_n(x)$ and we get

$$\int x^{n+1}e^x\,dx = P_{n+1}(x)e^x + C$$

as required.

Further Insights and Challenges

79. The Integration by Parts formula can be written

$$\int u(x)v(x)\,dx = u(x)V(x) - \int u'(x)V(x)\,dx \qquad \boxed{7}$$

where $V(x)$ satisfies $V'(x) = v(x)$.

(a) Show directly that the right-hand side of Eq. (7) does not change if $V(x)$ is replaced by $V(x) + C$, where C is a constant.

(b) Use $u = \tan^{-1}x$ and $v = x$ in Eq. (7) to calculate $\int x\tan^{-1}x\,dx$, but carry out the calculation twice: first with $V(x) = \frac{1}{2}x^2$ and then with $V(x) = \frac{1}{2}x^2 + \frac{1}{2}$. Which choice of $V(x)$ results in a simpler calculation?

SOLUTION

(a) Replacing $V(x)$ with $V(x) + C$ in the expression $u(x)V(x) - \int V(x)u'(x)\,dx$, we get

$$u(x)(V(x)+C) - \int (V(x)+C)u'(x)\,dx = u(x)V(x) + u(x)C - \int V(x)u'(x)\,dx - C\int u'(x)\,dx$$
$$= u(x)V(x) - \int V(x)u'(x)\,dx + C\left[u(x) - \int u'(x)\,dx\right]$$
$$= u(x)V(x) - \int V(x)u'(x)\,dx + C\,[u(x) - u(x)]$$
$$= u(x)V(x) - \int V(x)u'(x)\,dx.$$

(b) If we evaluate $\int x \tan^{-1} x\, dx$ with $u = \tan^{-1} x$ and $v' = x$, and if we don't add a constant to v, Integration by Parts gives us

$$\int x \tan^{-1} x\, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{x^2+1}.$$

The integral on the right requires algebraic manipulation in order to evaluate. But if we take $V(x) = \frac{1}{2}x^2 + \frac{1}{2}$ instead of $V(x) = \frac{1}{2}x^2$, then

$$\int x \tan^{-1} x\, dx = \left(\frac{1}{2}x^2 + \frac{1}{2}\right) \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1}{x^2+1}\, dx = \frac{1}{2}(x^2+1) \tan^{-1} x - \frac{1}{2}x + C$$
$$= \frac{1}{2}(x^2 \tan^{-1} x - x + \tan^{-1} x) + C.$$

80. Prove in two ways that

$$\int_0^a f(x)\, dx = af(a) - \int_0^a xf'(x)\, dx \qquad \boxed{8}$$

First use Integration by Parts. Then assume $f(x)$ is increasing. Use the substitution $u = f(x)$ to prove that $\int_0^a xf'(x)\, dx$ is equal to the area of the shaded region in Figure 1 and derive Eq. (8) a second time.

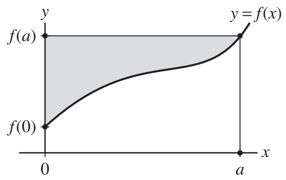


FIGURE 1

SOLUTION Let $u = f(x)$ and $v' = 1$. Then Integration by Parts gives

$$\int_0^a f(x)\, dx = xf(x)\Big|_0^a - \int_0^a xf'(x)\, dx = af(a) - \int_0^a xf'(x)\, dx.$$

Alternately, let $u = f(x)$. Then $du = f'(x)\, dx$, and if $f(x)$ is either increasing or decreasing, it has an inverse function, and $x = f^{-1}(u)$. Thus,

$$\int_{x=0}^{x=a} xf'(x)\, dx = \int_{f(0)}^{f(a)} f^{-1}(u)\, du$$

which is precisely the area of the shaded region in Figure 1 (integrating along the vertical axis). Since the area of the entire rectangle is $af(a)$, the difference between the areas of the two regions is $\int_0^a f(x)\, dx$.

81. Assume that $f(0) = f(1) = 0$ and that f'' exists. Prove

$$\int_0^1 f''(x) f(x)\, dx = -\int_0^1 f'(x)^2\, dx \qquad \boxed{9}$$

Use this to prove that if $f(0) = f(1) = 0$ and $f''(x) = \lambda f(x)$ for some constant λ, then $\lambda < 0$. Can you think of a function satisfying these conditions for some λ?

SOLUTION Let $u = f(x)$ and $v' = f''(x)$. Using Integration by Parts, we get

$$\int_0^1 f''(x) f(x)\, dx = f(x) f'(x)\Big|_0^1 - \int_0^1 f'(x)^2\, dx = f(1)f'(1) - f(0)f'(0) - \int_0^1 f'(x)^2\, dx = -\int_0^1 f'(x)^2\, dx.$$

Now assume that $f''(x) = \lambda f(x)$ for some constant λ. Then

$$\int_0^1 f''(x) f(x)\, dx = \lambda \int_0^1 [f(x)]^2\, dx = -\int_0^1 f'(x)^2\, dx < 0.$$

Since $\int_0^1 [f(x)]^2\, dx > 0$, we must have $\lambda < 0$. An example of a function satisfying these properties for some λ is $f(x) = \sin \pi x$.

82. Set $I(a,b) = \int_0^1 x^a(1-x)^b\,dx$, where a, b are whole numbers.

(a) Use substitution to show that $I(a,b) = I(b,a)$.

(b) Show that $I(a,0) = I(0,a) = \dfrac{1}{a+1}$.

(c) Prove that for $a \ge 1$ and $b \ge 0$,

$$I(a,b) = \frac{a}{b+1}\,I(a-1,b+1)$$

(d) Use (b) and (c) to calculate $I(1,1)$ and $I(3,2)$.

(e) Show that $I(a,b) = \dfrac{a!\,b!}{(a+b+1)!}$.

SOLUTION

(a) Let $u = 1-x$. Then $du = -dx$ and

$$I(a,b) = \int_{u=1}^{u=0} (1-u)^a u^b(-du) = \int_0^1 u^b(1-u)^a\,du = I(b,a).$$

(b) $I(a,0) = I(0,a)$ by part (a). Further,

$$I(a,0) = \int_0^1 x^a(1-x)^0\,dx = \int_0^1 x^a\,dx = \frac{1}{a+1}.$$

(c) Using Integration by Parts with $u = (1-x)^b$ and $v' = x^a$ gives

$$I(a,b) = (1-x)^b\frac{x^{a+1}}{a+1}\bigg|_0^1 + \frac{b}{a+1}\int_0^1 x^{a+1}(1-x)^{b-1}\,dx = \frac{b}{a+1}I(a+1,b-1).$$

The other equality arises from Integration by Parts with $u = x^a$ and $v' = (1-x)^b$.

(d)

$$I(1,1) = \frac{1}{1+1}I(1-1,1+1) = \frac{1}{2}I(0,2) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$

$$I(3,2) = \frac{1}{2}I(4,2) = \frac{1}{2}\cdot\frac{1}{5}I(5,0) = \frac{1}{10}\cdot\frac{1}{6} = \frac{1}{60}.$$

(e) We proceed as follows:

$$\begin{aligned}
I(a,b) &= \frac{a}{b+1}I(a-1,b+1) = \frac{a}{b+1}\cdot\frac{a-1}{b+2}I(a-2,b+2)\\
&\ \ \vdots\\
&= \frac{a}{b+1}\cdot\frac{a-1}{b+2}\cdots\frac{1}{b+a}I(0,b+a)\\
&= \frac{a(a-1)\cdots(1)}{(b+1)(b+2)\cdots(b+a)}\cdot\frac{1}{b+a+1}\\
&= \frac{b!\,a!}{b!\,(b+1)(b+2)\cdots(b+a)(b+a+1)} = \frac{a!\,b!}{(a+b+1)!}.
\end{aligned}$$

83. Let $I_n = \int x^n\cos(x^2)\,dx$ and $J_n = \int x^n\sin(x^2)\,dx$.

(a) Find a reduction formula that expresses I_n in terms of J_{n-2}. *Hint:* Write $x^n\cos(x^2)$ as $x^{n-1}(x\cos(x^2))$.

(b) Use the result of (a) to show that I_n can be evaluated explicitly if n is odd.

(c) Evaluate I_3.

SOLUTION

(a) Integration by Parts with $u = x^{n-1}$ and $v' = x\cos(x^2)\,dx$ yields

$$I_n = \frac{1}{2}x^{n-1}\sin(x^2) - \frac{n-1}{2}\int x^{n-2}\sin(x^2)\,dx = \frac{1}{2}x^{n-1}\sin(x^2) - \frac{n-1}{2}J_{n-2}.$$

(b) If n is odd, the reduction process will eventually lead to either

$$\int x\cos(x^2)\,dx \quad \text{or} \quad \int x\sin(x^2)\,dx,$$

both of which can be evaluated using the substitution $u = x^2$.
(c) Starting with the reduction formula from part (a), we find

$$I_3 = \frac{1}{2}x^2\sin(x^2) - \frac{2}{2}\int x\sin(x^2)\,dx = \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C.$$

7.2 Trigonometric Integrals

Preliminary Questions

1. Describe the technique used to evaluate $\int \sin^5 x\,dx$.

SOLUTION Because the sine function is raised to an odd power, rewrite $\sin^5 x = \sin x \sin^4 x = \sin x(1-\cos^2 x)^2$ and then substitute $u = \cos x$.

2. Describe a way of evaluating $\int \sin^6 x\; dx$.

SOLUTION Repeatedly use the reduction formula for powers of $\sin x$.

3. Are reduction formulas needed to evaluate $\int \sin^7 x\cos^2 x\,dx$? Why or why not?

SOLUTION No, a reduction formula is not needed because the sine function is raised to an odd power.

4. Describe a way of evaluating $\int \sin^6 x\cos^2 x\,dx$.

SOLUTION Because both trigonometric functions are raised to even powers, write $\cos^2 x = 1 - \sin^2 x$ and then apply the reduction formula for powers of the sine function.

5. Which integral requires more work to evaluate?

$$\int \sin^{798} x\cos x\,dx \qquad \text{or} \qquad \int \sin^4 x\cos^4 x\,dx$$

Explain your answer.

SOLUTION The first integral can be evaluated using the substitution $u = \sin x$, whereas the second integral requires the use of reduction formulas. The second integral therefore requires more work to evaluate.

Exercises

In Exercises 1–6, use the method for odd powers to evaluate the integral.

1. $\int \cos^3 x\,dx$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite the integrand:

$$\int \cos^3 x\,dx = \int \left(1-\sin^2 x\right)\cos x\,dx.$$

Now use the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \cos^3 x\,dx = \int \left(1-u^2\right)\,du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

2. $\int \sin^5 x\,dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite the integrand:

$$\int \sin^5 x\,dx = \int \left(\sin^2 x\right)^2 \sin x\,dx = \int \left(1-\cos^2 x\right)^2 \sin x\,dx.$$

Now use the substitution $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin^5 x\,dx &= -\int \left(1-u^2\right)^2 du = -\int \left(1-2u^2+u^4\right) du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C\\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

3. $\int \sin^3\theta \cos^2\theta \, d\theta$

SOLUTION Write $\sin^3\theta = \sin^2\theta \sin\theta = (1-\cos^2\theta)\sin\theta$. Then

$$\int \sin^3\theta\cos^2\theta\,d\theta = \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta.$$

Now use the substitution $u=\cos\theta$, $du = -\sin\theta\,d\theta$:

$$\int \sin^3\theta\cos^2\theta\,d\theta = -\int \left(1-u^2\right)u^2\,du = -\int \left(u^2-u^4\right)du$$
$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C.$$

4. $\int \sin^5 x \cos x \, dx$

SOLUTION Write $\sin^5 x = \sin^4 x \sin x = (1-\cos^2 x)^2 \sin x$. Then

$$\int \cos x \sin^5 x\,dx = \int \cos x \left(1-\cos^2 x\right)^2 \sin x\,dx.$$

Now use the substitution $u=\cos x$, $du = -\sin x\,dx$:

$$\int \cos x\sin^5 x\,dx = -\int u\left(1-u^2\right)^2 du = -\int u\left(1-2u^2+u^4\right)du = \int \left(-u+2u^3-u^5\right)du$$
$$= -\frac{1}{2}u^2 + \frac{1}{2}u^4 - \frac{1}{6}u^6 + C = -\frac{1}{2}\cos^2 x + \frac{1}{2}\cos^4 x - \frac{1}{6}\cos^6 x + C.$$

5. $\int \sin^3 t \cos^3 t \, dt$

SOLUTION Write $\sin^3 t = (1-\cos^2 t)\sin t\,dt$. Then

$$\int \sin^3 t\cos^3 t\,dt = \int (1-\cos^2 t)\cos^3 t\sin t\,dt = \int \left(\cos^3 t - \cos^5 t\right)\sin t\,dt.$$

Now use the substitution $u=\cos t$, $du=-\sin t\,dt$:

$$\int \sin^3 t\cos^3 t\,dt = -\int \left(u^3-u^5\right)du = -\frac{1}{4}u^4 + \frac{1}{6}u^6 + C = -\frac{1}{4}\cos^4 t + \frac{1}{6}\cos^6 t + C.$$

6. $\int \sin^2 x \cos^5 x \, dx$

SOLUTION Write $\cos^5 x = \cos^4 x\cos x = (1-\sin^2 x)^2\cos x$. Then

$$\int \sin^2 x\cos^5 x\,dx = \int \sin^2 x\left(1-\sin^2 x\right)^2 \cos x\,dx.$$

Now use the substitution $u=\sin x$, $du = \cos x\,dx$:

$$\int \sin^2 x\cos^5 x\,dx = \int u^2\left(1-u^2\right)^2 du = \int \left(u^2-2u^4+u^6\right)du$$
$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$$

7. Find the area of the shaded region in Figure 1.

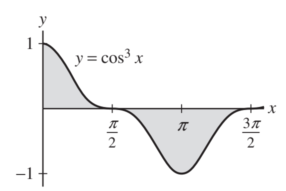


FIGURE 1 Graph of $y=\cos^3 x$.

SOLUTION First evaluate the indefinite integral by writing $\cos^3 x = (1 - \sin^2 x)\cos x$, and using the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \cos^3 x\,dx = \int \left(1 - \sin^2 x\right)\cos x\,dx = \int \left(1 - u^2\right) du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

The area is given by

$$\begin{aligned} A &= \int_0^{\pi/2} \cos^3 x\,dx - \int_{\pi/2}^{3\pi/2} \cos^3 x\,dx = \left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_0^{\pi/2} - \left(\sin x - \frac{1}{3}\sin^3 x\right)\Big|_{\pi/2}^{3\pi/2} \\ &= \left[\left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right) - 0\right] - \left[\left(\sin\frac{3\pi}{2} - \frac{1}{3}\sin^3\frac{3\pi}{2}\right) - \left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right)\right] \\ &= 1 - \frac{1}{3}(1)^3 - (-1) + \frac{1}{3}(-1)^3 + 1 - \frac{1}{3}(1)^3 = 2. \end{aligned}$$

8. Use the identity $\sin^2 x = 1 - \cos^2 x$ to write $\int \sin^2 x \cos^2 x\,dx$ as a sum of two integrals, and then evaluate using the reduction formula.

SOLUTION Using the identity $\sin^2 x = 1 - \cos^2 x$, we get

$$\int \sin^2 x \cos^2 x\,d = \int \left(1 - \cos^2 x\right)\cos^2 x\,dx = \int \cos^2 x\,dx - \int \cos^4 x\,dx.$$

Using the reduction formula for $\cos^m x$, we get

$$\int \cos^4 x\,dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx.$$

Thus,

$$\int \sin^2 x \cos^2 x\,dx = \int \cos^2 x - \frac{1}{4}\cos^3 x \sin x - \frac{3}{4}\int \cos^2 x\,dx = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{4}\int \cos^2 x\,dx.$$

Using the reduction formula again, we have

$$\int \sin^2 x \cos^2 x\,dx = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\int dx\right] = -\frac{1}{4}\cos^3 x \sin x + \frac{1}{8}\cos x \sin x + \frac{1}{8}x + C.$$

In Exercises 9–12, evaluate the integral using methods employed in Examples 3 and 4.

9. $\int \cos^4 y\,dy$

SOLUTION Using the reduction formula for $\cos^m y$, we get

$$\begin{aligned} \int \cos^4 y\,dy &= \frac{1}{4}\cos^3 y \sin y + \frac{3}{4}\int \cos^2 y\,dy = \frac{1}{4}\cos^3 y \sin y + \frac{3}{4}\left(\frac{1}{2}\cos y \sin y + \frac{1}{2}\int dy\right) \\ &= \frac{1}{4}\cos^3 y \sin y + \frac{3}{8}\cos y \sin y + \frac{3}{8}y + C. \end{aligned}$$

10. $\int \cos^2\theta \sin^2\theta\,d\theta$

SOLUTION First use the identity $\cos^2\theta = 1 - \sin^2\theta$ to write:

$$\int \cos^2\theta \sin^2\theta\,d\theta = \int \left(1 - \sin^2\theta\right)\sin^2\theta\,d\theta = \int \sin^2\theta\,d\theta - \int \sin^4\theta\,d\theta.$$

Using the reduction formula for $\sin^m\theta$, we get

$$\begin{aligned} \int \cos^2\theta \sin^2\theta\,d\theta &= \int \sin^2\theta\,d\theta - \left[-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta\,d\theta\right] = \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\int \sin^2\theta\,d\theta \\ &= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int d\theta\right) = \frac{1}{4}\sin^3\theta\cos\theta - \frac{1}{8}\sin\theta\cos\theta + \frac{1}{8}\theta + C. \end{aligned}$$

11. $\int \sin^4 x \cos^2 x\, dx$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to write:

$$\int \sin^4 x \cos^2 x\, dx = \int \sin^4 x \left(1 - \sin^2 x\right) dx = \int \sin^4 x\, dx - \int \sin^6 x\, dx.$$

Using the reduction formula for $\sin^m x$:

$$\begin{aligned}\int \sin^4 x \cos^2 x\, dx &= \int \sin^4 x\, dx - \left[-\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\int \sin^4 x\, dx\right]\\ &= \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\int \sin^4 x\, dx = \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\left(-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x\, dx\right)\\ &= \frac{1}{6}\sin^5 x \cos x - \frac{1}{24}\sin^3 x \cos x + \frac{1}{8}\int \sin^2 x\, dx\\ &= \frac{1}{6}\sin^5 x \cos x - \frac{1}{24}\sin^3 x \cos x + \frac{1}{8}\left(-\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx\right)\\ &= \frac{1}{6}\sin^5 x \cos x - \frac{1}{24}\sin^3 x \cos x - \frac{1}{16}\sin x \cos x + \frac{1}{16}x + C.\end{aligned}$$

12. $\int \sin^2 x \cos^6 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ to write

$$\int \sin^2 x \cos^6 x\, dx = \int (1 - \cos^2 x)\cos^6 x\, dx = \int \cos^6 x\, dx - \int \cos^8 x\, dx$$

Now use the reduction formula for $\cos^n x$:

$$\begin{aligned}\int \cos^6 x\, dx &= \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\int \cos^4 x\, dx\\ &= \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\left(\frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx\right)\\ &= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{24}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) + C\\ &= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{48}x + \frac{15}{96}\sin 2x + C\end{aligned}$$

and

$$\begin{aligned}\int \cos^8 x\, dx &= \frac{1}{8}\cos^7 x \sin x + \frac{7}{8}\int \cos^6 x\, dx\\ &= \frac{1}{8}\cos^7 x \sin x + \frac{7}{8}\left(\frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{48}x + \frac{15}{96}\sin 2x\right) + C\\ &= \frac{1}{8}\cos^7 x \sin x + \frac{7}{48}\cos^5 x \sin x + \frac{35}{192}\cos^3 x \sin x + \frac{105}{384}x + \frac{105}{768}\sin 2x + C\end{aligned}$$

so that

$$\int \sin^2 x \cos^6 x\, dx = -\frac{1}{8}\cos^7 x \sin x + \frac{1}{48}\cos^5 x \sin x + \frac{5}{192}\cos^3 x \sin x + \frac{5}{128}x + \frac{5}{256}\sin 2x + C$$

In Exercises 13 and 14, evaluate using Eq. (13).

13. $\int \sin^3 x \cos^2 x\, dx$

SOLUTION First rewrite $\sin^3 x = \sin x \cdot \sin^2 x = \sin x(1 - \cos^2 x)$, so that

$$\int \sin^3 x \cos^2 x\, dx = \int \sin x(1 - \cos^2 x)\cos^2 x\, dx = \int \sin x(\cos^2 x - \cos^4 x)\, dx$$

Now make the substitution $u = \cos x$, $du = -\sin x\, dx$:

$$\int \sin x(\cos^2 x - \cos^4 x)\, dx = -\int u^2 - u^4\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

14. $\int \sin^2 x \cos^4 x\, dx$

SOLUTION Using the formula for $\int \sin^m x \cos^n x\, dx$, we get

$$I = \int \sin^2 x \cos^4 x\, dx = \frac{1}{6}\sin^3 x \cos^3 x + \frac{3}{6}\int \sin^2 x \cos^2 x\, dx = \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{2}\int \sin^2 x \cos^2 x\, dx.$$

Applying the formula again on the remaining integral, we get

$$\int \sin^2 x \cos^2 x\, dx = \frac{1}{4}\sin^3 x \cos x + \frac{1}{4}\int \sin^2 x \cos^0 x\, dx = \frac{1}{4}\sin^3 x \cos x + \frac{1}{4}\int \sin^2 x\, dx.$$

The final result is

$$\begin{aligned} I &= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{2}\left(\frac{1}{4}\sin^3 x \cos x + \frac{1}{4}\int \sin^2 x\, dx\right) \\ &= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{8}\sin^3 x \cos x + \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{2}\sin x \cos x\right) + C \\ &= \frac{1}{6}\sin^3 x \cos^3 x + \frac{1}{8}\sin^3 x \cos x + \frac{1}{16}x - \frac{1}{16}\sin x \cos x + C. \end{aligned}$$

In Exercises 15–18, evaluate the integral using the method described on page 409 and the reduction formulas on page 410 as necessary.

15. $\int \tan^3 x \sec x\, dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^3 x \sec x = (\sec^2 x - 1)\sec x \tan x$. Then use the substitution $u = \sec x$, $du = \sec x \tan x\, dx$:

$$\int \tan^3 x \sec x\, dx = \int (\sec^2 x - 1)\sec x \tan x\, dx = \int u^2 - 1\, du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$$

16. $\int \tan^2 x \sec x\, dx$

SOLUTION First use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1)\sec x\, dx = \int \sec^3 x - \sec x\, dx = \int \sec^3 x\, dx - \ln|\sec x + \tan x|$$

To evaluate the remaining integral, we use the reduction formula:

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

so that finally, putting these together,

$$\int \tan^2 x \sec x\, dx = \int \sec^3 x\, dx - \ln|\sec x + \tan x| = \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$$

17. $\int \tan^2 x \sec^4 x\, dx$

SOLUTION First use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \sec^4 x\, dx = \int (\sec^2 x - 1)\sec^4 x\, dx = \int \sec^6 x - \sec^4 x\, dx = \int \sec^6 x\, dx - \int \sec^4 x, dx$$

We evaluate the second integral using the reduction formula:

$$\begin{aligned} \int \sec^4 x\, dx &= \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\int \sec^2 x\, dx \\ &= \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x \end{aligned}$$

Then

$$\int \sec^6 x\,dx = \frac{1}{5}\tan x\sec^4 x + \frac{4}{5}\int \sec^4 x\,dx$$
$$= \frac{1}{5}\tan x\sec^4 x + \frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x\right)$$
$$= \frac{1}{5}\tan x\sec^4 x + \frac{4}{15}\tan x\sec^2 x + \frac{8}{15}\tan x$$

so that

$$\int \tan^2 x\sec^4 x\,dx = \int \sec^6 x\,dx - \int \sec^4 x\,dx$$
$$= \frac{1}{5}\tan x\sec^4 x - \frac{1}{15}\tan x\sec^2 x - \frac{2}{15}\tan x + C$$

18. $\int \tan^8 x\sec^2 x\,dx$

SOLUTION Use the substitution $u = \tan x$, $du = \sec^2 x\,dx$; then

$$\int \tan^8 x\sec^2 x\,dx = \int u^8\,du = \frac{1}{9}u^9 = \frac{1}{9}\tan^9 x + C$$

In Exercises 19–22, evaluate using methods similar to those that apply to integral $\tan^m x\sec^n$.

19. $\int \cot^3 x\,dx$

SOLUTION Using the reduction formula for $\cot^m x$, we get

$$\int \cot^3 x\,dx = -\frac{1}{2}\cot^2 x - \int \cot x\,dx = -\frac{1}{2}\cot^2 x + \ln|\csc x| + C.$$

20. $\int \sec^3 x\,dx$

SOLUTION Using the reduction formula for $\sec^m x$, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\tan x\sec x + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\tan x\sec x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

21. $\int \cot^5 x\csc^2 x\,dx$

SOLUTION Make the substitution $u = \cot x$, $du = -\csc^2 x\,dx$; then

$$\int \cot^5 x\csc^2 x\,dx = -\int u^5\,du = -\frac{1}{6}u^6 = -\frac{1}{6}\cot^6 x + C$$

22. $\int \cot^4 x\csc x\,dx$

SOLUTION Use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^4 x\csc x\,dx = \int (\csc^2 x - 1)^2\csc x\,dx = \int \csc^5 x - 2\csc^3 x + \csc x\,dx$$

Now apply the reduction formula:

$$\int \csc^3 x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\cot x\csc x - \frac{1}{2}\ln|\csc x + \cot x| + C$$

so that

$$\int \csc^5 x\,dx = -\frac{1}{4}\cot x\csc^3 x + \frac{3}{4}\int \csc^3 x\,dx$$
$$= -\frac{1}{4}\cot x\csc^3 x - \frac{3}{4}\left(\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x + \cot x|\right) + C$$
$$= -\frac{1}{4}\cot x\csc^3 x - \frac{3}{8}\cot x\csc x - \frac{3}{8}\ln|\csc x + \cot x| + C$$

Putting all this together, we get

$$\begin{aligned}\int \cot^4 x \csc x\,dx &= \int \csc^5 x\,dx - 2\int \csc^3 x\,dx + \int \csc x\,dx\\ &= -\frac{1}{4}\cot x\csc^3 x - \frac{3}{8}\cot x\csc x - \frac{3}{8}\ln|\csc x + \cot x| + \cot x\csc x\\ &\quad + \ln|\csc x + \cot x| - \ln|\csc x + \cot x| + C\\ &= -\frac{1}{4}\cot x\csc^3 x + \frac{5}{8}\cot x\csc x - \frac{3}{8}\ln|\csc x + \cot x| + C\end{aligned}$$

In Exercises 23–46, evaluate the integral.

23. $\int \cos^5 x \sin x\,dx$

SOLUTION Use the substitution $u = \cos x$, $du = -\sin x\,dx$. Then

$$\int \cos^5 x \sin x\,dx = -\int u^5\,du = -\frac{1}{6}u^6 + C = -\frac{1}{6}\cos^6 x + C.$$

24. $\int \cos^3(2-x)\sin(2-x)\,dx$

SOLUTION Use the substitution $u = \cos(2-x)$, $du = \sin(2-x)\,dx$. Then

$$\int \cos^3(2-x)\sin(2-x)\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\cos^4(2-x) + C$$

25. $\int \cos^4(3x+2)\,dx$

SOLUTION First use the substitution $u = 3x+2$, $du = 3\,dx$ and then apply the reduction formula for $\cos^n x$:

$$\begin{aligned}\int \cos^4(3x+2)\,dx &= \frac{1}{3}\cos^4 u\,du = \frac{1}{3}\left(\frac{1}{4}\cos^3 u\sin u + \frac{3}{4}\int \cos^2 u\,du\right)\\ &= \frac{1}{12}\cos^3 u\sin u + \frac{1}{4}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + C\\ &= \frac{1}{12}\cos^3(3x+2)\sin(3x+2) + \frac{1}{8}(3x+2) + \frac{1}{16}\sin(6x+4) + C\end{aligned}$$

26. $\int \cos^7 3x\,dx$

SOLUTION Use the substitution $u = 3x$, $du = 3\,dx$, and the reduction formula for $\cos^m x$:

$$\begin{aligned}\int \cos^7 3x\,dx &= \frac{1}{3}\int \cos^7 u\,du = \frac{1}{21}\cos^6 u\sin u + \frac{6}{21}\int \cos^5 u\,du\\ &= \frac{1}{21}\cos^6 u\sin u + \frac{2}{7}\left(\frac{1}{5}\cos^4 u\sin u + \frac{4}{5}\int \cos^3 u\,du\right)\\ &= \frac{1}{21}\cos^6 u\sin u + \frac{2}{35}\cos^4 u\sin u + \frac{8}{35}\left(\frac{1}{3}\cos^2 u\sin u + \frac{2}{3}\int \cos u\,du\right)\\ &= \frac{1}{21}\cos^6 u\sin u + \frac{2}{35}\cos^4 u\sin u + \frac{8}{105}\cos^2 u\sin u + \frac{16}{105}\sin u + C\\ &= \frac{1}{21}\cos^6 3x\sin 3x + \frac{2}{35}\cos^4 3x\sin 3x + \frac{8}{105}\cos^2 3x\sin 3x + \frac{16}{105}\sin 3x + C.\end{aligned}$$

27. $\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta$

SOLUTION Use the substitution $u = \pi\theta$, $du = \pi\,d\theta$, and the identity $\cos^2 u = 1 - \sin^2 u$ to write

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)\,d\theta = \frac{1}{\pi}\int \cos^3 u\sin^4 u\,du = \frac{1}{\pi}\int \left(1 - \sin^2 u\right)\sin^4 u\cos u\,du.$$

Now use the substitution $w = \sin u$, $dw = \cos u\, du$:

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)\, d\theta = \frac{1}{\pi}\int \left(1 - w^2\right) w^4\, dw = \frac{1}{\pi}\int \left(w^4 - w^6\right) dw = \frac{1}{5\pi}w^5 - \frac{1}{7\pi}w^7 + C$$
$$= \frac{1}{5\pi}\sin^5(\pi\theta) - \frac{1}{7\pi}\sin^7(\pi\theta) + C.$$

28. $\int \cos^{498} y \sin^3 y\, dy$

SOLUTION Use the identity $\sin^2 y = 1 - \cos^2 y$ to write

$$\int \cos^{498} y \sin^3 y\, dy = \int \cos^{498} y \left(1 - \cos^2 y\right)\sin y\, dy.$$

Now use the substitution $u = \cos y$, $du = -\sin y\, dy$:

$$\int \cos^{498} y \sin^3 y\, dy = -\int u^{498}\left(1 - u^2\right) du = -\int \left(u^{498} - u^{500}\right) du$$
$$= -\frac{1}{499}u^{499} + \frac{1}{501}u^{501} + C = -\frac{1}{499}\cos^{499} y + \frac{1}{501}\cos^{501} y + C.$$

29. $\int \sin^4(3x)\, dx$

SOLUTION Use the substitution $u = 3x$, $du = 3\, dx$ and the reduction formula for $\sin^m x$:

$$\int \sin^4(3x)\, dx = \frac{1}{3}\int \sin^4 u\, du = -\frac{1}{12}\sin^3 u\cos u + \frac{1}{4}\int \sin^2 u\, du$$
$$= -\frac{1}{12}\sin^3 u\cos u + \frac{1}{4}\left(-\frac{1}{2}\sin u\cos u + \frac{1}{2}\int du\right)$$
$$= -\frac{1}{12}\sin^3 u\cos u - \frac{1}{8}\sin u\cos u + \frac{1}{8}u + C$$
$$= -\frac{1}{12}\sin^3(3x)\cos(3x) - \frac{1}{8}\sin(3x)\cos(3x) + \frac{3}{8}x + C.$$

30. $\int \sin^2 x\cos^6 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ and the reduction formula for $\cos^m x$:

$$\int \sin^2 x\cos^6 x\, dx = \int \cos^6 x\left(1 - \cos^2 x\right) dx = \int \cos^6 x\, dx - \int \cos^8 x\, dx$$
$$= \int \cos^6 x\, dx - \left(\frac{1}{8}\cos^7 x\sin x + \frac{7}{8}\int \cos^6 x\, dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{8}\int \cos^6 x\, dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{8}\left(\frac{1}{6}\cos^5 x\sin x + \frac{5}{6}\int \cos^4 x\, dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{48}\int \cos^4 x\, dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{48}\left(\frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\, dx\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{15}{192}\int \cos^2 x\, dx$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{15}{192}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right)$$
$$= -\frac{1}{8}\cos^7 x\sin x + \frac{1}{48}\cos^5 x\sin x + \frac{5}{192}\cos^3 x\sin x + \frac{5}{128}\cos x\sin x + \frac{5}{128}x + C.$$

31. $\int \csc^2(3-2x)\,dx$

SOLUTION First make the substitution $u = 3-2x$, $du = -2\,dx$, so that

$$\int \csc^2(3-2x)\,dx = \frac{1}{2}\int(-\csc^2 u)\,du = \frac{1}{2}\cot u + C = \frac{1}{2}\cot(3-2x) + C$$

32. $\int \csc^3 x\,dx$

SOLUTION Use the reduction formula for $\csc^m x$:

$$\int \csc^3 x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x| + C.$$

33. $\int \tan x\sec^2 x\,dx$

SOLUTION Use the substitution $u = \tan x$, $du = \sec^2 x\,dx$. Then

$$\int \tan x\sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

34. $\int \tan^3\theta\sec^3\theta\,d\theta$

SOLUTION Use the identity $\tan^2\theta = \sec^2\theta - 1$ to write

$$\int \tan^3\theta\sec^3\theta\,d\theta = \int\left(\sec^2\theta - 1\right)\sec^2\theta(\sec\theta\tan\theta\,d\theta).$$

Now use the substitution $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$:

$$\int \tan^3\theta\sec^3\theta\,d\theta = \int\left(u^2-1\right)u^2\,du = \int\left(u^4-u^2\right)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C.$$

35. $\int \tan^5 x\sec^4 x\,dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^5 x\sec^4 x\,dx = \int\left(\sec^2 x - 1\right)^2\sec^3 x(\sec x\tan x\,dx).$$

Now use the substitution $u = \sec x$, $du = \sec x\tan x\,dx$:

$$\begin{aligned}\int \tan^5 x\sec^4 x\,dx &= \int\left(u^2-1\right)^2u^3\,du = \int\left(u^7 - 2u^5 + u^3\right)du\\ &= \frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4 + C = \frac{1}{8}\sec^8 x - \frac{1}{3}\sec^6 x + \frac{1}{4}\sec^4 x + C.\end{aligned}$$

36. $\int \tan^4 x\sec x\,dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^4 x\sec x\,dx = \int\left(\sec^2 x - 1\right)^2\sec x\,dx = \int \sec^5 x\,dx - 2\int\sec^3 x\,dx + \int\sec x\,dx.$$

Now use the reduction formula for $\sec^m x$:

$$\begin{aligned}\int \tan^4 x\sec x\,dx &= \left(\frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\int\sec^3 x\,dx\right) - 2\int\sec^3 x\,dx + \int\sec x\,dx\\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{4}\int\sec^3 x\,dx + \int\sec x\,dx\\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{4}\left(\frac{1}{2}\tan x\sec x + \frac{1}{2}\int\sec x\,dx\right) + \int\sec x\,dx\\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{8}\tan x\sec x + \frac{3}{8}\int\sec x\,dx\\ &= \frac{1}{4}\tan x\sec^3 x - \frac{5}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C.\end{aligned}$$

37. $\int \tan^6 x \sec^4 x\,dx$

SOLUTION Use the identity $\sec^2 x = \tan^2 x + 1$ to write

$$\int \tan^6 x \sec^4 x\,dx = \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x\,dx.$$

Now use the substitution $u = \tan x$, $du = \sec^2 x\,dx$:

$$\int \tan^6 x \sec^4 x\,dx = \int u^6 \left(u^2+1\right) du = \int \left(u^8 + u^6\right) du = \frac{1}{9}u^9 + \frac{1}{7}u^7 + C = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.$$

38. $\int \tan^2 x \sec^3 x\,dx$

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^2 x \sec^3 x\,dx = \int \left(\sec^2 x - 1\right) \sec^3 x\,dx = \int \sec^5 x\,dx - \int \sec^3 x\,dx.$$

Now use the reduction formula for $\sec^m x$:

$$\begin{aligned}\int \tan^2 x \sec^3 x\,dx &= \frac{1}{4}\tan x \sec^3 x + \frac{3}{4}\int \sec^3 x\,dx - \int \sec^3 x\,dx \\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\int \sec^3 x\,dx \\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\left(\frac{1}{2}\tan x \sec x + \frac{1}{2}\int \sec x\,dx\right) \\ &= \frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\tan x \sec x - \frac{1}{8}\ln|\sec x + \tan x| + C.\end{aligned}$$

39. $\int \cot^5 x \csc^5 x\,dx$

SOLUTION First use the identity $\cot^2 x = \csc^2 x - 1$ to rewrite the integral:

$$\int \cot^5 x \csc^5 x\,dx = \int (\csc^2 x - 1)^2 \csc^4 x(\cot x \csc x)\,dx = \int (\csc^8 x - 2\csc^6 x + \csc^4 x)(\cot x \csc x)\,dx$$

Now use the substitution $u = \csc x$ and $du = -\cot x \csc x\,dx$ to get

$$\begin{aligned}\int \cot^5 x \csc^5 x\,dx &= -\int u^8 - 2u^6 + u^4\,du = -\frac{1}{9}u^9 + \frac{2}{7}u^7 - \frac{1}{5}u^5 + C \\ &= -\frac{1}{9}\csc^9 x + \frac{2}{7}\csc^7 x - \frac{1}{5}\csc^5 x + C\end{aligned}$$

40. $\int \cot^2 x \csc^4 x\,dx$

SOLUTION First rewrite using $\cot^2 x = \csc^2 x - 1$ and then use the reduction formula:

$$\begin{aligned}\int \cot^2 x \csc^4 x\,dx &= \int (\csc^2 x - 1)\csc^4 x\,dx = \int \csc^6 x\,dx - \int \csc^4 x\,dx \\ &= -\frac{1}{5}\cot x \csc^4 x + \frac{4}{5}\int \csc^4 x\,dx - \int \csc^4 x\,dx \\ &= -\frac{1}{5}\cot x \csc^4 x - \frac{1}{5}\int \csc^4 x\,dx \\ &= -\frac{1}{5}\cot x \csc^4 x - \frac{1}{5}\left(-\frac{1}{3}\cot x \csc^2 x + \frac{2}{3}\int \csc^2 x\,dx\right) \\ &= -\frac{1}{5}\cot x \csc^4 x + \frac{1}{15}\cot x \csc^2 x + \frac{2}{15}\cot x + C\end{aligned}$$

41. $\int \sin 2x \cos 2x\,dx$

SOLUTION Use the substitution $u = \sin 2x$, $du = 2\cos 2x\,dx$:

$$\int \sin 2x \cos 2x\,dx = \frac{1}{2}\int \sin 2x(2\cos 2x\,dx) = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2 2x + C.$$

42. $\int \cos 4x \cos 6x \, dx$

SOLUTION Use the formula for $\int \cos mx \cos nx \, dx$:

$$\int \cos 4x \cos 6x \, dx = \frac{\sin(4-6)x}{2(4-6)} + \frac{\sin(4+6)x}{2(4+6)} + C = \frac{\sin(-2x)}{-4} + \frac{\sin(10x)}{20} + C$$
$$= \frac{1}{4}\sin 2x + \frac{1}{20}\sin 10x + C.$$

Here we've used the fact that $\sin x$ is an odd function: $\sin(-x) = -\sin x$.

43. $\int t \cos^3(t^2) \, dt$

SOLUTION Use the substitution $u = t^2$, $du = 2t\,dt$, followed by the reduction formula for $\cos^m x$:

$$\int t\cos^3(t^2)\,dt = \frac{1}{2}\int \cos^3 u\,du = \frac{1}{6}\cos^2 u \sin u + \frac{1}{3}\int \cos u\,du$$
$$= \frac{1}{6}\cos^2 u \sin u + \frac{1}{3}\sin u + C = \frac{1}{6}\cos^2(t^2)\sin(t^2) + \frac{1}{3}\sin(t^2) + C.$$

44. $\int \frac{\tan^3(\ln t)}{t}\,dt$

SOLUTION Use the substitution $u = \ln t$, $du = \frac{1}{t}\,dt$, followed by the reduction formula for $\tan^n x$:

$$\int \frac{\tan^3(\ln t)}{t}\,dt = \int \tan^3 u\,du = \frac{1}{2}\tan^2 u - \int \tan u\,du$$
$$= \frac{1}{2}\tan^2 u - \ln|\sec u| + C = \frac{1}{2}\tan^2(\ln t) - \ln|\sec(\ln t)| + C.$$

45. $\int \cos^2(\sin t)\cos t\,dt$

SOLUTION Use the substitution $u = \sin t$, $du = \cos t\,dt$, followed by the reduction formula for $\cos^m x$:

$$\int \cos^2(\sin t)\cos t\,dt = \int \cos^2 u\,du = \frac{1}{2}\cos u\sin u + \frac{1}{2}\int du$$
$$= \frac{1}{2}\cos u \sin u + \frac{1}{2}u + C = \frac{1}{2}\cos(\sin t)\sin(\sin t) + \frac{1}{2}\sin t + C.$$

46. $\int e^x \tan^2(e^x)\,dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\,dx$ followed by the reduction formula for $\tan^m x$:

$$\int e^x\tan^2(e^x)\,dx = \int \tan^2 u\,du = \tan u - \int 1\,du = \tan u - u + C = \tan(e^x) - e^x + C$$

In Exercises 47–60, evaluate the definite integral.

47. $\int_0^{2\pi} \sin^2 x\,dx$

SOLUTION Use the formula for $\int \sin^2 x\,dx$:

$$\int_0^{2\pi}\sin^2 x\,dx = \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Bigg|_0^{2\pi} = \left(\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right) - \left(\frac{0}{2} - \frac{\sin 0}{4}\right) = \pi.$$

48. $\int_0^{\pi/2} \cos^3 x\,dx$

SOLUTION Use the reduction formula for $\cos^m x$:

$$\int_0^{\pi/2}\cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x\Bigg|_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\cos x\,dx = \left[\frac{1}{3}(0)(1) - \frac{1}{3}(1)(0)\right] + \frac{2}{3}\sin x\Bigg|_0^{\pi/2}$$
$$= 0 + \frac{2}{3}(1-0) = \frac{2}{3}.$$

49. $\int_0^{\pi/2} \sin^5 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ followed by the substitution $u = \cos x$, $du = -\sin x\, dx$ to get

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x) \sin x\, dx$$
$$= -\int_1^0 (1 - 2u^2 + u^4)\, du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_1^0 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

50. $\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$

SOLUTION Use the identity $\sin^2 x = 1 - \cos^2 x$ followed by the substitution $u = \sin x$, $du = \cos x\, dx$ to get

$$\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx = \int_0^{\pi/2} \sin^2 x (1 - \sin^2 x) \cos x\, dx = \int_0^{\pi/2} (\sin^2 x - \sin^4 x) \cos x\, dx$$
$$= \int_0^1 u^2 - u^4\, du = \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \frac{2}{15}$$

51. $\int_0^{\pi/4} \frac{dx}{\cos x}$

SOLUTION Use the definition of $\sec x$ to simplify the integral:

$$\int_0^{\pi/4} \frac{dx}{\cos x} = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left(\sqrt{2} + 1\right).$$

52. $\int_{\pi/4}^{\pi/2} \frac{dx}{\sin x}$

SOLUTION Use the definition of $\csc x$ to simplify the integral:

$$\int_{\pi/4}^{\pi/2} \frac{dx}{\sin x} = \int_{\pi/4}^{\pi/2} \csc x\, dx = \ln|\csc x - \cot x|\Big|_{\pi/4}^{\pi/2} = \ln|1 - 0| - \ln\left|\sqrt{2} - 1\right| = -\ln\left|\sqrt{2} - 1\right|$$
$$= \ln\left(\frac{1}{\sqrt{2} - 1}\right) = \ln\left(\frac{(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}\right) = \ln(\sqrt{2} + 1).$$

53. $\int_0^{\pi/3} \tan x\, dx$

SOLUTION Use the formula for $\int \tan x\, dx$:

$$\int_0^{\pi/3} \tan x\, dx = \ln|\sec x|\Big|_0^{\pi/3} = \ln 2 - \ln 1 = \ln 2.$$

54. $\int_0^{\pi/4} \tan^5 x\, dx$

SOLUTION First use the reduction formula for $\tan^m x$ to evaluate the indefinite integral:

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\, dx = \frac{1}{4}\tan^4 x - \left(\frac{1}{2}\tan^2 x - \int \tan x\, dx\right)$$
$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C.$$

Now compute the definite integral:

$$\int_0^{\pi/4} \tan^5 x\, dx = \left(\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x|\right)\Big|_0^{\pi/4}$$
$$= \left(\frac{1}{4}\left(1^4\right) - \frac{1}{2}\left(1^2\right) + \ln\sqrt{2}\right) - (0 - 0 + \ln 1)$$
$$= \frac{1}{4} - \frac{1}{2} + \ln\sqrt{2} - 0 = \frac{1}{2}\ln 2 - \frac{1}{4}.$$

55. $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^4 x\,dx$

SOLUTION First use the reduction formula for $\sec^m x$ to evaluate the indefinite integral:

$$\int \sec^4 x\,dx = \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\int \sec^2 x\,dx = \frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x + C.$$

Now compute the definite integral:

$$\int_{-\pi/4}^{\pi/4} \sec^4 x\,dx = \left(\frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x\right)\Bigg|_{-\pi/4}^{\pi/4}$$
$$= \left[\frac{1}{3}(1)\left(\sqrt{2}\right)^2 + \frac{2}{3}(1)\right] - \left[\frac{1}{3}(-1)\left(\sqrt{2}\right)^2 + \frac{2}{3}(-1)\right] = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}.$$

56. $\displaystyle\int_{\pi/4}^{3\pi/4} \cot^4 x \csc^2 x\,dx$

SOLUTION Use the substitution $u = \cot x$, $du = -\csc^2 x\,dx$. $x = \pi/4$ corresponds to $u = 1$, and $x = 3\pi/4$ corresponds to $u = -1$. We get

$$\int_{\pi/4}^{3\pi/4} \cot^4 x \csc^2 x\,dx = -\int_1^{-1} u^4\,du = -\frac{1}{5}u^5\Big|_1^{-1} = \frac{2}{5}$$

57. $\displaystyle\int_0^{\pi} \sin 3x \cos 4x\,dx$

SOLUTION Use the formula for $\int \sin mx \cos nx\,dx$:

$$\int_0^{\pi} \sin 3x \cos 4x\,dx = \left(-\frac{\cos(3-4)x}{2(3-4)} - \frac{\cos(3+4)x}{2(3+4)}\right)\Bigg|_0^{\pi} = \left(-\frac{\cos(-x)}{-2} - \frac{\cos 7x}{14}\right)\Bigg|_0^{\pi}$$
$$= \left(\frac{1}{2}\cos x - \frac{1}{14}\cos 7x\right)\Bigg|_0^{\pi} = \left[\frac{1}{2}(-1) - \frac{1}{14}(-1)\right] - \left[\frac{1}{2}(1) - \frac{1}{14}(1)\right] = -\frac{6}{7}.$$

58. $\displaystyle\int_0^{\pi} \sin x \sin 3x\,dx$

SOLUTION Use the formula for $\int \sin mx \sin nx\,dx$:

$$\int_0^{\pi} \sin x \sin 3x\,dx = \left(\frac{\sin(1-3)x}{2(1-3)} - \frac{\sin(1+3)x}{2(1+3)}\right)\Bigg|_0^{\pi} = \left(\frac{\sin(-2x)}{-4} - \frac{\sin 4x}{8}\right)\Bigg|_0^{\pi}$$
$$= \left(\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x\right)\Bigg|_0^{\pi} = 0 - 0 = 0.$$

59. $\displaystyle\int_0^{\pi/6} \sin 2x \cos 4x\,dx$

SOLUTION Using the formula for $\displaystyle\int \sin mx \cos nx\,dx$, we have

$$\int_0^{\pi/6} \sin 2x \cos 4x\,dx = \left(-\frac{1}{-4}\cos(-2x) - \frac{1}{2\cdot 6}\cos(6x)\right)\Bigg|_0^{\pi/6} = \left(\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x\right)\Bigg|_0^{\pi/6}$$
$$= \left(\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{12}\cdot(-1)\right) - \left(\frac{1}{4} - \frac{1}{12}\right) = \frac{1}{24}$$

Here we've used the fact that $\cos x$ is an even function: $\cos(-x) = \cos x$.

60. $\displaystyle\int_0^{\pi/4} \sin 7x \cos 2x\,dx$

SOLUTION Using the formula for $\displaystyle\int \sin mx \cos nx\,dx$, we have

$$\int_0^{\pi/4} \sin 7x \cos 2x\,dx = \left(-\frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x\right)\Bigg|_0^{\pi/4}$$
$$= \left(-\frac{1}{10}\cdot\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{18}\cdot\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{10} - \frac{1}{18}\right) = \frac{1}{45}(7+\sqrt{2})$$

61. Use the identities for $\sin 2x$ and $\cos 2x$ on page 407 to verify that the following formulas are equivalent.

$$\int \sin^4 x\,dx = \frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C$$
$$\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$$

SOLUTION First, observe

$$\begin{aligned}\sin 4x &= 2\sin 2x\cos 2x = 2\sin 2x(1 - 2\sin^2 x)\\ &= 2\sin 2x - 4\sin 2x\sin^2 x = 2\sin 2x - 8\sin^3 x\cos x.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C &= \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x\cos x + C\\ &= \frac{3}{8}x - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + C.\end{aligned}$$

62. Evaluate $\int \sin^2 x\cos^3 x\,dx$ using the method described in the text and verify that your result is equivalent to the following result produced by a computer algebra system.

$$\int \sin^2 x\cos^3 x\,dx = \frac{1}{30}(7 + 3\cos 2x)\sin^3 x + C$$

SOLUTION Use the identity $\cos^2 x = 1 - \sin^2 x$ to write

$$\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x(1 - \sin^2 x)\cos x\,dx.$$

Now use the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \sin^2 x\cos^3 x\,dx = \int u^2(1 - u^2)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$$

To show that this result matches that produced by the computer algebra system, we will make use of the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$. We find

$$\begin{aligned}\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C &= \sin^3 x\left(\frac{1}{3} - \frac{1}{5}\sin^2 x\right) + C = \sin^3 x\left(\frac{7}{30} + \frac{1}{10}\cos 2x\right) + C\\ &= \frac{1}{30}\sin^3 x(7 + 3\cos 2x) + C.\end{aligned}$$

63. Find the volume of the solid obtained by revolving $y = \sin x$ for $0 \le x \le \pi$ about the x-axis.

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^{\pi} \pi(\sin x)^2\,dx = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Big|_0^{\pi} = \pi\left[\left(\frac{\pi}{2} - 0\right) - (0)\right] = \frac{\pi^2}{2}.$$

64. Use Integration by Parts to prove Eqs. (1) and (2).

SOLUTION To prove the reduction formula for $\sin^n x$, use Integration by Parts with $u = \sin^{n-1} x$ and $v' = \sin x$. Then $u' = (n-1)\sin^{n-2} x\cos x$, $v = -\cos x$, and

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx\\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\left(1 - \sin^2 x\right)dx\\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

Solving this equation for $\int \sin^n x\,dx$, we get

$$\begin{aligned}n\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx\\ \int \sin^n x\,dx &= -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx\end{aligned}$$

To prove the reduction formula for $\cos^n x$, use Integration by Parts with $u = \cos^{n-1} x$ and $v' = \cos x$. Then $u' = -(n-1)\cos^{n-2} x \sin x$, $v = \sin x$, and

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

Solving this equation for $\int \cos^n x\,dx$, we get

$$\begin{aligned}n\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \\ \int \cos^n x\,dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\end{aligned}$$

In Exercises 65–68, use the following alternative method for evaluating the integral $J = \int \sin^m x \cos^n x\,dx$ when m and n are both even. Use the identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

to write $J = \frac{1}{4}\int (1-\cos 2x)^{m/2}(1+\cos 2x)^{n/2}\,dx$, and expand the right-hand side as a sum of integrals involving smaller powers of sine and cosine in the variable $2x$.

65. $\int \sin^2 x \cos^2 x\,dx$

SOLUTION Using the identities $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\begin{aligned}J &= \int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\,dx \\ &= \frac{1}{4}\int \left(1-\cos^2 2x\right)dx = \frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 2x\,dx.\end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$, and the formula for $\int \cos^2 u\,du$:

$$\begin{aligned}J &= \frac{1}{4}x - \frac{1}{8}\int \cos^2 u\,du = \frac{1}{4}x - \frac{1}{8}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) + C \\ &= \frac{1}{4}x - \frac{1}{16}(2x) - \frac{1}{16}\sin 2x \cos 2x + C = \frac{1}{8}x - \frac{1}{16}\sin 2x \cos 2x + C.\end{aligned}$$

66. $\int \cos^4 x\,dx$

SOLUTION Using the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\begin{aligned}J &= \int \cos^4 x\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right)dx \\ &= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2x(2\,dx) + \frac{1}{8}\int \cos^2 2x(2\,dx)\end{aligned}$$

Using the substitution $u = 2x$, $du = 2\,dx$, we get

$$\begin{aligned}J &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int \cos^2 u\,du = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) + C \\ &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{16}(2x) + \frac{1}{16}\sin 2x \cos 2x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{16}\sin 2x \cos 2x + C.\end{aligned}$$

67. $\int \sin^4 x \cos^2 x \, dx$

SOLUTION Using the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\begin{aligned} J = \int \sin^4 x \cos^2 x \, dx &= \frac{1}{8} \int (1 - \cos 2x)^2 (1 + \cos 2x) \, dx \\ &= \frac{1}{8} \int \left(1 - 2\cos 2x + \cos^2 2x\right)(1 + \cos 2x) \, dx \\ &= \frac{1}{8} \int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx. \end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$, together with the reduction formula for $\cos^m x$:

$$\begin{aligned} J &= \frac{1}{8}x - \frac{1}{16}\int \cos u \, du - \frac{1}{16}\int \cos^2 u \, du + \frac{1}{16}\int \cos^3 u \, du \\ &= \frac{1}{8}x - \frac{1}{16}\sin u - \frac{1}{16}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) + \frac{1}{16}\left(\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u \, du\right) \\ &= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{32}(2x) - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\cos^2 2x \sin 2x + \frac{1}{24}\sin 2x + C \\ &= \frac{1}{16}x - \frac{1}{48}\sin 2x - \frac{1}{32}\sin 2x \cos 2x + \frac{1}{48}\cos^2 2x \sin 2x + C. \end{aligned}$$

68. $\int \sin^6 x \, dx$

SOLUTION Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, we have

$$\begin{aligned} J = \int \sin^6 x \, dx &= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^3 dx = \frac{1}{8}\int (1 - \cos 2x)^3 \, dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x \, dx \end{aligned}$$

Now use the substitution $u = 2x$, $du = 2\,dx$ together with the reduction formula for $\cos^m x$:

$$\begin{aligned} J &= \frac{1}{8}x - \frac{3}{16}\int \cos u \, du + \frac{3}{16}\int \cos^2 u \, du - \frac{1}{16}\int \cos^3 u \, du \\ &= \frac{1}{8}x - \frac{3}{16}\sin u + \frac{3}{16}\left(\frac{u}{2} + \frac{1}{2}\sin u \cos u\right) - \frac{1}{16}\left(\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u \, du\right) \\ &= \frac{1}{8}x - \frac{3}{16}\sin u + \frac{3}{32}u + \frac{3}{32}\sin u \cos u - \frac{1}{48}\cos^2 u \sin u - \frac{1}{24}\sin u + C \\ &= \frac{1}{8}x - \frac{11}{48}\sin u + \frac{3}{32}u + \frac{3}{32}\sin u \cos u - \frac{1}{48}\cos^2 u \sin u + C \\ &= \frac{1}{8}x - \frac{11}{48}\sin 2x + \frac{3}{32}\cdot 2x + \frac{3}{32}\sin 2x \cos 2x - \frac{1}{48}\cos^2 2x \sin 2x + C \\ &= \frac{5}{16}x - \frac{11}{48}\sin 2x + \frac{3}{32}\sin 2x \cos 2x - \frac{1}{48}\cos^2 2x \sin 2x + C \end{aligned}$$

69. Prove the reduction formula

$$\int \tan^k x \, dx = \frac{\tan^{k-1} x}{k-1} - \int \tan^{k-2} x \, dx$$

Hint: $\tan^k x = (\sec^2 x - 1)\tan^{k-2} x$.

SOLUTION Use the identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^k x \, dx = \int \tan^{k-2} x \left(\sec^2 x - 1\right) dx = \int \tan^{k-2} x \sec^2 x \, dx - \int \tan^{k-2} x \, dx.$$

Now use the substitution $u = \tan x$, $du = \sec^2 x \, dx$:

$$\int \tan^k x \, dx = \int u^{k-2} \, du - \int \tan^{k-2} x \, dx = \frac{1}{k-1}u^{k-1} - \int \tan^{k-2} x \, dx = \frac{\tan^{k-1} x}{k-1} - \int \tan^{k-2} x \, dx.$$

70. Use the substitution $u = \csc x - \cot x$ to evaluate $\int \csc x \, dx$ (see Example 5).

SOLUTION Using the substitution $u = \csc x - \cot x$,

$$du = \left(-\csc x \cot x + \csc^2 x\right) dx = \csc x(\csc x - \cot x)\, dx,$$

we have

$$\int \csc x \, dx = \int \frac{\csc x(\csc x - \cot x)\, dx}{\csc x - \cot x} = \int \frac{du}{u} = \ln|u| + C = \ln|\csc x - \cot x| + C.$$

71. Let $I_m = \int_0^{\pi/2} \sin^m x \, dx$.

(a) Show that $I_0 = \frac{\pi}{2}$ and $I_1 = 1$.

(b) Prove that, for $m \geq 2$,

$$I_m = \frac{m-1}{m} I_{m-2}$$

(c) Use (a) and (b) to compute I_m for $m = 2, 3, 4, 5$.

SOLUTION

(a) We have

$$I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1$$

(b) Using the reduction formula for $\sin^m x$, we get for $m \geq 2$

$$\begin{aligned}
I_m &= \int_0^{\pi/2} \sin^m x \, dx = -\frac{1}{m} \sin^{m-1} x \cos x \Big|_0^{\pi/2} + \frac{m-1}{m} \int_0^{\pi/2} \sin^{m-2} x \, dx \\
&= -\frac{1}{m} \sin^{m-1}\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) + \frac{1}{m} \sin^{m-1}(0)\cos(0) + \frac{m-1}{m} I_{m-2} \\
&= \frac{1}{m}(-1 \cdot 0 + 0 \cdot 1) + \frac{m-1}{m} I_{m-2} \\
&= \frac{m-1}{m} I_{m-2}
\end{aligned}$$

(c)

$$\begin{aligned}
I_2 &= \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \\
I_3 &= \frac{2}{3} I_1 = \frac{2}{3} \\
I_4 &= \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3}{16}\pi \\
I_5 &= \frac{4}{5} I_3 = \frac{8}{15}
\end{aligned}$$

72. Evaluate $\int_0^{\pi} \sin^2 mx \, dx$ for m an arbitrary integer.

SOLUTION Use the substitution $u = mx$, $du = m\, dx$. Then

$$\begin{aligned}
\int_0^{\pi} \sin^2 mx \, dx &= \frac{1}{m} \int_{x=0}^{x=\pi} \sin^2 u \, du = \frac{1}{m}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Big|_{x=0}^{x=\pi} = \frac{1}{m}\left(\frac{mx}{2} - \frac{\sin 2mx}{4}\right)\Big|_0^{\pi} \\
&= \left(\frac{x}{2} - \frac{\sin 2mx}{4m}\right)\Big|_0^{\pi} = \left(\frac{\pi}{2} - \frac{\sin 2\pi m}{4}\right) - (0).
\end{aligned}$$

If m is an arbitrary integer, then $\sin 2m\pi = 0$. Thus

$$\int_0^{\pi} \sin^2 mx \, dx = \frac{\pi}{2}.$$

73. Evaluate $\int \sin x \ln(\sin x)\,dx$. *Hint:* Use Integration by Parts as a first step.

SOLUTION Start by using integration by parts with $u = \ln(\sin x)$ and $v' = \sin x$, so that $u' = \cot x$ and $v = -\cos x$. Then

$$I = \int \sin x \ln(\sin x)\,dx = -\cos x \ln(\sin x) + \int \cot x \cos x\,dx = -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x}\,dx$$
$$= -\cos x \ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx = -\cos x \ln(\sin x) - \int \sin x\,dx + \int \csc x\,dx$$
$$= -\cos x \ln(\sin x) + \cos x + \int \csc x\,dx$$

Using the table, $\int \csc x\,dx = \ln|\csc x - \cot x| + C$, so finally

$$I = -\cos x \ln(\sin x) + \cos x + \ln|\csc x - \cot x| + C$$

74. Total Energy A 100-W light bulb has resistance $R = 144\ \Omega$ (ohms) when attached to household current, where the voltage varies as $V = V_0 \sin(2\pi f t)$ ($V_0 = 110$ V, $f = 60$ Hz). The energy (in joules) expended by the bulb over a period of T seconds is

$$U = \int_0^T P(t)\,dt$$

where $P = V^2/R$ (J/s) is the power. Compute U if the bulb remains on for 5 hours.

SOLUTION After converting to seconds (5 hours = 18,000 seconds), the total energy expended is given by

$$U = \int_0^{18{,}000} P(t)\,dt = \int_0^{18{,}000} \frac{V^2}{R}\,dt = \frac{V_0^2}{R}\int_0^{18{,}000} \sin^2(2\pi f t)\,dt = \frac{110^2}{144}\int_0^{18{,}000} \sin^2(120\pi t)\,dt.$$

Now use the substitution $u = 120\pi t$, $du = 120\pi\,dt$:

$$U = \frac{110^2}{144}\left(\frac{1}{120\pi}\right)\int_{t=0}^{t=18{,}000} \sin^2 u\,du = \frac{110^2}{144\cdot 120\pi}\left[\frac{u}{2} - \frac{1}{2}\sin u \cos u\right]_{t=0}^{t=18{,}000}$$
$$= \frac{110^2}{144\cdot 120\pi}\left[60\pi t - \frac{1}{2}\sin(120\pi t)\cos(120\pi t)\right]_0^{18{,}000} = \frac{110^2}{144\cdot 120\pi}\left[\big(60\pi(18{,}000) - 0\big) - 0\right]$$
$$= \frac{(110^2)(60\pi)(18{,}000)}{(144)(120\pi)} = 756{,}260 \text{ joules.}$$

75. Let m, n be integers with $m \neq \pm n$. Use Eqs. (23)–(25) to prove the so-called **orthogonality relations** that play a basic role in the theory of Fourier Series (Figure 2):

$$\int_0^{\pi} \sin mx \sin nx\,dx = 0$$
$$\int_0^{\pi} \cos mx \cos nx\,dx = 0$$
$$\int_0^{2\pi} \sin mx \cos nx\,dx = 0$$

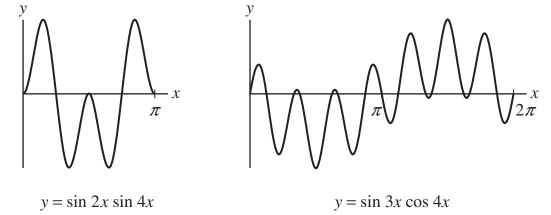


FIGURE 2 The integrals are zero by the orthogonality relations.

SOLUTION If m, n are integers, then $m-n$ and $m+n$ are integers, and therefore $\sin(m-n)\pi = \sin(m+n)\pi = 0$, since $\sin k\pi = 0$ if k is an integer. Thus we have

$$\int_0^{\pi} \sin mx \sin nx\, dx = \left(\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right)\Bigg|_0^{\pi} = \left(\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(m+n)\pi}{2(m+n)}\right) - 0 = 0;$$

$$\int_0^{\pi} \cos mx \cos nx\, dx = \left(\frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}\right)\Bigg|_0^{\pi} = \left(\frac{\sin(m-n)\pi}{2(m-n)} + \frac{\sin(m+n)\pi}{2(m+n)}\right) - 0 = 0.$$

If k is an integer, then $\cos 2k\pi = 1$. Using this fact, we have

$$\begin{aligned}\int_0^{2\pi} \sin mx \cos nx\, dx &= \left(-\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}\right)\Bigg|_0^{2\pi}\\ &= \left(-\frac{\cos(m-n)2\pi}{2(m-n)} - \frac{\cos(m+n)2\pi}{2(m+n)}\right) - \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right)\\ &= \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) - \left(-\frac{1}{2(m-n)} - \frac{1}{2(m+n)}\right) = 0.\end{aligned}$$

Further Insights and Challenges

76. Use the trigonometric identity

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

to prove Eq. (24) in the table of integrals on page 410.

SOLUTION Using the identity $\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$, we get

$$\int \sin mx \cos nx\, dx = \frac{1}{2}\int \sin(m-n)x\, dx + \frac{1}{2}\int \sin(m+n)x\, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C.$$

77. Use Integration by Parts to prove that (for $m \neq 1$)

$$\int \sec^m x\, dx = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\, dx$$

SOLUTION Using Integration by Parts with $u = \sec^{m-2} x$ and $v' = \sec^2 x$, we have $v = \tan x$ and

$$u' = (m-2)\sec^{m-3} x(\sec x \tan x) = (m-2)\tan x \sec^{m-2} x.$$

Then,

$$\begin{aligned}\int \sec^m x\, dx &= \tan x \sec^{m-2} x - (m-2)\int \tan^2 x \sec^{m-2} x\, dx\\ &= \tan x \sec^{m-2} x - (m-2)\int \left(\sec^2 x - 1\right)\sec^{m-2} x\, dx\\ &= \tan x \sec^{m-2} x - (m-2)\int \sec^m x\, dx + (m-2)\int \sec^{m-2} x\, dx.\end{aligned}$$

Solving this equation for $\int \sec^m x\, dx$, we get

$$\begin{aligned}(m-1)\int \sec^m x\, dx &= \tan x \sec^{m-2} x + (m-2)\int \sec^{m-2} x\, dx\\ \int \sec^m x\, dx &= \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\, dx.\end{aligned}$$

78. Set $I_m = \displaystyle\int_0^{\pi/2} \sin^m x\, dx$. Use Exercise 71 to prove that

$$\begin{aligned}I_{2m} &= \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}\\ I_{2m+1} &= \frac{2m}{2m+1}\frac{2m-2}{2m-1}\cdots\frac{2}{3}\end{aligned}$$

Conclude that

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots \frac{2m \cdot 2m}{(2m-1)(2m+1)} \frac{I_{2m}}{I_{2m+1}}$$

SOLUTION We'll use induction to show these results. Recall from Exercise 71 that

$$I_m = \frac{m-1}{m} I_{m-2}$$

when $m \geq 2$. Now, for I_{2m}, the result is true for $m = 1$ and $m = 2$ (again see Exercise 71). Now assume the result is true for $m = k - 1$:

$$I_{2(k-1)} = I_{2k-2} = \frac{2k-3}{2k-2} \cdot \frac{2k-5}{2k-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

Using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k} \cdot \left(\frac{2k-3}{2k-2} \cdot \frac{2k-5}{2k-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \right).$$

For I_{2m+1}, the result is true for $m = 1$. Now assume the result is true for $m = k - 1$:

$$I_{2(k-1)+1} = I_{2k-1} = \frac{2k-2}{2k-1} \cdot \frac{2k-4}{2k-3} \cdots \frac{2}{3}$$

Again using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$I_{2k+1} = \left(\frac{2k+1-1}{2k+1} \right) I_{2k-1} = \frac{2k}{2k+1} \left(\frac{2k-2}{2k-1} \cdot \frac{2k-4}{2k-3} \cdots \frac{2}{3} \right).$$

This establishes the explicit formulas for I_{2m} and I_{2m+1}. Now, divide these two results to obtain

$$\frac{I_{2m}}{I_{2m+1}} = \frac{(2m-1)(2m+1)}{2m \cdot 2m} \cdot \frac{(2m-3)(2m-1)}{(2m-2)(2m-2)} \cdots \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2}.$$

Solving for $\pi/2$, we get the desired result:

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots \frac{2m \cdot 2m}{(2m-1)(2m+1)} \cdot \frac{I_{2m}}{I_{2m+1}}.$$

79. This is a continuation of Exercise 78.

(a) Prove that $I_{2m+1} \leq I_{2m} \leq I_{2m-1}$. *Hint:* $\sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x$ for $0 \leq x \leq \frac{\pi}{2}$.

(b) Show that $\dfrac{I_{2m-1}}{I_{2m+1}} = 1 + \dfrac{1}{2m}$.

(c) Show that $1 \leq \dfrac{I_{2m}}{I_{2m+1}} \leq 1 + \dfrac{1}{2m}$.

(d) Prove that $\displaystyle\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1$.

(e) Finally, deduce the infinite product for $\frac{\pi}{2}$ discovered by English mathematician John Wallis (1616–1703):

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2m \cdot 2m}{(2m-1)(2m+1)}$$

SOLUTION

(a) For $0 \leq x \leq \frac{\pi}{2}$, $0 \leq \sin x \leq 1$. Multiplying this last inequality by $\sin x$, we obtain

$$0 \leq \sin^2 x \leq \sin x.$$

Continuing to multiply this inequality by $\sin x$, we obtain, more generally,

$$\sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x.$$

Integrating these functions over $[0, \frac{\pi}{2}]$, we get

$$\int_0^{\pi/2} \sin^{2m+1} x\, dx \leq \int_0^{\pi/2} \sin^{2m} x\, dx \leq \int_0^{\pi/2} \sin^{2m-1} x\, dx,$$

which is the same as

$$I_{2m+1} \leq I_{2m} \leq I_{2m-1}.$$

(b) Using the relation $I_m = ((m-1)/m)I_{m-2}$, we have

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{I_{2m-1}}{\left(\frac{2m}{2m+1}\right) I_{2m-1}} = \frac{2m+1}{2m} = \frac{2m}{2m} + \frac{1}{2m} = 1 + \frac{1}{2m}.$$

(c) First start with the inequality of part (a):

$$I_{2m+1} \le I_{2m} \le I_{2m-1}.$$

Divide through by I_{2m+1}:

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le \frac{I_{2m-1}}{I_{2m+1}}.$$

Use the result from part (b):

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le 1 + \frac{1}{2m}.$$

(d) Taking the limit of this inequality, and applying the Squeeze Theorem, we have

$$\lim_{m\to\infty} 1 \le \lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} \le \lim_{m\to\infty} \left(1 + \frac{1}{2m}\right).$$

Because

$$\lim_{m\to\infty} 1 = 1 \quad \text{and} \quad \lim_{m\to\infty} \left(1 + \frac{1}{2m}\right) = 1,$$

we obtain

$$1 \le \lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} \le 1.$$

Therefore

$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

(e) Take the limit of both sides of the equation obtained in Exercise 78(d):

$$\lim_{m\to\infty} \frac{\pi}{2} = \lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)} \frac{I_{2m}}{I_{2m+1}}$$

$$\frac{\pi}{2} = \left(\lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)}\right)\left(\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}}\right).$$

Finally, using the result from (d), we have

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdots \frac{2m\cdot 2m}{(2m-1)(2m+1)}.$$

7.3 Trigonometric Substitution

Preliminary Questions

1. State the trigonometric substitution appropriate to the given integral:

(a) $\int \sqrt{9-x^2}\,dx$ **(b)** $\int x^2(x^2-16)^{3/2}\,dx$

(c) $\int x^2(x^2+16)^{3/2}\,dx$ **(d)** $\int (x^2-5)^{-2}\,dx$

SOLUTION

(a) $x = 3\sin\theta$ **(b)** $x = 4\sec\theta$ **(c)** $x = 4\tan\theta$ **(d)** $x = \sqrt{5}\sec\theta$

2. Is trigonometric substitution needed to evaluate $\int x\sqrt{9-x^2}\,dx$?

SOLUTION No. There is a factor of x in the integrand outside the radical and the derivative of $9-x^2$ is $-2x$, so we may use the substitution $u = 9-x^2$, $du = -2x\,dx$ to evaluate this integral.

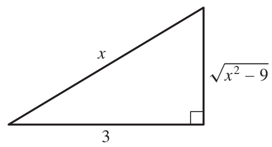
3. Express $\sin 2\theta$ in terms of $x = \sin\theta$.

SOLUTION First note that if $\sin\theta = x$, then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. Thus,

$$\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}.$$

4. Draw a triangle that would be used together with the substitution $x = 3\sec\theta$.

SOLUTION



Exercises

In Exercises 1–4, evaluate the integral by following the steps given.

1. $I = \displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

(a) Show that the substitution $x = 3\sin\theta$ transforms I into $\displaystyle\int d\theta$, and evaluate I in terms of θ.

(b) Evaluate I in terms of x.

SOLUTION

(a) Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$, and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta.$$

Thus,

$$I = \int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\,d\theta}{3\cos\theta} = \int d\theta = \theta + C.$$

(b) If $x = 3\sin\theta$, then $\theta = \sin^{-1}(\frac{x}{3})$. Thus,

$$I = \theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

2. $I = \displaystyle\int \frac{dx}{x^2\sqrt{x^2-2}}$

(a) Show that the substitution $x = \sqrt{2}\sec\theta$ transforms the integral I into $\displaystyle\frac{1}{2}\int \cos\theta d\theta$, and evaluate I in terms of θ.

(b) Use a right triangle to show that with the above substitution, $\sin\theta = \sqrt{x^2-2}/x$.

(c) Evaluate I in terms of x.

SOLUTION

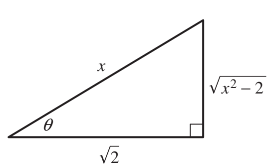
(a) Let $x = \sqrt{2}\sec\theta$. Then $dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-2} = \sqrt{2\sec^2\theta - 2} = \sqrt{2(\sec^2\theta - 1)} = \sqrt{2\tan^2\theta} = \sqrt{2}\tan\theta.$$

Thus,

$$I = \int \frac{dx}{x^2\sqrt{x^2-2}} = \int \frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{(2\sec^2\theta)(\sqrt{2}\tan\theta)} = \frac{1}{2}\int \frac{d\theta}{\sec\theta} = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C.$$

(b) Since $x = \sqrt{2}\sec\theta$, $\sec\theta = \frac{x}{\sqrt{2}}$, and we construct the following right triangle:



From this triangle we see that $\sin\theta = \sqrt{x^2-2}/x$.

(c) Combining the results from parts (a) and (b),

$$I = \frac{1}{2}\sin\theta + C = \frac{\sqrt{x^2-2}}{2x} + C.$$

3. $I = \int \frac{dx}{\sqrt{4x^2+9}}$

(a) Show that the substitution $x = \frac{3}{2}\tan\theta$ transforms I into $\frac{1}{2}\int \sec\theta\,d\theta$.

(b) Evaluate I in terms of θ (refer to the table of integrals on page 410 in Section 7.2 if necessary).

(c) Express I in terms of x.

SOLUTION

(a) If $x = \frac{3}{2}\tan\theta$, then $dx = \frac{3}{2}\sec^2\theta\,d\theta$, and

$$\sqrt{4x^2+9} = \sqrt{4\cdot\left(\frac{3}{2}\tan\theta\right)^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\sec^2\theta} = 3\sec\theta$$

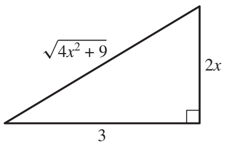
Thus,

$$I = \int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{\frac{3}{2}\sec^2\theta\,d\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta\,d\theta$$

(b)

$$I = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

(c) Since $x = \frac{3}{2}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{2x}{3}$:



From this triangle, we see that $\sec\theta = \frac{1}{3}\sqrt{4x^2+9}$, and therefore

$$\begin{aligned} I &= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\ln\left|\frac{1}{3}\sqrt{4x^2+9}+\frac{2x}{3}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}+2x}{3}\right| + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| - \frac{1}{2}\ln 3 + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| + C \end{aligned}$$

4. $I = \int \frac{dx}{(x^2+4)^2}$

(a) Show that the substitution $x = 2\tan\theta$ transforms the integral I into $\frac{1}{8}\int \cos^2\theta\,d\theta$.

(b) Use the formula $\int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$ to evaluate I in terms of θ.

(c) Show that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ and $\cos\theta = \frac{2}{\sqrt{x^2+4}}$.

(d) Express I in terms of x.

SOLUTION

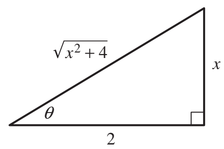
(a) If $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$, and

$$\begin{aligned} I &= \int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)^2} = \frac{2}{16}\int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2} \\ &= \frac{1}{8}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\,d\theta. \end{aligned}$$

(b) Using the formula $\int \cos^2\, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$, we get

$$I = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

(c) Since $x = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{2}$:



From this triangle we see that

$$\sin\theta = \frac{x}{\sqrt{x^2+4}} \quad \text{and} \quad \cos\theta = \frac{2}{\sqrt{x^2+4}}.$$

(d) Since $x = 2\tan\theta$, then $\theta = \tan^{-1}(\frac{x}{2})$, and

$$I = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right) + C = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{8(x^2+4)} + C.$$

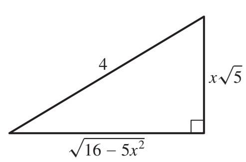
In Exercises 5–10, use the indicated substitution to evaluate the integral.

5. $\int \sqrt{16-5x^2}\, dx, \quad x = \frac{4}{\sqrt{5}}\sin\theta$

SOLUTION Let $x = \frac{4}{\sqrt{5}}\sin\theta$. Then $dx = \frac{4}{\sqrt{5}}\cos\theta\, d\theta$, and

$$\begin{aligned} I &= \int \sqrt{16-5x^2}\, dx = \int \sqrt{16 - 5\left(\frac{4}{\sqrt{5}}\sin\theta\right)^2} \cdot \frac{4}{\sqrt{5}}\cos\theta\, d\theta = \frac{4}{\sqrt{5}}\int \sqrt{16-16\sin^2\theta} \cdot \cos\theta\, d\theta \\ &= \frac{4}{\sqrt{5}} \cdot 4\int \cos\theta \cdot \cos\theta\, d\theta = \frac{16}{\sqrt{5}}\int \cos^2\theta\, d\theta \\ &= \frac{16}{\sqrt{5}}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C = \frac{8}{\sqrt{5}}(\theta + \sin\theta\cos\theta) + C \end{aligned}$$

Since $x = \frac{4}{\sqrt{5}}\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x\sqrt{5}}{4}$:



From this triangle we see that $\cos\theta = \frac{1}{4}\sqrt{16-5x^2}$, so we have

$$\begin{aligned} I &= \frac{8}{\sqrt{5}}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{8}{\sqrt{5}}\left(\sin^{-1}\left(\frac{x\sqrt{5}}{4}\right) + \frac{x\sqrt{5}}{4} \cdot \frac{1}{4}\sqrt{16-5x^2}\right) + C \\ &= \frac{8}{\sqrt{5}}\sin^{-1}\left(\frac{x\sqrt{5}}{4}\right) + \frac{1}{2}x\sqrt{16-5x^2} + C \end{aligned}$$

6. $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\, dx, \quad x = \sin\theta$

SOLUTION Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$, and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

Converting the limits of integration to θ, we find

$$x = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$x = 0 \Rightarrow \theta = \sin^{-1}(0) = 0$$

Therefore

$$I = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta) = \int_0^{\pi/6} \sin^2\theta\,d\theta = \left(\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right)\Bigg|_0^{\pi/6}$$
$$= \left[\frac{\pi}{12} - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right] - [0-0] = \frac{\pi}{12} - \frac{\sqrt{3}}{8} = \frac{2\pi - 3\sqrt{3}}{24}.$$

7. $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}, \quad x = 3\sec\theta$

SOLUTION Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta.$$

Thus,

$$\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{(3\sec\theta\tan\theta\,d\theta)}{(3\sec\theta)(3\tan\theta)} = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C.$$

Since $x = 3\sec\theta$, $\theta = \sec^{-1}(\frac{x}{3})$, and

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C.$$

8. $\displaystyle\int_{1/2}^{1} \frac{dx}{x^2\sqrt{x^2+4}}, \quad x = 2\tan\theta$

SOLUTION Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$, and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2\theta + 1} = 2\sqrt{\sec^2\theta} = 2\sec\theta.$$

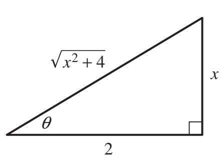
This gives us

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta(2\sec\theta)} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Now use substitution, with $u = \sin\theta$ and $du = \cos\theta\,d\theta$. Then

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\left(-u^{-1}\right) + C = -\frac{1}{4\sin\theta} + C.$$

Since $x = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{2}$:



From this triangle we see that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$. Thus

$$\int_{1/2}^{1} \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x}\Bigg|_{1/2}^{1} = -\frac{1}{4}\left[\sqrt{5} - \frac{\sqrt{\frac{1}{4}+4}}{\frac{1}{2}}\right] = \frac{1}{4}\left[\sqrt{17} - \sqrt{5}\right].$$

9. $\displaystyle\int \frac{dx}{(x^2-4)^{3/2}}, \quad x = 2\sec\theta$

SOLUTION Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta.$$

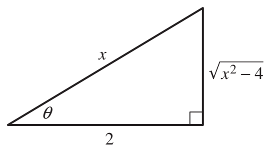
This gives

$$I = \int \frac{dx}{(x^2-4)^{3/2}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{(4\tan^2\theta)^{3/2}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{8\tan^3\theta} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Now use substitution with $u = \sin\theta$ and $du = \cos\theta\,d\theta$. Then

$$I = \frac{1}{4}\int u^{-2}\,du = -\frac{1}{4}u^{-1} + C = \frac{-1}{4\sin\theta} + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that $\sin\theta = \sqrt{x^2-4}/x$, so therefore

$$I = \frac{-1}{4(\sqrt{x^2-4}/x)} + C = \frac{-x}{4\sqrt{x^2-4}} + C.$$

10. $\displaystyle\int_0^1 \frac{dx}{(4+9x^2)^2}, \quad x = \frac{2}{3}\tan\theta$

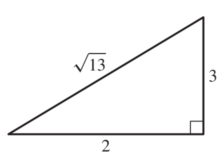
SOLUTION Let $x = \frac{2}{3}\tan\theta$. Then $dx = \frac{2}{3}\sec^2\theta\,d\theta$, and

$$4 + 9x^2 = 4 + 9\left(\frac{2}{3}\tan\theta\right)^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$$

This gives

$$\begin{aligned}\int \frac{dx}{(4+9x^2)^2} &= \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{24}\int \frac{d\theta}{\sec^2\theta}\\ &= \frac{1}{24}\int \cos^2\theta\,d\theta = \frac{1}{24}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C\\ &= \frac{1}{48}(\theta + \sin\theta\cos\theta) + C\end{aligned}$$

The limits of integration are from $x = 0$ to $x = 1$. $x = 0$ corresponds to $\theta = 0$, while $x = 1$ corresponds to the angle θ with $\tan\theta = \frac{3}{2}$. So we construct a right triangle with $\tan\theta = \frac{3}{2}$:



From this triangle we see that $\sin\theta = \frac{3}{\sqrt{13}}$ and $\cos\theta = \frac{2}{\sqrt{13}}$, so that

$$\begin{aligned}\int_0^1 \frac{dx}{(4+9x^2)^2} &= \frac{1}{48}(\theta + \sin\theta\cos\theta)\Big|_0^{\tan^{-1}(3/2)}\\ &= \frac{1}{48}\left(\tan^{-1}\left(\frac{3}{2}\right) + \frac{3}{\sqrt{13}}\cdot\frac{2}{\sqrt{13}} - 0 - 0\right) = \frac{1}{48}\tan^{-1}\left(\frac{3}{2}\right) + \frac{1}{104}\end{aligned}$$

11. Evaluate $\int \frac{x\,dx}{\sqrt{x^2-4}}$ in two ways: using the direct substitution $u = x^2 - 4$ and by trigonometric substitution.

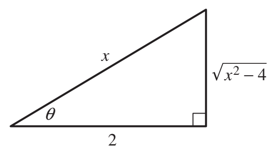
SOLUTION Let $u = x^2 - 4$. Then $du = 2x\,dx$, and

$$I_1 = \int \frac{x\,dx}{\sqrt{x^2-4}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{x^2-4} + C.$$

To use trigonometric substitution, let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, $x^2 - 4 = 4\sec^2\theta - 4 = 4\tan^2\theta$, and

$$I_1 = \int \frac{x\,dx}{\sqrt{x^2-4}} = \int \frac{2\sec\theta(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta} = 2\int \sec^2\theta\,d\theta = 2\tan\theta + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that

$$I_1 = 2\left(\frac{\sqrt{x^2-4}}{2}\right) + C = \sqrt{x^2-4} + C.$$

12. Is the substitution $u = x^2 - 4$ effective for evaluating the integral $\int \frac{x^2\,dx}{\sqrt{x^2-4}}$? If not, evaluate using trigonometric substitution.

SOLUTION If $u = x^2 - 4$, then $du = 2x\,dx$, $x^2 = u + 4$, $dx = du/2x = du/2\sqrt{u+4}$, and

$$I = \int \frac{x^2\,dx}{\sqrt{x^2-4}} = \int \frac{(u+4)}{\sqrt{u}}\left(\frac{du}{2\sqrt{u+4}}\right) = \frac{1}{2}\int \frac{u+4}{\sqrt{u^2+4u}}\,du$$

This substitution is clearly not effective for evaluating this integral.

Instead, use the trigonometric substitution $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta$,

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta,$$

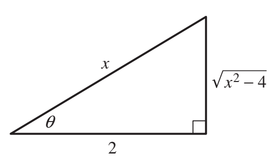
and we have

$$I = \int \frac{x^2\,dx}{\sqrt{x^2-4}} = \int \frac{4\sec^2\theta(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta} = 4\int \sec^3\theta\,d\theta.$$

Now use the reduction formula for $\int \sec^m x\,dx$ from Section 8.7.2:

$$4\int \sec^3\theta\,d\theta = 4\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right] = 2\tan\theta\sec\theta + 2\big[\ln|\sec\theta + \tan\theta|\big] + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this triangle we see that $\tan\theta = \frac{1}{2}\sqrt{x^2-4}$. Therefore

$$I = 2\left(\frac{1}{2}\sqrt{x^2-4}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2-4}\right| + C = \frac{1}{2}x\sqrt{x^2-4} + 2\ln\left|\frac{1}{2}\left(x + \sqrt{x^2-4}\right)\right| + C.$$

Finally, since

$$\ln\left|\frac{1}{2}(x + \sqrt{x^2-4})\right| = \ln\left(\frac{1}{2}\right) + \ln|x + \sqrt{x^2-4}|,$$

and $\ln(\frac{1}{2})$ is a constant, we can "absorb" this constant into the constant of integration, so that

$$I = \frac{1}{2}x\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + C.$$

13. Evaluate using the substitution $u = 1 - x^2$ or trigonometric substitution.

(a) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ **(b)** $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

(c) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$ **(d)** $\displaystyle\int \frac{x^4}{\sqrt{1-x^2}}\,dx$

SOLUTION

(a) Let $u = 1 - x^2$. Then $du = -2x\,dx$, and we have

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{u^{1/2}}.$$

(b) Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and so

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta(\cos\theta)\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta.$$

(c) Use the substitution $u = 1 - x^2$. Then $du = -2x\,dx$, $x^2 = 1 - u$, and so

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) = -\frac{1}{2}\int (1-u)u^{1/2}\,du.$$

(d) Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and so

$$\int \frac{x^4}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^4\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^4\theta\,d\theta.$$

14. Evaluate:

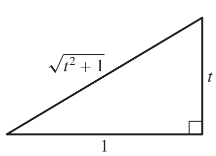
(a) $\displaystyle\int \frac{dt}{(t^2+1)^{3/2}}$ **(b)** $\displaystyle\int \frac{t\,dt}{(t^2+1)^{3/2}}$

SOLUTION

(a) Use the substitution $t = \tan\theta$, so that $dt = \sec^2\theta\,d\theta$. Then

$$\int \frac{dt}{(t^2+1)^{3/2}} = \int \frac{\sec^2\theta}{(\tan^2\theta+1)^{3/2}}\,d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C$$

Since $t = \tan\theta$, we construct a right triangle with $\tan\theta = t$:



From this we see that $\sin\theta = \frac{t}{\sqrt{t^2+1}}$, so that the integral is

$$\int \frac{dt}{(t^2+1)^{3/2}} = \sin\theta + C = \frac{t}{\sqrt{t^2+1}} + C$$

(b) Use the substitution $u = t^2 + 1$, $du = 2t\,dt$; then

$$\int \frac{t\,dt}{(t^2+1)^{3/2}} = \frac{1}{2}\int u^{-3/2}\,du = -u^{-1/2} + C = -\frac{1}{\sqrt{t^2+1}} + C$$

In Exercises 15–32, evaluate using trigonometric substitution. Refer to the table of trigonometric integrals as necessary.

15. $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

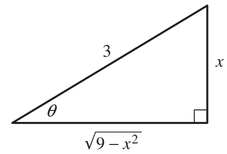
SOLUTION Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$,

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta,$$

and

$$I = \int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta(3\cos\theta\,d\theta)}{3\cos\theta} = 9\int \sin^2\theta\,d\theta = 9\left[\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $x = 3\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x}{3}$:



From this we see that $\cos\theta = \sqrt{9-x^2}/3$, and so

$$I = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C.$$

16. $\displaystyle\int \frac{dt}{(16-t^2)^{3/2}}$

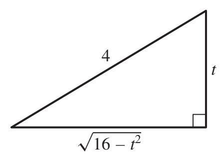
SOLUTION Let $t = 4\sin\theta$. Then $dt = 4\cos\theta\,d\theta$, and

$$(16-t^2)^{3/2} = (16-16\sin^2\theta)^{3/2} = (16\cos^2\theta)^{3/2} = (4\cos\theta)^3 = 64\cos^3\theta$$

so that

$$I = \int \frac{dt}{(16-t^2)^{3/2}} = \int \frac{4\cos\theta}{64\cos^3\theta}\,d\theta = \frac{1}{16}\int \sec^2\theta\,d\theta + C = \frac{1}{16}\tan\theta + C$$

Since $t = 4\sin\theta$, we construct a right triangle with $\sin\theta = \frac{t}{4}$:



From this, we see that $\tan\theta = \frac{t}{\sqrt{16-t^2}}$, so that

$$I = \frac{1}{16}\tan\theta + C = \frac{t}{16\sqrt{16-t^2}} + C$$

17. $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$

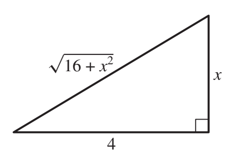
SOLUTION Use the substitution $x = 4\tan\theta$, so that $dx = 4\sec^2\theta\,d\theta$. Then

$$x\sqrt{x^2+16} = 4\tan\theta\sqrt{(4\tan\theta)^2+16} = 4\tan\theta\sqrt{16(\tan^2\theta+1)} = 16\tan\theta\sec\theta$$

so that

$$I = \int \frac{dx}{x\sqrt{x^2+16}} = \int \frac{4\sec^2\theta}{16\tan\theta\sec\theta}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta = -\frac{1}{4}\ln|\csc x + \cot x| + C$$

Since $x = 4\tan\theta$, we construct a right triangle with $\tan\theta = \frac{x}{4}$:



From this, we see that $\csc x = \frac{\sqrt{x^2+16}}{x}$ and $\cot x = \frac{4}{x}$, so that

$$I = -\frac{1}{4}\ln|\csc x + \cot x| + C = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x} + \frac{4}{x}\right| + C = -\frac{1}{4}\ln\left|\frac{4+\sqrt{x^2+16}}{x}\right| + C$$

18. $\int \sqrt{12+4t^2}\,dt$

SOLUTION First simplify the integral:

$$I = \int \sqrt{12+4t^2}\,dt = 2\int \sqrt{3+t^2}\,dt$$

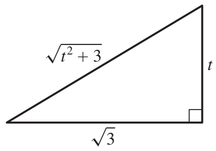
Now let $t = \sqrt{3}\tan\theta$. Then $dt = \sqrt{3}\sec^2\theta\,d\theta$,

$$3+t^2 = 3+3\tan^2\theta = 3(1+\tan^2\theta) = 3\sec^2\theta,$$

and

$$I = 2\int \sqrt{3\sec^2\theta}\left(\sqrt{3}\sec^2\theta\,d\theta\right) = 6\int \sec^3\theta\,d\theta = 6\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right]$$
$$= 3\tan\theta\sec\theta + 3\ln|\sec\theta+\tan\theta| + C.$$

Since $t = \sqrt{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{t}{\sqrt{3}}$:



From this we see that $\sec\theta = \sqrt{t^2+3}/\sqrt{3}$. Therefore,

$$I = 3\left(\frac{t}{\sqrt{3}}\right)\left(\frac{\sqrt{t^2+3}}{\sqrt{3}}\right) + 3\ln\left|\frac{\sqrt{t^2+3}}{\sqrt{3}} + \frac{t}{\sqrt{3}}\right| + C_1 = t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3}+t\right| + 3\ln\left(\frac{1}{\sqrt{3}}\right) + C_1$$
$$= t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3}+t\right| + C,$$

where $C = 3\ln(\frac{1}{\sqrt{3}}) + C_1$.

19. $\int \frac{dx}{\sqrt{x^2-9}}$

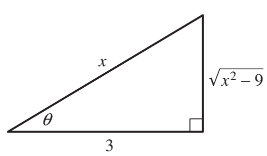
SOLUTION Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$,

$$x^2-9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta,$$

and

$$I = \int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C.$$

Since $x = 3\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{3}$:



From this we see that $\tan\theta = \sqrt{x^2-9}/3$, and so

$$I = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x+\sqrt{x^2-9}\right| + \ln\left(\frac{1}{3}\right) + C_1 = \ln\left|x+\sqrt{x^2-9}\right| + C,$$

where $C = \ln\left(\frac{1}{3}\right) + C_1$.

20. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-25}}$

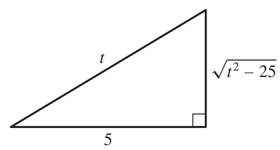
SOLUTION Let $t = 5\sec\theta$. Then $dt = 5\sec\theta\tan\theta\,d\theta$,

$$t^2 - 25 = 25\sec^2\theta - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta,$$

and

$$I = \int \frac{dt}{t^2\sqrt{t^2-25}} = \int \frac{5\sec\theta\tan\theta\,d\theta}{(25\sec^2\theta)(5\tan\theta)} = \frac{1}{25}\int \frac{d\theta}{\sec\theta} = \frac{1}{25}\int \cos\theta\,d\theta = \frac{1}{25}\sin\theta + C.$$

Since $t = 5\sec\theta$, we construct a right triangle with $\sec\theta = \frac{t}{5}$:



From this we see that $\sin\theta = \sqrt{t^2-25}/t$, and so

$$I = \frac{1}{25}\left(\frac{\sqrt{t^2-25}}{t}\right) + C = \frac{\sqrt{t^2-25}}{25t} + C.$$

21. $\displaystyle\int \frac{dy}{y^2\sqrt{5-y^2}}$

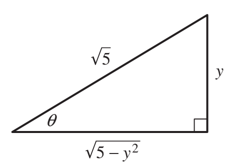
SOLUTION Let $y = \sqrt{5}\sin\theta$. Then $dy = \sqrt{5}\cos\theta\,d\theta$,

$$5 - y^2 = 5 - 5\sin^2\theta = 5(1-\sin^2\theta) = 5\cos^2\theta,$$

and

$$I = \int \frac{dy}{y^2\sqrt{5-y^2}} = \int \frac{\sqrt{5}\cos\theta\,d\theta}{(5\sin^2\theta)(\sqrt{5}\cos\theta)} = \frac{1}{5}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{5}\int \csc^2\theta\,d\theta = \frac{1}{5}(-\cot\theta) + C.$$

Since $y = \sqrt{5}\sin\theta$, we construct a right triangle with $\sin\theta = \frac{y}{\sqrt{5}}$:



From this we see that $\cot\theta = \sqrt{5-y^2}/y$, which gives us

$$I = \frac{1}{5}\left(\frac{-\sqrt{5-y^2}}{y}\right) + C = -\frac{\sqrt{5-y^2}}{5y} + C.$$

22. $\displaystyle\int x^3\sqrt{9-x^2}\,dx$

SOLUTION Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$,

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta,$$

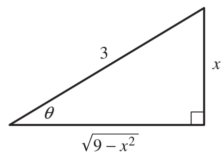
and

$$\begin{aligned} I &= \int x^3\sqrt{9-x^2}\,dx = \int (27\sin^3\theta)(3\cos\theta)(3\cos\theta\,d\theta) \\ &= 243\int \sin^3\theta\cos^2\theta\,d\theta = 243\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \\ &= 243\left[\int \cos^2\theta\sin\theta\,d\theta - \int \cos^4\theta\sin\theta\,d\theta\right]. \end{aligned}$$

Now use substitution, with $u = \cos\theta$ and $du = -\sin\theta\,d\theta$ for both integrals:

$$I = 243\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C.$$

Since $x = 3\sin\theta$, we construct a right triangle with $\sin\theta = \frac{x}{3}$:



From this we see that $\cos\theta = \sqrt{9-x^2}/3$. Thus

$$I = 243\left[-\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5\right] + C = -3(9-x^2)^{3/2} + \frac{1}{5}(9-x^2)^{5/2} + C.$$

Alternately, let $u = 9 - x^2$. Then

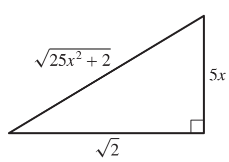
$$I = \int x^3\sqrt{9-x^2}\,dx = -\frac{1}{2}\int (9-u)\sqrt{u}\,du = -\frac{1}{2}\left(6u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$
$$= \frac{1}{5}u^{5/2} - 3u^{3/2} + C = \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C.$$

23. $\displaystyle\int \frac{dx}{\sqrt{25x^2+2}}$

SOLUTION Let $x = \frac{\sqrt{2}}{5}\tan\theta$. Then $dx = \frac{\sqrt{2}}{5}\sec^2\theta\,d\theta$, $25x^2 + 2 = 2\tan^2\theta + 2 = 2\sec^2\theta$, and

$$I = \int \frac{dx}{\sqrt{25x^2+2}} = \int \frac{\frac{\sqrt{2}}{5}\sec^2\theta\,d\theta}{\sqrt{2}\sec\theta} = \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C.$$

Since $x = \frac{\sqrt{2}}{5}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{5x}{\sqrt{2}}$:



From this we see that $\sec\theta = \frac{1}{\sqrt{2}}\sqrt{25x^2+2}$, so that

$$I = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C = \frac{1}{5}\ln\left|\frac{\sqrt{25x^2+2}}{\sqrt{2}} + \frac{5x}{\sqrt{2}}\right| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x + \sqrt{25x^2+2}}{\sqrt{2}}\right| + C = \frac{1}{5}\ln\left|5x + \sqrt{25x^2+2}\right| - \frac{1}{5}\ln\sqrt{2} + C$$
$$= \frac{1}{5}\ln\left|5x + \sqrt{25x^2+2}\right| + C$$

24. $\displaystyle\int \frac{dt}{(9t^2+4)^2}$

SOLUTION First factor out the t^2-coefficient:

$$I = \int \frac{dt}{(9t^2+4)^2} = \int \frac{dt}{\left[9\left(t^2+\frac{4}{9}\right)\right]^2} = \frac{1}{81}\int \frac{dt}{\left(t^2+\frac{4}{9}\right)^2}.$$

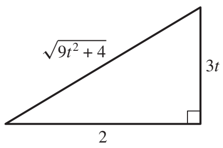
Now let $t = \frac{2}{3}\tan\theta$. Then $dt = \frac{2}{3}\sec^2\theta\,d\theta$,

$$t^2 + \frac{4}{9} = \frac{4}{9}\tan^2\theta + \frac{4}{9} = \frac{4}{9}(\tan^2\theta + 1) = \frac{4}{9}\sec^2\theta,$$

and

$$I = \frac{1}{81}\int \frac{\frac{2}{3}\sec^2 d\theta}{\frac{16}{81}\sec^4\theta\, d\theta} = \frac{1}{24}\int \cos^2\theta\, d\theta = \frac{1}{24}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $t = \frac{2}{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{3t}{2}$:



From this we see that $\sin\theta = 3t/\sqrt{9t^2+4}$ and $\cos\theta = 2/\sqrt{9t^2+4}$. Thus

$$I = \frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{1}{48}\left(\frac{3t}{\sqrt{9t^2+4}}\right)\left(\frac{2}{\sqrt{9t^2+4}}\right) + C = \frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{t}{8(9t^2+4)} + C.$$

25. $\displaystyle\int \frac{dz}{z^3\sqrt{z^2-4}}$

SOLUTION Let $z = 2\sec\theta$. Then $dz = 2\sec\theta\tan\theta\, d\theta$,

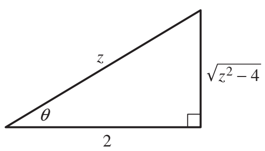
$$z^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta,$$

and

$$I = \int \frac{dz}{z^3\sqrt{z^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{(8\sec^3\theta)(2\tan\theta)} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{8}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

Since $z = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{z}{2}$:



From this we see that $\sin\theta = \sqrt{z^2-4}/z$ and $\cos\theta = 2/z$. Then

$$I = \frac{1}{16}\sec^{-1}\left(\frac{z}{2}\right) + \frac{1}{16}\left(\frac{\sqrt{z^2-4}}{z}\right)\left(\frac{2}{z}\right) + C = \frac{1}{16}\sec^{-1}\left(\frac{z}{2}\right) + \frac{\sqrt{z^2-4}}{8z^2} + C.$$

26. $\displaystyle\int \frac{dy}{\sqrt{y^2-9}}$

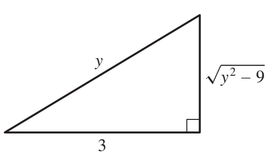
SOLUTION Let $y = 3\sec\theta$, so that $dy = 3\sec\theta\tan\theta\, d\theta$ and

$$y^2 - 9 = (3\sec\theta)^2 - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$$

so that

$$I = \int \frac{dy}{\sqrt{y^2-9}} = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Since $y = 3\sec\theta$, we construct a right triangle with $\sec\theta = \frac{y}{3}$:



From this, we see that $\tan\theta = \frac{1}{3}\sqrt{y^2-9}$, so that

$$I = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{y}{3} + \frac{\sqrt{y^2-9}}{3}\right| + C$$
$$= \ln\left|\frac{y+\sqrt{y^2-9}}{3}\right| + C = \ln\left|y+\sqrt{y^2-9}\right| - \ln 3 + C = \ln\left|y+\sqrt{y^2-9}\right| + C$$

27. $\displaystyle\int \frac{x^2\,dx}{(6x^2-49)^{1/2}}$

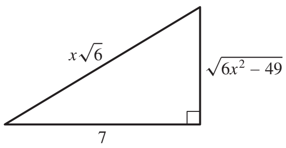
SOLUTION Let $x = \frac{7}{\sqrt{6}}\sec\theta$; then $dx = \frac{7}{\sqrt{6}}\sec\theta\tan\theta\,d\theta$, and

$$6x^2 - 49 = 6\left(\frac{7}{\sqrt{6}}\sec\theta\right)^2 - 49 = 49(\sec^2\theta - 1) = 49\tan^2\theta$$

so that

$$I = \int \frac{x^2\,dx}{(6x^2-49)^{1/2}} = \int \frac{\frac{49}{6}\sec^2\theta(\frac{7}{\sqrt{6}}\sec\theta\tan\theta)}{7\tan\theta}\,d\theta$$
$$= \frac{49}{6\sqrt{6}}\int \sec^3\theta\,d\theta = \frac{49}{6\sqrt{6}}\left(\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int\sec\theta\,d\theta\right)$$
$$= \frac{49}{12\sqrt{6}}(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|) + C$$

Since $x = \frac{7}{\sqrt{6}}\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x\sqrt{6}}{7}$:



From this we see that $\tan\theta = \frac{1}{7}\sqrt{6x^2-49}$, so that

$$I = \frac{49}{12\sqrt{6}}\left(\frac{x\sqrt{6}\sqrt{6x^2-49}}{49} + \ln\left|\frac{x\sqrt{6}+\sqrt{6x^2-49}}{7}\right|\right) + C$$
$$= \frac{49}{12\sqrt{6}}\left(\frac{x\sqrt{6}\sqrt{6x^2-49}}{49} + \ln\left|x\sqrt{6}+\sqrt{6x^2-49}\right| - \ln 7\right) + C$$
$$= \frac{1}{12\sqrt{6}}\left(x\sqrt{6}\sqrt{6x^2-49} + 49\ln\left|x\sqrt{6}+\sqrt{6x^2-49}\right|\right) + C$$

28. $\displaystyle\int \frac{dx}{(x^2-4)^2}$

SOLUTION Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$,

$$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta,$$

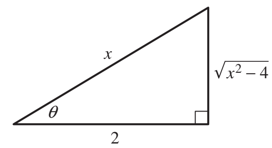
and

$$I = \int \frac{dx}{(x^2-4)^2} = \int \frac{2\sec\theta\tan\theta\,d\theta}{16\tan^4\theta} = \frac{1}{8}\int\frac{\sec\theta\,d\theta}{\tan^3\theta}$$
$$= \frac{1}{8}\int\frac{\cos^2\theta}{\sin^3\theta}\,d\theta = \frac{1}{8}\int\frac{1-\sin^2\theta}{\sin^3\theta}\,d\theta = \frac{1}{8}\int\csc^3\theta\,d\theta - \frac{1}{8}\int\csc\theta\,d\theta.$$

Now use the reduction formula for $\displaystyle\int\csc^3\theta\,d\theta$:

$$I = \frac{1}{8}\left[-\frac{\cot\theta\csc\theta}{2} + \frac{1}{2}\int\csc\theta\,d\theta\right] - \frac{1}{8}\int\csc\theta\,d\theta = -\frac{1}{16}\cot\theta\csc\theta - \frac{1}{16}\int\csc\theta\,d\theta$$
$$= -\frac{1}{16}\cot\theta\csc\theta - \frac{1}{16}\ln|\csc\theta - \cot\theta| + C.$$

Since $x = 2\sec\theta$, we construct a right triangle with $\sec\theta = \frac{x}{2}$:



From this we see that $\cot\theta = 2/\sqrt{x^2-4}$ and $\csc\theta = x/\sqrt{x^2-4}$. Thus

$$I = -\frac{1}{16}\left(\frac{2}{\sqrt{x^2-4}}\right)\left(\frac{x}{\sqrt{x^2-4}}\right) - \frac{1}{16}\ln\left|\frac{x}{\sqrt{x^2-4}} - \frac{2}{\sqrt{x^2-4}}\right| + C$$
$$= \frac{-x}{8(x^2-4)} - \frac{1}{16}\ln\left|\frac{x-2}{\sqrt{x^2-4}}\right| + C.$$

29. $\displaystyle\int \frac{dt}{(t^2+9)^2}$

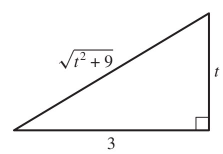
SOLUTION Let $t = 3\tan\theta$. Then $dt = 3\sec^2\theta\, d\theta$,

$$t^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$I = \int \frac{dt}{(t^2+9)^2} = \int \frac{3\sec^2\theta\, d\theta}{81\sec^4\theta} = \frac{1}{27}\int \cos^2\theta\, d\theta = \frac{1}{27}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Since $t = 3\tan\theta$, we construct a right triangle with $\tan\theta = \frac{t}{3}$:



From this we see that $\sin\theta = t/\sqrt{t^2+9}$ and $\cos\theta = 3/\sqrt{t^2+9}$. Thus

$$I = \frac{1}{54}\tan^{-1}\left(\frac{t}{3}\right) + \frac{1}{54}\left(\frac{t}{\sqrt{t^2+9}}\right)\left(\frac{3}{\sqrt{t^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{t}{3}\right) + \frac{t}{18(t^2+9)} + C.$$

30. $\displaystyle\int \frac{dx}{(x^2+1)^3}$

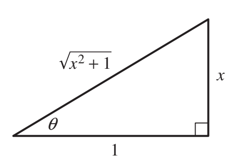
SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$, $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$I = \int \frac{dx}{(x^2+1)^3} = \int \frac{\sec^2\theta\, d\theta}{\sec^6\theta} = \int \cos^4\theta\, d\theta.$$

Using the reduction formula for $\displaystyle\int \cos^4\theta\, d\theta$, we get

$$I = \frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int \cos^2\theta\, d\theta = \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C.$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\sin\theta = x/\sqrt{x^2+1}$ and $\cos\theta = 1/\sqrt{x^2+1}$. Thus

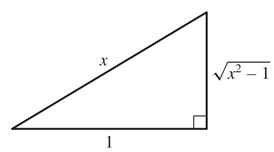
$$I = \frac{1}{4}\left(\frac{1}{\sqrt{x^2+1}}\right)^3\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{3}{8}\tan^{-1}x + \frac{3}{8}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$
$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1}x + C.$$

31. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{3/2}}$

SOLUTION Let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$, and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Thus

$$\begin{aligned} I &= \int \frac{x^2}{(x^2-1)^{3/2}}\,dx = \int \frac{\sec^2\theta}{(\tan^2\theta)^{3/2}}\sec\theta\tan\theta\,d\theta \\ &= \int \frac{\sec^2\theta\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta \\ &= \int \frac{\sec^2\theta}{\tan^2\theta}\sec\theta\,d\theta = \int \csc^2\theta\sec\theta\,d\theta = \int (1+\cot^2\theta)\sec\theta\,d\theta \\ &= \int \sec\theta + \cot\theta\csc\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \csc\theta + C \end{aligned}$$

Since $x = \sec\theta$, we construct the following right triangle:



From this we see that $\tan\theta = \sqrt{x^2-1}$ and that $\csc\theta = \frac{x}{\sqrt{x^2-1}}$, so that

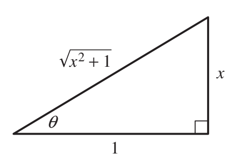
$$I = \ln\left|x + \sqrt{x^2-1}\right| - \frac{x}{\sqrt{x^2-1}} + C$$

32. $\displaystyle\int \frac{x^2\,dx}{(x^2+1)^{3/2}}$

SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$\begin{aligned} I &= \int \frac{x^2\,dx}{(x^2+1)^{3/2}} = \int \frac{\tan^2\theta(\sec^2\theta\,d\theta)}{(\sec^2\theta)^{3/2}} = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\ &= \int \frac{1}{\cos\theta}\,d\theta - \int \frac{\cos^2\theta}{\cos\theta}\,d\theta = \int \sec\theta\,d\theta - \int \cos\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C. \end{aligned}$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\sec\theta = \sqrt{x^2+1}$ and $\sin\theta = x/\sqrt{x^2+1}$. Thus

$$I = \ln\left|\sqrt{x^2+1} + x\right| - \frac{x}{\sqrt{x^2+1}} + C.$$

33. Prove for $a > 0$:

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + C$$

SOLUTION Let $x = \sqrt{a}\,u$. Then, $x^2 = au^2$, $dx = \sqrt{a}\,du$, and

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{a}}\tan^{-1}u + C = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + C.$$

34. Prove for $a > 0$:

$$\int \frac{dx}{(x^2+a)^2} = \frac{1}{2a}\left(\frac{x}{x^2+a} + \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right) + C$$

SOLUTION Let $x = \sqrt{a}\,u$. Then, $x^2 = au^2$, $dx = \sqrt{a}\,du$, and

$$\int \frac{dx}{(x^2+a)^2} = \frac{1}{a^{3/2}} \int \frac{du}{(u^2+1)^2}.$$

Now, let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{dx}{(x^2+a)^2} &= \frac{1}{a^{3/2}} \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \frac{1}{a^{3/2}} \int \cos^2\theta\,d\theta = \frac{1}{a^{3/2}}\left(\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right) + C\\ &= \frac{1}{2a^{3/2}}\left(\frac{u}{1+u^2} + \tan^{-1} u\right) + C = \frac{1}{2a^{3/2}}\left(\frac{x/\sqrt{a}}{1+(x/\sqrt{a})^2} + \tan^{-1}\left(\frac{x}{\sqrt{a}}\right)\right) + C\\ &= \frac{1}{2a}\left(\frac{x}{x^2+a} + \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)\right) + C.\end{aligned}$$

35. Let $I = \int \frac{dx}{\sqrt{x^2-4x+8}}$.

(a) Complete the square to show that $x^2 - 4x + 8 = (x-2)^2 + 4$.

(b) Use the substitution $u = x - 2$ to show that $I = \int \frac{du}{\sqrt{u^2+2^2}}$. Evaluate the u-integral.

(c) Show that $I = \ln\left|\sqrt{(x-2)^2+4} + x - 2\right| + C$.

SOLUTION

(a) Completing the square, we get

$$x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4.$$

(b) Let $u = x - 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2-4x+8}} = \int \frac{dx}{\sqrt{(x-2)^2+4}} = \int \frac{du}{\sqrt{u^2+4}}.$$

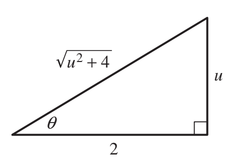
Now let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\,d\theta$,

$$u^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta,$$

and

$$I = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $u = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{u}{2}$:



From this we see that $\sec\theta = \sqrt{u^2+4}/2$. Thus

$$I = \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1 = \ln\left|\sqrt{u^2+4} + u\right| + \left(\ln\frac{1}{2} + C_1\right) = \ln\left|\sqrt{u^2+4} + u\right| + C.$$

(c) Substitute back for x in the result of part (b):

$$I = \ln\left|\sqrt{(x-2)^2+4} + x - 2\right| + C.$$

36. Evaluate $\int \frac{dx}{\sqrt{12x - x^2}}$. First complete the square to write $12x - x^2 = 36 - (x-6)^2$.

SOLUTION First complete the square:

$$12x - x^2 = -\left(x^2 - 12x + 36 - 36\right) = -\left(x^2 - 12x + 36\right) + 36 = 36 - (x-6)^2.$$

Now let $u = x - 6$, and $du = dx$. This gives us

$$I = \int \frac{dx}{\sqrt{12x - x^2}} = \int \frac{dx}{\sqrt{36 - (x-6)^2}} = \int \frac{du}{\sqrt{36 - u^2}}.$$

Next, let $u = 6\sin\theta$. Then $du = 6\cos\theta\, d\theta$,

$$36 - u^2 = 36 - 36\sin^2\theta = 36(1 - \sin^2\theta) = 36\cos^2\theta,$$

and

$$I = \int \frac{6\cos\theta\, d\theta}{6\cos\theta} = \int d\theta = \theta + C.$$

Substituting back, we find

$$I = \sin^{-1}\left(\frac{u}{6}\right) + C = \sin^{-1}\left(\frac{x-6}{6}\right) + C.$$

In Exercises 37–42, evaluate the integral by completing the square and using trigonometric substitution.

37. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x + 13}}$

SOLUTION First complete the square:

$$x^2 + 4x + 13 = x^2 + 4x + 4 + 9 = (x+2)^2 + 9.$$

Let $u = x + 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2 + 4x + 13}} = \int \frac{dx}{\sqrt{(x+2)^2 + 9}} = \int \frac{du}{\sqrt{u^2 + 9}}.$$

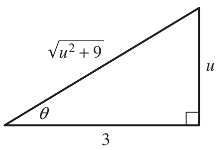
Now let $u = 3\tan\theta$. Then $du = 3\sec^2\theta\, d\theta$,

$$u^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$I = \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $u = 3\tan\theta$, we construct the following right triangle:



From this we see that $\sec\theta = \sqrt{u^2 + 9}/3$. Thus

$$\begin{aligned} I &= \ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + C_1 = \ln\left|\sqrt{u^2+9} + u\right| + \left(\ln\frac{1}{3} + C_1\right) \\ &= \ln\left|\sqrt{(x+2)^2 + 9} + x + 2\right| + C = \ln\left|\sqrt{x^2 + 4x + 13} + x + 2\right| + C. \end{aligned}$$

38. $\displaystyle\int \frac{dx}{\sqrt{2 + x - x^2}}$

SOLUTION First complete the square:

$$2 + x - x^2 = -\left(x^2 - x\right) + 2 = -\left(x^2 - x + \frac{1}{4}\right) + 2 + \frac{1}{4} = \frac{9}{4} - \left(x - \frac{1}{2}\right)^2.$$

Let $u = x - \frac{1}{2}$ and $du = dx$. This gives us

$$I = \int \frac{dx}{\sqrt{2 + x - x^2}} = \int \frac{dx}{\sqrt{\frac{9}{4} - (x - \frac{1}{2})^2}} = \int \frac{du}{\sqrt{\frac{9}{4} - u^2}}.$$

Now let $u = \frac{3}{2}\sin\theta$. Then $du = \frac{3}{2}\cos\theta\, d\theta$,

$$\frac{9}{4} - u^2 = \frac{9}{4} - \frac{9}{4}\sin^2\theta = \frac{9}{4}(1 - \sin^2\theta) = \frac{9}{4}\cos^2\theta,$$

and

$$I = \int \frac{\frac{3}{2}\cos\theta\, d\theta}{\frac{3}{2}\cos\theta} = \int d\theta = \theta + C = \sin^{-1}\left(\frac{2u}{3}\right) + C = \sin^{-1}\left(\frac{2(x - \frac{1}{2})}{3}\right) + C = \sin^{-1}\left(\frac{2x - 1}{3}\right) + C.$$

39. $\displaystyle\int \frac{dx}{\sqrt{x + 6x^2}}$

SOLUTION First complete the square:

$$6x^2 + x = \left(6x^2 + x + \frac{1}{24}\right) - \frac{1}{24} = \left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right)^2 - \frac{1}{24}$$

Let $u = \sqrt{6}x + \frac{1}{2\sqrt{6}}$ so that $du = \sqrt{6}\, dx$. Then

$$I = \int \frac{1}{\sqrt{x + 6x^2}}\, dx = \int \frac{1}{\sqrt{\left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right)^2 - \frac{1}{24}}}\, dx = \frac{1}{\sqrt{6}}\int \frac{1}{\sqrt{u^2 - \frac{1}{24}}}\, du$$

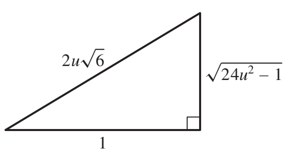
Now let $u = \frac{1}{2\sqrt{6}}\sec\theta$. Then $du = \frac{1}{2\sqrt{6}}\sec\theta\tan\theta$, and

$$u^2 - \frac{1}{24} = \frac{1}{24}(\sec^2\theta - 1) = \frac{1}{24}\tan^2\theta$$

so that

$$I = \frac{1}{\sqrt{6}}\int \frac{1}{\frac{1}{2\sqrt{6}}\tan\theta}\frac{1}{2\sqrt{6}}\sec\theta\tan\theta\, d\theta = \frac{1}{\sqrt{6}}\int \sec\theta\, d\theta = \frac{1}{\sqrt{6}}\ln|\sec\theta + \tan\theta| + C$$

Since $u = \frac{1}{2\sqrt{6}}\sec\theta$, we construct the following right triangle:



from which we see that $\tan\theta = \sqrt{24u^2 - 1}$ and $\sec\theta = 2u\sqrt{6}$. Thus

$$I = \frac{1}{\sqrt{6}}\ln\left|2u\sqrt{6} + \sqrt{24u^2 - 1}\right| + C = \frac{1}{\sqrt{6}}\ln\left|2\sqrt{6}\left(\sqrt{6}x + \frac{1}{2\sqrt{6}}\right) + \sqrt{24\left(6x^2 + x + \frac{1}{24}\right) - 1}\right| + C$$

$$= \frac{1}{\sqrt{6}}\ln\left|12x + 1 + \sqrt{144x^2 + 24x}\right| + C$$

40. $\displaystyle\int \sqrt{x^2 - 4x + 7}\, dx$

SOLUTION First complete the square:

$$x^2 - 4x + 7 = x^2 - 4x + 4 + 3 = (x - 2)^2 + 3.$$

Let $u = x - 2$. Then $du = dx$, and

$$I = \int \sqrt{x^2 - 4x + 7}\, dx = \int \sqrt{(x - 2)^2 + 3}\, dx = \int \sqrt{u^2 + 3}\, du.$$

Now let $u = \sqrt{3}\tan\theta$. Then $du = \sqrt{3}\sec^2\theta\, d\theta$,

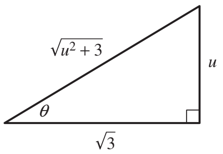
$$u^2 + 3 = 3\tan^2\theta + 3 = 3(\tan^2\theta + 1) = 3\sec^2\theta,$$

and

$$I = \int \sqrt{3\sec^2\theta}\sqrt{3}\sec^2\theta\, d\theta = 3\int \sec^3\theta\, d\theta = 3\left[\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta\right]$$

$$= \frac{3}{2}\tan\theta\sec\theta + \frac{3}{2}\ln|\sec\theta + \tan\theta| + C.$$

Since $u = \sqrt{3}\tan\theta$, we construct a right triangle with $\tan\theta = \frac{u}{\sqrt{3}}$:



From this we see that $\sec\theta = \sqrt{u^2+3}/3$. Thus

$$\begin{aligned} I &= \frac{3}{2}\left(\frac{u}{\sqrt{3}}\right)\left(\frac{\sqrt{u^2+3}}{\sqrt{3}}\right) + \frac{3}{2}\ln\left|\frac{\sqrt{u^2+3}}{\sqrt{3}} + \frac{u}{\sqrt{3}}\right| + C_1 \\ &= \frac{1}{2}u\sqrt{u^2+3} + \frac{3}{2}\ln\left|\sqrt{u^2+3}+u\right| + \left(\frac{3}{2}\ln\frac{1}{\sqrt{3}} + C_1\right) \\ &= \frac{1}{2}(x-2)\sqrt{(x-2)^2+3} + \frac{3}{2}\ln\left|\sqrt{(x-2)^2+3}+x-2\right| + C \\ &= \frac{1}{2}(x-2)\sqrt{x^2-4x+7} + \frac{3}{2}\ln\left|\sqrt{x^2-4x+7}+x-2\right| + C. \end{aligned}$$

41. $\int \sqrt{x^2-4x+3}\,dx$

SOLUTION First complete the square:

$$x^2-4x+3 = x^2-4x+4-1 = (x-2)^2-1.$$

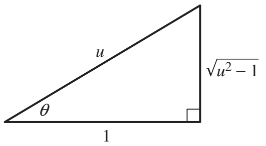
Let $u = x-2$. Then $du = dx$, and

$$I = \int \sqrt{x^2-4x+3}\,dx = \int \sqrt{(x-2)^2-1}\,dx = \int \sqrt{u^2-1}\,du.$$

Now let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, $u^2-1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$\begin{aligned} I &= \int \sqrt{\tan^2\theta}(\sec\theta\tan\theta\,d\theta) = \int \tan^2\theta\sec\theta\,d\theta = \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta \\ &= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta = \left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right) - \int \sec\theta\,d\theta \\ &= \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C. \end{aligned}$$

Since $u = \sec\theta$, we construct the following right triangle:



From this we see that $\tan\theta = \sqrt{u^2-1}$. Thus

$$\begin{aligned} I &= \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + C = \frac{1}{2}(x-2)\sqrt{(x-2)^2-1} - \frac{1}{2}\ln\left|x-2+\sqrt{(x-2)^2-1}\right| + C \\ &= \frac{1}{2}(x-2)\sqrt{x^2-4x+3} - \frac{1}{2}\ln\left|x-2+\sqrt{x^2-4x+3}\right| + C. \end{aligned}$$

42. $\int \frac{dx}{(x^2+6x+6)^2}$

SOLUTION First complete the square:

$$x^2+6x+6 = x^2+6x+9-3 = (x+3)^2-3.$$

Let $u = x+3$. Then $du = dx$, and

$$I = \int \frac{dx}{(x^2+6x+6)^2} = \int \frac{dx}{((x+3)^2-3)^2} = \int \frac{du}{(u^2-3)^2}.$$

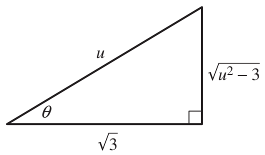
Now let $u = \sqrt{3}\sec\theta$. Then $du = \sqrt{3}\sec\theta\tan\theta$,

$$u^2 - 3 = 3\sec^2\theta - 3 = 3(\sec^2\theta - 1) = 3\tan^2\theta,$$

and

$$\begin{aligned} I &= \int \frac{\sqrt{3}\sec\theta\tan\theta\,d\theta}{9\tan^4\theta} = \frac{\sqrt{3}}{9}\int \frac{\sec\theta\,d\theta}{\tan^3\theta} = \frac{\sqrt{3}}{9}\int \frac{\cos^2\theta}{\sin^3\theta}\,d\theta = \frac{\sqrt{3}}{9}\int \frac{(1-\sin^2\theta)\,d\theta}{\sin^3\theta} \\ &= \frac{\sqrt{3}}{9}\left[\int \csc^3\theta\,d\theta - \int \csc\theta\,d\theta\right] = \frac{\sqrt{3}}{9}\left[\left(-\frac{\cot\theta\csc\theta}{2} + \frac{1}{2}\int \csc\theta\,d\theta\right) - \int \csc\theta\,d\theta\right] \\ &= \frac{\sqrt{3}}{9}\left[-\frac{1}{2}\cot\theta\csc\theta - \frac{1}{2}\int \csc\theta\,d\theta\right] = -\frac{\sqrt{3}}{18}\cot\theta\csc\theta - \frac{\sqrt{3}}{18}\ln|\csc\theta - \cot\theta| + C. \end{aligned}$$

Since $u = \sqrt{3}\sec\theta$, we construct a right triangle with $\sec\theta = \frac{u}{\sqrt{3}}$:



From this we see that $\cot\theta = \sqrt{3}/\sqrt{u^2-3}$ and $\csc\theta = u/\sqrt{u^2-3}$. Thus

$$\begin{aligned} I &= -\frac{\sqrt{3}}{18}\left(\frac{\sqrt{3}}{\sqrt{u^2-3}}\right)\left(\frac{u}{\sqrt{u^2-3}}\right) - \frac{\sqrt{3}}{18}\ln\left|\frac{u}{\sqrt{u^2-3}} - \frac{\sqrt{3}}{\sqrt{u^2-3}}\right| + C \\ &= \frac{-u}{6(u^2-3)} - \frac{\sqrt{3}}{18}\ln\left|\frac{u-\sqrt{3}}{\sqrt{u^2-3}}\right| + C = \frac{-(x+3)}{6((x+3)^2-3)} - \frac{\sqrt{3}}{18}\ln\left|\frac{x+3-\sqrt{3}}{\sqrt{(x+3)^2-3}}\right| + C \\ &= \frac{-(x+3)}{6(x^2+6x+6)} - \frac{\sqrt{3}}{18}\ln\left|\frac{x+3-\sqrt{3}}{\sqrt{x^2+6x+6}}\right| + C. \end{aligned}$$

In Exercises 43–52, indicate a good method for evaluating the integral (but do not evaluate). Your choices are: substitution (specify u and du), Integration by Parts (specify u and v′), a trigonometric method, or trigonometric substitution (specify). If it appears that these techniques are not sufficient, state this.

43. $\displaystyle\int \frac{x\,dx}{\sqrt{12-6x-x^2}}$

SOLUTION Complete the square so the the denominator is $\sqrt{15-(x+3)^2}$ and then use trigonometric substitution with $x + 3 = \sin\theta$.

44. $\displaystyle\int \sqrt{4x^2-1}\,dx$

SOLUTION Use trigonometric substitution, with $x = \frac{1}{2}\sec\theta$.

45. $\displaystyle\int \sin^3 x\cos^3 x\,dx$

SOLUTION Use one of the following trigonometric methods: rewrite $\sin^3 x = (1-\cos^2 x)\sin x$ and let $u = \cos x$, or rewrite $\cos^3 x = (1-\sin^2 x)\cos x$ and let $u = \sin x$.

46. $\displaystyle\int x\sec^2 x\,dx$

SOLUTION Use Integration by Parts, with $u = x$ and $v' = \sec^2 x$.

47. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

SOLUTION Either use the substitution $x = 3u$ and then recognize the formula for the inverse sine:

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C,$$

or use trigonometric substitution, with $x = 3\sin\theta$.

48. $\int \sqrt{1-x^3}\,dx$

SOLUTION Not solvable by any method yet considered. (In fact, this has no antiderivative using elementary functions).

49. $\int \sin^{3/2} x\,dx$

SOLUTION Not solvable by any method yet considered.

50. $\int x^2\sqrt{x+1}\,dx$

SOLUTION Use integration by parts twice, first with $u = x^2$ and then with $u = x$.

51. $\int \frac{dx}{(x+1)(x+2)^3}$

SOLUTION The techniques we have covered thus far are not sufficient to treat this integral. This integral requires a technique known as partial fractions.

52. $\int \frac{dx}{(x+12)^4}$

SOLUTION Use the substitution $u = x + 12$, and then recognize the formula

$$\int u^{-4}\,du = -\frac{1}{3u^3} + C.$$

In Exercises 53–56, evaluate using Integration by Parts as a first step.

53. $\int \sec^{-1} x\,dx$

SOLUTION Let $u = \sec^{-1} x$ and $v' = 1$. Then $v = x$, $u' = 1/x\sqrt{x^2-1}$, and

$$I = \int \sec^{-1} x\,dx = x\sec^{-1} x - \int \frac{x}{x\sqrt{x^2-1}}\,dx = x\sec^{-1} x - \int \frac{dx}{\sqrt{x^2-1}}.$$

To evaluate the integral on the right, let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$, $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{x^2-1}\right| + C.$$

Thus, the final answer is

$$I = x\sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C.$$

54. $\int \frac{\sin^{-1} x}{x^2}\,dx$

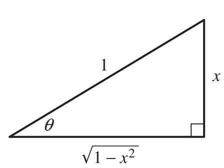
SOLUTION Let $u = \sin^{-1} x$ and $v' = x^{-2}$. Then $u' = 1/\sqrt{1-x^2}$, $v = -x^{-1}$, and

$$I = \int \frac{\sin^{-1} x}{x^2}\,dx = -\frac{\sin^{-1} x}{x} + \int \frac{dx}{x\sqrt{1-x^2}}.$$

To evaluate the integral on the right, let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$, and

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{(\sin\theta)(\cos\theta)} = \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Since $x = \sin\theta$, we construct the following right triangle:



From this we see that $\csc\theta = 1/x$ and $\cot\theta = \sqrt{1-x^2}/x$. Thus

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C.$$

The final answer is

$$I = -\frac{\sin^{-1} x}{x} + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C.$$

55. $\int \ln(x^2+1)\,dx$

SOLUTION Start by using integration by parts, with $u = \ln(x^2+1)$ and $v' = 1$; then $u' = \frac{2x}{x^2+1}$ and $v = x$, so that

$$\begin{aligned} I = \int \ln(x^2+1)\,dx &= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2\int\left(1 - \frac{1}{x^2+1}\right)dx \\ &= x\ln(x^2+1) - 2x + 2\int \frac{1}{x^2+1}\,dx \end{aligned}$$

To deal with the remaining integral, use the substitution $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{1}{x^2+1}\,dx = \int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int 1\,d\theta = \theta = \tan^{-1} x + C$$

so that finally

$$I = x\ln(x^2+1) - 2x + 2\tan^{-1} x + C$$

56. $\int x^2\ln(x^2+1)\,dx$

SOLUTION Start by using integration by parts with $u = \ln(x^2+1)$, $v' = x^2$; then $u' = \frac{2x}{x^2+1}$ and $v = \frac{1}{3}x^3$, so that

$$I = \int x^2\ln(x^2+1)\,dx = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx$$

To deal with the remaining integral, use the substitution $x = \tan\theta$; then $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{x^4}{x^2+1}\,dx = \int \frac{\tan^4\theta}{\tan^2\theta+1}\sec^2\theta\,d\theta = \int \frac{\tan^4\theta}{\sec^2\theta}\sec^2\theta\,d\theta = \int \tan^4\theta\,d\theta$$

Using the reduction formula for $\tan^n$ gives

$$\int \tan^4\theta\,d\theta = \frac{1}{3}\tan^3\theta - \int \tan^2\theta\,d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C$$

so that, substituting back for $x = \tan\theta$, we get

$$I = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}\left(\frac{1}{3}x^3 - x + \tan^{-1} x\right) + C = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{9}x^3 + \frac{2}{3}x - \frac{2}{3}\tan^{-1} x + C$$

57. Find the average height of a point on the semicircle $y = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

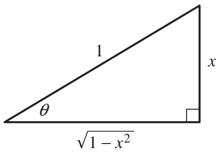
SOLUTION The average height is given by the formula

$$y_{\text{ave}} = \frac{1}{1-(-1)}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$, $1 - x^2 = \cos^2\theta$, and

$$\int \sqrt{1-x^2}\,dx = \int (\cos\theta)(\cos\theta\,d\theta) = \int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C.$$

Since $x = \sin\theta$, we construct the following right triangle:



From this we see that $\cos\theta = \sqrt{1-x^2}$. Therefore,

$$y_{\text{ave}} = \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}\right)\Big|_{-1}^{1} = \frac{1}{2}\left[\left(\frac{1}{2}\pi + 0\right) - \left(-\frac{1}{2}\pi + 0\right)\right] = \frac{\pi}{4}.$$

58. Find the volume of the solid obtained by revolving the graph of $y = x\sqrt{1-x^2}$ over [0, 1] about the y-axis.

SOLUTION Using the method of cylindrical shells, the volume is given by

$$V = 2\pi\int_0^1 x\left(x\sqrt{1-x^2}\right)dx = 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx.$$

To evaluate this integral, let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$,

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta,$$

and

$$I = \int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos^2\theta\,d\theta = \int\left(1-\cos^2\theta\right)\cos^2\theta\,d\theta = \int\cos^2\theta\,d\theta - \int\cos^4\theta\,d\theta.$$

Now use the reduction formula for $\int\cos^4\theta\,d\theta$:

$$I = \int\cos^2\theta\,d\theta - \left[\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int\cos^2\theta\,d\theta\right] = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\int\cos^2\theta\,d\theta$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{8}\theta + \frac{1}{8}\sin\theta\cos\theta + C.$$

Since $\sin\theta = x$, we know that $\cos\theta = \sqrt{1-x^2}$. Then we have

$$I = -\frac{1}{4}(1-x^2)^{3/2}x + \frac{1}{8}\sin^{-1}x + \frac{1}{8}x\sqrt{1-x^2} + C.$$

Now we can complete the volume:

$$V = 2\pi\left(-\frac{1}{4}x(1-x^2)^{3/2} + \frac{1}{8}\sin^{-1}x + \frac{1}{8}x\sqrt{1-x^2}\right)\Big|_0^1 = 2\pi\left[\left(0 + \frac{\pi}{16} + 0\right) - (0)\right] = \frac{\pi^2}{8}.$$

59. Find the volume of the solid obtained by revolving the region between the graph of $y^2 - x^2 = 1$ and the line $y = 2$ about the line $y = 2$.

SOLUTION First solve the equation $y^2 - x^2 = 1$ for y:

$$y = \pm\sqrt{x^2+1}.$$

The region in question is bounded in part by the top half of this hyperbola, which is the equation

$$y = \sqrt{x^2+1}.$$

The limits of integration are obtained by finding the points of intersection of this equation with $y = 2$:

$$2 = \sqrt{x^2+1} \Rightarrow x = \pm\sqrt{3}.$$

The radius of each disk is given by $2 - \sqrt{x^2+1}$; the volume is therefore given by

$$V = \int_{-\sqrt{3}}^{\sqrt{3}}\pi r^2\,dx = 2\pi\int_0^{\sqrt{3}}\left(2-\sqrt{x^2+1}\right)^2dx = 2\pi\int_0^{\sqrt{3}}\left[4 - 4\sqrt{x^2+1} + (x^2+1)\right]dx$$

$$= 8\pi\int_0^{\sqrt{3}}dx - 8\pi\int_0^{\sqrt{3}}\sqrt{x^2+1}\,dx + 2\pi\int_0^{\sqrt{3}}(x^2+1)\,dx.$$

To evaluate the integral $\int \sqrt{x^2+1}\,dx$, let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2+1 = \sec^2\theta$, and

$$\int \sqrt{x^2+1}\,dx = \int \sec^3\theta\,d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int \sec\theta\,d\theta$$
$$= \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right| + C.$$

Now we can compute the volume:

$$V = \left[8\pi x - 8\pi\left(\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right|\right) + \frac{2}{3}\pi x^3 + 2\pi x\right]\Bigg|_0^{\sqrt{3}}$$
$$= \left(10\pi x + \frac{2}{3}\pi x^3 - 4\pi x\sqrt{x^2+1} - 4\pi\ln\left|\sqrt{x^2+1}+x\right|\right)\Bigg|_0^{\sqrt{3}}$$
$$= \left(10\pi\sqrt{3} + 2\pi\sqrt{3} - 8\pi\sqrt{3} - 4\pi\ln\left|2+\sqrt{3}\right|\right) - (0) = 4\pi\left[\sqrt{3} - \ln\left|2+\sqrt{3}\right|\right].$$

60. Find the volume of revolution for the region in Exercise 59, but revolve around $y = 3$.

SOLUTION Using the washer method, the volume is given by

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left(R^2 - r^2\right)dx = 2\pi\int_0^{\sqrt{3}}\left[\left(3-\sqrt{x^2+1}\right)^2 - 1^2\right]dx$$
$$= 2\pi\int_0^{\sqrt{3}}\left(9 - 6\sqrt{x^2+1} + \left(x^2+1\right) - 1\right)dx = 2\pi\int_0^{\sqrt{3}}\left(9 - 6\sqrt{x^2+1} + x^2\right)dx$$
$$= 2\pi\left[9x - 6\left(\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right|\right) + \frac{1}{3}x^3\right]\Bigg|_0^{\sqrt{3}}$$
$$= 2\pi\left[\left(9\sqrt{3} - 3\sqrt{3}(2) - 3\ln\left|2+\sqrt{3}\right| + \sqrt{3}\right) - (0)\right] = 8\pi\sqrt{3} - 6\pi\ln\left|2+\sqrt{3}\right|.$$

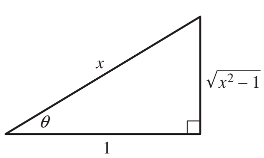
61. Compute $\int \frac{dx}{x^2-1}$ in two ways and verify that the answers agree: first via trigonometric substitution and then using the identity

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

SOLUTION Using trigonometric substitution, let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta d\theta$, $x^2-1 = \sec^2\theta - 1 = \tan^2\theta$, and

$$I = \int \frac{dx}{x^2-1} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^2\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \frac{d\theta}{\sin\theta} = \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Since $x = \sec\theta$, we construct the following right triangle:



From this we see that $\csc\theta = x/\sqrt{x^2-1}$ and $\cot\theta = 1/\sqrt{x^2-1}$. This gives us

$$I = \ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C.$$

Using the given identity, we get

$$I = \int \frac{dx}{x^2-1} = \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

To confirm that these answers agree, note that

$$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = \ln\sqrt{\left|\frac{x-1}{x+1}\right|} = \ln\left|\frac{\sqrt{x-1}}{\sqrt{x+1}}\cdot\frac{\sqrt{x-1}}{\sqrt{x-1}}\right| = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right|.$$

62. CAS You want to divide an 18-inch pizza equally among three friends using vertical slices at $\pm x$ as in Figure 6. Find an equation satisfied by x and find the approximate value of x using a computer algebra system.

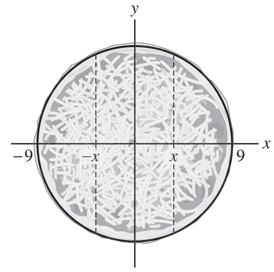


FIGURE 6 Dividing a pizza into three equal parts.

SOLUTION First find the value of x which divides evenly a pizza with a 1-inch radius. By proportionality, we can then take this answer and multiply by 9 to get the answer for the 18-inch pizza. The total area of a 1-inch radius pizza is $\pi \cdot 1^2 = \pi$ (in square inches). The three equal pieces will have an area of $\pi/3$. The center piece is further divided into 4 equal pieces, each of area $\pi/12$. From Example 1, we know that

$$\int_0^x \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}.$$

Setting this expression equal to $\pi/12$ and solving for x using a computer algebra system, we find $x = 0.265$. For the 18-inch pizza, the value of x should be

$$x = 9(0.265) = 2.385 \text{ inches.}$$

63. A charged wire creates an electric field at a point P located at a distance D from the wire (Figure 7). The component $E_\perp$ of the field perpendicular to the wire (in N/C) is

$$E_\perp = \int_{x_1}^{x_2} \frac{k\lambda D}{(x^2+D^2)^{3/2}}\,dx$$

where λ is the charge density (coulombs per meter), $k = 8.99 \times 10^9$ N·m^2/C^2 (Coulomb constant), and x_1, x_2 are as in the figure. Suppose that $\lambda = 6 \times 10^{-4}$ C/m, and $D = 3$ m. Find $E_\perp$ if (a) $x_1 = 0$ and $x_2 = 30$ m, and (b) $x_1 = -15$ m and $x_2 = 15$ m.

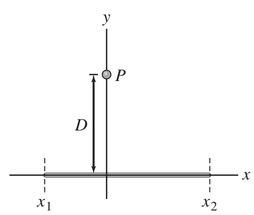


FIGURE 7

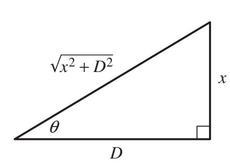
SOLUTION Let $x = D\tan\theta$. Then $dx = D\sec^2\theta\,d\theta$,

$$x^2 + D^2 = D^2\tan^2\theta + D^2 = D^2(\tan^2\theta + 1) = D^2\sec^2\theta,$$

and

$$\begin{aligned} E_\perp &= \int_{x_1}^{x_2} \frac{k\lambda D}{(x^2+D^2)^{3/2}}\,dx = k\lambda D\int_{x_1}^{x_2} \frac{D\sec^2\theta\,d\theta}{(D^2\sec^2\theta)^{3/2}} \\ &= \frac{k\lambda D^2}{D^3}\int_{x_1}^{x_2} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \frac{k\lambda}{D}\int_{x_1}^{x_2} \cos\theta\,d\theta = \frac{k\lambda}{D}\sin\theta\Big|_{x_1}^{x_2} \end{aligned}$$

Since $x = D\tan\theta$, we construct a right triangle with $\tan\theta = x/D$:



From this we see that $\sin\theta = x/\sqrt{x^2+D^2}$. Then

$$E_\perp = \frac{k\lambda}{D}\left(\frac{x}{\sqrt{x^2+D^2}}\right)\Bigg|_{x_1}^{x_2}$$

(a) Plugging in the values for the constants k, λ, D, and evaluating the antiderivative for $x_1 = 0$ and $x_2 = 30$, we get

$$E_\perp = \frac{(8.99\times 10^9)(6\times 10^{-4})}{3}\left[\frac{30}{\sqrt{30^2+3^2}} - 0\right] \approx 1.789\times 10^6\ \frac{V}{\text{m}}$$

(b) If $x_1 = -15$ m and $x_2 = 15$ m, we get

$$E_\perp = \frac{(8.99\times 10^9)(6\times 10^{-4})}{3}\left[\frac{15}{\sqrt{15^2+3^2}} - \frac{-15}{\sqrt{(-15)^2+3^2}}\right] \approx 3.526\times 10^6\ \frac{V}{\text{m}}$$

Further Insights and Challenges

64. Let $J_n = \int \frac{dx}{(x^2+1)^n}$. Use Integration by Parts to prove

$$J_{n+1} = \left(1-\frac{1}{2n}\right)J_n + \left(\frac{1}{2n}\right)\frac{x}{(x^2+1)^n}$$

Then use this recursion relation to calculate J_2 and J_3.

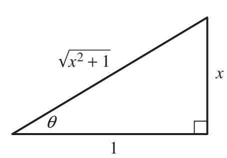
SOLUTION Let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $x^2+1 = \tan^2\theta+1 = \sec^2\theta$, and

$$J_{n+1} = \int \frac{dx}{(x^2+1)^{n+1}} = \int \frac{\sec^2\theta\,d\theta}{\sec^{2n+2}\theta} = \int \sec^{-2n}\theta\,d\theta = \int \cos^{2n}\theta\,d\theta.$$

Using the reduction formula for $\int \cos^m\theta\,d\theta$, we get

$$J_{n+1} = \frac{\cos^{2n-1}\theta\sin\theta}{2n} + \frac{2n-1}{2n}\int \cos^{2n-2}\theta\,d\theta.$$

Since $x = \tan\theta$, we construct the following right triangle:



From this we see that $\cos\theta = 1/\sqrt{x^2+1}$, and $\sin\theta = x/\sqrt{x^2+1}$. This gives us

$$J_{n+1} = \frac{1}{2n}\left(\frac{1}{\sqrt{x^2+1}}\right)^{2n-1}\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{2n-1}{2n}\int\left(\frac{1}{\sqrt{x^2+1}}\right)^{2n-2}\left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx.$$

Here we've used the fact that

$$d\theta = \frac{dx}{\sec^2\theta} = \cos^2\theta\,dx = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx.$$

Simplifying, we get

$$\begin{aligned} J_{n+1} &= \left(\frac{1}{2n}\right)\frac{x}{(\sqrt{x^2+1})^{2n}} + \frac{2n-1}{2n}\int\frac{dx}{(\sqrt{x^2+1})^{2n}} = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}\int\frac{dx}{(x^2+1)^n} \\ &= \frac{1}{2n}\frac{x}{(x^2+1)^n} + \left(1-\frac{1}{2n}\right)J_n. \end{aligned}$$

To use this formula, we first compute J_1:

$$J_1 = \int\frac{dx}{x^2+1} = \tan^{-1}x + C.$$

Now use the formula to compute J_2 and J_3:

$$J_2 = \frac{1}{2}\frac{x}{x^2+1} + \left(1 - \frac{1}{2}\right) J_1 = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C;$$

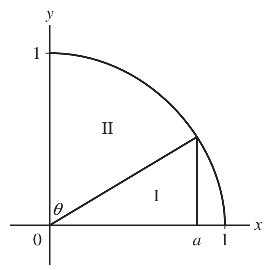
$$J_3 = \frac{1}{4}\frac{x}{(x^2+1)^2} + \left(1 - \frac{1}{4}\right) J_2 = \frac{1}{4}\left[\frac{x}{(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1} x\right] + C.$$

65. Prove the formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C$$

using geometry by interpreting the integral as the area of part of the unit circle.

SOLUTION The integral $\int_0^a \sqrt{1-x^2}\,dx$ is the area bounded by the unit circle, the x-axis, the y-axis, and the line $x = a$. This area can be divided into two regions as follows:



Region I is a triangle with base a and height $\sqrt{1-a^2}$. Region II is a sector of the unit circle with central angle $\theta = \frac{\pi}{2} - \cos^{-1} a = \sin^{-1} a$. Thus,

$$\int_0^a \sqrt{1-x^2}\,dx = \frac{1}{2}a\sqrt{1-a^2} + \frac{1}{2}\sin^{-1} a = \left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right)\Big|_0^a.$$

7.4 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions

Preliminary Questions

1. Which hyperbolic substitution can be used to evaluate the following integrals?

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$ **(b)** $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$ **(c)** $\displaystyle\int \frac{dx}{\sqrt{9x^2+1}}$

SOLUTION The appropriate hyperbolic substitutions are

(a) $x = \sinh t$
(b) $x = 3\sinh t$
(c) $3x = \sinh t$

2. Which two of the hyperbolic integration formulas differ from their trigonometric counterparts by a minus sign?

SOLUTION The integration formulas for $\sinh x$ and $\tanh x$ differ from their trigonometric counterparts by a minus sign.

3. Which antiderivative of $y = (1-x^2)^{-1}$ should we use to evaluate the integral $\int_3^5 (1-x^2)^{-1}\,dx$?

SOLUTION Because the integration interval lies outside $-1 < x < 1$, the appropriate antiderivative of $y = (1-x^2)^{-1}$ is $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$.

Exercises

In Exercises 1–16, calculate the integral.

1. $\displaystyle\int \cosh(3x)\,dx$

SOLUTION $\displaystyle\int \cosh(3x)\,dx = \frac{1}{3}\sinh 3x + C.$

2. $\displaystyle\int \sinh(x+1)\,dx$

SOLUTION $\displaystyle\int \sinh(x+1)\,dx = \cosh(x+1) + C.$

3. $\int x \sinh(x^2+1)\,dx$

SOLUTION $\int x \sinh(x^2+1)\,dx = \frac{1}{2}\cosh(x^2+1) + C.$

4. $\int \sinh^2 x \cosh x\,dx$

SOLUTION Let $u = \sinh x$. Then $du = \cosh x\,dx$ and

$$\int \sinh^2 x \cosh x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\sinh x)^3 + C.$$

5. $\int \operatorname{sech}^2(1-2x)\,dx$

SOLUTION $\int \operatorname{sech}^2(1-2x)\,dx = -\frac{1}{2}\tanh(1-2x) + C.$

6. $\int \tanh(3x)\operatorname{sech}(3x)\,dx$

SOLUTION $\int \tanh(3x)\operatorname{sech}(3x)\,dx = -\frac{1}{3}\operatorname{sech} 3x + C.$

7. $\int \tanh x \operatorname{sech}^2 x\,dx$

SOLUTION Let $u = \tanh x$. Then $du = \operatorname{sech}^2 x\,dx$ nd

$$\int \tanh x \operatorname{sech}^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{\tanh^2 x}{2} + C.$$

8. $\int \frac{\cosh x}{3\sinh x + 4}\,dx$

SOLUTION Let $u = 3\sinh x + 4$. Then $du = 3\cosh x\,dx$ and

$$\int \frac{\cosh x}{3\sinh x + 4}\,dx = \int \frac{du}{3u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3\sinh x + 4| + C.$$

9. $\int \tanh x\,dx$

SOLUTION $\int \tanh x\,dx = \ln\cosh x + C.$

10. $\int x \operatorname{csch}(x^2)\coth(x^2)\,dx$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ and

$$\int x \operatorname{csch}(x^2)\coth(x^2)\,dx = \frac{1}{2}\int \operatorname{csch} u \coth u\,du = -\frac{1}{2}\operatorname{csch} u + C = -\frac{1}{2}\operatorname{csch}(x^2) + C.$$

11. $\int \frac{\cosh x}{\sinh x}\,dx$

SOLUTION $\int \frac{\cosh x}{\sinh x}\,dx = \ln|\sinh x| + C.$

12. $\int \frac{\cosh x}{\sinh^2 x}\,dx$

SOLUTION $\int \frac{\cosh x}{\sinh^2 x}\,dx = \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C.$

13. $\int \sinh^2(4x-9)\,dx$

SOLUTION $\int \sinh^2(4x-9)\,dx = \frac{1}{2}\int (\cosh(8x-18) - 1)\,dx = \frac{1}{16}\sinh(8x-18) - \frac{1}{2}x + C.$

14. $\int \sinh^3 x \cosh^6 x\, dx$

SOLUTION Let $u = \cosh x$. Then $du = \sinh x\, dx$ and

$$\int \sinh^3 x \cosh^6 x\, dx = \int (\cosh^2 x - 1)\cosh^6 x \sinh x\, dx = \int (u^2 - 1)u^6\, du = \int (u^8 - u^6)\, du$$
$$= \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cosh^9 x - \frac{1}{7}\cosh^7 x + C.$$

15. $\int \sinh^2 x \cosh^2 x\, dx$

SOLUTION

$$\int \sinh^2 x \cosh^2 x\, dx = \frac{1}{4}\int \sinh^2 2x\, dx = \frac{1}{8}\int (\cosh 4x - 1)\, dx = \frac{1}{32}\sinh 4x - \frac{1}{8}x + C.$$

16. $\int \tanh^3 x\, dx$

SOLUTION

$$\int \tanh^3 x\, dx = \int (1 - \operatorname{sech}^2 x)\tanh x\, dx = \ln \cosh x - \int \tanh x \operatorname{sech}^2 x\, dx.$$

To evaluate the remaining integral, let $u = \tanh x$. Then $du = \operatorname{sech}^2 x\, dx$ and

$$\int \tanh x \operatorname{sech}^2 x\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\tanh^2 x + C.$$

Therefore,

$$\int \tanh^3 x\, dx = \ln \cosh x - \frac{1}{2}\tanh^2 x + C.$$

In Exercises 17–30, calculate the integral in terms of the inverse hyperbolic functions.

17. $\int \frac{dx}{\sqrt{x^2 - 1}}$

SOLUTION $\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C.$

18. $\int \frac{dx}{\sqrt{9x^2 - 4}}$

SOLUTION $\int \frac{dx}{\sqrt{9x^2 - 4}} = \frac{1}{3}\cosh^{-1}\left(\frac{3x}{2}\right) + C.$

19. $\int \frac{dx}{\sqrt{16 + 25x^2}}$

SOLUTION $\int \frac{dx}{\sqrt{16 + 25x^2}} = \frac{1}{5}\sinh^{-1}\left(\frac{5x}{4}\right) + C.$

20. $\int \frac{dx}{\sqrt{1 + 3x^2}}$

SOLUTION $\int \frac{dx}{\sqrt{1 + 3x^2}} = \frac{1}{\sqrt{3}}\sinh^{-1}(\sqrt{3}x) + C.$

21. $\int \sqrt{x^2 - 1}\, dx$

SOLUTION Let $x = \cosh t$. Then $dx = \sinh t\, dt$ and

$$\int \sqrt{x^2 - 1}\, dx = \int \sinh^2 t\, dt = \frac{1}{2}\int (\cosh 2t - 1)\, dt = \frac{1}{4}\sinh 2t - \frac{1}{2}t + C$$
$$= \frac{1}{2}\sinh t \cosh t - \frac{1}{2}t + C = \frac{1}{2}x\sqrt{x^2 - 1} - \frac{1}{2}\cosh^{-1} x + C.$$

22. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

SOLUTION Let $x = \sinh t$. Then $dx = \cosh t\,dt$ and

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \sinh^2 t\,dt = \frac{1}{2}\int (\cosh 2t - 1)\,dt = \frac{1}{4}\sinh 2t - \frac{1}{2}t + C = \frac{1}{2}\sinh t\cosh t - \frac{1}{2}t + C$$
$$= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\sinh^{-1} x + C.$$

23. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1-x^2}$

SOLUTION

$$\int_{-1/2}^{1/2} \frac{dx}{1-x^2} = \tanh^{-1} x\Big|_{-1/2}^{1/2} = \tanh^{-1}\left(\frac{1}{2}\right) - \tanh^{-1}\left(-\frac{1}{2}\right) = 2\tanh^{-1}\left(\frac{1}{2}\right).$$

24. $\displaystyle\int_4^5 \frac{dx}{1-x^2}$

SOLUTION

$$\int_4^5 \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big|_4^5 = \frac{1}{2}\left(\ln\frac{3}{2} - \ln\frac{5}{3}\right) = \frac{1}{2}\ln\frac{9}{10}.$$

25. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

SOLUTION $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}\Big|_0^1 = \sinh^{-1}(1) - \sinh^{-1}(0) = \sinh^{-1} 1.$

26. $\displaystyle\int_2^{10} \frac{dx}{4x^2-1}$

SOLUTION $\displaystyle\int_2^{10} \frac{dx}{4x^2-1} = -\frac{1}{2}\coth^{-1}(2x)\Big|_2^{10} = \frac{1}{2}(\coth^{-1} 4 - \coth^{-1} 20).$

27. $\displaystyle\int_{-3}^{-1} \frac{dx}{x\sqrt{x^2+16}}$

SOLUTION $\displaystyle\int_{-3}^{-1} \frac{dx}{x\sqrt{x^2+16}} = \frac{1}{4}\operatorname{csch}^{-1}\left(\frac{x}{4}\right)\Big|_{-3}^{-1} = \frac{1}{4}\left(\operatorname{csch}^{-1}\left(-\frac{1}{4}\right) - \operatorname{csch}^{-1}\left(-\frac{3}{4}\right)\right).$

28. $\displaystyle\int_{0.2}^{0.8} \frac{dx}{x\sqrt{1-x^2}}$

SOLUTION $\displaystyle\int_{0.2}^{0.8} \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x\Big|_{0.2}^{0.8} = \operatorname{sech}^{-1}(0.2) - \operatorname{sech}^{-1}(0.8)$

29. $\displaystyle\int \frac{\sqrt{x^2-1}\,dx}{x^2}$

SOLUTION Let $x = \cosh t$. Then $dx = \sinh t\,dt$ and

$$\int \frac{\sqrt{x^2-1}\,dx}{x^2} = \int \frac{\sinh^2 t}{\cosh^2 t}\,dt = \int \tanh^2 t\,dt = \int (1 - \operatorname{sech}^2 t)\,dt$$
$$= t - \tanh t + C = \cosh^{-1} x - \frac{\sqrt{x^2-1}}{x} + C.$$

30. $\displaystyle\int_1^9 \frac{dx}{x\sqrt{x^4+1}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ or $\frac{dx}{x} = \frac{1}{2}\frac{du}{x^2} = \frac{1}{2}\frac{du}{u}$. Hence,

$$\int_1^9 \frac{dx}{x\sqrt{x^4+1}} = \frac{1}{2}\int_1^{81} \frac{du}{u\sqrt{u^2+1}} = -\operatorname{csch}^{-1} u\Big|_1^{81} = \operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1} 81.$$

31. Verify the formulas

$$\sinh^{-1} x = \ln |x + \sqrt{x^2+1}|$$
$$\cosh^{-1} x = \ln |x + \sqrt{x^2-1}| \qquad (\text{for } x \geq 1)$$

SOLUTION Let $x = \sinh t$. Then

$$\cosh t = \sqrt{1+\sinh^2 t} = \sqrt{1+x^2}.$$

Moreover, because

$$\sinh t + \cosh t = \frac{e^t - e^{-t}}{2} + \frac{e^t + e^{-t}}{2} = e^t,$$

it follows that

$$\sinh^{-1} x = t = \ln(\sinh t + \cosh t) = \ln(x + \sqrt{x^2+1}).$$

Now, Let $x = \cosh t$. Then

$$\sinh t = \sqrt{\cosh^2 t - 1} = \sqrt{x^2-1}.$$

and

$$\cosh^{-1} x = t = \ln(\sinh t + \cosh t) = \ln(x + \sqrt{x^2-1}).$$

Because $\cosh t \geq 1$ for all t, this last expression is only valid for $x = \cosh t \geq 1$.

32. Verify that $\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ for $|x| < 1$.

SOLUTION Let $A = \tanh^{-1} x$. Then

$$x = \tanh A = \frac{\sinh A}{\cosh A} = \frac{e^A - e^{-A}}{e^A + e^{-A}}.$$

Solving for A yields

$$A = \frac{1}{2}\ln\frac{x+1}{1-x};$$

hence,

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{x+1}{1-x} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

for $|x| < 1$ (so that both $1+x$ and $1-x$ are positive).

33. Evaluate $\int \sqrt{x^2+16}\,dx$ using trigonometric substitution. Then use Exercise 31 to verify that your answer agrees with the answer in Example 3.

SOLUTION Let $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\int \sqrt{x^2+16}\,dx = 16\int \sec^3\theta\,d\theta = 8\tan\theta\sec\theta + 8\int \sec\theta\,d\theta = 8\tan\theta\sec\theta + 8\ln|\sec\theta + \tan\theta| + C$$
$$= 8\cdot\frac{x}{4}\cdot\frac{\sqrt{x^2+16}}{4} + 8\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$
$$= \frac{1}{2}x\sqrt{x^2+16} + 8\ln\left|\frac{x}{4} + \sqrt{\left(\frac{x}{4}\right)^2+1}\right| + C.$$

Using Exercise 31,

$$\ln\left|\frac{x}{4} + \sqrt{\left(\frac{x}{4}\right)^2+1}\right| = \sinh^{-1}\left(\frac{x}{4}\right),$$

so we can write the antiderivative as

$$\frac{1}{2}x\sqrt{x^2+16} + 8\sinh^{-1}\left(\frac{x}{4}\right) + C,$$

which agrees with the answer in Example 3.

34. Evaluate $\int \sqrt{x^2-9}\,dx$ in two ways: using trigonometric substitution and using hyperbolic substitution. Then use Exercise 31 to verify that the two answers agree.

SOLUTION First, let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$ and

$$\begin{aligned}\int \sqrt{x^2-9}\,dx &= 9\int \tan^2\theta\sec\theta\,d\theta = 9\int \sec^3\theta\,d\theta - 9\int \sec\theta\,d\theta\\ &= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\int \sec\theta\,d\theta - 9\int \sec\theta\,d\theta\\ &= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta| + C\\ &= \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} - \frac{9}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + C\\ &= \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left|\frac{x}{3}+\sqrt{\left(\frac{x}{3}\right)^2-1}\right| + C.\end{aligned}$$

Alternately, let $x = 3\cosh t$. Then $dx = 3\sinh t\,dt$ and

$$\begin{aligned}\int \sqrt{x^2-9}\,dx = 9\int \sinh^2 t\,dt &= \frac{9}{2}\int (\cosh 2t - 1)\,dt = \frac{9}{2}\sinh t\cosh t - \frac{9}{2}t + C\\ &= \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

Using Exercise 31,

$$\cosh^{-1}\left(\frac{x}{3}\right) = \ln\left|\frac{x}{3}+\sqrt{\left(\frac{x}{3}\right)^2-1}\right|,$$

so our two answers agree.

35. Prove the reduction formula for $n \ge 2$:

$$\int \cosh^n x\,dx = \frac{1}{n}\cosh^{n-1}x\sinh x + \frac{n-1}{n}\int \cosh^{n-2}x\,dx \qquad \boxed{5}$$

SOLUTION Using Integration by Parts with $u = \cosh^{n-1}x$ and $v' = \cosh x$, we have

$$\begin{aligned}\int \cosh^n x\,dx &= \cosh^{n-1}x\sinh x - (n-1)\int \cosh^{n-2}x\sinh^2 x\,dx\\ &= \cosh^{n-1}x\sinh x - (n-1)\int \cosh^n x\,dx + (n-1)\int \cosh^{n-2}x\,dx.\end{aligned}$$

Adding $(n-1)\int \cosh^n x\,dx$ to both sides then yields

$$n\int \cosh^n x\,dx = \cosh^{n-1}x\sinh x + (n-1)\int \cosh^{n-2}x\,dx.$$

Finally,

$$\int \cosh^n x\,dx = \frac{1}{n}\cosh^{n-1}x\sinh x + \frac{n-1}{n}\int \cosh^{n-2}x\,dx.$$

36. Use Eq. (5) to evaluate $\int \cosh^4 x\,dx$.

SOLUTION Using Eq. (5) twice,

$$\begin{aligned}\int \cosh^4 x\,dx &= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{4}\int \cosh^2 x\,dx\\ &= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{4}\left(\frac{1}{2}\cosh x\sinh x + \frac{1}{2}\int dx\right)\\ &= \frac{1}{4}\cosh^3 x\sinh x + \frac{3}{8}\cosh x\sinh x + \frac{3}{8}x + C.\end{aligned}$$

In Exercises 37–40, evaluate the integral.

37. $\int \frac{\tanh^{-1} x\, dx}{x^2 - 1}$

SOLUTION Let $u = \tanh^{-1} x$. Then $du = \frac{1}{1-x^2}\, dx = -\frac{1}{x^2-1}\, dx$ and

$$\int \frac{\tanh^{-1} x}{x^2 - 1}\, dx = -\int u\, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\left(\tanh^{-1} x\right)^2 + C.$$

38. $\int \sinh^{-1} x\, dx$

SOLUTION Using Integration by Parts with $u = \sinh^{-1} x$ and $v' = 1$,

$$\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \frac{x}{\sqrt{x^2+1}}\, dx = x \sinh^{-1} x - \sqrt{x^2+1} + C.$$

39. $\int \tanh^{-1} x\, dx$

SOLUTION Using Integration by Parts with $u = \tanh^{-1} x$ and $v' = 1$,

$$\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2}\, dx = x \tanh^{-1} x + \frac{1}{2} \ln |1 - x^2| + C.$$

40. $\int x \tanh^{-1} x\, dx$

SOLUTION Using Integration by Parts with $u = \tanh^{-1} x$ and $v' = x$,

$$\int x \tanh^{-1} x\, dx = \frac{1}{2}x^2 \tanh^{-1} x - \frac{1}{2}\int \frac{x^2}{1-x^2}\, dx = \frac{1}{2}x^2 \tanh^{-1} x - \frac{1}{2}\int \left(\frac{1}{1-x^2} - 1\right) dx$$
$$= \frac{1}{2}x^2 \tanh^{-1} x - \frac{1}{2}\tanh^{-1} x + \frac{1}{2}x + C.$$

Further Insights and Challenges

41. Show that if $u = \tanh(x/2)$, then

$$\cosh x = \frac{1+u^2}{1-u^2}, \qquad \sinh x = \frac{2u}{1-u^2}, \qquad dx = \frac{2du}{1-u^2}$$

Hint: For the first relation, use the identities

$$\sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}(\cosh x - 1), \qquad \cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}(\cosh x + 1)$$

SOLUTION Let $u = \tanh(x/2)$. Then

$$u = \frac{\sinh(x/2)}{\cosh(x/2)} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}.$$

Solving for $\cosh x$ yields

$$\cosh x = \frac{1+u^2}{1-u^2}.$$

Next,

$$\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{(1+u^2)^2 - (1-u^2)^2}{(1-u^2)^2}} = \frac{2u}{1-u^2}.$$

Finally, if $u = \tanh(x/2)$, then $x = 2\tanh^{-1} u$ and

$$dx = \frac{2\,du}{1-u^2}.$$

Exercises 42 and 43: evaluate using the substitution of Exercise 41.

42. $\int \operatorname{sech} x\, dx$

SOLUTION Let $u = \tanh(x/2)$. Then, by Exercise 41,

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1-u^2}{1+u^2} \quad \text{and} \quad dx = \frac{2\,du}{1-u^2},$$

so

$$\int \operatorname{sech} x\, dx = 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\left(\tanh\frac{x}{2}\right) + C.$$

43. $\int \frac{dx}{1+\cosh x}$

SOLUTION Let $u = \tanh(x/2)$. Then, by Exercise 41,

$$1+\cosh x = 1 + \frac{1+u^2}{1-u^2} = \frac{2}{1-u^2} \quad \text{and} \quad dx = \frac{2\,du}{1-u^2},$$

so

$$\int \frac{dx}{1+\cosh x} = \int du = u + C = \tanh\frac{x}{2} + C.$$

44. Suppose that $y = f(x)$ satisfies $y'' = y$. Prove:
(a) $f(x)^2 - (f'(x))^2$ is constant.
(b) If $f(0) = f'(0) = 0$, then $f(x)$ is the zero function.
(c) $f(x) = f(0)\cosh x + f'(0)\sinh x$.

SOLUTION
(a)

$$\frac{d}{dx}\left[f(x)^2 - (f'(x))^2\right] = 2f(x)f'(x) - 2f'(x)f''(x) = 2f(x)f'(x) - 2f'(x)f(x) = 0$$

so that $f(x)^2 - (f'(x))^2$ must be constant, since it has zero derivative everywhere.
(b) If $f(0) = f'(0) = 0$, then part (a) implies that $f(x)^2 - (f'(x))^2$ is the zero function, since it is constant and vanishes at 0. Thus $f(x) = \pm f'(x)$. But Theorem 1 in Section 5.8 states that the only function $y = f(x)$ with $y' = ky$ is $y = Ce^{kx}$; thus either $f(x) = Ce^x$ or $f(x) = Ce^{-x}$. But in either case, $f(0) = C = 0$, so we must have $C = 0$ and $f(x)$ is the zero function.
(c) Let $g(x) = f(x) - f(0)\cosh x - f'(0)\sinh x$. Then

$$\begin{aligned} g'(x) &= f'(x) - f(0)(\cosh x)' - f'(0)(\sinh x)' = f'(x) - f(0)\sinh x - f'(0)\cosh x \\ g''(x) &= f''(x) - f(0)(\sinh x)' - f'(0)(\cosh x)' = f''(x) - f(0)\cosh x - f'(0)\sinh x \\ &= f(x) - f(0)\cosh x - f'(0)\sinh x = g(x) \end{aligned}$$

since $f''(x) = f(x)$. But also

$$\begin{aligned} g(0) &= f(0) - f(0)\cosh 0 - f'(0)\sinh 0 = f(0) - f(0) = 0 \\ g'(0) &= f'(0) - f(0)\sinh 0 - f'(0)\cosh 0 = f'(0) - f'(0) = 0 \end{aligned}$$

Thus $g(x)$ satisfies the conditions the problem, and in particular of part (b) [replace f by g], so that $g(x)$ must be the zero function. But this means that $f(x) - f(0)\cosh x - f'(0)\sinh x = 0$ so that

$$f(x) = f(0)\cosh x + f'(0)\sinh x$$

*Exercises 45–48 refer to the function $gd(y) = \tan^{-1}(\sinh y)$, called the **gudermannian**. In a map of the earth constructed by Mercator projection, points located y radial units from the equator correspond to points on the globe of latitude $gd(y)$.*

45. Prove that $\frac{d}{dy} gd(y) = \operatorname{sech} y$.

SOLUTION Let $gd(y) = \tan^{-1}(\sinh y)$. Then

$$\frac{d}{dy} gd(y) = \frac{1}{1+\sinh^2 y}\cosh y = \frac{1}{\cosh y} = \operatorname{sech} y,$$

where we have used the identity $1 + \sinh^2 y = \cosh^2 y$.

46. Let $f(y) = 2\tan^{-1}(e^y) - \pi/2$. Prove that $gd(y) = f(y)$. *Hint:* Show that $gd'(y) = f'(y)$ and $f(0) = g(0)$.

SOLUTION Let $f(y) = 2\tan^{-1}\left(e^y\right) - \frac{\pi}{2}$. Then

$$f'(y) = \frac{2e^y}{1+e^{2y}} = \frac{2}{e^{-y}+e^y} = \frac{1}{\frac{e^y+e^{-y}}{2}} = \frac{1}{\cosh y} = \operatorname{sech} y.$$

In the previous exercise we found that $\frac{d}{dy}gd(y) = \operatorname{sech} y$; therefore, $gd'(y) = f'(y)$. Now, since the two functions have equal derivatives, they differ by a constant; that is,

$$gd(y) = f(y) + C.$$

To find C we substitute $y = 0$:

$$\begin{aligned}\tan^{-1}(\sinh 0) &= 2\tan^{-1}(e^0) - \frac{\pi}{2} + C\\ \tan^{-1}0 &= 2\tan^{-1}(1) - \frac{\pi}{2} + C\\ 0 &= 2\cdot\frac{\pi}{4} - \frac{\pi}{2} + C\\ C &= 0.\end{aligned}$$

Therefore,

$$gd(y) = f(y).$$

47. Let $t(y) = \sinh^{-1}(\tan y)$. Show that $t(y)$ is the inverse of $gd(y)$ for $0 \le y < \pi/2$.

SOLUTION Let $x = gd(y) = \tan^{-1}(\sinh y)$. Solving for y yields $y = \sinh^{-1}(\tan x)$. Therefore,

$$gd^{-1}(y) = \sinh^{-1}(\tan y).$$

48. Verify that $t(y)$ in Exercise 47 satisfies $t'(y) = \sec y$, and find a value of a such that

$$t(y) = \int_a^y \frac{dt}{\cos t}$$

SOLUTION Let $t(y) = \sinh^{-1}(\tan y)$. Then

$$t'(y) = \frac{1}{\cos^2 y\sqrt{\tan^2 y + 1}} = \frac{1}{\cos^2 y\sqrt{\frac{1}{\cos^2 y}}} = \frac{1}{\cos^2 y\cdot\frac{1}{|\cos y|}} = \frac{1}{|\cos y|} = |\sec y|.$$

For $0 \le y < \frac{\pi}{2}$, $\sec y > 0$; therefore $t'(y) = \sec y$. Integrating this last relation yields

$$t(y) - t(a) = \int_a^y \frac{1}{\cos t}\,dt.$$

For this to be of the desired form, we must have $t(a) = \sinh^{-1}(\tan a) = 0$. The only value for a that satisfies this equation is $a = 0$.

49. The relations $\cosh(it) = \cos t$ and $\sinh(it) = i\sin t$ were discussed in the Excursion. Use these relations to show that the identity $\cos^2 t + \sin^2 t = 1$ results from setting $x = it$ in the identity $\cosh^2 x - \sinh^2 x = 1$.

SOLUTION Let $x = it$. Then

$$\cosh^2 x = (\cosh(it))^2 = \cos^2 t$$

and

$$\sinh^2 x = (\sinh(it))^2 = i^2\sin^2 t = -\sin^2 t.$$

Thus,

$$1 = \cosh^2(it) - \sinh^2(it) = \cos^2 t - (-\sin^2 t) = \cos^2 t + \sin^2 t,$$

as desired.

7.5 The Method of Partial Fractions

Preliminary Questions

1. Suppose that $\int f(x)\,dx = \ln x + \sqrt{x+1} + C$. Can $f(x)$ be a rational function? Explain.

SOLUTION No, $f(x)$ cannot be a rational function because the integral of a rational function cannot contain a term with a non-integer exponent such as $\sqrt{x+1}$.

2. Which of the following are *proper* rational functions?

(a) $\dfrac{x}{x-3}$

(b) $\dfrac{4}{9-x}$

(c) $\dfrac{x^2+12}{(x+2)(x+1)(x-3)}$

(d) $\dfrac{4x^3-7x}{(x-3)(2x+5)(9-x)}$

SOLUTION

(a) No, this is not a proper rational function because the degree of the numerator is not less than the degree of the denominator.

(b) Yes, this is a proper rational function.

(c) Yes, this is a proper rational function.

(d) No, this is not a proper rational function because the degree of the numerator is not less than the degree of the denominator.

3. Which of the following quadratic polynomials are irreducible? To check, complete the square if necessary.

(a) x^2+5

(b) x^2-5

(c) x^2+4x+6

(d) x^2+4x+2

SOLUTION

(a) Square is already completed; irreducible.

(b) Square is already completed; factors as $(x-\sqrt{5})(x+\sqrt{5})$.

(c) $x^2+4x+6=(x+2)^2+2$; irreducible.

(d) $x^2+4x+2=(x+2)^2-2$; factors as $(x+2-\sqrt{2})(x+2+\sqrt{2})$.

4. Let $P(x)/Q(x)$ be a proper rational function where $Q(x)$ factors as a product of distinct linear factors $(x-a_i)$. Then

$$\int \frac{P(x)\,dx}{Q(x)}$$

(choose the correct answer):

(a) is a sum of logarithmic terms $A_i\ln(x-a_i)$ for some constants A_i.

(b) may contain a term involving the arctangent.

SOLUTION The correct answer is **(a)**: the integral is a sum of logarithmic terms $A_i\ln(x-a_i)$ for some constants A_i.

Exercises

1. Match the rational functions (a)–(d) with the corresponding partial fraction decompositions (i)–(iv).

(a) $\dfrac{x^2+4x+12}{(x+2)(x^2+4)}$

(b) $\dfrac{2x^2+8x+24}{(x+2)^2(x^2+4)}$

(c) $\dfrac{x^2-4x+8}{(x-1)^2(x-2)^2}$

(d) $\dfrac{x^4-4x+8}{(x+2)(x^2+4)}$

(i) $x-2+\dfrac{4}{x+2}-\dfrac{4x-4}{x^2+4}$

(ii) $\dfrac{-8}{x-2}+\dfrac{4}{(x-2)^2}+\dfrac{8}{x-1}+\dfrac{5}{(x-1)^2}$

(iii) $\dfrac{1}{x+2}+\dfrac{2}{(x+2)^2}+\dfrac{-x+2}{x^2+4}$

(iv) $\dfrac{1}{x+2}+\dfrac{4}{x^2+4}$

SOLUTION

(a) $\dfrac{x^2+4x+12}{(x+2)(x^2+4)} = \dfrac{1}{x+2} + \dfrac{4}{x^2+4}$.

(b) $\dfrac{2x^2+8x+24}{(x+2)^2(x^2+4)} = \dfrac{1}{x+2} + \dfrac{2}{(x+2)^2} + \dfrac{-x+2}{x^2+4}$.

(c) $\dfrac{x^2-4x+8}{(x-1)^2(x-2)^2} = \dfrac{-8}{x-2} + \dfrac{4}{(x-2)^2} + \dfrac{8}{x-1} + \dfrac{5}{(x-1)^2}$.

(d) $\dfrac{x^4-4x+8}{(x+2)(x^2+4)} = x-2+\dfrac{4}{x+2} - \dfrac{4x-4}{x^2+4}$.

2. Determine the constants A, B:

$$\frac{2x-3}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

SOLUTION Clearing denominators gives

$$2x-3 = A(x-4) + B(x-3).$$

Setting $x = 4$ then yields

$$8-3 = A(0) + B(1) \qquad \text{or} \qquad B = 5,$$

while setting $x = 3$ yields

$$6-3 = A(-1) + 0 \qquad \text{or} \qquad A = -3.$$

3. Clear denominators in the following partial fraction decomposition and determine the constant B (substitute a value of x or use the method of undetermined coefficients).

$$\frac{3x^2+11x+12}{(x+1)(x+3)^2} = \frac{1}{x+1} - \frac{B}{x+3} - \frac{3}{(x+3)^2}$$

SOLUTION Clearing denominators gives

$$3x^2+11x+12 = (x+3)^2 - B(x+1)(x+3) - 3(x+1).$$

Setting $x = 0$ then yields

$$12 = 9 - B(1)(3) - 3(1) \qquad \text{or} \qquad B = -2.$$

To use the method of undetermined coefficients, expand the right-hand side and gather like terms:

$$3x^2+11x+12 = (1-B)x^2 + (3-4B)x + (6-3B).$$

Equating x^2-coefficients on both sides, we find

$$3 = 1 - B \qquad \text{or} \qquad B = -2.$$

4. Find the constants in the partial fraction decomposition

$$\frac{2x+4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

SOLUTION Clearing denominators gives

$$2x+4 = A(x^2+4) + (Bx+C)(x-2).$$

Setting $x = 2$ then yields

$$4+4 = A(4+4) + 0 \qquad \text{or} \qquad A = 1.$$

To find B and C, expand the right side, gather like terms, and use the method of undetermined coefficients:

$$2x+4 = (B+1)x^2 + (-2B+C)x + (4-2C).$$

Equating x^2-coefficients, we find

$$0 = B+1 \qquad \text{or} \qquad B = -1,$$

while equating constants yields

$$4 = 4 - 2C \qquad \text{or} \qquad C = 0.$$

Thus, $A = 1$, $B = -1$, $C = 0$.

In Exercises 5–8, evaluate using long division first to write $f(x)$ as the sum of a polynomial and a proper rational function.

5. $\displaystyle\int \frac{x\,dx}{3x-4}$

SOLUTION Long division gives us

$$\frac{x}{3x-4} = \frac{1}{3} + \frac{4/3}{3x-4}$$

Therefore the integral is

$$\int \frac{x}{3x-4}\,dx = \int \frac{1}{3} - \frac{4}{9x-12}\,dx = \frac{1}{3}x - \frac{4}{9}\ln|9x-12| + C$$

6. $\displaystyle\int \frac{(x^2+2)\,dx}{x+3}$

SOLUTION Long division gives us

$$\frac{x^2+2}{x+3} = x - 3 + \frac{11}{x+3}.$$

Therefore the integral is

$$\int \frac{x^2+2}{x+3}\,dx = \int (x-3)\,dx + 11\int \frac{dx}{x+3} = \frac{x^2}{2} - 3x + 11\ln|x+3| + C.$$

7. $\displaystyle\int \frac{(x^3+2x^2+1)\,dx}{x+2}$

SOLUTION Long division gives us

$$\frac{x^3+2x^2+1}{x+2} = x^2 + \frac{1}{x+2}$$

Therefore the integral is

$$\int \frac{x^3+2x^2+1}{x+2}\,dx = \int x^2 + \frac{1}{x+2}\,dx = \frac{1}{3}x^3 + \ln|x+2| + C$$

8. $\displaystyle\int \frac{(x^3+1)\,dx}{x^2+1}$

SOLUTION Long division gives

$$\frac{x^3+1}{x^2+1} = x - \frac{x-1}{x^2+1}$$

Therefore the integral is

$$\begin{aligned}\int \frac{x^3+1}{x^2+1}\,dx &= \int x - \frac{x-1}{x^2+1}\,dx = \frac{1}{2}x^2 - \int \frac{x}{x^2+1}\,dx + \frac{1}{x^2+1}\,dx\\ &= \frac{1}{2}x^2 - \frac{1}{2}\int \frac{2x\,dx}{x^2+1} + \frac{1}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C\end{aligned}$$

In Exercises 9–44, evaluate the integral.

9. $\displaystyle\int \frac{dx}{(x-2)(x-4)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}.$$

Clearing denominators gives us

$$1 = A(x-4) + B(x-2).$$

Setting $x = 2$ then yields

$$1 = A(2-4) + 0 \qquad \text{or} \qquad A = -\frac{1}{2},$$

while setting $x = 4$ yields

$$1 = 0 + B(4-2) \qquad \text{or} \qquad B = \frac{1}{2}.$$

The result is:

$$\frac{1}{(x-2)(x-4)} = \frac{-\frac{1}{2}}{x-2} + \frac{\frac{1}{2}}{x-4}.$$

Thus,

$$\int \frac{dx}{(x-2)(x-4)} = -\frac{1}{2}\int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-4} = -\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x-4| + C.$$

10. $\displaystyle\int \frac{(x+3)\,dx}{x+4}$

SOLUTION Start with long division:

$$\frac{x+3}{x+4} = 1 - \frac{1}{x+4}$$

so that

$$\int \frac{x+3}{x+4}\,dx = \int 1 - \frac{1}{x+4}\,dx = x - \ln|x+4| + C$$

11. $\displaystyle\int \frac{dx}{x(2x+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}.$$

Clearing denominators gives us

$$1 = A(2x+1) + Bx.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = -\frac{1}{2}$ yields

$$1 = 0 + B\left(-\frac{1}{2}\right) \qquad \text{or} \qquad B = -2.$$

The result is:

$$\frac{1}{x(2x+1)} = \frac{1}{x} + \frac{-2}{2x+1}.$$

Thus,

$$\int \frac{dx}{x(2x+1)} = \int \frac{dx}{x} - \int \frac{2\,dx}{2x+1} = \ln|x| - \ln|2x+1| + C.$$

For the integral on the right, we have used the substitution $u = 2x+1$, $du = 2\,dx$.

12. $\int \frac{(2x-1)\,dx}{x^2-5x+6}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{2x-1}{x^2-5x+6} = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Clearing denominators gives us

$$2x-1 = A(x-3) + B(x-2).$$

Setting $x = 2$ then yields

$$3 = A(-1) + 0 \qquad \text{or} \qquad A = -3,$$

while setting $x = 3$ yields

$$5 = 0 + B(1) \qquad \text{or} \qquad B = 5.$$

The result is:

$$\frac{2x-1}{x^2-5x+6} = \frac{-3}{x-2} + \frac{5}{x-3}.$$

Thus,

$$\int \frac{(2x-1)\,dx}{x^2-5x+6} = -3\int \frac{dx}{x-2} + 5\int \frac{dx}{x-3} = -3\ln|x-2| + 5\ln|x-3| + C.$$

13. $\int \frac{x^2\,dx}{x^2+9}$

SOLUTION

$$\int \frac{x^2}{x^2+9}\,dx = \int 1 - \frac{9}{x^2+9}\,dx = x - 3\tan^{-1}\left(\frac{x}{3}\right) + C$$

14. $\int \frac{dx}{(x-2)(x-3)(x+2)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-2)(x-3)(x+2)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+2}.$$

Clearing denominators gives us

$$1 = A(x-3)(x+2) + B(x-2)(x+2) + C(x-2)(x-3).$$

Setting $x = 2$ then yields

$$1 = A(-1)(4) + 0 + 0 \qquad \text{or} \qquad A = -\frac{1}{4},$$

while setting $x = 3$ yields

$$1 = 0 + B(1)(5) + 0 \qquad \text{or} \qquad B = \frac{1}{5},$$

and setting $x = -2$ yields

$$1 = 0 + 0 + C(-4)(-5) \qquad \text{or} \qquad C = \frac{1}{20}.$$

The result is:

$$\frac{1}{(x-2)(x-3)(x+2)} = \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{5}}{x-3} + \frac{\frac{1}{20}}{x+2}.$$

Thus,

$$\begin{aligned}\int \frac{dx}{(x-2)(x-3)(x+2)} &= -\frac{1}{4}\int \frac{dx}{x-2} + \frac{1}{5}\int \frac{dx}{x-3} + \frac{1}{20}\int \frac{dx}{x+2} \\ &= -\frac{1}{4}\ln|x-2| + \frac{1}{5}\ln|x-3| + \frac{1}{20}\ln|x+2| + C.\end{aligned}$$

15. $\displaystyle\int \frac{(x^2+3x-44)\,dx}{(x+3)(x+5)(3x-2)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+3x-44}{(x+3)(x+5)(3x-2)} = \frac{A}{x+3} + \frac{B}{x+5} + \frac{C}{3x-2}.$$

Clearing denominators gives us

$$x^2+3x-44 = A(x+5)(3x-2) + B(x+3)(3x-2) + C(x+3)(x+5).$$

Setting $x=-3$ then yields

$$9-9-44 = A(2)(-11)+0+0 \qquad \text{or} \qquad A=2,$$

while setting $x=-5$ yields

$$25-15-44 = 0+B(-2)(-17)+0 \qquad \text{or} \qquad B=-1,$$

and setting $x=\frac{2}{3}$ yields

$$\frac{4}{9}+2-44 = 0+0+C\left(\frac{11}{3}\right)\left(\frac{17}{3}\right) \qquad \text{or} \qquad C=-2.$$

The result is:

$$\frac{x^2+3x-44}{(x+3)(x+5)(3x-2)} = \frac{2}{x+3} + \frac{-1}{x+5} + \frac{-2}{3x-2}.$$

Thus,

$$\begin{aligned}\int \frac{(x^2+3x-44)\,dx}{(x+3)(x+5)(3x-2)} &= 2\int\frac{dx}{x+3} - \int\frac{dx}{x+5} - 2\int\frac{dx}{3x-2}\\ &= 2\ln|x+3| - \ln|x+5| - \frac{2}{3}\ln|3x-2| + C.\end{aligned}$$

To evaluate the last integral, we have made the substitution $u=3x-2$, $du=3\,dx$.

16. $\displaystyle\int \frac{3\,dx}{(x+1)(x^2+x)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{(x+1)(x)(x+1)} = \frac{3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$3 = A(x+1)^2 + Bx(x+1) + Cx.$$

Setting $x=0$ then yields

$$3 = A(1)+0+0 \qquad \text{or} \qquad A=3,$$

while setting $x=-1$ yields

$$3 = 0+0+C(-1) \qquad \text{or} \qquad C=-3.$$

Now plug in $A=3$ and $C=-3$:

$$3 = 3(x+1)^2 + Bx(x+1) - 3x.$$

The constant B can be determined by plugging in for x any value other than 0 or -1. Plugging in $x=1$ gives us

$$3 = 3(4) + B(1)(2) - 3 \qquad \text{or} \qquad B=-3.$$

The result is

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2}.$$

Thus,

$$\int \frac{3\,dx}{(x+1)(x^2+x)} = 3\int\frac{dx}{x} - 3\int\frac{dx}{x+1} - 3\int\frac{dx}{(x+1)^2} = 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C.$$

17. $\displaystyle\int \frac{(x^2+11x)\,dx}{(x-1)(x+1)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$x^2+11x = A(x+1)^2 + B(x-1)(x+1) + C(x-1).$$

Setting $x=1$ then yields

$$12 = A(4)+0+0 \qquad \text{or} \qquad A=3,$$

while setting $x=-1$ yields

$$-10 = 0+0+C(-2) \qquad \text{or} \qquad C=5.$$

Plugging in these values results in

$$x^2+11x = 3(x+1)^2 + B(x-1)(x+1) + 5(x-1).$$

The constant B can be determined by plugging in for x any value other than 1 or -1. If we plug in $x=0$, we get

$$0 = 3 + B(-1)(1) + 5(-1) \qquad \text{or} \qquad B=-2.$$

The result is

$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{3}{x-1} + \frac{-2}{x+1} + \frac{5}{(x+1)^2}.$$

Thus,

$$\int \frac{(x^2+11x)\,dx}{(x-1)(x+1)^2} = 3\int \frac{dx}{x-1} - 2\int \frac{dx}{x+1} + 5\int \frac{dx}{(x+1)^2} = 3\ln|x-1| - 2\ln|x+1| - \frac{5}{x+1} + C.$$

18. $\displaystyle\int \frac{(4x^2-21x)\,dx}{(x-3)^2(2x+3)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{4x^2-21x}{(x-3)^2(2x+3)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{2x+3}.$$

Clearing denominators gives us

$$4x^2-21x = A(x-3)(2x+3) + B(2x+3) + C(x-3)^2.$$

Setting $x=3$ then yields

$$-27 = 0 + B(9) + 0 \qquad \text{or} \qquad B=-3,$$

while setting $x=-\frac{3}{2}$ yields

$$9 + \frac{63}{2} = 0+0+C\left(\frac{81}{4}\right) \qquad \text{or} \qquad C=2.$$

Plugging in these values results in

$$4x^2-21x = A(x-3)(2x+3) - 3(2x+3) + 2(x-3)^2.$$

Setting $x=0$ gives us

$$0 = A(-3)(3) - 9 + 18 \qquad \text{or} \qquad A=1.$$

The result is

$$\frac{4x^2-21x}{(x-3)^2(2x+3)} = \frac{1}{x-3} + \frac{-3}{(x-3)^2} + \frac{2}{2x+3}.$$

Thus,

$$\int \frac{(4x^2-21x)\,dx}{(x-3)^2(2x+3)} = \int \frac{dx}{x-3} - 3\int \frac{dx}{(x-3)^2} + \int \frac{2\,dx}{2x+3} = \ln|x-3| + \frac{3}{x-3} + \ln|2x+3| + C.$$

19. $\displaystyle\int \frac{dx}{(x-1)^2(x-2)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}.$$

Clearing denominators gives us

$$1 = A(x-1)(x-2)^2 + B(x-2)^2 + C(x-2)(x-1)^2 + D(x-1)^2.$$

Setting $x = 1$ then yields

$$1 = B(1) \qquad \text{or} \qquad B = 1,$$

while setting $x = 2$ yields

$$1 = D(1) \qquad \text{or} \qquad D = 1.$$

Plugging in these values gives us

$$1 = A(x-1)(x-2)^2 + (x-2)^2 + C(x-2)(x-1)^2 + (x-1)^2.$$

Setting $x = 0$ now yields

$$1 = A(-1)(4) + 4 + C(-2)(1) + 1 \qquad \text{or} \qquad -4 = -4A - 2C,$$

while setting $x = 3$ yields

$$1 = A(2)(1) + 1 + C(1)(4) + 4 \qquad \text{or} \qquad -4 = 2A + 4C.$$

Solving this system of two equations in two unknowns gives $A = 2$ and $C = -2$. The result is

$$\frac{1}{(x-1)^2(x-2)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{-2}{x-2} + \frac{1}{(x-2)^2}.$$

Thus,

$$\begin{aligned}\int \frac{dx}{(x-1)^2(x-2)^2} &= 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - 2\int \frac{dx}{x-2} + \int \frac{dx}{(x-2)^2} \\ &= 2\ln|x-1| - \frac{1}{x-1} - 2\ln|x-2| - \frac{1}{x-2} + C.\end{aligned}$$

20. $\displaystyle\int \frac{(x^2-8x)\,dx}{(x+1)(x+4)^3}$

SOLUTION The partial fraction decomposition is

$$\frac{x^2-8x}{(x+1)(x+4)^3} = \frac{A}{x+1} + \frac{B}{x+4} + \frac{C}{(x+4)^2} + \frac{D}{(x+4)^3}$$

Clearing fractions gives

$$x^2 - 8x = A(x+4)^3 + B(x+4)^2(x+1) + C(x+4)(x+1) + D(x+1)$$

Setting $x = -4$ gives $48 = -3D$ so that $D = -16$. Setting $x = -1$ gives $9 = 27A$ so that $A = \frac{1}{3}$. Thus

$$x^2 - 8x = \frac{1}{3}(x+4)^3 + B(x+4)^2(x+1) + C(x+4)(x+1) - 16(x+1)$$

The coefficient of x^3 on the right hand side must be zero; it is $\frac{1}{3} + B$, so that $B = -\frac{1}{3}$. Finally, the constant term on the right must be zero as well; substituting the known values of A, B, and D gives for the constant term

$$\frac{1}{3}\cdot 64 - \frac{1}{3}\cdot 16 + 4C - 16 = 4C$$

so that $C = 0$, and the partial fraction decomposition is

$$\frac{x^2-8x}{(x+1)(x+4)^3} = \frac{1}{3(x+1)} - \frac{1}{3(x+4)} - \frac{16}{(x+4)^3}$$

Thus

$$\int \frac{x^2-8x}{(x+1)(x+4)^3}\,dx = \frac{1}{3}\int \frac{1}{x+1}\,dx - \frac{1}{3}\int \frac{1}{x+4}\,dx - 16\int \frac{1}{(x+4)^3}\,dx$$
$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x+4| + 8(x+4)^{-2} + C = \frac{1}{3}\ln\left|\frac{x+1}{x+4}\right| + 8(x+4)^{-2} + C$$

21. $\int \frac{8\,dx}{x(x+2)^3}$

SOLUTION The partial fraction decomposition is

$$\frac{8}{x(x+2)^3} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Clearing fractions gives

$$8 = A(x+2)^3 + Bx(x+2)^2 + Cx(x+2) + Dx$$

Setting $x=0$ gives $8=8A$ so $A=1$; setting $x=-2$ gives $8=-2D$ so that $D=-4$; the result is

$$8 = (x+2)^3 + Bx(x+2)^2 + Cx(x+2) - 4x$$

The coefficient of x^3 on the right-hand side must be zero, since it is zero on the left. We compute it to be $1+B$, so that $B=-1$. Finally, we look at the coefficient of x^2 on the right-hand side; it must be zero as well. We compute it to be

$$3\cdot 2 - 4 + C = C + 2$$

so that $C=-2$ and the partial fraction decomposition is

$$\frac{8}{x(x+2)^3} = \frac{1}{x} - \frac{1}{x+2} - \frac{2}{(x+2)^2} - \frac{4}{(x+2)^3}$$

and

$$\int \frac{8}{x(x+2)^3}\,dx = \int \frac{1}{x}\,dx - \frac{1}{x+2}\,dx - 2\int (x+2)^{-2}\,dx - 4\int (x+2)^{-3}\,dx$$
$$= \ln|x| - \ln|x+2| + 2(x+2)^{-1} + 2(x+2)^{-2} + C = \ln\left|\frac{x}{x+2}\right| + \frac{2}{x+2} + \frac{2}{(x+2)^2} + C$$

22. $\int \frac{x^2\,dx}{x^2+3}$

SOLUTION

$$\int \frac{x^2}{x^2+3}\,dx = \int 1 - \frac{3}{x^2+3}\,dx = \int 1\,dx - 3\int \frac{1}{x^2+3}\,dx = x - \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

23. $\int \frac{dx}{2x^2-3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{1}{2x^2-3} = \frac{1}{(\sqrt{2}x-\sqrt{3})(\sqrt{2}x+\sqrt{3})} = \frac{A}{\sqrt{2}x-\sqrt{3}} + \frac{B}{\sqrt{2}x+\sqrt{3}}.$$

Clearing denominators, we get

$$1 = A\left(\sqrt{2}x+\sqrt{3}\right) + B\left(\sqrt{2}x-\sqrt{3}\right).$$

Setting $x=\sqrt{3}/\sqrt{2}$ then yields

$$1 = A\left(\sqrt{3}+\sqrt{3}\right) + 0 \qquad \text{or} \qquad A = \frac{1}{2\sqrt{3}},$$

while setting $x=-\sqrt{3}/\sqrt{2}$ yields

$$1 = 0 + B\left(-\sqrt{3}-\sqrt{3}\right) \qquad \text{or} \qquad B = \frac{-1}{2\sqrt{3}}.$$

The result is

$$\frac{1}{2x^2-3} = \frac{1/2\sqrt{3}}{\sqrt{2}x-\sqrt{3}} - \frac{1/2\sqrt{3}}{\sqrt{2}x+\sqrt{3}}.$$

Thus,

$$\int \frac{dx}{2x^2-3} = \frac{1}{2\sqrt{3}}\int \frac{dx}{\sqrt{2}x-\sqrt{3}} - \frac{1}{2\sqrt{3}}\int \frac{dx}{\sqrt{2}x+\sqrt{3}}.$$

For the first integral, let $u = \sqrt{2}x - \sqrt{3}$, $du = \sqrt{2}\,dx$, and for the second, let $w = \sqrt{2}x + \sqrt{3}$, $dw = \sqrt{2}\,dx$. Then we have

$$\int \frac{dx}{2x^2-3} = \frac{1}{2\sqrt{3}(\sqrt{2})}\int \frac{du}{u} - \frac{1}{2\sqrt{3}(\sqrt{2})}\int \frac{dw}{w} = \frac{1}{2\sqrt{6}}\ln\left|\sqrt{2}x-\sqrt{3}\right| - \frac{1}{2\sqrt{6}}\ln\left|\sqrt{2}x+\sqrt{3}\right| + C.$$

24. $\displaystyle\int \frac{dx}{(x-4)^2(x-1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-4)^2(x-1)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(x-1)}.$$

Clearing denominators, we get

$$1 = A(x-4)(x-1) + B(x-1) + C(x-4)^2.$$

Setting $x = 1$ then yields

$$1 = 0 + 0 + C(9) \qquad \text{or} \qquad C = \frac{1}{9},$$

while setting $x = 4$ yields

$$1 = 0 + B(3) + 0 \qquad \text{or} \qquad B = \frac{1}{3}.$$

Plugging in $B = \frac{1}{3}$ and $C = \frac{1}{9}$, and setting $x = 5$, we find

$$1 = A(1)(4) + \frac{1}{3}(4) + \frac{1}{9}(1) \qquad \text{or} \qquad A = -\frac{1}{9}.$$

The result is

$$\frac{1}{(x-4)^2(x-1)} = \frac{-\frac{1}{9}}{x-4} + \frac{\frac{1}{3}}{(x-4)^2} + \frac{\frac{1}{9}}{x-1}.$$

Thus,

$$\begin{aligned}\int \frac{dx}{(x-4)^2(x-1)} &= -\frac{1}{9}\int \frac{dx}{x-4} + \frac{1}{3}\int \frac{dx}{(x-4)^2} + \frac{1}{9}\int \frac{dx}{x-1}\\ &= -\frac{1}{9}\ln|x-4| - \frac{1}{3(x-4)} + \frac{1}{9}\ln|x-1| + C.\end{aligned}$$

25. $\displaystyle\int \frac{4x^2-20}{(2x+5)^3}\,dx$

SOLUTION The partial fraction decomposition is

$$\frac{4x^2-20}{(2x+5)^3} = \frac{A}{2x+5} + \frac{B}{(2x+5)^2} + \frac{C}{(2x+5)^3}$$

Clearing fractions gives

$$4x^2 - 20 = A(2x+5)^2 + B(2x+5) + C$$

Setting $x = -5/2$ gives $5 = C$ so that $C = 5$. The coefficient of x^2 on the left-hand side is 4, and on the right-hand side is $4A$, so that $A = 1$ and we have

$$4x^2 - 20 = (2x+5)^2 + B(2x+5) + 5$$

Considering the constant terms now gives $-20 = 25 + 5B + 5$ so that $B = -10$. Thus

$$\int \frac{4x^2 - 20}{(2x+5)^3} = \int \frac{1}{2x+5}\,dx - 10\int \frac{1}{(2x+5)^2}\,dx + 5\int \frac{1}{(2x+5)^3}\,dx$$
$$= \frac{1}{2}\ln|2x+5| + \frac{5}{2x+5} - \frac{5}{4(2x+5)^2} + C$$

26. $\displaystyle\int \frac{3x+6}{x^2(x-1)(x-3)}\,dx$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3x+6}{x^2(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-3}.$$

Clearing denominators gives us

$$3x + 6 = Ax(x-1)(x-3) + B(x-1)(x-3) + Cx^2(x-3) + Dx^2(x-1).$$

Setting $x = 0$, then yields

$$6 = 0 + B(-1)(-3) + 0 + 0 \qquad \text{or} \qquad B = 2,$$

while setting $x = 1$ yields

$$9 = 0 + 0 + C(1)(-2) + 0 \qquad \text{or} \qquad C = -\frac{9}{2},$$

and setting $x = 3$ yields

$$15 = 0 + 0 + 0 + D(9)(2) \qquad \text{or} \qquad D = \frac{5}{6}.$$

In order to find A, let's look at the x^3-coefficient on the right-hand side (which must equal 0, since there's no x^3 term on the left):

$$0 = A + C + D = A - \frac{9}{2} + \frac{5}{6}, \qquad \text{so} \qquad A = \frac{11}{3}.$$

The result is

$$\frac{3x+6}{x^2(x-1)(x-3)} = \frac{\frac{11}{3}}{x} + \frac{2}{x^2} + \frac{-\frac{9}{2}}{x-1} + \frac{\frac{5}{6}}{x-3}.$$

Thus,

$$\int \frac{(3x+6)\,dx}{x^2(x-1)(x-3)} = \frac{11}{3}\int \frac{dx}{x} + 2\int \frac{dx}{x^2} - \frac{9}{2}\int \frac{dx}{x-1} + \frac{5}{6}\int \frac{dx}{x-3}$$
$$= \frac{11}{3}\ln|x| - \frac{2}{x} - \frac{9}{2}\ln|x-1| + \frac{5}{6}\ln|x-3| + C.$$

27. $\displaystyle\int \frac{dx}{x(x-1)^3}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}.$$

Clearing denominators, we get

$$1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx.$$

Setting $x = 0$ then yields

$$1 = A(-1) + 0 + 0 + 0 \qquad \text{or} \qquad A = -1,$$

while setting $x = 1$ yields

$$1 = 0 + 0 + 0 + D(1) \qquad \text{or} \qquad D = 1.$$

Plugging in $A = -1$ and $D = 1$ gives us

$$1 = -(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + x.$$

Now, setting $x = 2$ yields

$$1 = -1 + 2B + 2C + 2 \qquad \text{or} \qquad 2B + 2C = 0,$$

and setting $x = 3$ yields

$$1 = -8 + 12B + 6C + 3 \qquad \text{or} \qquad 2B + C = 1.$$

Solving these two equations in two unknowns, we find $B = 1$ and $C = -1$. The result is

$$\frac{1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{1}{(x-1)^3}.$$

Thus,

$$\int \frac{dx}{x(x-1)^3} = -\int \frac{dx}{x} + \int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3}$$
$$= -\ln|x| + \ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + C.$$

28. $\displaystyle\int \frac{(3x^2-2)\,dx}{x-4}$

SOLUTION First we use long division to write

$$\frac{3x^2-2}{x-4} = 3x + 12 + \frac{46}{x-4}.$$

Then the integral becomes

$$\int \frac{(3x^2-2)\,dx}{x-4} = \int (3x+12)\,dx + 46\int \frac{dx}{x-4} = \frac{3}{2}x^2 + 12x + 46\ln|x-4| + C.$$

29. $\displaystyle\int \frac{(x^2-x+1)\,dx}{x^2+x}$

SOLUTION First use long division to write

$$\frac{x^2-x+1}{x^2+x} = 1 + \frac{-2x+1}{x^2+x} = 1 + \frac{-2x+1}{x(x+1)}.$$

The partial fraction decomposition of the term on the right has the form:

$$\frac{-2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Clearing denominators gives us

$$-2x + 1 = A(x+1) + Bx.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = -1$ yields

$$3 = 0 + B(-1) \qquad \text{or} \qquad B = -3.$$

The result is

$$\frac{-2x+1}{x(x+1)} = \frac{1}{x} + \frac{-3}{x+1}.$$

Thus,

$$\int \frac{(x^2-x+1)\,dx}{x^2+x} = \int dx + \int \frac{dx}{x} - 3\int \frac{dx}{x+1} = x + \ln|x| - 3\ln|x+1| + C.$$

30. $\displaystyle\int \frac{dx}{x(x^2+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$1 = A(x^2+1) + (Bx+C)x.$$

Setting $x = 0$ then yields

$$1 = A(1) + 0 \qquad \text{or} \qquad A = 1.$$

This gives us

$$1 = x^2 + 1 + Bx^2 + Cx = (B+1)x^2 + Cx + 1.$$

Equating x^2-coefficients, we find

$$B + 1 = 0 \qquad \text{or} \qquad B = -1;$$

while equating x-coefficients yields $C = 0$. The result is

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}.$$

Thus,

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1}.$$

For the integral on the right, use the substitution $u = x^2+1$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} = \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

31. $\displaystyle\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$3x^2 - 4x + 5 = A(x^2+1) + (Bx+C)(x-1).$$

Setting $x = 1$ then yields

$$3 - 4 + 5 = A(2) + 0 \qquad \text{or} \qquad A = 2.$$

This gives us

$$3x^2 - 4x + 5 = 2(x^2+1) + (Bx+C)(x-1) = (B+2)x^2 + (C-B)x + (2-C).$$

Equating x^2-coefficients, we find

$$3 = B + 2 \qquad \text{or} \qquad B = 1;$$

while equating constant coefficients yields

$$5 = 2 - C \qquad \text{or} \qquad C = -3.$$

The result is

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{x-3}{x^2+1}.$$

Thus,

$$\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)} = 2\int \frac{dx}{x-1} + \int \frac{(x-3)\,dx}{x^2+1} = 2\int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} - 3\int \frac{dx}{x^2+1}.$$

For the second integral, use the substitution $u = x^2+1$, $du = 2x\,dx$. The final answer is

$$\int \frac{(3x^2-4x+5)\,dx}{(x-1)(x^2+1)} = 2\ln|x-1| + \frac{1}{2}\ln|x^2+1| - 3\tan^{-1}x + C.$$

32. $\displaystyle\int \frac{x^2}{(x+1)(x^2+1)}\,dx$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators, we get

$$x^2 = A(x^2+1) + (Bx+C)(x+1).$$

Setting $x = -1$ then yields

$$1 = A(2) + 0 \qquad \text{or} \qquad A = \frac{1}{2}.$$

This gives us

$$x^2 = \frac{1}{2}x^2 + \frac{1}{2} + Bx^2 + Bx + Cx + C = \left(B + \frac{1}{2}\right)x^2 + (B+C)x + \left(C + \frac{1}{2}\right).$$

Equating x^2-coefficients, we find

$$1 = B + \frac{1}{2} \qquad \text{or} \qquad B = \frac{1}{2},$$

while equating constant coefficients yields

$$0 = C + \frac{1}{2} \qquad \text{or} \qquad C = -\frac{1}{2}.$$

The result is

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x - \frac{1}{2}}{x^2+1}.$$

Thus,

$$\int \frac{x^2\,dx}{(x+1)(x^2+1)} = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{(x-1)\,dx}{x^2+1} = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$
$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C.$$

Here we used $u = x^2+1$, $du = 2x\,dx$ for the second integral.

33. $\displaystyle\int \frac{dx}{x(x^2+25)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+25)} = \frac{A}{x} + \frac{Bx+C}{x^2+25}.$$

Clearing denominators, we get

$$1 = A(x^2+25) + (Bx+C)x.$$

Setting $x = 0$ then yields

$$1 = A(25) + 0 \qquad \text{or} \qquad A = \frac{1}{25}.$$

This gives us

$$1 = \frac{1}{25}x^2 + 1 + Bx^2 + Cx = \left(B + \frac{1}{25}\right)x^2 + Cx + 1.$$

Equating x^2-coefficients, we find

$$0 = B + \frac{1}{25} \qquad \text{or} \qquad B = -\frac{1}{25},$$

while equating x-coefficients yields $C = 0$. The result is

$$\frac{1}{x(x^2+25)} = \frac{\frac{1}{25}}{x} + \frac{-\frac{1}{25}x}{x^2+25}.$$

Thus,

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25}\int \frac{dx}{x} - \frac{1}{25}\int \frac{x\,dx}{x^2+25}.$$

For the integral on the right, use $u = x^2 + 25$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25}\ln|x| - \frac{1}{50}\ln|x^2+25| + C.$$

34. $\displaystyle\int \frac{dx}{x^2(x^2+25)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x^2(x^2+25)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+25}.$$

Clearing denominators, we get

$$1 = Ax(x^2+25) + B(x^2+25) + (Cx+D)x^2.$$

Setting $x = 0$ then yields

$$1 = 0 + B(25) + 0 \qquad \text{or} \qquad B = \frac{1}{25}.$$

This gives us

$$1 = Ax^3 + 25Ax + \frac{1}{25}x^2 + 1 + Cx^3 + Dx^2 = (A+C)x^3 + \left(D + \frac{1}{25}\right)x^2 + 25Ax + 1.$$

Equating x-coefficients yields

$$0 = 25A \qquad \text{or} \qquad A = 0,$$

while equating x^3-coefficients yields

$$0 = A + C = 0 + C \qquad \text{or} \qquad C = 0,$$

and equating x^2-coefficients yields

$$0 = D + \frac{1}{25} \qquad \text{or} \qquad D = \frac{-1}{25}.$$

The result is

$$\frac{1}{x^2(x^2+25)} = \frac{\frac{1}{25}}{x^2} + \frac{\frac{-1}{25}}{x^2+25}.$$

Thus,

$$\int \frac{dx}{x^2(x^2+25)} = \frac{1}{25}\int \frac{dx}{x^2} - \frac{1}{25}\int \frac{dx}{x^2+25} = -\frac{1}{25x} - \frac{1}{125}\tan^{-1}\left(\frac{x}{5}\right) + C.$$

35. $\displaystyle\int \frac{(6x^2+2)\,dx}{x^2+2x-3}$

SOLUTION Long division gives

$$\frac{6x^2+2}{x^2+2x-3} = 6 - \frac{12x-20}{x^2+2x-3} = 6 - \frac{12x-20}{(x+3)(x-1)}$$

The partial fraction decomposition of the second term is

$$\frac{12x-20}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Clear fractions to get

$$12x - 20 = A(x-1) + B(x+3)$$

Set $x = 1$ to get $-8 = 4B$ so that $B = -2$. Set $x = -3$ to get $-56 = -4A$ so that $A = 14$, and we have

$$\int \frac{6x^2+2}{x^2+2x-3} = \int 6 - \frac{14}{x+3} + \frac{2}{x-1}\,dx = \int 6\,dx - 14\int \frac{1}{x+3}\,dx + 2\int \frac{1}{x-1}\,dx$$
$$= 6x - 14\ln|x+3| + 2\ln|x-1| + C$$

36. $\displaystyle\int \frac{6x^2+7x-6}{(x^2-4)(x+2)}\,dx$

SOLUTION The partial fraction decomposition has the form:

$$\frac{6x^2+7x-6}{(x^2-4)(x+2)} = \frac{6x^2+7x-6}{(x-2)(x+2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

Clearing denominators, we get

$$6x^2+7x-6 = A(x+2)^2 + B(x-2)(x+2) + C(x-2).$$

Setting $x = 2$ then yields

$$24 + 14 - 6 = A(16) + 0 + 0 \qquad \text{or} \qquad A = 2,$$

while setting $x = -2$ yields

$$24 - 14 - 6 = 0 + 0 + C(-4) \qquad \text{or} \qquad C = -1.$$

This gives us

$$6x^2 + 7x - 6 = 2(x+2)^2 + B(x-2)(x+2) - (x-2).$$

Now, setting $x = 1$ yields

$$6 + 7 - 6 = 2(9) + B(-1)(3) - (-1) \qquad \text{or} \qquad B = 4.$$

The result is

$$\frac{6x^2+7x-6}{(x^2-4)(x+2)} = \frac{2}{x-2} + \frac{4}{x+2} + \frac{-1}{(x+2)^2}.$$

Thus,

$$\int \frac{(6x^2+7x-6)\,dx}{(x^2-4)(x+2)} = 2\int \frac{dx}{x-2} + 4\int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2} = 2\ln|x-2| + 4\ln|x+2| + \frac{1}{x+2} + C.$$

37. $\displaystyle\int \frac{10\,dx}{(x-1)^2(x^2+9)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{10}{(x-1)^2(x^2+9)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+9}.$$

Clearing denominators, we get

$$10 = A(x-1)(x^2+9) + B(x^2+9) + (Cx+D)(x-1)^2.$$

Setting $x = 1$ then yields

$$10 = 0 + B(10) + 0 \qquad \text{or} \qquad B = 1.$$

Expanding the right-hand side, we have

$$10 = (A+C)x^3 + (1 - A - 2C + D)x^2 + (9A + C - 2D)x + (9 - 9A + D).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} A + C &= 0 \\ 1 - A - 2C + D &= 0 \\ 9A + C - 2D &= 0 \\ 9 - 9A + D &= 10 \end{aligned}$$

From the first equation, we have $C = -A$, and from the fourth equation we have $D = 1 + 9A$. Substituting these into the second equation, we get

$$1 - A - 2(-A) + (1 + 9A) = 0 \qquad \text{or} \qquad A = -\frac{1}{5}.$$

Finally, $C = \frac{1}{5}$ and $D = -\frac{4}{5}$. The result is

$$\frac{10}{(x-1)^2(x^2+9)} = \frac{-\frac{1}{5}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{5}x - \frac{4}{5}}{x^2+9}.$$

Thus,

$$\begin{aligned} \int \frac{10\,dx}{(x-1)^2(x^2+9)} &= -\frac{1}{5}\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + \frac{1}{5}\int \frac{x\,dx}{x^2+9} - \frac{4}{5}\int \frac{dx}{x^2+9} \\ &= -\frac{1}{5}\ln|x-1| - \frac{1}{x-1} + \frac{1}{10}\ln|x^2+9| - \frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right) + C. \end{aligned}$$

38. $\displaystyle\int \frac{10\,dx}{(x+1)(x^2+9)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{10}{(x+1)(x^2+9)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2}.$$

Clearing denominators gives us

$$10 = A(x^2+9)^2 + (Bx+C)(x+1)(x^2+9) + (Dx+E)(x+1).$$

Setting $x = -1$ then yields

$$10 = A(100) + 0 + 0 \qquad \text{or} \qquad A = \frac{1}{10}.$$

Expanding the right-hand side, we find

$$10 = \left(B + \frac{1}{10}\right)x^4 + (B+C)x^3 + \left(9B + C + D + \frac{18}{10}\right)x^2(9B + 9C + D + E)x + \left(9C + E + \frac{81}{10}\right).$$

Equating x^4-coefficients yields

$$B + \frac{1}{10} = 0 \qquad \text{or} \qquad B = -\frac{1}{10},$$

while equating x^3-coefficients yields

$$-\frac{1}{10} + C = 0 \qquad \text{or} \qquad C = \frac{1}{10},$$

and equating x^2-coefficients yields

$$-\frac{9}{10} + \frac{1}{10} + D + \frac{18}{10} = 0 \qquad \text{or} \qquad D = -1.$$

Finally, equating constant coefficients, we find

$$10 = \frac{9}{10} + E + \frac{81}{10} \qquad \text{or} \qquad E = 1.$$

The result is

$$\frac{10}{(x+1)(x^2+9)^2} = \frac{\frac{1}{10}}{x+1} + \frac{-\frac{1}{10}x + \frac{1}{10}}{x^2+9} + \frac{-x+1}{(x^2+9)^2}.$$

Thus,

$$\int \frac{10\,dx}{(x+1)(x^2+9)^2} = \frac{1}{10}\int \frac{dx}{x+1} - \frac{1}{10}\int \frac{x\,dx}{x^2+9} + \frac{1}{10}\int \frac{dx}{x^2+9} - \int \frac{x\,dx}{(x^2+9)^2} + \int \frac{dx}{(x^2+9)^2}.$$

For the second and fourth integrals, use the substitution $u = x^2 + 9$, $du = 2x\,dx$. Then we have

$$\int \frac{10\,dx}{(x+1)(x^2+9)^2} = \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} + \int \frac{dx}{(x^2+9)^2}.$$

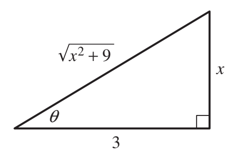
For the last integral, use the trigonometric substitution

$$x = 3\tan\theta, \qquad dx = 3\sec^2\theta\,d\theta, \qquad x^2+9 = 9\tan^2\theta + 9 = 9\sec^2\theta.$$

Then,

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta\,d\theta}{(9\sec^2\theta)^2} = \frac{1}{27}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{27}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C.$$

Now we construct a right triangle with $\tan\theta = \frac{x}{3}$:



From this we see that $\sin\theta = x/\sqrt{x^2+9}$ and $\cos\theta = 3/\sqrt{x^2+9}$. Thus

$$\int \frac{dx}{(x^2+9)^2} = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C.$$

Collecting all the terms, we obtain

$$\begin{aligned}\int \frac{10\,dx}{(x+1)(x^2+9)^2} &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} \\ &\quad + \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C \\ &= \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{7}{135}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x+9}{18(x^2+9)} + C.\end{aligned}$$

39. $\displaystyle\int \frac{dx}{x(x^2+8)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x(x^2+8)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+8} + \frac{Dx+E}{(x^2+8)^2}.$$

Clearing denominators, we get

$$1 = A(x^2+8)^2 + (Bx+C)x(x^2+8) + (Dx+E)x.$$

Expanding the right-hand side gives us

$$1 = (A+B)x^4 + Cx^3 + (16A+8B+D)x^2 + (8C+E)x + 64A.$$

Equating coefficients of like powers of x yields

$$\begin{aligned} A + B &= 0 \\ C &= 0 \\ 16A + 8B + D &= 0 \\ 8C + E &= 0 \\ 64A &= 1 \end{aligned}$$

The solution to this system of equations is

$$A = \frac{1}{64}, \qquad B = -\frac{1}{64}, \qquad C = 0, \qquad D = -\frac{1}{8}, \qquad E = 0.$$

Therefore

$$\frac{1}{x(x^2+8)^2} = \frac{\frac{1}{64}}{x} + \frac{-\frac{1}{64}x}{x^2+8} + \frac{-\frac{1}{8}x}{(x^2+8)^2},$$

and

$$\int \frac{dx}{x(x^2+8)^2} = \frac{1}{64}\int \frac{dx}{x} - \frac{1}{64}\int \frac{x\,dx}{x^2+8} - \frac{1}{8}\int \frac{x\,dx}{(x^2+8)^2}.$$

For the second and third integrals, use the substitution $u = x^2+8$, $du = 2x\,dx$. Then we have

$$\int \frac{dx}{x(x^2+8)^2} = \frac{1}{64}\ln|x| - \frac{1}{128}\ln|x^2+8| + \frac{1}{16(x^2+8)} + C.$$

40. $\displaystyle\int \frac{100x\,dx}{(x-3)(x^2+1)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{100x}{(x-3)(x^2+1)^2} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Clearing denominators, we get

$$100x = A(x^2+1)^2 + (Bx+C)(x-3)(x^2+1) + (Dx+E)(x-3).$$

Setting $x = 3$ then yields

$$300 = A(100) + 0 + 0 \qquad \text{or} \qquad A = 3.$$

Expanding the right-hand side, we find

$$100x = (B+3)x^4 + (C-3B)x^3 + (B-3C+D+6)x^2 + (C-3B-3D+E)x + (3-3C-3E).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} B+3 &= 0 \\ C-3B &= 0 \\ B-3C+D+6 &= 0 \\ C-3B-3D+E &= 100 \\ 3-3C-3E &= 0 \end{aligned}$$

The solution to this system of equations is

$$B = -3, \qquad C = -9, \qquad D = -30, \qquad E = 10.$$

Therefore

$$\frac{100x}{(x-3)(x^2+1)^2} = \frac{3}{x-3} + \frac{-3x-9}{x^2+1} + \frac{-30x+10}{(x^2+1)^2},$$

and

$$\begin{aligned} \int \frac{100x\,dx}{(x-3)(x^2+1)^2} &= 3\int \frac{dx}{x-3} + \int \frac{(-3x-9)\,dx}{x^2+1} + \int \frac{(-30x+10)\,dx}{(x^2+1)^2} \\ &= 3\int \frac{dx}{x-3} - 3\int \frac{x\,dx}{x^2+1} - 9\int \frac{dx}{x^2+1} - 30\int \frac{x\,dx}{(x^2+1)^2} + 10\int \frac{dx}{(x^2+1)^2}. \end{aligned}$$

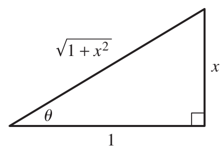
For the second and fourth integrals, use the substitution $u = x^2+1$, $du = 2x\,dx$. Then we have

$$\int \frac{100x\,dx}{(x-3)(x^2+1)^2} = 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 9\tan^{-1}x + \frac{15}{x^2+1} + 10\int \frac{dx}{(x^2+1)^2}.$$

For the last integral, use the trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$. Then $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, and

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C.$$

We construct the following right triangle with $\tan\theta = x$:



From this we see that $\sin\theta = x/\sqrt{1+x^2}$ and $\cos\theta = 1/\sqrt{1+x^2}$. Thus

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + C = \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C.$$

Collecting all the terms, we obtain

$$\begin{aligned}\int \frac{100x\, dx}{(x-3)(x^2+1)^2} &= 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 9\tan^{-1}x + \frac{15}{x^2+1} + 10\left(\frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)}\right) + C \\ &= 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 4\tan^{-1}x + \frac{5x+15}{x^2+1} + C.\end{aligned}$$

41. $\displaystyle\int \frac{dx}{(x+2)(x^2+4x+10)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x+2)(x^2+4x+10)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4x+10}.$$

Clearing denominators, we get

$$1 = A(x^2+4x+10) + (Bx+C)(x+2).$$

Setting $x = -2$ then yields

$$1 = A(6) + 0 \qquad \text{or} \qquad A = \frac{1}{6}.$$

Expanding the right-hand side gives us

$$1 = \left(\frac{1}{6} + B\right)x^2 + \left(\frac{2}{3} + 2B + C\right)x + \left(\frac{5}{3} + 2C\right).$$

Equating x^2-coefficients yields

$$0 = \frac{1}{6} + B \qquad \text{or} \qquad B = -\frac{1}{6},$$

while equating constant coefficients yields

$$1 = \frac{5}{3} + 2C \qquad \text{or} \qquad C = -\frac{1}{3}.$$

The result is

$$\frac{1}{(x+2)(x^2+4x+10)} = \frac{\frac{1}{6}}{x+2} + \frac{-\frac{1}{6}x - \frac{1}{3}}{x^2+4x+10}.$$

Thus,

$$\int \frac{dx}{(x+2)(x^2+4x+10)} = \frac{1}{6}\int \frac{dx}{x+2} - \frac{1}{6}\int \frac{(x+2)\,dx}{x^2+4x+10}.$$

For the second integral, let $u = x^2 + 4x + 10$. Then $du = (2x+4)\,dx$, and

$$\begin{aligned}\int \frac{dx}{(x+2)(x^2+4x+10)} &= \frac{1}{6}\ln|x+2| - \frac{1}{12}\int \frac{(2x+4)\,dx}{x^2+4x+10} \\ &= \frac{1}{6}\ln|x+2| - \frac{1}{12}\ln|x^2+4x+10| + C.\end{aligned}$$

42. $\displaystyle\int \frac{9\,dx}{(x+1)(x^2-2x+6)}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{9}{(x+1)(x^2-2x+6)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+6}.$$

Clearing denominators gives us

$$9 = A(x^2-2x+6) + (Bx+C)(x+1).$$

Setting $x=-1$ then yields

$$9 = A(9) + 0 \qquad \text{or} \qquad A = 1.$$

Expanding the right-hand side gives us

$$9 = (1+B)x^2 + (-2+B+C)x + (6+C).$$

Equating x^2-coefficients yields

$$0 = 1 + B \qquad \text{or} \qquad B = -1,$$

while equating constant coefficients yields

$$9 = 6 + C \qquad \text{or} \qquad C = 3.$$

The result is

$$\frac{9}{(x+1)(x^2-2x+6)} = \frac{1}{x+1} + \frac{-x+3}{x^2-2x+6}.$$

Thus,

$$\int \frac{9\,dx}{(x+1)(x^2-2x+6)} = \int \frac{dx}{x+1} + \int \frac{(-x+3)\,dx}{x^2-2x+6}.$$

To evaluate the integral on the right, we first write

$$\int \frac{(-x+3)\,dx}{x^2-2x+6} = -\int \frac{(x-1-2)\,dx}{x^2-2x+6} = -\int \frac{(x-1)\,dx}{x^2-2x+6} + 2\int \frac{dx}{x^2-2x+6}.$$

For the first integral, use the substitution $u = x^2-2x+6$, $du = (2x-2)\,dx$. Then

$$-\int \frac{(x-1)\,dx}{x^2-2x+6} = -\frac{1}{2}\int \frac{(2x-2)\,dx}{x^2-2x+6} = -\frac{1}{2}\ln|x^2-2x+6| + C.$$

For the second integral, we first complete the square:

$$2\int \frac{dx}{x^2-2x+6} = 2\int \frac{dx}{(x^2-2x+1)+5} = 2\int \frac{dx}{(x-1)^2+5}.$$

Now let $u = x-1$, $du = dx$. Then

$$2\int \frac{dx}{(x-1)^2+5} = 2\int \frac{du}{u^2+5} = 2\left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C.$$

Collecting all the terms, we have

$$\int \frac{9\,dx}{(x+1)(x^2-2x+6)} = \ln|x+1| - \frac{1}{2}\ln|x^2-2x+6| + \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C.$$

43. $\displaystyle\int \frac{25\,dx}{x(x^2+2x+5)^2}$

SOLUTION The partial fraction decomposition has the form

$$\frac{25}{x(x^2+2x+5)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2}.$$

Clearing denominators yields:

$$\begin{aligned} 25 &= A(x^2+2x+5)^2 + x(Bx+C)(x^2+2x+5) + x(Dx+E) \\ &= (Ax^4+4Ax^3+14Ax^2+20Ax+25A) + (Bx^4+Cx^3+2Bx^3+2Cx^2+5Bx^2+5Cx) + Dx^2 + Ex. \end{aligned}$$

Equating constant terms yields

$$25A = 25 \qquad \text{or} \qquad A = 1,$$

while equating x^4-coefficients yields

$$A + B = 0 \qquad \text{or} \qquad B = -A = -1.$$

Equating x^3-coefficients yields

$$4A + C + 2B = 0 \qquad \text{or} \qquad C = -2,$$

and equating x^2-coefficients yields

$$14A + 2C + 5B + D = 0 \qquad \text{or} \qquad D = -5.$$

Finally, equating x-coefficients yields

$$20A + 5C + E = 0 \qquad \text{or} \qquad E = -10.$$

Thus,

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \int \left(\frac{1}{x} - \frac{x+2}{x^2+2x+5} - 5\frac{x+2}{(x^2+2x+5)^2}\right) dx \\ &= \ln|x| - \int \frac{x+2}{x^2+2x+5}\,dx - 5\int \frac{x+2}{(x^2+2x+5)^2}\,dx.\end{aligned}$$

The two integrals on the right both require the substitution $u = x + 1$, so that $x^2 + 2x + 5 = (x+1)^2 + 4 = u^2 + 4$ and $du = dx$. This means:

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \int \frac{u+1}{u^2+4}\,du - 5\int \frac{u+1}{(u^2+4)^2}\,du \\ &= \ln|x| - \int \frac{u}{u^2+4}\,du - \int \frac{1}{u^2+4}\,du - 5\int \frac{u}{(u^2+4)^2}\,du - 5\int \frac{1}{(u^2+4)^2}\,du.\end{aligned}$$

For the first and third integrals, we make the substitution $w = u^2 + 4$, $dw = 2u\,du$. Then we have

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \frac{1}{2}\ln|u^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + \frac{5}{2(u^2+4)} - 5\int \frac{du}{(u^2+4)^2} \\ &= \ln|x| - \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{5}{2(x^2+2x+5)} - 5\int \frac{du}{(u^2+4)^2}.\end{aligned}$$

For the remaining integral, we use the trigonometric substitution $2\tan w = u$, so that $u^2 + 4 = 4\tan^2 w + 4 = 4\sec^2 w$ and $du = 2\sec^2 w\,dw$. This means

$$\begin{aligned}\int \frac{1}{(u^2+4)^2}\,du &= \frac{1}{8}\int \frac{1}{\sec^4 w}\sec^2 w\,dw = \frac{1}{8}\int \cos^2 w\,dw \\ &= \frac{1}{8}\left(\frac{1}{4}\sin 2w + \frac{w}{2}\right) + C = \left(\frac{1}{16}\sin w\cos w + \frac{w}{16}\right) + C \\ &= \frac{1}{16}\frac{u}{\sqrt{u^2+4}}\frac{2}{\sqrt{u^2+4}} + \frac{1}{16}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{8}\frac{u}{u^2+4} + \frac{1}{16}\tan^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{8}\frac{x+1}{x^2+2x+5} + \frac{1}{16}\tan^{-1}\left(\frac{x+1}{2}\right).\end{aligned}$$

Hence, the integral is

$$\begin{aligned}\int \frac{25\,dx}{x(x^2+2x+5)^2} &= \ln|x| - \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) \\ &\quad + \frac{5}{2(x^2+2x+5)} - \frac{5}{8}\frac{x+1}{x^2+2x+5} - \frac{5}{16}\tan^{-1}\left(\frac{x+1}{2}\right) \\ &= \ln|x| + \frac{15-5x}{8(x^2+2x+5)} - \frac{13}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln|x^2+2x+5| + C.\end{aligned}$$

44. $\displaystyle\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2}$

SOLUTION The partial fraction decomposition has the form:

$$\frac{x^2+3}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2}.$$

Clearing denominators gives us

$$x^2+3 = (Ax+B)(x^2+2x+3) + Cx + D.$$

Expanding the right-hand side, we get

$$x^2+3 = Ax^3 + (2A+B)x^2 + (3A+2B+C)x + (3B+D).$$

Equating coefficients of like powers of x then yields

$$\begin{aligned} A &= 0 \\ 2A + B &= 1 \\ 3A + 2B + C &= 0 \\ 3B + D &= 3 \end{aligned}$$

The solution to this system of equations is

$$A = 0, \qquad B = 1, \qquad C = -2, \qquad D = 0.$$

Therefore

$$\frac{x^2+3}{(x^2+2x+3)^2} = \frac{1}{x^2+2x+3} + \frac{-2x}{(x^2+2x+3)^2},$$

and

$$\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2} = \int \frac{dx}{x^2+2x+3} - \int \frac{2x\,dx}{(x^2+2x+3)^2}.$$

The first integral can be evaluated by completing the square:

$$\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{x^2+2x+1+2} = \int \frac{dx}{(x+1)^2+2}.$$

Now use the substitution $u = x+1$, $du = dx$. Then

$$\int \frac{dx}{x^2+2x+3} = \int \frac{du}{u^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

For the second integral, let $u = x^2+2x+3$. We want $du = (2x+2)\,dx$ to appear in the numerator, so we write

$$\begin{aligned} \int \frac{2x\,dx}{(x^2+2x+3)^2} &= \int \frac{(2x+2-2)\,dx}{(x^2+2x+3)^2} = \int \frac{(2x+2)\,dx}{(x^2+2x+3)^2} - 2\int \frac{dx}{(x^2+2x+3)^2} \\ &= \int \frac{du}{u^2} - 2\int \frac{dx}{(x^2+2x+3)^2} = -\frac{1}{u} - 2\int \frac{dx}{(x^2+2x+3)^2} \\ &= \frac{-1}{x^2+2x+3} - 2\int \frac{dx}{(x^2+3x+3)^2}. \end{aligned}$$

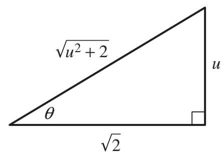
Finally, for this last integral, complete the square, then substitute $u = x+1$, $du = dx$:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{((x+1)^2+2)^2} = \int \frac{du}{(u^2+2)^2}.$$

Now use the trigonometric substitution $u = \sqrt{2}\tan\theta$. Then $du = \sqrt{2}\sec^2\theta\,d\theta$, and $u^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta$. Thus

$$\int \frac{du}{(u^2+2)^2} = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta} = \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{\sqrt{2}}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C.$$

We construct a right triangle with $\tan\theta = u/\sqrt{2}$:



From this we see that $\sin\theta = u/\sqrt{u^2+2}$ and $\cos\theta = \sqrt{2}/\sqrt{u^2+2}$. Therefore

$$\int \frac{du}{(u^2+2)^2} = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\left(\frac{u}{\sqrt{u^2+2}}\right)\left(\frac{\sqrt{2}}{\sqrt{u^2+2}}\right) + C$$

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{u}{4(u^2+2)} + C = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+1}{4(x^2+2x+3)} + C.$$

Collecting all the terms, we have

$$\int \frac{(x^2+3)\,dx}{(x^2+2x+3)^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \left[\frac{-1}{x^2+2x+3} - 2\left(\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+1}{4(x^2+2x+3)}\right)\right] + C$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{4}\right)\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{2+(x+1)}{2(x^2+2x+3)} + C$$

$$= \frac{3\sqrt{2}}{4}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{x+3}{2(x^2+2x+3)} + C.$$

In Exercises 45–48, evaluate by using first substitution and then partial fractions if necessary.

45. $\displaystyle\int \frac{x\,dx}{x^4+1}$

SOLUTION Use the substitution $u = x^2$ so that $du = 2x\,dx$, and

$$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1} u = \frac{1}{2}\tan^{-1}(x^2)$$

46. $\displaystyle\int \frac{x\,dx}{(x+2)^4}$

SOLUTION Use the substitution $u = x+2$ and $du = dx$; then

$$\int \frac{x}{(x+2)^4}\,dx = \int \frac{u-2}{u^4}\,du = \int \frac{1}{u^3}\,du - 2\int \frac{1}{u^4}\,du$$

$$= -\frac{1}{2u^2} + \frac{2}{3u^3} + C = \frac{2}{3(x+2)^3} - \frac{1}{2(x+2)^2} + C$$

47. $\displaystyle\int \frac{e^x\,dx}{e^{2x}-e^x}$

SOLUTION Use the substitution $u = e^x$. Then $du = e^x\,dx = u\,dx$ so that $dx = \frac{1}{u}\,du$. Then

$$\int \frac{e^x\,dx}{e^{2x}-e^x} = \int \frac{u\cdot\frac{1}{u}\,du}{u^2-u} = \int \frac{1}{u(u-1)}\,du$$

Using partial fractions, we have

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{(A+B)u - A}{u(u-1)}$$

Upon equating coefficients in the numerators, we have $A+B=0$, $A=-1$ so that $B=1$. Then

$$\int \frac{e^x\,dx}{e^{2x}-e^x} = -\int \frac{1}{u}\,du + \int \frac{1}{u-1}\,du = \ln|u-1| - \ln|u| + C = \ln|e^x-1| - \ln e^x + C$$

48. $\displaystyle\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta - 1}$

SOLUTION Let $u = \tan\theta$; then $du = \sec^2\theta\,d\theta$ and

$$\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta-1} = \int \frac{1}{u^2-1}\,du = -\int \frac{1}{1-u^2}\,du = -\tanh^{-1}(u) + C = -\tanh^{-1}(\tan\theta) + C$$

49. Evaluate $\int \frac{\sqrt{x}\,dx}{x-1}$. *Hint:* Use the substitution $u = \sqrt{x}$ (sometimes called a **rationalizing substitution**).

SOLUTION Let $u = \sqrt{x}$. Then $du = (1/2\sqrt{x})\,dx = (1/2u)\,dx$. Thus

$$\int \frac{\sqrt{x}\,dx}{x-1} = \int \frac{u(2u\,du)}{u^2-1} = 2\int \frac{u^2\,du}{u^2-1} = 2\int \frac{(u^2-1+1)\,du}{u^2-1}$$
$$= 2\int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right) du = 2\int du + \int \frac{2\,du}{u^2-1} = 2u + \int \frac{2\,du}{u^2-1}.$$

The partial fraction decomposition of the remaining integral has the form:

$$\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}.$$

Clearing denominators gives us

$$2 = A(u+1) + B(u-1).$$

Setting $u = 1$ yields $2 = A(2) + 0$ or $A = 1$, while setting $u = -1$ yields $2 = 0 + B(-2)$ or $B = -1$. The result is

$$\frac{2}{u^2-1} = \frac{1}{u-1} + \frac{-1}{u+1}.$$

Thus,

$$\int \frac{2\,du}{u^2-1} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln|u-1| - \ln|u+1| + C.$$

The final answer is

$$\int \frac{\sqrt{x}\,dx}{x-1} = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C.$$

50. Evaluate $\int \frac{dx}{x^{1/2} - x^{1/3}}$.

SOLUTION First use the substitution $u = x^{1/6}$. Then

$$du = \frac{1}{6}x^{-5/6}\,dx \quad \Rightarrow \quad 6x^{5/6}\,du = dx \quad \Rightarrow \quad 6u^5\,du = dx$$

and we have (using long division)

$$\int \frac{dx}{x^{1/2}-x^{1/3}} = \int \frac{6u^5}{u^3-u^2}\,du = 6\int \frac{u^3}{u-1}\,du = 6\int u^2 + u + 1 + \frac{1}{u-1}\,du$$
$$= 6\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2u^3 + 3u^2 + 6u + 6\ln|u-1| + C$$
$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6}-1\right| + C$$

51. Evaluate $\int \frac{dx}{x^2-1}$ in two ways: using partial fractions and using trigonometric substitution. Verify that the two answers agree.

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing denominators gives us

$$1 = A(x+1) + B(x-1).$$

Setting $x = 1$, we get $1 = A(2)$ or $A = \frac{1}{2}$; while setting $x = -1$, we get $1 = B(-2)$ or $B = -\frac{1}{2}$. The result is

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}.$$

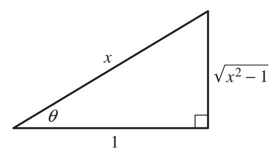
Thus,

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Using trigonometric substitution, let $x = \sec\theta$. Then $dx = \tan\theta\sec\theta\,d\theta$, and $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Thus

$$\int \frac{dx}{x^2-1} = \int \frac{\tan\theta\sec\theta\,d\theta}{\tan^2\theta} = \int \frac{\sec\theta\,d\theta}{\tan\theta} = \int \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta}$$
$$= \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Now we construct a right triangle with $\sec\theta = x$:



From this we see that $\csc\theta = x/\sqrt{x^2-1}$ and $\cot\theta = 1/\sqrt{x^2-1}$. Thus

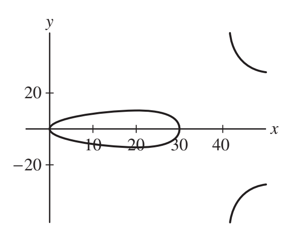
$$\int \frac{dx}{x^2-1} = \ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C.$$

To check that these two answers agree, we write

$$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}\left|\frac{x-1}{x+1}\right| = \ln\left|\sqrt{\frac{x-1}{x+1}}\right| = \ln\left|\frac{\sqrt{x-1}}{\sqrt{x+1}}\cdot\frac{\sqrt{x-1}}{\sqrt{x-1}}\right| = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right|.$$

52. GU Graph the equation $(x-40)y^2 = 10x(x-30)$ and find the volume of the solid obtained by revolving the region between the graph and the x-axis for $0 \le x \le 30$ around the x-axis.

SOLUTION The graph of $(x-40)y^2 = 10x(x-30)$ is shown below



Using the disk method, the volume is given by

$$V = \int_0^{30} \pi r^2\,dx = \pi\int_0^{30}\left(\sqrt{\frac{10x(x-30)}{x-40}}\right)^2 dx = \pi\int_0^{30}\frac{10x(x-30)\,dx}{x-40}.$$

To find the anti-derivative, expand the numerator and then use long division:

$$\frac{10x(x-30)}{x-40} = \frac{10x^2-300x}{x-40} = 10x + 100 + \frac{4000}{x-40}.$$

Thus,

$$\pi\int_0^{30}\frac{10x(x-30)\,dx}{x-40} = \pi\left[10\int_0^{30} x\,dx + 100\int_0^{30} dx + 4000\int_0^{30}\frac{dx}{x-40}\right]$$
$$= \pi\left(5x^2 + 100x + 4000\ln|x-40|\right)\Big|_0^{30}$$
$$= \pi\left[\left(4500 + 3000 + 4000\ln(10)\right) - \left(0 + 4000\ln(40)\right)\right]$$
$$= (7500 - 4000\ln 4)\pi.$$

In Exercises 53–66, evaluate the integral using the appropriate method or combination of methods covered thus far in the text.

53. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

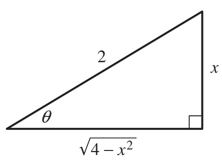
SOLUTION Use the trigonometric substitution $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$,

$$4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta,$$

and

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)} = \frac{1}{4}\int \csc^2\theta\,d\theta = -\frac{1}{4}\cot\theta + C.$$

Now construct a right triangle with $\sin\theta = x/2$:



From this we see that $\cot\theta = \sqrt{4-x^2}/x$. Thus

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right) + C = -\frac{\sqrt{4-x^2}}{4x} + C.$$

54. $\displaystyle\int \frac{dx}{x(x-1)^2}$

SOLUTION Using partial fractions, we first write

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators gives us

$$1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Setting $x = 0$ yields

$$1 = A(1) + 0 + 0 \qquad \text{or} \qquad A = 1,$$

while setting $x = 1$ yields

$$1 = 0 + 0 + C \qquad \text{or} \qquad C = 1,$$

and setting $x = 2$ yields

$$1 = 1 + 2B + 2 \qquad \text{or} \qquad B = -1.$$

The result is

$$\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}.$$

Thus,

$$\int \frac{dx}{x(x-1)^2} = \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C.$$

55. $\displaystyle\int \cos^2 4x\,dx$

SOLUTION Use the substitution $u = 4x$, $du = 4\,dx$. Then we have

$$\int \cos^2(4x)\,dx = \frac{1}{4}\int \cos^2(4x)4\,dx = \frac{1}{4}\int \cos^2 u\,du = \frac{1}{4}\left[\frac{1}{2}u + \frac{1}{2}\sin u\cos u\right] + C$$
$$= \frac{1}{8}u + \frac{1}{8}\sin u\cos u + C = \frac{1}{2}x + \frac{1}{8}\sin 4x\cos 4x + C.$$

56. $\int x \sec^2 x \, dx$

SOLUTION Use integration by parts, with $u = x$ and $v' = \sec^2 x$. Then $u' = 1$, $v = \tan x$, and

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x - \big(-\ln|\cos x| \big) + C = x \tan x + \ln|\cos x| + C.$$

57. $\int \frac{dx}{(x^2+9)^2}$

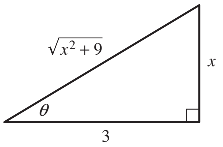
SOLUTION Use the trigonometric substitution $x = 3\tan\theta$. Then $dx = 3\sec^2\theta \, d\theta$,

$$x^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta,$$

and

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta \, d\theta}{(9\sec^2\theta)^2} = \frac{3}{81}\int \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \frac{1}{27}\int \cos^2\theta \, d\theta = \frac{1}{27}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C.$$

Now construct a right triangle with $\tan\theta = x/3$:



From this we see that $\sin\theta = x/\sqrt{x^2+9}$ and $\cos\theta = 3/\sqrt{x^2+9}$. Thus

$$\int \frac{dx}{\sqrt{x^2+9}^2} = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right) + C = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C.$$

58. $\int \theta \sec^{-1}\theta \, d\theta$

SOLUTION Use Integration by Parts, with $u = \sec^{-1}\theta$ and $v' = \theta$. Then $u' = 1/\theta\sqrt{\theta^2-1}$, $v = \theta^2/2$, and

$$\int \theta\sec^{-1}\theta \, d\theta = \frac{\theta^2}{2}\sec^{-1}\theta - \int \frac{\theta^2 \, d\theta}{2\theta\sqrt{\theta^2-1}} = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\int \frac{\theta \, d\theta}{\sqrt{\theta^2-1}}.$$

To evaluate the remaining integral, use the substitution $w = \theta^2 - 1$, $dw = 2\theta \, d\theta$. Then

$$\int \frac{\theta \, d\theta}{\sqrt{\theta^2-1}} = \frac{1}{2}\int \frac{2\theta \, d\theta}{\sqrt{\theta^2-1}} = \frac{1}{2}\int \frac{dw}{\sqrt{w}} = \frac{1}{2}\left(2\sqrt{w}\right) + C = \sqrt{\theta^2-1} + C.$$

The final answer is

$$\int \theta\sec^{-1}\theta \, d\theta = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\sqrt{\theta^2-1} + C.$$

59. $\int \tan^5 x \sec x \, dx$

SOLUTION Use the trigonometric identity $\tan^2 x = \sec^2 x - 1$ to write

$$\int \tan^5 x \sec x \, dx = \int \left(\sec^2 x - 1\right)^2 \tan x \sec x \, dx.$$

Now use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$:

$$\int \tan^5 x \sec x \, dx = \int (u^2-1)^2 \, du = \int \left(u^4 - 2u^2 + 1\right) du$$
$$= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C = \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C.$$

60. $\int \frac{(3x^2-1)\, dx}{x(x^2-1)}$

SOLUTION The denominator expands to $x^3 - x$, so if we let $u = x^3 - x$, then $du = (3x^2-1)\, dx$, which is the numerator. Thus

$$\int \frac{(3x^2-1)\, dx}{x(x^2-1)} = \int \frac{du}{u} = \ln|u| + C = \ln(x(x^2-1)) + C$$

61. $\int \ln(x^4 - 1)\,dx$

SOLUTION Apply integration by parts with $u = \ln(x^4-1)$, $v' = 1$; then $u' = \frac{4x^3}{x^4-1}$ and $v = x$, so after simplification,

$$\begin{aligned}\int \ln(x^4-1)\,dx &= x\ln(x^4-1) - 4\int \frac{x^4}{x^4-1}\,dx = x\ln(x^4-1) - 4\int 1 + \frac{1}{x^4-1}\,dx\\ &= x\ln(x^4-1) - 4\int 1\,dx - 4\int \frac{1}{x^4-1}\,dx\\ &= x\ln(x^4-1) - 4x - 4\int \frac{1}{2}\left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right)dx\\ &= x\ln(x^4-1) - 4x - 2\int \frac{1}{x^2-1}\,dx + 2\int \frac{1}{x^2+1}\,dx\\ &= x\ln(x^4-1) - 4x + 2\tanh^{-1} x + 2\tan^{-1} x + C\end{aligned}$$

62. $\int \frac{x\,dx}{(x^2-1)^{3/2}}$

SOLUTION Use the substitution $u = x^2 - 1$, $du = 2x\,dx$. Then we have

$$\int \frac{x\,dx}{(x^2-1)^{3/2}} = \frac{1}{2}\int \frac{2x\,dx}{(x^2-1)^{3/2}} = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}(-2)u^{-1/2} + C = \frac{-1}{\sqrt{u}} + C = \frac{-1}{\sqrt{x^2-1}} + C.$$

63. $\int \frac{x^2\,dx}{(x^2-1)^{3/2}}$

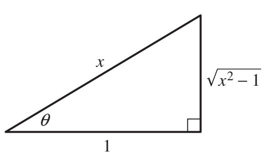
SOLUTION Use the trigonometric substitution $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$,

$$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta,$$

and

$$\begin{aligned}\int \frac{x^2\,dx}{(x^2-1)^{3/2}} &= \int \frac{(\sec^2\theta)\sec\theta\tan\theta\,d\theta}{(\tan^2\theta)^{3/2}} = \int \frac{\sec^3\theta\,d\theta}{\tan^2\theta} = \int \frac{(\tan^2\theta+1)\sec\theta\,d\theta}{\tan^2\theta}\\ &= \int \frac{\tan^2\theta\sec\theta\,d\theta}{\tan^2\theta} + \int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \int \sec\theta\,d\theta + \int \csc\theta\cot\theta\,d\theta\\ &= \ln|\sec\theta + \tan\theta| - \csc\theta + C.\end{aligned}$$

Now construct a right triangle with $\sec\theta = x$:



From this we see that $\tan\theta = \sqrt{x^2-1}$ and $\csc\theta = x/\sqrt{x^2-1}$. So the final answer is

$$\int \frac{x^2\,dx}{(x^2-1)^{3/2}} = \ln\left|x + \sqrt{x^2-1}\right| - \frac{x}{\sqrt{x^2-1}} + C.$$

64. $\int \frac{(x+1)\,dx}{(x^2+4x+8)^2}$

SOLUTION At first it might appear that one would use partial fractions to simplify this problem, but in fact it's already in simplified form. Instead, use the substitution $u = x^2 + 4x + 8$, $du = (2x+4)\,dx$. Then we have

$$\begin{aligned}\int \frac{(x+1)\,dx}{(x^2+4x+8)^2} &= \frac{1}{2}\int \frac{(2x+2)\,dx}{(x^2+4x+8)^2} = \frac{1}{2}\int \frac{(2x+2+2-2)\,dx}{(x^2+4x+8)^2}\\ &= \frac{1}{2}\int \frac{(2x+4)\,dx}{(x^2+4x+8)^2} - \int \frac{dx}{(x^2+4x+8)^2}\\ &= \frac{1}{2}\int \frac{du}{u^2} - \int \frac{dx}{(x^2+4x+8)^2} = \frac{-1}{2u} - \int \frac{dx}{(x^2+4x+8)^2}.\end{aligned}$$

To evaluate the remaining integral, complete the square, then let $w = x + 2$, $dw = dx$:

$$\int \frac{dx}{(x^2+4x+8)^2} = \int \frac{dx}{(x^2+4x+4+4)^2} = \int \frac{dx}{((x+2)^2+4)^2} = \int \frac{dw}{(w^2+4)^2}.$$

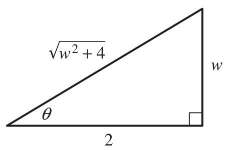
Next, let $w = 2\tan\theta$, $dw = 2\sec^2\theta\, d\theta$. Then

$$w^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta,$$

and we have

$$\int \frac{dw}{(w^2+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{8}\cos^2\theta\, d\theta = \frac{1}{8}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C.$$

Now construct a right triangle with $\tan\theta = w/2$:



From this we see that $\sin\theta = w/\sqrt{w^2+4}$ and $\cos\theta = 2/\sqrt{w^2+4}$. Thus

$$\int \frac{dw}{(w^2+4)^2} = \frac{1}{16}\tan^{-1}\left(\frac{w}{2}\right) + \frac{1}{16}\left(\frac{w}{\sqrt{w^2+4}}\right)\left(\frac{2}{\sqrt{w^2+4}}\right) + C = \frac{1}{16}\tan^{-1}\left(\frac{w}{2}\right) + \frac{w}{8(w^2+4)} + C.$$

In terms of x, we have

$$\int \frac{dx}{(x^2+4x+8)^2} = \int \frac{dw}{(w^2+4)^2} = \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{x+2}{8((x+2)^2+4)} + C.$$

Collecting all the terms, we have

$$\begin{aligned}\int \frac{(x+1)\,dx}{(x^2+4x+8)^2} &= \frac{-1}{2(x^2+4x+8)} - \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) - \frac{x+2}{8(x^2+4x+8)} + C \\ &= -\frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) - \frac{x+6}{8(x^2+4x+8)} + C.\end{aligned}$$

65. $\displaystyle\int \frac{\sqrt{x}\,dx}{x^3+1}$

SOLUTION Use the substitution $u = x^{3/2}$, $du = \frac{3}{2}x^{1/2}\,dx$. Then $x^3 = (x^{3/2})^2 = u^2$, so we have

$$\int \frac{\sqrt{x}\,dx}{x^3+1} = \frac{2}{3}\int \frac{du}{u^2+1} = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

66. $\displaystyle\int \frac{x^{1/2}\,dx}{x^{1/3}+1}$

SOLUTION Use the substitution $u = x^{1/6}$, $du = \frac{1}{6}x^{-5/6}\,dx$. Then $dx = 6x^{5/6}\,du = 6u^5\,du$, and we get

$$\int \frac{x^{1/2}\,dx}{x^{1/3}+1} = \int \frac{u^3(6u^5\,du)}{u^2+1} = 6\int \frac{u^8\,du}{u^2+1}.$$

By long division

$$\frac{u^8}{u^2+1} = u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1},$$

thus

$$\int \frac{u^8}{u^2+1}\,du = \int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right)du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1}u + C.$$

The final answer is

$$\int \frac{x^{1/2}}{x^{1/3}+1} = \frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C.$$

67. Show that the substitution $\theta = 2\tan^{-1} t$ (Figure 2) yields the formulas

$$\cos\theta = \frac{1-t^2}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2}, \qquad d\theta = \frac{2\,dt}{1+t^2} \qquad \boxed{10}$$

This substitution transforms the integral of any rational function of $\cos\theta$ and $\sin\theta$ into an integral of a rational function of t (which can then be evaluated using partial fractions). Use it to evaluate $\int \frac{d\theta}{\cos\theta + \frac{3}{4}\sin\theta}$.

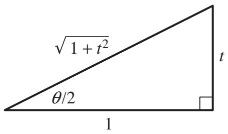


FIGURE 2

SOLUTION If $\theta = 2\tan^{-1} t$, then $d\theta = 2\,dt/(1+t^2)$. We also have that $\cos(\frac{\theta}{2}) = 1/\sqrt{1+t^2}$ and $\sin(\frac{\theta}{2}) = t/\sqrt{1+t^2}$. To find $\cos\theta$, we use the double angle identity $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$. This gives us

$$\cos\theta = 1 - 2\left(\frac{t}{\sqrt{1+t^2}}\right)^2 = 1 - \frac{2t^2}{1+t^2} = \frac{1+t^2-2t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.$$

To find $\sin\theta$, we use the double angle identity $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$. This gives us

$$\sin\theta = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}.$$

With these formulas, we have

$$\int \frac{d\theta}{\cos\theta + (3/4)\sin\theta} = \int \frac{\frac{2\,dt}{1+t^2}}{\left(\frac{1-t^2}{1+t^2}\right) + \frac{3}{4}\left(\frac{2t}{1+t^2}\right)} = \int \frac{8\,dt}{4(1-t^2)+3(2t)} = \int \frac{8\,dt}{4+6t-4t^2} = \int \frac{4\,dt}{2+3t-2t^2}.$$

The partial fraction decomposition has the form

$$\frac{4}{2+3t-2t^2} = \frac{A}{2-t} + \frac{B}{1+2t}.$$

Clearing denominators gives us

$$4 = A(1+2t) + B(2-t).$$

Setting $t = 2$ then yields

$$4 = A(5) + 0 \qquad \text{or} \qquad A = \frac{4}{5},$$

while setting $t = -\frac{1}{2}$ yields

$$4 = 0 + B\left(\frac{5}{2}\right) \qquad \text{or} \qquad B = \frac{8}{5}.$$

The result is

$$\frac{4}{2+3t-2t^2} = \frac{\frac{4}{5}}{2-t} + \frac{\frac{8}{5}}{1+2t}.$$

Thus,

$$\int \frac{4}{2+3t-2t^2}\,dt = \frac{4}{5}\int \frac{dt}{2-t} + \frac{8}{5}\int \frac{dt}{1+2t} = -\frac{4}{5}\ln|2-t| + \frac{4}{5}\ln|1+2t| + C.$$

The original substitution was $\theta = 2\tan^{-1} t$, which means that $t = \tan(\frac{\theta}{2})$. The final answer is then

$$\int \frac{d\theta}{\cos\theta + \frac{3}{4}\sin\theta} = -\frac{4}{5}\ln\left|2 - \tan\left(\frac{\theta}{2}\right)\right| + \frac{4}{5}\ln\left|1 + 2\tan\left(\frac{\theta}{2}\right)\right| + C.$$

68. Use the substitution of Exercise 67 to evaluate $\int \frac{d\theta}{\cos\theta + \sin\theta}$.

SOLUTION Using the substitution $\theta = 2\tan^{-1} t$, we get

$$\int \frac{d\theta}{\cos\theta + \sin\theta} = \int \frac{2\,dt/(1+t^2)}{(1-t^2)/(1+t^2) + 2t/(1+t^2)} = \int \frac{2\,dt}{1-t^2+2t} = -2\int \frac{dt}{t^2-2t-1}.$$

The partial fraction decomposition has the form

$$\frac{-2}{t^2-2t-1} = \frac{A}{t-1-\sqrt{2}} + \frac{B}{t-1+\sqrt{2}}.$$

Clearing denominators gives us

$$-2 = A(t-1+\sqrt{2}) + B(t-1-\sqrt{2}).$$

Setting $t = 1+\sqrt{2}$ then yields $A = -\frac{1}{\sqrt{2}}$, while setting $t = 1-\sqrt{2}$ yields $B = \frac{1}{\sqrt{2}}$. Thus,

$$\int \frac{d\theta}{\cos\theta + \sin\theta} = \frac{1}{\sqrt{2}}\int \frac{dt}{t-1+\sqrt{2}} - \frac{1}{\sqrt{2}}\int \frac{dt}{t-1-\sqrt{2}} = \frac{1}{\sqrt{2}}\ln|t-1+\sqrt{2}| - \frac{1}{\sqrt{2}}\ln|t-1-\sqrt{2}| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{\theta}{2}\right) - 1 + \sqrt{2}}{\tan\left(\frac{\theta}{2}\right) - 1 - \sqrt{2}}\right| + C.$$

Further Insights and Challenges

69. Prove the general formula

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\ln\frac{x-a}{x-b} + C$$

where a, b are constants such that $a \neq b$.

SOLUTION The partial fraction decomposition has the form:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Clearing denominators, we get

$$1 = A(x-b) + B(x-a).$$

Setting $x = a$ then yields

$$1 = A(a-b) + 0 \qquad \text{or} \qquad A = \frac{1}{a-b},$$

while setting $x = b$ yields

$$1 = 0 + B(b-a) \qquad \text{or} \qquad B = \frac{1}{b-a}.$$

The result is

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{x-a} + \frac{\frac{1}{b-a}}{x-b}.$$

Thus,

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\int \frac{dx}{x-a} + \frac{1}{b-a}\int \frac{dx}{x-b} = \frac{1}{a-b}\ln|x-a| + \frac{1}{b-a}\ln|x-b| + C$$

$$= \frac{1}{a-b}\ln|x-a| - \frac{1}{a-b}\ln|x-b| + C = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C.$$

70. The method of partial fractions shows that

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

The computer algebra system Mathematica evaluates this integral as $-\tanh^{-1} x$, where $\tanh^{-1} x$ is the inverse hyperbolic tangent function. Can you reconcile the two answers?

SOLUTION Let

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Solving for x in terms of y, we find

$$(e^x + e^{-x})y = e^x - e^{-x}$$
$$e^{-x}(1 + y) = e^x(1 - y)$$
$$e^{2x} = \frac{1+y}{1-y}$$
$$x = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|$$

Thus,

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|,$$

so

$$-\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right| = \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|,$$

as desired.

71. Suppose that $Q(x) = (x-a)(x-b)$, where $a \neq b$, and let $P(x)/Q(x)$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

(a) Show that $A = \dfrac{P(a)}{Q'(a)}$ and $B = \dfrac{P(b)}{Q'(b)}$.

(b) Use this result to find the partial fraction decomposition for $P(x) = 3x - 2$ and $Q(x) = x^2 - 4x - 12$.

SOLUTION

(a) Clearing denominators gives us

$$P(x) = A(x-b) + B(x-a).$$

Setting $x = a$ then yields

$$P(a) = A(a-b) + 0 \qquad \text{or} \qquad A = \frac{P(a)}{a-b},$$

while setting $x = b$ yields

$$P(b) = 0 + B(b-a) \qquad \text{or} \qquad B = \frac{P(b)}{b-a}.$$

Now use the product rule to differentiate $Q(x)$:

$$Q'(x) = (x-a)(1) + (1)(x-b) = x - a + x - b = 2x - a - b;$$

therefore,

$$Q'(a) = 2a - a - b = a - b$$
$$Q'(b) = 2b - a - b = b - a$$

Substituting these into the above results, we find

$$A = \frac{P(a)}{Q'(a)} \quad \text{and} \quad B = \frac{P(b)}{Q'(b)}.$$

(b) The partial fraction decomposition has the form:

$$\frac{P(x)}{Q(x)} = \frac{3x-2}{x^2-4x-12} = \frac{3x-2}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2};$$
$$A = \frac{P(6)}{Q'(6)} = \frac{3(6)-2}{2(6)-4} = \frac{16}{8} = 2;$$

$$B = \frac{P(-2)}{Q'(-2)} = \frac{3(-2) - 2}{2(-2) - 4} = \frac{-8}{-8} = 1.$$

The result is

$$\frac{3x - 2}{x^2 - 4x - 12} = \frac{2}{x - 6} + \frac{1}{x + 2}.$$

72. Suppose that $Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$, where the roots a_j are all distinct. Let $P(x)/Q(x)$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \cdots + \frac{A_n}{(x - a_n)}$$

(a) Show that $A_j = \dfrac{P(a_j)}{Q'(a_j)}$ for $j = 1, \ldots, n$.

(b) Use this result to find the partial fraction decomposition for $P(x) = 2x^2 - 1$, $Q(x) = x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3)$.

SOLUTION

(a) To differentiate $Q(x)$, first take the logarithm of both sides, and then differentiate:

$$\ln\big(Q(x)\big) = \ln\big[(x - a_1)(x - a_2) \cdots (x - a_n)\big] = \ln(x - a_1) + \ln(x - a_2) + \cdots + \ln(x - a_n)$$
$$\frac{d}{dx} \ln\big(Q(x)\big) = \frac{Q'(x)}{Q(x)} = \frac{1}{x - a_1} + \frac{1}{x - a_2} + \cdots + \frac{1}{x - a_n}$$

Multiplying both sides by $Q(x)$ gives us

$$\begin{aligned} Q'(x) &= Q(x)\left[\frac{1}{x - a_1} + \cdots + \frac{1}{x - a_n}\right] \\ &= (x - a_2)(x - a_3) \cdots (x - a_n) + (x - a_1)(x - a_3) \cdots (x - a_n) + \cdots + (x - a_1)(x - a_2) \cdots (x - a_{n-1}). \end{aligned}$$

In other words, the ith product in the formula for $Q'(x)$ has the $(x - a_i)$ factor removed. This means that

$$Q'(a_j) = (a_j - a_1) \cdots (a_j - a_{j-1})(a_j - a_{j+1}) \cdots (a_j - a_n).$$

Now clear denominators in the expression for $P(x)/Q(x)$:

$$\begin{aligned} P(x) &= \frac{A_1 Q(x)}{x - a_1} + \frac{A_2 Q(x)}{x - a_2} + \cdots + \frac{A_n Q(x)}{x - a_n} \\ &= A_1(x - a_2) \cdots (x - a_n) + (x - a_1)A_2(x - a_3) \cdots (x - a_n) + \cdots + (x - a_1)(x - a_2) \cdots (x - a_{n-1})A_n. \end{aligned}$$

Setting $x = a_j$, we get

$$P(a_j) = (a_j - a_1)(a_j - a_2) \cdots (a_j - a_{j-1})A_j(a_j - a_{j+1}) \cdots (a_j - a_n),$$

so that

$$A_j = \frac{P(a_j)}{(a_j - a_1) \cdots (a_j - a_{j-1})(a_j - a_{j+1}) \cdots (a_j - a_n)} = \frac{P(a_j)}{Q'(a_j)}.$$

(b) Let $P(x) = 2x^2 - 1$ and $Q(x) = (x + 1)(x - 2)(x - 3)$, so that $Q'(x) = 3x^2 - 8x + 1$. Then $a_1 = -1$, $a_2 = 2$, and $a_3 = 3$, so that

$$A_1 = P(-1)/Q'(-1) = \frac{1}{12};$$
$$A_2 = P(2)/Q'(2) = -\frac{7}{3};$$
$$A_3 = P(3)/Q'(3) = \frac{17}{4}.$$

Thus

$$\frac{P(x)}{Q(x)} = \frac{1}{12(x + 1)} - \frac{7}{1 = 3(x - 2)} + \frac{17}{4(x - 3)}.$$

7.6 Improper Integrals

Preliminary Questions

1. State whether the integral converges or diverges:

(a) $\int_1^{\infty} x^{-3}\,dx$

(b) $\int_0^1 x^{-3}\,dx$

(c) $\int_1^{\infty} x^{-2/3}\,dx$

(d) $\int_0^1 x^{-2/3}\,dx$

SOLUTION

(a) The integral is improper because one of the limits of integration is infinite. Because the power of x in the integrand is less than -1, this integral converges.
(b) The integral is improper because the integrand is undefined at $x = 0$. Because the power of x in the integrand is less than -1, this integral diverges.
(c) The integral is improper because one of the limits of integration is infinite. Because the power of x in the integrand is greater than -1, this integral diverges.
(d) The integral is improper because the integrand is undefined at $x = 0$. Because the power of x in the integrand is greater than -1, this integral converges.

2. Is $\int_0^{\pi/2} \cot x\,dx$ an improper integral? Explain.

SOLUTION Because the integrand $\cot x$ is undefined at $x = 0$, this is an improper integral.

3. Find a value of $b > 0$ that makes $\int_0^b \frac{1}{x^2 - 4}\,dx$ an improper integral.

SOLUTION Any value of b satisfying $|b| \geq 2$ will make this an improper integral.

4. Which comparison would show that $\int_0^{\infty} \frac{dx}{x + e^x}$ converges?

SOLUTION Note that, for $x > 0$,

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}.$$

Moreover

$$\int_0^{\infty} e^{-x}\,dx$$

converges. Therefore,

$$\int_0^{\infty} \frac{1}{x + e^x}\,dx$$

converges by the comparison test.

5. Explain why it is not possible to draw any conclusions about the convergence of $\int_1^{\infty} \frac{e^{-x}}{x}\,dx$ by comparing with the integral $\int_1^{\infty} \frac{dx}{x}$.

SOLUTION For $1 \leq x < \infty$,

$$\frac{e^{-x}}{x} < \frac{1}{x},$$

but

$$\int_1^{\infty} \frac{dx}{x}$$

diverges. Knowing that an integral is smaller than a divergent integral does not allow us to draw any conclusions using the comparison test.

Exercises

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

(a) $\int_0^2 \frac{dx}{x^{1/3}}$

(b) $\int_1^{\infty} \frac{dx}{x^{0.2}}$

(c) $\int_{-1}^{\infty} e^{-x}\,dx$

(d) $\int_0^1 e^{-x}\,dx$

(e) $\int_0^{\pi/2} \sec x\,dx$

(f) $\int_0^{\infty} \sin x\,dx$

(g) $\int_0^1 \sin x\,dx$ **(h)** $\int_0^1 \frac{dx}{\sqrt{3-x^2}}$ **(i)** $\int_1^\infty \ln x\,dx$

(j) $\int_0^3 \ln x\,dx$

SOLUTION

(a) Improper. The function $x^{-1/3}$ is infinite at 0.
(b) Improper. Infinite interval of integration.
(c) Improper. Infinite interval of integration.
(d) Proper. The function e^{-x} is continuous on the finite interval [0, 1].
(e) Improper. The function $\sec x$ is infinite at $\frac{\pi}{2}$.
(f) Improper. Infinite interval of integration.
(g) Proper. The function $\sin x$ is continuous on the finite interval [0, 1].
(h) Proper. The function $1/\sqrt{3-x^2}$ is continuous on the finite interval [0, 1].
(i) Improper. Infinite interval of integration.
(j) Improper. The function $\ln x$ is infinite at 0.

2. Let $f(x) = x^{-4/3}$.

(a) Evaluate $\int_1^R f(x)\,dx$.

(b) Evaluate $\int_1^\infty f(x)\,dx$ by computing the limit

$$\lim_{R\to\infty} \int_1^R f(x)\,dx$$

SOLUTION

(a) $\int_1^R x^{-4/3}\,dx = -3x^{-1/3}\Big|_1^R = -3R^{-1/3} - \big(-3(1)\big) = 3\left(1 - \frac{1}{R^{1/3}}\right)$.

(b) $\int_1^\infty x^{-4/3}\,dx = \lim_{R\to\infty} \int_1^R x^{-4/3}\,dx = \lim_{R\to\infty} 3\left(1 - \frac{1}{R^{1/3}}\right) = 3(1-0) = 3.$

3. Prove that $\int_1^\infty x^{-2/3}\,dx$ diverges by showing that

$$\lim_{R\to\infty} \int_1^R x^{-2/3}\,dx = \infty$$

SOLUTION First compute the proper integral:

$$\int_1^R x^{-2/3}\,dx = 3x^{1/3}\Big|_1^R = 3R^{1/3} - 3 = 3\left(R^{1/3} - 1\right).$$

Then show divergence:

$$\int_1^\infty x^{-2/3}\,dx = \lim_{R\to\infty} \int_1^R x^{-2/3}\,dx = \lim_{R\to\infty} 3\left(R^{1/3} - 1\right) = \infty.$$

4. Determine whether $\int_0^3 \frac{dx}{(3-x)^{3/2}}$ converges by computing

$$\lim_{R\to 3-} \int_0^R \frac{dx}{(3-x)^{3/2}}$$

SOLUTION First evaluate the integral on the interval $[0, R]$ for $0 < R < 3$:

$$\int_0^R \frac{dx}{(3-x)^{3/2}} = 2(3-x)^{-1/2}\Big|_0^R = \frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}.$$

Now compute the limit as $R \to 3^-$:

$$\int_0^3 \frac{dx}{(3-x)^{3/2}} = \lim_{R\to 3^-} \int_0^R \frac{dx}{(3-x)^{3/2}} = \lim_{R\to 3^-} \left(\frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}\right) = \infty;$$

thus, the integral diverges.

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

5. $\displaystyle\int_1^\infty \frac{dx}{x^{19/20}}$

SOLUTION First evaluate the integral over the finite interval $[1, R]$ for $R > 1$:

$$\int_1^R \frac{dx}{x^{19/20}} = 20x^{1/20}\Big|_1^R = 20R^{1/20} - 20.$$

Now compute the limit as $R \to \infty$:

$$\int_1^\infty \frac{dx}{x^{19/20}} = \lim_{R\to\infty}\int_1^R \frac{dx}{x^{19/20}} = \lim_{R\to\infty}\left(20R^{1/20} - 20\right) = \infty.$$

The integral does not converge.

6. $\displaystyle\int_1^\infty \frac{dx}{x^{20/19}}$

SOLUTION First evaluate the integral over the finite interval $[1, R]$ for $R > 1$:

$$\int_1^R \frac{dx}{x^{20/19}} = -19x^{-1/19}\Big|_1^R = \frac{-19}{R^{1/19}} - (-19) = 19 - \frac{19}{R^{1/19}}.$$

Now compute the limit as $R \to \infty$:

$$\int_1^\infty \frac{dx}{x^{20/19}} = \lim_{R\to\infty}\int_1^R \frac{dx}{x^{20/19}} = \lim_{R\to\infty}\left(19 - \frac{19}{R^{1/19}}\right) = 19 - 0 = 19.$$

7. $\displaystyle\int_{-\infty}^4 e^{0.0001t}\,dt$

SOLUTION First evaluate the integral over the finite interval $[R, 4]$ for $R < 4$:

$$\int_R^4 e^{(0.0001)t}\,dt = \frac{e^{(0.0001)t}}{0.0001}\Bigg|_R^4 = 10{,}000\left(e^{0.0004} - e^{(0.0001)R}\right).$$

Now compute the limit as $R \to -\infty$:

$$\begin{aligned}\int_{-\infty}^4 e^{(0.0001)t}\,dt &= \lim_{R\to-\infty}\int_R^4 e^{(0.0001)t}\,dt = \lim_{R\to-\infty} 10{,}000\left(e^{0.0004} - e^{(0.0001)R}\right)\\ &= 10{,}000\left(e^{0.0004} - 0\right) = 10{,}000e^{0.0004}.\end{aligned}$$

8. $\displaystyle\int_{20}^\infty \frac{dt}{t}$

SOLUTION First evaluate the integral over the finite interval $[20, R]$ for $20 < R$:

$$\int_{20}^R \frac{dt}{t} = \ln|t|\big|_{20}^R = \ln R - \ln 20.$$

Now compute the limit as $R \to \infty$:

$$\int_{20}^\infty \frac{dt}{t} = \lim_{R\to\infty}\int_{20}^R \frac{dt}{t} = \lim_{R\to\infty}(\ln R - \ln 20) = \infty;$$

thus, the integral does not converge.

9. $\displaystyle\int_0^5 \frac{dx}{x^{20/19}}$

SOLUTION The function $x^{-20/19}$ is infinite at the endpoint 0, so we'll first evaluate the integral on the finite interval $[R, 5]$ for $0 < R < 5$:

$$\int_R^5 \frac{dx}{x^{20/19}} = -19x^{-1/19}\Big|_R^5 = -19\left(5^{-1/19} - R^{-1/19}\right) = 19\left(\frac{1}{R^{1/19}} - \frac{1}{5^{1/19}}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^5 \frac{dx}{x^{20/19}} = \lim_{R\to 0^+} \int_R^5 \frac{dx}{x^{20/19}} = \lim_{R\to 0^+} 19\left(\frac{1}{R^{1/19}} - \frac{1}{5^{1/19}}\right) = \infty;$$

thus, the integral does not converge.

10. $\displaystyle\int_0^5 \frac{dx}{x^{19/20}}$

SOLUTION The function $x^{-19/20}$ is infinite at the endpoint 0, so we'll first evaluate the integral on the finite interval $[R, 5]$ for $0 < R < 5$:

$$\int_R^5 \frac{dx}{x^{19/20}} = 20x^{1/20}\Big|_R^5 = 20\left(5^{1/20} - R^{1/20}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^5 \frac{dx}{x^{19/20}} = \lim_{R\to 0^+} \int_R^5 \frac{dx}{x^{19/20}} = \lim_{R\to 0^+} 20\left(5^{1/20} - R^{1/20}\right) = 20\left(5^{1/20} - 0\right) = 20 \cdot 5^{1/20}.$$

11. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

SOLUTION The function $1/\sqrt{4-x}$ is infinite at $x = 4$, but is left-continuous at $x = 4$, so we'll first evaluate the integral on the interval $[0, R]$ for $0 < R < 4$:

$$\int_0^R \frac{dx}{\sqrt{4-x}} = -2\sqrt{4-x}\Big|_0^R = -2\sqrt{4-R} - (-2)\sqrt{4} = 4 - 2\sqrt{4-R}.$$

Now compute the limit as $R \to 4^-$:

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \int_0^R \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \left(4 - 2\sqrt{4-R}\right) = 4 - 0 = 4.$$

12. $\displaystyle\int_5^6 \frac{dx}{(x-5)^{3/2}}$

SOLUTION The function $(x-5)^{-3/2}$ is infinite at $x = 5$, but is right-continuous at $x = 5$, so we'll first evaluate the integral on the interval $[R, 6]$ for $5 < R < 6$:

$$\int_R^6 \frac{dx}{(x-5)^{3/2}} = 2(x-5)^{-1/2}\Big|_R^6 = \frac{-2}{\sqrt{1}} - \frac{-2}{\sqrt{R-5}} = \frac{2}{\sqrt{R-5}} - 2.$$

Now compute the limit as $R \to 5^+$:

$$\int_5^6 \frac{dx}{(x-5)^{-3/2}} = \lim_{R\to 5^+} \int_R^6 \frac{dx}{(x-5)^{3/2}} = \lim_{R\to 5^+} \left(\frac{2}{\sqrt{R-5}} - 2\right) = \infty;$$

thus, the integral does not converge.

13. $\displaystyle\int_2^\infty x^{-3}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $2 < R$:

$$\int_2^R x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_2^R = \frac{-1}{2R^2} - \frac{-1}{2(2^2)} = \frac{1}{8} - \frac{1}{2R^2}.$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty x^{-3}\,dx = \lim_{R\to\infty} \int_2^R x^{-3}\,dx = \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8}.$$

14. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \frac{dx}{(x+1)^3} = \frac{(x+1)^{-2}}{-2}\Big|_0^R = \frac{-1}{2(R+1)^2} - \frac{-1}{2(1)^2} = \frac{1}{2} - \frac{1}{2(R+1)^2}.$$

Now compute the limit as $R \to \infty$:

$$\int_0^\infty \frac{dx}{(x+1)^3} = \lim_{R\to\infty} \int_0^R \frac{dx}{(x+1)^3} = \lim_{R\to\infty} \left(\frac{1}{2} - \frac{1}{2(R+1)^2}\right) = \frac{1}{2}.$$

15. $\displaystyle\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}}$

SOLUTION First evaluate the integral on the finite interval $[-3, R]$ for $R > -3$:

$$\int_{-3}^R \frac{dx}{(x+4)^{3/2}} = -2(x+4)^{-1/2}\Big|_{-3}^R = \frac{-2}{\sqrt{R+4}} - \frac{-2}{\sqrt{1}} = 2 - \frac{2}{\sqrt{R+4}}.$$

Now compute the limit as $R \to \infty$:

$$\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}} = \lim_{R\to\infty} \int_{-3}^R \frac{dx}{(x+4)^{3/2}} = \lim_{R\to\infty} \left(2 - \frac{2}{\sqrt{R+4}}\right) = 2 - 0 = 2.$$

16. $\displaystyle\int_2^\infty e^{-2x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $R > 2$:

$$\int_2^R e^{-2x}\,dx = \frac{e^{-2x}}{-2}\Big|_2^R = -\frac{1}{2}\left(e^{-2R} - e^{-4}\right) = \frac{1}{2}\left(e^{-4} - e^{-2R}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty e^{-2x}\,dx = \lim_{R\to\infty} \int_2^R e^{-2x}\,dx = \lim_{R\to\infty} \left(e^{-4} - e^{-2R}\right) = \frac{1}{2}\left(e^{-4} - 0\right) = \frac{1}{2e^4}.$$

17. $\displaystyle\int_0^1 \frac{dx}{x^{0.2}}$

SOLUTION The function $x^{-0.2}$ is infinite at $x = 0$ and right-continuous at $x = 0$, so we'll first evaluate the integral on the interval $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{dx}{x^{0.2}} = \frac{x^{0.8}}{0.8}\Big|_R^1 = 1.25\left(1 - R^{0.8}\right).$$

Now compute the limit as $R \to 0^+$:

$$\int_0^1 \frac{dx}{x^{0.2}} = \lim_{R\to 0^+} \int_R^1 \frac{dx}{x^{0.2}} = \lim_{R\to 0^+} 1.25\left(1 - R^{0.8}\right) = 1.25(1-0) = 1.25.$$

18. $\displaystyle\int_2^\infty x^{-1/3}\,dx$

SOLUTION First evaluate the integral on the finite interval $[2, R]$ for $R > 2$:

$$\int_2^R x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\Big|_2^R = \frac{3}{2}\left(R^{2/3} - 2^{2/3}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_2^\infty x^{-1/3}\,dx = \lim_{R\to\infty} \int_2^R x^{-1/3}\,dx = \lim_{R\to\infty} \frac{3}{2}\left(R^{2/3} - 2^{2/3}\right) = \infty;$$

thus, the integral does not converge.

19. $\displaystyle\int_4^\infty e^{-3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[4, R]$ for $R > 4$:

$$\int_4^R e^{-3x}\,dx = \frac{e^{-3x}}{-3}\Big|_4^R = -\frac{1}{3}\left(e^{-3R} - e^{-12}\right) = \frac{1}{3}\left(e^{-12} - e^{-3R}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_4^\infty e^{-3x}\,dx = \lim_{R\to\infty}\int_4^R e^{-3x}\,dx = \lim_{R\to\infty}\frac{1}{3}\left(e^{-12} - e^{-3R}\right) = \frac{1}{3}\left(e^{-12} - 0\right) = \frac{1}{3e^{12}}.$$

20. $\displaystyle\int_4^\infty e^{3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[4, R]$ for $R > 4$:

$$\int_4^R e^{3x}\,dx = \left.\frac{e^{3x}}{3}\right|_4^R = \frac{1}{3}\left(e^{3R} - e^{12}\right).$$

Now compute the limit as $R \to \infty$:

$$\int_4^\infty e^{3x}\,dx = \lim_{R\to\infty}\int_4^R e^{3x}\,dx = \lim_{R\to\infty}\frac{1}{3}\left(e^{3R} - e^{12}\right) = \infty;$$

thus, the integral does not converge.

21. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$

SOLUTION First evaluate the integral on the finite interval $[R, 0]$ for $R < 0$:

$$\int_R^0 e^{3x}\,dx = \left.\frac{e^{3x}}{3}\right|_R^0 = \frac{1}{3} - \frac{e^{3R}}{3}.$$

Now compute the limit as $R \to -\infty$:

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{R\to-\infty}\int_R^0 e^{3x}\,dx = \lim_{R\to-\infty}\left(\frac{1}{3} - \frac{e^{3R}}{3}\right) = \frac{1}{3} - 0 = \frac{1}{3}.$$

22. $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$

SOLUTION The function $(x-1)^{-2}$ is infinite at $x = 1$ and is right-continuous at $x = 1$, so we first evaluate the integral on the interval $[R, 2]$ for $1 < R < 2$:

$$\int_R^2 \frac{dx}{(x-1)^2} = \left.\frac{(x-1)^{-1}}{-1}\right|_R^2 = \frac{-1}{1} - \frac{-1}{R-1} = \frac{1}{R-1} - 1.$$

Now compute the limit as $R \to 1^+$:

$$\int_1^2 \frac{dx}{(x-1)^2} = \lim_{R\to1^+}\int_R^2 \frac{dx}{(x-1)^2} = \lim_{R\to1^+}\left(\frac{1}{R-1} - 1\right) = \infty;$$

thus, the integral does not converge.

23. $\displaystyle\int_1^3 \frac{dx}{\sqrt{3-x}}$

SOLUTION The function $f(x) = 1/\sqrt{3-x}$ is infinite at $x = 3$ and is left continuous at $x = 3$, so we first evaluate the integral on the interval $[1, R]$ for $1 < R < 3$:

$$\int_1^R \frac{dx}{\sqrt{3-x}} = \left.-2\sqrt{3-x}\right|_1^R = -2\sqrt{3-R} + 2\sqrt{2}.$$

Now compute the limit as $R \to 3^-$:

$$\int_1^3 \frac{dx}{\sqrt{3-x}} = \lim_{R\to3-}\int_1^R \frac{dx}{\sqrt{3-x}} = 0 + 2\sqrt{2} = 2\sqrt{2}.$$

24. $\displaystyle\int_{-2}^4 \frac{dx}{(x+2)^{1/3}}$

SOLUTION The function $(x+2)^{-1/3}$ is infinite at $x = -2$ and right-continuous at $x = -2$, so we'll first evaluate the integral on the interval $[R, 4]$ for $-2 < R < 4$:

$$\int_R^4 \frac{dx}{(x+2)^{1/3}} = \left.\frac{3}{2}(x+2)^{2/3}\right|_R^4 = \frac{3}{2}\left(6^{2/3} - (R+2)^{2/3}\right).$$

Now compute the limit as $R \to -2^+$:

$$\int_{-2}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R\to -2^+} \int_{R}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R\to 2^+} \frac{3}{2}\left(6^{2/3} - (R+2)^{2/3}\right) = \frac{3}{2}\left(6^{2/3} - 0\right) = \frac{3\cdot 6^{2/3}}{2}.$$

25. $\displaystyle\int_0^\infty \frac{dx}{1+x}$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \frac{dx}{1+x} = \ln|1+x|\Big|_0^R = \ln|1+R| - \ln 1 = \ln|1+R|.$$

Now compute the limit as $R \to \infty$:

$$\int_0^\infty \frac{dx}{1+x} = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x} = \lim_{R\to\infty} \ln|1+R| = \infty;$$

thus, the integral does not converge.

26. $\displaystyle\int_{-\infty}^{0} xe^{-x^2}\,dx$

SOLUTION First evaluate the indefinite integral using substitution, with $u = -x^2$, $du = -2x\,dx$. This gives us

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

Next, evaluate the integral on the finite interval $[R, 0]$ for $R < 0$:

$$\int_R^0 xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_R^0 = -\frac{1}{2}\left(1 - e^{-R^2}\right).$$

Finally, compute the limit as $R \to -\infty$:

$$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{R\to-\infty} \int_R^0 xe^{-x^2}\,dx = \lim_{R\to-\infty} \frac{1}{2}\left(e^{-R^2} - 1\right) = \frac{1}{2}(0-1) = -\frac{1}{2}.$$

27. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$

SOLUTION First evaluate the indefinite integral, using the substitution $u = x^2$, $du = 2x\,dx$; then

$$\int \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{1}{(1+u)^2}\,du = -\frac{1}{2(u+1)} + C = -\frac{1}{2(x^2+1)} + C$$

Thus, for $R > 0$,

$$\int_0^R \frac{x\,dx}{(x^2+1)^2} = \left(-\frac{1}{2(x^2+1)}\right)\Bigg|_0^R = -\frac{1}{2(R^2+1)} + \frac{1}{2}$$

and thus in the limit

$$\int_0^\infty \frac{x\,dx}{(x^2+1)^2} = \lim_{R\to\infty} \int_0^R \frac{x\,dx}{(x^2+1)^2} = \frac{1}{2} - \lim_{R\to\infty} \frac{1}{2(R^2+1)} = \frac{1}{2}$$

28. $\displaystyle\int_3^6 \frac{x\,dx}{\sqrt{x-3}}$

SOLUTION First, evaluate the indefinite integral using the substitution $u = x - 3$, $du = dx$:

$$\int \frac{x}{\sqrt{x-3}}\,dx = \int \frac{u+3}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2} + 6u^{1/2} + C = \frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2} + C.$$

Next, evaluate the definite integral over the interval $[R, 6]$ for $R > 3$:

$$\int_R^6 \frac{x}{\sqrt{x-3}}\,dx = \left(\frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2}\right)\Big|_R^6 = \frac{2}{3}3^{3/2} + 6\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}$$
$$= 8\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}.$$

Finally, we compute the limit as $R \to 3^+$:

$$\int_3^6 \frac{x}{\sqrt{x-3}}\,dx = \lim_{R\to 3^+}\int_R^6 \frac{x}{\sqrt{x-3}}\,dx = \lim_{R\to 3^+}\left(8\sqrt{3} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}\right) = 8\sqrt{3}.$$

29. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$

SOLUTION First evaluate the indefinite integral using Integration by Parts, with $u = e^{-x}$, $v' = \cos x$. Then $u' = -e^{-x}$, $v = \sin x$, and

$$\int e^{-x}\cos x\,dx = e^{-x}\sin x - \int \sin x(-e^{-x})\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx.$$

Now use Integration by Parts again, with $u = e^{-x}$, $v' = \sin x$. Then $u' = -e^{-x}$, $v = -\cos x$, and

$$\int e^{-x}\cos x\,dx = e^{-x}\sin x + \left[-e^{-x}\cos x - \int e^{-x}\cos x\,dx\right].$$

Solving this equation for $\int e^{-x}\cos x\,dx$, we find

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C.$$

Thus,

$$\int_0^R e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^R = \frac{\sin R - \cos R}{2e^R} - \frac{\sin 0 - \cos 0}{2} = \frac{\sin R - \cos R}{2e^R} + \frac{1}{2},$$

and

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{R\to\infty}\left(\frac{\sin R - \cos R}{2e^R} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

30. $\displaystyle\int_1^\infty xe^{-2x}\,dx$

SOLUTION First evaluate the indefinite integral using Integration by Parts, with $u = x$ and $v' = e^{-2x}$. Then $u' = 1$, $v = -\frac{1}{2}e^{-2x}$, and

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int\left(-\frac{1}{2}\right)e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C = -\frac{1}{4}e^{-2x}(2x+1) + C = \frac{-(2x+1)}{4e^{2x}} + C.$$

Therefore,

$$\int_1^\infty xe^{-2x}\,dx = \lim_{R\to-\infty}\int_1^R xe^{-2x}\,dx = \lim_{R\to\infty}\left(\frac{-(2x+1)}{4e^{2x}}\Big|_1^R\right) = \lim_{R\to\infty}\left[\frac{-(2R+1)}{4e^{2R}} + \frac{3}{4e^2}\right].$$

Use L'Hôpital's Rule to evaluate the limit:

$$\int_1^\infty xe^{-2x}\,dx = \frac{3}{4e^2} - \lim_{R\to\infty}\frac{2}{8e^{2R}} = \frac{3}{4e^2} - 0 = \frac{3}{4e^2}.$$

31. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

SOLUTION The function $(9-x^2)^{-1/2}$ is infinite at $x = 3$, and is left-continuous at $x = 3$, so we'll first evaluate the integral on the interval $[0, R]$ for $0 < R < 3$:

$$\int_0^R \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^R = \sin^{-1}\frac{R}{3} - \sin^{-1}0 = \sin^{-1}\frac{R}{3}.$$

Thus,

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{R\to 3^-} \sin^{-1}\frac{R}{3} = \sin^{-1} 1 = \frac{\pi}{2}.$$

32. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-1/2}\,dx$. Then

$$\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2\int e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right) = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

The function $e^{\sqrt{x}}/\sqrt{x}$ is infinite and right-continuous at $x = 0$, so we first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2e^{\sqrt{x}}\Big|_R^1 = 2e - 2e^{\sqrt{R}}.$$

Now we compute the limit as $R \to 0+$:

$$\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to 0^+}\left(2e - 2e^{\sqrt{x}}\right) = 2e - 2(1) = 2(e-1).$$

33. $\displaystyle\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-1/2}\,dx$. Then

$$\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2\int e^{\sqrt{x}}\left(\frac{dx}{2\sqrt{x}}\right) = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C,$$

and

$$\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to\infty}\int_1^R \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \lim_{R\to\infty} 2e^{\sqrt{x}}\Big|_1^R = \lim_{R\to\infty}\left(2e^{\sqrt{R}} - 2e\right) = \infty.$$

The integral does not converge.

34. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$

SOLUTION First, evaluate the integral on the interval $[0, R]$ for $0 < R < \frac{\pi}{2}$:

$$\int_0^R \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^R = \ln|\sec R + \tan R|.$$

Now we compute the limit as $R \to \frac{\pi}{2}^-$:

$$\int_0^{\pi/2} \sec\theta\,d\theta = \lim_{R\to\pi/2^-}\int_0^R \sec\theta\,d\theta = \lim_{R\to\pi/2^-} \ln|\sec R + \tan R| = \infty.$$

The integral does not converge.

35. $\displaystyle\int_0^\infty \sin x\,dx$

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R \sin x\,dx = -\cos x\Big|_0^R = -\cos R + \cos 0 = 1 - \cos R.$$

Thus,

$$\int_0^R \sin x\,dx = \lim_{R\to\infty}(1 - \cos R) = 1 - \lim_{R\to\infty}\cos R.$$

This limit does not exist, since the value of $\cos R$ oscillates between 1 and -1 as R approaches infinity. Hence the integral does not converge.

36. $\displaystyle\int_0^{\pi/2} \tan x\, dx$

SOLUTION The function $\tan x$ is infinite and left-continuous at $x = \frac{\pi}{2}$, so we'll first evaluate the integral on $[0, R]$ for $0 < R < \frac{\pi}{2}$:

$$\int_0^R \tan x\, dx = \ln|\sec x|\Big|_0^R = \ln|\sec R|.$$

Thus,

$$\int_0^{\pi/2} \tan x\, dx = \lim_{R\to\frac{\pi}{2}^-} \int_0^R \tan x\, dx = \lim_{R\to\frac{\pi}{2}^-} \big(\ln|\sec R|\big) = \infty.$$

The integral does not converge.

37. $\displaystyle\int_0^1 \ln x\, dx$

SOLUTION The function $\ln x$ is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$. Use Integration by Parts with $u = \ln x$ and $v' = 1$. Then $u' = 1/x$, $v = x$, and we have

$$\int_R^1 \ln x\, dx = x\ln x\Big|_R^1 - \int_R^1 dx = (x\ln x - x)\Big|_R^1 = (\ln 1 - 1) - (R\ln R - R) = R - 1 - R\ln R.$$

Thus,

$$\int_0^1 \ln x\, dx = \lim_{R\to 0^+}(R - 1 - R\ln R) = -1 - \lim_{R\to 0^+} R\ln R.$$

To compute the limit, rewrite the function as a quotient and apply L'Hôpital's Rule:

$$\int_0^1 \ln x\, dx = -1 - \lim_{R\to 0^+} \frac{\ln R}{\frac{1}{R}} = -1 - \lim_{R\to 0^+} \frac{\frac{1}{R}}{\frac{-1}{R^2}} = -1 - \lim_{R\to 0^+}(-R) = -1 - 0 = -1.$$

38. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

SOLUTION Evaluate the indefinite integral using substitution, with $u = \ln x$, $du = (1/x)\,dx$. Then

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

Thus,

$$\int_R^2 \frac{dx}{x\ln x} = \ln|\ln x|\Big|_R^2 = \ln(\ln 2) - \ln(\ln R),$$

and

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{R\to 1^+}\big[\ln(\ln 2) - \ln(\ln R)\big] = \ln(\ln 2) - \lim_{R\to 1^+} \ln(\ln R) = \infty.$$

The integral does not converge.

39. $\displaystyle\int_0^1 \frac{\ln x}{x^2}\, dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = x^{-2}$. Then $u' = 1/x$, $v = -x^{-1}$, and

$$\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x + \int \frac{dx}{x^2} = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

The function is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_a^1 \frac{\ln x}{x^2}\, dx = \left(-\frac{1}{x}\ln x - \frac{1}{x}\right)\Bigg|_R^1 = \left(-\frac{1}{1}\ln 1 - \frac{1}{1}\right) - \left(-\frac{1}{R}\ln R - \frac{1}{R}\right) = \frac{1}{R}\ln R + \frac{1}{R} - 1.$$

Thus,

$$\int_0^1 \frac{\ln x}{x^2}\, dx = \lim_{R\to 0^+} \frac{1}{R}\ln R + \frac{1}{R} - 1 = -1 + \lim_{R\to 0^+} \frac{\ln R + 1}{R} = -\infty.$$

The integral does not converge.

40. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = x^{-2}$. Then $u' = x^{-1}$, $v = -x^{-1}$, and

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int x^{-2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

Thus,

$$\int_1^R \frac{\ln x}{x^2}\,dx = \left(-\frac{1}{x}\ln x - \frac{1}{x}\right)\Bigg|_1^R = \left(-\frac{1}{R}\ln R - \frac{1}{R}\right) - \left(-\frac{1}{1}\ln 1 - \frac{1}{1}\right) = 1 - \frac{1}{R}\ln R - \frac{1}{R}.$$

Use L'Hôpital's Rule to compute the limit:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\left(1 - \frac{1}{R}\ln R - \frac{1}{R}\right) = 1 - \lim_{R\to\infty}\left(\frac{\ln R}{R}\right) - 0 = 1 - \lim_{R\to\infty}\frac{\frac{1}{R}}{1} = 1 - \frac{0}{1} = 1.$$

41. Let $I = \int_4^\infty \frac{dx}{(x-2)(x-3)}$.

(a) Show that for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}$$

(b) Then show that $I = \ln 2$.

SOLUTION

(a) The partial fraction decomposition takes the form

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Clearing denominators gives us

$$1 = A(x-3) + B(x-2).$$

Setting $x = 2$ then yields $A = -1$, while setting $x = 3$ yields $B = 1$. Thus,

$$\int \frac{dx}{(x-2)(x-3)} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2} = \ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C,$$

and, for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{x-3}{x-2}\right|\Bigg|_4^R = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}.$$

(b) Using the result from part (a),

$$I = \lim_{R\to\infty}\left(\ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}\right) = \ln 1 - \ln\frac{1}{2} = \ln 2.$$

42. Evaluate the integral $I = \int_1^\infty \frac{dx}{x(2x+5)}$.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}.$$

Clearing denominators gives us

$$1 = A(2x+5) + Bx.$$

Setting $x = 0$ then yields $A = \frac{1}{5}$, while setting $x = -\frac{5}{2}$ yields $B = -\frac{2}{5}$. Thus,

$$\int \frac{dx}{x(2x+5)} = \frac{1}{5}\int \frac{dx}{x} - \frac{2}{5}\int \frac{dx}{2x+5} = \frac{1}{5}\ln|x| - \frac{1}{5}\ln|2x+5| + C = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C,$$

and, for $R > 1$,

$$\int_1^R \frac{dx}{x(2x+5)} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_1^R = \frac{1}{5}\ln\left|\frac{R}{2R+5}\right| - \frac{1}{5}\ln\frac{1}{7}.$$

Thus,

$$I = \lim_{R\to\infty}\left(\frac{1}{5}\ln\left|\frac{R}{2R+5}\right| - \frac{1}{5}\ln\frac{1}{7}\right) = \frac{1}{5}\ln\frac{1}{2} - \frac{1}{5}\ln\frac{1}{7} = \frac{1}{5}\ln\frac{7}{2}.$$

43. Evaluate $I = \int_0^1 \frac{dx}{x(2x+5)}$ or state that it diverges.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}.$$

Clearing denominators gives us

$$1 = A(2x+5) + Bx.$$

Setting $x = 0$ then yields $A = \frac{1}{5}$, while setting $x = -\frac{5}{2}$ yields $B = -\frac{2}{5}$. Thus,

$$\int \frac{dx}{x(2x+5)} = \frac{1}{5}\int\frac{dx}{x} - \frac{2}{5}\int\frac{dx}{2x+5} = \frac{1}{5}\ln|x| - \frac{1}{5}\ln|2x+5| + C = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C,$$

and, for $0 < R < 1$,

$$\int_R^1 \frac{dx}{x(2x+5)} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_R^1 = \frac{1}{5}\ln\frac{1}{7} - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right|.$$

Thus,

$$I = \lim_{R\to 0+}\left(\frac{1}{5}\ln\frac{1}{7} - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right|\right) = \infty.$$

The integral does not converge.

44. Evaluate $I = \int_2^\infty \frac{dx}{(x+3)(x+1)^2}$ or state that it diverges.

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{(x+3)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators gives us

$$1 = A(x+1)^2 + B(x+1)(x+3) + C(x+3).$$

Setting $x = -3$ then yields $A = \frac{1}{4}$, while setting $x = -1$ yields $C = \frac{1}{2}$. Setting $x = 0$ gives $1 = \frac{1}{4} + 3B + \frac{3}{2}$ or $B = -\frac{1}{4}$. Thus,

$$\int \frac{dx}{(x+3)(x+1)^2} = \frac{1}{4}\int\frac{dx}{x+3} - \frac{1}{4}\int\frac{dx}{x+1} + \frac{1}{2}\int\frac{dx}{(x+1)^2}$$
$$= \frac{1}{4}\ln|x+3| - \frac{1}{4}\ln|x+1| - \frac{1}{2(x+1)} + C = \frac{1}{4}\ln\left|\frac{x+3}{x+1}\right| - \frac{1}{2(x+1)} + C,$$

and, for $R > 2$,

$$\int_2^R \frac{dx}{(x+3)(x+1)^2} = \left(\frac{1}{4}\ln\left|\frac{x+3}{x+1}\right| - \frac{1}{2(x+1)}\right)\Bigg|_2^R = \frac{1}{4}\ln\left|\frac{R+3}{R+1}\right| - \frac{1}{2(R+1)} - \frac{1}{4}\ln\frac{5}{3} + \frac{1}{6}.$$

Thus

$$I = \lim_{R\to\infty}\left(\frac{1}{4}\ln\left|\frac{R+3}{R+1}\right| - \frac{1}{2(R+1)} - \frac{1}{4}\ln\frac{5}{3} + \frac{1}{6}\right) = \frac{1}{6} - \frac{1}{4}\ln\frac{5}{3}.$$

In Exercises 45–48, determine whether the doubly infinite improper integral converges and, if so, evaluate it. Use definition (2).

45. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2}$

SOLUTION Using the substitution $u = x^2 + 1$, $du = 2x\,dx$, we obtain

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(x^2+1) + C.$$

Thus,

$$\int_0^{\infty} \frac{x\,dx}{1+x^2} = \lim_{R\to\infty}\int_0^R \frac{x\,dx}{1+x^2} = \lim_{R\to\infty}\frac{1}{2}\ln(R^2+1) = \infty;$$

$$\int_{-\infty}^{0} \frac{x\,dx}{1+x^2} = \lim_{R\to-\infty}\int_R^0 \frac{x\,dx}{1+x^2} = \lim_{R\to-\infty}\frac{1}{2}\ln(R^2+1) = \infty;$$

It follows that

$$\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2}$$

diverges.

46. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

SOLUTION First, we find

$$\int_0^{\infty} e^{-|x|}\,dx = \int_0^{\infty} e^{-x}\,dx = \lim_{R\to\infty}\int_0^R e^{-x}\,dx = \lim_{R\to\infty}\left(1 - e^{-R}\right) = 1;$$

$$\int_{-\infty}^{0} e^{-|x|}\,dx = \int_{\infty}^{0} e^{x}\,dx = \lim_{R\to-\infty}\int_R^0 e^{x}\,dx = \lim_{R\to-\infty}\left(1 - e^{R}\right) = 1;$$

and

$$\int_{\infty}^{\infty} e^{-|x|}\,dx = 1 + 1 = 2.$$

47. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

SOLUTION First note that

$$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C.$$

Thus,

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty}\int_0^R xe^{-x^2}\,dx = \lim_{R\to\infty}\left(\frac{1}{2} - \frac{1}{2}e^{-R^2}\right) = \frac{1}{2};$$

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{R\to-\infty}\int_R^0 xe^{-x^2}\,dx = \lim_{R\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-R^2}\right) = -\frac{1}{2};$$

and

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \frac{1}{2} - \frac{1}{2} = 0.$$

48. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{3/2}}$

SOLUTION First, we evaluate the indefinite integral using the trigonometric substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. Then

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{1+x^2}} + C.$$

Thus,

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to\infty}\int_0^R \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to\infty}\frac{R}{\sqrt{1+R^2}} = 1;$$

$$\int_{-\infty}^0 \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to-\infty}\int_R^0 \frac{dx}{(1+x^2)^{3/2}} = \lim_{R\to-\infty} -\frac{R}{\sqrt{1+R^2}} = 1;$$

and

$$\int_\infty^\infty \frac{dx}{(1+x^2)^{(3/2)}} = 1+1 = 2.$$

49. Define $J = \int_{-1}^1 \frac{dx}{x^{1/3}}$ as the sum of the two improper integrals $\int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$. Show that J converges and that $J = 0$.

SOLUTION Note that since $x^{-1/3}$ is an odd function, one might expect this integral over a symmetric interval to be zero. To prove this, we start by evaluating the indefinite integral:

$$\int \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3} + C$$

Then

$$\int_{-1}^0 \frac{dx}{x^{1/3}} = \lim_{R\to0^-}\int_{-1}^R \frac{dx}{x^{1/3}} = \lim_{R\to0^-}\frac{3}{2}x^{2/3}\Big|_{-1}^R = \lim_{R\to0^-}\frac{3}{2}R^{2/3} - \frac{3}{2} = -\frac{3}{2}$$

$$\int_0^1 \frac{dx}{x^{1/3}} = \lim_{R\to0^+}\int_R^1 \frac{dx}{x^{1/3}} = \lim_{R\to0^+}\frac{3}{2}x^{2/3}\Big|_R^1 = \frac{3}{2} - \lim_{R\to0^+}\frac{3}{2}R^{2/3} = \frac{3}{2}$$

so that

$$J = \int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = -\frac{3}{2} + \frac{3}{2} = 0$$

50. Determine whether $J = \int_{-1}^1 \frac{dx}{x^2}$ (defined as in Exercise 49) converges.

SOLUTION We have

$$\int \frac{dx}{x^2} = -\frac{1}{x} + C$$

so that

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{R\to0^-}\int_{-1}^R \frac{dx}{x^2} = \lim_{R\to0^-}\left(-\frac{1}{x}\Big|_{-1}^R\right) = \lim_{R\to0^-}\left(-\frac{1}{R}+1\right) = 1 - \lim_{R\to0^-}\frac{1}{R} = \infty$$

$$\int_0^1 \frac{dx}{x^2} = \lim_{R\to0^+}\int_R^1 \frac{dx}{x^2} = \lim_{R\to0^+}\left(-\frac{1}{x}\Big|_R^1\right) = \lim_{R\to0^+}\left(-1+\frac{1}{R}\right) = -1 + \lim_{R\to0^+}\frac{1}{R} = \infty$$

so that the integral diverges.

51. For which values of a does $\int_0^\infty e^{ax}\,dx$ converge?

SOLUTION First evaluate the integral on the finite interval $[0, R]$ for $R > 0$:

$$\int_0^R e^{ax}\,dx = \frac{1}{a}e^{ax}\Big|_0^R = \frac{1}{a}\left(e^{aR}-1\right).$$

Thus,

$$\int_0^\infty e^{ax}\,dx = \lim_{R\to\infty}\frac{1}{a}\left(e^{aR}-1\right).$$

If $a > 0$, then $e^{aR} \to \infty$ as $R \to \infty$. If $a < 0$, then $e^{aR} \to 0$ as $R \to \infty$, and

$$\int_0^\infty e^{ax}\,dx = \lim_{R\to\infty}\frac{1}{a}\left(e^{aR}-1\right) = -\frac{1}{a}.$$

The integral converges for $a < 0$.

52. Show that $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

SOLUTION The function x^{-p} is infinite and right-continuous at $x = 0$, so we'll first evaluate the integral on $[R, 1]$ for $0 < R < 1$:

$$\int_R^1 \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_R^1 = \frac{1}{-p+1}\left(1 - R^{-p+1}\right).$$

If $p < 1$, then $-p + 1 = 1 - p > 0$, and

$$\int_0^1 \frac{dx}{x^p} = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - R^{1-p}\right) = \frac{1}{1-p}(1-0) = \frac{1}{1-p}.$$

If $p > 1$, then $-p + 1 < 0$, and

$$\int_0^1 \frac{dx}{x^p} = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - R^{1-p}\right) = \lim_{R\to 0^+} \frac{1}{1-p}\left(1 - \frac{1}{a^{p-1}}\right) = \infty.$$

If $p = 1$, then

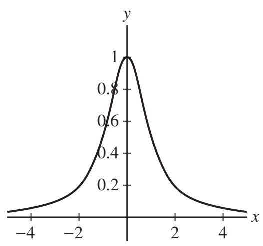
$$\int_R^1 \frac{dx}{x^p} = \int_R^1 \frac{dx}{x} = \ln x\Big|_R^1 = \ln 1 - \ln R = -\ln R; \text{ and}$$

$$\int_0^1 \frac{dx}{x} = \lim_{R\to 0^+}(-\ln R) = \infty.$$

Thus, the integral converges for $p < 1$ and diverges for $p \geq 1$.

53. Sketch the region under the graph of $f(x) = \frac{1}{1+x^2}$ for $-\infty < x < \infty$, and show that its area is π.

SOLUTION The graph is shown below.



Since $(1 + x^2)^{-1}$ is an even function, we can first compute the area under the graph for $x > 0$:

$$\int_0^R \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^R = \tan^{-1} R - \tan^{-1} 0 = \tan^{-1} R.$$

Thus,

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty} \tan^{-1} R = \frac{\pi}{2}.$$

By symmetry, we have

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

54. Show that $\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}$ for all x, and use this to prove that $\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$ converges.

SOLUTION Since $\sqrt{x^4+1} \geq \sqrt{x^4} = x^2$, it follows that

$$\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{x^2}.$$

The integral

$$\int_1^\infty \frac{dx}{x^2}$$

converges by Theorem 2, since $2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{\sqrt{x^4+1}} \text{ converges.}$$

55. Show that $\int_1^\infty \frac{dx}{x^3+4}$ converges by comparing with $\int_1^\infty x^{-3}\,dx$.

SOLUTION The integral $\int_1^\infty x^{-3}\,dx$ converges because $3 > 1$. Since $x^3+4 \ge x^3$, it follows that

$$\frac{1}{x^3+4} \le \frac{1}{x^3}.$$

Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{x^3+4} \text{ converges.}$$

56. Show that $\int_2^\infty \frac{dx}{x^3-4}$ converges by comparing with $\int_2^\infty 2x^{-3}\,dx$.

SOLUTION The integral $\int_1^\infty x^{-3}\,dx$ converges because $3 > 1$. If $\int_1^\infty x^{-3}\,dx = M < \infty$, then

$$\int_1^\infty 2x^{-3}\,dx = 2\int_1^\infty x^{-3}\,dx = 2M$$

also converges. If $x \ge 2$, then $x^3 \ge 8$ so $2x^3 - 8 \ge x^3$ and $x^3 - 4 \ge \frac{1}{2}x^3$. Then we have, for $x \ge 2$,

$$\frac{1}{x^3-4} \le \frac{2}{x^3}.$$

Therefore, by the comparison test:

$$\int_2^\infty \frac{2}{x^3-4} \text{ converges.}$$

57. Show that $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$ (Figure 10). Use the Comparison Test to show that $\int_0^\infty e^{-x^2}\,dx$ converges. *Hint:* It suffices (why?) to make the comparison for $x \ge 1$ because

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

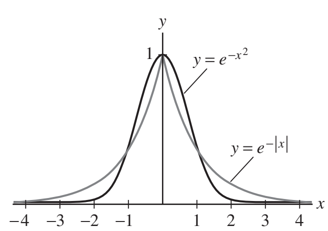


FIGURE 10 Comparison of $y = e^{-|x|}$ and $y = e^{-x^2}$.

SOLUTION For $x \ge 1$, $x^2 \ge x$, so $-x^2 \le -x$ and $e^{-x^2} \le e^{-x}$. Now

$$\int_1^\infty e^{-x}\,dx \text{ converges,} \quad \text{so} \quad \int_1^\infty e^{-x^2}\,dx \text{ converges}$$

by the comparison test. Finally, because e^{-x^2} is continuous on $[0, 1]$,

$$\int_0^\infty e^{-x^2}\,dx \text{ converges.}$$

We conclude that our integral converges by writing it as a sum:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

58. Prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges by comparing with $\int_{-\infty}^{\infty} e^{-|x|}\,dx$ (Figure 10).

SOLUTION From Figure 10, we see that for $|x| \ge 1$, $e^{-x^2} \le e^{-|x|}$. Now

$$\int_{-\infty}^{-1} e^{-|x|}\,dx \quad \text{and} \quad \int_{1}^{\infty} e^{-|x|}\,dx$$

both converge, so

$$\int_{-\infty}^{-1} e^{-x^2}\,dx \quad \text{and} \quad \int_{1}^{\infty} e^{-x^2}\,dx$$

must also converge by the comparison test. Because e^{-x^2} is continuous on $[-1, 1]$, it follows that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx$$

converges.

59. Show that $\int_{1}^{\infty} \frac{1-\sin x}{x^2}\,dx$ converges.

SOLUTION Let $f(x) = \frac{1-\sin x}{x^2}$. Since $f(x) \le \frac{2}{x^2}$ and $\int_{1}^{\infty} 2x^{-2}\,dx = 2$, it follows that

$$\int_{1}^{\infty} \frac{1-\sin x}{x^2}\,dx \text{ converges}$$

by the comparison test.

60. Let $a > 0$. Recall that $\lim_{x\to\infty} \frac{x^a}{\ln x} = \infty$ (by Exercise 64 in Section 4.5).

(a) Show that $x^a > 2\ln x$ for all x sufficiently large.

(b) Show that $e^{-x^a} < x^{-2}$ for all x sufficiently large.

(c) Show that $\int_{1}^{\infty} e^{-x^a}\,dx$ converges.

SOLUTION

(a) Since $\lim_{x\to\infty} x^a/\ln x = \infty$, there must be some number $M > 0$ such that, for all $x > M$,

$$\frac{x^a}{\ln x} > 2.$$

But this means that, for all $x > M$,

$$x^a > 2\ln x.$$

(b) For all $x > M$, we have $x^a > 2\ln x$. Then

$$-x^a < -2\ln x = \ln x^{-2}$$

so that

$$e^{-x^a} < e^{\ln x^{-2}} = x^{-2}.$$

(c) By the above calculations, we can use the comparison test on the interval $[M, \infty)$:

$$\int_{M}^{\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_{M}^{\infty} e^{-x^a}\,dx \text{ also converges.}$$

Since e^{-x^a} is continuous on $[1, M]$, we have that

$$\int_{M}^{\infty} e^{-x^a}\,dx \text{ converges} \Rightarrow \int_{1}^{\infty} e^{-x^a}\,dx \text{ also converges.}$$

In Exercises 61–74, use the Comparison Test to determine whether or not the integral converges.

61. $\int_{1}^{\infty} \frac{1}{\sqrt{x^5+2}}\,dx$

SOLUTION Since $\sqrt{x^5+2} \ge \sqrt{x^5} = x^{5/2}$, it follows that

$$\frac{1}{\sqrt{x^5+2}} \le \frac{1}{x^{5/2}}.$$

The integral $\int_1^{\infty} dx/x^{5/2}$ converges because $\frac{5}{2} > 1$. Therefore, by the comparison test:

$$\int_1^{\infty} \frac{dx}{\sqrt{x^5+2}} \text{ also converges.}$$

62. $\displaystyle\int_1^{\infty} \frac{dx}{(x^3+2x+4)^{1/2}}$

SOLUTION For all $x \geq 1$, $\sqrt{x^3+2x+4} \geq \sqrt{x^3} = x^{3/2}$. Thus

$$\frac{1}{\sqrt{x^3+2x+4}} \leq \frac{1}{x^{3/2}}.$$

The integral $\int_1^{\infty} dx/x^{3/2}$ converges because $\frac{3}{2} > 1$. Therefore, by the comparison test,

$$\int_1^{\infty} \frac{dx}{\sqrt{x^3+2x+4}} \text{ also converges.}$$

63. $\displaystyle\int_3^{\infty} \frac{dx}{\sqrt{x}-1}$

SOLUTION Since $\sqrt{x} \geq \sqrt{x}-1$, we have (for $x > 1$)

$$\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-1}.$$

The integral $\int_1^{\infty} dx/\sqrt{x} = \int_1^{\infty} dx/x^{1/2}$ diverges because $\frac{1}{2} < 1$. Since the function $x^{-1/2}$ is continuous (and therefore finite) on [1, 3], we also know that $\int_3^{\infty} dx/x^{1/2}$ diverges. Therefore, by the comparison test,

$$\int_3^{\infty} \frac{dx}{\sqrt{x}-1} \text{ also diverges.}$$

64. $\displaystyle\int_0^{5} \frac{dx}{x^{1/3}+x^3}$

SOLUTION For $0 \leq x \leq 5$, $x^{1/3}+x^3 \geq x^{1/3}$, so that

$$\frac{1}{x^{1/3}+x^3} \leq \frac{1}{x^{1/3}}.$$

The integral $\int_0^{5} x^{-1/3}\,dx$ converges; therefore, by the comparison test

$$\int_0^{5} \frac{dx}{x^{1/3}+x^3} \text{ also converges.}$$

65. $\displaystyle\int_1^{\infty} e^{-(x+x^{-1})}\,dx$

SOLUTION For all $x \geq 1$, $\frac{1}{x} > 0$ so $x + \frac{1}{x} \geq x$. Then

$$-\left(x+x^{-1}\right) \leq -x \qquad \text{and} \qquad e^{-(x+x^{-1})} \leq e^{-x}.$$

The integral $\int_1^{\infty} e^{-x}\,dx$ converges by direct computation:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{R\to\infty}\int_1^{R} e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_1^{R} = \lim_{R\to\infty} -e^{-R}+e^{-1} = 0+e^{-1} = e^{-1}.$$

Therefore, by the comparison test,

$$\int_1^{\infty} e^{-(x+x^{-1})} \text{ also converges.}$$

66. $\displaystyle\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx$

SOLUTION For all x, $|\sin x| \le 1$. Therefore, for $x \ne 0$,

$$\frac{|\sin x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

The integral

$$\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}}$$

converges, since $\frac{1}{2} < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx \text{ also converges.}$$

67. $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$

SOLUTION For $0 < x < 1$, $e^x > 1$, and therefore

$$\frac{1}{x^2} < \frac{e^x}{x^2}.$$

The integral $\displaystyle\int_0^1 dx/x^2$ diverges since $2 > 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{e^x}{x^2} \text{ also diverges.}$$

68. $\displaystyle\int_1^\infty \frac{1}{x^4 + e^x}\,dx$

SOLUTION For $x > 1$, $x^4 + e^x \ge x^4$, and

$$\frac{1}{x^4 + e^x} \le \frac{1}{x^4}.$$

The integral $\displaystyle\int_0^1 dx/x^4$ converges, since $4 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{dx}{x^4 + e^x} \text{ also converges.}$$

69. $\displaystyle\int_0^1 \frac{1}{x^4 + \sqrt{x}}\,dx$

SOLUTION For $0 < x < 1$, $x^4 + \sqrt{x} \ge \sqrt{x}$, and

$$\frac{1}{x^4 + \sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

The integral $\displaystyle\int_0^1 (1/\sqrt{x})\,dx$ converges, since $p = \frac{1}{2} < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{dx}{x^4 + \sqrt{x}} \text{ also converges.}$$

70. $\displaystyle\int_1^\infty \frac{\ln x}{\sinh x}\,dx$

SOLUTION For $x > 1$, $e^{-x} < \frac{1}{2}e^x$, so

$$\sinh x = \frac{e^x - e^{-x}}{2} \ge \frac{1}{4}e^x.$$

Similarly, $\ln x < x$ for all $x > 1$, so

$$\frac{\ln x}{\sinh x} \le \frac{4x}{e^x} \quad \text{for all } x \ge 1.$$

Because

$$\int_1^\infty 4xe^{-x}\,dx = -4xe^{-x}\Big|_1^\infty + \int_1^\infty 4e^{-x}\,dx = \frac{8}{e},$$

it follows by the comparison test that

$$\int_1^\infty \frac{\ln x}{\sinh x}\,dx \text{ converges.}$$

71. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^{1/3}+x^3}}$

SOLUTION For $x \ge 0$, $\sqrt{x^{1/3}+x^3} \ge \sqrt{x^3} = x^{3/2}$, so that

$$\frac{1}{\sqrt{x^{1/3}+x^3}} \le \frac{1}{x^{3/2}}$$

The integral $\displaystyle\int_1^\infty x^{-3/2}\,dx$ converges since $p = 3/2 > 1$. Therefore, by the comparison test,

$$\int \frac{1}{\sqrt{x^{1/3}+x^3}}\,dx \quad \text{also converges.}$$

72. $\displaystyle\int_0^1 \frac{dx}{(8x^2+x^4)^{1/3}}$

SOLUTION Clearly $8x^2 + x^4 \ge 8x^2$, so that

$$\frac{1}{(8x^2+x^4)^{1/3}} \le \frac{1}{(8x^2)^{1/3}}$$

Thus

$$\int_0^1 \frac{1}{(8x^2+x^4)}\,dx \le \int_0^1 \frac{1}{(8x^2)^{1/3}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x^{2/3}}\,dx$$

But $\displaystyle\int_0^1 x^{-2/3}\,dx$ converges since $p = 2/3 < 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{1}{(8x^2+x^4)^{1/3}}\,dx \quad \text{also converges.}$$

73. $\displaystyle\int_1^\infty \frac{dx}{(x+x^2)^{1/3}}$

SOLUTION For $x > 1$, $x < x^2$ so that $x + x^2 < 2x^2$; then

$$\int_1^\infty \frac{1}{(x+x^2)^{1/3}}\,dx \ge \int_1^\infty \frac{1}{(2x^2)^{1/3}}\,dx = \frac{1}{2^{1/3}}\int_1^\infty \frac{1}{x^{2/3}}\,dx$$

But $\displaystyle\int_1^\infty \frac{1}{x^{2/3}}\,dx$ diverges since $p = 2/3 < 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{1}{(x+x^2)^{1/3}}\,dx \quad \text{diverges as well.}$$

74. $\displaystyle\int_0^1 \frac{dx}{xe^x+x^2}$

SOLUTION $xe^x + x^2 = x(e^x + x)$; for $0 \le x \le 1$, $e^x \le e^1 = e$ and $x \le 1$, so that $x(e^x + x) \le x(e + 1)$. It follows that

$$\int_0^1 \frac{1}{xe^x+x^2}\,dx \ge \int_0^1 \frac{1}{x(e+1)}\,dx = \frac{1}{e+1}\int_0^1 \frac{1}{x}\,dx$$

But $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges since $p = 1$. Therefore, by the comparison test,

$$\int_0^1 \frac{1}{xe^x+x^2}\,dx \quad \text{diverges as well.}$$

Hint for Exercise 73: Show that for $x \geq 1$,

$$\frac{1}{(x+x^2)^{1/3}} \geq \frac{1}{2^{1/3}x^{2/3}}$$

Hint for Exercise 74: Show that for $0 \leq x \leq 1$,

$$\frac{1}{xe^x + x^2} \geq \frac{1}{(e+1)x}$$

75. Define $J = \int_0^\infty \frac{dx}{x^{1/2}(x+1)}$ as the sum of the two improper integrals

$$\int_0^1 \frac{dx}{x^{1/2}(x+1)} + \int_1^\infty \frac{dx}{x^{1/2}(x+1)}$$

Use the Comparison Test to show that J converges.

SOLUTION For the first integral, note that for $0 \leq x \leq 1$, we have $1 \leq 1 + x$, so that $x^{1/2}(x+1) \geq x^{1/2}$. It follows that

$$\int_0^1 \frac{1}{x^{1/2}(x+1)}\,dx \leq \int_0^1 \frac{1}{x^{1/2}}\,dx$$

which converges since $p = 1/2 < 1$. Thus the first integral converges by the comparison test. For the second integral, for $1 \leq x$, we have $x^{1/2}(x+1) = x^{3/2} + x^{1/2} \geq x^{3/2}$, so that

$$\int_1^\infty \frac{1}{x^{1/2}(x+1)}\,dx = \int_1^\infty \frac{1}{x^{3/2} + x^{1/2}}\,dx \leq \int_1^\infty \frac{1}{x^{3/2}}\,dx$$

which converges since $p = 3/2 > 1$. Thus the second integral converges as well by the comparison test, and therefore J, which is the sum of the two, converges.

76. Determine whether $J = \int_0^\infty \frac{dx}{x^{3/2}(x+1)}$ (defined as in Exercise 75) converges.

SOLUTION We have $x^{3/2}(x+1) = x^{5/2} + x^{3/2}$. For $0 \leq x \leq 1$, $x^{5/2} \leq x^{3/2}$, so that $x^{5/2} + x^{3/2} \leq 2x^{3/2}$. Then

$$\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx = \int_0^1 \frac{1}{x^{5/2} + x^{3/2}}\,dx \geq \int_0^1 \frac{1}{2x^{3/2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x^{3/2}}\,dx$$

But this integral diverges since $p = 3/2 > 1$. By the comparison test, $\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx$ diverges as well, so that J diverges.

77. An investment pays a dividend of \$250/year continuously forever. If the interest rate is 7%, what is the present value of the entire income stream generated by the investment?

SOLUTION The present value of the income stream after T years is

$$\int_0^T 250e^{-0.07t}\,dt = \left.\frac{250e^{-0.07t}}{-0.07}\right|_0^T = \frac{-250}{0.07}\left(e^{-0.07T} - 1\right) = \frac{250}{0.07}\left(1 - e^{-0.07T}\right).$$

Therefore the present value of the entire income stream is

$$\int_0^\infty 250e^{-0.07t} = \lim_{T\to\infty}\int_0^T 250e^{-0.07t} = \lim_{T\to\infty}\frac{250}{0.07}\left(1 - e^{-0.07T}\right) = \frac{250}{0.07}(1-0) = \frac{250}{0.07} = \$3571.43.$$

78. An investment is expected to earn profits at a rate of $10{,}000e^{0.01t}$ dollars per year forever. Find the present value of the income stream if the interest rate is 4%.

SOLUTION The present value of the income stream after T years is

$$\int_0^T \left(10{,}000e^{0.01t}\right)e^{-0.04t}\,dt = 10{,}000\int_0^T e^{-0.03t}\,dt = \left.\frac{10{,}000}{-0.03}e^{-0.03t}\right|_0^T = -333{,}333.33\left(e^{-0.03t} - 1\right).$$

Therefore the present value of the entire income stream is

$$\int_0^\infty 10{,}000e^{-0.03t} = \lim_{T\to\infty} 333{,}333.33\left(1 - e^{-0.03t}\right) = \$333{,}333.33.$$

79. Compute the present value of an investment that generates income at a rate of $5000te^{0.01t}$ dollars per year forever, assuming an interest rate of 6%.

SOLUTION The present value of the income stream after T years is

$$\int_0^T \left(5000te^{0.01t}\right) e^{-0.06t}\,dt = 5000\int_0^T te^{-0.05t}\,dt$$

Compute the indefinite integral using Integration by Parts, with $u = t$ and $v' = e^{-0.05t}$. Then $u' = 1$, $v = (-1/0.05)e^{-0.05t}$, and

$$\int te^{-0.05t}\,dt = \frac{-t}{0.05}e^{-0.05t} + \frac{1}{0.05}\int e^{-0.05t}\,dt = -20te^{-0.05t} + \frac{20}{-0.05}e^{-0.05t} + C$$
$$= e^{-0.05t}(-20t - 400) + C.$$

Thus,

$$5000\int_0^T te^{-0.05t}\,dt = 5000e^{-0.05t}(-20t-400)\Big|_0^T = 5000e^{-0.05T}(-20T-400) - 5000(-400)$$
$$= 2{,}000{,}000 - 5000e^{-0.05T}(20T+400).$$

Use L'Hôpital's Rule to compute the limit:

$$\lim_{T\to\infty}\left(2{,}000{,}000 - \frac{5000(20T+400)}{e^{0.05T}}\right) = 2{,}000{,}000 - \lim_{T\to\infty}\frac{5000(20)}{0.05e^{0.05T}} = 2{,}000{,}000 - 0 = \$2{,}000{,}000.$$

80. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x-axis for $0 \le x < \infty$.

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^\infty \pi\left(e^{-x}\right)^2\,dx = \pi\int_0^\infty e^{-2x}\,dx.$$

First compute the volume over a finite interval:

$$\pi\int_0^R e^{-2x}\,dx = \frac{-\pi}{2}e^{-2x}\Big|_0^R = \frac{-\pi}{2}\left(e^{-2R}-1\right) = \frac{\pi}{2}\left(1-e^{-2R}\right).$$

Thus,

$$V = \lim_{R\to\infty}\pi\int_0^R e^{-2x}\,dx = \lim_{R\to\infty}\frac{\pi}{2}\left(1-e^{-2R}\right) = \frac{\pi}{2}(1-0) = \frac{\pi}{2}.$$

81. The solid S obtained by rotating the region below the graph of $y = x^{-1}$ about the x-axis for $1 \le x < \infty$ is called **Gabriel's Horn** (Figure 11).

(a) Use the Disk Method (Section 6.3) to compute the volume of S. Note that the volume is finite even though S is an infinite region.

(b) It can be shown that the surface area of S is

$$A = 2\pi\int_1^\infty x^{-1}\sqrt{1+x^{-4}}\,dx$$

Show that A is infinite. If S were a container, you could fill its interior with a finite amount of paint, but you could not paint its surface with a finite amount of paint.

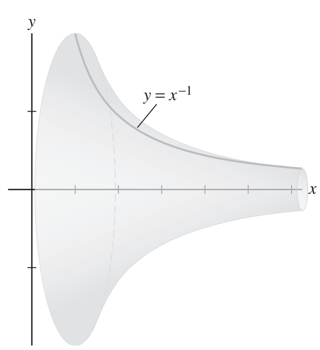


FIGURE 11

SOLUTION

(a) The volume is given by

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx.$$

First compute the volume over a finite interval:

$$\int_1^R \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^R x^{-2}\, dx = \pi \left.\frac{x^{-1}}{-1}\right|_1^R = \pi \left(\frac{-1}{R} - \frac{-1}{1}\right) = \pi \left(1 - \frac{1}{R}\right).$$

Thus,

$$V = \lim_{R\to\infty} \int_1^\infty \pi x^{-2}\, dx = \lim_{R\to\infty} \pi \left(1 - \frac{1}{R}\right) = \pi.$$

(b) For $x > 1$, we have

$$\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} = \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} = \frac{\sqrt{x^4+1}}{x^3} \geq \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \frac{1}{x}.$$

The integral $\int_1^\infty \frac{1}{x}\, dx$ diverges, since $p = 1 \geq 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx \text{ also diverges.}$$

Finally,

$$A = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

diverges.

82. Compute the volume of the solid obtained by rotating the region below the graph of $y = e^{-|x|/2}$ about the x-axis for $-\infty < x < \infty$.

SOLUTION The graph of y is symmetric around the y-axis, so it suffices to compute the volume for $0 \leq x \leq \infty$, where we have $y = e^{-x/2}$. Using the disk method,

$$\begin{aligned} V &= 2\int_0^\infty \pi \left(e^{-x/2}\right)^2 dx = 2\pi \int_0^\infty e^{-x}\, dx = 2\pi \lim_{R\to\infty} \int_0^R e^{-x}\, dx \\ &= -\lim_{R\to\infty} 2\pi e^{-x}\Big|_0^R = -2\pi \lim_{R\to\infty} (e^{-R} - 1) = 2\pi \end{aligned}$$

Therefore $V = 2\pi$.

83. When a capacitor of capacitance C is charged by a source of voltage V, the power expended at time t is

$$P(t) = \frac{V^2}{R}(e^{-t/RC} - e^{-2t/RC})$$

where R is the resistance in the circuit. The total energy stored in the capacitor is

$$W = \int_0^\infty P(t)\, dt$$

Show that $W = \frac{1}{2}CV^2$.

SOLUTION The total energy contained after the capacitor is fully charged is

$$W = \frac{V^2}{R}\int_0^\infty \left(e^{-t/RC} - e^{-2t/RC}\right) dt.$$

The energy after a finite amount of time ($t = T$) is

$$\frac{V^2}{R}\int_0^T \left(e^{-t/RC} - e^{-2t/RC}\right) dt = \frac{V^2}{R}\left(-RCe^{-t/RC} + \frac{RC}{2}e^{-2t/RC}\right)\Bigg|_0^T$$
$$= V^2 C\left[\left(-e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right) - \left(-1 + \frac{1}{2}\right)\right]$$
$$= CV^2\left(\frac{1}{2} - e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right).$$

Thus,

$$W = \lim_{T\to\infty} CV^2\left(\frac{1}{2} - e^{-T/RC} + \frac{1}{2}e^{-2T/RC}\right) = CV^2\left(\frac{1}{2} - 0 + 0\right) = \frac{1}{2}CV^2.$$

84. For which integers p does $\displaystyle\int_0^{1/2} \frac{dx}{x(\ln x)^p}$ converge?

SOLUTION If $p = 1$, the integral diverges. By substituting $u = \ln x$ and $du = dx/x$, we get

$$\int \frac{dx}{x(\ln x)} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C,$$

so

$$\int_0^{1/2} \frac{dx}{x(\ln x)} = \lim_{R\to 0+} (\ln|\ln x|)\Bigg|_R^{1/2} = \lim_{R\to 0+} (\ln|\ln(1/2)| - \ln|\ln R|),$$

which is infinite.

Now, suppose $p \neq 1$. Using the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$, the integral becomes

$$\int_R^{1/2} \frac{dx}{x(\ln x)^p} = \int_{x=R}^{x=1/2} \frac{du}{u^p} = \int_{x=R}^{x=1/2} u^{-p}\, du = \frac{1}{p-1}u^{-p+1}\Bigg|_{x=R}^{x=1/2}$$
$$= \frac{1}{p-1}(\ln x)^{-p+1}\Bigg|_R^{1/2} = \frac{1}{p-1}(\ln(1/2))^{-p+1} - \frac{1}{p-1}(\ln(R))^{-p+1}.$$

By definition,

$$\int_0^{1/2} \frac{dx}{x(\ln x)^p} = \lim_{R\to 0+}\int_R^{1/2} \frac{dx}{x(\ln x)^p} = \lim_{R\to 0+}\left[\frac{1}{p-1}(\ln(1/2))^{-p+1} - \frac{1}{p-1}(\ln R)^{-p+1}\right].$$

If $p > 1$, $\lim_{R\to 0+}(\ln R)^{-p+1} = \lim_{R\to 0}\frac{1}{(\ln R)^{p-1}} = 0$. If $p < 1$, $\lim_{R\to 0+}(\ln R)^{1-p} = \infty$. Therefore, the integral diverges if $p < 1$ or $p = 1$, and converges if $p > 1$.

85. Conservation of Energy can be used to show that when a mass m oscillates at the end of a spring with spring constant k, the period of oscillation is

$$T = 4\sqrt{m}\int_0^{\sqrt{2E/k}} \frac{dx}{\sqrt{2E - kx^2}}$$

where E is the total energy of the mass. Show that this is an improper integral with value $T = 2\pi\sqrt{m/k}$.

SOLUTION The integrand is infinite at the upper limit of integration, $x = \sqrt{2E/k}$, so the integral is improper. Now, let

$$T(R) = 4\sqrt{m}\int_0^R \frac{dx}{\sqrt{2E - kx^2}} = 4\sqrt{m}\frac{1}{\sqrt{2E}}\int_0^R \frac{dx}{\sqrt{1 - (\frac{k}{2E})x^2}}$$
$$= 4\sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\sin^{-1}\left(\sqrt{\frac{k}{2E}}R\right) = 4\sqrt{m/k}\sin^{-1}\left(\sqrt{\frac{k}{2E}}R\right).$$

Therefore

$$T = \lim_{R\to\sqrt{2E/k}} T(R) = 4\sqrt{\frac{m}{k}}\sin^{-1}(1) = 2\pi\sqrt{\frac{m}{k}}.$$

In Exercises 86–89, the ***Laplace transform*** *of a function $f(x)$ is the function $Lf(s)$ of the variable s defined by the improper integral (if it converges):*

$$Lf(s) = \int_0^\infty f(x)e^{-sx}\,dx$$

Laplace transforms are widely used in physics and engineering.

86. Show that if $f(x) = C$, where C is a constant, then $Lf(s) = C/s$ for $s > 0$.

SOLUTION If $f(x) = C$, a constant, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^\infty Ce^{-sx}\,dx = \lim_{R\to\infty} \frac{-C}{s}e^{-sx}\Big|_0^R = \lim_{R\to\infty} \frac{-C}{s}\left(e^{-sR} - 1\right) = \frac{-C}{s}(0-1) = \frac{C}{s}.$$

87. Show that if $f(x) = \sin \alpha x$, then $Lf(s) = \dfrac{\alpha}{s^2+\alpha^2}$.

SOLUTION If $f(x) = \sin\alpha x$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^\infty e^{-sx}\sin\alpha x\,dx$$

First evaluate the indefinite integral using Integration by Parts, with $u = \sin\alpha x$ and $v' = e^{-sx}$. Then $u' = \alpha\cos\alpha x$, $v = -\frac{1}{s}e^{-sx}$, and

$$\int e^{-sx}\sin\alpha x\,dx = -\frac{1}{s}e^{-sx}\sin\alpha x + \frac{\alpha}{s}\int e^{-sx}\cos\alpha x\,dx.$$

Use Integration by Parts again, with $u = \cos\alpha x$, $v' = e^{-sx}$. Then $u' = -\alpha\sin\alpha x$, $v = -\frac{1}{s}e^{-sx}$, and

$$\int e^{-sx}\cos\alpha x\,dx = -\frac{1}{s}e^{-sx}\cos\alpha x - \frac{\alpha}{s}\int e^{-sx}\sin\alpha x\,dx.$$

Substituting this into the first equation and solving for $\int e^{-sx}\sin\alpha x\,dx$, we get

$$\int e^{-sx}\sin\alpha x\,dx = -\frac{1}{s}e^{-sx}\sin\alpha x - \frac{\alpha}{s^2}e^{-sx}\cos\alpha x - \frac{\alpha^2}{s^2}\int e^{-sx}\sin\alpha x\,dx$$

$$\int e^{-sx}\sin\alpha x\,dx = \frac{-e^{-sx}\left(\frac{1}{s}\sin\alpha x + \frac{\alpha}{s^2}\cos\alpha x\right)}{\left(1+\frac{\alpha^2}{s^2}\right)} = \frac{-e^{-sx}(s\sin\alpha x + \alpha\cos\alpha x)}{s^2+\alpha^2}$$

Thus,

$$\int_0^R e^{-sx}\sin\alpha x\,dx = \frac{1}{s^2+\alpha^2}\left[\frac{s\sin\alpha R + \alpha\cos\alpha R}{-e^{sR}} - \frac{0+\alpha}{-1}\right] = \frac{1}{s^2+\alpha^2}\left[\alpha - \frac{s\sin\alpha R+\alpha\cos\alpha R}{e^{sR}}\right].$$

Finally we take the limit, noting the fact that, for all values of R, $|s\sin\alpha R + \alpha\cos\alpha R| \le s + |\alpha|$

$$Lf(s) = \lim_{R\to\infty}\frac{1}{s^2+\alpha^2}\left[\alpha - \frac{s\sin\alpha R+\alpha\cos\alpha R}{e^{sR}}\right] = \frac{1}{s^2+\alpha^2}(\alpha - 0) = \frac{\alpha}{s^2+\alpha^2}.$$

88. Compute $Lf(s)$, where $f(x) = e^{\alpha x}$ and $s > \alpha$.

SOLUTION If $f(x) = e^{\alpha x}$, where $s > \alpha$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^\infty e^{\alpha x}e^{-sx}\,dx = \int_0^\infty e^{-(s-\alpha)x}\,dx = \lim_{R\to\infty}\frac{-1}{s-\alpha}e^{-(s-\alpha)x}\Big|_0^R = \lim_{R\to\infty}\frac{-1}{s-\alpha}\left(e^{-(s-\alpha)R} - 1\right).$$

Because $s > \alpha$, $-(s-\alpha) < 0$, which gives us

$$\lim_{R\to\infty}\frac{1}{s-\alpha}\left(1 - e^{-(s-\alpha)R}\right) = \frac{1}{s-\alpha}(1-0) = \frac{1}{s-\alpha}.$$

The final answer is

$$Lf(s) = \frac{1}{s-\alpha}.$$

89. Compute $Lf(s)$, where $f(x) = \cos \alpha x$ and $s > 0$.

SOLUTION If $f(x) = \cos \alpha x$, then the Laplace transform of $f(x)$ is

$$Lf(x) = \int_0^\infty e^{-sx} \cos \alpha x \, dx$$

First evaluate the indefinite integral using Integration by Parts, with $u = \cos \alpha x$ and $v' - e^{-sx}$. Then $u' = -\alpha \sin \alpha x$, $v = -\frac{1}{s} e^{-sx}$, and

$$\int e^{-sx} \cos \alpha x \, dx = -\frac{1}{s} e^{-sx} \cos \alpha x - \frac{\alpha}{s} \int e^{-sx} \sin \alpha x \, dx.$$

Use Integration by Parts again, with $u = \sin \alpha x \, dx$ and $v' = -e^{-sx}$. Then $u' = \alpha \cos \alpha x$, $v = -\frac{1}{s} e^{-sx}$, and

$$\int e^{-sx} \sin \alpha x \, dx = -\frac{1}{s} e^{-sx} \sin \alpha x + \frac{\alpha}{s} \int e^{-sx} \cos \alpha x \, dx.$$

Substituting this into the first equation and solving for $\int e^{-sx} \cos \alpha x \, dx$, we get

$$\begin{aligned}
\int e^{-sx} \cos \alpha x \, dx &= -\frac{1}{s} e^{-sx} \cos \alpha x - \frac{\alpha}{s} \left[-\frac{1}{s} e^{-sx} \sin \alpha x + \frac{\alpha}{s} \int e^{-sx} \cos \alpha \, dx \right] \\
&= -\frac{1}{s} e^{-sx} \cos \alpha x + \frac{\alpha}{s^2} e^{-sx} \sin \alpha x - \frac{\alpha^2}{s^2} \int e^{-sx} \cos \alpha x \, dx \\
\int e^{-sx} \cos \alpha x \, dx &= \frac{e^{-sx} \left(\frac{\alpha}{s^2} \sin \alpha x - \frac{1}{s} \cos \alpha x \right)}{1 + \frac{\alpha^2}{s^2}} = \frac{e^{-sx}(\alpha \sin \alpha x - s \cos \alpha x)}{s^2 + \alpha^2}
\end{aligned}$$

Thus,

$$\int_0^R e^{-sx} \cos \alpha x \, dx = \frac{1}{s^2 + \alpha^2} \left[\frac{\alpha \sin \alpha R - s \cos \alpha R}{e^{sR}} - \frac{0 - s}{1} \right].$$

Finally we take the limit, noting the fact that, for all values of R, $|\alpha \sin \alpha R - s \cos \alpha R| \le |\alpha| + s$

$$Lf(s) = \lim_{R \to \infty} \frac{1}{s^2 + \alpha^2} \left[s + \frac{\alpha \sin \alpha R - s \cos \alpha R}{e^{sR}} \right] = \frac{1}{s^2 + \alpha^2}(s + 0) = \frac{s}{s^2 + \alpha^2}.$$

90. When a radioactive substance decays, the fraction of atoms present at time t is $f(t) = e^{-kt}$, where $k > 0$ is the decay constant. It can be shown that the *average* life of an atom (until it decays) is $A = -\int_0^\infty t f'(t) \, dt$. Use Integration by Parts to show that $A = \int_0^\infty f(t) \, dt$ and compute A. What is the average decay time of radon-222, whose half-life is 3.825 days?

SOLUTION Let $u = t$, $v' = f'(t)$. Then $u' = 1$, $v = f(t)$, and

$$A = -\int_0^\infty t f'(t) \, dt = -t f(t) \Big|_0^\infty + \int_0^\infty f(t) \, dt.$$

Since $f(t) = e^{-kt}$, we have

$$-t f(t) \Big|_0^\infty = \lim_{R \to \infty} -t e^{-kt} \Big|_0^R = \lim_{R \to \infty} -R e^{-Rt} + 0 = \lim_{R \to \infty} \frac{-R}{e^{Rt}} = \lim_{R \to \infty} \frac{-1}{R e^{Rt}} = 0.$$

Here we used L'Hôpital's Rule to compute the limit. Thus

$$A = \int_0^\infty f(t) \, dt = \int_0^\infty e^{-kt} \, dt.$$

Now,

$$\int_0^R e^{-kt} \, dt = -\frac{1}{k} e^{-kt} \Big|_0^R = -\frac{1}{k} \left(e^{-kR} - 1 \right) = \frac{1}{k} \left(1 - e^{-kR} \right),$$

so

$$A = \lim_{R \to \infty} \frac{1}{k} \left(1 - e^{-kR} \right) = \frac{1}{k}(1 - 0) = \frac{1}{k}.$$

Because k has units of $(\text{time})^{-1}$, A does in fact have the appropriate units of time. To find the average decay time of Radon-222, we need to determine the decay constant k, given the half-life of 3.825 days. Recall that

$$k = \frac{\ln 2}{t_n}$$

where t_n is the half-life. Thus,

$$A = \frac{1}{k} = \frac{t_n}{\ln 2} = \frac{3.825}{\ln 2} \approx 5.518 \text{ days.}$$

91. Let $J_n = \int_0^\infty x^n\, e^{-\alpha x}\, dx$, where $n \ge 1$ is an integer and $\alpha > 0$. Prove that

$$J_n = \frac{n}{\alpha} J_{n-1}$$

and $J_0 = 1/\alpha$. Use this to compute J_4. Show that $J_n = n!/\alpha^{n+1}$.

SOLUTION Using Integration by Parts, with $u = x^n$ and $v' = e^{-\alpha x}$, we get $u' = nx^{n-1}$, $v = -\frac{1}{\alpha} e^{-\alpha x}$, and

$$\int x^n e^{-\alpha x}\, dx = -\frac{1}{\alpha} x^n e^{-\alpha x} + \frac{n}{\alpha} \int x^{n-1} e^{-\alpha x}\, dx.$$

Thus,

$$J_n = \int_0^\infty x^n e^{-\alpha x}\, dx = \lim_{R\to\infty} \left(-\frac{1}{\alpha} x^n e^{-\alpha x} \right)\Bigg|_0^R + \frac{n}{\alpha} \int_0^\infty x^{n-1} e^{-\alpha x}\, dx = \lim_{R\to\infty} \frac{-R^n}{\alpha e^{\alpha R}} + 0 + \frac{n}{\alpha} J_{n-1}.$$

Use L'Hôpital's Rule repeatedly to compute the limit:

$$\lim_{R\to\infty} \frac{-R_n}{\alpha e^{\alpha R}} = \lim_{R\to\infty} \frac{-nR^{n-1}}{\alpha^2 e^{\alpha R}} = \lim_{R\to\infty} \frac{-n(n-1)R^{n-2}}{\alpha^3 e^{\alpha R}} = \cdots = \lim_{R\to\infty} \frac{-n(n-1)(n-2)\cdots(3)(2)(1)}{\alpha^{n+1} e^{\alpha R}} = 0.$$

Finally,

$$J_n = 0 + \frac{n}{\alpha} J_{n-1} = \frac{n}{\alpha} J_{n-1}.$$

J_0 can be computed directly:

$$J_0 = \int_0^\infty e^{-\alpha x}\, dx = \lim_{R\to\infty} \int_0^R e^{-\alpha x}\, dx = \lim_{R\to\infty} -\frac{1}{\alpha} e^{-\alpha x}\Bigg|_0^R = \lim_{R\to\infty} -\frac{1}{\alpha}\left(e^{-\alpha R} - 1\right) = -\frac{1}{\alpha}(0-1) = \frac{1}{\alpha}.$$

With this starting point, we can work up to J_4:

$$J_1 = \frac{1}{\alpha} J_0 = \frac{1}{\alpha}\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^2};$$

$$J_2 = \frac{2}{\alpha} J_1 = \frac{2}{\alpha}\left(\frac{1}{\alpha^2}\right) = \frac{2}{\alpha^3} = \frac{2!}{\alpha^{2+1}};$$

$$J_3 = \frac{3}{\alpha} J_2 = \frac{3}{\alpha}\left(\frac{2}{\alpha^3}\right) = \frac{6}{\alpha^4} = \frac{3!}{\alpha^{3+1}};$$

$$J_4 = \frac{4}{\alpha} J_3 = \frac{4}{\alpha}\left(\frac{6}{\alpha^4}\right) = \frac{24}{\alpha^5} = \frac{4!}{\alpha^{4+1}}.$$

We can use induction to prove the formula for J_n. If

$$J_{n-1} = \frac{(n-1)!}{\alpha^n},$$

then we have

$$J_n = \frac{n}{\alpha} J_{n-1} = \frac{n}{\alpha} \cdot \frac{(n-1)!}{\alpha^n} = \frac{n!}{\alpha^{n+1}}.$$

92. Let $a > 0$ and $n > 1$. Define $f(x) = \dfrac{x^n}{e^{ax} - 1}$ for $x \ne 0$ and $f(0) = 0$.

(a) Use L'Hôpital's Rule to show that $f(x)$ is continuous at $x = 0$.

(b) Show that $\int_0^\infty f(x)\, dx$ converges. *Hint:* Show that $f(x) \le 2x^n e^{-ax}$ if x is large enough. Then use the Comparison Test and Exercise 91.

SOLUTION

(a) Using L'Hôpital's Rule, we find

$$\lim_{x\to 0} \frac{x^n}{e^{\alpha x}-1} = \lim_{x\to 0} \frac{nx^{n-1}}{\alpha e^{\alpha x}} = \frac{0}{\alpha} = 0;$$

thus,

$$\lim_{x\to 0} f(x) = f(0),$$

and $f(x)$ is continuous at $x = 0$.

(b) Since $a > 0$, $\lim\limits_{x\to\infty} e^{ax} = \infty$. Therefore there will be some value of x, say $x = M$, such that, for all $x \geq M$, we'll have $e^{ax} \geq 2$. With this, we have

$$\frac{1}{e^{ax}} \leq \frac{1}{2} \quad \text{so} \quad \frac{1}{e^{ax}} + \frac{1}{2} \leq 1 \quad \text{and} \quad 1 - \frac{1}{e^{\alpha x}} \geq \frac{1}{2}.$$

Multiply this last inequality through by $e^{\alpha x}$ to obtain

$$e^{\alpha x} - 1 \geq \frac{e^{\alpha x}}{2} \quad \text{so} \quad \frac{1}{e^{\alpha x}-1} \leq \frac{2}{e^{\alpha x}} \quad \text{and} \quad \frac{x^n}{e^{\alpha x}-1} \leq \frac{2x^n}{e^{\alpha x}}.$$

From Exercise 91, we know that

$$\int_0^\infty x^n e^{-\alpha x}\,dx \text{ converges, so } \int_M^\infty 2x^n e^{-\alpha x}\,dx \text{ also converges.}$$

Therefore, by the comparison test,

$$\int_M^\infty \frac{x^n}{e^{\alpha x}-1}\,dx \text{ also converges.}$$

Now, from part (a), we know that $f(x)$ is continuous on $[0, M]$, so

$$\int_0^M \frac{x^n}{e^{\alpha x}-1}\,dx$$

exists and is finite. Thus we have shown

$$\int_0^\infty \frac{x^n}{e^{\alpha x}-1}\,dx = \int_0^M \frac{x^n}{e^{\alpha x}-1}\,dx + \int_M^\infty \frac{x^n}{e^{\alpha x}-1}\,dx \text{ converges.}$$

93. According to **Planck's Radiation Law**, the amount of electromagnetic energy with frequency between ν and $\nu + \Delta\nu$ that is radiated by a so-called black body at temperature T is proportional to $F(\nu)\,\Delta\nu$, where

$$F(\nu) = \left(\frac{8\pi h}{c^3}\right)\frac{\nu^3}{e^{h\nu/kT}-1}$$

where c, h, k are physical constants. Use Exercise 92 to show that the total radiated energy

$$E = \int_0^\infty F(\nu)\,d\nu$$

is finite. To derive his law, Planck introduced the quantum hypothesis in 1900, which marked the birth of quantum mechanics.

SOLUTION The total radiated energy E is given by

$$E = \int_0^\infty F(\nu)\,d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}\,d\nu.$$

Let $\alpha = h/kT$. Then

$$E = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{\alpha\nu}-1}\,d\nu.$$

Because $\alpha > 0$ and $8\pi h/c^3$ is a constant, we know E is finite by Exercise 92.

Further Insights and Challenges

94. Let $I = \int_0^1 x^p \ln x\, dx$.

(a) Show that I diverges for $p = -1$.

(b) Show that if $p \neq -1$, then

$$\int x^p \ln x\, dx = \frac{x^{p+1}}{p+1}\left(\ln x - \frac{1}{p+1}\right) + C$$

(c) Use L'Hôpital's Rule to show that I converges if $p > -1$ and diverges if $p < -1$.

SOLUTION

(a) If $p = -1$, then

$$I = \int_0^1 x^{-1} \ln x\, dx = \int_0^1 \frac{\ln x}{x}\, dx.$$

Let $u = \ln x$, $du = (1/x)\, dx$. Then

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

Thus,

$$\int_R^1 \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln 1)^2 - \frac{1}{2}(\ln R)^2 = -\frac{1}{2}(\ln R)^2,$$

and

$$I = \lim_{R\to 0^+} -\frac{1}{2}(\ln R)^2 = \infty.$$

The integral diverges for $p = -1$.

(b) If $p \neq 1$, then use Integration by Parts, with $u = \ln x$ and $v' = x^p$. Then $u' = 1/x$, $v = x^{p+1}/p + 1$, and

$$\begin{aligned}\int x^p \ln x\, dx &= \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\int \left(x^{p+1}\right)\left(\frac{1}{x}\right) dx = \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\int x^p\, dx \\ &= \frac{x^{p+1}}{p+1}\ln x - \frac{1}{p+1}\left(\frac{x^{p+1}}{p+1}\right) + C = \frac{x^{p+1}}{p+1}\left(\ln x - \frac{1}{p+1}\right) + C.\end{aligned}$$

(c) Let $p < -1$. Then

$$\begin{aligned}I &= \lim_{R\to 0^+}\int_R^1 x^p \ln x = \lim_{R\to 0^+}\left[\frac{1}{p+1}\left(\ln 1 - \frac{1}{p+1}\right) - \frac{R^{p+1}}{p+1}\left(\ln R - \frac{1}{p+1}\right)\right] \\ &= \lim_{R\to 0^+}\left(\frac{-1}{(p+1)^2} - \frac{R^{p+1}}{p+1}\ln R + \frac{R^{p+1}}{(p+1)^2}\right).\end{aligned}$$

Since $p < -1$, $p + 1 < 0$, and we have

$$I = \lim_{R\to 0^+}\left(\frac{-1}{(p+1)^2} - \frac{\ln R}{(p+1)R^{-p-1}} + \frac{1}{(p+1)^2 R^{-p-1}}\right) = \infty.$$

The integral diverges for $p < -1$. On the other hand, if $p > -1$, then $p + 1 > 0$, and

$$I = \frac{-1}{(p+1)^2} + \frac{1}{p+1}\lim_{R\to 0+} R^{p+1}\ln R + \frac{1}{(p+1)^2}\lim_{R\to 0^+} R^{p+1} = \frac{-1}{(p+1)^2} + 0 = \frac{-1}{(p+1)^2}.$$

95. Let

$$F(x) = \int_2^x \frac{dt}{\ln t} \qquad \text{and} \qquad G(x) = \frac{x}{\ln x}$$

Verify that L'Hôpital's Rule applies to the limit $L = \lim_{x\to\infty}\frac{F(x)}{G(x)}$ and evaluate L.

SOLUTION Because $\ln t < t$ for $t > 2$, we have $\frac{1}{\ln t} > \frac{1}{t}$ for $t > 2$, and so

$$F(x) = \int_2^x \frac{dt}{\ln t} > \int_2^x \frac{dt}{t} = \ln x - \ln 2$$

Thus, $F(x) \to \infty$ as $x \to \infty$. Moreover, by L'Hôpital's Rule

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Thus, $\lim_{x\to\infty} \dfrac{F(x)}{G(x)}$ is of the form ∞/∞, and L'Hôpital's Rule applies. Finally,

$$L = \lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}}{\frac{\ln x - 1}{(\ln x)^2}} = \lim_{x\to\infty} \frac{\ln x}{\ln x - 1} = \lim_{x\to\infty} \frac{1}{1 - (1/\ln x)} = 1.$$

*In Exercises 96–98, an improper integral $I = \int_a^\infty f(x)\,dx$ is called **absolutely convergent** if $\int_a^\infty |f(x)|\,dx$ converges. It can be shown that if I is absolutely convergent, then it is convergent.*

96. Show that $\displaystyle\int_1^\infty \frac{\sin x}{x^2}\,dx$ is absolutely convergent.

SOLUTION For all x, $|\sin x| \le 1$. This implies

$$\left|\frac{\sin x}{x^2}\right| = \frac{|\sin x|}{x^2} \le \frac{1}{x^2}.$$

The integral $\displaystyle\int_1^\infty x^{-2}\,dx$ converges because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \left|\frac{\sin x}{x^2}\right| dx \text{ also converges.}$$

Because the integral

$$\int_1^\infty \frac{\sin x}{x^2}\,dx$$

is absolutely convergent, it is also convergent.

97. Show that $\displaystyle\int_1^\infty e^{-x^2}\cos x\,dx$ is absolutely convergent.

SOLUTION By the result of Exercise 57, we know that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent. Then $\displaystyle\int_1^\infty e^{-x^2}\,dx$ is also convergent. Because $|\cos x| \le 1$ for all x, we have

$$\left|e^{-x^2}\cos x\right| = |\cos x|\left|e^{-x^2}\right| \le \left|e^{-x^2}\right| = e^{-x^2}.$$

Therefore, by the comparison test, we have

$$\int_1^\infty \left|e^{-x^2}\cos x\right| dx \text{ also converges.}$$

Since $\displaystyle\int_1^\infty e^{-x^2}\cos x\,dx$ converges absolutely, it itself converges.

98. Let $f(x) = \sin x/x$ and $I = \int_0^\infty f(x)\,dx$. We define $f(0) = 1$. Then $f(x)$ is continuous and I is not improper at $x = 0$.

(a) Show that

$$\int_1^R \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\bigg|_1^R - \int_1^R \frac{\cos x}{x^2}\,dx$$

(b) Show that $\int_1^\infty (\cos x/x^2)\,dx$ converges. Conclude that the limit as $R \to \infty$ of the integral in (a) exists and is finite.

(c) Show that I converges.

It is known that $I = \frac{\pi}{2}$. However, I is *not* absolutely convergent. The convergence depends on cancellation, as shown in Figure 12.

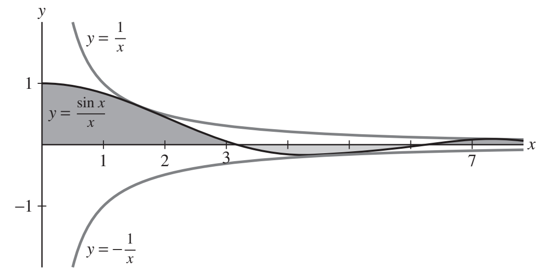


FIGURE 12 Convergence of $\int_1^\infty (\sin x/x)\,dx$ is due to the cancellation arising from the periodic change of sign.

SOLUTION

(a) Use Integration by Parts, with $u = \frac{1}{x}$ and $v' = \sin x$. Then $u' = -1/x^2$, $v = -\cos x$, and we have

$$\int_1^R \frac{\sin x}{x}\,dx = \frac{-\cos x}{x}\bigg|_1^R - \int_1^R \frac{\cos x}{x^2}\,dx.$$

(b) For all x, $|\cos x| \le 1$, and therefore

$$\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{x^2} \le \frac{1}{x^2}.$$

The integral $\displaystyle\int_1^\infty x^{-2}\,dx$ converges, because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx \text{ also converges.}$$

Because $\displaystyle\int_1^\infty (\cos x/x^2)\,dx$ converges absolutely, it also converges. By this result,

$$\lim_{R\to\infty} \int_1^R \frac{\sin x}{x}\,dx = \lim_{R\to\infty} \left[\frac{-\cos R}{R} + \frac{\cos 1}{1} - \int_1^R \frac{\cos x}{x^2}\,dx\right] = 0 + \frac{\cos 1}{1} - \int_0^\infty \frac{\cos x}{x^2}\,dx = \cos 1 - M,$$

where $M = \displaystyle\int_1^\infty (\cos x/x^2)\,dx$, the existence of which was shown in the argument above. Therefore the integral $\displaystyle\int_1^\infty (\sin x/x)\,dx$ converges to a finite value.

(c) The integral can be split up as follows:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx.$$

The second integral converges by part (b). For the first integral, if we define $f(0) = 1$, then the integrand is continuous on $[0, 1]$, and therefore

$$\int_0^1 \frac{\sin x}{x}\,dx = N$$

where N is some finite value. Thus, we have shown that I converges.

99. The **gamma function**, which plays an important role in advanced applications, is defined for $n \ge 1$ by

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$$

(a) Show that the integral defining $\Gamma(n)$ converges for $n \ge 1$ (it actually converges for all $n > 0$). *Hint:* Show that $t^{n-1}e^{-t} < t^{-2}$ for t sufficiently large.

(b) Show that $\Gamma(n+1) = n\Gamma(n)$ using Integration by Parts.

(c) Show that $\Gamma(n+1) = n!$ if $n \ge 1$ is an integer. *Hint:* Use (a) repeatedly. Thus, $\Gamma(n)$ provides a way of defining n-factorial when n is not an integer.

SOLUTION
(a) By repeated use of L'Hôpital's Rule, we can compute the following limit:

$$\lim_{t\to\infty} \frac{e^t}{t^{n+1}} = \lim_{t\to\infty} \frac{e^t}{(n+1)t^n} = \cdots = \lim_{t\to\infty} \frac{e^t}{(n+1)!} = \infty.$$

This implies that, for t sufficiently large, we have

$$e^t \ge t^{n+1};$$

therefore

$$\frac{e^t}{t^{n-1}} \ge \frac{t^{n+1}}{t^{n-1}} = t^2 \quad \text{or} \quad t^{n-1}e^{-t} \le t^{-2}.$$

The integral $\int_1^\infty t^{-2}\,dt$ converges because $p = 2 > 1$. Therefore, by the comparison test,

$$\int_M^\infty t^{n-1}e^{-t}\,dt \text{ also converges,}$$

where M is the value above which the above comparisons hold. Finally, because the function $t^{n-1}e^{-t}$ is continuous for all t, we know that

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \text{ converges} \quad \text{for all } n \ge 1.$$

(b) Using Integration by Parts, with $u = t^n$ and $v' - e^{-t}$, we have $u' = nt^{n-1}$, $v = -e^{-t}$, and

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,dt = -t^n e^{-t}\Big|_0^\infty + n\int_0^\infty t^{n-1}e^{-t}\,dt$$
$$= \lim_{R\to\infty}\left(\frac{-R^n}{e^R} - 0\right) + n\Gamma(n) = 0 + n\Gamma(n) = n\Gamma(n).$$

Here, we've computed the limit as in part (a) with repeated use of L'Hôpital's Rule.
(c) By the result of part (b), we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \cdots = n!\,\Gamma(1).$$

If $n = 1$, then

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{R\to\infty} -e^{-t}\Big|_0^R = \lim_{R\to\infty}\left(1 - e^{-R}\right) = 1.$$

Thus

$$\Gamma(n+1) = n!\,(1) = n!$$

100. Use the results of Exercise 99 to show that the Laplace transform (see Exercises 86–89 above) of x^n is $\frac{n!}{s^{n+1}}$.

SOLUTION If $f(x) = x^n$, then the Laplace transform of $f(x)$ is

$$Lf(s) = \int_0^\infty x^n e^{-sx}\,dx$$

Let $t = sx$. Then $dt = s\,dx$, and $x^n = t^n/s^n$. This gives us

$$Lf(s) = \int_0^\infty \frac{t^n}{s^n}e^{-t}\frac{dt}{s} = \frac{1}{s^{n+1}}\int_0^\infty t^n e^{-t}\,dt = \frac{1}{s^{n+1}}\Gamma(n+1) = \frac{n!}{s^{n+1}}.$$

7.7 Probability and Integration

Preliminary Questions

1. The function $p(x) = \cos x$ satisfies $\int_{-\pi/2}^{\pi} p(x)\,dx = 1$. Is $p(x)$ a probability density function on $[-\pi/2, \pi]$?

SOLUTION Since $p(x) = \cos x < 0$ for some points in $(-\pi/2, \pi)$, $p(x)$ is not a probability density function.

2. Estimate $P(2 \le X \le 2.1)$ assuming that the probability density function of X satisfies $p(2) = 0.2$.

SOLUTION $P(2 \le X \le 2.1) \approx p(2) \cdot (2.1 - 2) = 0.02$.

3. Which exponential probability density has mean $\mu = \frac{1}{4}$?

SOLUTION $\dfrac{1}{1/4} e^{-x/(1/4)} = 4e^{-4x}$.

Exercises

In Exercises 1–6, find a constant C such that $p(x)$ is a probability density function on the given interval, and compute the probability indicated.

1. $p(x) = \dfrac{C}{(x+1)^3}$ on $[0, \infty)$; $P(0 \le X \le 1)$

SOLUTION Compute the indefinite integral using the substitution $u = x + 1$, $du = dx$:

$$\int p(x)\,dx = \int \frac{C}{(x+1)^3}\,dx = -\frac{1}{2}C(x+1)^{-2} + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_0^\infty p(x)\,dx = -\frac{1}{2}C \lim_{R\to\infty} (x+1)^{-2}\Big|_0^R = \frac{1}{2}C - \frac{1}{2}C \lim_{R\to\infty}(R+1)^{-2} = \frac{1}{2}C$$

so that $C = 2$, and $p(x) = \frac{2}{(x+1)^3}$. Then using the indefinite integral above,

$$P(0 \le X \le 1) = \int_0^1 \frac{2}{(x+1)^3} = -\frac{1}{2} \cdot 2 \cdot (x+1)^{-2}\Big|_0^1 = -\frac{1}{4} + 1 = \frac{3}{4}$$

2. $p(x) = Cx(4-x)$ on $[0, 4]$; $P(3 \le X \le 4)$

SOLUTION Compute the indefinite integral:

$$\int p(x)\,dx = C\int x(4-x)\,dx = C\int 4x - x^2\,dx = C\left(2x^2 - \frac{1}{3}x^3\right) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_0^4 p(x)\,dx = C\left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = C\left(32 - \frac{64}{3}\right) = \frac{32}{3}C$$

so that $C = \frac{3}{32}$ and $p(x) = \frac{3}{32}x(4-x)$. Then using the indefinite integral above,

$$P(3 \le X \le 4) = \int_3^4 p(x)\,dx = \frac{3}{32}\left(2x^2 - \frac{1}{3}x^3\right)\Big|_3^4 = \frac{3}{32}\left(32 - \frac{64}{3} - 18 + 9\right) = \frac{5}{32}$$

3. $p(x) = \dfrac{C}{\sqrt{1-x^2}}$ on $(-1, 1)$; $P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right)$

SOLUTION Compute the indefinite integral:

$$\int p(x)\,dx = C\int \frac{1}{\sqrt{1-x^2}}\,dx = C\sin^{-1} x + K$$

valid for $-1 < x < 1$. For $p(x)$ to be a probability density function, we must have

$$\begin{aligned}1 = \int_{-1}^1 p(x)\,dx = \int_{-1}^0 p(x)\,dx + \int_0^1 p(x)\,dx &= C\left(\lim_{R\to -1^+} \sin^{-1} x\Big|_R^0 + \lim_{R\to 1^-} \sin^{-1} x\Big|_0^R\right) \\ &= C\left(\sin^{-1}(0) - \lim_{R\to -1^+} \sin^{-1}(R) + \lim_{R\to 1^-} \sin^{-1} R - \sin^{-1}(0)\right) \\ &= C\left(-\sin^{-1}(-1) + \sin^{-1}(1)\right) = \pi C\end{aligned}$$

so that $C = \frac{1}{\pi}$ and $p(x) = \frac{1}{\pi\sqrt{1-x^2}}$. Then using the indefinite integral above,

$$P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right) = \int_{-1/2}^{1/2} p(x)\,dx = \frac{1}{\pi}\sin^{-1} x\Big|_{-1/2}^{1/2} = \frac{1}{\pi}\left(\frac{\pi}{6} - \frac{-\pi}{6}\right) = \frac{1}{3}$$

4. $p(x) = \dfrac{Ce^{-x}}{1+e^{-2x}}$ on $(-\infty, \infty)$; $P(X \le -4)$

SOLUTION Compute the indefinite integral using the substitution $u = e^{-x}$; then $du = -e^{-x}\,dx = -u\,dx$ so that $dx = -\frac{1}{u}\,du$:

$$\int p(x)\,dx = \int \frac{Ce^{-x}}{1+e^{-2x}}\,dx = C\int \frac{u\cdot\left(-\frac{1}{u}\right)}{1+u^2}\,du = -C\int \frac{1}{1+u^2}\,du$$
$$= -C\tan^{-1}u + K = -C\tan^{-1}(e^{-x}) + K = C\tan^{-1}(e^x) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = C\lim_{R\to\infty}\tan^{-1}(e^x)\Big|_{-R}^{R} = C\lim_{R\to\infty}\left(\tan^{-1}(e^R) - \tan^{-1}(e^{-R})\right) = C\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}C$$

so that $C = \frac{2}{\pi}$ and $p(x) = \frac{2e^{-x}}{\pi(1+e^{-2x})}$. Then using the indefinite integral above,

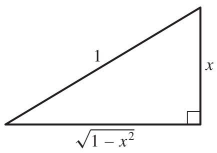
$$P(X \le -4) = \int_{-\infty}^{-4} p(x)\,dx = \lim_{R\to-\infty}\frac{2}{\pi}\tan^{-1}(e^x)\Big|_{R}^{-4} = \frac{2}{\pi}\tan^{-1}(e^{-4}) - \frac{2}{\pi}\lim_{R\to-\infty}\tan^{-1}(e^R)$$
$$= \frac{2}{\pi}\tan^{-1}(e^{-4}) \approx 0.0117$$

5. $p(x) = C\sqrt{1-x^2}$ on $(-1, 1)$; $P\left(-\frac{1}{2} \le X \le 1\right)$

SOLUTION Compute the indefinite integral using the substitution $x = \sin u$, so that $dx = \cos u\,du$:

$$\int p(x)\,dx = C\int\sqrt{1-x^2}\,dx = C\int\sqrt{1-\sin^2 u}\cos u\,du = C\int\cos^2 u\,du$$
$$= C\left(\frac{1}{2}u + \frac{1}{2}\cos u\sin u\right) + K$$

Since $x = \sin u$, we construct the following right triangle:



and we see that $\cos u = \sqrt{1-x^2}$, so that

$$\int p(x)\,dx = \frac{1}{2}C\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-1}^{1} p(x)\,dx = \frac{1}{2}C\left(\sin^{-1}x + x\sqrt{1-x^2}\right)\Big|_{-1}^{1} = \frac{1}{2}C(\sin^{-1}1 - \sin^{-1}(-1)) = \frac{\pi}{2}C$$

so that $C = \frac{2}{\pi}$ and $p(x) = \frac{2}{\pi}\sqrt{1-x^2}$. Then using the indefinite integral above,

$$P\left(-\frac{1}{2} \le X \le 1\right) = \int_{-1/2}^{1}\frac{2}{\pi}\sqrt{1-x^2}\,dx = \frac{1}{\pi}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)\Big|_{-1/2}^{1}$$
$$= \frac{1}{\pi}\left(\sin^{-1}1 + 0 - \sin^{-1}\left(-\frac{1}{2}\right) - \frac{-1}{2}\sqrt{1-\frac{1}{4}}\right)$$
$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{6} + \frac{\sqrt{3}}{4}\right) = \frac{2}{3} + \frac{\sqrt{3}}{4\pi} \approx 0.804$$

6. $p(x) = Ce^{-x}e^{-e^{-x}}$ on $(-\infty, \infty)$; $P(-4 \le X \le 4)$ This function, called the **Gumbel density**, is used to model extreme events such as floods and earthquakes.

SOLUTION Find the indefinite integral via the substitution $u = -e^{-x}$ so that $du = e^{-x}\,dx$; then

$$\int p(x)\,dx = C\int e^{-x}e^{-e^{-x}}\,dx = C\int e^u\,du = Ce^u = Ce^{-e^{-x}} + K$$

For $p(x)$ to be a probability density function, we must have

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = C\lim_{R\to\infty} e^{-e^{-x}}\Big|_{-R}^{R} = C\lim_{R\to\infty}\left(e^{-e^{-R}} - e^{-e^{R}}\right) = C$$

since $e^{-R} \to 0$ so that the first term approaches $e^0 = 1$, and $e^R \to \infty$ so that the second term approaches $e^{-\infty} = 0$. Thus $C = 1$ and $p(x) = e^{-x}e^{-e^{-x}}$. Then using the indefinite integral above,

$$P(-4 \le X \le 4) = e^{-e^{-4}} - e^{-e^{4}} \approx 0.982$$

7. Verify that $p(x) = 3x^{-4}$ is a probability density function on $[1, \infty)$ and calculate its mean value.

SOLUTION We have

$$\int_1^{\infty} 3x^{-4}\,dx = \lim_{R\to\infty}\left(-x^{-3}\right)\Big|_1^R = \lim_{R\to\infty}\left(-\frac{1}{R^3}\right) + 1 = 1$$

so that $p(x)$ is a probability density function on $[1, \infty)$. Its mean value is

$$\int_1^{\infty} x\cdot 3x^{-4}\,dx = \int_1^{\infty} 3x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_1^R = \lim_{R\to\infty}\left(-\frac{3}{2R^2}\right) + \frac{3}{2} = \frac{3}{2}$$

8. Show that the density function $p(x) = \dfrac{2}{\pi(x^2+1)}$ on $[0, \infty)$ has infinite mean.

SOLUTION To verify that $p(x)$ is a probability density function, note that

$$\int_0^{\infty} \frac{2}{\pi}\frac{1}{x^2+1}\,dx = \frac{2}{\pi}\lim_{R\to\infty}\tan^{-1}x\Big|_0^R = \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = 1$$

Its average value is (using the substitution $u = x^2 + 1$, $du = 2x\,dx$):

$$\frac{2}{\pi}\int_0^{\infty}\frac{x}{x^2+1}\,dx = \frac{1}{\pi}\int_0^{\infty}\frac{1}{u}\,du$$

The indefinite integral is $\ln u$, so the definite integral approaches $\infty - (-\infty) = \infty$, so this integral diverges and the mean is infinite.

9. Verify that $p(t) = \frac{1}{50}e^{-t/50}$ satisfies the condition
$\int_0^{\infty} p(t)\,dt = 1$.

SOLUTION Use the substitution $u = \frac{t}{50}$, so that $du = \frac{1}{50}\,dt$. Then

$$\int_0^{\infty} p(t)\,dt = \int_0^{\infty}\frac{1}{50}e^{-t/50}\,dt = \int_0^{\infty} e^{-u}\,du = \lim_{R\to\infty}(-e^{-u})\Big|_0^R = \lim_{R\to\infty} 1 - e^{-R} = 1$$

10. Verify that for all $r > 0$, the exponential density function $p(t) = \frac{1}{r}e^{-t/r}$ satisfies the condition $\int_0^{\infty} p(t)\,dt = 1$.

SOLUTION This is similar to the preceding problem. Use the substitution $u = \frac{t}{r}$, so that $du = \frac{1}{r}\,dt$. Then

$$\int_0^{\infty} p(t)\,dt = \int_0^{\infty}\frac{1}{r}e^{-t/r}\,dt = \int_0^{\infty} e^{-u}\,du = \lim_{R\to\infty}(e^{-u})\Big|_0^R = \lim_{R\to\infty} 1 - e^{-R} = 1$$

11. The life X (in hours) of a battery in constant use is a random variable with exponential density. What is the probability that the battery will last more than 12 hours if the average life is 8 hours?

SOLUTION If the average life is 8 hours, then the mean of the exponential distribution is 8, so that the distribution is

$$p(x) = \frac{1}{8}e^{-x/8}$$

The probability that the battery will last more than 12 hours is given by (using the substitution $u = x/8$, so that $du = 1/8\,dx$ and $x = 12$ corresponds to $u = 3/2$)

$$P(X \geq 12) = \int_{12}^{\infty} p(x)\,dx = \int_{12}^{\infty} \frac{1}{8}e^{-x/8}\,dx = \int_{3/2}^{\infty} e^{-u}\,du = \lim_{R\to\infty}(-e^{-u})\Big|_{3/2}^{R}$$
$$= e^{-3/2} - \lim_{R\to\infty} e^{-R} = e^{-3/2} \approx 0.223$$

12. The time between incoming phone calls at a call center is a random variable with exponential density. There is a 50% probability of waiting 20 seconds or more between calls. What is the average time between calls?

SOLUTION The distribution is exponential, so $p(x) = \frac{1}{r}e^{-x/r}$. Since there is a 50% probability of waiting 20 seconds or more between calls, this means that

$$\int_{20}^{\infty} \frac{1}{r}e^{-x/r}\,dx = \frac{1}{2}$$

But

$$\int_{20}^{\infty} \frac{1}{r}e^{-x/r}\,dx = e^{-x/r}\Big|_{20}^{\infty} = e^{-20/r}$$

Thus $\frac{1}{2} = e^{-20/r}$, so that $-\frac{20}{r} = \ln\frac{1}{2} = -\ln 2$; it follows that $r = \frac{20}{\ln 2}$, which is the mean value.

13. The distance r between the electron and the nucleus in a hydrogen atom (in its lowest energy state) is a random variable with probability density $p(r) = 4a_0^{-3}r^2e^{-2r/a_0}$ for $r \geq 0$, where a_0 is the Bohr radius (Figure 7). Calculate the probability P that the electron is within one Bohr radius of the nucleus. The value of a_0 is approximately 5.29×10^{-11} m, but this value is not needed to compute P.

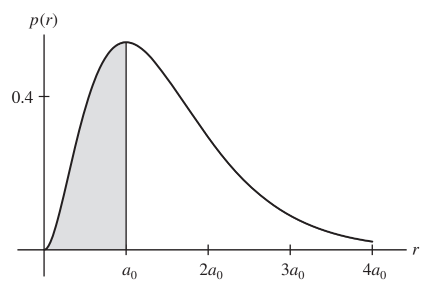


FIGURE 7 Probability density function $p(r) = 4a_0^{-3}r^2e^{-2r/a_0}$.

SOLUTION The probability P is the area of the shaded region in Figure 7. To calculate p, use the substitution $u = 2r/a_0$:

$$P = \int_0^{a_0} p(r)\,dr = \frac{4}{a_0^3}\int_0^{a_0} r^2e^{-2r/a_0}\,dr = \left(\frac{4}{a_0^3}\right)\left(\frac{a_0^3}{8}\right)\int_0^2 u^2e^{-u}\,du$$

The constant in front simplifies to $\frac{1}{2}$ and the formula in the margin gives us

$$P = \frac{1}{2}\int_0^2 u^2e^{-u}\,du = \frac{1}{2}\left(-(u^2+2u+2)e^{-u}\right)\Big|_0^2 = \frac{1}{2}\left(2 - 10e^{-2}\right) \approx 0.32$$

Thus, the electron within a distance a_0 of the nucleus with probability 0.32.

14. Show that the distance r between the electron and the nucleus in Exercise 13 has mean $\mu = 3a_0/2$.

SOLUTION The mean of the distribution is

$$\mu = \int_0^{\infty} rp(r)\,dr = \int_0^{\infty} r\cdot 4a_0^{-3}r^2e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\int_0^{\infty} r^3e^{-2r/a_0}\,dr$$

To calculate this integral, use as before the substitution $x = 2r/a_0$ to get

$$\mu = \frac{4}{a_0^3}\cdot\frac{a_0^3}{8}\cdot\frac{a_0}{2}\int_0^{\infty} x^3e^{-x}\,dx = \frac{a_0}{4}\int_0^{\infty} x^3e^{-x}\,dx$$

To calculate this integral, we use integration by parts, with $u = x^3$, $v' = e^{-x}$, so that $u' = 3x^2$ and $v = -e^{-x}$; then

$$\mu = \frac{a_0}{4}\left(-x^3e^{-x}\Big|_0^{\infty} + 3\int_0^{\infty} x^2e^{-x}\,dx\right)$$

The first term is evaluated as follows, using L'Hôpital's Rule multiple times:

$$-x^3e^{-x}\Big|_0^{\infty} = \lim_{R\to\infty}\left(-x^3e^{-x}\right)\Big|_0^R = \lim_{R\to\infty}\left(-\frac{R^3}{e^R}\right)$$
$$= \lim_{R\to\infty}\left(-\frac{3R^2}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6R}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6}{e^R}\right) = 0$$

The second term, by the marginal note in the previous problem, is

$$\int_0^{\infty} x^2e^{-x}\,dx = \lim_{R\to\infty}\left((-u^2+2u+2)e^{-u}\right)\Big|_0^R = \lim_{R\to\infty}\left(2-\frac{-R^2+2R+2}{e^R}\right) = 2$$

using L'Hôpital's Rule as in the previous formulas. Thus, finally,

$$\mu = \frac{a_0}{4}(0+3\cdot 2) = \frac{3}{2}a_0$$

In Exercises 15–21, $F(z)$ denotes the cumulative normal distribution function. Refer to a calculator, computer algebra system, or online resource to obtain values of $F(z)$.

15. Express the area of region A in Figure 8 in terms of $F(z)$ and compute its value.

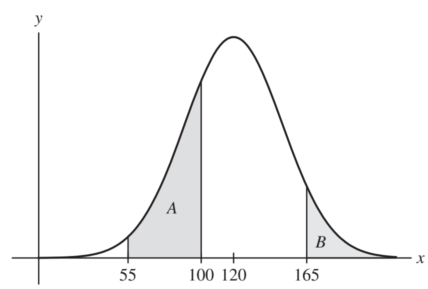


FIGURE 8 Normal density function with $\mu = 120$ and $\sigma = 30$.

SOLUTION The area of region A is $P(55 \le X \le 100)$. By Theorem 1, we have

$$P(55 \le X \le 100) = F\left(\frac{100-120}{30}\right) - F\left(\frac{55-120}{30}\right) = F\left(-\frac{2}{3}\right) - F\left(-\frac{13}{6}\right) \approx 0.237$$

16. Show that the area of region B in Figure 8 is equal to $1 - F(1.5)$ and compute its value. Verify numerically that this area is also equal to $F(-1.5)$ and explain why graphically.

SOLUTION The area of region B is $P(X \ge 165)$, and $P(X \ge 165) + P(X \le 165) = 1$. But by Theorem 1,

$$P(X \le 165) = F\left(\frac{165-120}{30}\right) = F(1.5)$$

so that

$$P(X \ge 165) = 1 - P(X \le 165) = 1 - F(1.5) \approx 0.0668$$

Using a computer algebra system, we also get $F(-1.5) \approx 0.0668$. Graphically, since the density function $p(x)$ is symmetric around $x = 120$, we see that the area to the right of $x = 165$ is equal to the area to the left of $x = 120 - (165 - 120) = 75$; this area is

$$F\left(\frac{75-120}{30}\right) = F\left(\frac{-45}{30}\right) = F(-1.5)$$

17. Assume X has a standard normal distribution ($\mu = 0, \sigma = 1$). Find:

(a) $P(X \le 1.2)$ **(b)** $P(X \ge -0.4)$

SOLUTION

(a) $P(X \le 1.2) = F(1.2) \approx 0.8849$

(b) $P(X \ge -0.4) = 1 - P(X \le -0.4) = 1 - F(-0.4) \approx 1 - 0.3446 \approx 0.6554$

18. Evaluate numerically: $\dfrac{1}{3\sqrt{2\pi}}\displaystyle\int_{14.5}^{\infty} e^{-(z-10)^2/18}\,dz$.

SOLUTION This is the area to the right of 14.5 under the cumulative distribution function for a normal distribution with $\mu = 10$ and $\sigma = 3$. In terms of the standard normal cumulative distribution function $F(z)$, this is

$$P(X \ge 14.5) = 1 - P(X \le 14.5) = 1 - F\left(\frac{14.5-10}{3}\right) = 1 - F(1.5) \approx 0.0668$$

19. Use a graph to show that $F(-z) = 1 - F(z)$ for all z. Then show that if $p(x)$ is a normal density function with mean μ and standard deviation σ, then for all $r \ge 0$,

$$P(\mu - r\sigma \le X \le \mu + r\sigma) = 2F(r) - 1$$

SOLUTION Consider the graph of the standard normal density function in Figure 5. This graph is symmetric around the y-axis, so that the area under the curve from z to ∞, which is $1 - F(z)$, is equal to the area under the curve from $-\infty$ to $-z$, which is $F(-z)$. Thus $1 - F(z) = F(-z)$. Now, if $p(x)$ is a normal density function with mean μ and standard deviation σ, then for $r \ge 0$ (so that the range $\mu - r\sigma \le X \le \mu + r\sigma$ is nonempty),

$$\begin{aligned} P(\mu - r\sigma \le X \le \mu + r\sigma) &= F\left(\frac{\mu + r\sigma - \mu}{\sigma}\right) - F\left(\frac{\mu - r\sigma - \mu}{\sigma}\right) \\ &= F(r) - F(-r) = F(r) - (1 - F(r)) = 2F(r) - 1 \end{aligned}$$

20. The average September rainfall in Erie, Pennsylvania, is a random variable X with mean $\mu = 102$ mm. Assume that the amount of rainfall is normally distributed with standard deviation $\sigma = 48$.

(a) Express $P(128 \le X \le 150)$ in terms of $F(z)$ and compute its value numerically.

(b) Let P be the probability that September rainfall will be at least 120 mm. Express P as an integral of an appropriate density function and compute its value numerically.

SOLUTION

(a)

$$P(128 \le X \le 150) = F\left(\frac{150-102}{48}\right) - F\left(\frac{128-102}{48}\right) = F(1) - F\left(\frac{13}{24}\right) \approx 0.135$$

(b) The cumulative density function associated with X is

$$f(z) = \frac{1}{48\sqrt{2\pi}}\int_{-\infty}^{z} e^{-(x-102)^2/(2\cdot 48^2)}\,dx$$

To compute the value numerically, we use the standard normal cumulative distribution $F(z)$. Recall that $P(X \ge 120) = 1 - P(X \le 120)$, and that

$$P(X \le 120) = F\left(\frac{120-102}{48}\right) = F\left(\frac{3}{8}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{3/8} e^{-x^2/2}\,dx \approx 0.646$$

so that $P(X \ge 120) \approx 1 - 0.646 \approx 0.354$.

21. A bottling company produces bottles of fruit juice that are filled, on average, with 32 ounces of juice. Due to random fluctuations in the machinery, the actual volume of juice is normally distributed with a standard deviation of 0.4 ounce. Let P be the probability of a bottle having less than 31 ounces. Express P as an integral of an appropriate density function and compute its value numerically.

SOLUTION The associated cumulative distribution function is

$$f(z) = \frac{1}{0.4\sqrt{2\pi}}\int_{-\infty}^{z} e^{-(x-32)^2/(2\cdot 0.4^2)}\,dx$$

To compute the value numerically, we use the standard normal cumulative distribution function $F(z)$:

$$P(X \le 31) = F\left(\frac{31-32}{0.4}\right) = F\left(-\frac{5}{2}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-5/2} e^{-x^2/2}\,dx \approx 0.0062$$

22. According to **Maxwell's Distribution Law**, in a gas of molecular mass m, the speed v of a molecule in a gas at temperature T (kelvins) is a random variable with density

$$p(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/(2kT)} \qquad (v \ge 0)$$

where k is Boltzmann's constant. Show that the average molecular speed is equal to $(8kT/\pi m)^{1/2}$. The average speed of oxygen molecules at room temperature is around 450 m/s.

SOLUTION The average speed $\bar{v}$ is given by

$$\bar{v} = \int_0^\infty vp(v)\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^3 e^{-mv^2/2kT}\,dv.$$

Let $\alpha = -m/2kT$. We'll first compute the indefinite integral

$$\int v^3 e^{\alpha v^2}\,dv.$$

Using Integration by Parts, let $u = v^2$, $v' = ve^{\alpha v^2}$. Then $u' = 2v$ and $v = \frac{1}{2\alpha}e^{\alpha v^2}$. This gives us

$$\int v^3 e^{\alpha v^2}\,dv = \frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{\alpha}\int ve^{\alpha v^2}\,dv.$$

To compute the remaining integral, let $w = \alpha v^2$, $dw = 2\alpha v\,dv$. The result is

$$\int v^3 e^{\alpha v^2}\,dv = \frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{2\alpha^2}e^{\alpha v^2} + C.$$

Thus,

$$\int_0^R vp(v)\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[\frac{e^{\alpha v^2}}{2\alpha}\left(v^2 - \frac{1}{\alpha}\right)\right]_0^R = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right],$$

and

$$\bar{v} = \lim_{R\to\infty} 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right] = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left[\lim_{R\to\infty} e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\right].$$

Use L'Hôpital's Rule to compute the limit, recalling that $\alpha = -m/2kT < 0$:

$$\lim_{R\to\infty} e^{\alpha R^2}\left(R^2 - \frac{1}{\alpha}\right) = \lim_{R\to\infty}\frac{R^2 - \frac{1}{\alpha}}{e^{-\alpha R^2}} = \lim_{R\to\infty}\frac{2R}{-2\alpha Re^{-\alpha R^2}} = \lim_{R\to\infty}\frac{-1}{\alpha e^{-\alpha R^2}} = 0.$$

Thus

$$\bar{v} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{1}{2\alpha}\left(0 + \frac{1}{\alpha}\right) = \frac{2\pi}{\alpha^2}\left(\frac{m}{2\pi kT}\right)^{3/2} = 2\pi\left(-\frac{2kT}{m}\right)^2\left(\frac{m}{2\pi kT}\right)\sqrt{\frac{m}{2\pi kT}}$$
$$= \frac{4kT}{m}\sqrt{\frac{m}{2\pi kT}} = \sqrt{\frac{8kT}{\pi m}}.$$

In Exercises 23–26, calculate μ and σ, where σ is the ***standard deviation****, defined by*

$$\sigma^2 = \int_{-\infty}^\infty (x-\mu)^2\,p(x)\,dx$$

The smaller the value of σ, the more tightly clustered are the values of the random variable X about the mean μ.

23. $p(x) = \dfrac{5}{2x^{7/2}}$ on $[1,\infty)$

SOLUTION The mean is

$$\int_1^\infty xp(x)\,dx = \int_1^\infty \frac{5}{2}x^{-5/2}\,dx = -\frac{5}{3}x^{-3/2}\Big|_1^\infty = \frac{5}{3}$$

and

$$\sigma^2 = \int_1^\infty (x-\mu)^2 p(x)\,dx = \int_1^\infty (x^2 - 2\mu x + \mu^2)\frac{5}{2}x^{-7/2}\,dx$$
$$= \frac{5}{2}\int_1^\infty x^{-3/2} - 2\mu x^{-5/2} + \mu^2 x^{-7/2}\,dx = \frac{5}{2}\left(-2x^{-1/2} + \frac{4}{3}\mu x^{-3/2} - \frac{2}{5}\mu^2 x^{-5/2}\right)\Big|_1^\infty$$
$$= \frac{5}{2}\left(2 - \frac{4}{3}\mu + \frac{2}{5}\mu^2\right) = \frac{5}{2}\left(2 - \frac{4}{3}\cdot\frac{5}{3} + \frac{2}{5}\cdot\frac{25}{9}\right) = \frac{20}{9}$$

24. $p(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$ on $(-1, 1)$

SOLUTION Use the substituion $u = 1 - x^2$ so that $du = -2x\,dx$. The mean is

$$\mu = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = -\frac{1}{2\pi}\int_{x=-1}^{1} \frac{-2x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2\pi}\int_{x=-1}^{1} \frac{1}{\sqrt{u}}\,du$$
$$= -\frac{1}{\pi}\sqrt{u}\bigg|_{x=-1}^{x=1} = -\frac{1}{\pi}\sqrt{1-x^2}\bigg|_{-1}^{1} = 0$$

To compute the standard deviation, use the substitution $x = \sin u$, $dx = \cos u\,du$; then $x = -1$ corresponds to $u = -\pi/2$ and $x = 1$ to $u = \pi/2$:

$$\sigma^2 = \int_{-1}^{1} (x-\mu)^2 p(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin^2 u}{\sqrt{1-\sin^2 u}}\cos u\,du$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin^2 u}{\cos u}\cos u\,du = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2 u\,du = \frac{1}{2\pi}(u - \cos u \sin u)\bigg|_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{1}{2}$$

25. $p(x) = \dfrac{1}{3}e^{-x/3}$ on $[0, \infty)$

SOLUTION This is an exponential density function with mean $\mu = 3$. The standard deviation is

$$\sigma^2 = \frac{1}{3}\int_0^\infty (x-3)^2 e^{-x/3}\,dx = \frac{1}{3}\int_0^\infty \left(x^2e^{-x/3} - 6xe^{-x/3} + 9e^{-x/3}\right)dx$$
$$= \frac{1}{3}\int_0^\infty x^2e^{-x/3}\,dx - 2\int_0^\infty xe^{-x/3}\,dx + 3\int_0^\infty e^{-x/3}\,dx$$

We tackle the third integral first:

$$\int_0^\infty e^{-x/3}\,dx = -3e^{-x/3}\bigg|_0^\infty = 3$$

For the second integral, use integration by parts with $u = x$, $v' = e^{-x/3}$ so that $u' = 1$ and $v = -3e^{-x/3}$. Then

$$\int_0^\infty xe^{-x/3}\,dx = -3xe^{-x/3}\bigg|_0^\infty + 3\int_0^\infty e^{-x/3}\,dx = 0 + 3\cdot 3 = 9$$

Finally, the first integral is solved using integration by parts with $u = x^2$, $v' = e^{-x/3}$ so that $u' = 2x$ and $v = -3e^{-x/3}$; then

$$\int_0^\infty x^2e^{-x/3}\,dx = -3x^2e^{-x/3}\bigg|_0^\infty + 6\int_0^\infty xe^{-x/3}\,dx = 0 + 6\cdot 9 = 54$$

and, finally,

$$\sigma^2 = \frac{1}{3}\int_0^\infty x^2e^{-x/3}\,dx - 2\int_0^\infty xe^{-x/3}\,dx + 3\int_0^\infty e^{-x/3}\,dx$$
$$= \frac{1}{3}\cdot 54 - 2\cdot 9 + 3\cdot 3 = 9$$

26. $p(x) = \dfrac{1}{r}e^{-x/r}$ on $[0, \infty)$, where $r > 0$

SOLUTION This is similar to the previous problem. We have an exponential density function with mean $\mu = r$. The standard deviation is

$$\sigma^2 = \frac{1}{r}\int_0^\infty (x-r)^2 e^{-x/r}\,dx = \frac{1}{r}\int_0^\infty \left(x^2e^{-x/r} - 2rxe^{-x/r} + r^2e^{-x/r}\right)dx$$
$$= \frac{1}{r}\int_0^\infty x^2e^{-x/r}\,dx - 2\int_0^\infty xe^{-x/r}\,dx + r\int_0^\infty e^{-x/r}\,dx$$

We tackle the third integral first:

$$\int_0^\infty e^{-x/r}\,dx = -re^{-x/r}\Big|_0^\infty = r$$

For the second integral, use integration by parts with $u = x$, $v' = e^{-x/r}$ so that $u' = 1$ and $v = -re^{-x/r}$. Then

$$\int_0^\infty xe^{-x/r}\,dx = -rxe^{-x/r}\Big|_0^\infty + r\int_0^\infty e^{-x/r}\,dx = 0 + r\cdot r = r^2$$

Finally, the first integral is solved using integration by parts with $u = x^2$, $v' = e^{-x/r}$ so that $u' = 2x$ and $v = -re^{-x/r}$; then

$$\int_0^\infty x^2e^{-x/r}\,dx = -rx^2e^{-x/r}\Big|_0^\infty + 2r\int_0^\infty xe^{-x/r}\,dx = 0 + 2r\cdot r^2 = 2r^3$$

and, finally,

$$\begin{aligned}\sigma^2 &= \frac{1}{r}\int_0^\infty x^2e^{-x/r}\,dx - 2\int_0^\infty xe^{-x/3}\,dx + r\int_0^\infty e^{-x/3}\,dx\\ &= \frac{1}{r}\cdot 2r^3 - 2\cdot r^2 + r\cdot r = r^2\end{aligned}$$

Further Insights and Challenges

27. The time to decay of an atom in a radioactive substance is a random variable X. The law of radioactive decay states that if N atoms are present at time $t = 0$, then $Nf(t)$ atoms will be present at time t, where $f(t) = e^{-kt}$ ($k > 0$ is the decay constant). Explain the following statements:

(a) The fraction of atoms that decay in a small time interval $[t, t + \Delta t]$ is approximately $-f'(t)\Delta t$.

(b) The probability density function of X is $-f'(t)$.

(c) The average time to decay is $1/k$.

SOLUTION

(a) The number of atoms that decay in the interval $[t, t + \Delta t]$ is just $f(t) - f(t + \Delta t)$; the statement simply says that $f(t) - f(t + \Delta t) \approx -f'(t)\Delta t$, which is the same as saying that

$$f'(t) \approx \frac{f(t) - f(t + \Delta t)}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

which is true by the definition of the derivative. Intuitively, since $f'(t)$ is the instantaneous rate of decay, we would expect that over a short interval, the number of atoms decaying is proportional to both $f'(t)$ and the size of the interval.

(b) The probability density function is defined by the property in (a): the probability that X lies in a small interval $[t, t + \Delta t]$ is approximately $p(t)\Delta t$, so that $p(t) = -f'(t)$.

(c) The average time to decay is the mean of the distribution, which is

$$\mu = \int_0^\infty t\cdot(-f'(t))\,dt = -\int_0^\infty tf'(t)\,dt$$

We compute this integral using integration by parts, with $u = t$, $v' = f'(t)$. Then $u' = 1$, $v = f(t)$, and

$$\mu = -\int_0^\infty tf'(t)\,dt = -tf(t)\Big|_0^\infty + \int_0^\infty f(t)\,dt.$$

Since $f(t) = e^{-kt}$, we have

$$-tf(t)\Big|_0^\infty = \lim_{R\to\infty} -te^{-kt}\Big|_0^R = \lim_{R\to\infty} -Re^{-Rt} + 0 = \lim_{R\to\infty}\frac{-R}{e^{Rt}} = \lim_{R\to\infty}\frac{-1}{Re^{Rt}} = 0.$$

Here we used L'Hôpital's Rule to compute the limit. Thus

$$\mu = \int_0^\infty f(t)\,dt = \int_0^\infty e^{-kt}\,dt.$$

Now,

$$\int_0^R e^{-kt}\,dt = -\frac{1}{k}e^{-kt}\Big|_0^R = -\frac{1}{k}\left(e^{-kR} - 1\right) = \frac{1}{k}\left(1 - e^{-kR}\right),$$

so

$$\mu = \lim_{R\to\infty} \frac{1}{k}\left(1 - e^{-kR}\right) = \frac{1}{k}(1-0) = \frac{1}{k}.$$

Because k has units of $(\text{time})^{-1}$, μ does in fact have the appropriate units of time.

28. The half-life of radon-222, is 3.825 days. Use Exercise 27 to compute:
(a) The average time to decay of a radon-222 atom.
(b) The probability that a given atom will decay in the next 24 hours.

SOLUTION
(a) The average decay time is just the mean, μ; to determine it, we must determine the decay constant k, given the half-life of 3.825 days. Recall that

$$k = \frac{\ln 2}{t_n}$$

where t_n is the half-life. Thus,

$$\mu = \frac{1}{k} = \frac{t_n}{\ln 2} = \frac{3.825}{\ln 2} \approx 5.518 \text{ days.}$$

(b) The probability that a particular atom will decay in the next 24 hours is the area under the probability density function between $t = 0$ and $t = 1$ (note that t is measured in days). Since $f(t) = e^{-kt}$, the probability density function is $-ke^{-kt}$; from part (a), $k \approx 0.1812$, so the required probability is

$$\int_0^1 (-f'(t))\,dt = f(0) - f(1) = 1 - e^{-0.1812} \approx 0.1657$$

7.8 Numerical Integration

Preliminary Questions

1. What are T_1 and T_2 for a function on $[0, 2]$ such that $f(0) = 3$, $f(1) = 4$, and $f(2) = 3$?

SOLUTION Using the given function values

$$T_1 = \frac{1}{2}(2)(3+3) = 6 \quad \text{and} \quad T_2 = \frac{1}{2}(1)(3+8+3) = 7.$$

2. For which graph in Figure 16 will T_N *overestimate* the integral? What about M_N?

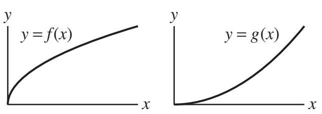


FIGURE 16

SOLUTION T_N overestimates the value of the integral when the integrand is concave up; thus, T_N will overestimate the integral of $y = g(x)$. On the other hand, M_N overestimates the value of the integral when the integrand is concave down; thus, M_N will overestimate the integral of $y = f(x)$.

3. How large is the error when the Trapezoidal Rule is applied to a linear function? Explain graphically.

SOLUTION The Trapezoidal Rule integrates linear functions exactly, so the error will be zero.

4. What is the maximum possible error if T_4 is used to approximate

$$\int_0^3 f(x)\,dx$$

where $|f''(x)| \le 2$ for all x.

SOLUTION The maximum possible error in T_4 is

$$\max|f''(x)|\frac{(b-a)^3}{12n^2} \le \frac{2(3-0)^3}{12(4)^2} = \frac{9}{32}.$$

5. What are the two graphical interpretations of the Midpoint Rule?

SOLUTION The two graphical interpretations of the Midpoint Rule are the sum of the areas of the midpoint rectangles and the sum of the areas of the tangential trapezoids.

Exercises

In Exercises 1–12, calculate T_N and M_N for the value of N indicated.

1. $\int_0^2 x^2\,dx, \quad N = 4$

SOLUTION Let $f(x) = x^2$. We divide $[0, 2]$ into 4 subintervals of width

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

with endpoints 0, 0.5, 1, 1.5, 2, and midpoints 0.25, 0.75, 1.25, 1.75. With this data, we get

$$T_4 = \frac{1}{2}\cdot\frac{1}{2}\left(0^2 + 2(0.5)^2 + 2(1)^2 + 2(1.5)^2 + 2^2\right) = 2.75; \text{ and}$$

$$M_4 = \frac{1}{2}\left(0.25^2 + 0.75^2 + 1.25^2 + 1.75^2\right) = 2.625.$$

2. $\int_0^4 \sqrt{x}\,dx, \quad N = 4$

SOLUTION Let $f(x) = \sqrt{x}$. We divide $[0, 4]$ into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1$$

with endpoints 0, 1, 2, 3, 4, and midpoints 0.5, 1.5, 2.5, 3.5. With this data, we get

$$T_4 = \frac{1}{2}\cdot 1\cdot\left(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}\right) \approx 5.14626; \text{ and}$$

$$M_4 = 1\cdot\left(\sqrt{0.5} + \sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5}\right) \approx 5.38382.$$

3. $\int_1^4 x^3\,dx, \quad N = 6$

SOLUTION Let $f(x) = x^3$. We divide $[1, 4]$ into 6 subintervals of width

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

with endpoints 1, 1.5, 2, 2.5, 3, 3.5, 4, and midpoints 1.25, 1.75, 2.25, 2.75, 3.25, 3.75. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(1^3 + 2(1.5)^3 + 2(2)^3 + 2(2.5)^3 + 2(3)^3 + 2(3.5)^3 + 4^3\right) = 64.6875; \text{ and}$$

$$M_6 = \frac{1}{2}\left(1.25^3 + 1.75^3 + 2.25^3 + 2.75^3 + 3.25^3 + 3.75^3\right) = 63.28125.$$

4. $\int_1^2 \sqrt{x^4+1}\,dx, \quad N = 5$

SOLUTION We divide $[1, 2]$ into 5 subintervals of width

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

with endpoints 1, 1.2, 1.4, 1.6, 1.8, 2, and midpoints 1.1, 1.3, 1.5, 1.7, 1.9. With this data, we have

$$T_5 = \frac{1}{2}\cdot\frac{1}{5}\left(\sqrt{1^4+1} + 2\sqrt{1.2^4+1} + 2\sqrt{1.4^4+1} + 2\sqrt{1.6^4+1} + 2\sqrt{1.8^4+1} + \sqrt{2^2+1}\right) \approx 2.57228$$

$$M_5 = \frac{1}{5}\left(\sqrt{1.1^4+1} + \sqrt{1.3^4+1} + \sqrt{1.5^4+1} + \sqrt{1.7^4+1} + \sqrt{1.9^4+1}\right) \approx 2.55994$$

5. $\int_1^4 \frac{dx}{x}, \quad N = 6$

SOLUTION Let $f(x) = 1/x$. We divide $[1, 4]$ into 6 subintervals of width

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

with endpoints 1, 1.5, 2, 2.5, 3, 3.5, 4, and midpoints 1.25, 1.75, 2.25, 2.75, 3.25, 3.75. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{1} + \frac{2}{1.5} + \frac{2}{2} + \frac{2}{2.5} + \frac{2}{3} + \frac{2}{3.5} + \frac{1}{4}\right) \approx 1.40536; \text{ and}$$

$$M_6 = \frac{1}{2}\left(\frac{1}{1.25} + \frac{1}{1.75} + \frac{1}{2.25} + \frac{1}{2.75} + \frac{1}{3.25} + \frac{1}{3.75}\right) \approx 1.37693.$$

6. $\displaystyle\int_{-2}^{-1} \frac{dx}{x}, \quad N = 5$

SOLUTION Let $f(x) = 1/x$. We divide $[-2, -1]$ into 5 subintervals of width

$$\Delta x = \frac{-1 - (-2)}{5} = \frac{1}{5} = 0.2$$

with endpoints $-2, -1.8, -1.6, -1.4, -1.2, -1$, and midpoints $-1.9, -1.7, -1.5, -1.3, -1.1$. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\frac{1}{-2} + \frac{2}{-1.8} + \frac{2}{-1.6} + \frac{2}{-1.4} + \frac{2}{-1.2} + \frac{1}{-1}\right) \approx -0.695635; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\frac{1}{-1.9} + \frac{1}{-1.7} + \frac{1}{-1.5} + \frac{1}{-1.3} + \frac{1}{-1.1}\right) \approx -0.691908.$$

7. $\displaystyle\int_{0}^{\pi/2} \sqrt{\sin x}\, dx, \quad N = 6$

SOLUTION Let $f(x) = \sqrt{\sin x}$. We divide $[0, \pi/2]$ into 6 subintervals of width

$$\Delta x = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

with endpoints

$$0, \frac{\pi}{12}, \frac{2\pi}{12}, \ldots, \frac{6\pi}{12} = \frac{\pi}{2},$$

and midpoints

$$\frac{\pi}{24}, \frac{3\pi}{24}, \ldots, \frac{11\pi}{24}.$$

With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{\pi}{12}\right)\left(\sqrt{\sin(0)} + 2\sqrt{\sin(\pi/12)} + \cdots + \sqrt{\sin(6\pi/12)}\right) \approx 1.17029; \text{ and}$$

$$M_6 = \frac{\pi}{12}\left(\sqrt{\sin(\pi/24)} + \sqrt{\sin(3\pi/24)} + \cdots + \sqrt{\sin(11\pi/24)}\right) \approx 1.20630.$$

8. $\displaystyle\int_{0}^{\pi/4} \sec x\, dx, \quad N = 6$

SOLUTION Let $f(x) = \sec x$. We divide $[0, \pi/4]$ into 6 subintervals of width

$$\Delta x = \frac{\frac{\pi}{4} - 0}{6} = \frac{\pi}{24}$$

with endpoints

$$0, \frac{\pi}{24}, \frac{2\pi}{24}, \ldots, \frac{6\pi}{24} = \frac{\pi}{4},$$

and midpoints

$$\frac{\pi}{48}, \frac{3\pi}{48}, \ldots, \frac{11\pi}{48}.$$

With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{\pi}{24}\right)\left(\sec(0) + 2\sec(\pi/24) + 2\sec(2\pi/24) + \cdots + \sec(6\pi/24)\right) \approx 0.883387; \text{ and}$$

$$M_6 = \frac{\pi}{24}\left(\sec(\pi/48) + \sec(3\pi/48) + \sec(5\pi/48) + \cdots + \sec(11\pi/48)\right) \approx 0.880369.$$

9. $\int_1^2 \ln x\,dx, \quad N = 5$

SOLUTION Let $f(x) = \ln x$. We divide $[1, 2]$ into 5 subintervals of width

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

with endpoints 1, 1.2, 1.4, 1.6, 1.8, 2, and midpoints 1.1, 1.3, 1.5, 1.7, 1.9. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\ln 1 + 2\ln 1.2 + 2\ln 1.4 + 2\ln 1.6 + 2\ln 1.8 + \ln 2\right) \approx 0.384632; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\ln 1.1 + \ln 1.3 + \ln 1.5 + \ln 1.7 + \ln 1.9\right) \approx 0.387124.$$

10. $\int_2^3 \frac{dx}{\ln x}, \quad N = 5$

SOLUTION Let $f(x) = 1/\ln x$. We divide $[2, 3]$ into 5 subintervals of width

$$\Delta x = \frac{3-2}{5} = \frac{1}{5} = 0.2$$

with endpoints 2, 2.2, 2.4, 2.6, 2.8, 3, and midpoints 2.1, 2.3, 2.5, 2.7, 2.9. With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(\frac{1}{\ln 2} + \frac{2}{\ln 2.2} + \frac{2}{\ln 2.4} + \frac{2}{\ln 2.6} + \frac{2}{\ln 2.8} + \frac{1}{\ln 3}\right) \approx 1.12096; \text{ and}$$

$$M_5 = \frac{1}{5}\left(\frac{1}{\ln 2.1} + \frac{1}{\ln 2.3} + \frac{1}{\ln 2.5} + \frac{1}{\ln 2.7} + \frac{1}{\ln 2.9}\right) \approx 1.11716.$$

11. $\int_0^1 e^{-x^2}\,dx, \quad N = 5$

SOLUTION Let $f(x) = e^{-x^2}$. We divide $[0, 1]$ into 5 subintervals of width

$$\Delta x = \frac{1-0}{5} = \frac{1}{5} = 0.2$$

with endpoints

$$0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$$

and midpoints

$$\frac{1}{10}, \frac{3}{10}, \frac{5}{10}, \frac{7}{10}, \frac{9}{10}.$$

With this data, we get

$$T_5 = \frac{1}{2}\left(\frac{1}{5}\right)\left(e^{-0^2} + 2e^{-(1/5)^2} + 2e^{-(2/5)^2} + 2e^{-(3/5)^2} + 2e^{-(4/5)^2} + e^{-1^2}\right) \approx 0.74437; \text{ and}$$

$$M_5 = \frac{1}{5}\left(e^{-(1/10)^2} + e^{-(3/10)^2} + e^{-(5/10)^2} + e^{-(7/10)^2} + e^{-(9/10)^2}\right) \approx 0.74805.$$

12. $\int_{-2}^1 e^{x^2}\,dx, \quad N = 6$

SOLUTION Let $f(x) = e^{x^2}$. We divide $[-2, 1]$ into 6 subintervals of width

$$\Delta x = \frac{1-(-2)}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

with endpoints $-2, -1.5, -1, -0.5, 0, 0.5, 1$, and midpoints $-1.75, -1.25, -0.75, -0.25, 0.25, 0.75$. With this data, we get

$$T_6 = \frac{1}{2}\left(\frac{1}{2}\right)\left(e^{(-2)^2} + 2e^{(-1.5)^2} + 2e^{(-1)^2} + 2e^{(-0.5)^2} + 2e^{0^2} + 2e^{(0.5)^2} + e^{1^2}\right) \approx 22.2161; \text{ and}$$

$$M_6 = \frac{1}{2}\left(e^{(-1.75)^2} + e^{(-1.25)^2} + e^{(-0.75)^2} + e^{(-0.25)^2} + e^{(0.25)^2} + e^{(0.75)^2}\right) \approx 15.8954.$$

In Exercises 13–22, calculate S_N given by Simpson's Rule for the value of N indicated.

13. $\int_0^4 \sqrt{x}\,dx, \quad N = 4$

SOLUTION Let $f(x) = \sqrt{x}$. We divide $[0, 4]$ into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1$$

with endpoints 0, 1, 2, 3, 4. With this data, we get

$$S_4 = \frac{1}{3}(1)\left(\sqrt{0} + 4\sqrt{1} + 2\sqrt{2} + 4\sqrt{3} + \sqrt{4}\right) \approx 5.25221.$$

14. $\int_3^5 (9 - x^2)\,dx, \quad N = 4$

SOLUTION Let $f(x) = 9 - x^2$. We divide $[3, 5]$ into 4 subintervals of length

$$\Delta x = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

with endpoints 3, 3.5, 4, 4.5, 5. With this data, we get

$$S_4 = \frac{1}{3}\left(\frac{1}{2}\right)\left[(9-3^2) + 4(9-3.5^2) + 2(9-4^2) + 4(9-4.5^2) + (9-5^2)\right] \approx -14.6667.$$

15. $\int_0^3 \frac{dx}{x^4+1}, \quad N = 6$

SOLUTION Let $f(x) = 1/(x^4 + 1)$. We divide $[0, 3]$ into 6 subintervals of length

$$\Delta x = \frac{3-0}{6} = \frac{1}{2} = 0.5$$

with endpoints 0, 0.5, 1, 1.5, 2, 2.5, 3. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{2}\right)\left[\frac{1}{0^4+1} + \frac{4}{0.5^4+1} + \frac{2}{1^4+1} + \frac{4}{1.5^4+1} + \frac{2}{2^4+1} + \frac{4}{2.5^4+1} + \frac{1}{3^4+1}\right] \approx 1.10903.$$

16. $\int_0^1 \cos(x^2)\,dx, \quad N = 6$

SOLUTION Let $f(x) = \cos(x^2)$. We divide $[0, 1]$ into 6 subintervals of length

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

with endpoints $0, \frac{1}{6}, \frac{2}{6}, \dots, \frac{6}{6} = 1$. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{6}\right)\left[\cos\left(0^2\right) + 4\cos\left(\left(\frac{1}{6}\right)^2\right) + 2\cos\left(\left(\frac{2}{6}\right)^2\right) + \cdots + 4\cos\left(\left(\frac{5}{6}\right)^2\right) + \cos\left(1^2\right)\right] \approx 0.904523.$$

17. $\int_0^1 e^{-x^2}\,dx, \quad N = 4$

SOLUTION Let $f(x) = e^{-x^2}$. We divide $[0, 1]$ into 4 subintervals of length

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

with endpoints $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1$. With this data, we get

$$S_4 = \frac{1}{3}\left(\frac{1}{4}\right)\left[e^{-0^2} + 4e^{-(1/4)^2} + 2e^{-(2/4)^2} + 4e^{-(3/4)^2} + e^{-(1)^2}\right] \approx 0.746855.$$

18. $\int_1^2 e^{-x}\,dx, \quad N = 6$

SOLUTION Let $f(x) = e^{-x}$. We divide $[1, 2]$ into 6 subintervals of width

$$\Delta x = \frac{2-1}{6} = \frac{1}{6}$$

with endpoints $1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \ldots, \frac{12}{6} = 2$. With this data, we get

$$S_6 = \frac{1}{3}\left(\frac{1}{6}\right)\left[e^{-1} + 4e^{-7/6} + 2e^{-8/6} + 4e^{-9/6} + 2e^{-10/6} + 4e^{-11/6} + e^{-12/6}\right] \approx 0.232545.$$

19. $\int_1^4 \ln x\,dx, \quad N = 8$

SOLUTION Let $f(x) = \ln x$. We divide [1, 4] into 8 subintervals of length

$$\Delta x = \frac{4-1}{8} = \frac{3}{8} = 0.375$$

with endpoints 1, 1.375, 1.75, 2.125, 2.5, 2.875, 3.25, 3.625, 4. With this data, we get

$$S_8 = \frac{1}{3}\left(\frac{3}{8}\right)\left[\ln 1 + 4\ln(1.375) + 2\ln(1.75) + \cdots + 4\ln(3.625) + \ln 4\right] \approx 2.54499.$$

20. $\int_2^4 \sqrt{x^4+1}\,dx, \quad N = 8$

SOLUTION Let $f(x) = \sqrt{x^4+1}$. We divide [2, 4] into 8 subintervals of width

$$\Delta x = \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$$

with endpoints 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5, 3.75, 4. With this data, we get

$$S_8 = \frac{1}{3}\left(\frac{1}{4}\right)\left[\sqrt{2^4+1} + 4\sqrt{(2.25)^4+1} + 2\sqrt{(2.5)^4+1} + \cdots + 4\sqrt{(3.75)^4+1} + \sqrt{4^4+1}\right] \approx 18.7909.$$

21. $\int_0^{\pi/4} \tan\theta\,d\theta, \quad N = 10$

SOLUTION Let $f(\theta) = \tan\theta$. We divide $[0, \frac{\pi}{4}]$ into 10 subintervals of width

$$\Delta\theta = \frac{\frac{\pi}{4} - 0}{10} = \frac{\pi}{40}$$

with endpoints $0, \frac{\pi}{40}, \frac{2\pi}{40}, \frac{3\pi}{40}, \ldots, \frac{10\pi}{40} = \frac{\pi}{4}$. With this data, we get

$$S_{10} = \frac{1}{3}\left(\frac{\pi}{40}\right)\left[\tan(0) + 4\tan\left(\frac{\pi}{40}\right) + 2\tan\left(\frac{2\pi}{40}\right) + \cdots + 4\tan\left(\frac{9\pi}{40}\right) + \tan\left(\frac{10\pi}{40}\right)\right] \approx 0.346576.$$

22. $\int_0^2 (x^2+1)^{-1/3}\,dx, \quad N = 10$

SOLUTION Let $f(x) = (x^2+1)^{-1/3}$. We divide [0, 2] into 10 subintervals of width

$$\Delta x = \frac{2-0}{10} = \frac{1}{5} = 0.2$$

with endpoints 0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2. With this data, we get

$$S_{10} = \frac{1}{3}\cdot\frac{1}{5}\Big[(0^2+1)^{-1/3} + 4(0.2^2+1)^{-1/3} + 2(0.4^2+1)^{-1/3} + \cdots + 4(1.8^2+1)^{-1/3} + (2^2+1)^{-1/3}\Big] \approx 1.598005$$

In Exercises 23–26, calculate the approximation to the volume of the solid obtained by rotating the graph around the given axis.

23. $y = \cos x; \quad [0, \frac{\pi}{2}]; \quad x\text{-axis}; \quad M_8$

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^{\pi/2} \pi r^2\,dx = \pi\int_0^{\pi/2} (\cos x)^2\,dx$$

which can be estimated as

$$\pi \int_0^{\pi/2} (\cos x)^2\, dx \approx \pi[M_8].$$

Let $f(x) = \cos^2 x$. We divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with midpoints

$$\frac{\pi}{32}, \frac{3\pi}{32}, \frac{5\pi}{32}, \dots, \frac{15\pi}{32}.$$

With this data, we get

$$V \approx \pi[M_8] = \pi\left[\Delta x(y_1 + y_2 + \cdots + y_8)\right] = \frac{\pi^2}{16}\left[\cos^2\left(\frac{\pi}{32}\right) + \cos^2\left(\frac{3\pi}{32}\right) + \cdots + \cos^2\left(\frac{15\pi}{32}\right)\right] \approx 2.46740.$$

24. $y = \cos x$; $\left[0, \frac{\pi}{2}\right]$; y-axis; S_8

SOLUTION Using the cylindrical shell method, the volume is given by

$$V = \int_0^{\pi/2} 2\pi r h\, dx = 2\pi \int_0^{\pi/2} x \cos x\, dx$$

where the radius of the cylinder is $r = x$ and the height is $h = \cos x$. This can be approximated as

$$V = 2\pi \int_0^{\pi/2} x \cos x\, dx \approx 2\pi\left[\, S_8 \,\right],$$

where $f(x) = x \cos x$. We divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with endpoints

$$0, \frac{\pi}{16}, \frac{2\pi}{16}, \dots, \frac{8\pi}{16}.$$

With this data, we get

$$V \approx 2\pi[S_8] = 2\pi\left[\frac{1}{3} \cdot \frac{\pi}{16}(y_0 + 4y_1 + 2y_2 + \cdots + 4y_7 + y_8)\right]$$

$$= \frac{\pi^2}{24}\left[0(\cos 0) + 4\frac{\pi}{16}\left(\cos\frac{\pi}{16}\right) + \cdots + \frac{8\pi}{16}\left(\cos\frac{8\pi}{16}\right)\right] \approx 3.58666.$$

25. $y = e^{-x^2}$; $[0, 1]$; x-axis; T_8

SOLUTION Using the disk method, the volume is given by

$$V = \int_0^1 \pi r^2\, dx = \pi \int_0^1 \left(e^{-x^2}\right)^2 dx = \pi \int_0^1 e^{-2x^2}\, dx.$$

We can use the approximation

$$V = \pi \int_0^1 e^{-2x^2}\, dx \approx \pi[T_8],$$

where $f(x) = e^{-2x^2}$. Divide $[0, 1]$ into 8 subintervals of length

$$\Delta x = \frac{1 - 0}{8} = \frac{1}{8},$$

with endpoints

$$0, \frac{1}{8}, \frac{2}{8}, \dots, 1.$$

With this data, we get

$$V \approx \pi[T_8] = \pi\left[\frac{1}{2} \cdot \frac{1}{8}\left(e^{-2(0^2)} + 2e^{-2(1/8)^2} + \cdots + 2e^{-2(7/8)^2} + e^{-2(1)^2}\right)\right] \approx 1.87691.$$

26. $y = e^{-x^2}$; $[0, 1]$; y-axis; S_8

SOLUTION Using the cylindrical shell method, the volume is given by

$$V = \int_0^1 2\pi r h\, dx = 2\pi \int_0^1 x e^{-x^2}\, dx$$

where $r = x$ and $h = e^{-x^2}$. We can use the approximation

$$V = 2\pi \int_0^1 x e^{-x^2}\, dx \approx 2\pi[S_8],$$

where $f(x) = xe^{-x^2}$. Divide $[0, 1]$ into 8 subintervals of length

$$\Delta x = \frac{1-0}{8} = \frac{1}{8},$$

with endpoints

$$0,\ \frac{1}{8},\ \frac{2}{8}, \ldots, 1.$$

With this data, we get

$$V \approx 2\pi[S_8] = 2\pi\left(\frac{1}{3}\right)\left(\frac{1}{8}\right)\left[(0)e^{-(0^2)} + 4\left(\frac{1}{8}\right)e^{-(1/8)^2} + \cdots + 4\left(\frac{7}{8}\right)e^{-(7/8)^2} + e^{-1^2}\right] \approx 1.98595.$$

27. An airplane's velocity is recorded at 5-min intervals during a 1-hour period with the following results, in miles per hour:

550,	575,	600,	580,	610,	640,	625,
595,	590,	620,	640,	640,	630	

Use Simpson's Rule to estimate the distance traveled during the hour.

SOLUTION The distance traveled is equal to the integral $\int_0^1 v(t)\, dt$, where t is in hours. Since 5 minutes is $1/12$ of an hour, we have $\Delta t = 1/12$. Simpson's Rule gives us

$$S_{12} = \frac{1}{3} \cdot \frac{1}{12}\Big[550 + 4 \cdot 575 + 2 \cdot 600 + 4 \cdot 580 + 2 \cdot 610 + \cdots + 4 \cdot 640 + 630\Big] \approx 608.611.$$

The distance traveled during the hour is approximately 608.6 miles.

28. Use Simpson's Rule to determine the average temperature in a museum over a 3-hour period, if the temperatures (in degrees Celsius), recorded at 15-min intervals, are

21,	21.3,	21.5,	21.8,	21.6,	21.2,	20.8,
20.6,	20.9,	21.2,	21.1,	21.3,	21.2	

SOLUTION If $T(t)$ represents the temperature at time t, then the average temperature T_{ave} from $t = 0$ to $t = 3$ hours is given by

$$T_{\text{ave}} = \frac{1}{3-0}\int_0^3 T(t)\, dt.$$

To use Simpson's Rule to approximate this, let $\Delta t = 1/4$ (15 minute intervals). Then we have

$$T_{\text{ave}} = \frac{1}{3}[S_{12}] = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{4}\Big[21 + 4 \cdot 21.3 + 2 \cdot 21.5 + \cdots + 4 \cdot 21.3 + 21.2\Big] \approx 21.2111.$$

The average temperature is approximately 21.2° C.

29. **Tsunami Arrival Times** Scientists estimate the arrival times of tsunamis (seismic ocean waves) based on the point of origin P and ocean depths. The speed s of a tsunami in miles per hour is approximately $s = \sqrt{15d}$, where d is the ocean depth in feet.

(a) Let $f(x)$ be the ocean depth x miles from P (in the direction of the coast). Argue using Riemann sums that the time T required for the tsunami to travel M miles toward the coast is

$$T = \int_0^M \frac{dx}{\sqrt{15 f(x)}}$$

(b) Use Simpson's Rule to estimate T if $M = 1000$ and the ocean depths (in feet), measured at 100-mile intervals starting from P, are

13,000, 11,500, 10,500, 9000, 8500,
7000, 6000, 4400, 3800, 3200, 2000

SOLUTION

(a) At a given distance from shore, say, x_i, the speed of the tsunami in mph is $s = \sqrt{15 f(x_i)}$. If we assume the speed s is constant over a small interval Δx, then the time to cover that interval at that speed is

$$t_i = \frac{\text{distance}}{\text{speed}} = \frac{\Delta x}{\sqrt{15 f(x_i)}}.$$

Now divide the interval $[0, M]$ into N subintervals of length Δx. The total time T is given by

$$T = \sum_{i=1}^{N} t_i = \sum_{i=1}^{N} \frac{\Delta x}{\sqrt{15 f(x_i)}}.$$

Taking the limit as $N \to \infty$, we get

$$T = \int_0^M \frac{dx}{\sqrt{15 f(x)}}.$$

(b) We have $\Delta x = 100$. Simpson's Rule gives us

$$S_{10} = \frac{1}{3} \cdot 100 \left[\frac{1}{\sqrt{15(13{,}000)}} + \frac{4}{\sqrt{15(11{,}500)}} + \cdots + \frac{1}{\sqrt{15(2000)}} \right] \approx 3.347.$$

It will take the tsunami about 3 hours and 21 minutes to reach shore.

30. Use S_8 to estimate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$, taking the value of $\dfrac{\sin x}{x}$ at $x = 0$ to be 1.

SOLUTION Divide $[0, \pi/2]$ into 8 subintervals of length

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with endpoints

$$0,\ \frac{\pi}{16},\ \frac{2\pi}{16},\ \ldots,\ \frac{8\pi}{16}.$$

Taking the value of $(\sin x)/x$ at $x = 0$ to be 1, we get

$$S_8 = \frac{1}{3}\left(\frac{\pi}{16}\right)\left[1 + 4\frac{\sin(\pi/16)}{\pi/16} + 2\frac{\sin(2\pi/16)}{2\pi/16} + \cdots + \frac{\sin(\pi/2)}{\pi/2}\right] \approx 1.37076.$$

31. Calculate T_6 for the integral $I = \displaystyle\int_0^2 x^3\, dx$.

(a) Is T_6 too large or too small? Explain graphically.

(b) Show that $K_2 = |f''(2)|$ may be used in the error bound and find a bound for the error.

(c) Evaluate I and check that the actual error is less than the bound computed in (b).

SOLUTION Let $f(x) = x^3$. Divide $[0, 2]$ into 6 subintervals of length $\Delta x = \frac{2-0}{6} = \frac{1}{3}$ with endpoints $0, \frac{1}{3}, \frac{2}{3}, \ldots, 2$. With this data, we get

$$T_6 = \frac{1}{2} \cdot \frac{1}{3}\left[0^3 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{2}{3}\right)^3 + 2\left(\frac{3}{3}\right)^3 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + (1)2^3\right] \approx 4.11111.$$

(a) Since x^3 is concave up on $[0, 2]$, T_6 is too large.

(b) We have $f'(x) = 3x^2$ and $f''(x) = 6x$. Since $|f''(x)| = |6x|$ is *increasing* on $[0, 2]$, its maximum value occurs at $x = 2$ and we may take $K_2 = |f''(2)| = 12$. Then

$$\text{Error}(T_6) \le \frac{K_2(b-a)^3}{12N^2} = \frac{12(2-0)^3}{12(6)^2} = \frac{2}{9} \approx 0.22222.$$

(c) The exact value is

$$\int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(16-0) = 4.$$

We can use this to compute the actual error:

$$\text{Error}(T_6) = |T_6 - 4| \approx |4.11111 - 4| \approx 0.11111.$$

Since $0.11111 < 0.22222$, the actual error is indeed less than the maximum possible error.

32. Calculate M_4 for the integral $I = \int_0^1 x\sin(x^2)\,dx$.

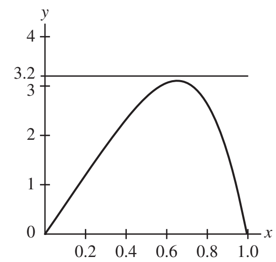
(a) GU Use a plot of $f''(x)$ to show that $K_2 = 3.2$ may be used in the error bound and find a bound for the error.

(b) CAS Evaluate I numerically and check that the actual error is less than the bound computed in (a).

SOLUTION Let $f(x) = x\sin(x^2)$. Divide [0, 1] into 4 subintervals of length $\Delta x = \frac{1-0}{4} = \frac{1}{4} = 0.25$, with endpoint 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 and midpoints $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$. With this data, we get

$$M_4 = \frac{1}{4}\left[\frac{1}{8}\sin((1/8)^2) + \frac{3}{8}\sin((3/8)^2) + \frac{5}{8}\sin((5/8)^2) + \frac{7}{8}\sin((7/8)^2)\right] \approx 0.224714$$

(a) Consider the following plot of $f''(x) = 6x\cos(x^2) - 4x^3\sin(x^2)$:



From this figure, it is clear that $f''(x)$ is bounded above (in absolute value) by 3.2, so we can choose $K_2 = 3.2$ in the error bound formula. With this choice, the bound for the error in the M_4 approximation is

$$\text{Error}(M_4) \le K_2 \cdot \frac{(b-a)^3}{24N^2} = 3.2 \cdot \frac{(1-0)^3}{24 \cdot 4^2} = \frac{3.2}{384} \approx 0.008333 \approx 8.333 \times 10^{-3}$$

(b) Using a computer algebra system, $I \approx 0.2298488$, so the actual error is

$$\approx 0.2298488 - 0.224714 = 0.005135 < 0.008333$$

In Exercises 33–36, state whether T_N or M_N underestimates or overestimates the integral and find a bound for the error (but do not calculate T_N or M_N).

33. $\int_1^4 \frac{1}{x}\,dx$, $\quad T_{10}$

SOLUTION Let $f(x) = \frac{1}{x}$. Then $f'(x) = \frac{-1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$ on [1, 4], so $f(x)$ is concave up, and T_{10} overestimates the integral. Since $|f''(x)| = |\frac{2}{x^3}|$ has its maximum value on [1, 4] at $x = 1$, we can take $K_2 = \frac{2}{1^3} = 2$, and

$$\text{Error}(T_{10}) \le \frac{K_2(4-1)^3}{12N^2} = \frac{2(3)^3}{12(10)^2} = 0.045.$$

34. $\int_0^2 e^{-x/4}\,dx$, $\quad T_{20}$

SOLUTION Let $f(x) = e^{-x/4}$. Then $f'(x) = -(1/4)e^{-x/4}$ and

$$f''(x) = \frac{1}{16}e^{-x/4} > 0$$

on [0, 2], so $f(x)$ is concave up, and T_{20} overestimates the integral. Since $|f''(x)| = |(1/16)e^{-x/4}|$ has its maximum value on [0, 2] at $x = 0$, we can take $K_2 = |(1/16)e^0| = 1/16$, and

$$\text{Error}(T_{20}) \le \frac{K_2(2-0)^3}{12N^2} = \frac{\frac{1}{16}(2)^3}{12(20)^2} = 1.04167 \times 10^{-4}.$$

35. $\int_1^4 \ln x\,dx, \quad M_{10}$

SOLUTION Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on $[1, 4]$, so $f(x)$ is concave down, and M_{10} overestimates the integral. Since $|f''(x)| = |-1/x^2|$ has its maximum value on $[1, 4]$ at $x = 1$, we can take $K_2 = |-1/1^2| = 1$, and

$$\text{Error}(M_{10}) \le \frac{K_2(4-1)^3}{24N^2} = \frac{(1)(3)^3}{24(10)^2} = 0.01125.$$

36. $\int_0^{\pi/4} \cos x, \quad M_{20}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and $f''(x) = -\cos x < 0$ on $[0, \pi/4]$, so $f(x)$ is concave down, and M_{20} overestimates the integral. Since $|f''(x)| = |-\cos x|$ has its maximum value on $[0, \pi/4]$ at $x = 0$, we can take $K_2 = |-\cos(0)| = 1$, and

$$\text{Error}(M_{20}) \le \frac{K_2(\pi/4-0)^3}{24N^2} = \frac{(1)(\pi/4)^3}{24(20)^2} = 5.04659 \times 10^{-5}.$$

CAS *In Exercises 37–40, use the error bound to find a value of N for which Error(T_N) $\le 10^{-6}$. If you have a computer algebra system, calculate the corresponding approximation and confirm that the error satisfies the required bound.*

37. $\int_0^1 x^4\,dx$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$ and $|f''(x)| = |12x^2|$, which has its maximum value on $[0, 1]$ at $x = 1$, so we can take $K_2 = |12(1)^2| = 12$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(1-0)^3}{12N^2} = \frac{12}{12N^2} = \frac{1}{N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{1}{N^2} \le \frac{1}{10^6}.$$

This gives $N^2 \ge 10^6$ or $N \ge 10^3$. Thus let $N = 1000$. The exact value of the integral is

$$\int_0^1 x^4\,dx = \left.\frac{x^5}{5}\right|_0^1 = \frac{1}{5} = 0.2.$$

Using a CAS, we find that

$$T_{1000} \approx 0.2000003333.$$

The actual error is approximately $|0.2000003333 - 0.2| \approx 3.333 \times 10^{-7}$, and is indeed less than 10^{-6}.

38. $\int_0^3 (5x^4 - x^5)\,dx$

SOLUTION Let $f(x) = 5x^4 - x^5$. Then $f'(x) = 20x^3 - 5x^4$ and $f''(x) = 60x^2 - 20x^3$. A plot reveals that $f''(x) \ge 0$ on $[0, 3]$; it achieves its maximum value where its derivative is zero, which is where $120x - 60x^2 = 0$, so $x = 2$. $|f''(2)| = |60 \cdot 2^2 - 20 \cdot 2^3| = 80$, so we may take $K_2 = 80$ in the error bound approximation. Then we have

$$\text{Error}(T_N) \le \frac{K_2(3-0)^3}{12N^2} = \frac{180}{N^2}$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{180}{N^2} \le 10^{-6}, \qquad \text{or} \quad N^2 \ge 180 \times 10^6 = 1.8 \times 10^8$$

Thus $N \ge \sqrt{1.8} \times 10^4 \approx 1.34164 \times 10^4$, so let $N = 13{,}417$. Using a computer algebra system, we get

$$T_{13417} \approx 121.5000006000$$

The true value of the integral is

$$I = \int_0^3 \left(5x^4 - x^5\right)\,dx = \left(x^5 - \frac{1}{6}x^6\right)\Big|_0^3 = 121.5$$

so that $T_{13417} - I \approx 0.0000006 = 6 \times 10^{-7} < 10^{-6}$.

39. $\displaystyle\int_2^5 \frac{1}{x}\,dx$

SOLUTION Let $f(x) = 1/x$. Then $f'(x) = -1/x^2$ and $|f''(x)| = |2/x^3|$, which has its maximum value on $[2, 5]$ at $x = 2$, so we can take $K_2 = |2/2^3| = 1/4$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(5-2)^3}{12N^2} = \frac{(1/4)3^3}{12N^2} = \frac{9}{16N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{9}{16N^2} \le \frac{1}{10^6}.$$

This gives us

$$N^2 \ge \frac{9 \cdot 10^6}{16} \Rightarrow N \ge \sqrt{\frac{9 \cdot 10^6}{16}} = 750.$$

Thus let $N = 750$. The exact value of the integral is

$$\int_2^5 \frac{1}{x}\,dx = \ln 5 - \ln 2 \approx 0.9162907314.$$

Using a CAS, we find that

$$T_{750} \approx 0.9162910119.$$

The error is approximately

$$|0.9162907314 - 0.9162910119| \approx 2.805 \times 10^{-7}$$

and is indeed less than 10^{-6}.

40. $\displaystyle\int_0^3 e^{-x}\,dx$

SOLUTION Let $f(x) = e^{-x}$. Then $f'(x) = -e^{-x}$ and $|f''(x)| = |e^{-x}| = e^{-x}$, which has its maximum value on $[0, 3]$ at $x = 0$, so we can take $K_2 = e^0 = 1$. Then we have

$$\text{Error}(T_N) \le \frac{K_2(3-0)^3}{12N^2} = \frac{(1)3^3}{12N^2} = \frac{9}{4N^2}.$$

To ensure that the error is at most 10^{-6}, we must choose N such that

$$\frac{9}{4N^2} \le \frac{1}{10^6}.$$

This gives us

$$N^2 \ge \frac{9 \cdot 10^6}{4} \Rightarrow N \ge \sqrt{\frac{9 \cdot 10^6}{4}} = 1500.$$

Thus let $N = 1500$. The exact value of the integral is

$$\int_0^3 e^{-x}\,dx = \left(-e^{-3}\right) - \left(-e^{-0}\right) = 1 - e^{-3} \approx 0.9502129316.$$

Using a CAS, we find that

$$T_{1500} \approx 0.9502132468.$$

The error is approximately

$$|0.9502129316 - 0.9502132468| \approx 3.152 \times 10^{-7}$$

and is indeed less than 10^{-6}.

41. Compute the error bound for the approximations T_{10} and M_{10} to $\int_0^3 (x^3+1)^{-1/2}\,dx$, using Figure 17 to determine a value of K_2. Then find a value of N such that the error in M_N is at most 10^{-6}.

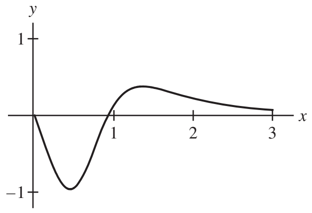


FIGURE 17 Graph of $f''(x)$, where $f(x) = (x^3+1)^{-1/2}$.

SOLUTION Clearly, in the range $0 \le x \le 3$, we have $|f''(x)| \le 1$, so we may choose $K_2 = 1$. Then

$$\text{Error}(T_{10}) \le \frac{K_2(3-0)^3}{12N^2} = \frac{27}{12\cdot 10^2} = \frac{27}{1200} = 0.0225$$

$$\text{Error}(M_{10}) \le \frac{K_2(3-0)^3}{24N^2} = \frac{27}{24\cdot 10^2} = \frac{27}{2400} = 0.01125$$

In order for the error in M_N to be at most 10^{-6}, we must have

$$\text{Error}(M_N) \le \frac{K_2(3-0)^3}{24N^2} = \frac{9}{8N^2} \le 10^{-6}$$

so that $8N^2 \ge 9\times 10^6$ and $N^2 \ge 1{,}125{,}000$. Thus we must choose $N \ge \sqrt{1{,}125{,}000} \approx 1060.7$, so that $N = 1061$.

42. (a) Compute S_6 for the integral $I = \displaystyle\int_0^1 e^{-2x}\,dx$.

(b) Show that $K_4 = 16$ may be used in the error bound and compute the error bound.

(c) Evaluate I and check that the actual error is less than the bound for the error computed in (b).

SOLUTION

(a) Let $f(x) = e^{-2x}$. We divide $[0,1]$ into six subintervals of length $\Delta x = (1-0)/6 = 1/6$, with endpoints $0, 1/6, \ldots, 5/6, 1$. With this data, we get

$$S_6 = \frac{1}{3}\cdot\frac{1}{6}\Big[e^{-2(0)} + 4e^{-2(1/6)} + 2e^{-2(2/6)} + \cdots + e^{-2(1)}\Big] \approx 0.432361.$$

(b) Taking derivatives, we get

$$f'(x) = -2e^{-2x},\ \ f''(x) = 4e^{-2x},\ \ f^{(3)}(x) = -8e^{-2x},\ \ f^{(4)}(x) = 16e^{-2x}.$$

Since $|f^{(4)}(x)| = |16e^{-2x}|$ assumes its maximum value on $[0,1]$ at $x=0$, we can set $K_4 = |16e^0| = 16$. Then we have

$$\text{Error}(S_6) \le \frac{K_4(1-0)^5}{180N^4} = \frac{16}{180\cdot 6^4} \approx 6.86\times 10^{-5}.$$

(c) The exact value of the integral is

$$\int_0^1 e^{-2x}\,dx = \left.\frac{e^{-2x}}{-2}\right|_0^1 = \frac{1-e^{-2}}{2} \approx 0.432332.$$

The actual error is

$$\text{Error}(S_6) \approx |0.432361 - 0.432332| \approx 2.9\times 10^{-5}.$$

The error is indeed less than the maximum possible error.

43. Calculate S_8 for $\int_1^5 \ln x\,dx$ and calculate the error bound. Then find a value of N such that S_N has an error of at most 10^{-6}.

SOLUTION Let $f(x) = \ln x$. We divide $[1,5]$ into eight subintervals of length $\Delta x = (5-1)/8 = 0.5$, with endpoints $1, 1.5, 2, \ldots, 5$. With this data, we get

$$S_8 = \frac{1}{3}\cdot\frac{1}{2}\Big[\ln 1 + 4\ln 1.5 + 2\ln 2 + \cdots + 4\ln 4.5 + \ln 5\Big] \approx 4.046655.$$

To find the maximum possible error, we first take derivatives:

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3},\ f^{(4)}(x) = -\frac{6}{x^4}.$$

Since $|f^{(4)}(x)| = |-6x^{-4}| = 6x^{-4}$, assumes its maximum value on $[1, 5]$ at $x = 1$, we can set $K_4 = 6(1)^{-4} = 6$. Then we have

$$\text{Error}(S_8) \le \frac{K_4(5-1)^5}{180N^4} = \frac{6 \cdot 4^5}{180 \cdot 8^4} \approx 0.0083333.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{6 \cdot 4^5}{180N^4} \le \frac{1}{10^6}.$$

This gives us

$$N^4 \ge \frac{6 \cdot 4^5 \cdot 10^6}{180} \Rightarrow N \ge \left(\frac{6 \cdot 4^5 \cdot 10^6}{180}\right)^{1/4} \approx 76.435.$$

Thus let $N = 78$ (remember that N must be even when using Simpson's Rule).

44. Find a bound for the error in the approximation S_{10} to $\int_0^3 e^{-x^2}\,dx$ (use Figure 18 to determine a value of K_4). Then find a value of N such that S_N has an error of at most 10^{-6}.

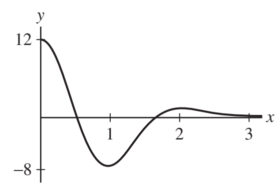


FIGURE 18 Graph of $f^{(4)}(x)$, where $f(x) = e^{-x^2}$.

SOLUTION From the graph, we see that $|f^{(4)}(x)| \le 12$, so we set $K_4 = 12$. This gives us

$$\text{Error}(S_{10}) \le \frac{K_4(3-0)^5}{180N^4} = \frac{12 \cdot 3^5}{180 \cdot 10^4} = 0.00162.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{12 \cdot 3^5}{180 \cdot N^4} \le \frac{1}{10^6}.$$

This gives us

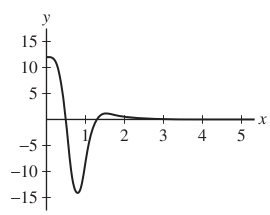
$$N^4 \ge \frac{12 \cdot 3^5 \cdot 10^6}{180} \Rightarrow N \ge \left(\frac{12 \cdot 3^5 \cdot 10^6}{180}\right)^{1/4} \approx 63.44.$$

Thus let $N = 64$.

45. CAS Use a computer algebra system to compute and graph $f^{(4)}(x)$ for $f(x) = \sqrt{1+x^4}$ and find a bound for the error in the approximation S_{40} to $\displaystyle\int_0^5 f(x)\,dx$.

SOLUTION From the graph of $f^{(4)}(x)$ shown below, we see that $|f^{(4)}(x)| \le 15$ on $[0, 5]$. Therefore we set $K_4 = 15$. Now we have

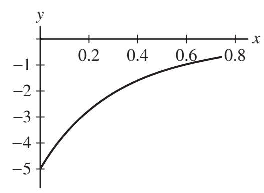
$$\text{Error}(S_{40}) \le \frac{15(5-0)^5}{180(40)^4} = \frac{5}{49152} \approx 1.017 \times 10^{-4}.$$



46. CAS Use a computer algebra system to compute and graph $f^{(4)}(x)$ for $f(x) = \tan x - \sec x$ and find a bound for the error in the approximation S_{40} to $\int_0^{\pi/4} f(x)\,dx$.

SOLUTION From the graph of $f^{(4)}(x)$ shown below, we see that $|f^{(x)}(x)| \le 5$ on $[0, \pi/4]$. Therefore we set $K_4 = 5$. Now we have

$$\text{Error}(S_{40}) \le \frac{5(\pi/4 - 0)^5}{180(40)^4} \approx 3.243 \times 10^{-9}.$$



In Exercises 47–50, use the error bound to find a value of N for which Error(S_N) $\le 10^{-9}$.

47. $\int_1^6 x^{4/3}\,dx$

SOLUTION Let $f(x) = x^{4/3}$. We start by taking derivatives:

$$f'(x) = \frac{4}{3}x^{1/3}$$
$$f''(x) = \frac{4}{9}x^{-2/3}$$
$$f'''(x) = -\frac{8}{27}x^{-5/3}$$
$$f^{(4)}(x) = \frac{40}{81}x^{-8/3}$$

For $x \ge 1$, $f^{(4)}(x)$ is a decreasing function of x, so it takes its maximum value on $[1, 6]$ at $x = 1$. That maximum value is $\frac{40}{81}$, which is quite close to (but smaller than) $\frac{1}{2}$. For simplicity, we take $K_4 = \frac{1}{2}$. Then

$$\text{Error}(S_N) \le \frac{K_4(b-a)^5}{180N^4} = \frac{(6-1)^5}{2 \cdot 180 \cdot N^4} = \frac{5^5}{360N^4} = \frac{625}{72N^4} \le 10^{-9}$$

Thus $72N^4 \ge 625 \times 10^9$, so that

$$N \ge \left(\frac{625 \times 10^9}{72}\right)^{1/4} \approx 305.24$$

so we can take $N = 306$.

48. $\int_0^4 xe^x\,dx$

SOLUTION Let $f(x) = xe^x$. To find K_4, we first take derivatives:

$$f'(x) = xe^x + e^x$$
$$f''(x) = xe^x + 2e^x$$
$$f^{(3)}(x) = xe^x + 3e^x$$
$$f^{(4)}(x) = xe^x + 4e^x.$$

On the interval $[0, 4]$,

$$|f^{(4)}(x)| = |xe^x + 4e^x| \le |4e^4 + 4e^4| = 8e^4.$$

Therefore we set $K_4 = 8e^4$, and we have

$$\text{Error}(S_N) \le \frac{K_4(4-0)^5}{180N^4} = \frac{8e^4 \cdot 4^5}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{8e^4 \cdot 4^5}{180N^4} \le \frac{1}{10^9}.$$

This gives us

$$N^4 \ge \frac{8e^4 \cdot 4^5 \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{8e^4 \cdot 4^5 \cdot 10^9}{180}\right)^{1/4} \approx 1255.52.$$

Thus let $N = 1256$.

49. $\int_0^1 e^{x^2}\,dx$

SOLUTION Let $f(x) = e^{x^2}$. To find K_4, we first take derivatives:

$$f'(x) = 2xe^{x^2}$$
$$f''(x) = 4x^2e^{x^2} + 2e^{x^2}$$
$$f^{(3)}(x) = 8x^3e^{x^2} + 12xe^{x^2}$$
$$f^{(4)}(x) = 16x^4e^{x^2} + 48x^2e^{x^2} + 12e^{x^2}.$$

On the interval [0, 1], $|f^{(4)}(x)|$ assumes its maximum value at $x = 1$. Therefore we set

$$K_4 = |f^{(4)}(1)| = 16e + 48e + 12e = 76e.$$

Now we have

$$\text{Error}(S_N) \le \frac{K_4(1-0)^5}{180N^4} = \frac{76e}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{76e}{180N^4} \le \frac{1}{10^9}.$$

This gives us

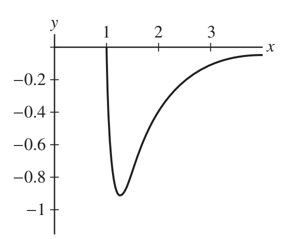
$$N^4 \ge \frac{76e \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{76e \cdot 10^9}{180}\right)^{1/4} \approx 184.06.$$

Thus we let $N = 186$ (remember that N must be even when using Simpson's Rule).

50. $\int_1^4 \sin(\ln x)\,dx$

SOLUTION Let $f(x) = \sin(\ln x)$. To find K_4, we first take derivatives:

$$f'(x) = \frac{\cos(\ln x)}{x}$$
$$f''(x) = \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$$
$$f^{(3)}(x) = \frac{\cos(\ln x) + 3\sin(\ln x)}{x^3}$$
$$f^{(4)}(x) = \frac{-10\sin(\ln x)}{x^4}$$



From the graph of $y = f^{(4)}(x)$ shown above, we can see that on the interval $[1, 4]$, $|f^{(4)}(x)| \le 1$. Therefore we set $K_4 = 1$. Now we have

$$\text{Error}(S_N) \le \frac{(1)(4-1)^5}{180N^4} = \frac{3^5}{180N^4}.$$

To ensure that S_N has error at most 10^{-9}, we must find N such that

$$\frac{3^5}{180N^4} \le \frac{1}{10^9}.$$

This gives us

$$N^4 \ge \frac{3^5 \cdot 10^9}{180} \Rightarrow N \ge \left(\frac{3^5 \cdot 10^9}{180}\right)^{1/4} \approx 191.7.$$

Thus we let $N = 192$.

51. CAS Show that $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$ [use Eq. (3) in Section 5.7].

(a) Use a computer algebra system to graph $f^{(4)}(x)$ for $f(x) = (1+x^2)^{-1}$ and find its maximum on $[0, 1]$.

(b) Find a value of N such that S_N approximates the integral with an error of at most 10^{-6}. Calculate the corresponding approximation and confirm that you have computed $\frac{\pi}{4}$ to at least four places.

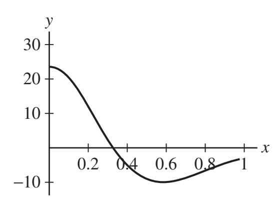
SOLUTION Recall from Section 3.9 that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

So then

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}.$$

(a) From the graph of $f^{(4)}(x)$ shown below, we can see that the maximum value of $|f^{(4)}(x)|$ on the interval $[0, 1]$ is 24.



(b) From part (a), we set $K_4 = 24$. Then we have

$$\text{Error}(S_N) \le \frac{24(1-0)^5}{180N^4} = \frac{2}{15N^4}.$$

To ensure that S_N has error at most 10^{-6}, we must find N such that

$$\frac{2}{15N^4} \le \frac{1}{10^6}.$$

This gives us

$$N^4 \ge \frac{2 \cdot 10^6}{15} \Rightarrow N \ge \left(\frac{2 \cdot 10^6}{15}\right)^{1/4} \approx 19.1.$$

Thus let $N = 20$. To compute S_{20}, let $\Delta x = (1-0)/20 = 0.05$. The endpoints of $[0, 1]$ are $0, \ 0.05, \ \dots, \ 1$. With this data, we get

$$S_{20} = \frac{1}{3}\left(\frac{1}{20}\right)\left[\frac{1}{1+0^2} + \frac{4}{1+(0.05)^2} + \frac{2}{1+(0.1)^2} + \cdots + \frac{1}{1+1^2}\right] \approx 0.785398163242.$$

The actual error is

$$|0.785398163242 - \pi/4| = |0.785398163242 - 0.785398163397| = 1.55 \times 10^{-10}.$$

52. Let $J = \int_0^\infty e^{-x^2}\,dx$ and $J_N = \int_0^N e^{-x^2}\,dx$. Although e^{-x^2} has no elementary antiderivative, it is known that $J = \sqrt{\pi}/2$. Let T_N be the Nth trapezoidal approximation to J_N. Calculate T_4 and show that T_4 approximates J to three decimal places.

SOLUTION T_4 is the 4$^{\text{th}}$ trapezoidal approximation to $J_4 = \int_0^4 e^{-x^2}\,dx$. We divide the interval [0, 4] into four subintervals, with endpoints 0, 1, 2, 3, and 4. Then

$$T_4 = \frac{1}{2}\cdot 1\left[e^{-0^2} + 2e^{-1^2} + 2e^{-2^2} + 2e^{-3^2} + e^{-4^2}\right] \approx 0.8863185$$

We have

$$T_4 - J \approx 0.8863185 - \frac{\sqrt{\pi}}{2} \approx 0.8863185 - 0.8862269 \approx 0.0000916$$

53. Let $f(x) = \sin(x^2)$ and $I = \int_0^1 f(x)\,dx$.

(a) Check that $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$. Then show that $|f''(x)| \le 6$ for $x \in [0, 1]$. *Hint:* Note that $|2\cos(x^2)| \le 2$ and $|4x^2\sin(x^2)| \le 4$ for $x \in [0, 1]$.

(b) Show that Error(M_N) is at most $\dfrac{1}{4N^2}$.

(c) Find an N such that $|I - M_N| \le 10^{-3}$.

SOLUTION

(a) Taking derivatives, we get

$$f'(x) = 2x\cos(x^2)$$
$$f''(x) = 2x(-\sin(x^2)\cdot 2x) + 2\cos(x^2) = 2\cos(x^2) - 4x^2\sin(x^2).$$

On the interval [0, 1],

$$|f''(x)| = |2\cos(x^2) - 4x^2\sin(x^2)| \le |2\cos(x^2)| + |4x^2\sin(x^2)| \le 2 + 4 = 6.$$

(b) Using $K_2 = 6$, we get

$$\text{Error}(M_N) \le \frac{K_2(1-0)^3}{24N^2} = \frac{6}{24N^2} = \frac{1}{4N^2}.$$

(c) To ensure that M_N has error at most 10^{-3}, we must find N such that

$$\frac{1}{4N^2} \le \frac{1}{10^3}.$$

This gives us

$$N^2 \ge \frac{10^3}{4} = 250 \Rightarrow N \ge \sqrt{250} \approx 15.81.$$

Thus let $N = 16$.

54. CAS The error bound for M_N is proportional to $1/N^2$, so the error bound decreases by $\frac{1}{4}$ if N is increased to $2N$. Compute the actual error in M_N for $\int_0^\pi \sin x\,dx$ for $N = 4$, 8, 16, 32, and 64. Does the actual error seem to decrease by $\frac{1}{4}$ as N is doubled?

SOLUTION The exact value of the integral is

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1) - (1) = 2.$$

To compute M_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and midpoints $\pi/8,\ 3\pi/8,\ 5\pi/8,\ 7\pi/8$. With this data, we get

$$M_4 = \frac{\pi}{4}\left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right] \approx 2.052344.$$

The values for M_8, M_{16}, M_{32}, and M_{64} are computed similarly:

$$M_8 = \frac{\pi}{8}\left[\sin\left(\frac{\pi}{16}\right) + \sin\left(\frac{3\pi}{16}\right) + \cdots + \sin\left(\frac{15\pi}{16}\right)\right] \approx 2.012909;$$
$$M_{16} = \frac{\pi}{16}\left[\sin\left(\frac{\pi}{32}\right) + \sin\left(\frac{3\pi}{32}\right) + \cdots + \sin\left(\frac{31\pi}{32}\right)\right] \approx 2.0032164;$$
$$M_{32} = \frac{\pi}{32}\left[\sin\left(\frac{\pi}{64}\right) + \sin\left(\frac{3\pi}{64}\right) + \cdots + \sin\left(\frac{63\pi}{64}\right)\right] \approx 2.00080342;$$
$$M_{64} = \frac{\pi}{64}\left[\sin\left(\frac{\pi}{128}\right) + \sin\left(\frac{3\pi}{128}\right) + \cdots + \sin\left(\frac{127\pi}{128}\right)\right] \approx 2.00020081.$$

Now we can compute the actual errors for each N:

$$\text{Error}(M_4) = |2 - 2.052344| = 0.052344$$
$$\text{Error}(M_8) = |2 - 2.012909| = 0.012909$$
$$\text{Error}(M_{16}) = |2 - 2.0032164| = 0.0032164$$
$$\text{Error}(M_{32}) = |2 - 2.00080342| = 0.00080342$$
$$\text{Error}(M_{64}) = |2 - 2.00020081| = 0.00020081$$

The actual error does in fact decrease by about 1/4 each time N is doubled.

55. CAS Observe that the error bound for T_N (which has 12 in the denominator) is twice as large as the error bound for M_N (which has 24 in the denominator). Compute the actual error in T_N for $\int_0^{\pi} \sin x\,dx$ for $N = 4, 8, 16, 32$, and 64 and compare with the calculations of Exercise 54. Does the actual error in T_N seem to be roughly twice as large as the error in M_N in this case?

SOLUTION The exact value of the integral is

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -(-1) - (1) = 2.$$

To compute T_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and endpoints $0,\ \pi/4,\ 2\pi/4,\ 3\pi/4,\ \pi$. With this data, we get

$$T_4 = \frac{1}{2}\cdot\frac{\pi}{4}\left[\sin(0) + 2\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{2\pi}{4}\right) + 2\sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right] \approx 1.896119.$$

The values for T_8, T_{16}, T_{32}, and T_{64} are computed similarly:

$$T_8 = \frac{1}{2}\cdot\frac{\pi}{8}\left[\sin(0) + 2\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{2\pi}{8}\right) + \cdots + 2\sin\left(\frac{7\pi}{8}\right) + \sin(\pi)\right] \approx 1.974232;$$
$$T_{16} = \frac{1}{2}\cdot\frac{\pi}{16}\left[\sin(0) + 2\sin\left(\frac{\pi}{16}\right) + 2\sin\left(\frac{2\pi}{16}\right) + \cdots + 2\sin\left(\frac{15\pi}{16}\right) + \sin(\pi)\right] \approx 1.993570;$$
$$T_{32} = \frac{1}{2}\cdot\frac{\pi}{32}\left[\sin(0) + 2\sin\left(\frac{\pi}{32}\right) + 2\sin\left(\frac{2\pi}{32}\right) + \cdots + 2\sin\left(\frac{31\pi}{32}\right) + \sin(\pi)\right] \approx 1.998393;$$
$$T_{64} = \frac{1}{2}\cdot\frac{\pi}{64}\left[\sin(0) + 2\sin\left(\frac{\pi}{64}\right) + 2\sin\left(\frac{2\pi}{64}\right) + \cdots + 2\sin\left(\frac{63\pi}{64}\right) + \sin(\pi)\right] \approx 1.999598.$$

Now we can compute the actual errors for each N:

$$\text{Error}(T_4) = |2 - 1.896119| = 0.103881$$
$$\text{Error}(T_8) = |2 - 1.974232| = 0.025768$$
$$\text{Error}(T_{16}) = |2 - 1.993570| = 0.006430$$
$$\text{Error}(T_{32}) = |2 - 1.998393| = 0.001607$$
$$\text{Error}(T_{64}) = |2 - 1.999598| = 0.000402$$

Comparing these results with the calculations of Exercise 54, we see that the actual error in T_N is in fact about twice as large as the error in M_N.

56. CAS Explain why the error bound for S_N decreases by $\frac{1}{16}$ if N is increased to $2N$. Compute the actual error in S_N for $\int_0^\pi \sin x\,dx$ for $N = 4, 8, 16, 32$, and 64. Does the actual error seem to decrease by $\frac{1}{16}$ as N is doubled?

SOLUTION If we plug in $2N$ for N in the formula for the error bound for S_N, we get

$$\frac{K_4(b-a)^5}{180(2N)^4} = \frac{K_4(b-a)^5}{180 \cdot 2^4 \cdot N^4} = \frac{1}{16}\left(\frac{K_4(b-a)^5}{180N^4}\right).$$

Thus we see that, since N is raised to the fourth power in the denominator, the Error Bound for S_N decreases by $1/16$ if N is increased to $2N$. The exact value of the integral is

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1) - (1) = 2.$$

To compute S_4, we have $\Delta x = (\pi - 0)/4 = \pi/4$, and endpoints $0,\ \pi/4,\ 2\pi/4,\ 3\pi/4,\ \pi$. With this data, we get

$$S_4 = \frac{1}{3}\cdot\frac{\pi}{4}\left[\sin(0) + 4\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{2\pi}{4}\right) + 4\sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right] \approx 2.004560.$$

The values for S_8, S_{16}, S_{32}, and S_{64} are computed similarly:

$$S_8 = \frac{1}{3}\cdot\frac{\pi}{8}\left[\sin(0) + 4\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{2\pi}{8}\right) + \cdots + 4\sin\left(\frac{7\pi}{8}\right) + \sin(\pi)\right] \approx 2.0002692;$$

$$S_{16} = \frac{1}{3}\cdot\frac{\pi}{16}\left[\sin(0) + 4\sin\left(\frac{\pi}{16}\right) + 2\sin\left(\frac{2\pi}{16}\right) + \cdots + 4\sin\left(\frac{15\pi}{16}\right) + \sin(\pi)\right] \approx 2.00001659;$$

$$S_{32} = \frac{1}{3}\cdot\frac{\pi}{32}\left[\sin(0) + 4\sin\left(\frac{\pi}{32}\right) + 2\sin\left(\frac{2\pi}{32}\right) + \cdots + 4\sin\left(\frac{31\pi}{32}\right) + \sin(\pi)\right] \approx 2.000001033;$$

$$S_{64} = \frac{1}{3}\cdot\frac{\pi}{64}\left[\sin(0) + 4\sin\left(\frac{\pi}{64}\right) + 2\sin\left(\frac{2\pi}{64}\right) + \cdots + 4\sin\left(\frac{63\pi}{64}\right) + \sin(\pi)\right] \approx 2.00000006453.$$

Now we can compute the actual errors for each N:

$$\text{Error}(S_4) = |2 - 2.004560| = 0.004560$$

$$\text{Error}(S_8) = |2 - 2.0002692| = 2.692 \times 10^{-4}$$

$$\text{Error}(S_{16}) = |2 - 2.00001659| = 1.659 \times 10^{-5}$$

$$\text{Error}(S_{32}) = |2 - 2.000001033| = 1.033 \times 10^{-6}$$

$$\text{Error}(S_{64}) = |2 - 2.00000006453| = 6.453 \times 10^{-8}$$

The actual error does in fact decrease by about $1/16$ each time N is doubled. For example, $0.004560/16 = 2.85 \times 10^{-4}$, which is roughly the same as 2.692×10^{-4}.

57. Verify that S_2 yields the exact value of $\displaystyle\int_0^1 (x - x^3)\,dx$.

SOLUTION Let $f(x) = x - x^3$. Clearly $f^{(4)}(x) = 0$, so we may take $K_4 = 0$ in the error bound estimate for S_2. Then

$$\text{Error}(S_2) \le \frac{K_4(1-0)^5}{180 \cdot 2^4} = 0 \cdot \frac{1}{2880} = 0$$

so that S_2 yields the exact value of the integral.

58. Verify that S_2 yields the exact value of $\displaystyle\int_a^b (x - x^3)\,dx$ for all $a < b$.

SOLUTION Let $f(x) = x - x^3$. Clearly $f^{(4)}(x) = 0$, so we may take $K_4 = 0$ in the error bound estimate for S_2. Then

$$\text{Error}(S_2) \le \frac{K_4(b-a)^5}{180 \cdot 2^4} = 0 \cdot \frac{(b-a)^5}{2880} = 0$$

so that S_2 yields the exact value of the integral.

Further Insights and Challenges

59. Show that if $f(x) = rx + s$ is a linear function (r, s constants), then $T_N = \int_a^b f(x)\,dx$ for all N and all endpoints a, b.

SOLUTION First, note that

$$\int_a^b (rx+s)\,dx = \frac{r(b^2-a^2)}{2} + s(b-a).$$

Now,

$$T_N(rx+s) = \frac{b-a}{2N}\left[f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b)\right] = \frac{r(b-a)}{2N}\left[a + 2\sum_{i=1}^{N-1} a + 2\frac{b-a}{N}\sum_{i=1}^{N-1} i + b\right] + s\frac{b-a}{2N}(2N)$$

$$= \frac{r(b-a)}{2N}\left[(2N-1)a + 2\frac{b-a}{N}\frac{(N-1)N}{2} + b\right] + s(b-a) = \frac{r(b^2-a^2)}{2} + s(b-a).$$

60. Show that if $f(x) = px^2 + qx + r$ is a quadratic polynomial, then $S_2 = \int_a^b f(x)\,dx$. In other words, show that

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left(y_0 + 4y_1 + y_2\right)$$

where $y_0 = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$, and $y_2 = f(b)$. *Hint:* Show this first for $f(x) = 1, x, x^2$ and use linearity.

SOLUTION For S_2, $\Delta x = (b-a)/2$, and the endpoints are $a,\ (a+b)/2,\ b$. Following the hint, let $f(x) = 1$. In this case,

$$S_2(1) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}(1 + 4(1) + 1) = \frac{b-a}{6}(6)$$

$$= b - a = \int_a^b 1\,dx.$$

If $f(x) = x$, then

$$S_2(x) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}\left(a + 4\left(\frac{a+b}{2}\right) + b\right) = \frac{b-a}{6}\left(\frac{6a+6b}{2}\right)$$

$$= \frac{b^2-a^2}{2} = \int_a^b x\,dx;$$

and if $f(x) = x^2$, then

$$S_2(x^2) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{b-a}{6}\left(a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right)$$

$$= \frac{b-a}{6}\left(a^2 + (a^2 + 2ab + b^2) + b^2\right) = \frac{b-a}{6}(2)(a^2 + ab + b^2) = \frac{b^3-a^3}{3} = \int_a^b x^2\,dx.$$

Now we use linearity:

$$\int_a^b (px^2 + qx + r)\,dx = p\int_a^b x^2\,dx + q\int_a^b x\,dx + r\int_a^b dx$$

$$= pS_2(x^2) + qS_2(x) + rS_2(1) = S_2(pa^2 + qa + r).$$

61. For N even, divide $[a, b]$ into N subintervals of width $\Delta x = \frac{b-a}{N}$. Set $x_j = a + j\,\Delta x$, $y_j = f(x_j)$, and

$$S_2^{2j} = \frac{b-a}{3N}\left(y_{2j} + 4y_{2j+1} + y_{2j+2}\right)$$

(a) Show that S_N is the sum of the approximations on the intervals $[x_{2j}, x_{2j+2}]$—that is, $S_N = S_2^0 + S_2^2 + \cdots + S_2^{N-2}$.

(b) By Exercise 60, $S_2^{2j} = \int_{x_{2j}}^{x_{2j+2}} f(x)\,dx$ if $f(x)$ is a quadratic polynomial. Use (a) to show that S_N is exact *for all N* if $f(x)$ is a quadratic polynomial.

SOLUTION

(a) This result follows because the even-numbered interior endpoints overlap:

$$\sum_{i=0}^{(N-2)/2} S_2^{2j} = \frac{b-a}{6}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots\right]$$

$$= \frac{b-a}{6}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{N-1} + y_N\right] = S_N.$$

(b) If $f(x)$ is a quadratic polynomial, then by part (a) we have

$$S_N = S_2^0 + S_2^2 + \cdots + S_2^{N-2} = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)\,dx = \int_a^b f(x)\,dx.$$

62. Show that S_2 also gives the exact value for $\int_a^b x^3\,dx$ and conclude, as in Exercise 61, that S_N is exact for all cubic polynomials. Show by counterexample that S_2 is not exact for integrals of x^4.

SOLUTION Let $f(x) = x^3$. Then $\Delta x = (b-a)/2$ and the endpoints are $a,\ (a+b)/2,\ b$. With this data, we get

$$S_2(x^3) = \frac{1}{3}\left(\frac{b-a}{2}\right)\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right] = \frac{b-a}{6}\left[a^3 + \frac{1}{2}(a^3 + 3a^2b + 3ab^2 + b^3) + b^3\right]$$

$$= \frac{b-a}{6}\left(\frac{3}{2}\right)[a^3 + a^2b + ab^2 + b^3] = \frac{1}{4}(b-a)(a^3 + a^2b + ab^2 + b^3) = \frac{b^4 - a^4}{4} = \int_a^b x^3\,dx.$$

By linearity, and using the result from Exercise 60, we have that

$$\int_a^b (sx^3 + px^2 + qx + r)\,dx = s\int_a^b x^3\,dx + \int_a^b (px^2 + qx + r)\,dx$$

$$= s\left(S_2(x^3)\right) + S_2(px^2 + qx + r)$$

$$= S_2(sx^3 + px^2 + qx + r).$$

For N even, we can now follow the procedure of Exercise 61; that is, divide $[a,b]$ into N subintervals and on each subinterval compute S_2. Then, for any cubic polynomial $f(x)$, we have

$$\int_a^b f(x)\,dx = \int_a^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{b} f(x)\,dx = S_2^0 + S_2^2 + \cdots + S_2^{N-2} = S_N.$$

However, S_2 is not exact for polynomials of degree 4. For example,

$$\int_0^1 x^4\,dx = \frac{1}{5}$$

but

$$S_2 = \frac{1}{3}\left(\frac{1}{2}\right)\left[0^5 + 4(0.5)^5 + 1^5\right] = \frac{1}{6}\left(\frac{33}{32}\right) = \frac{11}{64} \neq \frac{1}{5}.$$

63. Use the error bound for S_N to obtain another proof that Simpson's Rule is exact for all cubic polynomials.

SOLUTION Let $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$, be any cubic polynomial. Then, $f^{(4)}(x) = 0$, so we can take $K_4 = 0$. This yields

$$\text{Error}(S_N) \le \frac{0}{180N^4} = 0.$$

In other words, S_N is exact for all cubic polynomials for all N.

64. **Sometimes, Simpson's Rule Performs Poorly** Calculate M_{10} and S_{10} for the integral $\int_0^1 \sqrt{1-x^2}\,dx$, whose value we know to be $\frac{\pi}{4}$ (one-quarter of the area of the unit circle).

(a) We usually expect S_N to be more accurate than M_N. Which of M_{10} and S_{10} is more accurate in this case?

(b) How do you explain the result of part (a)? *Hint:* The error bounds are not valid because $|f''(x)|$ and $|f^{(4)}(x)|$ tend to ∞ as $x \to 1$, but $|f^{(4)}(x)|$ goes to infinity faster.

SOLUTION Let $f(x) = \sqrt{1-x^2}$. Divide [0, 1] into 10 subintervals of length $\Delta x = (1-0)/10 = 0.1$ Then we have

$$M_{10} = \frac{1}{10}\left[\sqrt{1-(0.05)^2} + \sqrt{1-(0.15)^2} + \cdots + \sqrt{1-(0.95)^2}\right] \approx 0.788103;$$

$$S_{10} = \frac{1}{3}\left(\frac{1}{10}\right)\left[\sqrt{1-0^2} + 4\sqrt{1-(0.1)^2} + 2\sqrt{1-(0.2)^2} + \cdots + \sqrt{1-1^2}\right] \approx 0.781752.$$

(a) Since $\pi/4 = 0.785389$, we have

$$\text{Error}(M_{10}) = 0.0027;$$
$$\text{Error}(S_{10}) = 0.00365.$$

Thus, M_{10} is more accurate.

(b) These results can be explained by looking at the derivatives:

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}$$
$$f''(x) = \frac{-1}{(1-x^2)^{3/2}}$$
$$f^{(3)}(x) = \frac{-3x}{(1-x^2)^{5/2}}$$
$$f^{(4)}(x) = \frac{-3(x^2+1)}{(1-x^2)^{7/2}}$$

Both $|f''(x)|$ and $|f^{(4)}(x)|$ tend to ∞ as $x \to 1$, but $|f^{(4)}(x)|$ tends to ∞ faster due to the $7/2$ exponent in the denominator.

CHAPTER REVIEW EXERCISES

1. Match the integrals (a)–(e) with their antiderivatives (i)–(v) on the basis of the general form (do not evaluate the integrals).

(a) $\displaystyle\int \frac{x\,dx}{x^2-4}$

(b) $\displaystyle\int \frac{(2x+9)\,dx}{x^2+4}$

(c) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

(d) $\displaystyle\int \frac{dx}{x\sqrt{16x^2-1}}$

(e) $\displaystyle\int \frac{16\,dx}{x(x-4)^2}$

(i) $\sec^{-1} 4x + C$

(ii) $\log|x| - \log|x-4| - \dfrac{4}{x-4} + C$

(iii) $\dfrac{1}{30}(3\cos^5 x - 3\cos^3 x \sin^2 x - 7\cos^3 x) + C$

(iv) $\dfrac{9}{2}\tan^{-1}\dfrac{x}{2} + \ln(x^2+4) + C$

(v) $\sqrt{x^2-4} + C$

SOLUTION

(a) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-4}}$

Since x is a constant multiple of the derivative of x^2-4, the substitution method implies that the integral is a constant multiple of $\int \frac{du}{\sqrt{u}}$ where $u = x^2-4$, that is a constant multiple of $\sqrt{u} = \sqrt{x^2-4}$. It corresponds to the function in **(v)**.

(b) $\displaystyle\int \frac{(2x+9)\,dx}{x^2+4}$

The part $\int \frac{2x}{x^2+4}\,dx$ corresponds to $\ln(x^2+4)$ in **(iv)** and the part $\int \frac{9}{x^2+4}\,dx$ corresponds to $\frac{9}{2}\tan^{-1}\frac{x}{2}$. Hence the integral corresponds to the function in **(iv)**.

(c) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

The reduction formula for $\int \sin^m x \cos^n x\,dx$ shows that this integral is equal to a sum of constant multiples of products in the form $\cos^i x \sin^j x$ as in **(iii)**.

(d) $\displaystyle\int \frac{dx}{x\sqrt{16x^2-1}}$

Since $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$, we expect the integral $\int \frac{dx}{x\sqrt{16x^2-1}}$ to be equal to the function in **(i)**.

(e) $\displaystyle\int \frac{16\,dx}{x(x-4)^2}$

The partial fraction decomposition of the integrand has the form:

$$\frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

The term $\frac{A}{x}$ contributes the function $A\ln|x|$ to the integral, the term $\frac{B}{x-4}$ contributes $B\ln|x-4|$ and the term $\frac{C}{(x-4)^2}$ contributes $-\frac{C}{x-4}$. Therefore, we expect the integral to be equal to the function in **(ii)**.

2. Evaluate $\displaystyle\int \frac{x\,dx}{x+2}$ in two ways: using substitution and using the Method of Partial Fractions.

SOLUTION Using substitution, write $u = x+2$; then $du = dx$ and

$$\int \frac{x}{x+2}\,dx = \int \frac{u-2}{u}\,du = \int 1\,du - 2\int \frac{1}{u}\,du = u - 2\ln|u| + C_1$$
$$= x + 2 - 2\ln|x+2| + C_1 = x - 2\ln|x+2| + C$$

Using partial fractions, first do long division to get

$$\frac{x}{x+2} = 1 - \frac{2}{x+2}$$

Then

$$\int \frac{x}{x+2}\,dx = \int \left(1 - \frac{2}{x+2}\right)\,dx = \int 1\,dx - 2\int \frac{1}{x+2}\,dx = x - 2\ln|x+2| + C$$

In Exercises 3–12, evaluate using the suggested method.

3. $\displaystyle\int \cos^3\theta\sin^8\theta\,d\theta$ [write $\cos^3\theta$ as $\cos\theta(1-\sin^2\theta)$]

SOLUTION We use the identity $\cos^2\theta = 1 - \sin^2\theta$ to rewrite the integral:

$$\int \cos^3\theta\sin^8\theta\,d\theta = \int \cos^2\theta\sin^8\theta\cos\theta\,d\theta = \int \left(1-\sin^2\theta\right)\sin^8\theta\cos\theta\,d\theta.$$

Now, we use the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$:

$$\int \cos^3\theta\sin^8\theta\,d\theta = \int \left(1-u^2\right)u^8\,du = \int \left(u^8 - u^{10}\right)\,du = \frac{u^9}{9} - \frac{u^{11}}{11} + C = \frac{\sin^9\theta}{9} - \frac{\sin^{11}\theta}{11} + C.$$

4. $\displaystyle\int xe^{-12x}\,dx$ (Integration by Parts)

SOLUTION We use Integration by Parts with $u = x$ and $v' = e^{-12x}$. Then $u' = 1$, $v = -\frac{1}{12}e^{-12x}$, and we obtain:

$$\int xe^{-12x}\,dx = -\frac{xe^{-12x}}{12} + \int \frac{1}{12}e^{-12x}\,dx = -\frac{xe^{-12x}}{12} - \frac{1}{144}e^{-12x} + C = -\frac{e^{-12x}}{144}(12x+1) + C.$$

5. $\displaystyle\int \sec^3\theta\tan^4\theta\,d\theta$ (trigonometric identity, reduction formula)

SOLUTION We use the identity $1 + \tan^2\theta = \sec^2\theta$ to write $\tan^4\theta = \left(\sec^2\theta - 1\right)^2$ and to rewrite the integral as

$$\int \sec^3\theta\tan^4\theta\,d\theta \int \sec^3\theta\left(1-\sec^2\theta\right)^2 d\theta = \int \sec^3\theta\left(1 - 2\sec^2\theta + \sec^4\theta\right)\,d\theta$$
$$= \int \sec^7\theta\,d\theta - 2\int \sec^5\theta\,d\theta + \int \sec^3\theta\,d\theta.$$

Now we use the reduction formula

$$\int \sec^m\theta\, d\theta = \frac{\tan\theta \sec^{m-2}\theta}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2}\theta\, d\theta.$$

We have

$$\int \sec^5\theta\, d\theta = \frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta + C,$$

and

$$\int \sec^7\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{6}\int \sec^5\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{6}\left(\frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta\right) + C$$

$$= \frac{\tan\theta\sec^5\theta}{6} + \frac{5}{24}\tan\theta\sec^3\theta + \frac{5}{8}\int \sec^3\theta\, d\theta + C.$$

Therefore,

$$\int \sec^3\theta\tan^4\theta\, d\theta = \left(\frac{\tan\theta\sec^5\theta}{6} + \frac{5}{24}\tan\theta\sec^3\theta + \frac{5}{8}\int \sec^3\theta\, d\theta\right)$$

$$- 2\left(\frac{\tan\theta\sec^3\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta\right) + \int \sec^3\theta\, d\theta$$

$$= \frac{\tan\theta\sec^5\theta}{6} - \frac{7\tan\theta\sec^3\theta}{24} + \frac{1}{8}\int \sec^3\theta\, d\theta.$$

We again use the reduction formula to compute

$$\int \sec^3\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

Finally,

$$\int \sec^3\theta\tan^4\theta\, d\theta = \frac{\tan\theta\sec^5\theta}{6} - \frac{7\tan\theta\sec^3\theta}{24} + \frac{\tan\theta\sec\theta}{16} + \frac{1}{16}\ln|\sec\theta + \tan\theta| + C.$$

6. $\displaystyle\int \frac{4x+4}{(x-5)(x+3)}\, dx$ (partial fractions)

SOLUTION The following partial fraction decomposition takes the form

$$\frac{4x+4}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}.$$

Clearing denominators gives us

$$4x + 4 = A(x+3) + B(x-5).$$

Setting $x = 5$ then yields $A = 3$, while setting $x = -3$ yields $B = 1$. Hence,

$$\int \frac{4x+4}{(x-5)(x+3)}\, dx = \int \frac{3}{x-5}\, dx + \int \frac{1}{x+3}\, dx = 3\ln|x-5| + \ln|x+3| + C.$$

7. $\displaystyle\int \frac{dx}{x(x^2-1)^{3/2}}\, dx$ (trigonometric substitution)

SOLUTION Substitute $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$. Then,

$$\left(x^2-1\right)^{3/2} = \left(\sec^2\theta - 1\right)^{3/2} = \left(\tan^2\theta\right)^{3/2} = \tan^3\theta,$$

and

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan^3\theta} = \int \frac{d\theta}{\tan^2\theta} = \int \cot^2\theta\, d\theta.$$

Using a reduction formula we find that:

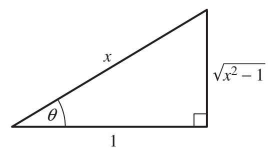
$$\int \cot^2\theta\, d\theta = -\cot\theta - \theta + C$$

so

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = -\cot\theta - \theta + C.$$

We now must return to the original variable x. We use the relation $x = \sec\theta$ and the figure to obtain:

$$\int \frac{dx}{x\left(x^2-1\right)^{3/2}} = -\frac{1}{\sqrt{x^2-1}} - \sec^{-1}x + C.$$

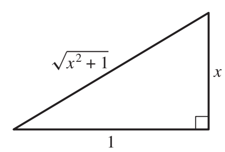


8. $\int (1+x^2)^{-3/2}dx$ (trigonometric substitution)

SOLUTION Use the substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. Then

$$\int (1+x^2)^{-3/2}\,dx = \int (1+\tan^2\theta)^{-3/2}\sec^2\theta\,d\theta = \int (\sec^2\theta)^{-3/2}\sec^2\theta\,d\theta = \int \frac{1}{\sec\theta}\,d\theta$$
$$= \int \cos\theta\,d\theta = \sin\theta + C$$

Since $x = \tan\theta$, draw the following right triangle:



From the figure, we see that $\sin\theta = \frac{x}{\sqrt{x^2+1}}$, so that

$$\int (1+x^2)^{-3/2}\,dx = x(1+x^2)^{-1/2} + C$$

9. $\int \frac{dx}{x^{3/2}+x^{1/2}}$ (substitution)

SOLUTION Let $t = x^{1/2}$. Then $dt = \frac{1}{2}x^{-1/2}\,dx$ or $dx = 2x^{1/2}\,dt = 2t\,dt$. Therefore,

$$\int \frac{dx}{x^{3/2}+x^{1/2}} = \int \frac{2t\,dt}{t^3+t} = \int \frac{2\,dt}{t^2+1} = 2\tan^{-1}t + C = 2\tan^{-1}\sqrt{x} + C.$$

10. $\int \frac{dx}{x+x^{-1}}$ (rewrite integrand)

SOLUTION We rewrite the integrand as follows:

$$\int \frac{dx}{x+x^{-1}} = \int \frac{x\,dx}{x^2+1}.$$

Now, we substitute $u = x^2+1$. Then $du = 2x\,dx$ and

$$\int \frac{dx}{x+x^{-1}} = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C.$$

11. $\int x^{-2}\tan^{-1}x\,dx$ (Integration by Parts)

SOLUTION We use Integration by Parts with $u = \tan^{-1}x$ and $v' = x^{-2}$. Then $u' = \frac{1}{1+x^2}$, $v = -x^{-1}$ and

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x\left(1+x^2\right)}.$$

For the remaining integral, the partial fraction decomposition takes the form

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}.$$

Clearing denominators gives us

$$1 = A(1+x^2) + (Bx+C)x.$$

Setting $x=0$ then yields $A=1$. Next, equating the x^2-coefficients gives

$$0 = A+B \qquad \text{so} \qquad B=-1,$$

while equating x-coefficients gives $C=0$. Hence,

$$\frac{1}{x\left(1+x^2\right)} = \frac{1}{x} - \frac{x}{1+x^2},$$

and

$$\int \frac{dx}{x(1+x^2)} = \int \frac{1}{x}\,dx - \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

Therefore,

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

12. $\displaystyle\int \frac{dx}{x^2+4x-5}$ (complete the square, substitution, partial fractions)

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5}.$$

Clearing denominators gives us

$$1 = A(x+5) + B(x-1).$$

Setting $x=1$ then yields $A=\frac{1}{6}$, while setting $x=-5$ yields $B=-\frac{1}{6}$. Therefore,

$$\int \frac{dx}{x^2+4x-5} = \frac{1}{6}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{dx}{x+5} = \frac{1}{6}\ln|x-1| - \frac{1}{6}\ln|x+5| + C = \frac{1}{6}\ln\left|\frac{x-1}{x+5}\right| + C.$$

In Exercises 13–64, evaluate using the appropriate method or combination of methods.

13. $\displaystyle\int_0^1 x^2 e^{4x}\,dx$

SOLUTION We evaluate the indefinite integral using Integration by Parts with $u=x^2$ and $v'=e^{4x}$. Then $u'=2x$, $v=\frac{1}{4}e^{4x}$ and

$$\int x^2 e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\int xe^{4x}\,dx.$$

We compute the resulting integral using Integration by Parts again, this time with $u=x$ and $v'=e^{4x}$. Then $u'=1$, $v=\frac{1}{4}e^{4x}$ and

$$\int xe^{4x}\,dx = x\cdot\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\,dx = \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C.$$

Therefore,

$$\int x^2 e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\left(\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right) + C = \frac{e^{4x}}{32}\left(8x^2-4x+1\right) + C.$$

Finally,

$$\int_0^1 x^2 e^{4x}\,dx = \left(\frac{e^{4x}}{32}\left(8x^2-4x+1\right)\right)\Bigg|_0^1 = \frac{e^4}{32}(8-4+1) - \frac{1}{32}(1) = \frac{5e^4-1}{32}$$

14. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

SOLUTION Substitute $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then

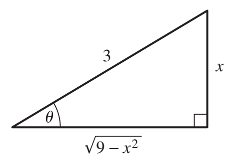
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\left(1-\sin^2\theta\right)} = \sqrt{9\cos^2\theta} = 3\cos\theta,$$

and

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{3\cos\theta} = 9\int \sin^2\theta\,d\theta$$
$$= 9\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{9\theta}{2} - \frac{9\sin\theta\cos\theta}{2} + C.$$

We now must return to the original variable x. Since $x = 3\sin\theta$, we have $t = \sin^{-1}\frac{x}{3}$. Using the figure we obtain

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.$$



15. $\displaystyle\int \cos^9 6\theta\sin^3 6\theta\,d\theta$

SOLUTION We use the identity $\sin^2 6\theta = 1 - \cos^2 6\theta$ to rewrite the integral:

$$\int \cos^9 6\theta \sin^3 6\theta\,d\theta = \int \cos^9 6\theta \sin^2 6\theta\sin 6\theta\,d\theta = \int \cos^9 6\theta\left(1-\cos^2 6\theta\right)\sin 6\theta\,d\theta.$$

Now, we use the substitution $u = \cos 6\theta$, $du = -6\sin 6\theta\,d\theta$:

$$\int \cos^9 6\theta\sin^3 6\theta\,d\theta = \int u^9\left(1-u^2\right)\left(-\frac{du}{6}\right) = -\frac{1}{6}\int\left(u^9 - u^{11}\right)du$$
$$= -\frac{1}{6}\left(\frac{u^{10}}{10} - \frac{u^{12}}{12}\right) + C = \frac{\cos^{12}6\theta}{72} - \frac{\cos^{10}6\theta}{60} + C.$$

16. $\displaystyle\int \sec^2\theta\tan^4\theta\,d\theta$

SOLUTION We substitute $u = \tan\theta$, $du = \sec^2\theta\,d\theta$ to obtain

$$\int \sec^2\theta\tan^4\theta\,d\theta = \int u^4\,du = \frac{u^5}{5} + C = \frac{\tan^5\theta}{5} + C.$$

17. $\displaystyle\int \frac{(6x+4)\,dx}{x^2-1}$

SOLUTION The partial fraction decomposition takes the form

$$\frac{6x+4}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing the denominators gives us

$$6x+4 = A(x+1) + B(x-1).$$

Setting $x = 1$ then yields $A = 5$, while setting $x = -1$ yields $B = 1$. Hence,

$$\int \frac{(6x+4)dx}{x^2-1} = \int \frac{5}{x-1}\,dx + \int \frac{1}{x+1}\,dx = 5\ln|x-1| + \ln|x+1| + C.$$

18. $\displaystyle\int_4^9 \frac{dt}{(t^2-1)^2}$

SOLUTION First evaluate the indefinite integral. Substitute $t = \sin\theta$, $dt = \cos\theta\, d\theta$. Then

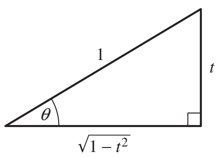
$$\left(t^2-1\right)^2 = \left(1-t^2\right)^2 = \left(1-\sin^2\theta\right)^2 = \left(\cos^2\theta\right)^2 = \cos^4\theta,$$

and

$$\int \frac{dt}{\left(t^2-1\right)^2} = \int \frac{\cos\theta\, d\theta}{\cos^4\theta} = \int \frac{d\theta}{\cos^3\theta} = \int \sec^3\theta\, d\theta.$$

We use a reduction formula to compute the resulting integral:

$$\int \frac{dt}{\left(t^2-1\right)^2} = \int \sec^3\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\, d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$



We now must return to the original variable t. Using the relation $t = \sin\theta$ and the accompanying figure,

$$\begin{aligned}\int \frac{dt}{\left(t^2-1\right)^2} &= \frac{1}{2}\cdot\frac{t}{\sqrt{1-t^2}}\cdot\frac{1}{\sqrt{1-t^2}} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{1-t^2}} + \frac{t}{\sqrt{1-t^2}}\right| + C \\ &= \frac{1}{2}\left(\frac{t}{1-t^2} + \ln\left|\frac{1+t}{\sqrt{1-t^2}}\right|\right) + C = \frac{1}{2}\left(\frac{t}{1-t^2} + \ln\left|\sqrt{\frac{1+t}{1-t}}\right|\right) + C \\ &= \frac{1}{2}\frac{t}{1-t^2} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| + C\end{aligned}$$

Finally,

$$\begin{aligned}\int_4^9 \frac{dt}{(t^2-1)^2} &= \left(\frac{1}{2}\frac{t}{1-t^2} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|\right)\Bigg|_4^9 \\ &= \frac{1}{2}\cdot\frac{9}{-80} + \frac{1}{4}\ln\frac{10}{8} - \frac{1}{2}\cdot\frac{4}{-15} - \frac{1}{4}\ln\frac{5}{3} = -\frac{9}{160} + \frac{2}{15} + \frac{1}{4}\left(\ln\frac{5}{4} - \ln\frac{5}{3}\right) \\ &= \frac{37}{480} + \frac{1}{4}\ln\frac{3}{4} = \frac{37}{480} + \frac{1}{4}\ln 3 - \frac{1}{2}\ln 2\end{aligned}$$

19. $\displaystyle\int \frac{d\theta}{\cos^4\theta}$

SOLUTION We use the identity $1 + \tan^2\theta = \sec^2\theta$ to rewrite the integral:

$$\int \frac{d\theta}{\cos^4\theta} = \int \sec^4\theta\, d\theta = \int \left(1+\tan^2\theta\right)\sec^2\theta\, d\theta.$$

Now, we substitute $u = \tan\theta$. Then, $du = \sec^2\theta\, d\theta$ and

$$\int \frac{d\theta}{\cos^4\theta} = \int \left(1+u^2\right) du = u + \frac{u^3}{3} + C = \frac{\tan^3\theta}{3} + \tan\theta + C.$$

20. $\displaystyle\int \sin 2\theta \sin^2\theta\, d\theta$

SOLUTION We use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ to rewrite the integral:

$$\int \sin 2\theta \sin^2\theta\, d\theta = \int 2\sin\theta\cos\theta\sin^2\theta\, d\theta = \int 2\sin^3\theta\cos\theta\, d\theta.$$

Now, we substitute $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \sin 2\theta\sin^2\theta\, d\theta = 2\int u^3\, du = \frac{u^4}{2} + C = \frac{\sin^4\theta}{2} + C.$$

21. $\int_0^1 \ln(4-2x)\,dx$

SOLUTION Note that $\ln(4-2x) = \ln(2(2-x)) = \ln 2 + \ln(2-x)$. Use integration by parts to integrate $\ln(2-x)$, with $u = \ln(2-x)$, $v' = 1$, so that $u' = -\frac{1}{2-x}$ and $v = x$. Then

$$I = \int_0^1 \ln(4-2x)\,dx = \int_0^1 \ln 2\,dx + \int_0^1 \ln(2-x)\,dx = \ln 2 + (x\ln(2-x))\Big|_0^1 + \int_0^1 \frac{x}{2-x}\,dx$$

Now use long division on the remaining integral, and the substitution $u = 2-x$:

$$\begin{aligned} I &= \ln 2 + (x\ln(2-x))\Big|_0^1 + \int_0^1 \left(-1 + \frac{2}{2-x}\right) dx \\ &= \ln 2 + 1\ln 1 - \int_0^1 1\,dx + 2\int_0^1 \frac{1}{2-x}\,dx = \ln 2 - 1 - 2\int_2^1 \frac{1}{u}\,du \\ &= \ln 2 - 1 - 2\ln u\Big|_2^1 = \ln 2 - 1 + 2\ln 2 = 3\ln 2 - 1 \end{aligned}$$

22. $\int (\ln(x+1))^2\,dx$

SOLUTION First, substitute $w = x+1$, $dw = dx$. Then

$$\int (\ln(x+1))^2\,dx = \int (\ln w)^2\,dw.$$

Now, we use Integration by Parts with $u = (\ln w)^2$ and $v' = 1$. We find $u' = 2\frac{\ln w}{w}$, $v = w$, and

$$\int (\ln w)^2\,dw = w(\ln w)^2 - 2\int \ln w\,dw.$$

We use Integration by Parts again, this time with $u = \ln w$ and $v' = 1$. We find $u' = \frac{1}{w}$, $v = w$, and

$$\int \ln w\,dx = w\ln w - \int dw = w\ln w - w + C.$$

Thus,

$$\int (\ln w)^2\,dw = w(\ln w)^2 - 2w\ln w + 2w + C,$$

and

$$\int (\ln(x+1))^2\,dx = (x+1)\,[\ln(x+1)]^2 - 2(x+1)\ln(x+1) + 2(x+1) + C.$$

23. $\int \sin^5\theta\,d\theta$

SOLUTION We use the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$ to rewrite the integral:

$$\int \sin^5\theta\,d\theta = \int \sin^4\theta\sin\theta\,d\theta = \int \left(1-\cos^2\theta\right)^2 \sin\theta\,d\theta.$$

Now, we substitute $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and

$$\begin{aligned} \int \sin^5\theta\,d\theta &= \int \left(1-u^2\right)^2(-du) = -\int \left(1 - 2u^2 + u^4\right) du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C = -\frac{\cos^5\theta}{5} + \frac{2\cos^3\theta}{3} - \cos\theta + C. \end{aligned}$$

24. $\int \cos^4(9x-2)\,dx$

SOLUTION We substitute $u = 9x-2$, $du = 9\,dx$ and then use a reduction formula to evaluate the resulting integral. We obtain:

$$\int \cos^4(9x-2)\,dx = \frac{1}{9}\int \cos^4 u\,du = \frac{1}{9}\left(\frac{\cos^3 u\sin u}{4} + \frac{3}{4}\int \cos^2 u\,du\right)$$

$$= \frac{\cos^3 u \sin u}{36} + \frac{1}{12}\int \cos^2 u\,du = \frac{\cos^3 u \sin u}{36} + \frac{1}{12}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + C$$
$$= \frac{\cos^3(9x-2)\sin(9x-2)}{36} + \frac{9x-2}{24} + \frac{\sin(18x-4)}{48} + C.$$

25. $\displaystyle\int_0^{\pi/4} \sin 3x \cos 5x\,dx$

SOLUTION First compute the indefinite integral, using the trigonometric identity:

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right).$$

For $\alpha = 3x$ and $\beta = 5x$ we get:

$$\sin 3x \cos 5x = \frac{1}{2}\left(\sin 8x + \sin(-2x)\right) = \frac{1}{2}(\sin 8x - \sin 2x).$$

Hence,

$$\int \sin 3x \cos 5x\,dx = \frac{1}{2}\int \sin 8x\,dx - \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C.$$

Then

$$\int_0^{\pi/4} \sin 3x \cos 5x\,dx = \left(\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x\right)\Big|_0^{\pi/4} = \frac{1}{4}\cos\frac{\pi}{2} - \frac{1}{16}\cos 2\pi - \frac{1}{4}\cos 0 + \frac{1}{16}\cos 0 = -\frac{1}{4}$$

26. $\displaystyle\int \sin 2x \sec^2 x\,dx$

SOLUTION We use the trigonometric identity $\sin 2x = 2\cos x \sin x$ to rewrite the integrand:

$$\sin 2x \sec^2 x = 2\sin x \cos x \sec^2 x = \frac{2\sin x \cos x}{\cos^2 x} = \frac{2\sin x}{\cos x} = 2\tan x.$$

Hence,

$$\int \sin 2x \sec^2 x\,dx = \int 2\tan x\,dx = 2\ln|\sec x| + C.$$

27. $\displaystyle\int \sqrt{\tan x}\,\sec^2 x\,dx$

SOLUTION We substitute $u = \tan x$. Then $du = \sec^2 x\,dx$ and we obtain:

$$\int \sqrt{\tan x}\,\sec^2 x\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C.$$

28. $\displaystyle\int (\sec x + \tan x)^2\,dx$

SOLUTION We rewrite the integrand as

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x = 2\sec x \tan x + 2\sec^2 x - 1.$$

Therefore,

$$\int (\sec x + \tan x)^2\,dx = 2\int \sec x \tan x\,dx + 2\int \sec^2 x\,dx - \int dx = 2\sec x + 2\tan x - x + C.$$

29. $\displaystyle\int \sin^5\theta \cos^3\theta\,d\theta$

SOLUTION We use the identity $\cos^2\theta = 1 - \sin^2\theta$ to rewrite the integral:

$$\int \sin^5\theta\cos^3\theta\,d\theta = \int \sin^5\theta\cos^2\theta\cos\theta\,d\theta = \int \sin^5\theta\left(1 - \sin^2\theta\right)\cos\theta\,d\theta.$$

Now, we use the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$:

$$\int \sin^5\theta\cos^3\theta\,d\theta = \int u^5\left(1 - u^2\right)du = \int\left(u^5 - u^7\right)du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6\theta}{6} - \frac{\sin^8\theta}{8} + C.$$

30. $\displaystyle\int \cot^3 x \csc x\, dx$

SOLUTION Use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^3 x \csc x\, dx = \int \left(\csc^2 x - 1\right) \csc x \cot x\, dx.$$

Now use the substitution $u = \csc x$, $du = -\csc x \cot x\, dx$:

$$\int \cot^3 x \csc x\, dx = -\int \left(u^2 - 1\right) du = \int \left(1 - u^2\right) du = u - \frac{1}{3}u^3 + C = \csc x - \frac{1}{3}\csc^3 x + C.$$

31. $\displaystyle\int \cot^2 x \csc^2 x\, dx$

SOLUTION Use the substitution $u = \cot x$, $du = -\csc^2 x\, dx$:

$$\int \cot^2 x \csc^2 x\, dx = -\int \cot^2 x \left(-\csc^2 x\, dx\right) = -\int u^2\, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cot^3 x + C.$$

32. $\displaystyle\int_{\pi/2}^{\pi} \cot^2 \frac{\theta}{2}\, d\theta$

SOLUTION To compute the indefinite integral, substitute $u = \frac{\theta}{2}$. Then $du = \frac{1}{2}\, d\theta$ and

$$\int \cot^2 \frac{\theta}{2}\, d\theta = 2\int \cot^2 u\, du.$$

Now, we use a reduction formula to compute

$$\int \cot^2 \frac{\theta}{2}\, d\theta = 2\int \cot^2 u\, du = 2(-\cot u - u) + C = -2\cot\frac{\theta}{2} - \theta + C.$$

Then

$$\int_{\pi/2}^{\pi} \cot^2 \frac{\theta}{2}\, d\theta = \left(-2\cot\frac{\theta}{2} - \theta\right)\Bigg|_{\pi/2}^{\pi} = -2\cot\frac{\pi}{2} - \pi + 2\cot\frac{\pi}{4} + \frac{\pi}{2} = 0 - \pi + 2 + \frac{\pi}{2} = 2 - \frac{\pi}{2}$$

33. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^2 x \csc^3 x\, dx$

SOLUTION To compute the indefinite integral, use the identity $\cot^2 x = \csc^2 x - 1$ to write

$$\int \cot^2 x \csc^3 x\, dx = \int \left(\csc^2 x - 1\right) \csc^3 x\, dx = \int \csc^5 x\, dx - \int \csc^3 x\, dx.$$

Now use the reduction formula for $\csc^m x$:

$$\begin{aligned}
\int \cot^2 x \csc^3 x\, dx &= \left(-\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\int \csc^3 x\, dx\right) - \int \csc^3 x\, dx \\
&= -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\int \csc^3 x\, dx \\
&= -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\left(-\frac{1}{2}\cot x \csc x + \frac{1}{2}\int \csc x\, dx\right) \\
&= -\frac{1}{4}\cot x \csc^3 x + \frac{1}{8}\cot x \csc x - \frac{1}{8}\ln|\csc x - \cot x| + C.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cos^2 x \csc^3 x\, dx &= \left(-\frac{1}{4}\cot x \csc^3 x + \frac{1}{8}\cot x \csc x - \frac{1}{8}\ln|\csc x - \cot x|\right)\Bigg|_{\pi/4}^{\pi/2} \\
&= -\frac{1}{4}\cot\frac{\pi}{2}\csc^3\frac{\pi}{2} + \frac{1}{8}\cot\frac{\pi}{2}\csc\frac{\pi}{2} - \frac{1}{8}\ln\left|\csc\frac{\pi}{2} - \cot\frac{\pi}{2}\right| \\
&\quad + \frac{1}{4}\cot\frac{\pi}{4}\csc^3\frac{\pi}{4} - \frac{1}{8}\cot\frac{\pi}{4}\csc\frac{\pi}{4} + \frac{1}{8}\ln\left|\csc\frac{\pi}{4} - \cot\frac{\pi}{4}\right| \\
&= 0 + 0 - \frac{1}{8}\ln|1 - 0| + \frac{1}{4}\cdot 1\cdot(\sqrt{2})^3 - \frac{1}{8}\cdot 1\cdot\sqrt{2} + \frac{1}{8}\ln\left|\sqrt{2} - 1\right| \\
&= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} + \frac{1}{8}\ln(\sqrt{2} - 1) = \frac{3}{8}\sqrt{2} + \frac{1}{8}\ln(\sqrt{2} - 1)
\end{aligned}$$

34. $\displaystyle\int_4^6 \frac{dt}{(t-3)(t+4)}$

SOLUTION The partial fraction decomposition takes the form

$$\frac{1}{(t-3)(t+4)} = \frac{A}{t-3} + \frac{B}{t+4}.$$

Clearing denominators gives us

$$1 = A(t+4) + B(t-3) = (A+B)t + 4A - 3B.$$

Setting $t = 3$ then yields $A = \frac{1}{7}$, while setting $t = -4$ yields $B = -\frac{1}{7}$. Hence,

$$\int_4^6 \frac{dt}{(t-3)(t+4)} = \frac{1}{7}\int_4^6 \frac{dt}{t-3} - \frac{1}{7}\int_4^6 \frac{dt}{t+4} = \left(\frac{1}{7}\ln|t-3| - \frac{1}{7}\ln|t+4|\right)\Big|_4^6$$
$$= \left(\frac{1}{7}\ln\left|\frac{t-3}{t+4}\right|\right)\Big|_4^6 = \frac{1}{7}\left(\ln\frac{3}{10} - \ln\frac{1}{8}\right) = \frac{1}{7}\ln\frac{12}{5}$$

35. $\displaystyle\int \frac{dt}{(t-3)^2(t+4)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{1}{(t-3)^2(t+4)} = \frac{A}{t+4} + \frac{B}{t-3} + \frac{C}{(t-3)^2}.$$

Clearing denominators gives us

$$1 = A(t-3)^2 + B(t-3)(t+4) + C(t+4).$$

Setting $t = 3$ then yields $C = \frac{1}{7}$, while setting $t = -4$ yields $A = \frac{1}{49}$. Lastly, setting $t = 0$ yields

$$1 = 9A - 12B + 4C \qquad \text{or} \qquad B = -\frac{1}{49}.$$

Hence,

$$\int \frac{dt}{(t-3)^2(t+4)} = \frac{1}{49}\int \frac{dt}{t+4} - \frac{1}{49}\int \frac{dt}{t-3} + \frac{1}{7}\int \frac{dt}{(t-3)^2}$$
$$= \frac{1}{49}\ln|t+4| - \frac{1}{49}\ln|t-3| + \frac{1}{7}\cdot\frac{-1}{t-3} + C = \frac{1}{49}\ln\left|\frac{t+4}{t-3}\right| - \frac{1}{7}\cdot\frac{1}{t-3} + C.$$

36. $\displaystyle\int \sqrt{x^2+9}\,dx$

SOLUTION Substitute $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$. Then

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9\left(\tan^2\theta + 1\right)} = 3\sqrt{\sec^2\theta} = 3\sec\theta,$$

and

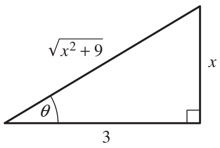
$$\int \sqrt{x^2+9}\,dx = \int 3\sec\theta \cdot 3\sec^2\theta\,d\theta = 9\int \sec^3\theta\,d\theta.$$

We use a reduction formula to compute the resulting integral:

$$\int \sqrt{x^2+9}\,dx = 9\int \sec^3\theta\,d\theta = 9\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int \sec\theta\,d\theta\right) = \frac{9\tan\theta\sec\theta}{2} + \frac{9}{2}\ln|\sec\theta + \tan\theta| + C.$$

We now return to the original variable x. Since $x = 3\tan\theta$, we have $\theta = \tan^{-1}\frac{x}{3}$. We also use the figure to obtain:

$$\int \sqrt{x^2+9}\,dx = \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2+9}}{3} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C = \frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C.$$



37. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$

SOLUTION Substitute $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$. Then

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\left(\sec^2\theta - 1\right)} = \sqrt{4\tan^2\theta} = 2\tan\theta,$$

and

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{2\sec\theta \cdot 2\tan\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C.$$

Now, return to the original variable x. Since $x = 2\sec\theta$, we have $\sec\theta = \frac{x}{2}$ or $\theta = \sec^{-1}\frac{x}{2}$. Thus,

$$\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\sec^{-1}\frac{x}{2} + C.$$

38. $\displaystyle\int_8^{27} \frac{dx}{x + x^{2/3}}$

SOLUTION We rewrite the integrand:

$$\int_8^{27} \frac{dx}{x + x^{2/3}} = \int_8^{27} \frac{dx}{x^{2/3}\left(x^{1/3}+1\right)} = \int_8^{27} t\,\frac{x^{-2/3}\,dx}{1 + x^{1/3}}.$$

Now, use the substitution $u = 1 + x^{1/3}$, $du = \frac{1}{3}x^{-2/3}\,dx$. $x = 8$ corresponds to $u = 3$, and $x = 27$ corresponds to $u = 4$. Then

$$\int_8^{27} \frac{dx}{x + x^{2/3}} = \int_8^{27} \frac{x^{-2/3}\,dx}{1 + x^{1/3}} = 3\int_3^4 \frac{du}{u} = 3\left(\ln|u|\right)\Big|_3^4 = 3(\ln 4 - \ln 3)$$

39. $\displaystyle\int \frac{dx}{x^{3/2} + ax^{1/2}}$

SOLUTION Let $u = x^{1/2}$ or $x = u^2$. Then $dx = 2u\,du$ and

$$\int \frac{dx}{x^{3/2} + ax^{1/2}} = \int \frac{2u\,du}{u^3 + au} = 2\int \frac{du}{u^2 + a}.$$

If $a > 0$, then

$$\int \frac{dx}{x^{3/2} + ax^{1/2}} = 2\int \frac{du}{u^2 + a} = \frac{2}{\sqrt{a}}\tan^{-1}\left(\frac{u}{\sqrt{a}}\right) + C = \frac{2}{\sqrt{a}}\tan^{-1}\sqrt{\frac{x}{a}} + C.$$

If $a = 0$, then

$$\int \frac{dx}{x^{3/2}} = -\frac{2}{\sqrt{x}} + C.$$

Finally, if $a < 0$, then

$$\int \frac{du}{u^2 + a} = \int \frac{du}{u^2 - \left(\sqrt{-a}\right)^2},$$

and the partial fraction decomposition takes the form

$$\frac{1}{u^2 - \left(\sqrt{-a}\right)^2} = \frac{A}{u - \sqrt{-a}} + \frac{B}{u + \sqrt{-a}}.$$

Clearing denominators gives us

$$1 = A(u + \sqrt{-a}) + B(u - \sqrt{-a}).$$

Setting $u = \sqrt{-a}$ then yields $A = \frac{1}{2\sqrt{-a}}$, while setting $u = -\sqrt{-a}$ yields $B = -\frac{1}{2\sqrt{-a}}$. Hence,

$$\begin{aligned}
\int \frac{dx}{x^{3/2} + ax^{1/2}} = 2\int \frac{du}{u^2 + a} &= \frac{1}{\sqrt{-a}}\int \frac{du}{u - \sqrt{-a}} - \frac{1}{\sqrt{-a}}\int \frac{du}{u + \sqrt{-a}} \\
&= \frac{1}{\sqrt{-a}}\ln|u - \sqrt{-a}| - \frac{1}{\sqrt{-a}}\ln\left|u + \sqrt{-a}\right| + C \\
&= \frac{1}{\sqrt{-a}}\ln\left|\frac{u - \sqrt{-a}}{u + \sqrt{-a}}\right| + C = \frac{1}{\sqrt{-a}}\ln\left|\frac{\sqrt{x} - \sqrt{-a}}{\sqrt{x} + \sqrt{-a}}\right| + C.
\end{aligned}$$

In summary,

$$\int \frac{dx}{x^{3/2}+ax^{1/2}} = \begin{cases} \frac{2}{\sqrt{a}}\tan^{-1}\sqrt{\frac{x}{a}}+C & a>0 \\ \frac{1}{\sqrt{-a}}\ln\left|\frac{\sqrt{x}-\sqrt{-a}}{\sqrt{x}+\sqrt{-a}}\right|+C & a<0 \\ -\frac{2}{\sqrt{x}}+C & a=0 \end{cases}$$

40. $\int \frac{dx}{(x-b)^2+4}$

SOLUTION Substitute $u = x - b$, $du = dx$. Then

$$\int \frac{dx}{(x-b)^2+4} = \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{x-b}{2}\right) + C.$$

41. $\int \frac{(x^2-x)\,dx}{(x+2)^3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2-x}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}.$$

Clearing denominators gives us

$$x^2 - x = A(x+2)^2 + B(x+2) + C.$$

Setting $x = -2$ then yields $C = 6$. Equating x^2-coefficients gives us $A = 1$, and equating x-coefficients yields $4A + B = -1$, or $B = -5$. Thus,

$$\int \frac{x^2-x}{(x+2)^3}\,dx = \int \frac{dx}{x+2} + \int \frac{-5\,dx}{(x+2)^2} + \int \frac{6\,dx}{(x+2)^3} = \ln|x+2| + \frac{5}{x+2} - \frac{3}{(x+2)^2} + C.$$

42. $\int \frac{(7x^2+x)\,dx}{(x-2)(2x+1)(x+1)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{7x^2+x}{(x-2)(2x+1)(x+1)} = \frac{A}{x-2} + \frac{B}{2x+1} + \frac{C}{x+1}.$$

Clearing denominators gives us

$$7x^2 + x = A(2x+1)(x+1) + B(x-2)(x+1) + C(x-2)(2x+1).$$

Setting $x = 2$ then yields $A = 2$, while setting $x = -\frac{1}{2}$ yields $B = -1$, and setting $x = -1$ yields $C = 2$. Hence,

$$\int \frac{7x^2+x}{(x-2)(2x+1)(x+1)}\,dx = 2\int \frac{dx}{x-2} - \int \frac{dx}{2x+1} + 2\int \frac{dx}{x+1}$$

$$= 2\ln|x-2| - \frac{1}{2}\ln|2x+1| + 2\ln|x+1| + C.$$

43. $\int \frac{16\,dx}{(x-2)^2(x^2+4)}$

SOLUTION The partial fraction decomposition has the form

$$\frac{16}{(x-2)^2\left(x^2+4\right)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4}.$$

Clearing denominators gives us

$$16 = A(x-2)\left(x^2+4\right) + B\left(x^2+4\right) + (Cx+D)(x-2)^2.$$

Setting $x = 2$ then yields $B = 2$. With $B = 2$,

$$16 = A\left(x^3 - 2x^2 + 4x - 8\right) + 2\left(x^2+4\right) + Cx^3 + (D-4C)x^2 + (4C-4D)x + 4D$$

$$16 = (A+C)x^3 + (-2A + 2 + D - 4C)\,x^2 + (4A + 4C - 4D)x + (-8A + 8 + 4D)$$

Equating coefficients of like powers of x now gives us the system of equations

$$\begin{aligned} A + C &= 0 \\ -2A - 4C + D + 2 &= 0 \\ 4A + 4C - 4D &= 0 \\ -8A + 4D + 8 &= 1 \end{aligned}$$

whose solution is

$$A = -1,\ C = 1,\ D = 0.$$

Thus,

$$\begin{aligned} \int \frac{dx}{(x-2)^2\left(x^2+4\right)} &= -\int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2} + \int \frac{x}{x^2+4}\,dx \\ &= -\ln|x-2| - 2\frac{1}{x-2} + \frac{1}{2}\ln\left(x^2+4\right) + C. \end{aligned}$$

44. $\displaystyle\int \frac{dx}{(x^2+25)^2}$

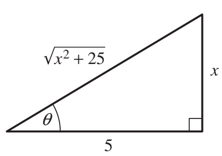
SOLUTION Use the trigonometric substitution $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$,

$$x^2 + 25 = (5\tan\theta)^2 + 25 = 25\left(\tan^2\theta + 1\right) = 25\sec^2\theta.$$

Then,

$$\begin{aligned} \int \frac{dx}{\left(x^2+25\right)^2} &= \int \frac{5\sec^2\theta\,d\theta}{\left(25\sec^2\theta\right)^2} = \int \frac{d\theta}{125\sec^2\theta} = \frac{1}{125}\int \cos^2\theta\,d\theta \\ &= \frac{1}{125}\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\theta\right) + C = \frac{1}{250}(\cos\theta\sin\theta + \theta) + C. \end{aligned}$$

To return to the original variable x we use the relation $x = 5\tan\theta$ and the accompanying figure.



Thus,

$$\int \frac{dx}{\left(x^2+25\right)^2} = \frac{1}{250}\left(\frac{5}{\sqrt{x^2+25}} \cdot \frac{x}{\sqrt{x^2+25}} + \tan^{-1}\left(\frac{x}{5}\right)\right) + C = \frac{1}{50}\frac{x}{x^2+25} + \frac{1}{250}\tan^{-1}\left(\frac{x}{5}\right) + C.$$

45. $\displaystyle\int \frac{dx}{x^2+8x+25}$

SOLUTION Complete the square to rewrite the denominator as

$$x^2 + 8x + 25 = (x+4)^2 + 9.$$

Now, let $u = x + 4$, $du = dx$. Then,

$$\int \frac{dx}{x^2+8x+25} = \int \frac{du}{u^2+9} = \frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{3}\tan^{-1}\left(\frac{x+4}{3}\right) + C.$$

46. $\displaystyle\int \frac{dx}{x^2+8x+4}$

SOLUTION Use the method of partial fractions. To facilitate the computations we first complete the square in the denominator:

$$\frac{1}{x^2+8x+4} = \frac{1}{(x+4)^2 - 12}.$$

Now we substitute $t = x + 4$. Then $dt = dx$ and

$$\int \frac{dx}{x^2 + 8x + 4} = \int \frac{dt}{t^2 - 12} = \int \frac{dt}{\left(t - 2\sqrt{3}\right)\left(t + 2\sqrt{3}\right)}.$$

We use the following partial fraction decomposition of the integrand:

$$\frac{1}{\left(t - 2\sqrt{3}\right)\left(t + 2\sqrt{3}\right)} = \frac{A}{t - 2\sqrt{3}} + \frac{B}{t + 2\sqrt{3}}.$$

Clearing denominators gives us

$$1 = A\left(t + 2\sqrt{3}\right) + B\left(t - 2\sqrt{3}\right).$$

Setting $t = 2\sqrt{3}$ then yields $A = \frac{1}{4\sqrt{3}}$, while setting $t = -2\sqrt{3}$ yields $B = -\frac{1}{4\sqrt{3}}$. Hence,

$$\int \frac{dx}{x^2 + 8x + 4} = \frac{1}{4\sqrt{3}} \int \frac{dt}{t - 2\sqrt{3}} - \frac{1}{4\sqrt{3}} \int \frac{dt}{t + 2\sqrt{3}} = \frac{1}{4\sqrt{3}} \ln|t - 2\sqrt{3}| - \frac{1}{4\sqrt{3}} \ln|t + 2\sqrt{3}| + C$$

$$= \frac{1}{4\sqrt{3}} \ln\left|\frac{t - 2\sqrt{3}}{t + 2\sqrt{3}}\right| + C = \frac{1}{4\sqrt{3}} \ln\left|\frac{x + 42\sqrt{3}}{x + 4 + 2\sqrt{3}}\right| + C.$$

47. $\displaystyle\int \frac{(x^2 - x)\,dx}{(x + 2)^3}$

SOLUTION The partial fraction decomposition has the form

$$\frac{x^2 - x}{(x + 2)^3} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{(x + 2)^3}.$$

Clearing denominators gives us

$$x^2 - x = A(x + 2)^2 + B(x + 2) + C.$$

Setting $x = -2$ then yields $C = 6$. Equating x^2-coefficients gives us $A = 1$, and equating x-coefficients yields $4A + B = -1$, or $B = -5$. Thus,

$$\int \frac{x^2 - x}{(x + 2)^3}\,dx = \int \frac{dx}{x + 2} + \int \frac{-5\,dx}{(x + 2)^2} + \int \frac{6\,dx}{(x + 2)^3} = \ln|x + 2| + \frac{5}{x + 2} - \frac{3}{(x + 2)^2} + C.$$

48. $\displaystyle\int_0^1 t^2\sqrt{1 - t^2}\,dt$

SOLUTION First compute the indefinite integral by using the substitution $t = \sin\theta$, $dt = \cos\theta\,d\theta$. We have

$$\sqrt{1 - t^2} = \sqrt{1 - \sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta,$$

and

$$\int t^2\sqrt{1 - t^2}\,dt = \int \sin^2\theta\cos\theta\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta$$

$$= \int (1 - \cos^2\theta)\cos^2\theta\,d\theta = \int \cos^2\theta\,d\theta - \int \cos^4\theta\,d\theta$$

$$= \int \cos^2\theta\,d\theta - \left(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int \cos^2\theta\,d\theta\right)$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\int \cos^2\theta\,d\theta$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right) + C$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{8}\cos\theta\sin\theta + \frac{1}{8}\theta + C.$$

Now, return to the original variable t. Since $t = \sin\theta$, $\cos\theta = \sqrt{1 - t^2}$ and

$$\int t^2\sqrt{1 - t^2}\,dt = -\frac{t(1 - t^2)^{3/2}}{4} + \frac{t\sqrt{1 - t^2}}{8} + \frac{\sin^{-1} t}{8} + C = \frac{t^3\sqrt{1 - t^2}}{4} + \frac{\sin^{-1} t}{8} - \frac{t\sqrt{1 - t^2}}{8} + C.$$

Then

$$\int_0^1 t^2\sqrt{1-t^2}\,dt = \left(\frac{t^3\sqrt{1-t^2}}{4} + \frac{\sin^{-1} t}{8} - \frac{t\sqrt{1-t^2}}{8}\right)\Bigg|_0^1$$

$$= 0 + \frac{1}{8}\sin^{-1} 1 - 0 - 0 + \frac{1}{8}\sin^{-1} 0 + 0 = \frac{\sin^{-1} 1}{8} = \frac{\pi}{16}$$

49. $\int \frac{dx}{x^4\sqrt{x^2+4}}$

SOLUTION Substitute $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$. Then

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\left(\tan^2\theta + 1\right)} = 2\sqrt{\sec^2\theta} = 2\sec\theta,$$

and

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{16\tan^4\theta \cdot 2\sec\theta} = \int \frac{\sec\theta\,d\theta}{16\tan^4\theta}.$$

We have

$$\frac{\sec\theta}{\tan^4\theta} = \frac{\cos^3\theta}{\sin^4\theta}.$$

Hence,

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \frac{1}{16}\int \frac{\cos^3\theta\,d\theta}{\sin^4\theta} = \frac{1}{16}\int \frac{\cos^2\theta\cos\theta\,d\theta}{\sin^4\theta} = \frac{1}{16}\int \frac{\left(1-\sin^2\theta\right)\cos\theta\,d\theta}{\sin^4\theta}.$$

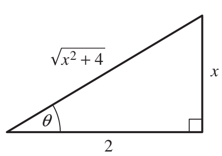
Now substitute $u = \sin\theta$ and $du = \cos\theta\,d\theta$ to obtain

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = \frac{1}{16}\int \frac{1-u^2}{u^4}\,du = \frac{1}{16}\int \left(u^{-4} - u^{-2}\right)du = -\frac{1}{48u^3} + \frac{1}{16}\frac{1}{u} + C$$

$$= -\frac{1}{48}\cdot\frac{1}{\sin^3\theta} + \frac{1}{16}\frac{1}{\sin\theta} + C = -\frac{1}{48}\csc^3\theta + \frac{1}{16}\csc\theta + C.$$

Finally, return to the original to the original variable x using the relation $x = 2\tan\theta$ and the figure below.

$$\int \frac{dx}{x^4\sqrt{x^2+4}} = -\frac{1}{48}\left(\frac{\sqrt{x^2+4}}{x}\right)^3 + \frac{1}{16}\frac{\sqrt{x^2+4}}{x} + C = -\frac{\left(x^2+4\right)^{3/2}}{48x^3} + \frac{\sqrt{x^2+4}}{16x} + C.$$



50. $\int \frac{dx}{(x^2+5)^{3/2}}$

SOLUTION Substitute $x = \sqrt{5}\tan\theta$. Then $dx = \sqrt{5}\sec^2\theta\,d\theta$,

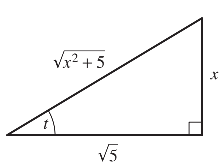
$$x^2 + 5 = 5\tan^2\theta + 5 = 5(\tan^2\theta + 1) = 5\sec^2\theta,$$

and

$$\int \frac{dx}{\left(x^2+5\right)^{3/2}} = \frac{1}{5}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{5}\int \cos\theta\,d\theta = \frac{1}{5}\sin\theta + C.$$

We now return to the original variable x using the relation $x = \sqrt{5}\tan\theta$ and the figure below. Thus,

$$\int \frac{dx}{\left(x^2+5\right)^{3/2}} = \frac{1}{5}\cdot\frac{x}{\sqrt{x^2+5}} + C.$$



51. $\int (x+1)e^{4-3x}\,dx$

SOLUTION We compute the integral using Integration by Parts with $u = x+1$ and $v' = e^{4-3x}$. Then $u' = 1$, $v = -\frac{1}{3}e^{4-3x}$ and

$$\int (x+1)e^{4-3x}\,dx = -\frac{1}{3}(x+1)e^{4-3x} + \frac{1}{3}\int e^{4-3x}\,dx = -\frac{1}{3}(x+1)e^{4-3x} + \frac{1}{3}\cdot\left(-\frac{1}{3}\right)e^{4-3x} + C$$
$$= -\frac{1}{9}e^{4-3x}(3x+4) + C.$$

52. $\int x^{-2}\tan^{-1}x\,dx$

SOLUTION We use Integration by Parts with $u = \tan^{-1}x$ and $v' = x^{-2}$. Then $u' = \frac{1}{1+x^2}$, $v = -x^{-1}$ and

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x\left(1+x^2\right)}.$$

For the remaining integral, the partial fraction decomposition takes the form

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}.$$

Clearing denominators gives us

$$1 = A(1+x^2) + (Bx+C)x.$$

Setting $x = 0$ then yields $A = 1$. Next, equating the x^2-coefficients gives

$$0 = A + B \qquad \text{so} \qquad B = -1,$$

while equating x-coefficients gives $C = 0$. Hence,

$$\frac{1}{x\left(1+x^2\right)} = \frac{1}{x} - \frac{x}{1+x^2},$$

and

$$\int \frac{dx}{x\left(1+x^2\right)} = \int \frac{1}{x}\,dx - \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

Therefore,

$$\int x^{-2}\tan^{-1}x\,dx = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

53. $\int x^3\cos(x^2)\,dx$

SOLUTION Substitute $t = x^2$, $dt = 2x\,dx$. Then

$$\int x^3\cos\left(x^2\right)\,dx = \frac{1}{2}\int t\cos t\,dt.$$

We compute the resulting integral using Integration by Parts with $u = t$ and $v' = \cos t$. Then $u' = 1$, $v = \sin t$ and

$$\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t + C.$$

Thus,

$$\int x^3\cos\left(x^2\right)\,dx = \frac{1}{2}x^2\sin x^2 + \frac{1}{2}\cos x^2 + C.$$

54. $\int x^2(\ln x)^2\,dx$

SOLUTION We use Integration by Parts with $u = (\ln x)^2$ and $v' = x^2$. Then $u' = \frac{2\ln x}{x}$, $v = \frac{x^3}{3}$ and

$$\int x^2(\ln x)^2dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2\ln x\,dx.$$

To calculate the resulting integral, we again use Integration by Parts, this time with $u = \ln x$ and $v' = x^2$. Then, $u' = \frac{1}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2 \ln x\, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

Finally,

$$\int x^2 (\ln x)^2 dx = \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \left(\frac{x^3}{3} \ln x - \frac{x^3}{9} \right) + C = \frac{x^3}{3} \left((\ln x)^2 - \frac{2}{3} \ln x + \frac{2}{9} \right) + C.$$

55. $\int x \tanh^{-1} x\, dx$

SOLUTION We use Integration by Parts with $u = \tanh^{-1} x$ and $v' = x$. Then $u' = \frac{1}{1-x^2}$, $v = \frac{x^2}{2}$ and

$$\int x \tanh^{-1} x\, dx = \frac{x^2}{2} \tanh^{-1} x - \frac{1}{2} \int \frac{x^2}{1 - x^2}\, dx.$$

Now

$$\frac{x^2}{1-x^2} = \frac{x^2 - 1 + 1}{1 - x^2} = -1 + \frac{1}{1-x^2},$$

and the partial fraction decomposition for the remaining fraction takes the form

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}.$$

Clearing denominators gives us

$$1 = A(1+x) + B(1-x).$$

Setting $x = 1$ then yields $A = \frac{1}{2}$, while setting $x = -1$ yields $B = \frac{1}{2}$. Thus,

$$\begin{aligned} \int \frac{x^2}{1-x^2} &= -\int dx + \frac{1}{2} \int \frac{1}{1-x}\, dx + \frac{1}{2} \int \frac{1}{1+x}\, dx \\ &= -x - \frac{1}{2} \ln |1-x| + \frac{1}{2} \ln |1+x| + C = -x + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C. \end{aligned}$$

Therefore,

$$\int x \tanh^{-1} x\, dx = \frac{x^2}{2} \tanh^{-1} x - \frac{1}{2} \left(-x + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \right) + C = \frac{x^2}{2} \tanh^{-1} x + \frac{x}{2} - \frac{1}{4} \ln \left| \frac{1+x}{1-x} \right| + C.$$

56. $\int \frac{\tan^{-1} t\, dt}{1+t^2}$

SOLUTION Substitute $u = \tan^{-1} t$. Then, $du = \frac{dt}{1+t^2}$ and

$$\int \frac{\tan^{-1} t\, dt}{1+t^2} = \int u\, du = \frac{1}{2} u^2 + C = \frac{1}{2} \left(\tan^{-1} t \right)^2 + C.$$

57. $\int \ln(x^2 + 9)\, dx$

SOLUTION We compute the integral using Integration by Parts with $u = \ln \left(x^2 + 9 \right)$ and $v' = 1$. Then $u' = \frac{2x}{x^2+9}$, $v = x$, and

$$\int \ln \left(x^2 + 9 \right)\, dx = x \ln \left(x^2 + 9 \right) - \int \frac{2x^2}{x^2 + 9}\, dx.$$

To compute this integral we write:

$$\frac{x^2}{x^2+9} = \frac{\left(x^2 + 9 \right) - 9}{x^2 + 9} = 1 - \frac{9}{x^2+9};$$

hence,

$$\int \frac{x^2}{x^2+9}\,dx = \int 1\,dx - 9\int \frac{dx}{x^2+9} = x - 3\tan^{-1}\frac{x}{3} + C.$$

Therefore,

$$\int \ln\left(x^2+9\right)\,dx = x\ln\left(x^2+9\right) - 2x + 6\tan^{-1}\left(\frac{x}{3}\right) + C.$$

58. $\int (\sin x)(\cosh x)\,dx$

SOLUTION We compute the integral using Integration by Parts with $u = \sin x$ and $v' = \cosh x$. Then $u' = \cos x$, $v = \sinh x$ and

$$\int \sin x \cosh x\,dx = \sin x \sinh x - \int \cos x \sinh x\,dx.$$

We compute the resulting integral using Integration by Parts, this time with $u = \cos x$ and $v' = \sinh x$. Then $u' = -\sin x$, $v = \cosh x$ and

$$\int \cos x \sinh x\,dx = \cos x \cosh x + \int \sin x \cosh x\,dx.$$

Therefore,

$$\int \sin x \cosh x\,dx = \sin x \sinh x - \cos x \cosh x - \int \sin x \cosh x\,dx.$$

Solving for $\int (\sin x)(\cosh x)\,dx$, we find

$$2\int \sin x \cosh x\,dx = \sin x \sinh x - \cos x \cosh x + C$$
$$\int \sin x \cosh x\,dx = \frac{1}{2}\sin x \sinh x - \frac{1}{2}\cos x \cosh x + C$$

59. $\int_0^1 \cosh 2t\,dt$

SOLUTION $\int_0^1 \cosh 2t\,dt = \frac{1}{2}\sinh 2t\Big|_0^1 = \frac{1}{2}\sinh 2.$

60. $\int \sinh^3 x \cosh x\,dx$

SOLUTION Let $u = \sinh x$. Then $du = \cosh x\,dx$ and

$$\int \sinh^3 x \cosh x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sinh^4 x + C.$$

61. $\int \coth^2(1-4t)\,dt$

SOLUTION $\int \coth^2(1-4t)\,dt = \int \left(1 + \operatorname{csch}^2(1-4t)\right)\,dt = t + \frac{1}{4}\coth(1-4t) + C.$

62. $\int_{-0.3}^{0.3} \frac{dx}{1-x^2}$

SOLUTION $\int_{-0.3}^{0.3} \frac{dx}{1-x^2} = \tanh^{-1} x\Big|_{-0.3}^{0.3} = 2\tanh^{-1}(0.3).$

63. $\int_0^{3\sqrt{3}/2} \frac{dx}{\sqrt{9-x^2}}$

SOLUTION $\int_0^{3\sqrt{3}/2} \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^{3\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}.$

64. $\displaystyle\int \frac{\sqrt{x^2+1}\,dx}{x^2}$

SOLUTION Let $x = \sinh t$. Then $dx = \cosh t\,dt$ and

$$\int \frac{\sqrt{x^2+1}\,dx}{x^2} = \int \frac{\cosh^2 t}{\sinh^2 t}\,dt = \int \coth^2 t\,dt = \int (1+\operatorname{csch}^2 t)\,dt = t - \coth t + C$$
$$= \sinh^{-1} x - \frac{\sqrt{x^2+1}}{x} + C.$$

65. Use the substitution $u = \tanh t$ to evaluate $\displaystyle\int \frac{dt}{\cosh^2 t + \sinh^2 t}$.

SOLUTION Let $u = \tanh t$. Then $du = \operatorname{sech}^2 t\,dt$ and

$$\int \frac{dt}{\cosh^2 t + \sinh^2 t} = \int \frac{\operatorname{sech}^2 t}{1+\tanh^2 t}\,dt = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(\tanh x) + C.$$

66. Find the volume obtained by rotating the region enclosed by $y = \ln x$ and $y = (\ln x)^2$ about the y-axis.

SOLUTION The curves meet at $(1, 0)$ and at $(e, 1)$. We compute the volume of the solid using the method of cylindrical shells:

$$V = \int_1^e 2\pi x \cdot (\ln x - (\ln x)^2)\,dx = 2\pi \int_1^e x \ln x\,dx - 2\pi \int_0^1 x(\ln x)^2\,dx$$

For the second integral, use integration by parts, with $u = (\ln x)^2$ and $v' = x$, so that $u' = \frac{2\ln x}{x}$ and $v = \frac{1}{2}x^2$. Then

$$V = 2\pi \int_1^e x \ln x\,dx - 2\pi \left(\frac{1}{2}x^2(\ln x)^2 \Big|_1^e - \int_1^e x \ln x\,dx \right) = -\pi e^2 + 4\pi \int_1^e x \ln x\,dx$$

Again apply integration by parts, with $u = \ln x$ and $v' = x$, so that $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. Then

$$V = -\pi e^2 + 4\pi \int_1^e x \ln x\,dx = -\pi e^2 + 4\pi \left(\frac{1}{2}x^2 \ln x \Big|_1^e - \frac{1}{2}\int_1^e x\,dx \right) = -\pi e^2 + 4\pi \left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} \right) = \pi$$

67. Let $I_n = \displaystyle\int \frac{x^n\,dx}{x^2+1}$.

(a) Prove that $I_n = \dfrac{x^{n-1}}{n-1} - I_{n-2}$.

(b) Use (a) to calculate I_n for $0 \le n \le 5$.

(c) Show that, in general,

$$I_{2n+1} = \frac{x^{2n}}{2n} - \frac{x^{2n-2}}{2n-2} + \cdots + (-1)^{n-1}\frac{x^2}{2} + (-1)^n \frac{1}{2}\ln(x^2+1) + C$$
$$I_{2n} = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots + (-1)^{n-1}x + (-1)^n \tan^{-1} x + C$$

SOLUTION

(a) $I_n = \displaystyle\int \frac{x^n}{x^2+1}\,dx = \int \frac{x^{n-2}(x^2+1-1)}{x^2+1}\,dx = \int x^{n-2}\,dx - \int \frac{x^{n-2}}{x^2+1}\,dx = \frac{x^{n-1}}{n-1} - I_{n-2}.$

(b) First compute I_0 and I_1 directly:

$$I_0 = \int \frac{x^0\,dx}{x^2+1} = \int \frac{dx}{x^2+1} = \tan^{-1}x + C \quad \text{and} \quad I_1 = \int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln\left(x^2+1\right) + C.$$

We now use the equality obtained in part (a) to compute I_2, I_3, I_4 and I_5:

$$I_2 = \frac{x^{2-1}}{2-1} - I_{2-2} = x - I_0 = x - \tan^{-1}x + C;$$
$$I_3 = \frac{x^{3-1}}{3-1} - I_{3-2} = \frac{x^2}{2} - I_1 = \frac{x^2}{2} - \frac{1}{2}\ln\left(x^2+1\right) + C;$$
$$I_4 = \frac{x^{4-1}}{4-1} - I_{4-2} = \frac{x^3}{3} - I_2 = \frac{x^3}{3} - \left(x - \tan^{-1}x\right) + C = \frac{x^3}{3} - x + \tan^{-1}x + C;$$
$$I_5 = \frac{x^{5-1}}{5-1} - I_{5-2} = \frac{x^4}{4} - I_3 = \frac{x^4}{4} - \left(\frac{x^2}{2} - \frac{1}{2}\ln\left(x^2+1\right)\right) + C = \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln\left(x^2+1\right) + C.$$

(c) We prove the two identities using mathematical induction. We first prove that for $n \geq 1$:

$$I_{2n+1} = \frac{x^{2n}}{2n} - \frac{x^{2n-2}}{2n-2} + \cdots + (-1)^n \cdot \frac{1}{2} \ln\left(x^2+1\right) + C.$$

We verify the equality for $n = 1$. Setting $n = 1$, we find

$$I_3 = \frac{x^2}{2} + (-1)^1 \cdot \frac{1}{2} \ln\left(x^2+1\right) + C = \frac{x^2}{2} - \frac{1}{2} \ln\left(x^2+1\right) + C,$$

which agrees with the value obtained in part (b). We now assume that for $n = k$:

$$I_{2k+1} = \frac{x^{2k}}{2k} - \frac{x^{2k-2}}{2k-2} + \cdots + (-1)^k \cdot \frac{1}{2} \ln\left(x^2+1\right) + C.$$

We use this assumption to prove the equality for $n = k + 1$. By part (a) and the induction hypothesis

$$\begin{aligned} I_{2k+3} = \frac{x^{2k+2}}{2k+2} - I_{2k+1} &= \frac{x^{2k+2}}{2k+2} - \frac{x^{2k}}{2k} + \frac{x^{2k-2}}{2k-2} - \cdots - (-1)^k \cdot \frac{1}{2} \ln\left(x^2+1\right) + C \\ &= \frac{x^{2k+2}}{2k+2} - \frac{x^{2k}}{2k} + \cdots + (-1)^{k+1} \cdot \frac{1}{2} \ln\left(x^2+1\right) + C \end{aligned}$$

as required. We now prove the second identity for $n \geq 1$:

$$I_{2n} = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots + (-1)^n \tan^{-1} x + C.$$

We verify this equality for $n = 1$:

$$I_2 = x - \tan^{-1} x + C,$$

which agrees with the value obtained in part (b). We now assume that for $n = k$

$$I_{2k} = \frac{x^{2k-1}}{2k-1} - \frac{x^{2k-3}}{2k-3} + \cdots + (-1)^k \tan^{-1} x + C.$$

We use this assumption to prove the equality for $n = k + 1$. By part (a) and the induction hypothesis

$$\begin{aligned} I_{2k+2} = \frac{x^{2k+1}}{2k+1} - I_{2k} &= \frac{x^{2k+1}}{2k+1} - \frac{x^{2k-1}}{2k-1} + \frac{x^{2k-3}}{2k-3} - \cdots - (-1)^k \cdot \tan^{-1} x + C \\ &= \frac{x^{2k+1}}{2k+1} - \frac{x^{2k-1}}{2k-1} + \cdots + (-1)^{k+1} \cdot \tan^{-1} x + C \end{aligned}$$

as required.

68. Let $J_n = \int x^n e^{-x^2/2}\, dx$.

(a) Show that $J_1 = -e^{-x^2/2}$.

(b) Prove that $J_n = -x^{n-1} e^{-x^2/2} + (n-1) J_{n-2}$.

(c) Use (a) and (b) to compute J_3 and J_5.

SOLUTION

(a) Let $u = -\frac{x^2}{2}$. Then $du = -x\, dx$ and

$$J_1 = \int x e^{-x^2/2}\, dx = -\int e^u\, du = -e^u + C = -e^{-x^2/2} + C.$$

(b) Using Integration by Parts with $u = x^{n-1}$ and $v' = x e^{-x^2/2}$, we find

$$J_n = -x^{n-1} e^{-x^2/2} + (n-1) \int x^{n-2} e^{-x^2/2}\, dx = -x^{n-1} e^{-x^2/2} + (n-1) J_{n-2}.$$

(c) Using the results from parts (a) and (b),

$$\begin{aligned} J_3 &= -x^{3-1} e^{-x^2/2} + (3-1) J_{3-2} = -x^2 e^{-x^2/2} + 2 J_1 \\ &= -x^2 e^{-x^2/2} - 2e^{-x^2/2} + C = -e^{-x^2/2}(x^2+2) + C \end{aligned}$$

and then

$$J_5 = -x^{5-1}e^{-x^2/2} + (5-1)J_{5-2} = -x^4e^{-x^2/2} + 4J_3$$
$$= -x^4e^{-x^2/2} - 4e^{-x^2/2}(x^2+2) + C = -e^{-x^2/2}(x^4+4x^2+8) + C.$$

69. Compute $p(X \le 1)$, where X is a continuous random variable with probability density $p(x) = \dfrac{1}{\pi(x^2+1)}$.

SOLUTION

$$P(X \le 1) = \int_{-\infty}^{1} p(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{1} \frac{1}{x^2+1}\,dx = \frac{1}{\pi}\tan^{-1} x\Big|_{-\infty}^{1} = \frac{1}{\pi}\cdot\left(\frac{\pi}{4} - \frac{-\pi}{2}\right) = \frac{3}{4}$$

70. Show that $p(x) = \frac{1}{4}e^{-x/2} + \frac{1}{6}e^{-x/3}$ is a probability density on $[0, \infty)$ and find its mean.

SOLUTION To show that $p(x)$ is a probability density, we must show that its integral over $[0, \infty)$ is 1:

$$\int_0^\infty p(x)\,dx = \int_0^\infty \left(\frac{1}{4}e^{-x/2} + \frac{1}{6}e^{-x/3}\right)dx = \left(-\frac{1}{2}e^{-x/2} - \frac{1}{2}e^{-x/3}\right)\Big|_0^\infty = 0 + 0 + \frac{1}{2} + \frac{1}{2} = 1$$

The mean of $p(x)$ is

$$\mu = \int_0^\infty xp(x)\,dx = \int_0^\infty \left(\frac{1}{4}xe^{-x/2} + \frac{1}{6}xe^{-x/3}\right)dx$$

Now, for a positive constant a, using integration by parts with $u = x$, $v' = e^{-x/a}$, we have $u' = 1$, $v = -ae^{-x/a}$, and

$$\int_0^\infty xe^{-x/a}\,dx = -axe^{-x/a}\Big|_0^\infty + a\int_0^\infty e^{-x/a}\,dx = -a^2\left(e^{-x/a}\right)\Big|_0^\infty = a^2$$

so that

$$\mu = \frac{1}{4}\int_0^\infty xe^{-x/2}\,dx + \frac{1}{6}\int_0^\infty xe^{-x/3}\,dx = \frac{1}{4}\cdot 4 + \frac{1}{6}\cdot 9 = \frac{5}{2}$$

71. Find a constant C such that $p(x) = Cx^3e^{-x^2}$ is a probability density and compute $p(0 \le X \le 1)$.

SOLUTION We first find the indefinite integral of $p(x)$ using integration by parts, with $u = x^2$, $v' = xe^{-x^2}$, so that $u' = 2x$ and $v = -\frac{1}{2}e^{-x^2}$:

$$\int Cx^3e^{-x^2}\,dx = C\left(-\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx\right) = C\left(-\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2}\right) = -\frac{C}{2}e^{-x^2}(x^2+1)$$

To determine the constant C, the value of the integral on the interval $[0, \infty)$ must be 1:

$$1 = \int_0^\infty Cx^3e^{-x^2}\,dx = -\frac{C}{2}e^{-x/2}(x^2+1)\Big|_0^\infty = -\frac{C}{2}\left(\lim_{R\to\infty}\frac{x^2+1}{e^{x/2}} - 1\right) = \frac{C}{2}$$

so that $C = 2$. Then

$$P(0 \le X \le 1) = \int_0^1 2x^3e^{-x^2}\,dx = -e^{-x^2}(x^2+1)\Big|_0^1 = 1 - 2e^{-1} \approx 0.13212$$

72. The interval between patient arrivals in an emergency room is a random variable with exponential density function $p(t) = 0.125e^{-0.125t}$ (t in minutes). What is the average time between patient arrivals? What is the probability of two patients arriving within 3 minutes of each other?

SOLUTION The mean of the distribution is (using integration by parts with $u = t$, $v' = 0.125e^{-0.125t}$):

$$\int_0^\infty tp(t)\,dt = \int_0^\infty 0.125te^{-0.125t}\,dt = te^{-0.125t}\Big|_0^\infty + \int_0^\infty e^{-0.125t}\,dt = -8e^{-0.125t}\Big|_0^\infty = 8$$

Since the distribution gives the waiting time between arrivals, it follows that the probability of two patients arriving within 3 minutes of each other is

$$\int_0^3 p(t)\,dt = \int_0^3 0.125e^{-0.125t}\,dt = -e^{-0.125t}\Big|_0^3 = 1 - e^{-0.375} \approx 1 - 0.68729 \approx 0.31271$$

73. Calculate the following probabilities, assuming that X is normally distributed with mean $\mu = 40$ and $\sigma = 5$.

(a) $p(X \geq 45)$ **(b)** $p(0 \leq X \leq 40)$

SOLUTION Let F be the standard normal cumulative distribution function. Then by Theorem 1 in Section 7.7,

(a)

$$p(X \geq 45) = 1 - p(X \leq 45) = 1 - F\left(\frac{45-40}{5}\right) = 1 - F(1) \approx 1 - 0.8413 \approx 0.1587$$

(b)

$$p(0 \leq X \leq 40) = p(X \leq 40) - p(X \leq 0) = F\left(\frac{40-40}{5}\right) - F\left(\frac{0-40}{5}\right)$$
$$= F(0) - F(-8) = \frac{1}{2} - F(-8) \approx \frac{1}{2} - 0 = \frac{1}{2}$$

Note that $p(X \leq 40)$ is exactly $\frac{1}{2}$ since 40 is the mean. Also, since -8 is so far to the left in the standard normal distribution, the probability of its occurrence is quite small (approximately 8×10^{-11}).

74. According to kinetic theory, the molecules of ordinary matter are in constant random motion. The energy E of a molecule is a random variable with density function $p(E) = \frac{1}{kT}e^{-E/(kT)}$, where T is the temperature (in kelvins) and k is Boltzmann's constant. Compute the *mean* kinetic energy $\overline{E}$ in terms of k and T.

SOLUTION By definition,

$$\int_0^\infty Ee^{-E/kT}\,dE = \lim_{R\to\infty}\int_0^R Ee^{-E/kT}\,dE.$$

We compute the definite integral using Integration by Parts with $u = E$, $v' = e^{-E/kT}$. Then $u' = 1$, $v = -kTe^{-E/kT}$ and

$$\int_0^R Ee^{-E/kT}\,dE = -kTe^{-E/kT}E\Big|_{E=0}^R + \int_0^R kTe^{-E/kT}\,dE = -kTe^{-R/kT}R - (kT)^2e^{-E/kT}\Big|_{E=0}^R$$
$$= -kTRe^{-R/kT} - \left(k^2T^2e^{-R/kT} - k^2T^2e^0\right) = k^2T^2 - kTRe^{-R/kT} - k^2T^2e^{-R/kT}.$$

We now let $R \to \infty$, obtaining:

$$\int_0^\infty Ee^{-E/RT}\,dE = \lim_{R\to\infty}\int_0^R Ee^{-E/RT}\,dE = \lim_{R\to\infty}\left(k^2T^2 - kTRe^{-R/kT} - k^2T^2e^{-R/kT}\right)$$
$$= k^2T^2 - kT\lim_{R\to\infty} Re^{-R/kT} - 0 = k^2T^2 - kT\lim_{R\to\infty} Re^{-R/kT}.$$

We compute the remaining limit using L'Hôpital's Rule:

$$\lim_{R\to\infty} Re^{-R/kT} = \lim_{R\to\infty}\frac{R}{e^{R/kT}} = \lim_{R\to\infty}\frac{\frac{dR}{dR}}{\frac{d}{dR}\left(e^{R/kT}\right)} = \lim_{R\to\infty}\frac{1}{\frac{1}{kT}e^{R/kT}} = 0.$$

Thus,

$$\int_0^\infty Ee^{-E/RT}\,dE = k^2T^2,$$

and

$$\overline{E} = \frac{1}{kT}\int_0^\infty Ee^{-E/kT}\,dE = \frac{1}{kT}\cdot k^2T^2 = kT.$$

In Exercises 75–84, determine whether the improper integral converges and, if so, evaluate it.

75. $\displaystyle\int_0^\infty \frac{dx}{(x+2)^2}$

SOLUTION

$$\int_0^\infty \frac{dx}{(x+2)^2} = \lim_{R\to\infty}\int_0^R \frac{dx}{(x+2)^2} = \lim_{R\to\infty} -\frac{1}{x+2}\Big|_0^R$$
$$= \lim_{R\to\infty}\left(-\frac{1}{R+2} + \frac{1}{0+2}\right) = \lim_{R\to\infty}\left(-\frac{1}{R+2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

76. $\displaystyle\int_4^\infty \frac{dx}{x^{2/3}}$

SOLUTION The integral $\displaystyle\int_a^\infty \frac{dx}{x^p}$ $(a > 0)$ converges if $p > 1$ and diverges if $p \le 1$. Here, $p = \frac{2}{3} < 1$, hence the integral diverges.

77. $\displaystyle\int_0^4 \frac{dx}{x^{2/3}}$

SOLUTION

$$\int_0^4 \frac{dx}{x^{2/3}} = \lim_{R\to 0+} \int_R^4 \frac{dx}{x^{2/3}} = \lim_{R\to 0+} 3x^{1/3}\Big|_R^4 = \lim_{R\to 0+} \left(3\cdot 4^{1/3} - 3\cdot R^{1/3}\right) = 3\sqrt[3]{4}.$$

78. $\displaystyle\int_9^\infty \frac{dx}{x^{12/5}}$

SOLUTION

$$\int_9^\infty \frac{dx}{x^{12/5}} = \lim_{R\to\infty} \int_9^R \frac{dx}{x^{12/5}} = \lim_{R\to\infty} -\frac{5}{7}x^{-7/5}\Big|_9^R = \lim_{R\to\infty}\left(-\frac{5}{7}R^{-7/5} + \frac{5}{7}\cdot 9^{-7/5}\right)$$
$$= 0 + \frac{5}{7}\cdot 9^{-7/5} = \frac{5}{7\cdot 9\cdot 9^{2/5}} = \frac{5}{63\cdot 9^{2/5}}.$$

79. $\displaystyle\int_{-\infty}^0 \frac{dx}{x^2+1}$

SOLUTION

$$\int_{-\infty}^0 \frac{dx}{x^2+1} = \lim_{R\to-\infty} \int_R^0 \frac{dx}{x^2+1} = \lim_{R\to-\infty} \tan^{-1}x\Big|_R^0 = \lim_{R\to-\infty}\left(\tan^{-1}0 - \tan^{-1}R\right)$$
$$= \lim_{R\to-\infty}\left(-\tan^{-1}R\right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

80. $\displaystyle\int_{-\infty}^9 e^{4x}\,dx$

SOLUTION

$$\int_{-\infty}^9 e^{4x}\,dx = \lim_{R\to-\infty}\int_R^9 e^{4x}\,dx = \lim_{R\to-\infty}\frac{1}{4}e^{4x}\Big|_R^9 = \lim_{R\to-\infty}\frac{1}{4}e^{36} - \frac{1}{4}e^{4R} = \frac{e^{36}}{4}.$$

81. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

SOLUTION

$$\int_0^{\pi/2}\cot\theta\,d\theta = \lim_{R\to 0+}\int_R^{\pi/2}\cot\theta\,d\theta = \lim_{R\to 0+}\ln|\sin\theta|\Big|_R^{\pi/2} = \lim_{R\to 0+}\left(\ln\left(\sin\frac{\pi}{2}\right) - \ln(\sin R)\right)$$
$$= \lim_{R\to 0+}(\ln 1 - \ln(\sin R)) = \lim_{R\to 0+}\ln\left(\frac{1}{\sin R}\right) = \infty.$$

We conclude that the improper integral diverges.

82. $\displaystyle\int_1^\infty \frac{dx}{(x+2)(2x+3)}$

SOLUTION First, evaluate the indefinite integral. The following partial fraction decomposition has the form

$$\frac{1}{(x+2)(2x+3)} = -\frac{1}{x+2} + \frac{2}{2x+3}.$$

Clearing denominators gives us

$$1 = A(2x+3) + B(x+2).$$

Setting $x = -2$ then yields $A = -1$, while setting $x = -\frac{3}{2}$ yields $B = 2$. Hence,

$$\int \frac{dx}{(x+2)(2x+3)} = -\int \frac{dx}{x+2} + 2\int \frac{dx}{2x+3} = -\ln|x+2| + \ln|2x+3| + C = \ln\left|\frac{2x+3}{x+2}\right| + C.$$

Now, for $R > 1$,

$$\int_1^R \frac{dx}{(x+2)(2x+3)} = \ln\left|\frac{2x+3}{x+2}\right|\Bigg|_1^R = \ln\frac{2R+3}{R+2} - \ln\frac{5}{3},$$

and

$$\int_1^\infty \frac{dx}{(x+2)(2x+3)} = \lim_{R\to\infty}\left(\ln\frac{2R+3}{R+2}\right) - \ln\frac{5}{3} = \ln 2 + \ln\frac{3}{5} = \ln\frac{6}{5}.$$

83. $\int_0^\infty (5+x)^{-1/3}\,dx$

SOLUTION

$$\int_0^\infty (5+x)^{-1/3}\,dx = \lim_{R\to\infty}\int_0^R (5+x)^{-1/3}\,dx = \lim_{R\to\infty} \frac{3}{2}(5+x)^{2/3}\Big|_0^R$$
$$= \lim_{R\to\infty}\left(\frac{3}{2}(5+R)^{2/3} - \frac{3}{2}5^{2/3}\right) = \infty.$$

We conclude that the improper integral diverges.

84. $\int_2^5 (5-x)^{-1/3}\,dx$

SOLUTION

$$\int_2^5 (5-x)^{-1/3}\,dx = \lim_{R\to 5-}\int_2^R (5-x)^{-1/3}\,dx = \lim_{R\to 5-} -\frac{3}{2}(5-x)^{2/3}\Big|_2^R$$
$$= \lim_{R\to 5-} -\frac{3}{2}\left((5-R)^{2/3} - 3^{2/3}\right) = -\frac{3}{2}\left(0 - 3^{2/3}\right) = \frac{3^{5/3}}{2}.$$

In Exercises 85–90, use the Comparison Test to determine whether the improper integral converges or diverges.

85. $\int_8^\infty \frac{dx}{x^2-4}$

SOLUTION For $x \ge 8$, $\frac{1}{2}x^2 \ge 4$, so that

$$-\frac{1}{2}x^2 \le -4$$
$$\frac{1}{2}x^2 \le x^2 - 4$$

and

$$\frac{1}{x^2-4} \le \frac{2}{x^2}.$$

Now, $\int_1^\infty \frac{dx}{x^2}$ converges, so $\int_8^\infty \frac{2}{x^2}\,dx$ also converges. Therefore, by the comparison test,

$$\int_8^\infty \frac{dx}{x^2-4} \text{ converges.}$$

86. $\int_8^\infty (\sin^2 x)e^{-x}\,dx$

SOLUTION The following inequality holds for all x,

$$0 \le \left(\sin^2 x\right)e^{-x} \le e^{-x}.$$

We use direct computation to show that the improper integral of e^{-x} over the interval $[8, \infty)$ converges:

$$\int_8^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_8^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_8^R = \lim_{R\to\infty}\left(-e^{-R}+e^{-8}\right) = 0 + e^{-8} = e^{-8}.$$

Therefore, by the Comparison Test, the improper integral $\int_8^\infty (\sin^2 x)e^{-x}\,dx$ also converges.

87. $\displaystyle\int_3^\infty \frac{dx}{x^4+\cos^2 x}$

SOLUTION For $x \ge 1$, we have

$$\frac{1}{x^4+\cos^2 x} \le \frac{1}{x^4}.$$

Since $\int_1^\infty \frac{dx}{x^4}$ converges, the Comparison Test guarantees that $\int_1^\infty \frac{dx}{x^4+\cos^2 x}$ also converges. The integral $\int_1^3 \frac{dx}{x^4+\cos^2 x}$ has a finite value (notice that $x^4+\cos^2 x \ne 0$) hence we conclude that the integral $\int_3^\infty \frac{dx}{x^4+\cos^2 x}$ also converges.

88. $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}+x^{2/3}}$

SOLUTION If $x \ge 1$, then $x^{1/3} \ge 1$; therefore,

$$x^{1/3}+x^{2/3} = x^{1/3}\left(1+x^{1/3}\right) \le x^{1/3}\left(x^{1/3}+x^{1/3}\right) = x^{1/3}\cdot 2x^{1/3} = 2x^{2/3}.$$

Hence,

$$\frac{1}{x^{1/3}+x^{2/3}} \ge \frac{1}{2x^{2/3}}.$$

The integral $\int_1^\infty \frac{dx}{x^{2/3}}$ diverges; hence $\int_1^\infty \frac{dx}{2x^{2/3}}$ also diverges. Therefore, by the Comparison Test, the improper integral $\int_1^\infty \frac{dx}{x^{1/3}+x^{2/3}}$ also diverges.

89. $\displaystyle\int_0^1 \frac{dx}{x^{1/3}+x^{2/3}}$

SOLUTION For $0 \le x \le 1$,

$$x^{1/3}+x^{2/3} \ge x^{1/3} \quad \text{so} \quad \frac{1}{x^{1/3}+x^{2/3}} \le \frac{1}{x^{1/3}}.$$

Now, $\int_0^1 x^{-1/3}\,dx$ converges. Therefore, by the Comparison Test, the improper integral $\int_0^1 \frac{dx}{x^{1/3}+x^{2/3}}$ also converges.

90. $\displaystyle\int_0^\infty e^{-x^3}\,dx$

SOLUTION For $x > 1$, $e^x \ge x$; hence $e^{x^3} \ge x^3$, therefore $0 \le e^{-x^3} \le x^{-3}$. Since $\int_1^\infty \frac{dx}{x^3}$ converges, the integral $\int_1^\infty e^{-x^3}\,dx$ also converges by the Comparison Test. We write

$$\int_0^\infty e^{-x^3}\,dx = \int_0^1 e^{-x^3}\,dx + \int_1^\infty e^{-x^3}\,dx.$$

The first integral on the right hand side has a finite value and the second integral converges. We conclude that the integral $\int_0^\infty e^{-x^3}\,dx$ converges.

91. Calculate the volume of the infinite solid obtained by rotating the region under $y = (x^2+1)^{-2}$ for $0 \le x < \infty$ about the y-axis.

SOLUTION Using the Shell Method, the volume of the infinite solid obtained by rotating the region under the graph of $y = \left(x^2+1\right)^{-2}$ over the interval $[0, \infty)$ about the y-axis is

$$V = 2\pi\int_0^\infty \frac{x}{\left(x^2+1\right)^2}\,dx.$$

Now,

$$\int_0^\infty \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{x\,dx}{\left(x^2+1\right)^2}$$

We substitute $t = x^2 + 1$, $dt = 2x\,dx$. The new limits of integration are $t = 1$ and $t = R^2 + 1$. Thus,

$$\int_0^R \frac{x\,dx}{\left(x^2+1\right)^2} = \int_1^{R^2+1} \frac{\frac{1}{2}\,dt}{t^2} = -\frac{1}{2t}\bigg|_1^{R^2+1} = \frac{1}{2}\left(1 - \frac{1}{R^2+1}\right).$$

Taking the limit as $R \to \infty$ yields:

$$\int_0^\infty \frac{x\,dx}{\left(x^2+1\right)^2} = \lim_{R\to\infty}\frac{1}{2}\left(1-\frac{1}{R^2+1}\right) = \frac{1}{2}(1-0) = \frac{1}{2}.$$

Therefore,

$$V = 2\pi\cdot\frac{1}{2} = \pi.$$

92. Let R be the region under the graph of $y = (x+1)^{-1}$ for $0 \le x < \infty$. Which of the following quantities is finite?

(a) The area of R

(b) The volume of the solid obtained by rotating R about the x-axis

(c) The volume of the solid obtained by rotating R about the y-axis

SOLUTION

(a) The area of R is

$$\int_0^\infty \frac{dx}{x+1} = \lim_{R\to\infty}\int_0^R \frac{dx}{x+1} = \lim_{R\to\infty} \ln|x+1|\,\bigg|_0^R = \lim_{R\to\infty}\big(\ln(R+1) - \ln 1\big) = \infty.$$

Hence, the area of R is not finite.

(b) Using the Disk Method, the volume of the solid obtained by rotating R about the x-axis is

$$\pi\int_0^\infty \frac{dx}{(x+1)^2} = \pi\lim_{R\to\infty}\int_0^R \frac{dx}{(x+1)^2} = \pi\lim_{R\to\infty} -\frac{1}{x+1}\bigg|_0^R = \pi\lim_{R\to\infty}\left(-\frac{1}{R+1}+1\right) = \pi.$$

Hence, the volume of the solid obtained by rotating R about the x-axis is finite.

(c) Using the Shell Method, the volume of the solid obtained by rotating R about the y-axis is

$$2\pi\int_0^\infty \frac{x}{x+1}\,dx = 2\pi\lim_{R\to\infty}\int_0^R \frac{x\,dx}{x+1}.$$

Now,

$$\int_0^R \frac{x\,dx}{x+1} = \int_0^R \frac{(x+1)-1}{x+1}\,dx = \int_0^R \left(1 - \frac{1}{x+1}\right)dx = (x - \ln(x+1))\bigg|_0^R$$
$$= R - (\ln(R+1) - \ln 1) = R - \ln(R+1).$$

Thus,

$$2\pi\lim_{R\to\infty}\int_0^R \frac{x\,dx}{x+1} = 2\pi\lim_{R\to\infty}(R - \ln(R+1)) = 2\pi\lim_{R\to\infty} R\left(1 - \frac{\ln(R+1)}{R}\right) = \infty.$$

Hence, the volume of the solid obtained by rotating R about the y-axis is not finite.

93. Show that $\int_0^\infty x^n e^{-x^2}\,dx$ converges for all $n > 0$. *Hint:* First observe that $x^n e^{-x^2} < x^n e^{-x}$ for $x > 1$. Then show that $x^n e^{-x} < x^{-2}$ for x sufficiently large.

SOLUTION For $x > 1$, $x^2 > x$; hence $e^{x^2} > e^x$, and $0 < e^{-x^2} < e^{-x}$. Therefore, for $x > 1$ the following inequality holds:

$$x^{n+2}e^{-x^2} < x^{n+2}e^{-x}.$$

Now, using L'Hôpital's Rule $n+2$ times, we find

$$\lim_{x\to\infty} x^{n+2}e^{-x} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = \lim_{x\to\infty} \frac{(n+2)x^{n+1}}{e^x} = \lim_{x\to\infty} \frac{(n+2)(n+1)x^n}{e^x}$$
$$= \cdots = \lim_{x\to\infty} \frac{(n+2)!}{e^x} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{n+2}e^{-x^2} = 0$$

by the Squeeze Theorem, and there exists a number $R > 1$ such that, for all $x > R$:

$$x^{n+2}e^{-x^2} < 1 \quad \text{or} \quad x^n e^{-x^2} < x^{-2}.$$

Finally, write

$$\int_0^\infty x^n e^{-x^2}\,dx = \int_0^R x^n e^{-x^2}\,dx + \int_R^\infty x^n e^{-x^2}\,dx.$$

The first integral on the right-hand side has finite value since the integrand is a continuous function. The second integral converges since on the interval of integration, $x^n e^{-x^2} < x^{-2}$ and we know that $\int_R^\infty x^{-2}\,dx = \int_R^\infty \frac{dx}{x^2}$ converges. We conclude that the integral $\int_0^\infty x^n e^{-x^2}\,dx$ converges.

94. Compute the Laplace transform $Lf(s)$ of the function $f(x) = x$ for $s > 0$. See Exercises 86–89 in Section 7.6 for the definition of $Lf(s)$.

SOLUTION The Laplace transform of $f(x) = x$ is the following integral:

$$L(x)(s) = \int_0^\infty xe^{-sx}\,dx = \lim_{R\to\infty}\int_0^R xe^{-sx}\,dx.$$

We compute the definite integral using Integration by Parts with $u = x$ and $v' = e^{-sx}$. Then $u' = 1$, $v = -\frac{1}{s}e^{-sx}$ and

$$\int_0^R xe^{-sx}\,dx = -\frac{1}{s}xe^{-sx}\Big|_0^R + \int_0^R \frac{1}{s}e^{-sx}\,dx = \left(-\frac{1}{s}Re^{-sR} - \frac{1}{s^2}e^{-sx}\right)\Big|_0^R$$
$$= -\frac{1}{s}Re^{-sR} - \frac{1}{s^2}\left(e^{-sR} - e^0\right) = \frac{1}{s^2} - \frac{1}{s^2}e^{-sR} - \frac{1}{s}Re^{-sR}.$$

Therefore,

$$L(x)(s) = \lim_{R\to\infty}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-sR} - \frac{1}{s}Re^{-sR}\right) = \frac{1}{s^2} - \frac{1}{s^2}\lim_{R\to\infty} e^{-sR} - \frac{1}{s}\lim_{R\to\infty} Re^{-sR}.$$

Since $s > 0$, we have $\lim_{R\to\infty} e^{-sR} = 0$. Also by L'Hôpital's Rule:

$$\lim_{R\to\infty} Re^{-sR} = \lim_{R\to\infty}\frac{R}{e^{sR}} = \lim_{R\to\infty}\frac{1}{se^{sR}} = 0.$$

Finally,

$$L(x)(s) = \frac{1}{s^2} - 0 - 0 = \frac{1}{s^2}.$$

95. Compute the Laplace transform $Lf(s)$ of the function $f(x) = x^2e^{\alpha x}$ for $s > \alpha$.

SOLUTION The Laplace transform is the following integral:

$$L\left(x^2e^{\alpha x}\right)(s) = \int_0^\infty x^2e^{\alpha x}e^{-sx}\,dx = \int_0^\infty x^2e^{(\alpha-s)x}\,dx = \lim_{R\to\infty}\int_0^R x^2e^{(\alpha-s)x}\,dx.$$

We compute the definite integral using Integration by Parts with $u = x^2$, $v' = e^{(\alpha-s)x}$. Then $u' = 2x$, $v = \frac{1}{\alpha-s}e^{(\alpha-s)x}$ and

$$\int_0^R x^2e^{(\alpha-s)x}\,dx = \frac{1}{\alpha-s}x^2e^{(\alpha-s)x}\Big|_{x=0}^R - \int_0^R 2x\cdot\frac{1}{\alpha-s}e^{(\alpha-s)x}\,dx$$
$$= \frac{1}{\alpha-s}R^2e^{(\alpha-s)R} - \frac{2}{\alpha-s}\int_0^R xe^{(\alpha-s)x}\,dx.$$

We compute the resulting integral using Integration by Parts again, this time with $u = x$ and $v' = e^{(\alpha-s)x}$. Then $u' = 1$, $v = \frac{1}{\alpha-s}e^{(\alpha-s)x}$ and

$$\int_0^R xe^{(\alpha-s)x}\,dx = x \cdot \frac{1}{\alpha-s}e^{(\alpha-s)x}\Big|_{x=0}^{R} - \frac{1}{\alpha-s}\int_0^R e^{(\alpha-s)x}\,dx = \left(\frac{x}{\alpha-s}e^{(\alpha-s)x} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)x}\right)\Big|_{x=0}^{R}$$

$$= \frac{R}{\alpha-s}e^{(\alpha-s)R} - \frac{1}{(\alpha-s)^2}\left(e^{(\alpha-s)R} - e^0\right) = \frac{1}{(\alpha-s)^2} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)R} + \frac{R}{\alpha-s}e^{(\alpha-s)R}.$$

Thus,

$$\int_0^R x^2e^{(\alpha-s)x}\,dx = \frac{1}{\alpha-s}R^2e^{(\alpha-s)R} - \frac{2}{\alpha-s}\left(\frac{1}{(\alpha-s)^2} - \frac{1}{(\alpha-s)^2}e^{(\alpha-s)R} + \frac{R}{\alpha-s}e^{(\alpha-s)R}\right)$$

$$= \frac{1}{\alpha-s}R^2e^{(\alpha-s)R} - \frac{2}{(\alpha-s)^3} + \frac{2}{(\alpha-s)^3}e^{(\alpha-s)R} - \frac{2R}{(\alpha-s)^2}e^{(\alpha-s)R},$$

and

$$L\left(x^2e^{\alpha x}\right)(s) = \frac{2}{(s-\alpha)^3} - \frac{1}{s-\alpha}\lim_{R\to\infty} R^2e^{-(s-\alpha)R} - \frac{2}{(s-\alpha)^3}\lim_{R\to\infty} e^{-(s-\alpha)R} - \frac{2}{(s-\alpha)^2}\lim_{R\to\infty} Re^{-(s-\alpha)R}.$$

Now, since $s > \alpha$, $\lim_{R\to\infty} e^{-(s-\alpha)R} = 0$. We use L'Hôpital's Rule to compute the other two limits:

$$\lim_{R\to\infty} Re^{-(s-\alpha)R} = \lim_{R\to\infty} \frac{R}{e^{(s-\alpha)R}} = \lim_{R\to\infty} \frac{1}{(s-\alpha)e^{(s-\alpha)R}} = 0;$$

$$\lim_{R\to\infty} R^2e^{-(s-\alpha)R} = \lim_{R\to\infty} \frac{R^2}{e^{(s-\alpha)R}} = \lim_{R\to\infty} \frac{2R}{(s-\alpha)e^{(s-\alpha)R}} = \lim_{R\to\infty} \frac{2}{(s-\alpha)^2e^{(s-\alpha)R}} = 0.$$

Finally,

$$L\left(x^2e^{\alpha x}\right)(s) = \frac{2}{(s-\alpha)^3} - 0 - 0 - 0 = \frac{2}{(s-\alpha)^3}.$$

96. Estimate $\int_2^5 f(x)\,dx$ by computing T_2, M_3, T_6, and S_6 for a function $f(x)$ taking on the values in the following table:

x	2	2.5	3	3.5	4	4.5	5
$f(x)$	$\frac{1}{2}$	2	1	0	$-\frac{3}{2}$	-4	-2

SOLUTION To calculate T_2, divide [2, 5] into two subintervals of length $\Delta x = \frac{3}{2}$ with endpoints $x_0 = 2$, $x_1 = 3.5$, $x_2 = 5$. Then

$$T_2 = \frac{1}{2}\cdot\frac{3}{2}\left(f(2) + 2f(3.5) + f(5)\right) = 0.75\left(\frac{1}{2} + 2\cdot 0 + (-2)\right) = -\frac{9}{8}.$$

To calculate M_3, divide [2, 5] into three subintervals of length $\Delta x = 1$ with midpoints $c_1 = 2.5$, $c_2 = 3.5$, $c_3 = 4.5$. Then

$$M_3 = 1\cdot\left(f(2.5) + f(3.5) + f(4.5)\right) = 2 + 0 - 4 = -2.$$

To calculate T_6, divide [2, 5] into 6 subintervals of length $\frac{5-2}{6} = \frac{1}{2}$ with endpoints $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, $x_4 = 4$, $x_5 = 4.5$, $x_6 = 5$. Then

$$T_6 = \frac{1}{2}\cdot\frac{1}{2}\left(f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\right)$$

$$= \frac{1}{4}\left(\frac{1}{2} + 2\cdot 2 + 2\cdot 1 + 2\cdot 0 + 2\cdot\left(-\frac{3}{2}\right) + 2(-4) + (-2)\right) = -\frac{13}{8}.$$

Finally, to calculate S_6, divide [2, 5] into 6 subintervals of length $\Delta x = \frac{5-2}{6} = \frac{1}{2}$ with endpoints $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, $x_4 = 4$, $x_5 = 4.5$, $x_6 = 5$. Then

$$S_6 = \frac{1}{3}\cdot\frac{1}{2}\left(f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)\right)$$

$$= \frac{1}{6}\left(\frac{1}{2} + 4\cdot 2 + 2\cdot 1 + 4\cdot 0 + 2\cdot\left(-\frac{3}{2}\right) + 4(-4) + (-2)\right) = -\frac{7}{4}.$$

97. State whether the approximation M_N or T_N is larger or smaller than the integral.

(a) $\int_0^\pi \sin x\,dx$ **(b)** $\int_\pi^{2\pi} \sin x\,dx$

(c) $\int_1^8 \frac{dx}{x^2}$ **(d)** $\int_2^5 \ln x\,dx$

SOLUTION

(a) Because $f(x) = \sin x$ is concave down on the interval $[0, \pi]$,

$$T_N \le \int_0^\pi \sin x\,dx \le M_N;$$

that is, T_N is smaller and M_N is larger than the integral.

(b) On the interval $[\pi, 2\pi]$, the function $f(x) = \sin x$ is concave up, therefore

$$M_N \le \int_\pi^{2\pi} \sin x\,dx \le T_N;$$

that is, M_N is smaller and T_N is larger than the integral.

(c) The function $f(x) = \frac{1}{x^2}$ is concave up on the interval $[1, 8]$; therefore,

$$M_N \le \int_1^8 \frac{dx}{x^2} \le T_N;$$

that is, M_N is smaller and T_N is larger than the integral.

(d) The integrand $y = \ln x$ is concave down on the interval $[2, 5]$; hence,

$$T_N \le \int_2^5 \ln x\,dx \le M_N;$$

that is, T_N is smaller and M_N is larger than the integral.

98. The rainfall rate (in inches per hour) was measured hourly during a 10-hour thunderstorm with the following results:

0, 0.41, 0.49, 0.32, 0.3, 0.23,
0.09, 0.08, 0.05, 0.11, 0.12

Use Simpson's Rule to estimate the total rainfall during the 10-hour period.

SOLUTION We have 10 subintervals of length $\Delta x = 1$. Thus, the total rainfall during the 10-hour period is approximately

$$\begin{aligned} S_{10} &= \frac{1}{3} \cdot 1[0 + 4 \cdot 0.41 + 2 \cdot 0.49 + 4 \cdot 0.32 + 2 \cdot 0.3 + 4 \cdot 0.23 + 2 \cdot 0.09 + 4 \cdot 0.08 + 2 \cdot 0.05 \\ &\qquad + 4 \cdot 0.11 + 0.12] \\ &= 2.19 \text{ inches.} \end{aligned}$$

In Exercises 99–104, compute the given approximation to the integral.

99. $\int_0^1 e^{-x^2}\,dx, \quad M_5$

SOLUTION Divide the interval $[0, 1]$ into 5 subintervals of length $\Delta x = \frac{1-0}{5} = \frac{1}{5}$, with midpoints $c_1 = \frac{1}{10}$, $c_2 = \frac{3}{10}$, $c_3 = \frac{1}{2}$, $c_4 = \frac{7}{10}$, and $c_5 = \frac{9}{10}$. Then

$$\begin{aligned} M_5 &= \Delta x \left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right) \right] \\ &= \frac{1}{5} \left[e^{-(1/10)^2} + e^{-(3/10)^2} + e^{-(1/2)^2} + e^{-(7/10)^2} + e^{-(9/10)^2} \right] = 0.748053. \end{aligned}$$

100. $\displaystyle\int_2^4 \sqrt{6t^3+1}\,dt, \quad T_3$

SOLUTION Divide the interval [2, 4] into 3 subintervals of length $\Delta x = \frac{4-2}{3} = \frac{2}{3}$, with endpoints 2, $\frac{8}{3}$, $\frac{10}{3}$, 4. Then,

$$T_3 = \frac{1}{2}\Delta x\left(f(2) + 2f\left(\frac{8}{3}\right) + 2f\left(\frac{10}{3}\right) + f(4)\right)$$
$$= \frac{1}{2}\cdot\frac{2}{3}\left(\sqrt{6\cdot 2^3+1} + 2\sqrt{6\cdot\left(\frac{8}{3}\right)^3+1} + 2\sqrt{6\cdot\left(\frac{10}{3}\right)^3+1} + \sqrt{6\cdot 4^3+1}\right) = 25.976514.$$

101. $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{\sin\theta}\,d\theta, \quad M_4$

SOLUTION Divide the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ into 4 subintervals of length $\Delta x = \frac{\frac{\pi}{2}-\frac{\pi}{4}}{4} = \frac{\pi}{16}$ with midpoints $\frac{9\pi}{32}$, $\frac{11\pi}{32}$, $\frac{13\pi}{32}$, and $\frac{15\pi}{32}$. Then

$$M_4 = \Delta x\left(f\left(\frac{9\pi}{32}\right) + f\left(\frac{11\pi}{32}\right) + f\left(\frac{13\pi}{32}\right) + f\left(\frac{15\pi}{32}\right)\right)$$
$$= \frac{\pi}{16}\left(\sqrt{\sin\frac{9\pi}{32}} + \sqrt{\sin\frac{11\pi}{32}} + \sqrt{\sin\frac{13\pi}{32}} + \sqrt{\sin\frac{15\pi}{32}}\right) = 0.744978.$$

102. $\displaystyle\int_1^4 \frac{dx}{x^3+1}, \quad T_6$

SOLUTION Divide the interval [1, 4] into 6 subintervals of length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ with endpoints 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4. Then

$$T_6 = \frac{1}{2}\Delta x\left(f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right)$$
$$= \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{1^3+1} + 2\frac{1}{\left(\frac{3}{2}\right)^3+1} + 2\frac{1}{2^3+1} + 2\frac{1}{\left(\frac{5}{2}\right)^3+1} + 2\frac{1}{3^3+1} + 2\frac{1}{(\frac{7}{2})^2+1} + \frac{1}{4^3+1}\right) = 0.358016.$$

103. $\displaystyle\int_0^1 e^{-x^2}\,dx, \quad S_4$

SOLUTION Divide the interval [0, 1] into 4 subintervals of length $\Delta x = \frac{1}{4}$ with endpoints 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. Then

$$S_6 = \frac{1}{3}\Delta x\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right)$$
$$= \frac{1}{3}\cdot\frac{1}{4}\left(e^{-0^2} + 4e^{-(1/4)^2} + 2e^{-(1/2)^2} + 4e^{-(3/4)^2} + e^{-1^2}\right) = 0.746855.$$

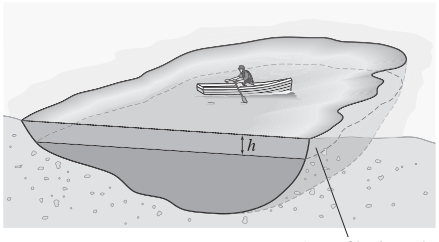
104. $\displaystyle\int_5^9 \cos(x^2)\,dx, \quad S_8$

SOLUTION Divide the interval [5, 9] into 8 subintervals of length $\Delta x = \frac{9-5}{8} = \frac{1}{2}$ with endpoints 5, $\frac{11}{2}$, 6, $\frac{13}{2}$, 7, $\frac{15}{2}$, 8, $\frac{17}{2}$, 9. Then

$$S_8 = \frac{1}{3}\Delta x\left(f(5) + 4f\left(\frac{11}{2}\right) + 2f(6) + 4f\left(\frac{13}{2}\right) + 2f(7) + 4f\left(\frac{15}{2}\right) + 2f(8) + 4f\left(\frac{17}{2}\right) + f(9)\right)$$
$$= \frac{1}{3}\cdot\frac{1}{2}\left(\cos(5^2) + 4\cos(5.5^2) + 2\cos(6^2) + 4\cos(6.5^2)\right.$$
$$\left. + 2\cos(7^2) + 4\cos(7.5^2) + 2\cos(8^2) + 4\cos(8.5^2) + \cos(9^2)\right)$$
$$= 0.608711.$$

105. The following table gives the area $A(h)$ of a horizontal cross section of a pond at depth h. Use the Trapezoidal Rule to estimate the volume V of the pond (Figure 1).

h (ft)	$A(h)$ (acres)	h (ft)	$A(h)$ (acres)
0	2.8	10	0.8
2	2.4	12	0.6
4	1.8	14	0.2
6	1.5	16	0.1
8	1.2	18	0



Area of horizontal cross section is $A(h)$

FIGURE 1

SOLUTION The volume of the pond is the following integral:

$$V = \int_0^{18} A(h)dh$$

We approximate the integral using the trapezoidal approximation T_9. The interval of depth [0, 18] is divided to 9 subintervals of length $\Delta x = 2$ with endpoints 0, 2, 4, 6, 8, 10, 12, 14, 16, 18. Thus,

$$\begin{aligned} V \approx T_9 &= \frac{1}{2} \cdot 2(2.8 + 2 \cdot 2.4 + 2 \cdot 1.8 + 2 \cdot 1.5 + 2 \cdot 1.2 + 2 \cdot 0.8 + 2 \cdot 0.6 + 2 \cdot 0.2 + 2 \cdot 0.1 + 0) \\ &= 20 \text{ acre} \cdot \text{ft} = 871{,}200 \text{ ft}^3, \end{aligned}$$

where we have used the fact that 1 acre $= 43{,}560$ ft^2.

106. Suppose that the second derivative of the function $A(h)$ in Exercise 105 satisfies $|A''(h)| \le 1.5$. Use the error bound to find the maximum possible error in your estimate of the volume V of the pond.

SOLUTION The Error Bound for the Trapezoidal Rule states that

$$\text{Error } (T_N) \le \frac{K_2(b-a)^3}{12N^2},$$

where K_2 is a number such that $\left|f''(x)\right| \le K_2$ for all $x \in [a, b]$. We estimated the volume of the pond by T_9; hence $N = 9$. The interval of depth is [0, 18] hence $b - a = 18 - 0 = 18$. Since $\left|A''(h)\right| \le 1.5$ acres/ft^2 we may take $K_2 = 1.5$, to find that the error cannot exceed

$$\frac{K_2(b-a)^3}{12N^2} = \frac{1.5 \cdot 18^3}{12 \cdot 9^2} = 9 \text{ acre} \cdot \text{ft} = 392{,}040 \text{ ft}^3,$$

where we have used the fact that 1 acre $= 43{,}560$ ft^2.

107. Find a bound for the error $\left|M_{16} - \int_1^3 x^3\,dx\right|$.

SOLUTION The Error Bound for the Midpoint Rule states that

$$\left|M_N - \int_a^b f(x)\,dx\right| \le \frac{K_2(b-a)^3}{24N^2},$$

where K_2 is a number such that $\left|f''(x)\right| \le K_2$ for all $x \in [1, 3]$. Here $b - a = 3 - 1 = 2$ and $N = 16$. Therefore,

$$\left|M_{16} - \int_1^3 x^3\,dx\right| \le \frac{K_2 \cdot 2^3}{24 \cdot 16^2} = \frac{K_2}{768}.$$

To find K_2, we differentiate $f(x) = x^3$ twice:

$$f'(x) = 3x^2 \qquad \text{and} \qquad f''(x) = 6x.$$

On the interval $[1, 3]$ we have $\left|f''(x)\right| = 6x \le 6 \cdot 3 = 18$; hence, we may take $K_2 = 18$. Thus,

$$\left|M_{16} - \int_1^3 x^3\,dx\right| \le \frac{18}{768} = \frac{3}{128} = 0.0234375.$$

108. $\boxed{\text{GU}}$ Let $f(x) = \sin(x^3)$. Find a bound for the error

$$\left|T_{24} - \int_0^{\pi/2} f(x)\,dx\right|$$

Hint: Find a bound K_2 for $|f''(x)|$ by plotting $f''(x)$ with a graphing utility.

SOLUTION Using the error bound for T_{24} we obtain:

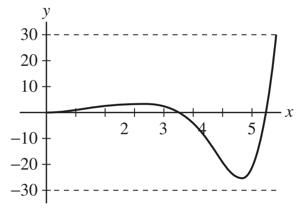
$$\left|T_{24} - \int_0^{\pi/2} f(x)\,dx\right| \le \frac{K_2\left(\frac{\pi}{2} - 0\right)^3}{12 \cdot 24^2} = \frac{K_2\pi^3}{55,296},$$

where K_2 is a number such that $\left|f''(x)\right| < k_2$ for all $x \in \left[0, \frac{\pi}{2}\right]$. We compute the first and second derivative of $f(x) = \sin(x^3)$:

$$f'(x) = 3x^2\cos(x^3)$$
$$f''(x) = 6x\cos(x^3) + 3x^2 \cdot 3x^2\left(-\sin(x^3)\right) = 6x\cos(x^3) - 9x^4\sin(x^3)$$

The graph of $f''(x) = 6x\cos(x^3) - 9x^4\sin(x^3)$ on the interval $\left[0, \frac{\pi}{2}\right]$ shows that $\left|f''(x)\right| \le 30$ on this interval. We may choose $K_2 = 30$ and find

$$\left|T_{24} - \int_0^{\pi/2} f(x)\,dx\right| \le \frac{30\pi^3}{55,296} = \frac{5\pi^3}{9216} = 0.0168220.$$



109. Find a value of N such that

$$\left|M_N - \int_0^{\pi/4} \tan x\,dx\right| \le 10^{-4}$$

SOLUTION To use the Error Bound we must find the second derivative of $f(x) = \tan x$. We differentiate f twice to obtain:

$$f'(x) = \sec^2 x$$
$$f''(x) = 2\sec x\tan x = \frac{2\sin x}{\cos^2 x}$$

For $0 \le x \le \frac{\pi}{4}$, we have $\sin x \le \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos x \ge \frac{1}{\sqrt{2}}$ or $\cos^2 x \ge \frac{1}{2}$. Therefore, for $0 \le x \le \frac{\pi}{4}$ we have:

$$f''(x) = \frac{2\sin x}{\cos^2 x} \le \frac{2 \cdot \frac{1}{\sqrt{2}}}{\frac{1}{2}} = 2\sqrt{2}.$$

Using the Error Bound with $b = \frac{\pi}{4}$, $a = 0$ and $K_2 = 2\sqrt{2}$ we have:

$$\left|M_N - \int_0^{\pi/4} \tan x\,dx\right| \le \frac{2\sqrt{2} \cdot \left(\frac{\pi}{4} - 0\right)^3}{24N^2} = \frac{\pi^3\sqrt{2}}{768N^2}.$$

We must choose a value of N such that:

$$\frac{\pi^3\sqrt{2}}{768N^2} \le 10^{-4}$$

$$N^2 \ge \frac{10^4 \cdot \sqrt{2}\pi^3}{768}$$

$$N \ge 23.9$$

The smallest integer that is needed to obtain the required precision is $N = 24$.

110. Find a value of N such that S_N approximates $\int_2^5 x^{-1/4}\,dx$ with an error of at most 10^{-2} (but do not calculate S_N).

SOLUTION To use the error bound we must find the fourth derivative $f^{(4)}(x)$. We differentiate $f(x) = x^{-1/4}$ four times to obtain:

$$f'(x) = -\frac{1}{4}x^{-5/4}, \ f''(x) = \frac{5}{16}x^{-9/4}, \ f'''(x) = -\frac{45}{64}x^{-13/4}, \ f^{(4)}(x) = \frac{585}{256}x^{-17/4}.$$

For $2 \le x \le 5$ we have:

$$\left|f^{(4)}(x)\right| = \frac{585}{256x^{17/4}} \le \frac{585}{256 \cdot 2^{17/4}} = 0.120099.$$

Using the error bound with $b = 5$, $a = 2$ and $K_4 = 0.120099$ we have:

$$\text{Error}\,(S_N) \le \frac{0.120099(5-2)^5}{180N^4} = \frac{0.162134}{N^4}.$$

We must choose a value of N such that:

$$\frac{0.162134}{N^4} \le 10^{-2}$$

$$N^4 \ge 16.2134$$

$$N \ge 2.00664$$

The smallest even value of N that is needed to obtain the required precision is $N = 4$.

8 FURTHER APPLICATIONS OF THE INTEGRAL AND TAYLOR POLYNOMIALS

8.1 Arc Length and Surface Area

Preliminary Questions

1. Which integral represents the length of the curve $y = \cos x$ between 0 and π?

$$\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx, \qquad \int_0^{\pi} \sqrt{1+\sin^2 x}\, dx$$

SOLUTION Let $y = \cos x$. Then $y' = -\sin x$, and $1 + (y')^2 = 1 + \sin^2 x$. Thus, the length of the curve $y = \cos x$ between 0 and π is

$$\int_0^{\pi} \sqrt{1+\sin^2 x}\, dx.$$

2. Use the formula for arc length to show that for any constant C, the graphs $y = f(x)$ and $y = f(x) + C$ have the same length over every interval $[a, b]$. Explain geometrically.

SOLUTION The graph of $y = f(x) + C$ is a vertical translation of the graph of $y = f(x)$; hence, the two graphs should have the same arc length. We can explicitly establish this as follows:

$$\text{length of } y = f(x) + C = \int_a^b \sqrt{1 + \left[\frac{d}{dx}(f(x)+C)\right]^2}\, dx = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \text{length of } y = f(x).$$

3. Use the formula for arc length to show that the length of a graph over $[1, 4]$ cannot be less than 3.

SOLUTION Note that $f'(x)^2 \geq 0$, so that $\sqrt{1+[f'(x)]^2} \geq \sqrt{1} = 1$. Then the arc length of the graph of $f(x)$ on $[1, 4]$ is

$$\int_1^4 \sqrt{1+[f'(x)]^2}\, dx \geq \int_1^4 1\, dx = 3$$

Exercises

1. Express the arc length of the curve $y = x^4$ between $x = 2$ and $x = 6$ as an integral (but do not evaluate).

SOLUTION Let $y = x^4$. Then $y' = 4x^3$ and

$$s = \int_2^6 \sqrt{1+(4x^3)^2}\, dx = \int_2^6 \sqrt{1+16x^6}\, dx.$$

2. Express the arc length of the curve $y = \tan x$ for $0 \leq x \leq \frac{\pi}{4}$ as an integral (but do not evaluate).

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$, and

$$s = \int_0^{\pi/4} \sqrt{1+(\sec^2 x)^2}\, dx = \int_0^{\pi/4} \sqrt{1+\sec^4 x}\, dx.$$

3. Find the arc length of $y = \frac{1}{12}x^3 + x^{-1}$ for $1 \leq x \leq 2$. *Hint:* Show that $1 + (y')^2 = \left(\frac{1}{4}x^2 + x^{-2}\right)^2$.

SOLUTION Let $y = \dfrac{1}{12}x^3 + x^{-1}$. Then $y' = \dfrac{x^2}{4} x^{-2}$, and

$$(y')^2 + 1 = \left(\frac{x^2}{4} - x^{-2}\right)^2 + 1 = \frac{x^4}{16} - \frac{1}{2} + x^{-4} + 1 = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2.$$

Thus,

$$s=\int_1^2\sqrt{1+(y')^2}\,dx=\int_1^2\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx=\int_1^2\left|\frac{x^2}{4}+\frac{1}{x^2}\right|dx$$
$$=\int_1^2\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\quad\text{since}\quad\frac{x^2}{4}+\frac{1}{x^2}>0$$
$$=\left(\frac{x^3}{12}-\frac{1}{x}\right)\bigg|_1^2=\frac{13}{12}.$$

4. Find the arc length of $y=\left(\frac{x}{2}\right)^4+\frac{1}{2x^2}$ over $[1,4]$. *Hint:* Show that $1+(y')^2$ is a perfect square.

SOLUTION Let $y=\left(\frac{x}{2}\right)^4+\frac{1}{2x^2}$. Then

$$y'=4\left(\frac{x}{2}\right)^3\left(\frac{1}{2}\right)-\frac{1}{x^3}=\frac{x^3}{4}-\frac{1}{x^3}$$

and

$$(y')^2+1=\left(\frac{x^3}{4}-\frac{1}{x^3}\right)^2+1=\frac{x^6}{16}-\frac{1}{2}+\frac{1}{x^6}+1=\frac{x^6}{16}+\frac{1}{2}+\frac{1}{x^6}=\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2.$$

Hence,

$$s=\int_1^4\sqrt{1+y'^2}\,dx=\int_1^4\sqrt{\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2}\,dx=\int_1^4\left|\frac{x^3}{4}+\frac{1}{x^3}\right|dx$$
$$=\int_1^4\left(\frac{x^3}{4}+\frac{1}{x^3}\right)dx\quad\text{since}\quad\frac{x^3}{4}+\frac{1}{x^3}>0\text{ on }[1,4]$$
$$=\left(\frac{x^4}{16}+\frac{x^{-2}}{-2}\right)\bigg|_1^4=\frac{525}{32}.$$

In Exercises 5–10, calculate the arc length over the given interval.

5. $y=3x+1,\quad[0,3]$

SOLUTION Let $y=3x+1$. Then $y'=3$, and $s=\int_0^3\sqrt{1+9}\,dx=3\sqrt{10}$.

6. $y=9-3x,\quad[1,3]$

SOLUTION Let $y=9-3x$. Then $y'=-3$, and $s=\int_1^3\sqrt{1+9}\,dx=3\sqrt{10}-\sqrt{10}=2\sqrt{10}$.

7. $y=x^{3/2},\quad[1,2]$

SOLUTION Let $y=x^{3/2}$. Then $y'=\frac{3}{2}x^{1/2}$, and

$$s=\int_1^2\sqrt{1+\frac{9}{4}x}\,dx=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\bigg|_1^2=\frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right)=\frac{1}{27}\left(22\sqrt{22}-13\sqrt{13}\right).$$

8. $y=\frac{1}{3}x^{3/2}-x^{1/2},\quad[2,8]$

SOLUTION Let $y=\frac{1}{3}x^{3/2}-x^{1/2}$. Then

$$y'=\frac{1}{2}x^{1/2}-\frac{1}{2}x^{-1/2},$$

and

$$1+(y')^2=1+\left(\frac{1}{2}x^{1/2}-\frac{1}{2}x^{-1/2}\right)^2=\frac{1}{4}x+\frac{1}{2}+\frac{1}{4}x^{-1}=\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2.$$

Hence,

$$\begin{aligned} s &= \int_2^8 \sqrt{1+(y')^2}\,dx = \int_2^8 \sqrt{\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2}\,dx = \int_2^8 \left|\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right|\,dx \\ &= \int_2^8 \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)dx \quad \text{since} \quad \frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2} > 0 \\ &= \left(\frac{1}{3}x^{3/2}+x^{1/2}\right)\Big|_2^8 = \frac{17\sqrt{2}}{3}. \end{aligned}$$

9. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad [1, 2e]$

SOLUTION Let $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$. Then

$$y' = \frac{x}{2} - \frac{1}{2x},$$

and

$$1+(y')^2 = 1 + \left(\frac{x}{2}-\frac{1}{2x}\right)^2 = \frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2} = \left(\frac{x}{2}+\frac{1}{2x}\right)^2.$$

Hence,

$$\begin{aligned} s &= \int_1^{2e} \sqrt{1+(y')^2}\,dx = \int_1^{2e} \sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = \int_1^{2e} \left|\frac{x}{2}+\frac{1}{2x}\right|\,dx \\ &= \int_1^{2e} \left(\frac{x}{2}+\frac{1}{2x}\right)dx \quad \text{since} \quad \frac{x}{2}+\frac{1}{2x} > 0 \text{ on } [1, 2e] \\ &= \left(\frac{x^2}{4}+\frac{1}{2}\ln x\right)\Big|_1^{2e} = e^2 + \frac{\ln 2}{2} + \frac{1}{4}. \end{aligned}$$

10. $y = \ln(\cos x), \quad \left[0, \frac{\pi}{4}\right]$

SOLUTION Let $y = \ln(\cos x)$. Then $y' = -\tan x$ and $1+(y')^2 = 1+\tan^2 x = \sec^2 x$. Hence,

$$\begin{aligned} s &= \int_0^{\pi/4} \sqrt{1+(y')^2}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx \\ &= \int_0^{\pi/4} \sec x\,dx \quad \text{since} \quad \sec x > 0 \text{ on } \left[0, \frac{\pi}{4}\right] \\ &= \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln(\sqrt{2}+1). \end{aligned}$$

In Exercises 11–14, approximate the arc length of the curve over the interval using the Trapezoidal Rule T_N, the Midpoint Rule M_N, or Simpson's Rule S_N as indicated.

11. $y = \frac{1}{4}x^4, \quad [1, 2], \quad T_5$

SOLUTION Let $y = \frac{1}{4}x^4$. Then

$$1+(y')^2 = 1+(x^3)^2 = 1+x^6.$$

Therefore, the arc length over $[1, 2]$ is

$$\int_1^2 \sqrt{1+x^6}\,dx.$$

Now, let $f(x) = \sqrt{1+x^6}$. With $n = 5$,

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} \quad \text{and} \quad \{x_i\}_{i=0}^5 = \left\{1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2\right\}.$$

Using the Trapezoidal Rule,

$$\int_1^2 \sqrt{1+x^6}\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2\sum_{i=1}^{4} f(x_i)+f(x_5)\right]=3.957736.$$

The arc length is approximately 3.957736 units.

12. $y=\sin x$, $\left[0,\frac{\pi}{2}\right]$, M_8

SOLUTION Let $y=\sin x$. Then

$$1+y'^2=1+\cos^2 x.$$

Therefore, the arc length over $[0,\pi/2]$ is

$$\int_0^{\pi/2}\sqrt{1+\cos^2 x}\,dx.$$

Now, let $f(x)=\sqrt{1+\cos^2 x}$. With $n=8$, we have:

$$\Delta x=\frac{\pi/2}{8}=\frac{\pi}{16} \quad\text{and}\quad \{x_i^*\}_{i=1}^8=\left\{\frac{\pi}{32},\frac{3\pi}{32},\frac{5\pi}{32},\frac{7\pi}{32},\frac{9\pi}{32},\frac{11\pi}{32},\frac{13\pi}{32},\frac{15\pi}{32}\right\}.$$

Using the Midpoint Rule,

$$\int_0^{\pi/2}\sqrt{1+\cos^2 x}\,dx \approx \Delta x\sum_{i=1}^{8} f(x_i^*)=1.910099.$$

The arc length is approximately 1.910099 units.

13. $y=x^{-1}$, $[1,2]$, S_8

SOLUTION Let $y=x^{-1}$. Then $y'=-x^{-2}$ and

$$1+(y')^2=1+\frac{1}{x^4}.$$

Therefore, the arc length over $[1,2]$ is

$$\int_1^2\sqrt{1+\frac{1}{x^4}}\,dx.$$

Now, let $f(x)=\sqrt{1+\frac{1}{x^4}}$. With $n=8$,

$$\Delta x=\frac{2-1}{8}=\frac{1}{8} \quad\text{and}\quad \{x_i\}_{i=0}^8=\left\{1,\frac{9}{8},\frac{5}{4},\frac{11}{8},\frac{3}{2},\frac{13}{8},\frac{7}{4},\frac{15}{8},2\right\}.$$

Using Simpson's Rule,

$$\int_1^2\sqrt{1+\frac{1}{x^4}}\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4\sum_{i=1}^{4} f(x_{2i-1})+2\sum_{i=1}^{3} f(x_{2i})+f(x_8)\right]=1.132123.$$

The arc length is approximately 1.132123 units.

14. $y=e^{-x^2}$, $[0,2]$, S_8

SOLUTION Let $y=e^{-x^2}$. Then

$$1+(y')^2=1+4x^2e^{-2x^2}.$$

Therefore, the arc length over $[0,2]$ is

$$\int_0^2\sqrt{1+4x^2e^{-2x^2}}\,dx.$$

Now, let $f(x)=\sqrt{1+4x^2e^{-2x^2}}$. With $n=8$,

$$\Delta x=\frac{2-0}{8}=\frac{1}{4} \quad\text{and}\quad \{x_i\}_{i=0}^8=\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2\right\}.$$

Using Simpson's Rule,

$$\int_0^2 \sqrt{1+4x^2e^{-2x^2}}\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4\sum_{i=1}^{4} f(x_{2i-1})+2\sum_{i=1}^{3} f(x_{2i})+f(x_8)\right]=2.280718.$$

The arc length is approximately 2.280718 units.

15. Calculate the length of the astroid $x^{2/3}+y^{2/3}=1$ (Figure 11).

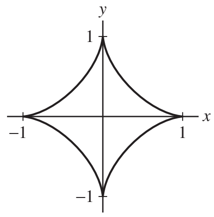


FIGURE 11 Graph of $x^{2/3}+y^{2/3}=1$.

SOLUTION We will calculate the arc length of the portion of the asteroid in the first quadrant and then multiply by 4. By implicit differentiation

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0,$$

so

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}.$$

Thus

$$1+(y')^2=1+\frac{y^{2/3}}{x^{2/3}}=\frac{x^{2/3}+y^{2/3}}{x^{2/3}}=\frac{1}{x^{2/3}},$$

and

$$s=\int_0^1 \frac{1}{x^{1/3}}\,dx=\frac{3}{2}.$$

The total arc length is therefore $4\cdot\frac{3}{2}=6$.

16. Show that the arc length of the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$ (for $a>0$) is proportional to a.

SOLUTION We will calculate the arc length of the portion of the asteroid in the first quadrant and then multiply by 4. By implicit differentiation

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0,$$

so

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}.$$

Thus

$$1+(y')^2=1+\frac{y^{2/3}}{x^{2/3}}=\frac{x^{2/3}+y^{2/3}}{x^{2/3}}=\frac{a^{2/3}}{x^{2/3}},$$

and

$$s=\int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx=a^{1/3}\left(\frac{3}{2}a^{2/3}\right)=\frac{3}{2}a.$$

The total arc length is therefore $4\cdot\frac{3}{2}a=6a$, which is proportional to a.

17. Let $a, r>0$. Show that the arc length of the curve $x^r+y^r=a^r$ for $0\le x\le a$ is proportional to a.

SOLUTION Using implicit differentiation, we find $y'=-(x/y)^{r-1}$ and

$$1+(y')^2=1+(x/y)^{2r-2}=\frac{x^{2r-1}+y^{2r-2}}{y^{2r-2}}=\frac{x^{2r-2}+(a^r-x^r)^{2-2/r}}{(a^r-x^r)^{2-2/r}}.$$

The arc length is then

$$s = \int_0^a \sqrt{\frac{x^{2r-2} + (a^r - x^r)^{2-2/r}}{(a^r - x^r)^{2-2/r}}}\, dx.$$

Using the substitution $x = au$, we obtain

$$s = a\int_0^1 \sqrt{\frac{u^{2r-2} + (1 - u^r)^{2-2/r}}{(1 - u^r)^{2-2/r}}}\, du,$$

where the integral is independent of a.

18. Find the arc length of the curve shown in Figure 12.

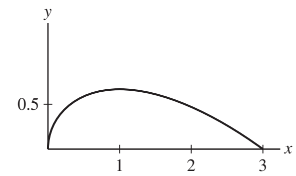


FIGURE 12 Graph of $9y^2 = x(x-3)^2$.

SOLUTION Using implicit differentiation,

$$18yy' = x(2)(x-3) + (x-3)^2 = 3(x-3)(x-1)$$

Hence,

$$(y')^2 = \frac{(x-3)^2(x-1)^2}{36y^2} = \frac{(x-3)^2(x-1)^2}{4(9y^2)} = \frac{(x-3)^2(x-1)^2}{4x(x-3)^2} = \frac{(x-1)^2}{4x}$$

and

$$1 + (y')^2 = \frac{(x-1)^2 + 4x}{4x} = \frac{(x+1)^2}{4x}.$$

Finally,

$$\begin{aligned} s &= \int_0^3 \sqrt{\frac{(x+1)^2}{4x}}\, dx = \int_0^3 \frac{|x+1|}{2\sqrt{x}}\, dx \\ &= \int_0^3 \frac{x+1}{2\sqrt{x}}\, dx \quad \text{since} \quad x+1 > 0 \text{ on } [0,3] \\ &= \int_0^3 \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx = \left(\frac{1}{3}x^{3/2} + x^{1/2}\right)\Big|_0^3 = 2\sqrt{3}. \end{aligned}$$

19. Find the value of a such that the arc length of the *catenary* $y = \cosh x$ for $-a \le x \le a$ equals 10.

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$ and

$$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x.$$

Thus,

$$s = \int_{-a}^{a} \cosh x\, dx = \sinh(a) - \sinh(-a) = 2\sinh a.$$

Setting this expression equal to 10 and solving for a yields $a = \sinh^{-1}(5) = \ln(5 + \sqrt{26})$.

20. Calculate the arc length of the graph of $f(x) = mx + r$ over $[a, b]$ in two ways: using the Pythagorean theorem (Figure 13) and using the arc length integral.

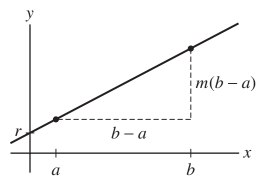


FIGURE 13

SOLUTION Let h denote the length of the hypotenuse. Then, by Pythagoras' Theorem,

$$h^2 = (b-a)^2 + m^2(b-a)^2 = (b-a)^2(1+m^2),$$

or

$$h = (b-a)\sqrt{1+m^2}$$

since $b > a$. Moreover, $(f'(x))^2 = m^2$, so

$$s = \int_a^b \sqrt{1+m^2}\,dx = (b-a)\sqrt{1+m^2} = h.$$

21. Show that the circumference of the unit circle is equal to

$$2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \quad \text{(an improper integral)}$$

Evaluate, thus verifying that the circumference is 2π.

SOLUTION Note the circumference of the unit circle is twice the arc length of the upper half of the curve defined by $x^2 + y^2 = 1$. Thus, let $y = \sqrt{1-x^2}$. Then

$$y' = -\frac{x}{\sqrt{1-x^2}} \quad \text{and} \quad 1 + (y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

Finally, the circumference of the unit circle is

$$2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1} x\Big|_{-1}^{1} = \pi - (-\pi) = 2\pi.$$

22. Generalize the result of Exercise 21 to show that the circumference of the circle of radius r is $2\pi r$.

SOLUTION Let $y = \sqrt{r^2 - x^2}$ denote the upper half of a circle of radius r centered at the origin. Then

$$1 + (y')^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2} = \frac{1}{1 - \frac{x^2}{r^2}},$$

and the circumference of the circle is given by

$$C = 2\int_{-r}^{r} \frac{dx}{\sqrt{1 - x^2/r^2}}.$$

Using the substitution $u = x/r$, $du = dx/r$, we find

$$C = 2r\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = 2r\sin^{-1} u\Big|_{-1}^{1}$$
$$= 2r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi r$$

23. Calculate the arc length of $y = x^2$ over $[0, a]$. *Hint:* Use trigonometric substitution. Evaluate for $a = 1$.

SOLUTION Let $y = x^2$. Then $y' = 2x$ and

$$s = \int_0^a \sqrt{1+4x^2}\,dx.$$

Using the substitution $2x = \tan\theta$, $2\,dx = \sec^2\theta\,d\theta$, we find

$$s = \frac{1}{2}\int_{x=0}^{x=a} \sec^3\theta\,d\theta.$$

Next, using a reduction formula for the integral of $\sec^3\theta$, we see that

$$s = \left(\frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta|\right)\Big|_{x=0}^{x=a} = \left(\frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln|\sqrt{1+4x^2} + 2x|\right)\Big|_0^a$$
$$= \frac{a}{2}\sqrt{1+4a^2} + \frac{1}{4}\ln|\sqrt{1+4a^2} + 2a|$$

Thus, when $a = 1$,

$$s = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(\sqrt{5} + 2) \approx 1.478943.$$

24. Express the arc length of $g(x) = \sqrt{x}$ over $[0, 1]$ as a definite integral. Then use the substitution $u = \sqrt{x}$ to show that this arc length is equal to the arc length of x^2 over $[0, 1]$ (but do not evaluate the integrals). Explain this result graphically.

SOLUTION Let $g(x) = \sqrt{x}$. Then

$$1 + g'(x)^2 = \frac{1 + 4x}{4x} \quad \text{and} \quad s = \int_0^1 \sqrt{\frac{1 + 4x}{4x}}\, dx = \int_0^1 \frac{\sqrt{1 + 4x}}{2\sqrt{x}}\, dx.$$

With the substitution $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\, dx$, this becomes

$$s = \int_0^1 \sqrt{1 + 4u^2}\, du.$$

Now, let $f(x) = x^2$. Then $1 + f'(x)^2 = 1 + 4x^2$, and

$$s = \int_0^1 \sqrt{1 + 4x^2}\, dx.$$

Thus, the two arc lengths are equal. This is explained graphically by the fact that for $x \geq 0$, x^2 and $\sqrt{x}$ are inverses of each other. This means that the two graphs are symmetric with respect to the line $y = x$. Moreover, the graphs of x^2 and $\sqrt{x}$ intersect at $x = 0$ and at $x = 1$. Thus, it is clear that the arc length of the two graphs on $[0, 1]$ are equal.

25. Find the arc length of $y = e^x$ over $[0, a]$. *Hint:* Try the substitution $u = \sqrt{1 + e^{2x}}$ followed by partial fractions.

SOLUTION Let $y = e^x$. Then $1 + (y')^2 = 1 + e^{2x}$, and the arc length over $[0, a]$ is

$$\int_0^a \sqrt{1 + e^{2x}}\, dx.$$

Now, let $u = \sqrt{1 + e^{2x}}$. Then

$$du = \frac{1}{2} \cdot \frac{2e^{2x}}{\sqrt{1 + e^{2x}}}\, dx = \frac{u^2 - 1}{u}\, dx$$

and the arc length is

$$\begin{aligned}
\int_0^a \sqrt{1 + e^{2x}}\, dx &= \int_{x=0}^{x=a} u \cdot \frac{u}{u^2 - 1}\, du = \int_{x=0}^{x=a} \frac{u^2}{u^2 - 1}\, du = \int_{x=0}^{x=a} \left(1 + \frac{1}{u^2 - 1}\right) du \\
&= \int_{x=0}^{x=a} \left(1 + \frac{1}{2}\frac{1}{u - 1} - \frac{1}{2}\frac{1}{u + 1}\right) du = \left(u + \frac{1}{2}\ln(u - 1) - \frac{1}{2}\ln(u + 1)\right)\Bigg|_{x=0}^{x=a} \\
&= \left[\sqrt{1 + e^{2x}} + \frac{1}{2}\ln\left(\frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1}\right)\right]\Bigg|_0^a \\
&= \sqrt{1 + e^{2a}} + \frac{1}{2}\ln\frac{\sqrt{1 + e^{2a}} - 1}{\sqrt{1 + e^{2a}} + 1} - \sqrt{2} + \frac{1}{2}\ln\frac{1 + \sqrt{2}}{\sqrt{2} - 1} \\
&= \sqrt{1 + e^{2a}} + \frac{1}{2}\ln\frac{\sqrt{1 + e^{2a}} - 1}{\sqrt{1 + e^{2a}} + 1} - \sqrt{2} + \ln(1 + \sqrt{2}).
\end{aligned}$$

26. Show that the arc length of $y = \ln(f(x))$ for $a \leq x \leq b$ is

$$\int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)}\, dx$$ 4

SOLUTION Let $y = \ln(f(x))$. Then

$$y' = \frac{f'(x)}{f(x)} \qquad \text{and} \qquad 1 + (y')^2 = \frac{f(x)^2 + f'(x)^2}{f(x)^2}.$$

Therefore,

$$s = \int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)}\,dx$$

since $f(x) > 0$ in order for $y = \ln(f(x))$ to be defined on $[a, b]$.

27. Use Eq. (4) to compute the arc length of $y = \ln(\sin x)$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

SOLUTION With $f(x) = \sin x$, Eq. (4) yields

$$s = \int_{\pi/4}^{\pi/2} \frac{\sqrt{\sin^2 x + \cos^2 x}}{\sin x}\,dx = \int_{\pi/4}^{\pi/2} \csc x\,dx = \ln(\csc x - \cot x)\Big|_{\pi/4}^{\pi/2}$$

$$= \ln 1 - \ln(\sqrt{2} - 1) = \ln \frac{1}{\sqrt{2} - 1} = \ln(\sqrt{2} + 1).$$

28. Use Eq. (4) to compute the arc length of $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$ over $[1, 3]$.

SOLUTION With $f(x) = \frac{e^x + 1}{e^x - 1}$,

$$f'(x) = \frac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2} = -\frac{2e^x}{(e^x - 1)^2}$$

and

$$f(x)^2 + f'(x)^2 = \left(\frac{e^x + 1}{e^x - 1}\right)^2 + \frac{4e^{2x}}{(e^x - 1)^4} = \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^x - 1)^4} = \frac{(e^{2x} + 1)^2}{(e^x - 1)^4}.$$

Thus, by Eq. (4),

$$s = \int_1^3 \frac{e^{2x} + 1}{(e^x - 1)^2} \cdot \frac{e^x - 1}{e^x + 1}\,dx = \int_1^3 \frac{e^{2x} + 1}{e^{2x} - 1}\,dx.$$

Observe that

$$\frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{(e^x + e^{-x})/2}{(e^x - e^{-x})/2} = \frac{\cosh x}{\sinh x}.$$

Therefore,

$$s = \int_1^3 \frac{\cosh x}{\sinh x}\,dx = \ln(\sinh x)\Big|_1^3 = \ln(\sinh 3) - \ln(\sinh 1).$$

29. Show that if $0 \le f'(x) \le 1$ for all x, then the arc length of $y = f(x)$ over $[a, b]$ is at most $\sqrt{2}(b - a)$. Show that for $f(x) = x$, the arc length equals $\sqrt{2}(b - a)$.

SOLUTION If $0 \le f'(x) \le 1$ for all x, then

$$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx \le \int_a^b \sqrt{1 + 1}\,dx = \sqrt{2}(b - a).$$

If $f(x) = x$, then $f'(x) = 1$ and

$$s = \int_a^b \sqrt{1 + 1}\,dx = \sqrt{2}(b - a).$$

30. Use the Comparison Theorem (Section 5.2) to prove that the arc length of $y = x^{4/3}$ over $[1, 2]$ is not less than $\frac{5}{3}$.

SOLUTION Note that $f'(x) = \frac{4}{3}x^{1/3}$; for $x \in [1, 2]$, we have $x^{1/3} \ge 1$ so that $f'(x) \ge \frac{4}{3}$. Then

$$\sqrt{1 + f'(x)^2} \ge \sqrt{1 + \left(\frac{4}{3}\right)^2} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

and then the arc length is

$$\int_1^2 \sqrt{1 + f'(x)^2}\,dx \ge \int_1^2 \frac{5}{3}\,dx = \frac{5}{3}$$

31. Approximate the arc length of one-quarter of the unit circle (which we know is $\frac{\pi}{2}$) by computing the length of the polygonal approximation with $N = 4$ segments (Figure 14).

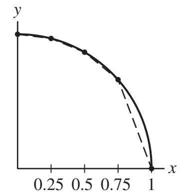


FIGURE 14 One-quarter of the unit circle

SOLUTION With $y = \sqrt{1-x^2}$, the five points along the curve are

$$P_0(0,1),\ P_1(1/4, \sqrt{15}/4),\ P_2(1/2, \sqrt{3}/2),\ P_3(3/4, \sqrt{7}/4),\ P_4(1,0)$$

Then

$$\overline{P_0P_1} = \sqrt{\frac{1}{16} + \left(\frac{4-\sqrt{15}}{4}\right)^2} \approx 0.252009$$

$$\overline{P_1P_2} = \sqrt{\frac{1}{16} + \left(\frac{2\sqrt{3}-\sqrt{15}}{4}\right)^2} \approx 0.270091$$

$$\overline{P_2P_3} = \sqrt{\frac{1}{16} + \left(\frac{2\sqrt{3}-\sqrt{7}}{4}\right)^2} \approx 0.323042$$

$$\overline{P_3P_4} = \sqrt{\frac{1}{16} + \frac{7}{16}} \approx 0.707108$$

and the total approximate distance is 1.552250 whereas $\pi/2 \approx 1.570796$.

32. CAS A merchant intends to produce specialty carpets in the shape of the region in Figure 15, bounded by the axes and graph of $y = 1 - x^n$ (units in yards). Assume that material costs \$50/yd^2 and that it costs $50L$ dollars to cut the carpet, where L is the length of the curved side of the carpet. The carpet can be sold for $150A$ dollars, where A is the carpet's area. Using numerical integration with a computer algebra system, find the whole number n for which the merchant's profits are maximal.

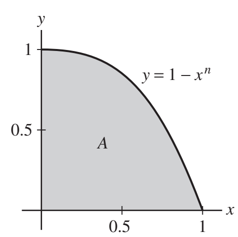


FIGURE 15

SOLUTION The area of the carpet is

$$A = \int_0^1 (1 - x^n)dx = \left(x - \frac{x^{n+1}}{n+1}\right)\bigg|_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

while the length of the curved side of the carpet is

$$L = \int_0^1 \sqrt{1 + (nx^{n-1})^2}\,dx = \int_0^1 \sqrt{1 + n^2x^{2n-2}}\,dx.$$

Using these formulas, we find that the merchant's profit is given by

$$150A - (50A + 50L) = 100A - 50L = \frac{100n}{n+1} - 50\int_0^1 \sqrt{1 + n^2x^{2n-2}}\,dx.$$

Using a CAS, we find that the merchant's profit is maximized (approximately \$3.31 per carpet) when $n = 13$. The table below lists the profit for $1 \le n \le 15$.

n	Profit	n	Profit
1	−20.71067810	9	3.06855532
2	−7.28047621	10	3.18862208
3	−2.39328273	11	3.25953632
4	−0.01147138	12	3.29668137
5	1.30534545	13	3.31024566
6	2.08684099	14	3.30715476
7	2.57017349	15	3.29222024
8	2.87535925		

In Exercises 33–40, compute the surface area of revolution about the x-axis over the interval.

33. $y = x$, $[0, 4]$

SOLUTION $1 + (y')^2 = 2$ so that

$$SA = 2\pi \int_0^4 x\sqrt{2}\,dx = 2\pi\sqrt{2}\frac{1}{2}x^2\bigg|_0^4 = 16\pi\sqrt{2}$$

34. $y = 4x + 3$, $[0, 1]$

SOLUTION Let $y = 4x + 3$. Then $1 + (y')^2 = 17$ and

$$SA = 2\pi \int_0^1 (4x+3)\sqrt{17}\,dx = 2\pi\sqrt{17}\left(2x^2 + 3x\right)\bigg|_0^1 = 10\pi\sqrt{17}.$$

35. $y = x^3$, $[0, 2]$

SOLUTION $1 + (y')^2 = 1 + 9x^4$, so that

$$SA = 2\pi \int_0^2 x^3\sqrt{1+9x^4}\,dx = \frac{2\pi}{36}\int_0^2 36x^3\sqrt{1+9x^4}\,dx = \frac{\pi}{18}(1+9x^4)^{3/2}\bigg|_0^2 = \frac{\pi}{18}\left(145^{3/2} - 1\right)$$

36. $y = x^2$, $[0, 4]$

SOLUTION Let $y = x^2$. Then $y' = 2x$ and

$$SA = 2\pi \int_0^4 x^2\sqrt{1+4x^2}\,dx.$$

Using the substitution $2x = \tan\theta$, $2\,dx = \sec^2\theta\,d\theta$, we find that

$$\begin{aligned}
\int x^2\sqrt{1+4x^2}\,dx &= \frac{1}{8}\int \sec^3\theta\tan^2\theta\,d\theta = \frac{1}{8}\int\left(\sec^5\theta - \sec^3\theta\right)d\theta \\
&= \frac{1}{8}\left(\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| - \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C \\
&= \frac{x}{16}(1+4x^2)^{3/2} - \frac{x}{32}\sqrt{1+4x^2} - \frac{1}{64}\ln|\sqrt{1+4x^2} + 2x| + C.
\end{aligned}$$

Finally,

$$\begin{aligned}
SA &= 2\pi\left(\frac{x}{16}(1+4x^2)^{3/2} - \frac{x}{32}\sqrt{1+4x^2} - \frac{1}{64}\ln|\sqrt{1+4x^2}+2x|\right)\bigg|_0^4 \\
&= 2\pi\left(\frac{1}{4}65^{3/2} - \frac{\sqrt{65}}{8} - \frac{1}{64}\ln(8+\sqrt{65})\right) = \frac{129\sqrt{65}}{4}\pi - \frac{\pi}{32}\ln(8+\sqrt{65}).
\end{aligned}$$

37. $y = (4 - x^{2/3})^{3/2}$, $[0, 8]$

SOLUTION Let $y = (4 - x^{2/3})^{3/2}$. Then

$$y' = -x^{-1/3}(4 - x^{2/3})^{1/2},$$

and

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}.$$

Therefore,

$$SA = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \left(\frac{2}{x^{1/3}}\right) dx.$$

Using the substitution $u = 4 - x^{2/3}$, $du = -\frac{2}{3}x^{-1/3}\,dx$, we find

$$SA = 2\pi \int_4^0 u^{3/2}(-3)\,du = 6\pi \int_0^4 u^{3/2}\,du = \frac{12}{5}\pi u^{5/2}\Big|_0^4 = \frac{384\pi}{5}.$$

38. $y = e^{-x}$, $[0, 1]$

SOLUTION Let $y = e^{-x}$. Then $y' = -e^{-x}$ and

$$SA = 2\pi \int_0^1 e^{-x}\sqrt{1 + e^{-2x}}\,dx.$$

Using the substitution $e^{-x} = \tan\theta$, $-e^{-x}\,dx = \sec^2\theta\,d\theta$, we find that

$$\begin{aligned}\int e^{-x}\sqrt{1+e^{-2x}}\,dx &= -\int \sec^3\theta\,d\theta = -\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\\ &= -\frac{1}{2}e^{-x}\sqrt{1+e^{-2x}} - \frac{1}{2}\ln|\sqrt{1+e^{-2x}} + e^{-x}| + C.\end{aligned}$$

Finally,

$$\begin{aligned}SA &= \left(-\pi e^{-x}\sqrt{1+e^{-2x}} - \pi\ln|\sqrt{1+e^{-2x}} + e^{-x}|\right)\Big|_0^1\\ &= -\pi e^{-1}\sqrt{1+e^{-2}} - \pi\ln(\sqrt{1+e^{-2}} + e^{-1}) + \pi\sqrt{2} + \pi\ln(\sqrt{2}+1)\\ &= \pi\sqrt{2} - \pi e^{-1}\sqrt{1+e^{-2}} + \pi\ln\left(\frac{\sqrt{2}+1}{\sqrt{1+e^{-2}}+e^{-1}}\right).\end{aligned}$$

39. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $[1, e]$

SOLUTION We have $y' = \frac{x}{2} - \frac{1}{2x}$, and

$$1 + (y')^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Thus,

$$\begin{aligned}SA &= 2\pi \int_1^e \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^e \frac{x^3}{8} + \frac{x}{8} - \frac{x\ln x}{4} - \frac{\ln x}{4x}\,dx\\ &= 2\pi\left(\frac{x^4}{32} + \frac{x^2}{16} - \frac{x^2\ln x}{8} + \frac{x^2}{16} - \frac{(\ln x)^2}{8}\right)\Big|_1^e\\ &= 2\pi\left(\frac{e^4}{32} + \frac{e^2}{16} - \frac{e^2}{8} + \frac{e^2}{16} - \frac{1}{8} - \left(\frac{1}{32} + \frac{1}{16} + 0 + \frac{1}{16} - 0\right)\right)\\ &= 2\pi\left(\frac{e^4}{32} - \frac{1}{8} - \frac{1}{32} - \frac{1}{16} - \frac{1}{16}\right)\\ &= \frac{\pi}{16}(e^4 - 9)\end{aligned}$$

40. $y = \sin x$, $[0, \pi]$

SOLUTION Let $y = \sin x$. Then $y' = \cos x$, and

$$SA = 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx.$$

Using the substitution $\cos x = \tan\theta$, $-\sin x\,dx = \sec^2\theta\,d\theta$, we find that

$$\int \sin x\sqrt{1+\cos^2 x}\,dx = -\int \sec^3\theta\,d\theta = -\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= -\frac{1}{2}\cos x\sqrt{1+\cos^2 x} - \frac{1}{2}\ln\left|\sqrt{1+\cos^2 x} + \cos x\right| + C.$$

Finally,

$$SA = 2\pi\left(-\frac{1}{2}\cos x\sqrt{1+\cos^2 x} - \frac{1}{2}\ln\left|\sqrt{1+\cos^2 x} + \cos x\right|\right)\Big|_0^{\pi}$$
$$= 2\pi\left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}-1) + \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right) = 2\pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).$$

CAS *In Exercises 41–44, use a computer algebra system to find the approximate surface area of the solid generated by rotating the curve about the x-axis.*

41. $y = x^{-1}$, $[1, 3]$

SOLUTION

$$SA = 2\pi\int_1^3 \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx = 2\pi\int_1^3 \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \approx 7.603062807$$

using Maple.

42. $y = x^4$, $[0, 1]$

SOLUTION

$$SA = 2\pi\int_0^1 x^4\sqrt{1+(4x^3)^2}\,dx = 2\pi\int_0^1 x^4\sqrt{1+16x^6}\,dx \approx 3.436526697$$

using Maple.

43. $y = e^{-x^2/2}$, $[0, 2]$

SOLUTION

$$SA = 2\pi\int_0^2 e^{-x^2/2}\sqrt{1+(-xe^{-x^2/2})^2}\,dx = 2\pi\int_0^2 e^{-x^2/2}\sqrt{1+x^2e^{-x^2}}\,dx \approx 8.222695606$$

using Maple.

44. $y = \tan x$, $\left[0, \frac{\pi}{4}\right]$

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$, $1 + (y')^2 = 1 + \sec^4 x$, and

$$SA = 2\pi\int_0^{\pi/4} \tan x\sqrt{1+\sec^4 x}\,dx.$$

Using a computer algebra system to approximate the value of the definite integral, we find

$$SA \approx 3.83908.$$

45. Find the area of the surface obtained by rotating $y = \cosh x$ over $[-\ln 2, \ln 2]$ around the x-axis.

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$, and

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x.$$

Therefore,

$$SA = 2\pi\int_{-\ln 2}^{\ln 2} \cosh^2 x\,dx = \pi\int_{-\ln 2}^{\ln 2} (1+\cosh 2x)\,dx = \pi\left(x + \frac{1}{2}\sinh 2x\right)\Big|_{-\ln 2}^{\ln 2}$$
$$= \pi\left(\ln 2 + \frac{1}{2}\sinh(2\ln 2) + \ln 2 - \frac{1}{2}\sinh(-2\ln 2)\right) = 2\pi\ln 2 + \pi\sinh(2\ln 2).$$

We can simplify this answer by recognizing that

$$\sinh(2\ln 2) = \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} = \frac{4-\frac{1}{4}}{2} = \frac{15}{8}.$$

Thus,

$$SA = 2\pi\ln 2 + \frac{15\pi}{8}.$$

46. Show that the surface area of a spherical cap of height h and radius R (Figure 16) has surface area $2\pi Rh$.

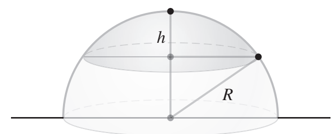


FIGURE 16

SOLUTION To determine the surface area of the cap, we will rotate a portion of a circle of radius R, centered at the origin, about the y-axis. Since the equation of the right half of the circle is $x = \sqrt{R^2 - y^2}$,

$$1 + (x')^2 = 1 + \frac{y^2}{R^2 - y^2} = \frac{R^2}{R^2 - y^2},$$

and

$$SA = 2\pi \int_{R-h}^{R} \sqrt{R^2 - y^2}\left(\frac{R}{\sqrt{R^2 - y^2}}\right) dy = 2\pi R\,(R - (R - h)) = 2\pi Rh.$$

47. Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = a^2$ about the x-axis (Figure 17).

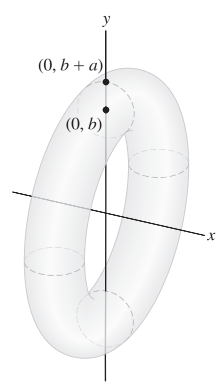


FIGURE 17 Torus obtained by rotating a circle about the x-axis.

SOLUTION $y = b + \sqrt{a^2 - x^2}$ gives the top half of the circle and $y = b - \sqrt{a^2 - x^2}$ gives the bottom half. Note that in each case,

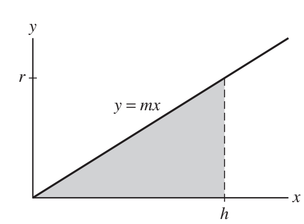
$$1 + (y')^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Rotating the two halves of the circle around the x-axis then yields

$$\begin{aligned} SA &= 2\pi \int_{-a}^{a} (b + \sqrt{a^2 - x^2})\frac{a}{\sqrt{a^2 - x^2}}\,dx + 2\pi \int_{-a}^{a} (b - \sqrt{a^2 - x^2})\frac{a}{\sqrt{a^2 - x^2}}\,dx \\ &= 2\pi \int_{-a}^{a} 2b\frac{a}{\sqrt{a^2 - x^2}}\,dx = 4\pi ba \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx \\ &= 4\pi ba \cdot \sin^{-1}\left(\frac{x}{a}\right)\Big|_{-a}^{a} = 4\pi ba\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 4\pi^2 ba. \end{aligned}$$

48. Show that the surface area of a right circular cone of radius r and height h is $\pi r\sqrt{r^2 + h^2}$. *Hint:* Rotate a line $y = mx$ about the x-axis for $0 \le x \le h$, where m is determined suitably by the radius r.

SOLUTION



From the figure, we see that $m = \frac{r}{h}$, so $y = \frac{rx}{h}$. Thus

$$SA = 2\pi \int_0^h \frac{rx}{h}\sqrt{1+\frac{r^2}{h^2}}\,dx = \frac{2\pi r}{h}\sqrt{1+\frac{r^2}{h^2}}\int_0^h x\,dx = \pi r\sqrt{h^2+r^2}.$$

Further Insights and Challenges

49. Find the surface area of the ellipsoid obtained by rotating the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ about the x-axis.

SOLUTION Taking advantage of symmetry, we can find the surface area of the ellipsoid by doubling the surface area obtained by rotating the portion of the ellipse in the first quadrant about the x-axis. The equation for the portion of the ellipse in the first quadrant is

$$y = \frac{b}{a}\sqrt{a^2-x^2}.$$

Thus,

$$1+(y')^2 = 1+\frac{b^2x^2}{a^2(a^2-x^2)} = \frac{a^4+(b^2-a^2)x^2}{a^2(a^2-x^2)},$$

and

$$SA = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4+(b^2-a^2)x^2}}{a\sqrt{a^2-x^2}}\,dx = 4\pi b\int_0^a \sqrt{1+\left(\frac{b^2-a^2}{a^4}\right)x^2}\,dx.$$

We now consider two cases. If $b^2 > a^2$, then we make the substitution

$$\frac{\sqrt{b^2-a^2}}{a^2}x = \tan\theta, \quad dx = \frac{a^2}{\sqrt{b^2-a^2}}\sec^2\theta\,d\theta,$$

and find that

$$\begin{aligned} SA &= 4\pi b\frac{a^2}{\sqrt{b^2-a^2}}\int_{x=0}^{x=a}\sec^3\theta\,d\theta = 2\pi b\frac{a^2}{\sqrt{b^2-a^2}}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_{x=0}^{x=a} \\ &= \left(2\pi bx\sqrt{1+\left(\frac{b^2-a^2}{a^4}\right)x^2} + 2\pi b\frac{a^2}{\sqrt{b^2-a^2}}\ln\left|\sqrt{1+\left(\frac{b^2-a^2}{a^4}\right)x^2}+\frac{\sqrt{b^2-a^2}}{a^2}x\right|\right)\Bigg|_0^a \\ &= 2\pi b^2 + 2\pi b\frac{a^2}{\sqrt{b^2-a^2}}\ln\left(\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}\right). \end{aligned}$$

On the other hand, if $a^2 > b^2$, then we make the substitution

$$\frac{\sqrt{a^2-b^2}}{a^2}x = \sin\theta, \quad dx = \frac{a^2}{\sqrt{a^2-b^2}}\cos\theta\,d\theta,$$

and find that

$$\begin{aligned} SA &= 4\pi b\frac{a^2}{\sqrt{a^2-b^2}}\int_{x=0}^{x=a}\cos^2\theta\,d\theta = 2\pi b\frac{a^2}{\sqrt{a^2-b^2}}\left(\theta+\sin\theta\cos\theta\right)\Big|_{x=0}^{x=a} \\ &= \left[2\pi bx\sqrt{1-\left(\frac{a^2-b^2}{a^4}\right)x^2} + 2\pi b\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a^2}x\right)\right]\Bigg|_0^a \\ &= 2\pi b^2 + 2\pi b\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right). \end{aligned}$$

Observe that in both cases, as a approaches b, the value of the surface area of the ellipsoid approaches $4\pi b^2$, the surface area of a sphere of radius b.

50. Show that if the arc length of $f(x)$ over $[0, a]$ is proportional to a, then $f(x)$ must be a linear function.

SOLUTION

$$s = \int_0^a \sqrt{1 + f'(x)^2}\,dx$$

For s to be proportional to a, $\sqrt{1 + f'(x)^2}$ must be a constant, which implies $f'(x)$ is a constant. This, in turn, requires $f(x)$ be linear.

51. CAS Let L be the arc length of the upper half of the ellipse with equation

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

(Figure 18) and let $\eta = \sqrt{1 - (b^2/a^2)}$. Use substitution to show that

$$L = a\int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2 \sin^2\theta}\,d\theta$$

Use a computer algebra system to approximate L for $a = 2$, $b = 1$.

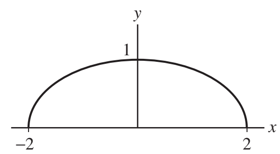


FIGURE 18 Graph of the ellipse $y = \frac{1}{2}\sqrt{4 - x^2}$.

SOLUTION Let $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Then

$$1 + (y')^2 = \frac{b^2x^2 + a^2(a^2 - x^2)}{a^2(a^2 - x^2)}$$

and

$$s = \int_{-a}^a \sqrt{\frac{b^2x^2 + a^2(a^2 - x^2)}{a^2(a^2 - x^2)}}\,dx.$$

With the substitution $x = a\sin t$, $dx = a\cos t\,dt$, $a^2 - x^2 = a^2\cos^2 t$ and

$$s = a\int_{-\pi/2}^{\pi/2} \cos t\sqrt{\frac{a^2b^2\sin^2 t + a^2a^2\cos^2 t}{a^2(a^2\cos^2 t)}}\,dt = a\int_{\pi/2}^{\pi/2} \sqrt{\frac{b^2\sin^2 t}{a^2} + \cos^2 t}\,dt$$

Because

$$\eta = \sqrt{1 - \frac{b^2}{a^2}},\ \eta^2 = 1 - \frac{b^2}{a^2}$$

we then have

$$1 - \eta^2\sin^2 t = 1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2 t = 1 - \sin^2 t + \frac{b^2}{a^2}\sin^2 t = \cos^2 t + \frac{b^2}{a^2}\sin^2 t$$

which is the same as the expression under the square root above. Substituting, we get

$$s = a\int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2\sin^2 t}\,dt$$

When $a = 2$ and $b = 1$, $\eta^2 = \frac{3}{4}$. Using a computer algebra system to approximate the value of the definite integral, we find $s \approx 4.84422$.

52. Prove that the portion of a sphere of radius R seen by an observer located at a distance d above the North Pole has area $A = 2\pi d R^2/(d+R)$. *Hint:* According to Exercise 46, the cap has surface area is $2\pi Rh$. Show that $h = dR/(d+R)$ by applying the Pythagorean Theorem to the three right triangles in Figure 19.

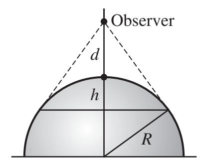
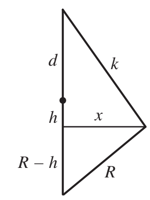


FIGURE 19 Spherical cap observed from a distance d above the North Pole.

SOLUTION Label distances as shown in the figure below.



By repeated application of the Pythagorean Theorem, we find

$$(d+R)^2 = R^2 + k^2 = R^2 + (d+h)^2 + x^2 = R^2 + (d+h)^2 + R^2 - (R-h)^2.$$

Solving for h yields

$$\begin{aligned} d^2 + 2dR + R^2 &= R^2 + d^2 + 2dh + h^2 + R^2 - R^2 + 2Rh - h^2 \\ 2dR &= 2dh + 2Rh \\ dR &= (d+R)h \\ h &= \frac{dR}{d+R} \end{aligned}$$

and thus

$$SA = 2\pi R\left(\frac{dR}{d+R}\right).$$

53. Suppose that the observer in Exercise 52 moves off to infinity—that is, $d \to \infty$. What do you expect the limiting value of the observed area to be? Check your guess by calculating the limit using the formula for the area in the previous exercise.

SOLUTION We would assume the observed surface area would approach $2\pi R^2$ which is the surface area of a hemisphere of radius R. To verify this, observe:

$$\lim_{d\to\infty} SA = \lim_{d\to\infty} \frac{2\pi R^2 d}{R+d} = \lim_{d\to\infty} \frac{2\pi R^2}{1} = 2\pi R^2.$$

54. Let M be the total mass of a metal rod in the shape of the curve $y = f(x)$ over $[a,b]$ whose mass density $\rho(x)$ varies as a function of x. Use Riemann sums to justify the formula

$$M = \int_a^b \rho(x)\sqrt{1 + f'(x)^2}\,dx$$

SOLUTION Divide the interval $[a,b]$ into n subintervals, which we shall denote by $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \ldots, n$. On each subinterval, we will assume that the mass density of the rod is constant; hence, the mass of the corresponding segment of the rod will be approximately equal to the product of the mass density of the segment and the length of the segment. Specifically, let c_j be any point in the jth subinterval and approximate the mass of the segment by

$$\rho(c_j)\sqrt{1 + f'(c_j)^2}\,\Delta x,$$

where $\sqrt{1 + f'(c_j)^2}\,\Delta x$ is the approximate length of the segment. Thus,

$$M \approx \sum_{j=1}^{n} \rho(c_j)\sqrt{1 + f'(c_j)^2}\,\Delta x.$$

As $n \to \infty$, this Riemann sum approaches a definite integral, and we have

$$M = \int_a^b \rho(x)\sqrt{1 + f'(x)^2}\, dx.$$

55. Let $f(x)$ be an increasing function on $[a, b]$ and let $g(x)$ be its inverse. Argue on the basis of arc length that the following equality holds:

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_{f(a)}^{f(b)} \sqrt{1 + g'(y)^2}\, dy \qquad \boxed{5}$$

Then use the substitution $u = f(x)$ to prove Eq. (5).

SOLUTION Since the graphs of $f(x)$ and $g(x)$ are symmetric with respect to the line $y = x$, the arc length of the curves will be equal on the respective domains. Since the domain of g is the range of f, on $f(a)$ to $f(b)$, $g(x)$ will have the same arc length as $f(x)$ on a to b. If $g(x) = f^{-1}(x)$ and $u = f(x)$, then $x = g(u)$ and $du = f'(x)\, dx$. But

$$g'(u) = \frac{1}{f'(g(u))} = \frac{1}{f'(x)} \Rightarrow f'(x) = \frac{1}{g'(u)}$$

Now substituting $u = f(x)$,

$$s = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{1}{g'(u)}\right)^2}\, g'(u)\, du = \int_{f(a)}^{f(b)} \sqrt{g'(u)^2 + 1}\, du$$

8.2 Fluid Pressure and Force

Preliminary Questions

1. How is pressure defined?

SOLUTION Pressure is defined as force per unit area.

2. Fluid pressure is proportional to depth. What is the factor of proportionality?

SOLUTION The factor of proportionality is the weight density of the fluid, $w = \rho g$, where ρ is the mass density of the fluid.

3. When fluid force acts on the side of a submerged object, in which direction does it act?

SOLUTION Fluid force acts in the direction perpendicular to the side of the submerged object.

4. Why is fluid pressure on a surface calculated using thin horizontal strips rather than thin vertical strips?

SOLUTION Pressure depends only on depth and does not change horizontally at a given depth.

5. If a thin plate is submerged horizontally, then the fluid force on one side of the plate is equal to pressure times area. Is this true if the plate is submerged vertically?

SOLUTION When a plate is submerged vertically, the pressure is not constant along the plate, so the fluid force is not equal to the pressure times the area.

Exercises

1. A box of height 6 m and square base of side 3 m is submerged in a pool of water. The top of the box is 2 m below the surface of the water.

(a) Calculate the fluid force on the top and bottom of the box.

(b) Write a Riemann sum that approximates the fluid force on a side of the box by dividing the side into N horizontal strips of thickness $\Delta y = 6/N$.

(c) To which integral does the Riemann sum converge?

(d) Compute the fluid force on a side of the box.

SOLUTION

(a) At a depth of 2 m, the pressure on the top of the box is $\rho g h = 10^3 \cdot 9.8 \cdot 2 = 19{,}600$ Pa. The top has area 9 m^2, and the pressure is constant, so the force on the top of the box is $19{,}600 \cdot 9 = 176{,}400 N$. At a depth of 8 m, the pressure on the bottom of the box is $\rho g h = 10^3 \cdot 9.8 \cdot 8 = 78{,}400$ Pa, so the force on the bottom of the box is $78{,}400 \cdot 9 = 705{,}600 N$.

(b) Let y_j denote the depth of the j^{th} strip, for $j = 1, 2, 3, \ldots, N$; the pressure at this depth is $10^3 \cdot 9.8 \cdot y_j = 9800 y_j$ Pa. The strip has thickness Δy m and length 3 m, so has area $3\Delta y$ m^2. Thus the force on the strip is $29{,}400 y_j \Delta y$ N. Sum over all the strips to conclude that the force on one side of the box is approximately

$$F \approx \sum_{j=1}^{N} 29{,}400 y_j \Delta y.$$

(c) As $N \to \infty$, the Riemann sum in part (b) converges to the definite integral $29{,}400 \int_2^8 y\,dy$.

(d) Using the result from part (c), the fluid force on one side of the box is

$$29{,}400 \int_2^8 y\,dy = 14{,}700 y^2 \Big|_2^8 = 882{,}000\ N$$

2. A plate in the shape of an isosceles triangle with base 1 m and height 2 m is submerged vertically in a tank of water so that its vertex touches the surface of the water (Figure 7).

(a) Show that the width of the triangle at depth y is $f(y) = \frac{1}{2}y$.

(b) Consider a thin strip of thickness Δy at depth y. Explain why the fluid force on a side of this strip is approximately equal to $\rho g \frac{1}{2} y^2 \Delta y$.

(c) Write an approximation for the total fluid force F on a side of the plate as a Riemann sum and indicate the integral to which it converges.

(d) Calculate F.

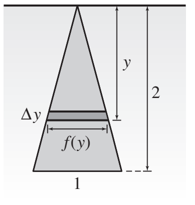


FIGURE 7

SOLUTION

(a) By similar triangles, $\frac{y}{2} = \frac{f(y)}{1}$ so $f(y) = \frac{y}{2}$.

(b) The pressure at a depth of y feet is $\rho g y$ Pa, and the area of the strip is approximately $f(y)\,\Delta y = \frac{1}{2} y \Delta y$ m^2. Therefore, the fluid force on this strip is approximately

$$\rho g y \left(\frac{1}{2} y \Delta y\right) = \frac{1}{2} \rho g y^2 \Delta y.$$

(c) $F \approx \sum_{j=1}^{N} \rho g \frac{y_j^2}{2} \Delta y$. As $N \to \infty$, the Riemann sum converges to the definite integral

$$\frac{\rho g}{2} \int_0^2 y^2\,dy.$$

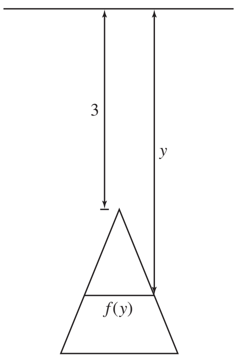
(d) Using the result of part (c),

$$F = \frac{\rho g}{2} \int_0^2 y^2\,dy = \frac{\rho g}{2}\left(\frac{y^3}{3}\right)\Bigg|_0^2 = \frac{9800}{2} \cdot \frac{8}{3} = \frac{39200}{3}\ \text{N}.$$

3. Repeat Exercise 2, but assume that the top of the triangle is located 3 m below the surface of the water.

SOLUTION

(a) Examine the figure below. By similar triangles, $\frac{y-3}{2} = \frac{f(y)}{1}$ so $f(y) = \frac{y-3}{2}$.



(b) The pressure at a depth of y feet is $\rho g y$ lb/ Pa, and the area of the strip is approximately $f(y)\,\Delta y = \frac{1}{2}(y-3)\Delta y$ m^2. Therefore, the fluid force on this strip is approximately

$$\rho g y\left(\frac{1}{2}(y-3)\Delta y\right) = \frac{1}{2}\rho g y(y-3)\Delta y \text{ N}.$$

(c) $F \approx \sum_{j=1}^{N} \rho g \frac{y_j^2 - 3y_j}{2}\Delta y$. As $N \to \infty$, the Riemann sum converges to the definite integral

$$\frac{\rho g}{2}\int_3^5 (y^2 - 3y)\,dy.$$

(d) Using the result of part (c),

$$F = \frac{\rho g}{2}\int_3^5 (y^2-3y)\,dy = \frac{\rho g}{2}\left(\frac{y^3}{3} - \frac{3y^2}{2}\right)\Bigg|_3^5 = \frac{9800}{2}\left[\left(\frac{125}{3} - \frac{75}{2}\right) - \left(9 - \frac{27}{2}\right)\right] = \frac{127{,}400}{3} \text{ N}.$$

4. The plate R in Figure 8, bounded by the parabola $y = x^2$ and $y = 1$, is submerged vertically in water (distance in meters).

(a) Show that the width of R at height y is $f(y) = 2\sqrt{y}$ and the fluid force on a side of a horizontal strip of thickness Δy at height y is approximately $(\rho g)2y^{1/2}(1-y)\Delta y$.

(b) Write a Riemann sum that approximates the fluid force F on a side of R and use it to explain why

$$F = \rho g \int_0^1 2y^{1/2}(1-y)\,dy$$

(c) Calculate F.

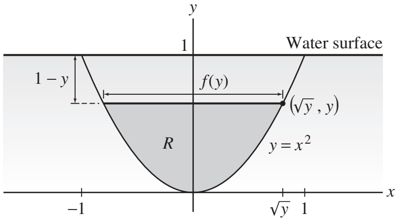


FIGURE 8

SOLUTION

(a) At height y, the thin plate R extends from the point $(-\sqrt{y}, y)$ on the left to the point $(\sqrt{y}, y)$ on the right; thus, the width of the plate is $f(y) = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$. Moreover, the area of a horizontal strip of thickness Δy at height y is $f(y)\,\Delta y = 2\sqrt{y}\,\Delta y$. Because the water surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Consequently, the fluid force on the strip is approximately

$$\rho g(1-y) \times 2\sqrt{y}\Delta y = 2\rho g y^{1/2}(1-y)\Delta y.$$

(b) If the plate is divided into N strips with y_j being the representative height of the jth strip (for $j = 1, 2, 3, \ldots, N$), then the total fluid force exerted on the plate is

$$F \approx 2\rho g \sum_{j=1}^{N} (1 - y_j)\sqrt{y_j}\Delta y.$$

As $N \to \infty$, the Riemann sum converges to the definite integral

$$2\rho g \int_0^1 (1-y)\sqrt{y}\,dy.$$

(c) Using the result from part (b),

$$F = 2\rho g \int_0^1 (1-y)\sqrt{y}\,dy = 2\rho g\left(\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^1 = \frac{8}{15}\rho g.$$

Now, $\rho g = 9800\ \text{N/m}^3$ so that $F = \frac{15680}{3}$ N.

5. Let F be the fluid force on a side of a semicircular plate of radius r meters, submerged vertically in water so that its diameter is level with the water's surface (Figure 9).

(a) Show that the width of the plate at depth y is $2\sqrt{r^2 - y^2}$.

(b) Calculate F as a function of r using Eq. (2).

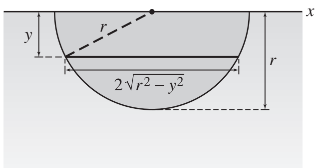


FIGURE 9

SOLUTION

(a) Place the origin at the center of the semicircle and point the positive y-axis downward. The equation for the edge of the semicircular plate is then $x^2 + y^2 = r^2$. At a depth of y, the plate extends from the point $(-\sqrt{r^2 - y^2}, y)$ on the left to the point $(\sqrt{r^2 - y^2}, y)$ on the right. The width of the plate at depth y is then

$$\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.$$

(b) With $w = 9800\ \text{N/m}^3$,

$$F = 2w \int_0^r y\sqrt{r^2 - y^2}\,dy = -\frac{19{,}600}{3}(r^2 - y^2)^{3/2}\Big|_0^r = \frac{19{,}600 r^3}{3}\ \text{N}.$$

6. Calculate the force on one side of a circular plate with radius 2 m, submerged vertically in a tank of water so that the top of the circle is tangent to the water surface.

SOLUTION Place the origin at the point where the top of the circle is tangent to the water surface and orient the positive y-axis pointing downward. The equation of the circle is then $x^2 + (y-2)^2 = 4$, and the width at any depth y is $2\sqrt{4 - (y-2)^2}$. Thus,

$$F = 2\rho g \int_0^4 y\sqrt{4 - (y-2)^2}\,dy,$$

Using the substitution $y - 2 = 2\sin\theta$, $dy = 2\cos\theta\,d\theta$, the limits of integration become $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so we find

$$\begin{aligned}
F &= 2\rho g \int_0^4 y\sqrt{4 - (y-2)^2}\,dy \\
&= 2\rho g \int_{-\pi/2}^{\pi/2} (2 + 2\sin\theta)(2\cos\theta)(2\cos\theta\,d\theta) = 16\rho g \int_{-\pi/2}^{\pi/2} \cos^2\theta + \sin\theta\cos^2\theta\,d\theta \\
&= 16\rho g\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_{-\pi/2}^{\pi/2} \\
&= 16\rho g\left(\frac{\pi}{4} + 0 - 0 - (-\frac{\pi}{4} + 0 - 0)\right) = 8\rho g\pi = 78{,}400\pi\ \text{N}.
\end{aligned}$$

7. A semicircular plate of radius r meters, oriented as in Figure 9, is submerged in water so that its diameter is located at a depth of m meters. Calculate the fluid force on one side of the plate in terms of m and r.

SOLUTION Place the origin at the center of the semicircular plate with the positive y-axis pointing downward. The water surface is then at $y = -m$. Moreover, at location y, the width of the plate is $2\sqrt{r^2 - y^2}$ and the depth is $y + m$. Thus,

$$F = 2\rho g \int_0^r (y+m)\sqrt{r^2 - y^2}\,dy.$$

Now,

$$\int_0^r y\sqrt{r^2 - y^2}\,dy = -\frac{1}{3}(r^2 - y^2)^{3/2}\Big|_0^r = \frac{1}{3}r^3.$$

Geometrically,

$$\int_0^r \sqrt{r^2 - y^2}\,dy$$

represents the area of one quarter of a circle of radius r, and thus has the value $\frac{\pi r^2}{4}$. Bringing these results together, we find that

$$F = 2\rho g\left(\frac{1}{3}r^3 + \frac{\pi}{4}r^2\right) = \frac{19{,}600}{3}r^3 + 4900mr^2 \text{ N}.$$

8. A plate extending from depth $y = 2$ m to $y = 5$ m is submerged in a fluid of density $\rho = 850$ kg/m^3. The horizontal width of the plate at depth y is $f(y) = 2(1 + y^2)^{-1}$. Calculate the fluid force on one side of the plate.

SOLUTION The fluid force on one side of the plate is given by

$$F = \rho g \int_2^5 y f(y)\,dy = \rho g \int_2^5 2y(1 + y^2)^{-1}\,dy = \rho g \ln(1 + y^2)\Big|_2^5 = \rho g(\ln 26 - \ln 5)$$
$$= 8330 \ln \frac{26}{5} \approx 13733.32 \text{ N}.$$

9. Figure 10 shows the wall of a dam on a water reservoir. Use the Trapezoidal Rule and the width and depth measurements in the figure to estimate the fluid force on the wall.

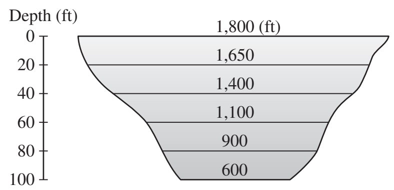


FIGURE 10

SOLUTION Let $f(y)$ denote the width of the dam wall at depth y feet. Then the force on the dam wall is

$$F = w\int_0^{100} y f(y)\,dy.$$

Using the Trapezoidal Rule and the width and depth measurements in the figure,

$$F \approx w\frac{20}{2}[0 \cdot f(0) + 2 \cdot 20 \cdot f(20) + 2 \cdot 40 \cdot f(40) + 2 \cdot 60 \cdot f(60) + 2 \cdot 80 \cdot f(80) + 100 \cdot f(100)]$$
$$= 10w(0 + 66{,}000 + 112{,}000 + 132{,}000 + 144{,}000 + 60{,}000) = 321{,}250{,}000 \text{ lb}.$$

10. Calculate the fluid force on a side of the plate in Figure 11(A), submerged in water.

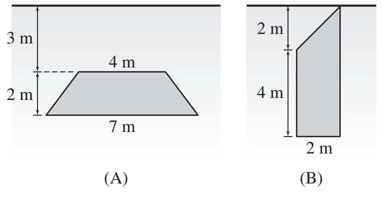


FIGURE 11

SOLUTION The width of the plate varies linearly from 4 meters at a depth of 3 meters to 7 meters at a depth of 5 meters. Thus, at depth y, the width of the plate is

$$4 + \frac{3}{2}(y - 3) = \frac{3}{2}y - \frac{1}{2}.$$

Finally, the force on a side of the plate is

$$F = w\int_3^5 y\left(\frac{3}{2}y - \frac{1}{2}\right)dy = w\left(\frac{1}{2}y^3 - \frac{1}{4}y^2\right)\Big|_3^5 = 45w = 441{,}000 \text{ N}.$$

11. Calculate the fluid force on a side of the plate in Figure 11(B), submerged in a fluid of mass density $\rho = 800$ kg/m^3.

SOLUTION Because the fluid has a mass density of $\rho = 800$ kg/m^3,

$$w = (800)(9.8) = 7840 \text{ N/m}^3.$$

For depths up to 2 meters, the width of the plate at depth y is y; for depths from 2 meters to 6 meters, the width of the plate is a constant 2 meters. Thus,

$$F = w\int_0^2 y(y)\,dy + w\int_2^6 2y\,dy = w\frac{y^3}{3}\bigg|_0^2 + wy^2\Big|_2^6 = \frac{8w}{3} + 32w = \frac{104w}{3} = \frac{815{,}360}{3} \text{ N}.$$

12. Find the fluid force on the side of the plate in Figure 12, submerged in a fluid of density $\rho = 1200$ kg/m^3. The top of the place is level with the fluid surface. The edges of the plate are the curves $y = x^{1/3}$ and $y = -x^{1/3}$.

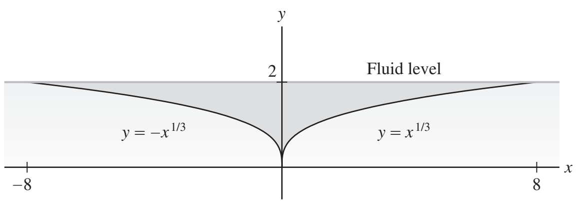


FIGURE 12

SOLUTION At height y, the plate extends from the point $(-y^3, y)$ on the left to the point (y^3, y) on the right; thus, the width of the plate is $f(y) = y^3 - (-y^3) = 2y^3$. Because the water surface is at height $y = 2$, the horizontal strip at height y is at a depth of $2 - y$. Consequently,

$$F = \rho g\int_0^2 (2-y)(2y^3)\,dy = 2\rho g\left(\frac{1}{2}y^4 - \frac{1}{5}y^5\right)\bigg|_0^2 = \frac{16\rho g}{5} = \frac{16\cdot 1200\cdot 9.8}{5} = 37{,}632 \text{ N}.$$

13. Let R be the plate in the shape of the region under $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$ in Figure 13(A). Find the fluid force on a side of R if it is rotated counterclockwise by 90° and submerged in a fluid of density 1100 kg/m^3 with its top edge level with the surface of the fluid as in (B).

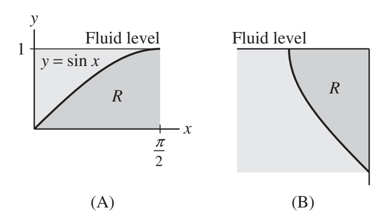


FIGURE 13

SOLUTION Place the origin at the bottom corner of the plate with the positive y-axis pointing upward. The fluid surface is then at height $y = \frac{\pi}{2}$, and the horizontal strip of the plate at height y is at a depth of $\frac{\pi}{2} - y$ and has a width of $\sin y$. Now, using integration by parts we find

$$F = \rho g\int_0^{\pi/2}\left(\frac{\pi}{2} - y\right)\sin y\,dy = \rho g\left[-\left(\frac{\pi}{2} - y\right)\cos y - \sin y\right]\bigg|_0^{\pi/2} = \rho g\left(\frac{\pi}{2} - 1\right)$$
$$= 1100\cdot 9.8\left(\frac{\pi}{2} - 1\right) \approx 6153.184 \text{ N}.$$

14. In the notation of Exercise 13, calculate the fluid force on a side of the plate R if it is oriented as in Figure 13(A). You may need to use Integration by Parts and trigonometric substitution.

SOLUTION Place the origin at the lower left corner of the plate. Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of

$$\frac{\pi}{2} - \sin^{-1} y = \cos^{-1} y.$$

Thus,

$$F = \rho g\int_0^1 (1-y)\cos^{-1} y\,dy.$$

Starting with integration by parts, we find

$$\int_0^1 (1-y)\cos^{-1} y\,dy = \left(y - \frac{1}{2}y^2\right)\cos^{-1} y\Big|_0^1 + \int_0^1 \frac{y - \frac{1}{2}y^2}{\sqrt{1-y^2}}\,dy$$

$$= \frac{1}{2}\cos^{-1} 1 + \int_0^1 \frac{y - \frac{1}{2}y^2}{\sqrt{1-y^2}}\,dy = \int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy - \frac{1}{2}\int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy.$$

Now,

$$\int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy = \; -\sqrt{1-y^2}\Big|_0^1 = 1.$$

For the remaining integral, we use the trigonometric substitution $y = \sin\theta$, $dy = \cos\theta\,d\theta$ and find

$$\frac{1}{2}\int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy = \frac{1}{2}\int_{y=0}^{y=1} \sin^2\theta\,d\theta = \frac{1}{4}(\theta - \sin\theta\cos\theta)\Big|_{y=0}^{y=1}$$

$$= \frac{1}{4}\left(\sin^{-1} y - y\sqrt{1-y^2}\right)\Big|_0^1 = \frac{\pi}{8}.$$

Finally,

$$F = \rho g\left(1 - \frac{\pi}{8}\right) = 1100 \cdot 9.8\left(1 - \frac{\pi}{8}\right) \approx 6546.70 \text{ N}.$$

15. Calculate the fluid force on one side of a plate in the shape of region A shown Figure 14. The water surface is at $y = 1$, and the fluid has density $\rho = 900$ kg/m^3.

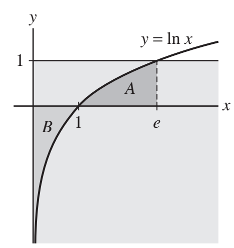


FIGURE 14

SOLUTION Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of $e - e^y$. Thus,

$$F = \rho g \int_0^1 (1-y)(e - e^y)\,dy = e\rho g \int_0^1 (1-y)\,dy - \rho g \int_0^1 (1-y)e^y\,dy.$$

Now,

$$\int_0^1 (1-y)\,dy = \left(y - \frac{1}{2}y^2\right)\Big|_0^1 = \frac{1}{2},$$

and using integration by parts

$$\int_0^1 (1-y)e^y\,dy = \left((1-y)e^y + e^y\right)\Big|_0^1 = e - 2.$$

Combining these results, we find that

$$F = \rho g\left(\frac{1}{2}e - (e-2)\right) = \rho g\left(2 - \frac{1}{2}e\right) = 900 \cdot 9.8\left(2 - \frac{1}{2}e\right) \approx 5652.37 \text{ N}.$$

16. Calculate the fluid force on one side of the “infinite” plate B in Figure 14, assuming the fluid has density $\rho = 900$ kg/m^3.

SOLUTION Because the fluid surface is at height $y = 1$, the horizontal strip at height y is at a depth of $1 - y$. Moreover, this strip has a width of e^y. Thus,

$$F = \rho g \int_{-\infty}^0 (1-y)e^y\,dy.$$

Using integration by parts, we find

$$\int_{-\infty}^{0} (1-y)e^y \, dy = \left[(1-y)e^y + e^y\right]\Big|_{-\infty}^{0} = 2.$$

Thus, $F = 2\rho g = 2 \cdot 900 \cdot 9.8 = 17{,}640$ N.

17. Figure 15(A) shows a ramp inclined at 30° leading into a swimming pool. Calculate the fluid force on the ramp.

SOLUTION A horizontal strip at depth y has length 6 and width

$$\frac{\Delta y}{\sin 30^\circ} = 2\Delta y.$$

Thus,

$$F = 2\rho g \int_0^4 6y \, dy = 96\rho g.$$

If distances are in feet, then $\rho g = w = 62.5$ lb/ft^3 and $F = 6000$ lb; if distances are in meters, then $\rho g = 9800$ N/m^3 and $F = 940{,}800$ N.

18. Calculate the fluid force on one side of the plate (an isosceles triangle) shown in Figure 15(B).

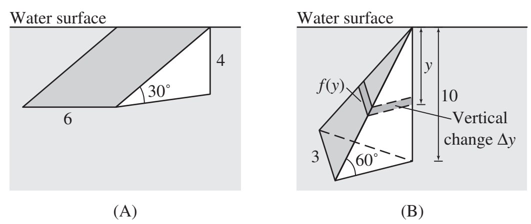


FIGURE 15

SOLUTION A horizontal strip at depth y has length $f(y) = \frac{3}{10}y$ and width

$$\frac{\Delta y}{\sin 60^\circ} = \frac{2}{\sqrt{3}}\Delta y.$$

Thus,

$$F = \frac{\sqrt{3}}{5} w \int_0^{10} y^2 \, dy = \frac{200\sqrt{3}}{3} w.$$

If distances are in feet, then $w = 62.5$ lb/ft^3 and $F \approx 7216.88$ lb; if distances are in meters, then $w = 9800$ N/m^3 and $F \approx 1{,}131{,}606.5$ N.

19. The massive Three Gorges Dam on China's Yangtze River has height 185 m (Figure 16). Calculate the force on the dam, assuming that the dam is a trapezoid of base 2000 m and upper edge 3000 m, inclined at an angle of 55° to the horizontal (Figure 17).

FIGURE 16 Three Gorges Dam on the Yangtze River

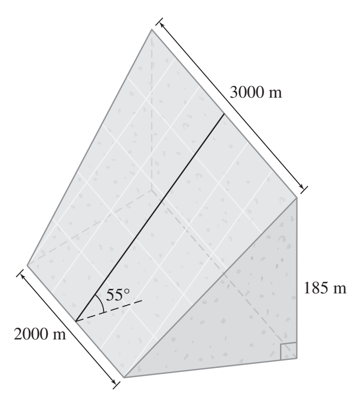


FIGURE 17

SOLUTION Let $y = 0$ be at the bottom of the dam, so that the top of the dam is at $y = 185$. Then the width of the dam at height y is $2000 + \frac{1000y}{185}$. The dam is inclined at an angle of 55° to the horizontal, so the height of a horizontal strip is

$$\frac{\Delta y}{\sin 55^\circ} \approx 1.221\Delta y$$

so that the area of such a strip is

$$1.221\left(2000+\frac{1000y}{185}\right)\Delta y$$

Then

$$F=\rho g\int_0^{185}1.221y\left(2000+\frac{1000y}{185}\right)dy=\rho g\int_0^{185}2442y+6.6y^2\,dy=\rho g(1221y^2+2.2y^3)\Big|_0^{185}$$
$$=55{,}718{,}300\rho g=55{,}718{,}300\cdot 9800=5.460393400\times 10^{11}\text{ N}.$$

20. A square plate of side 3 m is submerged in water at an incline of 30° with the horizontal. Calculate the fluid force on one side of the plate if the top edge of the plate lies at a depth of 6 m.

SOLUTION Because the plate is 3 meters on a side, is submerged at a horizontal angle of 30°, and has its top edge located at a depth of 6 meters, the bottom edge of the plate is located at a depth of $6+3\sin 30°=\frac{15}{2}$ meters. Let y denote the depth at any point of the plate. The width of each horizontal strip of the plate is then

$$\frac{\Delta y}{\sin 30°}=2\Delta y,$$

and

$$F=\rho g\int_6^{15/2}(2)3y\,dy=(\rho g)\frac{243}{4}=595{,}350\text{ N}.$$

21. The trough in Figure 18 is filled with corn syrup, whose weight density is 90 lb/ft^3. Calculate the force on the front side of the trough.

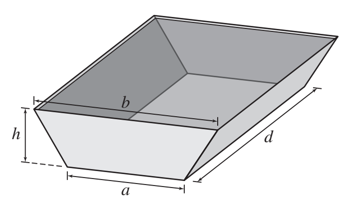


FIGURE 18

SOLUTION Place the origin along the top edge of the trough with the positive y-axis pointing downward. The width of the front side of the trough varies linearly from b when $y=0$ to a when $y=h$; thus, the width of the front side of the trough at depth y feet is given by

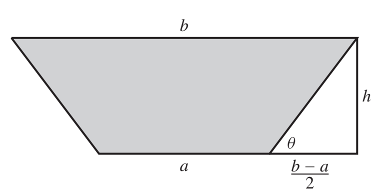
$$b+\frac{a-b}{h}y.$$

Now,

$$F=w\int_0^h y\left(b+\frac{a-b}{h}y\right)dy=w\left(\frac{1}{2}by^2+\frac{a-b}{3h}y^3\right)\Big|_0^h=w\left(\frac{b}{6}+\frac{a}{3}\right)h^2=(15b+30a)h^2\text{ lb}.$$

22. Calculate the fluid pressure on one of the slanted sides of the trough in Figure 18 when it is filled with corn syrup as in Exercise 21.

SOLUTION



The diagram above displays a side view of the trough. From this diagram, we see that

$$\sin\theta=\frac{h}{\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}}.$$

Thus,

$$F=\frac{w}{\sin\theta}\int_0^h d\cdot y\,dy=\frac{90\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}}{h}\frac{dh^2}{2}=45dh\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}.$$

Further Insights and Challenges

23. The end of the trough in Figure 19 is an equilateral triangle of side 3. Assume that the trough is filled with water to height H. Calculate the fluid force on each side of the trough as a function of H and the length l of the trough.

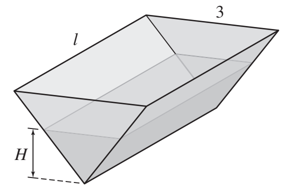


FIGURE 19

SOLUTION Place the origin at the lower vertex of the trough and orient the positive y-axis pointing upward. First, consider the faces at the front and back ends of the trough. A horizontal strip at height y has a length of $\frac{2y}{\sqrt{3}}$ and is at a depth of $H - y$. Thus,

$$F = w\int_0^H (H-y)\frac{2y}{\sqrt{3}}\,dy = w\left(\frac{H}{\sqrt{3}}y^2 - \frac{2}{3\sqrt{3}}y^3\right)\Big|_0^H = \frac{\sqrt{3}}{9}wH^3.$$

For the slanted sides, we note that each side makes an angle of 60° with the horizontal. If we let ℓ denote the length of the trough, then

$$F = \frac{2w\ell}{\sqrt{3}}\int_0^H (H-y)\,dy = \frac{\sqrt{3}}{3}\ell w H^2.$$

24. A rectangular plate of side ℓ is submerged vertically in a fluid of density w, with its top edge at depth h. Show that if the depth is increased by an amount Δh, then the force on a side of the plate increases by $wA\Delta h$, where A is the area of the plate.

SOLUTION Let F_1 be the force on a side of the plate when its top edge is at depth h and F_2 be the force on a side of the plate when its top edge is at depth $h + \Delta h$. Further, let b denote the width of the rectangular plate. Then

$$F_1 = w\int_h^{h+\ell} yb\,dy = bw\left(\frac{y^2}{2}\right)\Big|_h^{h+\ell} = bw\left(\frac{\ell^2 + 2\ell h}{2}\right)$$

$$F_2 = w\int_{h+\Delta h}^{h+\ell+\Delta h} yb\,dy = bw\left(\frac{y^2}{2}\right)\Big|_{h+\Delta h}^{h+\ell+\Delta h} = bw\frac{\ell^2 + 2\ell h + 2\ell\Delta h}{2}$$

and $F_2 - F_1 = bw\ell\Delta h = wA\Delta h$.

25. Prove that the force on the side of a rectangular plate of area A submerged vertically in a fluid is equal to $p_0 A$, where p_0 is the fluid pressure at the center point of the rectangle.

SOLUTION Let ℓ denote the length of the vertical side of the rectangle, x denote the length of the horizontal side of the rectangle, and suppose the top edge of the rectangle is at depth $y = m$. The pressure at the center of the rectangle is then

$$p_0 = w\left(m + \frac{\ell}{2}\right),$$

and the force on the side of the rectangular plate is

$$F = \int_m^{\ell+m} wxy\,dy = \frac{wx}{2}\left[(\ell+m)^2 - m^2\right] = \frac{wx\ell}{2}(\ell + 2m) = Aw\left(\frac{\ell}{2} + m\right) = Ap_0.$$

26. If the density of a fluid varies with depth, then the pressure at depth y is a function $p(y)$ (which need not equal wy as in the case of constant density). Use Riemann sums to argue that the total force F on the flat side of a submerged object submerged vertically is $F = \int_a^b f(y)p(y)\,dy$, where $f(y)$ is the width of the side at depth y.

SOLUTION Suppose the object extends from a depth of $y = a$ to a depth of $y = b$. Divide the object into N horizontal strips, each of width Δy. Let $p(y)$ denote the pressure within the fluid at depth y and $f(y)$ denote the width of the flat side of the submerged object at depth y. The approximate force on the jth strip ($j = 1, 2, 3, \ldots, N$) is

$$p(y_j)f(y_j)\Delta y,$$

where y_j is a depth associated with the jth strip. Summing over all of the strips,

$$F \approx \sum_{j=1}^{N} p(y_j) f(y_j) \Delta y.$$

As $N \to \infty$, this Riemann sum converges to a definite integral, and

$$F = \int_a^b p(y) f(y)\, dy.$$

8.3 Center of Mass

Preliminary Questions

1. What are the x- and y-moments of a lamina whose center of mass is located at the origin?

SOLUTION Because the center of mass is located at the origin, it follows that $M_x = M_y = 0$.

2. A thin plate has mass 3. What is the x-moment of the plate if its center of mass has coordinates $(2, 7)$?

SOLUTION The x-moment of the plate is the product of the mass of the plate and the y-coordinate of the center of mass. Thus, $M_x = 3(7) = 21$.

3. The center of mass of a lamina of total mass 5 has coordinates $(2, 1)$. What are the lamina's x- and y-moments?

SOLUTION The x-moment of the plate is the product of the mass of the plate and the y-coordinate of the center of mass, whereas the y-moment is the product of the mass of the plate and the x-coordinate of the center of mass. Thus, $M_x = 5(1) = 5$, and $M_y = 5(2) = 10$.

4. Explain how the Symmetry Principle is used to conclude that the centroid of a rectangle is the center of the rectangle.

SOLUTION Because a rectangle is symmetric with respect to both the vertical line and the horizontal line through the center of the rectangle, the Symmetry Principle guarantees that the centroid of the rectangle must lie along both of these lines. The only point in common to both lines of symmetry is the center of the rectangle, so the centroid of the rectangle must be the center of the rectangle.

Exercises

1. Four particles are located at points $(1, 1)$, $(1, 2)$, $(4, 0)$, $(3, 1)$.

(a) Find the moments M_x and M_y and the center of mass of the system, assuming that the particles have equal mass m.

(b) Find the center of mass of the system, assuming the particles have masses 3, 2, 5, and 7, respectively.

SOLUTION

(a) Because each particle has mass m,

$$M_x = m(1) + m(2) + m(0) + m(1) = 4m;$$
$$M_y = m(1) + m(1) + m(4) + m(3) = 9m;$$

and the total mass of the system is $4m$. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{9m}{4m}, \frac{4m}{4m}\right) = \left(\frac{9}{4}, 1\right).$$

(b) With the indicated masses of the particles,

$$M_x = 3(1) + 2(2) + 5(0) + 7(1) = 14;$$
$$M_y = 3(1) + 2(1) + 5(4) + 7(3) = 46;$$

and the total mass of the system is 17. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{46}{17}, \frac{14}{17}\right).$$

2. Find the center of mass for the system of particles of masses 4, 2, 5, 1 located at $(1, 2)$, $(-3, 2)$, $(2, -1)$, $(4, 0)$.

SOLUTION With the indicated masses and locations of the particles

$$M_x = 4(2) + 2(2) + 5(-1) + 1(0) = 7;$$

$$M_y = 4(1) + 2(-3) + 5(2) + 1(4) = 12;$$

and the total mass of the system is 12. Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(1, \frac{7}{12}\right).$$

3. Point masses of equal size are placed at the vertices of the triangle with coordinates $(a, 0)$, $(b, 0)$, and $(0, c)$. Show that the center of mass of the system of masses has coordinates $\left(\frac{1}{3}(a+b), \frac{1}{3}c\right)$.

SOLUTION Let each particle have mass m. The total mass of the system is then $3m$. and the moments are

$$M_x = 0(m) + 0(m) + c(m) = cm; \text{ and}$$

$$M_y = a(m) + b(m) + 0(m) = (a+b)m.$$

Thus, the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(a+b)m}{3m}, \frac{cm}{3m}\right) = \left(\frac{a+b}{3}, \frac{c}{3}\right).$$

4. Point masses of mass m_1, m_2, and m_3 are placed at the points $(-1, 0)$, $(3, 0)$, and $(0, 4)$.

(a) Suppose that $m_1 = 6$. Find m_2 such that the center of mass lies on the y-axis.

(b) Suppose that $m_1 = 6$ and $m_2 = 4$. Find the value of m_3 such that $y_{CM} = 2$.

SOLUTION With the given masses and locations, we find

$$M_x = m_1(0) + m_2(0) + m_3(4) = 4m_3;$$

$$M_y = m_1(-1) + m_2(3) + m_3(0) = 3m_2 - m_1;$$

and the total mass of the system is $m_1 + m_2 + m_3$. Thus, the coordinates of the center of mass are

$$\left(\frac{3m_2 - m_1}{m_1 + m_2 + m_3}, \frac{4m_3}{m_1 + m_2 + m_3}\right).$$

(a) For the center of mass to lie on the y-axis, we must have $3m_2 - m_1 = 0$, or $m_2 = \frac{1}{3}m_1$. Given $m_1 = 6$, it follows that $m_2 = 2$.

(b) To have $y_{CM} = 2$ requires

$$\frac{4m_3}{m_1 + m_2 + m_3} = 2 \quad \text{or} \quad m_3 = m_1 + m_2.$$

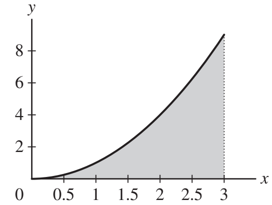
Given $m_1 = 6$ and $m_2 = 4$, it follows that $m_3 = 10$.

5. Sketch the lamina S of constant density $\rho = 3$ g/cm^2 occupying the region beneath the graph of $y = x^2$ for $0 \le x \le 3$.

(a) Use Eqs. (1) and (2) to compute M_x and M_y.

(b) Find the area and the center of mass of S.

SOLUTION A sketch of the lamina is shown below



(a) Using Eq. (2),

$$M_x = 3\int_0^9 y(3 - \sqrt{y})\, dy = \left(\frac{9y^2}{2} - \frac{6}{5}y^{5/2}\right)\Bigg|_0^9 = \frac{729}{10}.$$

Using Eq. (1),

$$M_y = 3\int_0^3 x(x^2)\, dx = \frac{3x^4}{4}\Bigg|_0^3 = \frac{243}{4}.$$

(b) The area of the lamina is

$$A = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9\text{ cm}^2.$$

With a constant density of $\rho = 3\text{ g/cm}^2$, the mass of the lamina is $M = 27$ grams, and the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{243/4}{27}, \frac{729/10}{27}\right) = \left(\frac{9}{4}, \frac{27}{10}\right).$$

6. Use Eqs. (1) and (3) to find the moments and center of mass of the lamina S of constant density $\rho = 2\text{ g/cm}^2$ occupying the region between $y = x^2$ and $y = 9x$ over $[0, 3]$. Sketch S, indicating the location of the center of mass.

SOLUTION With $\rho = 2\text{ g/cm}^2$,

$$M_x = \frac{1}{2}(2)\int_0^3 \left((9x)^2 - (x^2)^2\right)dx = \frac{3402}{5},$$

and

$$M_y = 2\int_0^3 x(9x - x^2)\,dx = \frac{243}{2}.$$

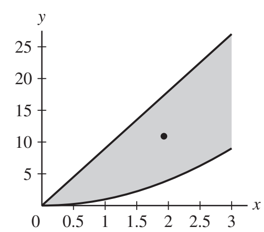
The mass of the lamina is

$$M = 2\int_0^3 (9x - x^2)\,dx = 63\text{ g},$$

so the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{243}{126}, \frac{3402}{315}\right).$$

A sketch of the lamina, with the location of the center of mass indicated, is shown below.



7. Find the moments and center of mass of the lamina of uniform density ρ occupying the region underneath $y = x^3$ for $0 \le x \le 2$.

SOLUTION With uniform density ρ,

$$M_x = \frac{1}{2}\rho\int_0^2 (x^3)^2\,dx = \frac{64\rho}{7} \quad\text{and}\quad M_y = \rho\int_0^2 x(x^3)\,dx = \frac{32\rho}{5}.$$

The mass of the lamina is

$$M = \rho\int_0^2 x^3\,dx = 4\rho,$$

so the coordinates of the center of mass are

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{8}{5}, \frac{16}{7}\right).$$

8. Calculate M_x (assuming $\rho = 1$) for the region underneath the graph of $y = 1 - x^2$ for $0 \le x \le 1$ in two ways, first using Eq. (2) and then using Eq. (3).

SOLUTION By Eq. (2),

$$M_x = \int_0^1 y\sqrt{1-y}\,dy.$$

Using the substitution $u = 1 - y$, $du = -dy$, we find

$$M_x = \int_0^1 (1-u)\sqrt{u}\,du = \left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^1 = \frac{4}{15}.$$

By Eq. (3),

$$M_x = \frac{1}{2}\int_0^1 (1-x^2)^2\,dx = \frac{1}{2}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{4}{15}.$$

9. Let T be the triangular lamina in Figure 17.

(a) Show that the horizontal cut at height y has length $4 - \frac{2}{3}y$ and use Eq. (2) to compute M_x (with $\rho = 1$).

(b) Use the Symmetry Principle to show that $M_y = 0$ and find the center of mass.

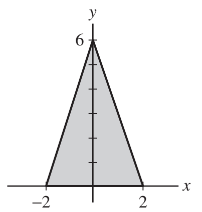


FIGURE 17 Isosceles triangle.

SOLUTION

(a) The equation of the line from (2, 0) to (0, 6) is $y = -3x + 6$, so

$$x = 2 - \frac{1}{3}y.$$

The length of the horizontal cut at height y is then

$$2\left(2 - \frac{1}{3}y\right) = 4 - \frac{2}{3}y,$$

and

$$M_x = \int_0^6 y\left(4 - \frac{2}{3}y\right)dy = 24.$$

(b) Because the triangular lamina is symmetric with respect to the y-axis, $x_{cm} = 0$, which implies that $M_y = 0$. The total mass of the lamina is

$$M = 2\int_0^2 (-3x+6)\,dx = 12,$$

so $y_{cm} = 24/12$. Finally, the coordinates of the center of mass are (0, 2).

In Exercises 10–17, find the centroid of the region lying underneath the graph of the function over the given interval.

10. $f(x) = 6 - 2x$, $[0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (6-2x)^2\,dx = 18 \quad \text{and} \quad M_y = \int_0^3 x(6-2x)\,dx = 9.$$

The area of the region is

$$A = \int_0^3 (6-2x)\,dx = 9,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = (1, 2).$$

11. $f(x) = \sqrt{x}$, $[1, 4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^4 x\,dx = \frac{15}{4} \quad \text{and} \quad M_y = \int_1^4 x\sqrt{x}\,dx = \frac{62}{5}.$$

The area of the region is

$$A = \int_1^4 \sqrt{x}\,dx = \frac{14}{3},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{93}{35}, \frac{45}{56}\right).$$

12. $f(x) = x^3$, $[0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 x^6\,dx = \frac{1}{14} \quad \text{and} \quad M_y = \int_0^1 x^4\,dx = \frac{1}{5}.$$

The area of the region is

$$A = \int_0^1 x^3\,dx = \frac{1}{4},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{4}{5}, \frac{2}{7}\right).$$

13. $f(x) = 9 - x^2$, $[0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (9 - x^2)^2\,dx = \frac{324}{5} \quad \text{and} \quad M_y = \int_0^3 x(9 - x^2)\,dx = \frac{81}{4}.$$

The area of the region is

$$A = \int_0^3 (9 - x^2)\,dx = 18,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{9}{8}, \frac{18}{5}\right).$$

14. $f(x) = (1 + x^2)^{-1/2}$, $[0, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 \frac{1}{1 + x^2}\,dx = \left.\frac{\tan^{-1} x}{2}\right|_0^3 = \frac{1}{2}\tan^{-1} 3 \quad \text{and} \quad M_y = \int_0^3 \frac{x}{\sqrt{1 + x^2}}\,dx = \sqrt{10} - 1.$$

The area of the region is

$$A = \int_0^3 \frac{1}{\sqrt{1 + x^2}}\,dx = \ln|x + \sqrt{1 + x^2}|\Big|_0^3 = \ln(3 + \sqrt{10}),$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{\sqrt{10} - 1}{\ln(3 + \sqrt{10})}, \frac{\tan^{-1} 3}{2\ln(3 + \sqrt{10})}\right).$$

15. $f(x) = e^{-x}$, $[0, 4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^4 e^{-2x}\,dx = \frac{1}{4}\left(1 - e^{-8}\right) \quad \text{and} \quad M_y = \int_0^4 xe^{-x}\,dx = -e^{-x}(x + 1)\Big|_0^4 = 1 - 5e^{-4}.$$

The area of the region is

$$A = \int_0^4 e^{-x}\,dx = 1 - e^{-4},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{1-5e^{-4}}{1-e^{-4}}, \frac{1-e^{-8}}{4(1-e^{-4})}\right).$$

16. $f(x) = \ln x, \quad [1, 2]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^2 (\ln x)^2\,dx = \frac{1}{2}(x(\ln x)^2 - 2x\ln x + 2x)\Big|_1^2 = (\ln 2)^2 - 2\ln 2 + 1; \text{ and}$$

$$M_y = \int_1^2 x\ln x\,dx = \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^2 = 2\ln 2 - \frac{3}{4}.$$

The area of the region is

$$A = \int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 1,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1}, \frac{(\ln 2)^2 - 2\ln 2 + 1}{2\ln 2 - 1}\right).$$

17. $f(x) = \sin x, \quad [0, \pi]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{1}{4}(x - \sin x\cos x)\Big|_0^\pi = \frac{\pi}{4}; \text{ and}$$

$$M_y = \int_0^\pi x\sin x\,dx = (-x\cos x + \sin x)\Big|_0^\pi = \pi.$$

The area of the region is

$$A = \int_0^\pi \sin x\,dx = 2,$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

18. Calculate the moments and center of mass of the lamina occupying the region between the curves $y = x$ and $y = x^2$ for $0 \le x \le 1$.

SOLUTION The moments of the lamina are

$$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{15} \quad \text{and} \quad M_y = \int_0^1 x(x - x^2)\,dx = \frac{1}{12}.$$

The area of the lamina is

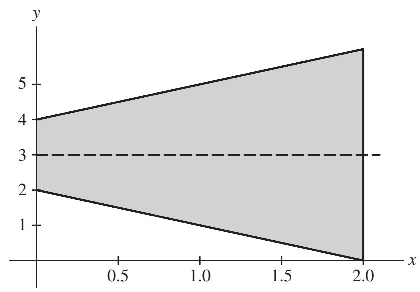
$$A = \int_0^1 (x - x^2)\,dx = \frac{1}{6},$$

so the coordinates of the centroid are

$$\left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{1}{2}, \frac{2}{5}\right).$$

19. Sketch the region between $y = x + 4$ and $y = 2 - x$ for $0 \le x \le 2$. Using symmetry, explain why the centroid of the region lies on the line $y = 3$. Verify this by computing the moments and the centroid.

SOLUTION A sketch of the region is shown below.



The region is clearly symmetric about the line $y = 3$, so we expect the centroid of the region to lie along this line. We find

$$M_x = \frac{1}{2}\int_0^2 \left((x+4)^2 - (2-x)^2\right) dx = 24;$$

$$M_y = \int_0^2 x\,((x+4) - (2-x))\,dx = \frac{28}{3}; \text{ and}$$

$$A = \int_0^2 ((x+4) - (2-x))\,dx = 8.$$

Thus, the coordinates of the centroid are $\left(\frac{7}{6}, 3\right)$.

In Exercises 20–25, find the centroid of the region lying between the graphs of the functions over the given interval.

20. $y = x, \quad y = \sqrt{x}, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (x - x^2)\,dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x)\,dx = \frac{1}{15}.$$

The area of the region is

$$A = \int_0^1 (\sqrt{x} - x)\,dx = \frac{1}{6},$$

so the coordinates of the centroid are

$$\left(\frac{6}{15}, \frac{1}{2}\right).$$

21. $y = x^2, \quad y = \sqrt{x}, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (x - x^4)\,dx = \frac{3}{20} \quad \text{and} \quad M_y = \int_0^1 x(\sqrt{x} - x^2)\,dx = \frac{3}{20}.$$

The area of the region is

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \frac{1}{3},$$

so the coordinates of the centroid are

$$\left(\frac{9}{20}, \frac{9}{20}\right).$$

Note: This makes sense, since the functions are inverses of each other. This makes the region symmetric with respect to the line $y = x$. Thus, by the symmetry principle, the center of mass must lie on that line.

22. $y = x^{-1}, \quad y = 2 - x, \quad [1, 2]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^2 \left[\left(\frac{1}{x}\right)^2 - (2-x)^2\right] dx = \frac{1}{12} \quad \text{and} \quad M_y = \int_1^2 x\left(\frac{1}{x} - (2-x)\right) dx = \frac{1}{3}.$$

The area of the region is

$$A = \int_1^2 \left(\frac{1}{x} - (2-x)\right) dx = \ln 2 - \frac{1}{2},$$

so the coordinates of the centroid are

$$\left(\frac{2}{6\ln 2 - 3}, \frac{1}{12\ln 2 - 6}\right).$$

23. $y = e^x, \quad y = 1, \quad [0, 1]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^1 (e^{2x} - 1)\, dx = \frac{e^2 - 3}{4} \quad \text{and} \quad M_y = \int_0^1 x(e^x - 1)\, dx = \left(xe^x - e^x - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{2}.$$

The area of the region is

$$A = \int_0^1 (e^x - 1)\, dx = e - 2,$$

so the coordinates of the centroid are

$$\left(\frac{1}{2(e-2)}, \frac{e^2 - 3}{4(e-2)}\right).$$

24. $y = \ln x, \quad y = x - 1, \quad [1, 3]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_1^3 \left[(x-1)^2 - (\ln x)^2\right] dx = \left(\frac{1}{3}x^3 - x^2 - x - x(\ln x)^2 + 2x\ln x\right)\Big|_1^3 = 3\ln 3 - \frac{3}{2}(\ln 3)^2 - \frac{2}{3}; \text{ and}$$

$$M_y = \int_1^3 x((x-1) - \ln x)\, dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^3 = \frac{20}{3} - \frac{9}{2}\ln 3.$$

The area of the region is

$$A = \int_1^3 (x - 1 - \ln x)\, dx = \left(\frac{1}{2}x^2 - x\ln x\right)\Big|_1^3 = 4 - 3\ln 3,$$

so the coordinates of the centroid are

$$\left(\frac{40 - 27\ln 3}{24 - 18\ln 3}, \frac{18\ln 3 - 9(\ln 3)^2 - 4}{24 - 18\ln 3}\right).$$

25. $y = \sin x, \quad y = \cos x, \quad [0, \pi/4]$

SOLUTION The moments of the region are

$$M_x = \frac{1}{2}\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x\, dx = \frac{1}{4}; \text{ and}$$

$$M_y = \int_0^{\pi/4} x(\cos x - \sin x)\, dx = [(x-1)\sin x + (x+1)\cos x]\Big|_0^{\pi/4} = \frac{\pi\sqrt{2}}{4} - 1.$$

The area of the region is

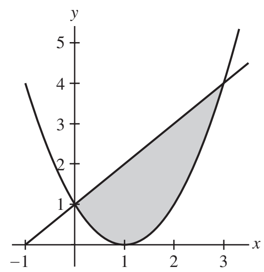
$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \sqrt{2} - 1,$$

so the coordinates of the centroid are

$$\left(\frac{\pi\sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)}\right).$$

26. Sketch the region enclosed by $y = x + 1$, and $y = (x - 1)^2$, and find its centroid.

SOLUTION A sketch of the region is shown below.



The moments of the region are

$$M_x = \frac{1}{2}\int_0^3 (x+1)^2 - (x-1)^4\,dx = \frac{1}{2}\left(\frac{1}{3}(x+1)^3 - \frac{1}{5}(x-1)^5\right)\Big|_0^3 = \frac{1}{2}\left(\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right) = \frac{36}{5}$$

$$M_y = \int_0^3 x((x+1) - (x-1)^2)\,dx = \int_0^3 3x^2 - x^3\,dx = \left(x^3 - \frac{1}{4}x^4\right)\Big|_0^3 = \frac{27}{4}$$

The area of the region is

$$A = \int_0^3 (x+1) - (x-1)^2\,dx = \int_0^3 -x^2 + 3x\,dx = \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 = \frac{9}{2}$$

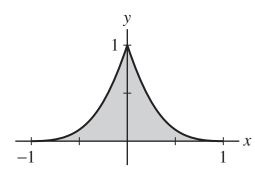
so that the coordinates of the centroid are

$$\left(\frac{27}{4}\cdot\frac{2}{9}, \frac{36}{5}\cdot\frac{2}{9}\right) = \left(\frac{3}{2}, \frac{8}{5}\right)$$

s

27. Sketch the region enclosed by $y = 0$, $y = (x + 1)^3$, and $y = (1 - x)^3$, and find its centroid.

SOLUTION A sketch of the region is shown below.



The moments of the region are

$$M_x = \frac{1}{2}\left(\int_{-1}^0 (x+1)^6\,dx + \int_0^1 (1-x)^6\,dx\right) = \frac{1}{7};\ \text{and}$$

$M_y = 0$ by the Symmetry Principle.

The area of the region is

$$A = \int_{-1}^0 (x+1)^3\,dx + \int_0^1 (1-x)^3\,dx = \frac{1}{2},$$

so the coordinates of the centroid are $\left(0, \frac{2}{7}\right)$.

In Exercises 28–32, find the centroid of the region.

28. Top half of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

SOLUTION The equation of the top half of the ellipse is $y = \sqrt{16 - 4x^2}$. Thus,

$$M_x = \frac{1}{2}\int_{-2}^2 \left(\sqrt{16 - 4x^2}\right)^2\,dx = \frac{64}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of an ellipse with major axis 4 and minor axis 2; i.e., $\frac{1}{2}\pi(4)(2) = 4\pi$. Finally, the coordinates of the centroid are

$$\left(0, \frac{16}{3\pi}\right).$$

29. Top half of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for arbitrary $a, b > 0$

SOLUTION The equation of the top half of the ellipse is

$$y = \sqrt{b^2 - \frac{b^2x^2}{a^2}}$$

Thus,

$$M_x = \frac{1}{2}\int_{-a}^{a}\left(\sqrt{b^2 - \frac{b^2x^2}{a^2}}\right)^2 dx = \frac{2ab^2}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of an ellipse with axes of length a and b; i.e., $\frac{1}{2}\pi ab$. Finally, the coordinates of the centroid are

$$\left(0, \frac{4b}{3\pi}\right).$$

30. Semicircle of radius r with center at the origin

SOLUTION The equation of the top half of the circle is $y = \sqrt{r^2 - x^2}$. Thus,

$$M_x = \frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{2r^3}{3}.$$

By the Symmetry Principle, $M_y = 0$. The area of the region is one-half the area of a circle of radius r; i.e., $\frac{1}{2}\pi r^2$. Finally, the coordinates of the centroid are

$$\left(0, \frac{4r}{3\pi}\right).$$

31. Quarter of the unit circle lying in the first quadrant

SOLUTION By the Symmetry Principle, the center of mass must lie on the line $y = x$ in the first quadrant. Therefore, we need only calculate one of the moments of the region. With $y = \sqrt{1 - x^2}$, we find

$$M_y = \int_0^1 x\sqrt{1 - x^2}\, dx = \frac{1}{3}.$$

The area of the region is one-quarter of the area of a unit circle; i.e., $\frac{1}{4}\pi$. Thus, the coordinates of the centroid are

$$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right).$$

32. Triangular plate with vertices $(-c, 0)$, $(0, c)$, (a, b), where $a, b, c > 0$, and $b < c$

SOLUTION By symmetry, the center of mass must lie on the line connecting $(-c, 0)$ and the midpoint $(a/2, (b + c)/2)$ of the opposite side:

$$\ell_1 : y = \frac{b + c}{a + 2c}(x + c)$$

Also by symmetry, the center of mass must lie on the line connecting $(0, c)$ and the midpoint $((a - c)/2, b/2)$ of the opposite side:

$$\ell_2 : y = \frac{b - 2c}{a - c}x + c$$

These lines intersect at one point (x_{cm}, y_{cm}). Equating the formulas for the two lines and solving for x yields

$$x = \frac{a - c}{3}.$$

Substituting this value for x into the equation for ℓ_2 gives

$$y = \frac{b - 2c}{a - c}\frac{a - c}{3} + c = \frac{b + c}{3}.$$

Hence, the coordinates of the centroid are

$$\left(\frac{a - c}{3}, \frac{b + c}{3}\right).$$

33. Find the centroid for the shaded region of the semicircle of radius r in Figure 18. What is the centroid when $r = 1$ and $h = \frac{1}{2}$? *Hint:* Use geometry rather than integration to show that the *area* of the region is $r^2 \sin^{-1}(\sqrt{1 - h^2/r^2}) - h\sqrt{r^2 - h^2}$).

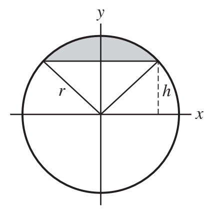


FIGURE 18

SOLUTION From the symmetry of the region, it is obvious that the centroid lies along the y-axis. To determine the y-coordinate of the centroid, we must calculate the moment about the x-axis and the area of the region. Now, the length of the horizontal cut of the semicircle at height y is

$$\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}.$$

Therefore, taking $\rho = 1$, we find

$$M_x = 2\int_h^r y\sqrt{r^2 - y^2}\,dy = \frac{2}{3}(r^2 - h^2)^{3/2}.$$

Observe that the region is comprised of a sector of the circle with the triangle between the two radii removed. The angle of the sector is 2θ, where $\theta = \sin^{-1}\sqrt{1 - h^2/r^2}$, so the area of the sector is $\frac{1}{2}r^2(2\theta) = r^2\sin^{-1}\sqrt{1 - h^2/r^2}$. The triangle has base $2\sqrt{r^2 - h^2}$ and height h, so the area is $h\sqrt{r^2 - h^2}$. Therefore,

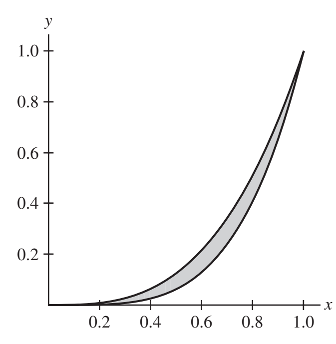
$$Y_{CM} = \frac{M_x}{A} = \frac{\frac{2}{3}(r^2 - h^2)^{3/2}}{r^2\sin^{-1}\sqrt{1 - h^2/r^2} - h\sqrt{r^2 - h^2}}.$$

When $r = 1$ and $h = 1/2$, we find

$$Y_{CM} = \frac{\frac{2}{3}(3/4)^{3/2}}{\sin^{-1}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}} = \frac{3\sqrt{3}}{4\pi - 3\sqrt{3}}.$$

34. Sketch the region between $y = x^n$ and $y = x^m$ for $0 \le x \le 1$, where $m > n \ge 0$ and find the COM of the region. Find a pair (n, m) such that the COM lies outside the region.

SOLUTION A sketch of the region for x^3 and x^4 is below.



Since $m > n \ge 0$, the graph of x^n lies above that of x^m for x between 0 and 1. Thus the moments are

$$\begin{aligned} M_x &= \frac{1}{2}\int_0^1 x^{2n} - x^{2m}\,dx = \frac{1}{2}\left(\frac{1}{2n+1}x^{2n+1} - \frac{1}{2m+1}x^{2m+1}\right)\Big|_0^1 \\ &= \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right) = \frac{m-n}{(2n+1)(2m+1)} \\ M_y &= \int_0^1 x(x^n - x^m)\,dx = \int_0^1 x^{n+1} - x^{m+1}\,dx = \left(\frac{1}{n+2}x^{n+2} - \frac{1}{m+2}x^{m+2}\right)\Big|_0^1 \\ &= \frac{1}{n+2} - \frac{1}{m+2} = \frac{m-n}{(n+2)(m+2)} \end{aligned}$$

The area of the region is

$$A = \int_0^1 x^n - x^m \, dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$$

Thus the center of mass has coordinates

$$\left(\frac{(n+1)(m+1)}{(n+2)(m+2)}, \frac{(n+1)(m+1)}{(2n+1)(2m+1)} \right)$$

In the case graphed above, for $n = 3$, $m = 4$, the center of mass is

$$\left(\frac{20}{30}, \frac{20}{63} \right) = \left(\frac{2}{3}, \frac{20}{63} \right)$$

and

$$\left(\frac{2}{3} \right)^3 = \frac{8}{27} < \frac{20}{63}$$

Thus the point $\left(\frac{2}{3}, \frac{8}{27} \right)$ lies on $y = x^3$ and then the curve $y = x^3$ lies below the center of mass of the region.

In Exercises 35–37, use the additivity of moments to find the COM of the region.

35. Isosceles triangle of height 2 on top of a rectangle of base 4 and height 3 (Figure 19)

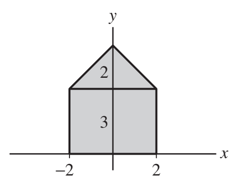


FIGURE 19

SOLUTION The region is symmetric with respect to the y-axis, so $M_y = 0$ by the Symmetry Principle. The moment about the x-axis for the rectangle is

$$M_x^{\text{rect}} = \frac{1}{2} \int_{-2}^{2} 3^2 \, dx = 18,$$

whereas the moment about the x-axis for the triangle is

$$M_x^{\text{triangle}} = \int_3^5 y(10 - 2y) \, dy = \frac{44}{3}.$$

The total moment about the x-axis is then

$$M_x = M_x^{\text{rect}} + M_x^{\text{triangle}} = 18 + \frac{44}{3} = \frac{98}{3}.$$

Because the area of the region is $12 + 4 = 16$, the coordinates of the center of mass are

$$\left(0, \frac{49}{24} \right).$$

36. An ice cream cone consisting of a semicircle on top of an equilateral triangle of side 6 (Figure 20)

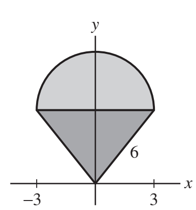


FIGURE 20

SOLUTION The region is symmetric with respect to the y-axis, so $M_y = 0$ by the Symmetry Principle. The moment about the x-axis for the triangle is

$$M_x^{\text{triangle}} = \frac{2}{\sqrt{3}} \int_0^{3\sqrt{3}} y^2\,dy = 54.$$

For the semicircle, first note that the center is $(0, 3\sqrt{3})$, so the equation is $x^2 + (y - 3\sqrt{3})^2 = 9$, and

$$M_x^{\text{semi}} = 2\int_{3\sqrt{3}}^{3+3\sqrt{3}} y\sqrt{9 - (y - 3\sqrt{3})^2}\,dy.$$

Using the substitution $w = y - 3\sqrt{3}$, $dw = dy$, we find

$$\begin{aligned} M_x^{\text{semi}} &= 2\int_0^3 (w + 3\sqrt{3})\sqrt{9 - w^2}\,dw \\ &= 2\int_0^3 w\sqrt{9 - w^2}\,dw + 6\sqrt{3}\int_0^3 \sqrt{9 - w^2}\,dw = 18 + \frac{27\pi\sqrt{3}}{2}, \end{aligned}$$

where we have used the fact that $\int_0^3 \sqrt{9 - w^2}\,dw$ represents the area of one-quarter of a circle of radius 3. The total moment about the x-axis is then

$$M_x = M_x^{\text{triangle}} + M_x^{\text{semi}} = 72 + \frac{27\pi\sqrt{3}}{2}.$$

Because the area of the region is $9\sqrt{3} + \frac{9\pi}{2}$, the coordinates of the center of mass are

$$\left(0, \frac{16 + 3\pi\sqrt{3}}{\pi + 2\sqrt{3}}\right).$$

37. Three-quarters of the unit circle (remove the part in the fourth quadrant)

SOLUTION By the Symmetry Principle, the center of mass must lie on the line $y = -x$. Let region 1 be the semicircle above the x-axis and region 2 be the quarter circle in the third quadrant. Because region 1 is symmetric with respect to the y-axis, $M_y^1 = 0$ by the Symmetry Principle. Furthermore

$$M_y^2 = \int_{-1}^0 x\sqrt{1 - x^2}\,dx = -\frac{1}{3}.$$

Thus, $M_y = M_y^1 + M_y^2 = 0 + (-\frac{1}{3}) = -\frac{1}{3}$. The area of the region is $3\pi/4$, so the coordinates of the centroid are

$$\left(-\frac{4}{9\pi}, \frac{4}{9\pi}\right).$$

38. Let S be the lamina of mass density $\rho = 1$ obtained by removing a circle of radius r from the circle of radius $2r$ shown in Figure 21. Let M_x^S and M_y^S denote the moments of S. Similarly, let M_y^{big} and M_y^{small} be the y-moments of the larger and smaller circles.

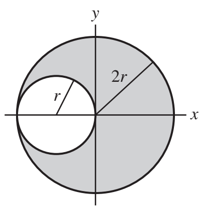


FIGURE 21

(a) Use the Symmetry Principle to show that $M_x^S = 0$.

(b) Show that $M_y^S = M_y^{\text{big}} - M_y^{\text{small}}$ using the additivity of moments.

(c) Find M_y^{big} and M_y^{small} using the fact that the COM of a circle is its center. Then compute M_y^S using (b).

(d) Determine the COM of S.

SOLUTION

(a) Because S is symmetric with respect to the x-axis, $M_x^S = 0$.

(b) Because the small circle together with the region S comprise the big circle, by the additivity of moments,

$$M_y^S + M_y^{\text{small}} = M_y^{\text{big}}.$$

Thus $M_y^S = M_y^{\text{big}} - M_y^{\text{small}}$.

(c) The center of the big circle is the origin, so $x_{\text{cm}}^{\text{big}} = 0$; consequently, $M_y^{\text{big}} = 0$. On the other hand, the center of the small circle is $(-r, 0)$, so $x_{\text{cm}}^{\text{small}} = -r$; consequently

$$M_y^{\text{small}} = x_{\text{cm}}^{\text{small}} \cdot A^{\text{small}} = -r \cdot \pi r^2 = -\pi r^3.$$

By the result of part (b), it follows that $M_y^S = 0 - (-\pi r^3) = \pi r^3$.

(d) The area of the region S is $4\pi r^2 - \pi r^2 = 3\pi r^2$. The coordinates of the center of mass of the region S are then

$$\left(\frac{\pi r^3}{3\pi r^2}, 0\right) = \left(\frac{r}{3}, 0\right).$$

39. Find the COM of the laminas in Figure 22 obtained by removing squares of side 2 from a square of side 8.

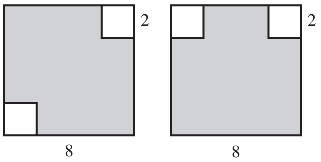


FIGURE 22

SOLUTION Start with the square on the left. Place the square so that the bottom left corner is at (0, 0). By the Symmetry Principle, the center of mass must lie on the lines $y = x$ and $y = 8 - x$. The only point in common to these two lines is (4, 4), so the center of mass is (4, 4).

Now consider the square on the right. Place the square as above. By the symmetry principle, $x_{cm} = 4$. Now, let $s1$ denote the square in the upper left, $s2$ denote the square in the upper right, and B denote the entire square. Then

$$M_x^{s1} = \frac{1}{2}\int_0^2 (8^2 - 6^2)\,dx = 28;$$

$$M_x^{s2} = \frac{1}{2}\int_6^8 (8^2 - 6^2)\,dx = 28; \text{ and}$$

$$M_x^{B} = \frac{1}{2}\int_0^8 8^2\,dx = 256.$$

By the additivity of moments, $M_x = 256 - 28 - 28 = 200$. Finally, the area of the region is $A = 64 - 4 - 4 = 56$, so the coordinates of the center of mass are

$$\left(4, \frac{200}{56}\right) = \left(4, \frac{25}{7}\right).$$

Further Insights and Challenges

40. A **median** of a triangle is a segment joining a vertex to the midpoint of the opposite side. Show that the centroid of a triangle lies on each of its medians, at a distance two-thirds down from the vertex. Then use this fact to prove that the three medians intersect at a single point. *Hint:* Simplify the calculation by assuming that one vertex lies at the origin and another on the x-axis.

SOLUTION Orient the triangle by placing one vertex at (0, 0) and the long side of the triangle along the x-axis. Label the vertices $(0, 0)$, $(a, 0)$, (b, c). Thus, the equations of the short sides are $y = \frac{cx}{b}$ and $y = \frac{cx}{b-a} - \frac{ac}{b-a}$. Now,

$$M_x = \frac{1}{2}\int_0^b (cx/b)^2\,dx + \frac{1}{2}\int_b^a \left(\frac{cx - ac}{b - a}\right)^2 dx = \frac{ac^2}{6};$$

$$M_y = \int_0^b x(cx/b)\,dx + \int_b^a x\left(\frac{cx - ac}{b - a}\right) dx = \frac{ac(a + b)}{6}; \text{ and}$$

$$M = \frac{ac}{2}.$$

so the center of mass is $\left(\dfrac{a+b}{3}, \dfrac{c}{3}\right)$. To show that the centroid lies on each median, let y_1 be the median from (b, c), y_2 the median from $(0, 0)$ and y_3 the median from $(a, 0)$. We find

$$y_1(x) = \frac{2c}{2b-a}(x - a/2), \qquad \text{so} \qquad y_1\left(\frac{a+b}{3}\right) = \frac{c}{3};$$

$$y_2(x) = \frac{c}{a+b}x, \qquad \text{so} \qquad y_2\left(\frac{a+b}{3}\right) = \frac{c}{3};$$

$$y_3(x) = \frac{c}{b-2a}(x - a), \qquad \text{so} \qquad y_3\left(\frac{a+b}{3}\right) = \frac{c}{3}.$$

This shows that the center of mass lies on each median. We now show that the center of mass is $\frac{2}{3}$ of the way from each vertex. For y_1, note that $x = b$ gives the vertex and $x = \frac{a}{2}$ gives the midpoint of the opposite side, so two-thirds of this distance is

$$x = b + \frac{2}{3}\left(\frac{a}{2} - b\right) = \frac{a+b}{3},$$

the x-coordinate of the center of mass. Likewise, for y_2, two-thirds of the distance from $x = 0$ to $x = \frac{a+b}{2}$ is $\frac{a+b}{3}$, and for y_3, the two-thirds point is

$$x = a + \frac{2}{3}\left(\frac{b}{2} - a\right) = \frac{a+b}{3}.$$

A similar method shows that the y-coordinate is also two-thirds of the way along the median. Thus, since the centroid lies on all three medians, we can conclude that all three medians meet at a single point, namely the centroid.

41. Let P be the COM of a system of two weights with masses m_1 and m_2 separated by a distance d. Prove Archimedes' Law of the (weightless) Lever: P is the point on a line between the two weights such that $m_1L_1 = m_2L_2$, where L_j is the distance from mass j to P.

SOLUTION Place the lever along the x-axis with mass m_1 at the origin. Then $M_y = m_2d$ and the x-coordinate of the center of mass, P, is

$$\frac{m_2d}{m_1 + m_2}.$$

Thus,

$$L_1 = \frac{m_2d}{m_1+m_2}, \quad L_2 = d - \frac{m_2d}{m_1+m_2} = \frac{m_1d}{m_1+m_2},$$

and

$$L_1m_1 = m_1\frac{m_2d}{m_1+m_2} = m_2\frac{m_1d}{m_1+m_2} = L_2m_2.$$

42. Find the COM of a system of two weights of masses m_1 and m_2 connected by a lever of length d whose mass density ρ is uniform. *Hint:* The moment of the system is the sum of the moments of the weights and the lever.

SOLUTION Let A be the cross-sectional area of the rod. Place the rod with m_1 at the origin and rod lying on the positive x-axis. The y-moment of the rod is $M_y = \frac{1}{2}\rho Ad^2$, the y-moment of the mass m_2 is $M_y = m_2d$, and the total mass of the system is $M = m_1 + m_2 + \rho Ad$. Therefore, the x-coordinate of the center of mass is

$$\frac{m_2d + \frac{1}{2}\rho Ad^2}{m_1 + m_2 + \rho Ad}.$$

43. **Symmetry Principle** Let $\mathcal{R}$ be the region under the graph of $f(x)$ over the interval $[-a, a]$, where $f(x) \geq 0$. Assume that $\mathcal{R}$ is symmetric with respect to the y-axis.

(a) Explain why $f(x)$ is even—that is, why $f(x) = f(-x)$.

(b) Show that $xf(x)$ is an *odd* function.

(c) Use (b) to prove that $M_y = 0$.

(d) Prove that the COM of $\mathcal{R}$ lies on the y-axis (a similar argument applies to symmetry with respect to the x-axis).

SOLUTION

(a) By the definition of symmetry with respect to the y-axis, $f(x) = f(-x)$, so f is even.

(b) Let $g(x) = xf(x)$ where f is even. Then

$$g(-x) = -xf(-x) = -xf(x) = -g(x),$$

and thus g is odd.

(c) $M_y = \rho \int_{-a}^{a} x f(x)\,dx = 0$ since $x f(x)$ is an odd function.

(d) By part (c), $x_{cm} = \dfrac{M_y}{M} = \dfrac{0}{M} = 0$ so the center of mass lies along the y-axis.

44. Prove directly that Eqs. (2) and (3) are equivalent in the following situation. Let $f(x)$ be a positive decreasing function on $[0, b]$ such that $f(b) = 0$. Set $d = f(0)$ and $g(y) = f^{-1}(y)$. Show that

$$\frac{1}{2}\int_0^b f(x)^2\,dx = \int_0^d y g(y)\,dy$$

Hint: First apply the substitution $y = f(x)$ to the integral on the left and observe that $dx = g'(y)\,dy$. Then apply Integration by Parts.

SOLUTION $f(x) \geq 0$ and $f'(x) < 0$ shows that f has an inverse g on $[a, b]$. Because $f(b) = 0$, $f(0) = d$, and $f^{-1}(x) = g(x)$, it follows that $g(d) = 0$ and $g(0) = b$. If we let $x = g(y)$, then $dx = g'(y)\,dy$. Thus, with $y = f(x)$,

$$\frac{1}{2}\int_0^b f(x)^2\,dx = \frac{1}{2}\int_0^b y^2\,dx = \frac{1}{2}\int_d^0 y^2 g'(y)\,dy.$$

Using Integration by Parts with $u = y^2$ and $v' = g'(y)\,dy$, we find

$$\frac{1}{2}\int_d^0 y^2 g'(y)\,dy = \frac{1}{2}\left[y^2 g(y)\Big|_d^0 - 2\int_d^0 y g(y)\,dy\right] = \frac{1}{2}\left[0 - d^2 g(d)\right] - \int_d^0 y g(y)\,dy = \int_0^d y g(y)\,dy.$$

45. Let R be a lamina of uniform density submerged in a fluid of density w (Figure 23). Prove the following law: The fluid force on one side of R is equal to the area of R times the fluid pressure on the centroid. *Hint:* Let $g(y)$ be the horizontal width of R at depth y. Express both the fluid pressure [Eq. (2) in Section 8.2] and y-coordinate of the centroid in terms of $g(y)$.

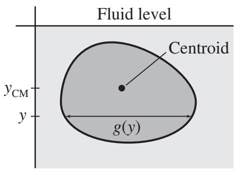


FIGURE 23

SOLUTION Let ρ denote the uniform density of the submerged lamina. Then

$$M_x = \rho\int_a^b y g(y)\,dy,$$

and the mass of the lamina is

$$M = \rho\int_a^b g(y)\,dy = \rho A,$$

where A is the area of the lamina. Thus, the y-coordinate of the centroid is

$$y_{\text{cm}} = \frac{\rho\int_a^b y g(y)\,dy}{\rho A} = \frac{\int_a^b y g(y)\,dy}{A}.$$

Now, the fluid force on the lamina is

$$F = w\int_a^b y g(y)\,dy = w\frac{\int_a^b y g(y)\,dy}{A}A = w y_{\text{cm}} A.$$

In other words, the fluid force on the lamina is equal to the fluid pressure at the centroid of the lamina times the area of the lamina.

8.4 Taylor Polynomials

Preliminary Questions

1. What is $T_3(x)$ centered at $a = 3$ for a function $f(x)$ such that $f(3) = 9$, $f'(3) = 8$, $f''(3) = 4$, and $f'''(3) = 12$?

SOLUTION In general, with $a = 3$,

$$T_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{6}(x-3)^3.$$

Using the information provided, we find

$$T_3(x) = 9 + 8(x-3) + 2(x-3)^2 + 2(x-3)^3.$$

2. The dashed graphs in Figure 9 are Taylor polynomials for a function $f(x)$. Which of the two is a Maclaurin polynomial?

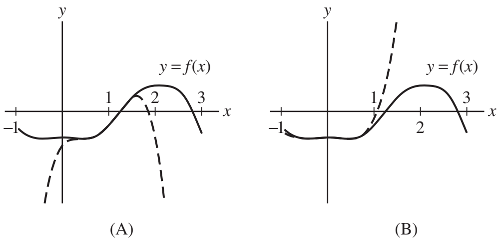


FIGURE 9

SOLUTION A Maclaurin polynomial always gives the value of $f(0)$ exactly. This is true for the Taylor polynomial sketched in (B); thus, this is the Maclaurin polynomial.

3. For which value of x does the Maclaurin polynomial $T_n(x)$ satisfy $T_n(x) = f(x)$, no matter what $f(x)$ is?

SOLUTION A Maclaurin polynomial always gives the value of $f(0)$ exactly.

4. Let $T_n(x)$ be the Maclaurin polynomial of a function $f(x)$ satisfying $|f^{(4)}(x)| \le 1$ for all x. Which of the following statements follow from the error bound?

(a) $|T_4(2) - f(2)| \le \frac{2}{3}$

(b) $|T_3(2) - f(2)| \le \frac{2}{3}$

(c) $|T_3(2) - f(2)| \le \frac{1}{3}$

SOLUTION For a function $f(x)$ satisfying $|f^{(4)}(x)| \le 1$ for all x,

$$|T_3(2) - f(2)| \le \frac{1}{24}|f^{(4)}(x)|2^4 \le \frac{16}{24} < \frac{2}{3}.$$

Thus, **(b)** is the correct answer.

Exercises

In Exercises 1–14, calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x = a$ for the given function and value of a.

1. $f(x) = \sin x, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \sin x & f(a) &= 0 \\ f'(x) &= \cos x & f'(a) &= 1 \\ f''(x) &= -\sin x & f''(a) &= 0 \\ f'''(x) &= -\cos x & f'''(a) &= -1 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-1}{6}(x-0)^3 = x - \frac{1}{6}x^3. \end{aligned}$$

2. $f(x) = \sin x, \quad a = \frac{\pi}{2}$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \sin x & f(a) &= 1 \\ f'(x) &= \cos x & f'(a) &= 0 \\ f''(x) &= -\sin x & f''(a) &= -1 \\ f'''(x) &= -\cos x & f'''(a) &= 0 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= 1 + 0\left(x - \frac{\pi}{2}\right) + \frac{-1}{2}\left(x - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 1 + 0\left(x - \frac{\pi}{2}\right) + \frac{-1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{0}{6}\left(x - \frac{\pi}{2}\right)^3 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2. \end{aligned}$$

3. $f(x) = \frac{1}{1+x}, \quad a = 2$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{1}{1+x} & f(a) &= \frac{1}{3} \\ f'(x) &= \frac{-1}{(1+x)^2} & f'(a) &= -\frac{1}{9} \\ f''(x) &= \frac{2}{(1+x)^3} & f''(a) &= \frac{2}{27} \\ f'''(x) &= \frac{-6}{(1+x)^4} & f'''(a) &= -\frac{2}{27} \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2/27}{2!}(x-2)^2 \\ &= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2/27}{2!}(x-2)^2 - \frac{2/27}{3!}(x-2)^3 = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 \end{aligned}$$

4. $f(x) = \frac{1}{1+x^2}, \quad a = -1$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{1}{1+x^2} & f(a) &= 1/2 \\ f'(x) &= \frac{-2x}{(x^2+1)^2} & f'(a) &= 1/2 \\ f''(x) &= \frac{2(3x^2-1)}{(x^2+1)^3} & f''(a) &= 1/2 \\ f'''(x) &= \frac{-24x(x^2-1)}{(x^2+1)^4} & f'''(a) &= 0 \end{aligned}$$

Now,

$$\begin{aligned}T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2\\ &= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1/2}{2}(x+1)^2 = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+1)^2; \text{ and}\\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3\\ &= \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1/2}{2}(x+1)^2 + \frac{0}{6}(x+1)^3 = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+1)^2.\end{aligned}$$

5. $f(x) = x^4 - 2x, \quad a = 3$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= x^4 - 2x & f(a) &= 75\\ f'(x) &= 4x^3 - 2 & f'(a) &= 106\\ f''(x) &= 12x^2 & f''(a) &= 108\\ f'''(x) &= 24x & f'''(a) &= 72\end{aligned}$$

Now,

$$\begin{aligned}T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 75 + 106(x-3) + \frac{108}{2}(x-3)^2\\ &= 75 + 106(x-3) + 54(x-3)^2\\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3\\ &= 75 + 106(x-3) + \frac{108}{2}(x-3)^2 + \frac{72}{3!}(x-3)^3\\ &= 75 + 106(x-3) + 54(x-3)^2 + 12(x-3)^3\end{aligned}$$

6. $f(x) = \dfrac{x^2+1}{x+1}, \quad a = -2$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{x^2+1}{x+1} & f(a) &= -5\\ f'(x) &= \frac{x^2+2x-1}{(x+1)^2} & f'(a) &= -1\\ f''(x) &= \frac{4}{(x+1)^3} & f''(a) &= -4\\ f'''(x) &= \frac{-12}{(x+1)^4} & f'''(a) &= -12\end{aligned}$$

Now,

$$\begin{aligned}T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = -5 - (x+2) + \frac{-4}{2}(x+2)^2\\ &= -5 - (x+2) - 2(x+2)^2\\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3\\ &= -5 - (x+2) + \frac{-4}{2}(x+2)^2 + \frac{-12}{3!}(x+2)^3\\ &= -5 - (x+2) - 2(x+2)^2 - 2(x+2)^3\end{aligned}$$

7. $f(x) = \tan x, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \tan x & f(a) &= 0 \\ f'(x) &= \sec^2 x & f'(a) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(a) &= 0 \\ f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x & f'''(a) &= 2 \end{aligned}$$

Now,

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x; \text{ and}$$

$$\begin{aligned} T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{2}{6}(x-0)^3 = x + \frac{1}{3}x^3. \end{aligned}$$

8. $f(x) = \tan x, \quad a = \frac{\pi}{4}$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \tan x & f(a) &= 1 \\ f'(x) &= \sec^2 x & f'(a) &= 2 \\ f''(x) &= 2\sec^2 x \tan x & f''(a) &= 4 \\ f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x & f'''(a) &= 16 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{6}\left(x - \frac{\pi}{4}\right)^3 = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3. \end{aligned}$$

9. $f(x) = e^{-x} + e^{-2x}, \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= e^{-x} + e^{-2x} & f(a) &= 2 \\ f'(x) &= -e^{-x} - 2e^{-2x} & f'(a) &= -3 \\ f''(x) &= e^{-x} + 4e^{-2x} & f''(a) &= 5 \\ f'''(x) &= -e^{-x} - 8e^{-2x} & f'''(a) &= -9 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 = 2 - 3x + \frac{5}{2}x^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 2 + (-3)(x-0) + \frac{5}{2}(x-0)^2 + \frac{-9}{6}(x-0)^3 = 2 - 3x + \frac{5}{2}x^2 - \frac{3}{2}x^3. \end{aligned}$$

10. $f(x) = e^{2x}, \quad a = \ln 2$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= e^{2x} & f(a) &= 4 \\ f'(x) &= 2e^{2x} & f'(a) &= 8 \\ f''(x) &= 4e^{2x} & f''(a) &= 16 \\ f'''(x) &= 8e^{2x} & f'''(a) &= 32 \end{aligned}$$

Now

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 4 + 8(x - \ln 2) + \frac{16}{2!}(x - \ln 2)^2 \\ &= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 4 + 8(x - \ln 2) + \frac{16}{2!}(x - \ln 2)^2 + \frac{32}{6}(x - \ln 2)^3 \\ &= 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 + \frac{16}{3}(x - \ln 2)^3 \end{aligned}$$

11. $f(x) = x^2e^{-x}, \quad a = 1$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= x^2e^{-x} & f(a) &= 1/e \\ f'(x) &= (2x - x^2)e^{-x} & f'(a) &= 1/e \\ f''(x) &= (x^2 - 4x + 2)e^{-x} & f''(a) &= -1/e \\ f'''(x) &= (-x^2 + 6x - 6)e^{-x} & f'''(a) &= -1/e \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 = \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) + \frac{-1/e}{2}(x-1)^2 + \left(\frac{-1/e}{6}\right)(x-1)^3 \\ &= \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3. \end{aligned}$$

12. $f(x) = \cosh 2x, \quad a = 0$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \cosh 2x & f(a) &= 1 \\ f'(x) &= 2\sinh 2x & f'(a) &= 0 \\ f''(x) &= 4\cosh 2x & f''(a) &= 4 \\ f'''(x) &= 8\sinh 2x & f'''(a) &= 0 \end{aligned}$$

so that

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 1 + 0(x-0) + \frac{4}{2!}(x-0)^2 \\ &= 1 + 2x^2 \end{aligned}$$

$$\begin{aligned} T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 1 + 0(x-0) + 2(x-0)^2 + \frac{0}{3!}(x-0)^3 \\ &= 1 + 2x^2 \end{aligned}$$

13. $f(x) = \dfrac{\ln x}{x}, \quad a = 1$

SOLUTION First calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \frac{\ln x}{x} & f(a) &= 0 \\ f'(x) &= \frac{1 - \ln x}{x^2} & f(a) &= 1 \\ f''(x) &= \frac{-3 + 2\ln x}{x^3} & f(a) &= -3 \\ f'''(x) &= \frac{11 - 6\ln x}{x^4} & f(a) &= 11 \end{aligned}$$

so that

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 0 + 1(x-1) + \frac{-3}{2!}(x-1)^2 \\ &= (x-1) - \frac{3}{2}(x-1)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &= 0 + 1(x-1) + \frac{-3}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 \\ &= (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 \end{aligned}$$

14. $f(x) = \ln(x+1), \quad a = 0$

SOLUTION First, we calculate and evaluate the needed derivatives:

$$\begin{aligned} f(x) &= \ln(x+1) & f(a) &= 0 \\ f'(x) &= \frac{1}{x+1} & f'(a) &= 1 \\ f''(x) &= \frac{-1}{(x+1)^2} & f''(a) &= -1 \\ f'''(x) &= \frac{2}{(x+1)^3} & f'''(a) &= 2 \end{aligned}$$

Now,

$$\begin{aligned} T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1(x-0) + \frac{-1}{2}(x-0)^2 = x - \frac{1}{2}x^2; \text{ and} \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 \\ &= 0 + 1(x-0) + \frac{-1}{2}(x-0)^2 + \frac{2}{6}(x-0)^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3. \end{aligned}$$

15. Show that the nth Maclaurin polynomial for e^x is

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

SOLUTION With $f(x) = e^x$, it follows that $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Thus,

$$T_n(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \cdots + \frac{1}{n!}(x-0)^n = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

16. Show that the nth Taylor polynomial for $\dfrac{1}{x+1}$ at $a=1$ is

$$T_n(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} + \cdots + (-1)^n \frac{(x-1)^n}{2^{n+1}}$$

SOLUTION Let $f(x) = \frac{1}{x+1}$. Then

$$\begin{aligned}
f(x) &= \frac{1}{x+1} & f(1) &= \frac{1}{2} = \frac{(-1)^0 0!}{2^{0+1}}\\
f'(x) &= \frac{-1}{(x+1)^2} & f'(1) &= -\frac{1}{4} = \frac{(-1)^1 1!}{2^{1+1}}\\
f''(x) &= \frac{2}{(x+1)^3} & f''(1) &= \frac{1}{4} = \frac{(-1)^2 2!}{2^{2+1}}\\
&\vdots & &\vdots\\
f^{(n)}(x) &= \frac{(-1)^n n!}{(x+1)^{n+1}} & f^{(n)}(1) &= \frac{(-1)^n n!}{2^{n+1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_n(x) &= \frac{1}{2} + \left(-\frac{1}{4}\right)(x-1) + \frac{1}{4}\frac{(x-1)^2}{2!} + \cdots + \frac{(-1)^n n!}{2^{n+1}}\frac{(x-1)^n}{n!}\\
&= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{(x-1)^2}{8} + \cdots + (-1)^n \frac{(x-1)^n}{2^{n+1}}.
\end{aligned}$$

17. Show that the Maclaurin polynomials for $\sin x$ are

$$T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

SOLUTION Let $f(x) = \sin x$. Then

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f'''(x) &= -\cos x & f'''(0) &= -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1\\
&\vdots & &\vdots
\end{aligned}$$

Consequently,

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and

$$T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + 0 = T_{2n+1}(x).$$

18. Show that the Maclaurin polynomials for $\ln(1+x)$ are

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

SOLUTION Let $f(x) = \ln(1+x)$. Then

$$\begin{aligned}
f(x) &= \ln(1+x) & f(0) &= 0\\
f'(x) &= (1+x)^{-1} & f'(0) &= 1
\end{aligned}$$

$$\begin{aligned} f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\ f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 \\ f^{(4)}(x) &= -3!(1+x)^{-4} & f^{(4)}(0) &= -6 \\ f^{(5)}(x) &= 4!(1+x)^{-5} & f^{(5)}(0) &= 24 \end{aligned}$$

so that in general

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Thus

$$T_n(x) = x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

In Exercises 19–24, find $T_n(x)$ at $x = a$ for all n.

19. $f(x) = \dfrac{1}{1+x}, \quad a = 0$

SOLUTION We have

$$\frac{1}{1+x} = (\ln(1+x))'$$

so that from Exercise 18, letting $g(x) = \ln(1+x)$,

$$f^{(n)}(x) = g^{(n+1)}(x) = (-1)^n n!(x+1)^{-1-n} \quad \text{and} \quad f^{(n)}(0) = (-1)^n n!$$

Then

$$\begin{aligned} T_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 - x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \cdots + (-1)^n\frac{n!}{n!}x^n \\ &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n \end{aligned}$$

20. $f(x) = \dfrac{1}{x-1}, \quad a = 4$

SOLUTION Let $f(x) = \frac{1}{x-1}$. Then

$$\begin{aligned} f(x) &= \frac{1}{x-1} & f(4) &= \frac{1}{3} = \frac{(-1)^0 0!}{3^{0+1}} \\ f'(x) &= \frac{-1}{(x-1)^2} & f'(4) &= -\frac{1}{9} = \frac{(-1)^1 1!}{3^{1+1}} \\ f''(x) &= \frac{2}{(x-1)^3} & f''(4) &= \frac{2}{27} = \frac{(-1)^2 2!}{3^{2+1}} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(x-1)^{n+1}} & f^{(n)}(4) &= \frac{(-1)^n n!}{3^{n+1}} \end{aligned}$$

Therefore,

$$\begin{aligned} T_n(x) &= \frac{1}{3} + \left(-\frac{1}{9}\right)(x-4) + \frac{2/27}{2}(x-4)^2 + \cdots + \frac{(-1)^n n!}{3^{n+1}}\frac{(x-4)^n}{n!} \\ &= \frac{1}{3} - \frac{1}{9}(x-4) + \frac{1}{27}(x-4)^2 + \cdots + \frac{(-1)^n}{3^{n+1}}(x-4)^n. \end{aligned}$$

21. $f(x) = e^x, \quad a = 1$

SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(1) = e$ for all n. Therefore,

$$T_n(x) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots + \frac{e}{n!}(x-1)^n.$$

22. $f(x) = x^{-2}, \quad a = 2$

SOLUTION We have

$$\begin{aligned} f(x) &= x^{-2} & f(2) &= \frac{1}{4} \\ f'(x) &= -2x^{-3} & f'(2) &= -\frac{1}{4} \\ f''(x) &= 6x^{-4} & f''(2) &= \frac{3}{8} \\ f'''(x) &= -24x^{-5} & f'''(2) &= -\frac{3}{4} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^n (n+1)! x^{-n-2} & f^{(n)}(2) &= (-1)^n \frac{(n+1)!}{2^{n+2}} \end{aligned}$$

so that

$$\begin{aligned} T_n(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 + \cdots + (-1)^n \frac{n+1}{2^{n+2}}(x-2)^n \end{aligned}$$

23. $f(x) = \cos x, \quad a = \frac{\pi}{4}$

SOLUTION Let $f(x) = \cos x$. Then

$$\begin{aligned} f(x) &= \cos x & f(\pi/4) &= \frac{1}{\sqrt{2}} \\ f'(x) &= -\sin x & f'(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f''(x) &= -\cos x & f''(\pi/4) &= -\frac{1}{\sqrt{2}} \\ f'''(x) &= \sin x & f'''(\pi/4) &= \frac{1}{\sqrt{2}} \end{aligned}$$

This pattern of four values repeats indefinitely. Thus,

$$f^{(n)}(\pi/4) = \begin{cases} (-1)^{(n+1)/2} \dfrac{1}{\sqrt{2}}, & n \text{ odd} \\ (-1)^{n/2} \dfrac{1}{\sqrt{2}}, & n \text{ even} \end{cases}$$

and

$$T_n(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 \cdots.$$

In general, the coefficient of $(x - \pi/4)^n$ is

$$\pm \frac{1}{(\sqrt{2})n!}$$

with the pattern of signs $+, -, -, +, +, -, -, \ldots$.

24. $f(\theta) = \sin 3\theta, \quad a = 0$

SOLUTION We have

$$\begin{aligned} f(\theta) &= \sin 3\theta & f(0) &= 0 \\ f'(\theta) &= 3\cos 3\theta & f'(0) &= 3 \\ f''(\theta) &= -9\sin 3\theta & f''(0) &= 0 \\ f'''(\theta) &= -27\cos 3\theta & f'''(0) &= -27 \\ f^{(4)}(\theta) &= 81\sin 3\theta & f^{(4)}(0) &= 0 \end{aligned}$$

and in general

$$f^{(2n)}(\theta) = (-1)^n 3^{2n} \sin 3\theta \qquad f^{(2n)}(0) = 0$$
$$f^{(2n+1)}(\theta) = (-1)^n 3^{2n+1} \cos 3\theta \qquad f^{(2n+1)}(0) = (-1)^n 3^{2n+1}$$

Thus

$$T_n(x) = 3\theta - \frac{27}{3!}\theta^3 + \frac{243}{5!}\theta^5 - \dots$$

where the coefficient of θ^{2n+1} is $(-1)^n \frac{3^{2n+1}}{(2n+1)!}$.

In Exercises 25–28, find $T_2(x)$ and use a calculator to compute the error $|f(x) - T_2(x)|$ for the given values of a and x.

25. $y = e^x, \quad a = 0, \quad x = -0.5$

SOLUTION Let $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, $f(a) = 1$, $f'(a) = 1$ and $f''(a) = 1$. Therefore

$$T_2(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{1}{2}x^2,$$

and

$$T_2(-0.5) = 1 + (-0.5) + \frac{1}{2}(-0.5)^2 = 0.625.$$

Using a calculator, we find

$$f(-0.5) = \frac{1}{\sqrt{e}} = 0.606531,$$

so

$$|T_2(-0.5) - f(-0.5)| = 0.0185.$$

26. $y = \cos x, \quad a = 0, \quad x = \frac{\pi}{12}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f(a) = 1$, $f'(a) = 0$, and $f''(a) = -1$. Therefore

$$T_2(x) = 1 + 0(x-0) + \frac{-1}{2}(x-0)^2 = 1 - \frac{1}{2}x^2,$$

and

$$T_2\left(\frac{\pi}{12}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{12}\right)^2 \approx 0.965731.$$

Using a calculator, we find

$$f\left(\frac{\pi}{12}\right) = 0.965926,$$

so

$$\left|T_2\left(\frac{\pi}{12}\right) - f\left(\frac{\pi}{12}\right)\right| = 0.000195.$$

27. $y = x^{-2/3}, \quad a = 1, \quad x = 1.2$

SOLUTION Let $f(x) = x^{-2/3}$. Then $f'(x) = -\frac{2}{3}x^{-5/3}$, $f''(x) = \frac{10}{9}x^{-8/3}$, $f(1) = 1$, $f'(1) = -\frac{2}{3}$, and $f''(1) = \frac{10}{9}$. Thus

$$T_2(x) = 1 - \frac{2}{3}(x-1) + \frac{10}{2\cdot 9}(x-1)^2 = 1 - \frac{2}{3}(x-1) + \frac{5}{9}(x-1)^2$$

and

$$T_2(1.2) = 1 - \frac{2}{3}(0.2) + \frac{5}{9}(0.2)^2 = \frac{8}{9} \approx 0.88889$$

Using a calculator, $f(1.2) = (1.2)^{-2/3} \approx 0.88555$ so that

$$|T_2(1.2) - f(1.2)| \approx 0.00334$$

28. $y = e^{\sin x}, \quad a = \frac{\pi}{2}, \quad x = 1.5$

SOLUTION Let $f(x) = e^{\sin x}$. Then $f'(x) = \cos x e^{\sin x}$, $f''(x) = \cos^2 x e^{\sin x} - \sin x e^{\sin x}$, $f(a) = e$, $f'(a) = 0$ and $f''(a) = -e$. Therefore

$$T_2(x) = e + 0\left(x - \frac{\pi}{2}\right) + \frac{-e}{2}\left(x - \frac{\pi}{2}\right)^2 = e - \frac{e}{2}\left(x - \frac{\pi}{2}\right)^2,$$

and

$$T_2(1.5) = e - \frac{e}{2}\left(1.5 - \frac{\pi}{2}\right)^2 \approx 2.711469651.$$

Using a calculator, we find $f(1.5) = 2.711481018$, so

$$|T_2(1.5) - f(1.5)| = 1.14 \times 10^{-5}.$$

29. GU Compute $T_3(x)$ for $f(x) = \sqrt{x}$ centered at $a = 1$. Then use a plot of the error $|f(x) - T_3(x)|$ to find a value $c > 1$ such that the error on the interval $[1, c]$ is at most 0.25.

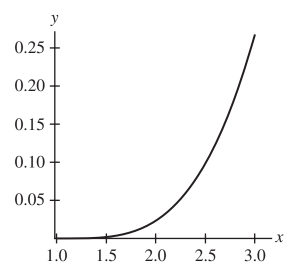
SOLUTION We have

$$\begin{aligned} f(x) &= x^{1/2} & f(1) &= 1 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(1) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} & f''(1) &= -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(1) &= \frac{3}{8} \end{aligned}$$

Therefore

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

A plot of $|f(x) - T_3(x)|$ is below.



It appears that for $x \in [1, 2.9]$ that the error does not exceed 0.25. The error at $x = 3$ appears to just exceed 0.25.

30. CAS Plot $f(x) = 1/(1+x)$ together with the Taylor polynomials $T_n(x)$ at $a = 1$ for $1 \le n \le 4$ on the interval $[-2, 8]$ (be sure to limit the upper plot range).

(a) Over which interval does $T_4(x)$ appear to approximate $f(x)$ closely?

(b) What happens for $x < -1$?

(c) Use your computer algebra system to produce and plot T_{30} together with $f(x)$ on $[-2, 8]$. Over which interval does T_{30} appear to give a close approximation?

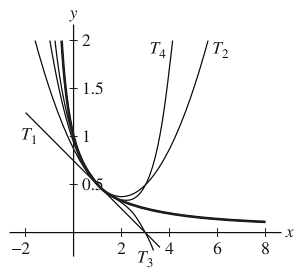
SOLUTION Let $f(x) = \frac{1}{1+x}$. Then

$$\begin{aligned} f(x) &= \frac{1}{1+x} & f(1) &= \frac{1}{2} \\ f'(x) &= -\frac{1}{(1+x)^2} & f'(1) &= -\frac{1}{4} \\ f''(x) &= \frac{2}{(1+x)^3} & f''(1) &= \frac{1}{4} \\ f'''(x) &= -\frac{6}{(1+x)^4} & f'''(1) &= -\frac{3}{8} \\ f^{(4)}(x) &= \frac{24}{(1+x)^5} & f^{(4)}(1) &= \frac{3}{4} \end{aligned}$$

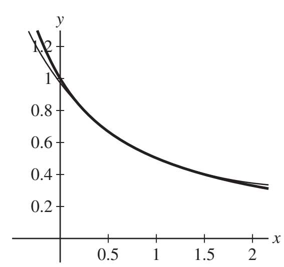
and

$$T_1(x) = \frac{1}{2} - \frac{1}{4}(x-1);$$
$$T_2(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2;$$
$$T_3(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3; \text{ and}$$
$$T_4(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4.$$

A plot of $f(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$ is shown below.

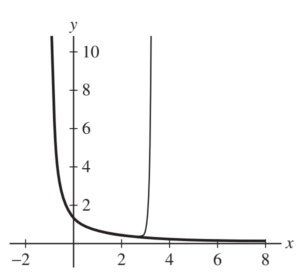


(a) The graph below displays $f(x)$ and $T_4(x)$ over the interval $[-0.5, 2.5]$. It appears that $T_4(x)$ gives a close approximation to $f(x)$ over the interval $(0.1, 2)$.



(b) For $x < -1$, $f(x)$ is negative, but the Taylor polynomials are positive; thus, the Taylor polynomials are poor approximations for $x < -1$.

(c) The graph below displays $f(x)$ and $T_{30}(x)$ over the interval $[-2, 8]$. It appears that $T_{30}(x)$ gives a close approximation to $f(x)$ over the interval $(-1, 3)$.



31. Let $T_3(x)$ be the Maclaurin polynomial of $f(x) = e^x$. Use the error bound to find the maximum possible value of $|f(1.1) - T_3(1.1)|$. Show that we can take $K = e^{1.1}$.

SOLUTION Since $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for all n; since e^x is increasing, the maximum value of e^x on the interval $[0, 1.1]$ is $K = e^{1.1}$. Then by the error bound,

$$\left|e^{1.1} - T_3(1.1)\right| \le K\frac{(1.1-0)^4}{4!} = \frac{e^{1.1}1.1^4}{24} \approx 0.183$$

32. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of the error $|f(3.9) - T_2(3.9)|$.

SOLUTION We have $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f'''(x) = \frac{3}{8}x^{-5/2}$. This is a decreasing function of x, so its maximum value on $[3.9, 4]$ is achieved at $x = 3.9$; that value is $\frac{3}{8 \cdot 3.9^{5/2}} \approx 0.0125$, so we can take $K = 0.0125$. Then

$$|f(x) - T_2(x)| \le K\frac{|3.9-4|^3}{3!} = 0.0125\frac{0.001}{6} \approx 2.08 \times 10^{-6}$$

In Exercises 33–36, compute the Taylor polynomial indicated and use the error bound to find the maximum possible size of the error. Verify your result with a calculator.

33. $f(x) = \cos x, \quad a = 0; \quad |\cos 0.25 - T_5(0.25)|$

SOLUTION The Maclaurin series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

so that

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
$$T_5(0.25) \approx 0.9689127604$$

In addition, $f^{(6)}(x) = -\cos x$ so that $|f^{(6)}(x)| \le 1$ and we may take $K = 1$ in the error bound formula. Then

$$|\cos 0.25 - T_5(0.25)| \le K\frac{0.25^6}{6!} = \frac{1}{2^{12} \cdot 6!} \approx 3.390842014 \cdot 10^{-7}$$

(The true value is $\cos 0.25 \approx 0.9689124217$ and the difference is in fact $\approx 3.387 \cdot 10^{-7}$.)

34. $f(x) = x^{11/2}, \quad a = 1; \quad |f(1.2) - T_4(1.2)|$

SOLUTION Let $f(x) = x^{11/2}$. Then

$$\begin{aligned} f(x) &= x^{11/2} & f(1) &= 1 \\ f'(x) &= \frac{11}{2}x^{9/2} & f'(1) &= \frac{11}{2} \\ f''(x) &= \frac{99}{4}x^{7/2} & f''(1) &= \frac{99}{4} \\ f'''(x) &= \frac{693}{8}x^{5/2} & f'''(1) &= \frac{693}{8} \\ f^{(4)}(x) &= \frac{3465}{16}x^{3/2} & f^{(4)}(1) &= \frac{3465}{16} \end{aligned}$$

and

$$T_4(x) = 1 + \frac{11}{2}(x-1) + \frac{99}{8}(x-1)^2 + \frac{231}{16}(x-1)^3 + \frac{1155}{128}(x-1)^4.$$

Using the Error Bound,

$$|f(1.2) - T_4(1.2)| \le \frac{K|1.2-1|^5}{5!} = \frac{K}{375{,}000},$$

where K is a number such that $|f^{(5)}(x)| \le K$ for x between 1 and 1.2. Now,

$$f^{(5)}(x) = \frac{10{,}395}{32}x^{1/2},$$

which is increasing for $x > 1$. Consequently, on the interval $[1, 1.2]$, $f^{(5)}(x)$ is maximized at $x = 1.2$. We can therefore take $K = \frac{10{,}395}{32}\sqrt{1.2}$, and then

$$|f(1.2) - T_4(1.2)| \le \frac{10{,}395}{(32)(375{,}000)}\sqrt{1.2} \approx 9.489 \times 10^{-4}.$$

35. $f(x) = x^{-1/2}$, $a = 4$; $|f(4.3) - T_3(4.3)|$

SOLUTION We have

$$\begin{aligned} f(x) &= x^{-1/2} & f(4) &= \frac{1}{2} \\ f'(x) &= -\frac{1}{2}x^{-3/2} & f'(4) &= -\frac{1}{16} \\ f''(x) &= \frac{3}{4}x^{-5/2} & f''(4) &= \frac{3}{128} \\ f'''(x) &= -\frac{15}{8}x^{-7/2} & f'''(4) &= -\frac{15}{1024} \\ f^{(4)}(x) &= \frac{105}{16}x^{-9/2} \end{aligned}$$

so that

$$T_3(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2 - \frac{5}{2048}(x-4)^3$$

Using the error bound formula,

$$|f(4.3) - T_3(4.3)| \le K\frac{|4.3-4|^4}{4!} = \frac{27K}{80{,}000}$$

where K is a number such that $|f^{(4)}(x)| \le K$ for x between 4 and 4.3. Now, $f^{(4)}(x)$ is a decreasing function for $x > 1$, so it takes its maximum value on $[4, 4.3]$ at $x = 4$; there, its value is

$$K = \frac{105}{16}4^{-9/2} = \frac{105}{8192}$$

so that

$$|f(4.3) - T_3(4.3)| \le \frac{27\frac{105}{8192}}{80{,}000} = \frac{27 \cdot 105}{8192 \cdot 80{,}000} \approx 4.3258667 \cdot 10^{-6}$$

36. $f(x) = \sqrt{1+x}$, $a = 8$; $|\sqrt{9.02} - T_3(8.02)|$

SOLUTION Let $f(x) = \sqrt{1+x}$. Then

$$\begin{aligned} f(x) &= \sqrt{1+x} & f(8) &= 3 \\ f'(x) &= \frac{1}{2}(x+1)^{-1/2} & f'(8) &= \frac{1}{6} \\ f''(x) &= \frac{-1}{4}(x+1)^{-3/2} & f''(8) &= \frac{-1}{108} \\ f'''(x) &= \frac{3}{8}(x+1)^{-5/2} & f'''(8) &= \frac{1}{648} \end{aligned}$$

and

$$T_3(x) = 3 + \frac{1}{6}(x-8) - \frac{1}{108 \cdot 2!}(x-8)^2 + \frac{1}{648 \cdot 3!}(x-8)^3 = 3 + \frac{1}{6}(x-8) - \frac{1}{216}(x-8)^2 + \frac{1}{3888}(x-8)^3.$$

Therefore

$$T_3(8.02) = 3 + \frac{1}{6}(0.02) - \frac{1}{216}(0.02)^2 + \frac{1}{3888}(0.02)^3 = 3.003331484.$$

Using the Error Bound, we have

$$|\sqrt{9.02} - T_3(8.02)| \le K\frac{|8.02-8|^4}{4!} = \frac{K}{150{,}000{,}000},$$

where K is a number such that $|f^{(4)}(x)| \le K$ for x between 8 and 8.02. Now

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2},$$

which is a decreasing function for $8 \le x \le 8.02$, so we may take

$$K = \frac{15}{16}9^{-7/2} = \frac{15}{34992}.$$

Thus,

$$|\sqrt{9.02} - T_3(8.02)| \le \frac{15/34992}{150{,}000{,}000} \approx 2.858 \times 10^{-12}.$$

37. Calculate the Maclaurin polynomial $T_3(x)$ for $f(x) = \tan^{-1} x$. Compute $T_3\left(\frac{1}{2}\right)$ and use the error bound to find a bound for the error $\left|\tan^{-1} \frac{1}{2} - T_3\left(\frac{1}{2}\right)\right|$. Refer to the graph in Figure 10 to find an acceptable value of K. Verify your result by computing $\left|\tan^{-1} \frac{1}{2} - T_3\left(\frac{1}{2}\right)\right|$ using a calculator.

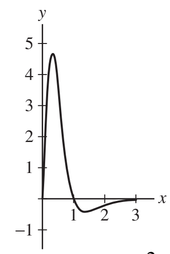


FIGURE 10 Graph of $f^{(4)}(x) = \dfrac{-24x(x^2-1)}{(x^2+1)^4}$, where $f(x) = \tan^{-1} x$.

SOLUTION Let $f(x) = \tan^{-1} x$. Then

$$\begin{aligned} f(x) &= \tan^{-1} x & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x^2} & f'(0) &= 1 \\ f''(x) &= \frac{-2x}{(1+x^2)^2} & f''(0) &= 0 \\ f'''(x) &= \frac{(1+x^2)^2(-2) - (-2x)(2)(1+x^2)(2x)}{(1+x^2)^4} & f'''(0) &= -2 \end{aligned}$$

and

$$T_3(x) = 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-2}{6}(x-0)^3 = x - \frac{x^3}{3}.$$

Since $f^{(4)}(x) \le 5$ for $x \ge 0$, we may take $K = 5$ in the error bound; then,

$$\left|\tan^{-1}\left(\frac{1}{2}\right) - T_3\left(\frac{1}{2}\right)\right| \le \frac{5(1/2)^4}{4!} = \frac{5}{384}.$$

38. Let $f(x) = \ln(x^3 - x + 1)$. The third Taylor polynomial at $a = 1$ is

$$T_3(x) = 2(x-1) + (x-1)^2 - \frac{7}{3}(x-1)^3$$

Find the maximum possible value of $|f(1.1) - T_3(1.1)|$, using the graph in Figure 11 to find an acceptable value of K. Verify your result by computing $|f(1.1) - T_3(1.1)|$ using a calculator.

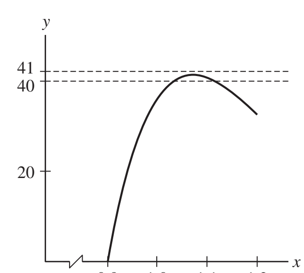


FIGURE 11 Graph of $f^{(4)}(x)$, where $f(x) = \ln(x^3 - x + 1)$.

SOLUTION The maximum value of $f^{(4)}(x)$ on $[1.0, 1.1]$ is less than 41, so we may take $K = 41$. Then

$$|f(1.1) - T_3(1.1)| \le K\frac{|1.1-1|^4}{4!} = \frac{41}{24 \cdot 10{,}000} \approx 0.00017083$$

In fact, we have

$$f(1.1) = \ln(1.1^3 - 1.1 + 1) = \ln(1.231) \approx 0.2078268472$$

$$T_3(1.1) = 2(1.1 - 1) + (1.1 - 1)^2 - \frac{7}{3}(1.1 - 1)^3 \approx 0.2076666667$$

$$|f(1.1) - T_3(1.1)| \approx 0.2078268472 - 0.2076666667 = 0.0001601805$$

which is in accordance with the error bound above.

39. $\boxed{\text{GU}}$ Calculate the $T_3(x)$ at $a = 0.5$ for $f(x) = \cos(x^2)$, and use the error bound to find the maximum possible value of $|f(0.6) - T_3(0.6)|$. Plot $f^{(4)}(x)$ to find an acceptable value of K.

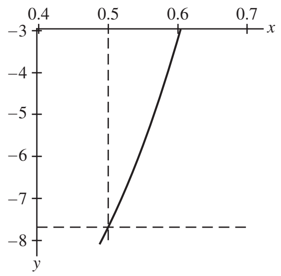
SOLUTION We have

$$\begin{aligned} f(x) &= \cos(x^2) & f(0.5) &= \cos(0.25) \approx 0.9689 \\ f'(x) &= -2x\sin(x^2) & f'(0.5) &= -\sin(0.25) \approx -0.2474039593 \\ f''(x) &= -4x^2\cos(x^2) - 2\sin(x^2) & f''(0.5) &= -\cos(0.25) - 2\sin(0.25) \approx -1.463720340 \\ f'''(x) &= 8x^3\sin(x^2) - 12x\cos(x^2) & f'''(0.5) &= \sin(0.25) - 6\cos(0.25) \approx -5.566070571 \\ f^{(4)}(x) &= 16x^4\cos(x^2) + 48x^2\sin(x^2) - 12\cos(x^2) \end{aligned}$$

so that

$$T_3(x) = 0.9689 - 0.2472039593(x - 0.5) - 0.73186017(x - 0.5)^2 - 0.92767843(x - 0.5)^3$$

and $T_3(0.6) \approx 0.9359257453$. A graph of $f^{(4)}(x)$ for x near 0.5 is below.



Clearly the maximum value of $|f^{(4)}(x)|$ on $[0.5, 0.6]$ is bounded by 8 (near $x = 0.5$), so we may take $K = 8$; then

$$|f(0.6) - T_3(0.6)| \le K\frac{|0.6 - 0.5|^4}{4!} = \frac{8}{240{,}000} \approx 0.000033333$$

40. $\boxed{\text{GU}}$ Calculate the Maclaurin polynomial $T_2(x)$ for $f(x) = \operatorname{sech} x$ and use the error bound to find the maximum possible value of $\left|f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right|$. Plot $f'''(x)$ to find an acceptable value of K.

SOLUTION To compute $T_2(x)$ for $f(x) = \operatorname{sech} x$, we take the first two derivatives:

$$\begin{aligned} f(x) &= \operatorname{sech} x & f(0) &= 1 \\ f'(x) &= -\operatorname{sech} x \tanh x & f'(0) &= 0 \\ f''(x) &= \operatorname{sech} x \tanh^2 x - \operatorname{sech}^3 x & f''(0) &= -1 \end{aligned}$$

From this,

$$T_2(x) = 1 - \frac{1}{2}x^2,$$

and

$$T_2\left(\frac{1}{2}\right) = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{8} = \frac{7}{8}.$$

Using the Error Bound, we have

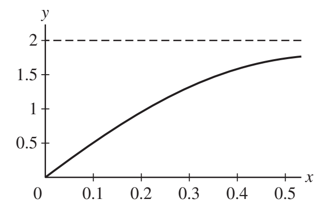
$$\left|f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right)\right| \le K\frac{|1/2|^3}{6} = \frac{K}{48},$$

where K is a number such that $|f'''(x)| \le K|$ for x between 0 and $\frac{1}{2}$. Here,

$$\begin{aligned} f'''(x) &= -\operatorname{sech} x \tanh^3 x + 2\operatorname{sech}^3 x \tanh x + 3\operatorname{sech}^2 x(\operatorname{sech} x \tanh x) \\ &= 5\operatorname{sech}^2 x \tanh x - \operatorname{sech} x \tanh^3 x. \end{aligned}$$

A plot of $f'''(x)$ is given below. From the plot, we see that $|f'''(x)| \le 2$ for all x between 0 and $1/2$. Thus,

$$\left| f\left(\frac{1}{2}\right) - T_2\left(\frac{1}{2}\right) \right| \le \frac{2}{48} = \frac{1}{24}.$$



In Exercises 41–44, use the error bound to find a value of n for which the given inequality is satisfied. Then verify your result using a calculator.

41. $|\cos 0.1 - T_n(0.1)| \le 10^{-7}, \quad a = 0$

SOLUTION Using the error bound with $K = 1$ (every derivative of $f(x) = \cos x$ is $\pm \sin x$ or $\pm \cos x$, so $|f^{(n)}(x)| \le 1$ for all n), we have

$$|T_n(0.1) - \cos 0.1| \le \frac{(0.1)^{n+1}}{(n+1)!}.$$

With $n = 3$,

$$\frac{(0.1)^4}{4!} \approx 4.17 \times 10^{-6} > 10^{-7},$$

but with $n = 4$,

$$\frac{(0.1)^5}{5!} \approx 8.33 \times 10^{-8} < 10^{-7},$$

so we choose $n = 4$. Now,

$$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4,$$

so

$$T_4(0.1) = 1 - \frac{1}{2}(0.1)^2 + \frac{1}{24}(0.1)^4 = 0.995004166.$$

Using a calculator, $\cos 0.1 = 0.995004165$, so

$$|T_4(0.1) - \cos 0.1| = 1.387 \times 10^{-8} < 10^{-7}.$$

42. $|\ln 1.3 - T_n(1.3)| \le 10^{-4}, \quad a = 1$

SOLUTION Let $f(x) = \ln x$. Then $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, etc. In general,

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}.$$

Now, $|f^{(n+1)}(x)|$ is decreasing on the interval $[1, 1.3]$, so $|f^{(n+1)}(x)| \le |f^{(n+1)}(1)| = n!$ for all $x \in [1, 1.3]$. We can therefore take $K = n!$ in the error bound, and

$$|\ln 1.3 - T_n(1.3)| \le n!\frac{|1.3 - 1|^{n+1}}{(n+1)!} = \frac{(0.3)^{n+1}}{n+1}.$$

With $n = 5$,

$$\frac{(0.3)^6}{6} = 1.215 \times 10^{-4} > 10^{-4},$$

but with $n = 6$,

$$\frac{(0.3)^7}{7} = 3.124 \times 10^{-5} < 10^{-4}.$$

Therefore, the error is guaranteed to be below 10^{-4} for $n = 6$. Now,

$$T_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6$$

and $T_6(1.3) \approx 0.2623395000$. Using a calculator, $\ln(1.3) \approx 0.2623642645$; the difference is

$$\ln(1.3) - T_6(1.3) \approx 0.0000247645 < 10^{-4}$$

43. $|\sqrt{1.3} - T_n(1.3)| \le 10^{-6}, \quad a = 1$

SOLUTION Using the Error Bound, we have

$$|\sqrt{1.3} - T_n(1.3)| \le K\frac{|1.3-1|^{n+1}}{(n+1)!} = K\frac{|0.3|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(x)| \le K$ for x between 1 and 1.3. For $f(x) = \sqrt{x}$, $|f^{(n)}(x)|$ is decreasing for $x > 1$, hence the maximum value of $|f^{(n+1)}(x)|$ occurs at $x = 1$. We may therefore take

$$\begin{aligned} K = |f^{(n+1)}(1)| &= \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}} \\ &= \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\cdot\frac{2\cdot 4\cdot 6\cdots(2n+2)}{2\cdot 4\cdot 6\cdots(2n+2)} = \frac{(2n+2)!}{(n+1)!2^{2n+2}}. \end{aligned}$$

Then

$$|\sqrt{1.3} - T_n(1.3)| \le \frac{(2n+2)!}{(n+1)!2^{2n+2}}\cdot\frac{|0.3|^{n+1}}{(n+1)!} = \frac{(2n+2)!}{[(n+1)!]^2}(0.075)^{n+1}.$$

With $n = 9$

$$\frac{(20)!}{[(10)!]^2}(0.075)^{10} = 1.040 \times 10^{-6} > 10^{-6},$$

but with $n = 10$

$$\frac{(22)!}{[(11)!]^2}(0.075)^{11} = 2.979 \times 10^{-7} < 10^{-6}.$$

Hence, $n = 10$ will guarantee the desired accuracy. Using technology to compute and evaluate $T_{10}(1.3)$ gives

$$T_{10}(1.3) \approx 1.140175414, \qquad \sqrt{1.3} \approx 1.140175425$$

and

$$|\sqrt{1.3} - T_{10}(1.3)| \approx 1.1 \times 10^{-8} < 10^{-6}$$

44. $|e^{-0.1} - T_n(-0.1)| \le 10^{-6}, \quad a = 0$

SOLUTION Using the Error Bound, we have

$$|e^{-0.1} - T_n(-0.1)| \le K\frac{|-0.1-0|^{n+1}}{(n+1)!} = K\frac{1}{10^{n+1}(n+1)!}$$

where K is a number such that $|f^{(n+1)}(x)| \le K$ for x between -0.1 and 0. Since $f(x) = e^x$, $f^{(n)}(x) = e^x$ for all n; this is an increasing function, so it takes its maximum value at $x = 0$; this value is 1. So we may take $K = 1$ and then

$$|e^{-0.1} - T_n(-0.1)| \le \frac{1}{10^{n+1}(n+1)!}$$

With $n = 3$

$$\frac{1}{10^4\cdot 24} = \frac{1}{240{,}000} \approx 4.166666667 \times 10^{-6} > 10^{-6}$$

but with $n = 4$

$$\frac{1}{10^5\cdot 120} = \frac{1}{12{,}000{,}000} \approx 8.333333333 \times 10^{-8} < 10^{-6}$$

Thus $n = 4$ will guarantee the desired accuracy. Using technology to compute $T_4(x)$ and evalute,

$$T_4(-0.1) \approx 0.9048375000, \qquad e^{-0.1} \approx 0.9048374180$$

and

$$|e^{-0.1} - T_4(-0.1)| \approx 8.2 \times 10^{-8} < 10^{-6}$$

45. Let $f(x) = e^{-x}$ and $T_3(x) = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$. Use the error bound to show that for all $x \geq 0$,

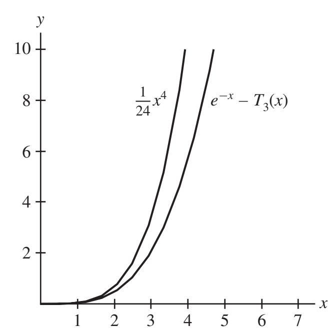
$$|f(x) - T_3(x)| \leq \frac{x^4}{24}$$

If you have a GU, illustrate this inequality by plotting $f(x) - T_3(x)$ and $x^4/24$ together over [0, 1].

SOLUTION Note that $f^{(n)}(x) = \pm e^{-x}$, so that $|f^{(n)}(x)| = f(x)$. Now, $f(x)$ is a decreasing function for $x \geq 0$, so that for any $c > 0$, $|f^{(n)}(x)|$ takes its maximum value at $x = 0$; this value is $e^0 = 1$. Thus we may take $K = 1$ in the error bound equation. Thus for any x,

$$|f(x) - T_3(x)| \leq K\frac{|x-0|^4}{4!} = \frac{x^4}{24}$$

A plot of $f(x) - T_3(x)$ and $\frac{x^4}{24}$ is shown below.



46. Use the error bound with $n = 4$ to show that

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{|x|^5}{120} \quad \text{(for all } x\text{)}$$

SOLUTION Note that all derivatives of $\sin x$ are either $\pm\cos x$ or $\pm\sin x$ so are bounded in absolute value by 1. Thus we may take $K = 1$ in the Error Bound. Now,

$$T_4(x) = x - \frac{x^3}{3!}$$

so that

$$|\sin x - T_4(x)| = \left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq K\frac{|x-0|^5}{5!} = \frac{|x|^5}{120}$$

47. Let $T_n(x)$ be the Taylor polynomial for $f(x) = \ln x$ at $a = 1$, and let $c > 1$. Show that

$$|\ln c - T_n(c)| \leq \frac{|c-1|^{n+1}}{n+1}$$

Then find a value of n such that $|\ln 1.5 - T_n(1.5)| \leq 10^{-2}$.

SOLUTION With $f(x) = \ln x$, we have

$$f'(x) = x^{-1},\ f''(x) = -x^{-2},\ f'''(x) = 2x^{-3},\ f^{(4)}(x) = -6x^{-4},$$

and, in general,

$$f^{(k+1)}(x) = (-1)^k k!\, x^{-k-1}.$$

Notice that $|f^{(k+1)}(x)| = k!|x|^{-k-1}$ is a decreasing function for $x > 0$. Therefore, the maximum value of $|f^{(k+1)}(x)|$ on $[1, c]$ is $|f^{(k+1)}(1)|$. Using the Error Bound, we have

$$|\ln c - T_n(c)| \leq K\frac{|c-1|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(x)| \leq K$ for x between 1 and c. From part (a), we know that we may take $K = |f^{(n+1)}(1)| = n!$. Then

$$|\ln c - T_n(c)| \leq n!\frac{|c-1|^{n+1}}{(n+1)!} = \frac{|c-1|^{n+1}}{n+1}.$$

Evaluating at $c = 1.5$ gives

$$|\ln 1.5 - T_n(1.5)| \leq \frac{|1.5-1|^{n+1}}{n+1} = \frac{(0.5)^{n+1}}{n+1}.$$

With $n = 3$,

$$\frac{(0.5)^4}{4} = 0.015625 > 10^{-2}.$$

but with $n = 4$,

$$\frac{(0.5)^5}{5} = 0.00625 < 10^{-2}.$$

Hence, $n = 4$ will guarantee the desired accuracy.

48. Let $n \geq 1$. Show that if $|x|$ is small, then

$$(x+1)^{1/n} \approx 1 + \frac{x}{n} + \frac{1-n}{2n^2}x^2$$

Use this approximation with $n = 6$ to estimate $1.5^{1/6}$.

SOLUTION Let $f(x) = (x+1)^{1/n}$. Then

$$\begin{aligned} f(x) &= (x+1)^{1/n} & f(0) &= 1 \\ f'(x) &= \frac{1}{n}(x+1)^{1/n-1} & f'(0) &= \frac{1}{n} \\ f''(x) &= \frac{1}{n}\left(\frac{1}{n}-1\right)(x+1)^{1/n-2} & f''(0) &= \frac{1}{n}\left(\frac{1}{n}-1\right) \end{aligned}$$

and

$$T_2(x) = 1 + \frac{1}{n}(x) + \left(\frac{1}{n^2} - \frac{1}{n}\right)\frac{x^2}{2} = 1 + \frac{x}{n} + \left(\frac{1-n}{2n^2}\right)x^2.$$

With $n = 6$ and $x = 0.5$,

$$1.5^{1/6} \approx T_2(0.5) = \frac{307}{288} \approx 1.065972.$$

49. Verify that the third Maclaurin polynomial for $f(x) = e^x \sin x$ is equal to the product of the third Maclaurin polynomials of e^x and $\sin x$ (after discarding terms of degree greater than 3 in the product).

SOLUTION Let $f(x) = e^x \sin x$. Then

$$\begin{aligned} f(x) &= e^x \sin x & f(0) &= 0 \\ f'(x) &= e^x(\cos x + \sin x) & f'(0) &= 1 \\ f''(x) &= 2e^x \cos x & f''(0) &= 2 \\ f'''(x) &= 2e^x(\cos x - \sin x) & f'''(0) &= 2 \end{aligned}$$

and

$$T_3(x) = 0 + (1)x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 = x + x^2 + \frac{x^3}{3}.$$

Now, the third Maclaurin polynomial for e^x is $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, and the third Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6}$. Multiplying these two polynomials, and then discarding terms of degree greater than 3, yields

$$e^x \sin x \approx x + x^2 + \frac{x^3}{3},$$

which agrees with the Maclaurin polynomial obtained from the definition.

50. Find the fourth Maclaurin polynomial for $f(x) = \sin x \cos x$ by multiplying the fourth Maclaurin polynomials for $f(x) = \sin x$ and $f(x) = \cos x$.

SOLUTION The fourth Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6}$, and the fourth Maclaurin polynomial for $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{24}$. Multiplying these two polynomials, and then discarding terms of degree greater than 4, we find that the fourth Maclaurin polynomial for $f(x) = \sin x \cos x$ is

$$T_4(x) = x - \frac{2x^3}{3}.$$

51. Find the Maclaurin polynomials $T_n(x)$ for $f(x) = \cos(x^2)$. You may use the fact that $T_n(x)$ is equal to the sum of the terms up to degree n obtained by substituting x^2 for x in the nth Maclaurin polynomial of $\cos x$.

SOLUTION The Maclaurin polynomials for $\cos x$ are of the form

$$T_{2n}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Accordingly, the Maclaurin polynomials for $\cos(x^2)$ are of the form

$$T_{4n}(x) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!}.$$

52. Find the Maclaurin polynomials of $1/(1 + x^2)$ by substituting $-x^2$ for x in the Maclaurin polynomials of $1/(1 - x)$.

SOLUTION The Maclaurin polynomials for $\frac{1}{1-x}$ are of the form

$$T_n(x) = 1 + x + x^2 + \cdots + x^n.$$

Accordingly, the Maclaurin polynomials for $\frac{1}{1+x^2}$ are of the form

$$T_{2n}(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n.$$

53. Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_j(x)$ for $j = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.

SOLUTION Let $f(x) = 3x^3 + 2x^2 - x - 4$. Then

$$\begin{array}{lll}
f(x) = 3x^3 + 2x^2 - x - 4 & f(0) = -4 & f(1) = 0 \\
f'(x) = 9x^2 + 4x - 1 & f'(0) = -1 & f'(1) = 12 \\
f''(x) = 18x + 4 & f''(0) = 4 & f''(1) = 22 \\
f'''(x) = 18 & f'''(0) = 18 & f'''(1) = 18 \\
f^{(4)}(x) = 0 & f^{(4)}(0) = 0 & f^{(4)}(1) = 0 \\
f^{(5)}(x) = 0 & f^{(5)}(0) = 0 & f^{(5)}(1) = 0
\end{array}$$

At $a = 0$,

$$\begin{aligned}
&T_1(x) = -4 - x; \\
&T_2(x) = -4 - x + 2x^2; \\
&T_3(x) = -4 - x + 2x^2 + 3x^3 = f(x); \\
&T_4(x) = T_3(x); \text{ and} \\
&T_5(x) = T_3(x).
\end{aligned}$$

At $a = 1$,

$$T_1(x) = 12(x-1);$$
$$T_2(x) = 12(x-1) + 11(x-1)^2;$$
$$T_3(x) = 12(x-1) + 11(x-1)^2 + 3(x-1)^3 = -4 - x + 2x^2 + 3x^3 = f(x);$$
$$T_4(x) = T_3(x); \text{ and}$$
$$T_5(x) = T_3(x).$$

54. Let $T_n(x)$ be the nth Taylor polynomial at $x = a$ for a polynomial $f(x)$ of degree n. Based on the result of Exercise 53, guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

SOLUTION Based on Exercise 53, we expect $|f(x) - T_n(x)| = 0$. From the Error Bound,

$$|f(x) - T_n(x)| \le K \frac{|x-a|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(u)| \le K$ for u between a and x. Since $f^{(n+1)}(x) = 0$ for an nth degree polynomial, we may take $K = 0$; the Error Bound then becomes $|f(x) - T_n(x)| = 0$.

55. Let $s(t)$ be the distance of a truck to an intersection. At time $t = 0$, the truck is 60 meters from the intersection, is traveling at a velocity of 24 m/s, and begins to slow down with an acceleration of $a = -3$ m/s^2. Determine the second Maclaurin polynomial of $s(t)$, and use it to estimate the truck's distance from the intersection after 4 s.

SOLUTION Place the origin at the intersection, so that $s(0) = 60$ (the truck is traveling away from the intersection). The second Maclaurin polynomial of $s(t)$ is

$$T_2(t) = s(0) + s'(0)t + \frac{s''(0)}{2}t^2$$

The conditions of the problem tell us that $s(0) = 60$, $s'(0) = 24$, and $s''(0) = -3$. Thus

$$T_2(t) = 60 + 24t - \frac{3}{2}t^2$$

so that after 4 seconds,

$$T_2(4) = 60 + 24 \cdot 4 - \frac{3}{2} \cdot 4^2 = 132 \text{ m}$$

The truck is 132 m past the intersection.

56. A bank owns a portfolio of bonds whose value $P(r)$ depends on the interest rate r (measured in percent; for example, $r = 5$ means a 5% interest rate). The bank's quantitative analyst determines that

$$P(5) = 100{,}000, \qquad \left.\frac{dP}{dr}\right|_{r=5} = -40{,}000, \qquad \left.\frac{d^2P}{dr^2}\right|_{r=5} = 50{,}000$$

In finance, this second derivative is called **bond convexity**. Find the second Taylor polynomial of $P(r)$ centered at $r = 5$ and use it to estimate the value of the portfolio if the interest rate moves to $r = 5.5\%$.

SOLUTION The second Taylor polynomial of $P(r)$ at $r = 5$ is

$$T_2(r) = P(5) + P'(5)(r-5) + \frac{P''(5)}{2}(r-5)^2$$

From the conditions of the problem, $P(5) = 100{,}000$, $P'(5) = -40{,}000$, and $P''(5) = 50{,}000$, so that

$$T_2(r) = 100{,}000 - 40{,}000(r-5) + 25{,}000(r-5)^2$$

If the interest rate moves to 5.5%, then the value of the portfolio can be estimated by

$$T_2(5.5) = 100{,}000 - 40{,}000(0.5) + 25{,}000(0.5)^2 = 86{,}250$$

57. A narrow, negatively charged ring of radius R exerts a force on a positively charged particle P located at distance x above the center of the ring of magnitude

$$F(x) = -\frac{kx}{(x^2+R^2)^{3/2}}$$

where $k > 0$ is a constant (Figure 12).

(a) Compute the third-degree Maclaurin polynomial for $F(x)$.

(b) Show that $F \approx -(k/R^3)x$ to second order. This shows that when x is small, $F(x)$ behaves like a restoring force similar to the force exerted by a spring.

(c) Show that $F(x) \approx -k/x^2$ when x is large by showing that

$$\lim_{x\to\infty} \frac{F(x)}{-k/x^2} = 1$$

Thus, $F(x)$ behaves like an inverse square law, and the charged ring looks like a point charge from far away.

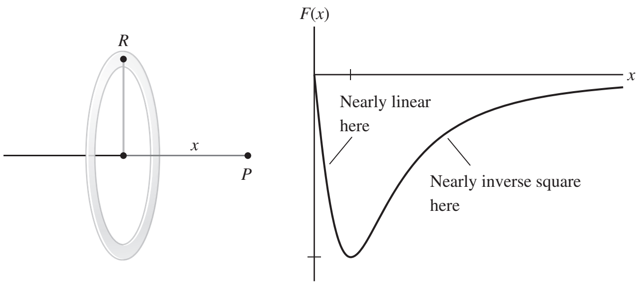


FIGURE 12

SOLUTION

(a) Start by computing and evaluating the necessary derivatives:

$$F(x) = -\frac{kx}{(x^2+R^2)^{3/2}} \qquad F(0) = 0$$

$$F'(x) = \frac{k(2x^2-R^2)}{(x^2+R^2)^{5/2}} \qquad F'(0) = -\frac{k}{R^3}$$

$$F''(x) = \frac{3kx(3R^2-2x^2)}{(x^2+R^2)^{7/2}} \qquad F''(0) = 0$$

$$F'''(x) = \frac{3k(8x^4-24x^2R^2+3R^4)}{(x^2+R^2)^{9/2}} \qquad F'''(0) = \frac{9k}{R^5}$$

so that

$$T_3(x) = F(0) + F'(0)x + \frac{F''(0)}{2!}x^2 + \frac{F'''(0)}{3!}x^3 = -\frac{k}{R^3}x + \frac{3k}{2R^5}x^3$$

(b) To degree 2, $F(x) \approx T_3(x) \approx -\frac{k}{R^3}x$ as we may ignore the x^3 term of $T_3(x)$.

(c) We have

$$\lim_{x\to\infty}\frac{F(x)}{-k/x^2} = \lim_{x\to\infty}\left(-\frac{x^2}{k}\cdot\frac{-kx}{(x^2+R^2)^{3/2}}\right) = \lim_{x\to\infty}\frac{x^3}{(x^2+R^2)^{3/2}}$$

$$= \lim_{x\to\infty}\frac{1}{x^{-3}(x^2+R^2)^{3/2}} = \lim_{x\to\infty}\frac{1}{(1+R^2/x^2)^{3/2}}$$

$$= 1$$

Thus as x grows large, $F(x)$ looks like an inverse square function.

58. A light wave of wavelength λ travels from A to B by passing through an aperture (circular region) located in a plane that is perpendicular to $\overline{AB}$ (see Figure 13 for the notation). Let $f(r) = d' + h'$; that is, $f(r)$ is the distance $AC + CB$ as a function of r.

(a) Show that $f(r) = \sqrt{d^2+r^2} + \sqrt{h^2+r^2}$, and use the Maclaurin polynomial of order 2 to show that

$$f(r) \approx d + h + \frac{1}{2}\left(\frac{1}{d}+\frac{1}{h}\right)r^2$$

(b) The **Fresnel zones**, used to determine the optical disturbance at B, are the concentric bands bounded by the circles of radius R_n such that $f(R_n) = d + h + n\lambda/2$. Show that $R_n \approx \sqrt{n\lambda L}$, where $L = (d^{-1}+h^{-1})^{-1}$.

(c) Estimate the radii R_1 and R_{100} for blue light ($\lambda = 475 \times 10^{-7}$ cm) if $d = h = 100$ cm.

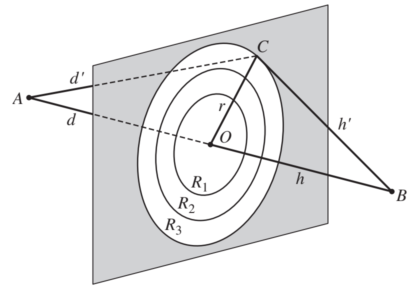


FIGURE 13 The Fresnel zones are the regions between the circles of radius R_n.

SOLUTION

(a) From the diagram, we see that $\overline{AC} = \sqrt{d^2 + r^2}$ and $\overline{CB} = \sqrt{h^2 + r^2}$. Therefore, $f(r) = \sqrt{d^2 + r^2} + \sqrt{h^2 + r^2}$. Moreover,

$$f'(r) = \frac{r}{\sqrt{d^2 + r^2}} + \frac{r}{\sqrt{h^2 + r^2}}, \quad f''(r) = \frac{d^2}{(d^2 + r^2)^{3/2}} + \frac{h^2}{(h^2 + r^2)^{3/2}},$$

$f(0) = d + h$, $f'(0) = 0$ and $f''(0) = d^{-1} + h^{-1}$. Thus,

$$f(r) \approx T_2(r) = d + h + \frac{1}{2}\left(\frac{1}{d} + \frac{1}{h}\right) r^2.$$

(b) Solving

$$f(R_n) \approx d + h + \frac{1}{2}\left(\frac{1}{d} + \frac{1}{h}\right) R_n^2 = d + h + \frac{n\lambda}{2}$$

yields

$$R_n = \sqrt{n\lambda(d^{-1} + h^{-1})^{-1}} = \sqrt{n\lambda L},$$

where $L = (d^{-1} + h^{-1})^{-1}$.

(c) With $d = h = 100$ cm, $L = 50$ cm. Taking $\lambda = 475 \times 10^{-7}$ cm, it follows that

$$R_1 \approx \sqrt{\lambda L} = 0.04873 \text{ cm; and}$$
$$R_{100} \approx \sqrt{100\lambda L} = 0.4873 \text{ cm.}$$

59. Referring to Figure 14, let a be the length of the chord $\overline{AC}$ of angle θ of the unit circle. Derive the following approximation for the excess of the arc over the chord.

$$\theta - a \approx \frac{\theta^3}{24}$$

Hint: Show that $\theta - a = \theta - 2\sin(\theta/2)$ and use the third Maclaurin polynomial as an approximation.

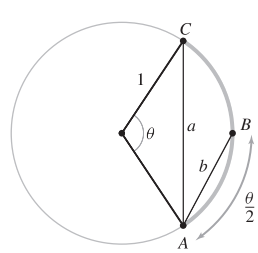


FIGURE 14 Unit circle.

SOLUTION Draw a line from the center O of the circle to B, and label the point of intersection of this line with AC as D. Then $CD = \frac{a}{2}$, and the angle COB is $\frac{\theta}{2}$. Since $CO = 1$, we have

$$\sin\frac{\theta}{2} = \frac{a}{2}$$

so that $a = 2\sin(\theta/2)$. Thus $\theta - a = \theta - 2\sin(\theta/2)$. Now, the third Maclaurin polynomial for $f(\theta) = \sin(\theta/2)$ can be computed as follows: $f(0) = 0$, $f'(x) = \frac{1}{2}\cos(\theta/2)$ so that $f'(0) = \frac{1}{2}$. $f''(x) = -\frac{1}{4}\sin(\theta/2)$ and $f''(0) = 0$. Finally, $f'''(x) = -\frac{1}{8}\cos(\theta/2)$ and $f'''(0) = -\frac{1}{8}$. Thus

$$T_3(\theta) = f(0) + f'(0)\theta + \frac{f''(0)}{2!}\theta^2 + \frac{f'''(0)}{3!}\theta^3 = \frac{1}{2}\theta - \frac{1}{48}\theta^3$$

Finally,

$$\theta - a = \theta - 2\sin\frac{\theta}{2} \approx \theta - 2T_3(\theta) = \theta - \left(\theta - \frac{1}{24}\theta^3\right) = \frac{\theta^3}{24}$$

60. To estimate the length θ of a circular arc of the unit circle, the seventeenth-century Dutch scientist Christian Huygens used the approximation $\theta \approx (8b - a)/3$, where a is the length of the chord $\overline{AC}$ of angle θ and b is length of the chord $\overline{AB}$ of angle $\theta/2$ (Figure 14).

(a) Prove that $a = 2\sin(\theta/2)$ and $b = 2\sin(\theta/4)$, and show that the Huygens approximation amounts to the approximation

$$\theta \approx \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}$$

(b) Compute the fifth Maclaurin polynomial of the function on the right.

(c) Use the error bound to show that the error in the Huygens approximation is less than $0.00022|\theta|^5$.

SOLUTION

(a) By the Law of Cosines and the identity $\sin^2(\theta/2) = (1 - \cos\theta)/2$:

$$a^2 = 1^2 + 1^2 - 2\cos\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$$

and so $a = 2\sin(\theta/2)$. Similarly, $b = 2\sin(\theta/4)$. Substituting these expressions for a and b into the Huygens approximation yields

$$\theta \approx \frac{8}{3}\cdot 2\sin\frac{\theta}{4} - \frac{1}{3}\cdot 2\sin\frac{\theta}{2} = \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}.$$

(b) The fifth Maclaurin polynomial for $\sin x$ is $x - \frac{x^3}{6} + \frac{x^5}{120}$; therefore, the fifth Maclaurin polynomial for $\sin(\theta/2)$ is

$$\frac{\theta}{2} - \frac{(\theta/2)^3}{6} + \frac{(\theta/2)^5}{120} = \frac{\theta}{2} - \frac{\theta^3}{48} + \frac{\theta^5}{3840},$$

and the fifth Maclaurin polynomial for $\sin(\theta/4)$ is

$$\frac{\theta}{4} - \frac{(\theta/4)^3}{6} + \frac{(\theta/4)^5}{120} = \frac{\theta}{4} - \frac{\theta^3}{384} + \frac{\theta^5}{122{,}880}.$$

Thus, the fifth Maclaurin polynomial for $f(\theta) = \frac{16}{3}\sin\frac{\theta}{4} - \frac{2}{3}\sin\frac{\theta}{2}$ is

$$\theta - \frac{1}{7680}\theta^5.$$

(c) Based on the result from part (b), the Huygens approximation for θ is equal to the fourth Maclaurin polynomial $T_4(\theta)$ for $f(\theta)$, and the error is at most $K|\theta|^5/5!$, where K is the maximum value of the absolute value of the fifth derivative $f^{(5)}(\theta)$. Because

$$f^{(5)}(\theta) = \frac{1}{192}\cos\frac{\theta}{4} - \frac{1}{48}\cos\frac{\theta}{2},$$

we may take $K = 1/48 + 1/192 = 0.0260417$, so the error is at most $|\theta|^5$ times the constant

$$\frac{0.0261}{5!} = 0.00022.$$

Further Insights and Challenges

61. Show that the nth Maclaurin polynomial of $f(x) = \arcsin x$ for n odd is

$$T_n(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdot 5\cdots(n-2)}{2\cdot 4\cdot 6\cdots(n-1)}\frac{x^n}{n}$$

SOLUTION Let $f(x) = \sin^{-1} x$. Then

$$\begin{aligned}
f(x) &= \sin^{-1} x & f(0) &= 0 \\
f'(x) &= \frac{1}{\sqrt{1-x^2}} & f'(0) &= 1 \\
f''(x) &= -\frac{1}{2}(1-x^2)^{-3/2}(-2x) & f''(0) &= 0 \\
f'''(x) &= \frac{2x^2+1}{(1-x^2)^{5/2}} & f'''(0) &= 1 \\
f^{(4)}(x) &= \frac{-3x(2x^2+3)}{(1-x^2)^{7/2}} & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \frac{24x^4+72x^2+9}{(1-x^2)^{9/2}} & f^{(5)}(0) &= 9 \\
&\vdots & &\vdots \\
& & f^{(7)}(0) &= 225
\end{aligned}$$

and

$$T_7(x) = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \frac{225x^7}{7!} = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}.$$

Thus, we can infer that

$$T_n(x) = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots + \frac{1}{2}\frac{3}{4}\cdots\frac{n-2}{n-1}\frac{x^n}{n}.$$

62. Let $x \ge 0$ and assume that $f^{(n+1)}(t) \ge 0$ for $0 \le t \le x$. Use Taylor's Theorem to show that the nth Maclaurin polynomial $T_n(x)$ satisfies

$$T_n(x) \le f(x) \quad \text{for all } x \ge 0$$

SOLUTION From Taylor's Theorem,

$$R_n(x) = f(x) - T_n(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du.$$

If $f^{(n+1)}(t) \ge 0$ for all t then

$$\frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du \ge 0$$

since $(x-u)^n \ge 0$ for $0 \le u \le x$. Thus, $f(x) - T_n(x) \ge 0$, or $f(x) \ge T_n(x)$.

63. Use Exercise 62 to show that for $x \ge 0$ and all n,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Sketch the graphs of e^x, $T_1(x)$, and $T_2(x)$ on the same coordinate axes. Does this inequality remain true for $x < 0$?

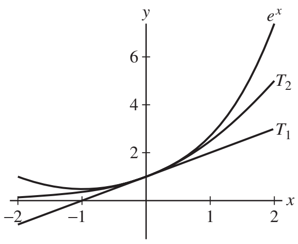
SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for all n. Because $e^x > 0$ for all x, it follows from Exercise 62 that $f(x) \ge T_n(x)$ for all $x \ge 0$ and for all n. For $f(x) = e^x$,

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

thus,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

From the figure below, we see that the inequality does not remain true for $x < 0$, as $T_2(x) \ge e^x$ for $x < 0$.



64. This exercise is intended to reinforce the proof of Taylor's Theorem.

(a) Show that $f(x) = T_0(x) + \int_a^x f'(u)\,du$.

(b) Use Integration by Parts to prove the formula

$$\int_a^x (x-u) f^{(2)}(u)\,du = -f'(a)(x-a) + \int_a^x f'(u)\,du$$

(c) Prove the case $n = 2$ of Taylor's Theorem:

$$f(x) = T_1(x) + \int_a^x (x-u) f^{(2)}(u)\,du.$$

SOLUTION

(a)

$$T_0(x) + \int_a^x f'(u)\,du = T_0(x) + f(x) - f(a) \quad \text{(from FTC2)}$$
$$= f(a) + f(x) - f(a) = f(x).$$

(b) Using Integration by Parts with $w = x - u$ and $v' = f''(u)\,du$,

$$\int_a^x (x-u) f''(u)\,du = f'(u)(x-u)\Big|_a^x + \int_a^x f'(u)\,du$$
$$= f'(x)(x-x) - f'(a)(x-a) + \int_a^x f'(u)\,du$$
$$= -f'(a)(x-a) + \int_a^x f'(u)\,du.$$

(c)

$$T_1(x) + \int_a^x (x-u) f''(u)\,du = f(a) + f'(a)(x-a) + \big(-f'(a)(x-a)\big) + \int_a^x f'(u)\,du$$
$$= f(a) + f(x) - f(a) = f(x).$$

In Exercises 65–69, we estimate integrals using Taylor polynomials. Exercise 66 is used to estimate the error.

65. Find the fourth Maclaurin polynomial $T_4(x)$ for $f(x) = e^{-x^2}$, and calculate $I = \int_0^{1/2} T_4(x)\,dx$ as an estimate $\int_0^{1/2} e^{-x^2}\,dx$. A CAS yields the value $I \approx 0.4794255$. How large is the error in your approximation? *Hint:* $T_4(x)$ is obtained by substituting $-x^2$ in the second Maclaurin polynomial for e^x.

SOLUTION Following the hint, since the second Maclaurin polynomial for e^x is

$$1 + x + \frac{x^2}{2}$$

we substitute $-x^2$ for x to get the fourth Maclaurin polynomial for e^{x^2}:

$$T_4(x) = 1 - x^2 + \frac{x^4}{2}$$

Then

$$\int_0^{1/2} e^{-x^2}\,dx \approx \int_0^{1/2} T_4(x)\,dx = \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right)\Big|_0^{1/2} = \frac{443}{960} \approx 0.4614583333$$

Using a CAS, we have $\int_0^{1/2} e^{-x^2}\,dx \approx 0.4612810064$, so the error is about 1.77×10^{-4}.

66. Approximating Integrals Let $L > 0$. Show that if two functions $f(x)$ and $g(x)$ satisfy $|f(x) - g(x)| < L$ for all $x \in [a, b]$, then

$$\left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < L(b-a)$$

SOLUTION Because $f(x) - g(x) \le |f(x) - g(x)|$, it follows that

$$\left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| = \left| \int_a^b (f(x) - g(x))\,dx \right| \le \int_a^b |f(x) - g(x)|\,dx$$
$$< \int_a^b L\,dx = L(b-a).$$

67. Let $T_4(x)$ be the fourth Maclaurin polynomial for $\cos x$.

(a) Show that $|\cos x - T_4(x)| \le \left(\frac{1}{2}\right)^6/6!$ for all $x \in \left[0, \frac{1}{2}\right]$. *Hint:* $T_4(x) = T_5(x)$.

(b) Evaluate $\int_0^{1/2} T_4(x)\,dx$ as an approximation to $\int_0^{1/2} \cos x\,dx$. Use Exercise 66 to find a bound for the size of the error.

SOLUTION

(a) Let $f(x) = \cos x$. Then

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Moreover, with $a = 0$, $T_4(x) = T_5(x)$ and

$$|\cos x - T_4(x)| \le K\frac{|x|^6}{6!},$$

where K is a number such that $|f^{(6)}(u)| \le K$ for u between 0 and x. Now $|f^{(6)}(u)| = |\cos u| \le 1$, so we may take $K = 1$. Finally, with the restriction $x \in [0, \frac{1}{2}]$,

$$|\cos x - T_4(x)| \le \frac{(1/2)^6}{6!} \approx 0.000022.$$

(b)

$$\int_0^{1/2} \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) dx = \frac{1841}{3840} \approx 0.479427.$$

By (a) and Exercise 66, the error associated with this approximation is less than or equal to

$$\frac{(1/2)^6}{6!}\left(\frac{1}{2} - 0\right) = \frac{1}{92{,}160} \approx 1.1 \times 10^{-5}.$$

Note that $\displaystyle\int_0^{1/2} \cos x\,dx \approx 0.4794255$, so the actual error is roughly 1.5×10^{-6}.

68. Let $Q(x) = 1 - x^2/6$. Use the error bound for $\sin x$ to show that

$$\left| \frac{\sin x}{x} - Q(x) \right| \le \frac{|x|^4}{5!}$$

Then calculate $\int_0^1 Q(x)\,dx$ as an approximation to $\int_0^1 (\sin x/x)\,dx$ and find a bound for the error.

SOLUTION The third Maclaurin polynomial for $\sin x$ is

$$T_3(x) = x - \frac{1}{3!}x^3 = x - \frac{1}{6}x^3 = xQ(x)$$

Additionally, this is also $T_4(x)$ since $(\sin x)^{(4)}(0) = 0$. All derivatives of $\sin x$ are either $\pm\sin x$ or $\pm\cos x$, which are bounded in absolute value by 1. Thus we may take $K = 1$ in the Error Bound, so

$$|\sin x - xQ(x)| = |\sin x - T_3(x)| = |\sin x - T_4(x)| \le K\frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Divide both sides of this inequality by $|x|$ to get

$$\left|\frac{\sin x}{x} - Q(x)\right| \le \frac{|x|^4}{5!}$$

We can thus estimate $\int_0^1 (\sin x/x)\,dx$ by

$$\int_0^1 Q(x)\,dx = \int_0^1 1 - \frac{x^2}{6}\,dx = \left(x - \frac{x^3}{18}\right)\Bigg|_0^1 = \frac{17}{18} \approx 0.9444444444$$

The error in this approximation is at most

$$\frac{|1|^4}{5!} = \frac{1}{120} \approx 0.008333333333$$

The true value of the integral is approximately 0.9460830704, which is consistent with the error bound.

69. **(a)** Compute the sixth Maclaurin polynomial $T_6(x)$ for $\sin(x^2)$ by substituting x^2 in $P(x) = x - x^3/6$, the third Maclaurin polynomial for $\sin x$.

(b) Show that $|\sin(x^2) - T_6(x)| \le \dfrac{|x|^{10}}{5!}$.

Hint: Substitute x^2 for x in the error bound for $|\sin x - P(x)|$, noting that $P(x)$ is also the fourth Maclaurin polynomial for $\sin x$.

(c) Use $T_6(x)$ to approximate $\displaystyle\int_0^{1/2} \sin(x^2)\,dx$ and find a bound for the error.

SOLUTION Let $s(x) = \sin x$ and $f(x) = \sin(x^2)$. Then

(a) The third Maclaurin polynomial for $\sin x$ is

$$S_3(x) = x - \frac{x^3}{6}$$

so, substituting x^2 for x, we see that the sixth Maclaurin polynomial for $\sin(x^2)$ is

$$T_6(x) = x^2 - \frac{x^6}{6}$$

(b) Since all derivatives of $s(x)$ are either $\pm\cos x$ or $\pm\sin x$, they are bounded in magnitude by 1, so we may take $K = 1$ in the Error Bound for $\sin x$. Since the third Maclaurin polynomial $S_3(x)$ for $\sin x$ is also the fourth Maclaurin polynomial $S_4(x)$, we have

$$|\sin x - S_3(x)| = |\sin x - S_4(x)| \le K\frac{|x|^5}{5!} = \frac{|x|^5}{5!}$$

Now substitute x^2 for x in the above inequality and note from part (a) that $S_3(x^2) = T_6(x)$ to get

$$|\sin(x^2) - S_3(x^2)| = |\sin(x^2) - T_6(x)| \le \frac{|x^2|^5}{5!} = \frac{|x|^{10}}{5!}$$

(c)

$$\int_0^{1/2} \sin(x^2)\,dx \approx \int_0^{1/2} T_6(x)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{42}x^7\right)\Bigg|_0^{1/2} \approx 0.04148065476$$

From part (b), the error is bounded by

$$\frac{x^{10}}{5!} = \frac{(1/2)^{10}}{120} = \frac{1}{1024 \cdot 120} \approx 8.138020833 \times 10^{-6}$$

The true value of the integral is approximately 0.04148102420, which is consistent with the computed error bound.

70. Prove by induction that for all k,

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right) = \frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!}$$

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right)\Bigg|_{x=a} = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \ne j \end{cases}$$

Use this to prove that $T_n(x)$ agrees with $f(x)$ at $x = a$ to order n.

SOLUTION The first formula is clearly true for $j = 0$. Suppose the formula is true for an arbitrary j. Then

$$\frac{d^{j+1}}{dx^{j+1}}\left(\frac{(x-a)^k}{k!}\right) = \frac{d}{dx}\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right) = \frac{d}{dx}\left(\frac{k(k-1)\cdots(k-j+1)(x-a)^{k-j}}{k!}\right)$$
$$= \frac{k(k-1)\cdots(k-j+1)(k-(j+1)+1)(x-a)^{k-(j+1)}}{k!}$$

as desired. Note that if $k = j$, then the numerator is $k!$, the denominator is $k!$ and the value of the derivative is 1; otherwise, the value of the derivative is 0 at $x = a$. In other words,

$$\frac{d^j}{dx^j}\left(\frac{(x-a)^k}{k!}\right)\bigg|_{x=a} = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}$$

Applying this latter formula, it follows that

$$\frac{d^j}{dx^j}T_n(a)\bigg|_{x=a} = \sum_{k=0}^{n}\frac{d^j}{dx^j}\left(\frac{f^{(k)}(a)}{k!}(x-a)^k\right)\bigg|_{x=a} = f^{(j)}(a)$$

as required.

71. Let a be any number and let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0$$

be a polynomial of degree n or less.

(a) Show that if $P^{(j)}(a) = 0$ for $j = 0, 1, \ldots, n$, then $P(x) = 0$, that is, $a_j = 0$ for all j. *Hint:* Use induction, noting that if the statement is true for degree $n-1$, then $P'(x) = 0$.

(b) Prove that $T_n(x)$ is the only polynomial of degree n or less that agrees with $f(x)$ at $x = a$ to order n. *Hint:* If $Q(x)$ is another such polynomial, apply (a) to $P(x) = T_n(x) - Q(x)$.

SOLUTION

(a) Note first that if $n = 0$, i.e. if $P(x) = a_0$ is a constant, then the statement holds: if $P^{(0)}(a) = P(a) = 0$, then $a_0 = 0$ so that $P(x) = 0$. Next, assume the statement holds for all polynomials of degree $n-1$ or less, and let $P(x)$ be a polynomial of degree at most n with $P^{(j)}(a) = 0$ for $j = 0, 1, \ldots, n$. If $P(x)$ has degree less than n, then we know $P(x) = 0$ by induction, so assume the degree of $P(x)$ is exactly n. Then

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$; also,

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

Note that $P^{(j+1)}(a) = (P')^{(j)}(a)$ for $j = 0, 1, \ldots, n-1$. But then

$$0 = P^{(j+1)}(a) = (P')^{(j)}(a) \quad \text{for all } j = 0, 1, \ldots, n-1$$

Since $P'(x)$ has degree at most $n-1$, it follows by induction that $P'(x) = 0$. Thus $a_n = a_{n-1} = \cdots = a_1 = 0$ so that $P(x) = a_0$. But $P(a) = 0$ so that $a_0 = 0$ as well and thus $P(x) = 0$.

(b) Suppose $Q(x)$ is a polynomial of degree at most n that agrees with $f(x)$ at $x = a$ up to order n. Let $P(x) = T_n(x) - Q(x)$. Note that $P(x)$ is a polynomial of degree at most n since both $T_n(x)$ and $Q(x)$ are. Since both $T_n(x)$ and $Q(x)$ agree with $f(x)$ at $x = a$ to order n, we have

$$T_n^{(j)}(a) = f^{(j)}(a) = Q^{(j)}(a), \quad j = 0, 1, 2, \ldots, n$$

Thus

$$P^{(j)}(a) = T_n^{(j)}(a) - Q^{(j)}(a) = 0 \quad \text{for } j = 0, 1, 2, \ldots, n$$

But then by part (a), $P(x) = 0$ so that $T_n(x) = Q(x)$.

CHAPTER REVIEW EXERCISES

In Exercises 1–4, calculate the arc length over the given interval.

1. $y = \dfrac{x^5}{10} + \dfrac{x^{-3}}{6}, \quad [1, 2]$

SOLUTION Let $y = \dfrac{x^5}{10} + \dfrac{x^{-3}}{6}$. Then

$$1 + (y')^2 = 1 + \left(\frac{x^4}{2} - \frac{x^{-4}}{2}\right)^2 = 1 + \frac{x^8}{4} - \frac{1}{2} + \frac{x^{-8}}{4}$$
$$= \frac{x^8}{4} + \frac{1}{2} + \frac{x^{-8}}{4} = \left(\frac{x^4}{2} + \frac{x^{-4}}{2}\right)^2.$$

Because $\frac{1}{2}(x^4 + x^{-4}) > 0$ on $[1, 2]$, the arc length is

$$s = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(\frac{x^4}{2} + \frac{x^{-4}}{2}\right) dx = \left(\frac{x^5}{10} - \frac{x^{-3}}{6}\right)\bigg|_1^2 = \frac{779}{240}.$$

2. $y = e^{x/2} + e^{-x/2}, \quad [0, 2]$

SOLUTION Let $y = e^{x/2} + e^{-x/2} = 2\cosh\frac{x}{2}$. Then, $y' = \sinh\frac{x}{2}$ and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \sinh^2\frac{x}{2}} = \sqrt{\cosh^2\left(\frac{x}{2}\right)} = \cosh\frac{x}{2}.$$

Thus,

$$s = \int_0^2 \cosh\left(\frac{x}{2}\right) dx = 2\sinh\left(\frac{x}{2}\right)\bigg|_0^2 = 2\left(\sinh\left(\frac{2}{2}\right) - \sinh(0)\right) = 2\sinh(1).$$

Alternately, $y' = \frac{1}{2}(e^{x/2} - e^{-x/2})$, so

$$1 + (y')^2 = \frac{1}{4}(e^x - 2 + e^{-x}) + 1 = \frac{1}{4}(e^x + 2 + e^{-x}) = \left[\frac{1}{2}(e^{x/2} + e^{-x/2})\right]^2.$$

Because $\frac{1}{2}(e^{x/2} + e^{-x/2}) > 0$ on $[0, 2]$,

$$s = \int_0^2 \frac{1}{2}(e^{x/2} + e^{-x/2})\, dx = (e^{x/2} - e^{-x/2})\bigg|_0^2 = e - e^{-1} = 2\sinh(1).$$

3. $y = 4x - 2, \quad [-2, 2]$

SOLUTION Let $y = 4x - 2$. Then

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4^2} = \sqrt{17}.$$

Hence,

$$s = \int_{-2}^2 \sqrt{17}\, dx = 4\sqrt{17}.$$

4. $y = x^{2/3}, \quad [1, 8]$

SOLUTION Let $y = x^{2/3}$. Then $y' = \frac{2}{3}x^{-1/3}$, and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4}{9}x^{-2/3}} = \sqrt{\frac{4}{9}x^{-2/3}\left(\frac{9}{4}x^{2/3} + 1\right)} = \frac{2}{3}x^{-1/3}\sqrt{1 + \frac{9}{4}x^{2/3}}.$$

The arc length is

$$s = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \frac{2}{3}x^{-1/3}\sqrt{1 + \frac{9}{4}x^{2/3}}\, dx.$$

Now, we make the substitution $u = 1 + \frac{9}{4}x^{2/3}$, $du = \frac{3}{2}x^{-1/3}\,dx$. Then

$$s = \int_{13/4}^{10} \sqrt{u} \cdot \frac{4}{9}\,du = \frac{8}{27}u^{3/2}\bigg|_{13/4}^{10} = \frac{8}{27}\left[10^{3/2} - \left(\frac{\sqrt{13}}{2}\right)^3\right]$$

$$= \frac{8}{27}\left(10\sqrt{10} - \frac{13\sqrt{13}}{8}\right) \approx 7.633705415.$$

5. Show that the arc length of $y = 2\sqrt{x}$ over $[0, a]$ is equal to $\sqrt{a(a+1)} + \ln(\sqrt{a} + \sqrt{a+1})$. *Hint:* Apply the substitution $x = \tan^2\theta$ to the arc length integral.

SOLUTION Let $y = 2\sqrt{x}$. Then $y' = \frac{1}{\sqrt{x}}$, and

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{1}{\sqrt{x}}\sqrt{x+1}.$$

Thus,

$$s = \int_0^a \frac{1}{\sqrt{x}}\sqrt{1+x}\,dx.$$

We make the substitution $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$. Then

$$s = \int_{x=0}^{x=a} \frac{1}{\tan\theta}\sec\theta \cdot 2\tan\theta\sec^2\theta\,d\theta = 2\int_{x=0}^{x=a}\sec^3\theta\,d\theta.$$

We use a reduction formula to obtain

$$s = 2\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\bigg|_{x=0}^{x=a} = (\sqrt{x}\sqrt{1+x} + \ln|\sqrt{1+x} + \sqrt{x}|)\bigg|_0^a$$

$$= \sqrt{a}\sqrt{1+a} + \ln|\sqrt{1+a} + \sqrt{a}| = \sqrt{a(a+1)} + \ln\left(\sqrt{a} + \sqrt{a+1}\right).$$

6. CAS Compute the trapezoidal approximation T_5 to the arc length s of $y = \tan x$ over $\left[0, \frac{\pi}{4}\right]$.

SOLUTION Let $y = \tan x$. With $N = 5$, the subintervals are $\left[(i-1)\frac{\pi}{20}, i\frac{\pi}{20}\right]$, $i = 1, 2, 3, 4, 5$. Now,

$$1 + (y')^2 = 1 + (\sec^2 x)^2 = 1 + \sec^4 x$$

so the arc length is approximately

$$s = \int_1^{\pi/4}\sqrt{1 + \sec^4 x}\,dx$$

$$\approx \frac{\pi}{40}\left(\sqrt{1+\sec^4 0} + 2\sqrt{1+\sec^4\frac{\pi}{20}} + 2\sqrt{1+\sec^4\frac{\pi}{10}} + 2\sqrt{1+\sec^4\frac{3\pi}{20}} + 2\sqrt{1+\sec^4\frac{\pi}{5}}\right.$$
$$\left. + \sqrt{1+\sec^4\frac{\pi}{4}}\right)$$

$$\approx \frac{\pi}{40}(1.41421356 + 2\cdot 1.43206164 + 2\cdot 1.49073513 + 2\cdot 1.60830125 + 2\cdot 1.82602534 + 2.23606797)$$

$$\approx 1.285267058$$

In Exercises 7–10, calculate the surface area of the solid obtained by rotating the curve over the given interval about the x-axis.

7. $y = x + 1$, $[0, 4]$

SOLUTION Let $y = x + 1$. Then $y' = 1$, and

$$y\sqrt{1 + y'^2} = (x+1)\sqrt{1+1} = \sqrt{2}(x+1).$$

Thus,

$$SA = 2\pi\int_0^4 \sqrt{2}(x+1)\,dx = 2\sqrt{2}\pi\left(\frac{x^2}{2} + x\right)\bigg|_0^4 = 24\sqrt{2}\pi.$$

8. $y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}$, $[0, 1]$

SOLUTION Let $y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}$. Then

$$y' = \frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2},$$

and

$$1 + (y')^2 = 1 + \left(\frac{x^{-1/4}}{2} - \frac{x^{1/4}}{2}\right)^2 = \frac{x^{-1/2}}{4} + \frac{1}{2} + \frac{x^{1/2}}{4} = \left(\frac{x^{-1/4}}{2} + \frac{x^{1/4}}{2}\right)^2.$$

Because $\frac{1}{2}(x^{-1/4} + x^{1/4}) \geq 0$, the surface area is

$$\begin{aligned}2\pi \int_0^1 y\sqrt{1 + (y')^2}\, dy &= 2\pi \int_0^1 \left(\frac{2x^{3/4}}{3} - \frac{2x^{5/4}}{5}\right)\left(\frac{x^{1/4}}{2} + \frac{x^{-1/4}}{2}\right) dx \\ &= 2\pi \int_0^1 \left(-\frac{x^{3/2}}{5} - \frac{x}{5} + \frac{x}{3} + \frac{\sqrt{x}}{3}\right) dx \\ &= 2\pi \left(-\frac{2x^{5/2}}{25} + \frac{x^2}{15} + \frac{2x^{3/2}}{9}\right)\Bigg|_0^1 = \frac{94}{225}\pi.\end{aligned}$$

9. $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$, $[1, 2]$

SOLUTION Let $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$. Then

$$y' = \sqrt{x} - \frac{1}{4\sqrt{x}},$$

and

$$1 + \left(y'\right)^2 = 1 + \left(\sqrt{x} - \frac{1}{4\sqrt{x}}\right)^2 = 1 + \left(x - \frac{1}{2} + \frac{1}{16x}\right) = x + \frac{1}{2} + \frac{1}{16x} = \left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right)^2.$$

Because $\sqrt{x} + \frac{1}{\sqrt{x}} \geq 0$, the surface area is

$$\begin{aligned}2\pi \int_a^b y\sqrt{1 + \left(y'\right)^2}\, dx &= 2\pi \int_1^2 \left(\frac{2}{3}x^{3/2} - \frac{\sqrt{x}}{2}\right)\left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right) dx \\ &= 2\pi \int_1^2 \left(\frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{2}x - \frac{1}{8}\right) dx = 2\pi \left(\frac{2x^3}{9} - \frac{x^2}{6} - \frac{1}{8}x\right)\Bigg|_1^2 = \frac{67}{36}\pi.\end{aligned}$$

10. $y = \frac{1}{2}x^2$, $[0, 2]$

SOLUTION Let $y = \frac{1}{2}x^2$. Then $y' = x$ and

$$SA = 2\pi \int_0^2 \frac{1}{2}x^2\sqrt{1 + x^2}\, dx = \pi \int_0^2 x^2\sqrt{1 + x^2}\, dx.$$

Using the substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, we find that

$$\begin{aligned}\int x^2\sqrt{1 + x^2}\, dx &= \int \sec^3\theta \tan^2\theta\, d\theta = \int \left(\sec^5\theta - \sec^3\theta\right) d\theta \\ &= \left(\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| - \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C \\ &= \frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8}\sqrt{1 + x^2} - \frac{1}{8}\ln|\sqrt{1 + x^2} + x| + C.\end{aligned}$$

Finally,

$$\begin{aligned}SA &= \pi \left(\frac{x}{4}(1 + x^2)^{3/2} - \frac{x}{8}\sqrt{1 + x^2} - \frac{1}{8}\ln|\sqrt{1 + x^2} + x|\right)\Bigg|_0^2 \\ &= \pi \left(\frac{5\sqrt{5}}{2} - \frac{\sqrt{5}}{4} - \frac{1}{8}\ln(2 + \sqrt{5})\right) = \frac{9\sqrt{5}}{4}\pi - \frac{\pi}{8}\ln(2 + \sqrt{5}).\end{aligned}$$

11. Compute the total surface area of the coin obtained by rotating the region in Figure 1 about the x-axis. The top and bottom parts of the region are semicircles with a radius of 1 mm.

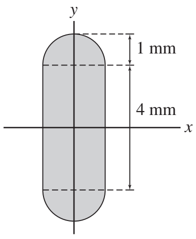


FIGURE 1

SOLUTION The generating half circle of the edge is $y = 2 + \sqrt{1 - x^2}$. Then,

$$y' = \frac{-2x}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}},$$

and

$$1 + (y')^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}.$$

The surface area of the edge of the coin is

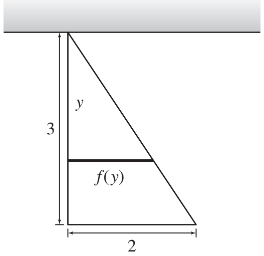
$$\begin{aligned} 2\pi \int_{-1}^{1} y\sqrt{1 + \left(y'\right)^2}\,dx &= 2\pi \int_{-1}^{1} \left(2 + \sqrt{1 - x^2}\right) \frac{1}{\sqrt{1 - x^2}}\,dx \\ &= 2\pi \left(2\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} + \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}\,dx\right) \\ &= 2\pi \left(2 \arcsin x\big|_{-1}^{1} + \int_{-1}^{1} dx\right) \\ &= 2\pi(2\pi + 2) = 4\pi^2 + 4\pi. \end{aligned}$$

We now add the surface area of the two sides of the disk, which are circles of radius 2. Hence the surface area of the coin is:

$$\left(4\pi^2 + 4\pi\right) + 2\pi \cdot 2^2 = 4\pi^2 + 12\pi.$$

12. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex at the surface of the water.

SOLUTION To find the fluid force, we must find an expression for the horizontal width $f(y)$ of the triangle at depth y.



By similar triangles we have:

$$\frac{y}{f(y)} = \frac{3}{2} \qquad \text{so} \qquad f(y) = \frac{2y}{3}.$$

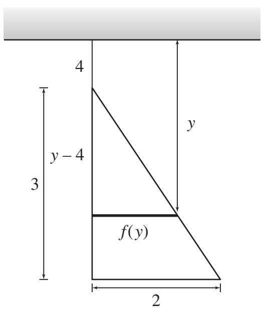
Therefore, the fluid force on the side of the triangle is

$$F = \rho g \int_0^3 y f(y)\,dy = \rho g \int_0^3 \frac{2y^2}{3}\,dy = \rho g \cdot \frac{2y^3}{9}\bigg|_0^3 = 6\rho g.$$

For water, $\rho = 10^3$; $g = 9.8$, so $F = 6 \cdot 9800 = 58{,}800$ N.

13. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex located at a depth of 4 m.

SOLUTION We need to find an expression for the horizontal width $f(y)$ at depth y.



By similar triangles we have:

$$\frac{f(y)}{y-4} = \frac{2}{3} \qquad \text{so} \qquad f(y) = \frac{2(y-4)}{3}.$$

Hence, the force on the side of the triangle is

$$F = \rho g \int_4^7 y f(y)\, dy = \frac{2\rho g}{3} \int_4^7 \left(y^2 - 4y\right)\, dy = \frac{2\rho g}{3}\left(\frac{y^3}{3} - 2y^2\right)\Big|_4^7 = 18\rho g.$$

For water, $\rho = 10^3$; $g = 9.8$, so $F = 18 \cdot 9800 = 176{,}400$ N.

14. A plate in the shape of the shaded region in Figure 2 is submerged in water. Calculate the fluid force on a side of the plate if the water surface is $y = 1$.

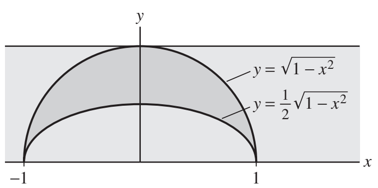


FIGURE 2

SOLUTION Here, we can proceed as follows: Calculate the force that would be exerted on the entire semicircle and then subtract the force that would be exerted on the "missing" portion of the ellipse. The force on the semicircle is

$$2w \int_0^1 (1-y)\sqrt{1-y^2}\, dy = 2w \int_0^1 \sqrt{1-y^2}\, dy - 2w \int_0^1 y\sqrt{1-y^2}\, dy.$$

The first integral can be interpreted as the area of one-quarter of a circle of radius 1. Hence,

$$\int_0^1 \sqrt{1-y^2}\, dy = \frac{\pi}{4}.$$

On the other hand,

$$\int_0^1 y\sqrt{1-y^2}\, dy = -\frac{1}{3}(1-y^2)^{3/2}\Big|_0^1 = \frac{1}{3}.$$

Thus, the force on the semicircle is

$$2w\left(\frac{\pi}{4} - \frac{1}{3}\right).$$

Now for the ellipse. The force that would be exerted on the upper half of the ellipse is

$$2w \int_0^{1/2} (1-y)\sqrt{1-4y^2}\, dy = 2w \int_0^{1/2} \sqrt{1-4y^2}\, dy - 2w \int_0^{1/2} y\sqrt{1-4y^2}\, dy.$$

Using the substitution $w = 2y$, $dw = 2\, dy$, it follows that

$$\int_0^{1/2} \sqrt{1-4y^2}\, dy = \frac{1}{2}\int_0^1 \sqrt{1-w^2}\, dw = \frac{\pi}{8},$$

and

$$\int_0^{1/2} y\sqrt{1-4y^2}\,dy = \frac{1}{4}\int_0^1 w\sqrt{1-w^2}\,dw = \frac{1}{12}.$$

Thus, the force on the "missing" ellipse is

$$2w\left(\frac{\pi}{8}-\frac{1}{12}\right).$$

Finally, the force exerted on the plate shown in Figure 2 is

$$F = 2w\left(\frac{\pi}{4}-\frac{1}{3}\right) - 2w\left(\frac{\pi}{8}-\frac{1}{12}\right) = \frac{\pi-2}{4}w.$$

15. Figure 3 shows an object whose face is an equilateral triangle with 5-m sides. The object is 2 m thick and is submerged in water with its vertex 3 m below the water surface. Calculate the fluid force on both a triangular face and a slanted rectangular edge of the object.

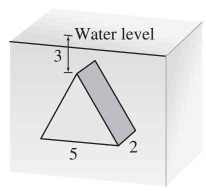


FIGURE 3

SOLUTION Start with each triangular face of the object. Place the origin at the upper vertex of the triangle, with the positive y-axis pointing downward. Note that because the equilateral triangle has sides of length 5 feet, the height of the triangle is $\frac{5\sqrt{3}}{2}$ feet. Moreover, the width of the triangle at location y is $\frac{2y}{\sqrt{3}}$. Thus,

$$F = \frac{2\rho g}{\sqrt{3}}\int_0^{5\sqrt{3}/2}(y+3)y\,dy = \frac{2\rho g}{\sqrt{3}}\left(\frac{1}{3}y^3+\frac{3}{2}y^2\right)\Big|_0^{5\sqrt{3}/2} = \frac{\rho g}{4}(125+75\sqrt{3}) \approx 624{,}514 \text{ N}.$$

Now, consider the slanted rectangular edges of the object. Each edge is a constant 2 feet wide and makes an angle of 60° with the horizontal. Therefore,

$$F = \frac{\rho g}{\sin 60^\circ}\int_0^{5\sqrt{3}/2} 2(y+3)\,dy = \frac{2\rho g}{\sqrt{3}}\left(y^2+6y\right)\Big|_0^{5\sqrt{3}/2} = \rho g\left(\frac{25\sqrt{3}}{2}+30\right) \approx 506{,}176 \text{ N}.$$

The force on the bottom face can be computed without calculus:

$$F = \left(3+\frac{5\sqrt{3}}{2}\right)(2)(5)\rho g \approx 718{,}352 \text{ N}.$$

16. The end of a horizontal oil tank is an ellipse (Figure 4) with equation $(x/4)^2+(y/3)^2=1$ (length in meters). Assume that the tank is filled with oil of density 900 kg/m^3.

(a) Calculate the total force F on the end of the tank when the tank is full.

(b) Would you expect the total force on the lower half of the tank to be greater than, less than, or equal to $\frac{1}{2}F$? Explain. Then compute the force on the lower half exactly and confirm (or refute) your expectation.

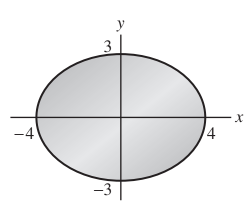


FIGURE 4

SOLUTION

(a) Solving the equation of the ellipse for x yields

$$x = \frac{4}{3}\sqrt{9-y^2}.$$

Therefore, a horizontal strip of the ellipse at height y has width $\frac{8}{3}\sqrt{9-y^2}$. This strip is at a depth of $3-y$, so the total force on the end of the tank is

$$F = \rho g \int_{-3}^{3} (3-y)\cdot\frac{8}{3}\sqrt{9-y^2}\,dy = 8\rho g \int_{-3}^{3}\sqrt{9-y^2}\,dy - \frac{8}{3}\rho g\int_{-3}^{3} y\sqrt{9-y^2}\,dy.$$

The first integral can be interpreted as the area of one-half of a circle of radius 3, so the value of this integral is $\frac{9\pi}{2}$. The second integral is zero, since the integrand is an odd function and the interval of integration is symmetric about zero. Hence,

$$F = 8\rho g\frac{9\pi}{2} - \frac{8}{3}\rho g(0) = 8\cdot 900\cdot 9.8\cdot\frac{9\pi}{2} \approx 997{,}518\text{ N}.$$

(b) The oil in the lower half of the tank is at a greater depth than the oil in the upper half, therefore we expect the total force F_l on the lower half of the tank to be greater than the total force F_u on the upper half. We compute the two forces to verify our expectation. Now,

$$F_l = \rho g \int_{-3}^{0} (3-y)\cdot\frac{8}{3}\sqrt{9-y^2}\,dy = 8\rho g \int_{-3}^{0}\sqrt{9-y^2}\,dy - \frac{8}{3}\rho g\int_{-3}^{0} y\sqrt{9-y^2}\,dy.$$

Similarly,

$$F_u = 8\rho g \int_{0}^{3}\sqrt{9-y^2}\,dy - \frac{8}{3}\rho g\int_{0}^{3} y\sqrt{9-y^2}\,dy.$$

The first integral in each expression,

$$\int_{-3}^{0}\sqrt{9-y^2}\,dy \qquad\text{and}\qquad \int_{0}^{3}\sqrt{9-y^2}\,dy,$$

can be interpreted as the area of one-quarter of a circle of radius 3, so both integrals have the value $\frac{9\pi}{4}$. Using the substitution $u = 9-y^2$, $du = -2y\,dy$ we find

$$\int_{-3}^{0} y\sqrt{9-y^2}\,dy = \int_{0}^{9}\sqrt{u}\left(-\frac{1}{2}\right)du = -\frac{1}{3}u^{3/2}\Big|_{0}^{9} = -9.$$

Moreover, since the integrand is an odd function, we have

$$\int_{0}^{3} y\sqrt{9-y^2}\,dy = -\int_{-3}^{0} y\sqrt{9-y^2}\,dy = 9.$$

Thus,

$$F_l = 8\rho g\frac{9\pi}{4} - \frac{8}{3}\rho g(-9) = (18\pi+24)\rho g;\ \text{and}$$
$$F_u = 8\rho g\frac{9\pi}{4} - \frac{8}{3}\rho g(9) = (18\pi-24)\rho g.$$

We see that $F_l > F_u$. That is, the total force on the lower half of the tank is greater than the total force on the upper half, as expected.

17. Calculate the moments and COM of the lamina occupying the region under $y = x(4-x)$ for $0 \le x \le 4$, assuming a density of $\rho = 1200\text{ kg/m}^3$.

SOLUTION Because the lamina is symmetric with respect to the vertical line $x = 2$, by the symmetry principle, we know that $x_{\rm cm} = 2$. Now,

$$M_x = \frac{\rho}{2}\int_0^4 f(x)^2\,dx = \frac{1200}{2}\int_0^4 x^2(4-x)^2\,dx = \frac{1200}{2}\left(\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right)\Big|_0^4 = 20{,}480.$$

Moreover, the mass of the lamina is

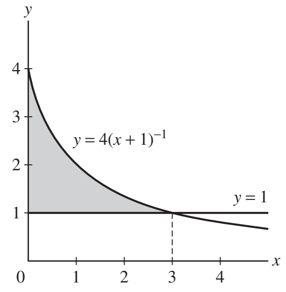
$$M = \rho\int_0^4 f(x)\,dx = 1200\int_0^4 x(4-x)\,dx = 1200\left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = 12{,}800.$$

Thus, the coordinates of the center of mass are

$$\left(2, \frac{20{,}480}{12{,}800}\right) = \left(2, \frac{8}{5}\right).$$

18. Sketch the region between $y = 4(x+1)^{-1}$ and $y = 1$ for $0 \le x \le 3$, and find its centroid.

SOLUTION



First, we calculate the moments:

$$M_x = \frac{1}{2}\int_0^3 \left(\frac{16}{(x+1)^2} - 1\right) dx = \frac{1}{2}\left(-\frac{16}{x+1} - x\right)\Big|_0^3 = \frac{9}{2},$$

and

$$\begin{aligned} M_y &= \int_0^3 x\left(4(x+1)^{-1} - 1\right) dx = \int_0^3 \left(\frac{4x}{x+1} - x\right) dx \\ &= \int_0^3 \left(\frac{4(x+1)-4}{x+1} - x\right) dx = \int_0^3 \left(4 - \frac{4}{x+1} - x\right) dx \\ &= \left(4x - \frac{x^2}{2} - 4\ln(x+1)\right)\Big|_0^3 = \frac{15}{2} - 4\ln 4. \end{aligned}$$

The area of the region is

$$A = \int_0^3 \left(\frac{4}{x+1} - 1\right) dx = \left(4\ln(x+1) - x\right)\big|_0^3 = 4\ln 4 - 3,$$

so the coordinates of the centroid are:

$$\left(\frac{15 - 8\ln 4}{8\ln 4 - 6}, \frac{9}{8\ln 4 - 6}\right).$$

19. Find the centroid of the region between the semicircle $y = \sqrt{1-x^2}$ and the top half of the ellipse $y = \frac{1}{2}\sqrt{1-x^2}$ (Figure 2).

SOLUTION Since the region is symmetric with respect to the y-axis, the centroid lies on the y-axis. To find $y_{\rm cm}$ we calculate

$$\begin{aligned} M_x &= \frac{1}{2}\int_{-1}^1 \left[\left(\sqrt{1-x^2}\right)^2 - \left(\frac{\sqrt{1-x^2}}{2}\right)^2\right] dx \\ &= \frac{1}{2}\int_{-1}^1 \frac{3}{4}\left(1-x^2\right) dx = \frac{3}{8}\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^1 = \frac{1}{2}. \end{aligned}$$

The area of the lamina is $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so the coordinates of the centroid are

$$\left(0, \frac{1/2}{\pi/4}\right) = \left(0, \frac{2}{\pi}\right).$$

20. Find the centroid of the shaded region in Figure 5 bounded on the left by $x = 2y^2 - 2$ and on the right by a semicircle of radius 1. *Hint:* Use symmetry and additivity of moments.

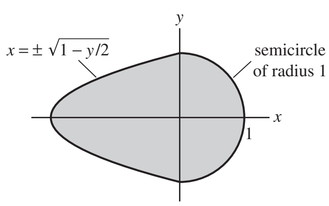


FIGURE 5

SOLUTION The region is symmetric with respect to the x-axis, hence the centroid lies on the x-axis; that is, $y_{\text{cm}} = 0$. To compute the area and the moment with respect to the y-axis, we treat the left side and the right side of the region separately. Starting with the left side, we find

$$M_y^{\text{left}} = 2\int_{-2}^{0} x\sqrt{\frac{x}{2}+1}\,dx \qquad \text{and} \qquad A^{\text{left}} = 2\int_{-2}^{0} \sqrt{\frac{x}{2}+1}\,dx.$$

In each integral we make the substitution $u = \frac{x}{2} + 1$, $du = \frac{1}{2}\,dx$, and find

$$M_y^{\text{left}} = 8\int_0^1 (u-1)u^{1/2}\,du = 8\int_0^1 \left(u^{3/2} - u^{1/2}\right)du = 8\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_0^1 = -\frac{32}{15}$$

and

$$A^{\text{left}} = 4\int_0^1 u^{1/2}\,du = \frac{8}{3}u^{3/2}\Big|_0^1 = \frac{8}{3}.$$

On the right side of the region

$$M_y^{\text{right}} = 2\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{2}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{2}{3},$$

and $A^{\text{right}} = \frac{\pi}{2}$ (because the right side of the region is one-half of a circle of radius 1). Thus,

$$M_y = M_y^{\text{left}} + M_y^{\text{right}} = -\frac{32}{15} + \frac{2}{3} = -\frac{22}{15};$$
$$A = A^{\text{left}} + A^{\text{right}} = \frac{8}{3} + \frac{\pi}{2} = \frac{16+3\pi}{6};$$

and the coordinates of the centroid are

$$\left(\frac{-22/15}{(16+3\pi)/6}, 0\right) = \left(-\frac{44}{80+15\pi}, 0\right).$$

In Exercises 21–26, find the Taylor polynomial at $x = a$ for the given function.

21. $f(x) = x^3, \quad T_3(x), \quad a = 1$

SOLUTION We start by computing the first three derivatives of $f(x) = x^3$:

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$
$$f'''(x) = 6$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = 1,\ f'(1) = 3,\ f''(1) = 6,\ f'''(1) = 6.$$

Therefore,

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-2)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 + 3(x-1) + \frac{6}{2!}(x-2)^2 + \frac{6}{3!}(x-1)^3 \\ &= 1 + 3(x-1) + 3(x-2)^2 + (x-1)^3. \end{aligned}$$

22. $f(x) = 3(x+2)^3 - 5(x+2), \quad T_3(x), \quad a = -2$

SOLUTION $T_3(x)$ is the Taylor polynomial of f consisting of powers of $(x+2)$ up to three. Since $f(x)$ is already in this form we conclude that $T_3(x) = f(x)$.

23. $f(x) = x\ln(x), \quad T_4(x), \quad a = 1$

SOLUTION We start by computing the first four derivatives of $f(x) = x\ln x$:

$$\begin{aligned}
f'(x) &= \ln x + x \cdot \frac{1}{x} = \ln x + 1\\
f''(x) &= \frac{1}{x}\\
f'''(x) &= -\frac{1}{x^2}\\
f^{(4)}(x) &= \frac{2}{x^3}
\end{aligned}$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = 1,\ f'''(1) = -1,\ f^{(4)}(1) = 2.$$

Therefore,

$$\begin{aligned}
T_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4\\
&= 0 + 1(x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 + \frac{2}{4!}(x-1)^4\\
&= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4.
\end{aligned}$$

24. $f(x) = (3x+2)^{1/3}, \quad T_3(x), \quad a = 2$

SOLUTION We start by computing the first three derivatives of $f(x) = (3x+2)^{1/3}$:

$$\begin{aligned}
f'(x) &= \frac{1}{3}(3x+2)^{-2/3} \cdot 3 = (3x+2)^{-2/3}\\
f''(x) &= -\frac{2}{3}(3x+2)^{-5/3} \cdot 3 = -2(3x+2)^{-5/3}\\
f'''(x) &= \frac{10}{3}(3x+2)^{-8/3} \cdot 3 = 10(3x+2)^{-8/3}
\end{aligned}$$

Evaluating the function and its derivatives at $x = 2$, we find

$$f(2) = 2,\ f'(2) = \frac{1}{4},\ f''(2) = -\frac{1}{16},\ f'''(2) = \frac{5}{128}.$$

Therefore,

$$\begin{aligned}
T_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3\\
&= 2 + \frac{1}{4}(x-2) + \frac{-1/16}{2!}(x-2)^2 + \frac{5/128}{3!}(x-2)^3\\
&= 2 + \frac{1}{4}(x-2) - \frac{1}{32}(x-2)^2 - \frac{5}{768}(x-2)^3.
\end{aligned}$$

25. $f(x) = xe^{-x^2}, \quad T_4(x), \quad a = 0$

SOLUTION We start by computing the first four derivatives of $f(x) = xe^{-x^2}$:

$$\begin{aligned}
f'(x) &= e^{-x^2} + x \cdot (-2x)e^{-x^2} = (1-2x^2)e^{-x^2}\\
f''(x) &= -4xe^{-x^2} + (1-2x^2) \cdot (-2x)e^{-x^2} = (4x^3 - 6x)e^{-x^2}\\
f'''(x) &= (12x^2 - 6)e^{-x^2} + (4x^3 - 6x) \cdot (-2x)e^{-x^2} = (-8x^4 + 24x^2 - 6)e^{-x^2}\\
f^{(4)}(x) &= (-32x^3 + 48x)e^{-x^2} + (-8x^4 + 24x^2 - 6) \cdot (-2x)e^{-x^2} = (16x^5 - 80x^3 + 60x)e^{-x^2}
\end{aligned}$$

Evaluating the function and its derivatives at $x = 0$, we find

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -6,\ f^{(4)}(0) = 0.$$

Therefore,

$$T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 0 + x + 0 \cdot x^2 - \frac{6}{3!}x^3 + 0 \cdot x^4 = x - x^3.$$

26. $f(x) = \ln(\cos x)$, $T_3(x)$, $a = 0$

SOLUTION We start by computing the first three derivatives of $f(x) = \ln(\cos x)$:

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x$$
$$f''(x) = -\sec^2 x$$
$$f'''(x) = -2\sec^2 x \tan x$$

Evaluating the function and its derivatives at $x = 0$, we find

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0.$$

Therefore,

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 = -\frac{x^2}{2}.$$

27. Find the nth Maclaurin polynomial for $f(x) = e^{3x}$.

SOLUTION We differentiate the function $f(x) = e^{3x}$ repeatedly, looking for a pattern:

$$f'(x) = 3e^{3x} = 3^1 e^{3x}$$
$$f''(x) = 3 \cdot 3e^{3x} = 3^2 e^{3x}$$
$$f'''(x) = 3 \cdot 3^2 e^{3x} = 3^3 e^{3x}$$

Thus, for general n, $f^{(n)}(x) = 3^n e^{3x}$ and $f^{(n)}(0) = 3^n$. Substituting into the formula for the nth Taylor polynomial, we obtain:

$$T_n(x) = 1 + \frac{3x}{1!} + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{3^4x^4}{4!} + \cdots + \frac{3^nx^n}{n!}$$
$$= 1 + 3x + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \cdots + \frac{1}{n!}(3x)^n.$$

28. Use the fifth Maclaurin polynomial of $f(x) = e^x$ to approximate $\sqrt{e}$. Use a calculator to determine the error.

SOLUTION Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Hence,

$$T_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

For $x = \frac{1}{2}$ we have

$$T_5\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \frac{\left(\frac{1}{2}\right)^5}{5!}$$
$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} = 1.648697917$$

Using a calculator, we find that $\sqrt{e} = 1.648721271$. The error in the Taylor polynomial approximation is

$$|1.648697917 - 1.648721271| = 2.335 \times 10^{-5}.$$

29. Use the third Taylor polynomial of $f(x) = \tan^{-1} x$ at $a = 1$ to approximate $f(1.1)$. Use a calculator to determine the error.

SOLUTION We start by computing the first three derivatives of $f(x) = \tan^{-1} x$:

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{\left(1+x^2\right)^2}$$

$$f'''(x) = \frac{-2\left(1+x^2\right)^2 + 2x \cdot 2\left(1+x^2\right) \cdot 2x}{\left(1+x^2\right)^4} = \frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^3}$$

Evaluating the function and its derivatives at $x = 1$, we find

$$f(1) = \frac{\pi}{4},\ f'(1) = \frac{1}{2},\ f''(1) = -\frac{1}{2},\ f'''(1) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3. \end{aligned}$$

Setting $x = 1.1$ yields:

$$T_3(1.1) = \frac{\pi}{4} + \frac{1}{2}(0.1) - \frac{1}{4}(0.1)^2 + \frac{1}{12}(0.1)^3 = 0.832981496.$$

Using a calculator, we find $\tan^{-1} 1.1 = 0.832981266$. The error in the Taylor polynomial approximation is

$$\left|T_3(1.1) - \tan^{-1} 1.1\right| = |0.832981496 - 0.832981266| = 2.301 \times 10^{-7}.$$

30. Let $T_4(x)$ be the Taylor polynomial for $f(x) = \sqrt{x}$ at $a = 16$. Use the error bound to find the maximum possible size of $|f(17) - T_4(17)|$.

SOLUTION Using the Error Bound, we have

$$|f(17) - T_4(17)| \le K\frac{(17-16)^5}{5!} = \frac{K}{5!},$$

where K is a number such that $\left|f^{(5)}(x)\right| \le K$ for all $16 \le x \le 17$. Starting from $f(x) = \sqrt{x}$ we find

$$f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = -\frac{1}{4}x^{-3/2},\ f'''(x) = \frac{3}{8}x^{-5/2},\ f^{(4)}(x) = -\frac{15}{16}x^{-7/2},$$

and

$$f^{(5)}(x) = \frac{105}{32}x^{-9/2}.$$

For $16 \le x \le 17$,

$$\left|f^{(5)}(x)\right| = \frac{105}{32x^{9/2}} \le \frac{105}{32 \cdot 16^{9/2}} = \frac{105}{8{,}388{,}608}.$$

Therefore, we may take

$$K = \frac{105}{8{,}388{,}608}.$$

Finally,

$$|f(17) - T_4(17)| \le \frac{105}{8{,}388{,}608} \cdot \frac{1}{5!} \approx 1.044 \cdot 10^{-7}.$$

31. Find n such that $|e - T_n(1)| < 10^{-8}$, where $T_n(x)$ is the nth Maclaurin polynomial for $f(x) = e^x$.

SOLUTION Using the Error Bound, we have

$$|e - T_n(1)| \le K\frac{|1-0|^{n+1}}{(n+1)!} = \frac{K}{(n+1)!}$$

where K is a number such that $\left|f^{(n+1)}(x)\right| = e^x \le K$ for all $0 \le x \le 1$. Since e^x is increasing, the maximum value on the interval $0 \le x \le 1$ is attained at the endpoint $x = 1$. Thus, for $0 \le u \le 1$, $e^u \le e^1 < 2.8$. Hence we may take $K = 2.8$ to obtain:

$$|e - T_n(1)| \le \frac{2.8}{(n+1)!}$$

We now choose n such that

$$\frac{2.8}{(n+1)!} < 10^{-8}$$
$$\frac{(n+1)!}{2.8} > 10^{8}$$
$$(n+1)! > 2.8 \times 10^{8}$$

For $n = 10$, $(n+1)! = 3.99 \times 10^7 < 2.8 \times 10^8$ and for $n = 11$, $(n+1)! = 4.79 \times 10^8 > 2.8 \times 10^8$. Hence, to make the error less than 10^{-8}, $n = 11$ is sufficient; that is,

$$|e - T_{11}(1)| < 10^{-8}.$$

32. Let $T_4(x)$ be the Taylor polynomial for $f(x) = x\ln x$ at $a = 1$ computed in Exercise 23. Use the error bound to find a bound for $|f(1.2) - T_4(1.2)|$.

SOLUTION Using the Error Bound, we have

$$|f(1.2) - T_4(1.2)| \le K\frac{(1.2-1)^5}{5!} = \frac{(0.2)^5}{120}K,$$

where K is a number such that $\left|f^{(5)}x\right| \le K$ for all $1 \le x \le 1.2$. Starting from $f(x) = x\ln x$, we find

$$f'(x) = \ln x + x\frac{1}{x} = \ln x + 1,\ f''(x) = \frac{1}{x},\ f'''(x) = -\frac{1}{x^2},\ f^{(4)}(x) = \frac{2}{x^3},$$

and

$$f^{(5)}(x) = \frac{-6}{x^4}.$$

For $1 \le x \le 1.2$,

$$\left|f^{(5)}(x)\right| = \frac{6}{x^4} \le \frac{6}{1^4} = 6.$$

Hence we may take $K = 6$ to obtain:

$$|f(1.2) - T_4(1.2)| \le \frac{(0.2)^5}{120}6 = 1.6 \times 10^{-5}.$$

33. Verify that $T_n(x) = 1 + x + x^2 + \cdots + x^n$ is the nth Maclaurin polynomial of $f(x) = 1/(1-x)$. Show using substitution that the nth Maclaurin polynomial for $f(x) = 1/(1 - x/4)$ is

$$T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n$$

What is the nth Maclaurin polynomial for $g(x) = \dfrac{1}{1+x}$?

SOLUTION Let $f(x) = (1-x)^{-1}$. Then, $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 3!(1-x)^{-4}$, and, in general, $f^{(n)}(x) = n!(1-x)^{-(n+1)}$. Therefore, $f^{(n)}(0) = n!$ and

$$T_n(x) = 1 + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \cdots + \frac{n!}{n!}x^n = 1 + x + x^2 + \cdots + x^n.$$

Upon substituting $x/4$ for x, we find that the nth Maclaurin polynomial for $f(x) = \dfrac{1}{1-x/4}$ is

$$T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n.$$

Substituting $-x$ for x, the nth Maclaurin polynomial for $g(x) = \dfrac{1}{1+x}$ is

$$T_n(x) = 1 - x + x^2 - x^3 + - \cdots + (-x)^n.$$

34. Let $f(x) = \dfrac{5}{4+3x-x^2}$ and let a_k be the coefficient of x^k in the Maclaurin polynomial $T_n(x)$ of for $k \le n$.

(a) Show that $f(x) = \left(\dfrac{1/4}{1-x/4} + \dfrac{1}{1+x}\right)$.

(b) Use Exercise 33 to show that $a_k = \dfrac{1}{4^{k+1}} + (-1)^k$.

(c) Compute $T_3(x)$.

SOLUTION

(a) Start by factoring the denominator and writing the form of the partial fraction decomposition:

$$f(x) = \frac{5}{4+3x-x^2} = \frac{5}{(x+1)(4-x)} = \frac{A}{x+1} + \frac{B}{4-x}.$$

Multiplying through by $(x+1)(4-x)$, we obtain:

$$5 = A(4-x) + B(x+1).$$

Substituting $x = 4$ yields $5 = A(0) + B(5)$, so $B = 1$; substituting $x = -1$ yields $5 = A(5) + B(0)$, so $A = 1$. Thus,

$$f(x) = \frac{1}{x+1} + \frac{1}{4-x} = \frac{1}{x+1} + \frac{\frac{1}{4}}{1-\frac{x}{4}}.$$

(b) The nth Maclaurin polynomial for $f(x) = \frac{\frac{1}{4}}{1-\frac{x}{4}} + \frac{1}{x+1}$ is the sum of the nth Maclaurin polynomials for the functions $g(x) = \frac{1}{4} \cdot \frac{1}{1-\frac{x}{4}}$ and $h(x) = \frac{1}{1+x}$. In Exercise 33, we found that the nth Maclaurin polynomials $P_n(x)$ and $Q_n(x)$ for g and h are

$$P_n(x) = \frac{1}{4}\left(1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \cdots + \frac{1}{4^n}x^n\right) = \frac{1}{4} + \frac{1}{4^2}x + \frac{1}{4^3}x^2 + \cdots + \frac{1}{4^{n+1}}x^n = \sum_{k=0}^{n} \frac{x^k}{4^{k+1}}$$

$$Q_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$$

Therefore,

$$T_n(x) = P_n(x) + Q_n(x) = \sum_{k=0}^{n} \frac{x^k}{4^{k+1}} + \sum_{k=0}^{n} (-1)^k x^k = \sum_{k=0}^{n} \left[\frac{1}{4^{k+1}} + (-1)^k\right] x^k;$$

that is, the coefficient of x^k in T_n for $k \le n$ is

$$a_k = \frac{1}{4^{k+1}} + (-1)^k.$$

(c) From part (b),

$$a_0 = \frac{1}{4} + 1, \quad a_1 = \frac{1}{4^2} - 1, \quad a_2 = \frac{1}{4^3} + 1, \quad a_3 = \frac{1}{4^4} - 1$$

so that

$$T_3(x) = \frac{5}{4} - \frac{15}{16}x + \frac{65}{64}x^2 - \frac{255}{256}x^3$$

35. Let $T_n(x)$ be the nth Maclaurin polynomial for the function $f(x) = \sin x + \sinh x$.
(a) Show that $T_5(x) = T_6(x) = T_7(x) = T_8(x)$.
(b) Show that $|f^n(x)| \le 1 + \cosh x$ for all n. *Hint:* Note that $|\sinh x| \le |\cosh x|$ for all x.
(c) Show that $|T_8(x) - f(x)| \le \frac{2.6}{9!}|x|^9$ for $-1 \le x \le 1$.

SOLUTION

(a) Let $f(x) = \sin x + \sinh x$. Then

$$\begin{aligned} f'(x) &= \cos x + \cosh x \\ f''(x) &= -\sin x + \sinh x \\ f'''(x) &= -\cos x + \cosh x \\ f^{(4)}(x) &= \sin x + \sinh x. \end{aligned}$$

From this point onward, the pattern of derivatives repeats indefinitely. Thus

$$\begin{aligned} f(0) = f^{(4)}(0) = f^{(8)}(0) &= \sin 0 + \sinh 0 = 0 \\ f'(0) = f^{(5)}(0) &= \cos 0 + \cosh 0 = 2 \\ f''(0) = f^{(6)}(0) &= -\sin 0 + \sinh 0 = 0 \\ f'''(0) = f^{(7)}(0) &= -\cos 0 + \cosh 0 = 0. \end{aligned}$$

Consequently,

$$T_5(x) = f'(0)x + \frac{f^{(5)}(0)}{5!}x^5 = 2x + \frac{1}{60}x^5,$$

and, because $f^{(6)}(0) = f^{(7)}(0) = f^{(8)}(0) = 0$, it follows that

$$T_6(x) = T_7(x) = T_8(x) = T_5(x) = 2x + \frac{1}{60}x^5.$$

(b) First note that $|\sin x| \le 1$ and $|\cos x| \le 1$ for all x. Moreover,

$$|\sinh x| = \left|\frac{e^x - e^{-x}}{2}\right| \le \frac{e^x + e^{-x}}{2} = \cosh x.$$

Now, recall from part (a), that all derivatives of $f(x)$ contain two terms: the first is $\pm\sin x$ or $\pm\cos x$, while the second is either $\sinh x$ or $\cosh x$. In absolute value, the trigonometric term is always less than or equal to 1, while the hyperbolic term is always less than or equal to $\cosh x$. Thus, for all n,

$$f^{(n)}(x) \le 1 + \cosh x.$$

(c) Using the Error Bound, we have

$$|T_8(x) - f(x)| \le \frac{K|x-0|^9}{9!} = \frac{K|x|^9}{9!},$$

where K is a number such that $\left|f^{(9)}(u)\right| \le K$ for all u between 0 and x. By part (b), we know that

$$f^{(9)}(u) \le 1 + \cosh u.$$

Now, $\cosh u$ is an even function that is increasing on $(0, \infty)$. The maximum value for u between 0 and x is therefore $\cosh x$. Moreover, for $-1 \le x \le 1$, $\cosh x \le \cosh 1 \approx 1.543 < 1.6$. Hence, we may take $K = 1 + 1.6 = 2.6$, and

$$|T_8(x) - f(x)| \le \frac{2.6}{9!}|x|^9.$$

9 INTRODUCTION TO DIFFERENTIAL EQUATIONS

9.1 Solving Differential Equations

Preliminary Questions

1. Determine the order of the following differential equations:

(a) $x^5 y' = 1$

(b) $(y')^3 + x = 1$

(c) $y''' + x^4 y' = 2$

(d) $\sin(y'') + x = y$

SOLUTION

(a) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(b) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(c) The highest order derivative that appears in this equation is a third derivative, so this is a third order equation.

(d) The highest order derivative that appears in this equation is a second derivative, so this is a second order equation.

2. Is $y'' = \sin x$ a linear differential equation?

SOLUTION Yes.

3. Give an example of a nonlinear differential equation of the form $y' = f(y)$.

SOLUTION One possibility is $y' = y^2$.

4. Can a nonlinear differential equation be separable? If so, give an example.

SOLUTION Yes. An example is $y' = y^2$.

5. Give an example of a linear, nonseparable differential equation.

SOLUTION One example is $y' + y = x$.

Exercises

1. Which of the following differential equations are first-order?

(a) $y' = x^2$

(b) $y'' = y^2$

(c) $(y')^3 + yy' = \sin x$

(d) $x^2 y' - e^x y = \sin y$

(e) $y'' + 3y' = \dfrac{y}{x}$

(f) $yy' + x + y = 0$

SOLUTION

(a) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(b) The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

(c) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(d) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

(e) The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

(f) The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

2. Which of the equations in Exercise 1 are linear?

SOLUTION

(a) Linear; $(1)y' - x^2 = 0$.

(b) Not linear; y^2 is not a linear function of y.

(c) Not linear; $(y')^3$ is not a linear function of y'.

(d) Not linear; $\sin y$ is not a linear function of y.

(e) Linear; $(1)y'' + (3)y' - \frac{1}{x}y = 0$.

(f) Not linear. yy' cannot be expressed as $a(x)y^{(n)}$.

In Exercises 3–8, verify that the given function is a solution of the differential equation.

3. $y' - 8x = 0, \quad y = 4x^2$

SOLUTION Let $y = 4x^2$. Then $y' = 8x$ and

$$y' - 8x = 8x - 8x = 0.$$

4. $yy' + 4x = 0, \quad y = \sqrt{12 - 4x^2}$

SOLUTION Let $y = \sqrt{12 - 4x^2}$. Then

$$y' = \frac{-4x}{\sqrt{12 - 4x^2}},$$

and

$$yy' + 4x = \sqrt{12 - 4x^2}\,\frac{-4x}{\sqrt{12 - 4x^2}} + 4x = -4x + 4x = 0.$$

5. $y' + 4xy = 0, \quad y = 25e^{-2x^2}$

SOLUTION Let $y = 25e^{-2x^2}$. Then $y' = -100xe^{-2x^2}$, and

$$y' + 4xy = -100xe^{-2x^2} + 4x(25e^{-2x^2}) = 0.$$

6. $(x^2 - 1)y' + xy = 0, \quad y = 4(x^2 - 1)^{-1/2}$

SOLUTION Let $y = 4(x^2 - 1)^{-1/2}$. Then $y' = -4x(x^2 - 1)^{-3/2}$, and

$$\begin{aligned}(x^2 - 1)y' + xy &= (x^2 - 1)(-4x)(x^2 - 1)^{-3/2} + 4x(x^2 - 1)^{-1/2}\\ &= -4x(x^2 - 1)^{-1/2} + 4x(x^2 - 1)^{-1/2} = 0.\end{aligned}$$

7. $y'' - 2xy' + 8y = 0, \quad y = 4x^4 - 12x^2 + 3$

SOLUTION Let $y = 4x^4 - 12x^2 + 3$. Then $y' = 16x^3 - 24x$, $y'' = 48x^2 - 24$, and

$$\begin{aligned}y'' - 2xy' + 8y &= (48x^2 - 24) - 2x(16x^3 - 24x) + 8(4x^4 - 12x^2 + 3)\\ &= 48x^2 - 24 - 32x^4 + 48x^2 + 32x^4 - 96x^2 + 24 = 0.\end{aligned}$$

8. $y'' - 2y' + 5y = 0, \quad y = e^x \sin 2x$

SOLUTION Let $y = e^x \sin 2x$. Then

$$\begin{aligned}y' &= 2e^x \cos 2x + e^x \sin 2x,\\ y'' &= -4e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x + e^x \sin 2x = -3e^x \sin 2x + 4e^x \cos 2x,\end{aligned}$$

and

$$\begin{aligned}y'' - 2y' + 5y &= -3e^x \sin 2x + 4e^x \cos 2x - 4e^x \cos 2x - 2e^x \sin 2x + 5e^x \sin 2x\\ &= (-3e^x - 2e^x + 5e^x)\sin 2x + (4e^x - 4e^x)\cos 2x = 0.\end{aligned}$$

9. Which of the following equations are separable? Write those that are separable in the form $y' = f(x)g(y)$ (but do not solve).

(a) $xy' - 9y^2 = 0$

(b) $\sqrt{4 - x^2}\,y' = e^{3y} \sin x$

(c) $y' = x^2 + y^2$

(d) $y' = 9 - y^2$

SOLUTION

(a) $xy' - 9y^2 = 0$ is separable:

$$\begin{aligned}xy' - 9y^2 &= 0\\ xy' &= 9y^2\\ y' &= \frac{9}{x}y^2\end{aligned}$$

(b) $\sqrt{4-x^2}y' = e^{3y}\sin x$ is separable:

$$\sqrt{4-x^2}y' = e^{3y}\sin x$$
$$y' = e^{3y}\frac{\sin x}{\sqrt{4-x^2}}.$$

(c) $y' = x^2 + y^2$ is not separable; y' is already isolated, but is not equal to a product $f(x)g(y)$.
(d) $y' = 9 - y^2$ is separable: $y' = (1)(9-y^2)$.

10. The following differential equations appear similar but have very different solutions.

$$\frac{dy}{dx} = x, \qquad \frac{dy}{dx} = y$$

Solve both subject to the initial condition $y(1) = 2$.

SOLUTION For the first differential equation, we have $y' = x$ so that, integrating,

$$y = \frac{x^2}{2} + C$$

Since $y(1) = 2$, $C = \frac{3}{2}$, so that

$$y = \frac{x^2+3}{2}$$

The second equation is separable: $y^{-1}\,dy = 1\,dx$, so that $\ln|y| = x + C$ and $y = Ce^x$. Since $y(1) = 2$, we have $2 = Ce$ or $C = 2e^{-1}$. Thus $y = 2e^{x-1}$.

11. Consider the differential equation $y^3y' - 9x^2 = 0$.
(a) Write it as $y^3\,dy = 9x^2\,dx$.
(b) Integrate both sides to obtain $\frac{1}{4}y^4 = 3x^3 + C$.
(c) Verify that $y = (12x^3 + C)^{1/4}$ is the general solution.
(d) Find the particular solution satisfying $y(1) = 2$.

SOLUTION Solving $y^3y' - 9x^2 = 0$ for y' gives $y' = 9x^2y^{-3}$.
(a) Separating variables in the equation above yields

$$y^3\,dy = 9x^2\,dx$$

(b) Integrating both sides gives

$$\frac{y^4}{4} = 3x^3 + C$$

(c) Simplify the equation above to get $y^4 = 12x^3 + C$, or $y = (12x^3 + C)^{1/4}$.
(d) Solve $2 = (12\cdot 1^3 + C)^{1/4}$ to get $16 = 12 + C$, or $C = 4$. Thus the particular solution is $y = (12x^3 + 4)^{1/4}$.

12. Verify that $x^2y' + e^{-y} = 0$ is separable.
(a) Write it as $e^y\,dy = -x^{-2}\,dx$.
(b) Integrate both sides to obtain $e^y = x^{-1} + C$.
(c) Verify that $y = \ln(x^{-1} + C)$ is the general solution.
(d) Find the particular solution satisfying $y(2) = 4$.

SOLUTION Solving $x^2y' + e^{-y} = 0$ for y' yields

$$y' = -x^{-2}e^{-y}.$$

(a) Separating variables in the last equation yields

$$e^y dy = -x^{-2}dx.$$

(b) Integrating both sides of the result of part (a), we find

$$\int e^y dy = -\int x^{-2}dx$$
$$e^y + C_1 = x^{-1} + C_2$$
$$e^y = x^{-1} + C$$

(c) Solving the last expression from part (b) for y, we find

$$y = \ln\left|x^{-1} + C\right|$$

(d) $y(2) = 4$ yields $4 = \ln\left|\frac{1}{2} + C\right|$, or $e^4 = C + \frac{1}{2}$. Thus the particular solution is

$$y = \ln\left|\frac{1}{x} - \frac{1}{2} + e^4\right|$$

In Exercises 13–28, use separation of variables to find the general solution.

13. $y' + 4xy^2 = 0$

SOLUTION Rewrite

$$y' + 4xy^2 = 0 \quad \text{as} \quad \frac{dy}{dx} = -4xy^2 \quad \text{and then as} \quad y^{-2}\,dy = -4x\,dx$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int x\,dx$$
$$-y^{-1} = -2x^2 + C$$
$$y^{-1} = 2x^2 + C$$

Solving for y gives

$$y = \frac{1}{2x^2 + C}$$

where C is an arbitrary constant.

14. $y' + x^2y = 0$

SOLUTION Rewrite

$$y' + x^2y = 0 \quad \text{as} \quad \frac{dy}{dx} = -x^2y \quad \text{and then as} \quad y^{-1}\,dy = -x^2\,dx$$

Integrating both sides of this equation gives

$$\int y^{-1}\,dy = -\int x^2\,dx$$
$$\ln|y| = -\frac{x^3}{3} + C_1$$

Solve for y to get

$$y = \pm e^{-x^3/3 + C_1} = Ce^{-x^3/3}$$

where $C = \pm e^{C_1}$ is an arbitrary constant.

15. $\frac{dy}{dt} - 20t^4e^{-y} = 0$

SOLUTION Rewrite

$$\frac{dy}{dt} - 20t^4e^{-y} = 0 \quad \text{as} \quad \frac{dy}{dt} = 20t^4e^{-y} \quad \text{and then as} \quad e^y\,dy = 20t^4\,dt$$

Integrating both sides of this equation gives

$$\int e^y\,dy = \int 20t^4\,dt$$
$$e^y = 4t^5 + C$$

Solve for y to get $y = \ln(4t^5 + C)$, where C is an arbitrary constant.

16. $t^3y' + 4y^2 = 0$

SOLUTION Rewrite

$$t^3y' + 4y^2 = 0 \quad \text{as} \quad \frac{dy}{dt} = -4y^2t^{-3} \quad \text{and then as} \quad y^{-2}\,dy = -4t^{-3}\,dt$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int t^{-3}\,dt$$
$$-y^{-1} = 2t^{-2} + C$$

Solve for y to get

$$y = \frac{-1}{2t^{-2} + C} = \frac{-t^2}{2 + Ct^2}$$

where C is an arbitrary constant.

17. $2y' + 5y = 4$

SOLUTION Rewrite

$$2y' + 5y = 4 \quad \text{as} \quad y' = 2 - \frac{5}{2}y \quad \text{and then as} \quad (4 - 5y)^{-1}\,dy = \frac{1}{2}\,dx$$

Integrating both sides and solving for y gives

$$\begin{aligned} \int \frac{dy}{4 - 5y} &= \frac{1}{2}\int 1\,dx \\ -\frac{1}{5}\ln|4 - 5y| &= \frac{1}{2}x + C_1 \\ \ln|4 - 5y| &= C_2 - \frac{5}{2}x \\ 4 - 5y &= C_3 e^{-5x/2} \\ 5y &= 4 - C_3 e^{-5x/2} \\ y &= Ce^{-5x/2} + \frac{4}{5} \end{aligned}$$

where C is an arbitrary constant.

18. $\dfrac{dy}{dt} = 8\sqrt{y}$

SOLUTION Rewrite

$$\frac{dy}{dt} = 8\sqrt{y} \qquad \text{as} \qquad \frac{dy}{\sqrt{y}} = 8\,dt.$$

Integrating both sides of this equation yields

$$\begin{aligned} \int \frac{dy}{\sqrt{y}} &= 8\int dt \\ 2\sqrt{y} &= 8t + C. \end{aligned}$$

Solving for y, we find

$$\begin{aligned} \sqrt{y} &= 4t + C \\ y &= (4t + C)^2, \end{aligned}$$

where C is an arbitrary constant.

19. $\sqrt{1 - x^2}\,y' = xy$

SOLUTION Rewrite

$$\sqrt{1 - x^2}\,\frac{dy}{dx} = xy \qquad \text{as} \qquad \frac{dy}{y} = \frac{x}{\sqrt{1 - x^2}}\,dx.$$

Integrating both sides of this equation yields

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{x}{\sqrt{1 - x^2}}\,dx \\ \ln|y| &= -\sqrt{1 - x^2} + C. \end{aligned}$$

Solving for y, we find

$$\begin{aligned} |y| &= e^{-\sqrt{1-x^2}+C} = e^C e^{-\sqrt{1-x^2}} \\ y &= \pm e^C e^{-\sqrt{1-x^2}} = Ae^{-\sqrt{1-x^2}}, \end{aligned}$$

where A is an arbitrary constant.

20. $y' = y^2(1 - x^2)$

SOLUTION Rewrite

$$\frac{dy}{dx} = y^2(1 - x^2) \qquad \text{as} \qquad \frac{dy}{y^2} = (1 - x^2)\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{dy}{y^2} = \int (1 - x^2)\,dx$$
$$-y^{-1} = x - \frac{1}{3}x^3 + C.$$

Solving for y, we find

$$y^{-1} = \frac{1}{3}x^3 - x + C$$
$$y = \frac{1}{\frac{1}{3}x^3 - x + C},$$

where C is an arbitrary constant.

21. $yy' = x$

SOLUTION Rewrite

$$y\frac{dy}{dx} = x \qquad \text{as} \qquad y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$$

Solving for y, we find

$$y^2 = x^2 + 2C$$
$$y = \pm\sqrt{x^2 + A},$$

where $A = 2C$ is an arbitrary constant.

22. $(\ln y)y' - ty = 0$

SOLUTION Rewrite

$$(\ln y)y' - ty = 0 \quad \text{as} \quad (\ln y)\frac{dy}{dt} = ty \quad \text{and then as} \quad \frac{\ln y}{y}\,dy = t\,dt$$

Integrating both sides of this equation gives

$$\int \frac{\ln y}{y}\,dy = \int t\,dt$$
$$\frac{1}{2}\ln^2 y = \frac{1}{2}t^2 + C_1$$
$$\ln^2 y = t^2 + C$$
$$\ln y = \pm\sqrt{t^2 + C}$$
$$y = e^{\pm\sqrt{t^2+C}}$$

23. $\dfrac{dx}{dt} = (t + 1)(x^2 + 1)$

SOLUTION Rewrite

$$\frac{dx}{dt} = (t + 1)(x^2 + 1) \qquad \text{as} \qquad \frac{1}{x^2 + 1}\,dx = (t + 1)\,dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{x^2+1}\,dx = \int (t+1)\,dt$$
$$\tan^{-1} x = \frac{1}{2}t^2 + t + C.$$

Solving for x, we find

$$x = \tan\left(\frac{1}{2}t^2 + t + C\right).$$

where $A = \tan C$ is an arbitrary constant.

24. $(1+x^2)y' = x^3y$

SOLUTION Rewrite

$$(1+x^2)\frac{dy}{dx} = x^3y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{x^3}{1+x^2}\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\,dy = \int \frac{x^3}{1+x^2}\,dx.$$

To integrate $\frac{x^3}{1+x^2}$, note

$$\frac{x^3}{1+x^2} = \frac{(x^3+x)-x}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C$$
$$|y| = e^C\frac{e^{x^2/2}}{\sqrt{x^2+1}}$$
$$y = \pm e^C\frac{e^{x^2/2}}{\sqrt{x^2+1}} = A\frac{e^{x^2/2}}{\sqrt{x^2+1}},$$

where $A = \pm e^C$ is an arbitrary constant.

25. $y' = x\sec y$

SOLUTION Rewrite

$$\frac{dy}{dx} = x\sec y \qquad \text{as} \qquad \cos y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int \cos y\,dy = \int x\,dx$$
$$\sin y = \frac{1}{2}x^2 + C.$$

Solving for y, we find

$$y = \sin^{-1}\left(\frac{1}{2}x^2 + C\right),$$

where C is an arbitrary constant.

26. $\dfrac{dy}{d\theta} = \tan y$

SOLUTION Rewrite

$$\frac{dy}{d\theta} = \tan y \qquad \text{as} \qquad \cot y\,dy = d\theta.$$

Integrating both sides of this equation yields

$$\int \frac{\cos y}{\sin y}\,dy = \int d\theta$$
$$\ln|\sin y| = \theta + C.$$

Solving for y, we have

$$|\sin y| = e^{\theta+C} = e^C e^\theta$$
$$\sin y = \pm e^C e^\theta$$
$$y = \sin^{-1}\left(Ae^\theta\right),$$

where $A = \pm e^C$ is an arbitrary constant.

27. $\dfrac{dy}{dt} = y\tan t$

SOLUTION Rewrite

$$\frac{dy}{dt} = y\tan t \qquad \text{as} \qquad \frac{1}{y}\,dy = \tan t\,dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\,dy = \int \tan t\,dt$$
$$\ln|y| = \ln|\sec t| + C.$$

Solving for y, we find

$$|y| = e^{\ln|\sec t|+C} = e^C\,|\sec t|$$
$$y = \pm e^C \sec t = A\sec t,$$

where $A = \pm e^C$ is an arbitrary constant.

28. $\dfrac{dx}{dt} = t\tan x$

SOLUTION Rewrite

$$\frac{dx}{dt} = t\tan x \qquad \text{as} \qquad \cot x\,dx = t\,dt.$$

Integrating both sides of this equation yields

$$\int \cot x\,dx = \int t\,dt$$
$$\ln|\sin x| = \frac{1}{2}t^2 + C.$$

Solving for y, we find

$$|\sin x| = e^{\frac{1}{2}t^2+C} = e^C e^{\frac{1}{2}t^2}$$
$$\sin x = \pm e^C e^{\frac{1}{2}t^2}$$
$$x = \sin^{-1}\left(Ae^{\frac{1}{2}t^2}\right),$$

where $A = \pm e^C$ is an arbitrary constant.

In Exercises 29–42, solve the initial value problem.

29. $y' + 2y = 0, \quad y(\ln 5) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} + 2y = 0 \qquad \text{as} \qquad \frac{1}{y}\,dy = -2\,dx,$$

and then integrate to obtain

$$\ln|y| = -2x + C.$$

Thus,

$$y = Ae^{-2x},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 5) = 3$ allows us to determine the value of A.

$$3 = Ae^{-2(\ln 5)}; \quad 3 = A\frac{1}{25}; \quad \text{so} \quad 75 = A.$$

Finally,

$$y = 75e^{-2x}.$$

30. $y' - 3y + 12 = 0, \quad y(2) = 1$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} - 3y + 12 = 0 \qquad \text{as} \qquad \frac{1}{3y-12}\,dy = 1\,dx,$$

and then integrate to obtain

$$\frac{1}{3}\ln|3y - 12| = x + C.$$

Thus,

$$y = Ae^{3x} + 4,$$

where $A = \pm\frac{1}{3}e^{3C}$ is an arbitrary constant. The initial condition $y(2) = 1$ allows us to determine the value of A.

$$1 = Ae^6 + 4; \quad -3 = Ae^6; \quad \text{so} \quad -3e^{-6} = A.$$

Finally,

$$y = -3e^{-6}e^{3x} + 4 = -3e^{3x-6} + 4$$

31. $yy' = xe^{-y^2}, \quad y(0) = -2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$y\frac{dy}{dx} = xe^{-y^2} \qquad \text{as} \qquad ye^{y^2}\,dy = x\,dx,$$

and then integrate to obtain

$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2 + C.$$

Thus,

$$y = \pm\sqrt{\ln(x^2 + A)},$$

where $A = 2C$ is an arbitrary constant. The initial condition $y(0) = -2$ allows us to determine the value of A. Since $y(0) < 0$, we have $y = -\sqrt{\ln(x^2 + A)}$, and

$$-2 = -\sqrt{\ln(A)}; \quad 4 = \ln(A); \quad \text{so} \quad e^4 = A.$$

Finally,

$$y = -\sqrt{\ln(x^2 + e^4)}.$$

32. $y^2\frac{dy}{dx} = x^{-3}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$y^2\frac{dy}{dx} = x^{-3} \qquad \text{as} \qquad y^2\,dy = x^{-3}\,dx,$$

and then integrate to obtain

$$\frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + C.$$

Thus,

$$y = \left(A - \frac{3}{2}x^{-2}\right)^{1/3},$$

where $A = 3C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of A.

$$0 = \left(A - \frac{3}{2}1^{-2}\right)^{1/3}; \quad 0 = \left(A - \frac{3}{2}\right)^{1/3}; \quad \text{so} \quad A = \frac{3}{2}.$$

Finally,

$$y = \left(\frac{3}{2} - \frac{3}{2}x^{-2}\right)^{1/3}.$$

33. $y' = (x-1)(y-2), \quad y(2) = 4$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 4$ allows us to determine the value of A.

$$4 = Ae^0 + 2 \quad \text{so} \quad A = 2.$$

Finally,

$$y = 2e^{(1/2)x^2 - x} + 2.$$

34. $y' = (x-1)(y-2), \quad y(2) = 2$

SOLUTION First (as in the previous problem), we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 2$ allows us to determine the value of A.

$$2 = Ae^0 + 2 \quad \text{so} \quad A = 0.$$

Finally,

$$y = 2.$$

35. $y' = x(y^2 + 1), \quad y(0) = 0$

SOLUTION First, find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = x(y^2 + 1) \qquad \text{as} \qquad \frac{1}{y^2 + 1}\,dy = x\,dx$$

and integrate to obtain

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

so that

$$y = \tan\left(\frac{1}{2}x^2 + C\right)$$

where C is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of C: $0 = \tan(C)$, so $C = 0$. Finally,

$$y = \tan\left(\frac{1}{2}x^2\right)$$

36. $(1-t)\dfrac{dy}{dt} - y = 0, \quad y(2) = -4$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$(1-t)\frac{dy}{dt} = y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{-1}{t-1}\,dt,$$

and then integrate to obtain

$$\ln|y| = -\ln|t-1| + C.$$

Thus,

$$y = \frac{A}{t-1},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = -4$ allows us to determine the value of A.

$$-4 = \frac{A}{2-1} = A.$$

Finally,

$$y = \frac{-4}{t-1}.$$

37. $\dfrac{dy}{dt} = ye^{-t}, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = ye^{-t} \qquad \text{as} \qquad \frac{1}{y}\,dy = e^{-t}\,dt,$$

and then integrate to obtain

$$\ln|y| = -e^{-t} + C.$$

Thus,

$$y = Ae^{-e^{-t}},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(0) = 1$ allows us to determine the value of A.

$$1 = Ae^{-1} \quad \text{so} \quad A = e.$$

Finally,

$$y = (e)e^{-e^{-t}} = e^{1-e^{-t}}.$$

38. $\dfrac{dy}{dt} = te^{-y}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = te^{-y} \qquad \text{as} \qquad e^y\,dy = t\,dt,$$

and then integrate to obtain

$$e^y = \frac{1}{2}t^2 + C.$$

Thus,

$$y = \ln\left(\frac{1}{2}t^2 + C\right),$$

where C is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of C.

$$0 = \ln\left(\frac{1}{2} + C\right); \quad 1 = \frac{1}{2} + C; \quad \text{so} \quad C = \frac{1}{2}.$$

Finally,

$$y = \ln\left(\frac{1}{2}t^2 + \frac{1}{2}\right).$$

39. $t^2\dfrac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$t^2\frac{dy}{dt} = 1 + t + y + ty = (1+t)(1+y)$$

as

$$\frac{1}{1+y}\,dy = \frac{1+t}{t^2}\,dt,$$

and then integrate to obtain

$$\ln|1+y| = -t^{-1} + \ln|t| + C.$$

Thus,

$$y = A\frac{t}{e^{1/t}} - 1,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of A.

$$0 = A\left(\frac{1}{e}\right) - 1 \quad \text{so} \quad A = e.$$

Finally,

$$y = \frac{et}{e^{1/t}} - 1.$$

40. $\sqrt{1-x^2}\,y' = y^2 + 1, \quad y(0) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\sqrt{1-x^2}\,\frac{dy}{dx} = y^2 + 1 \qquad \text{as} \qquad \frac{1}{y^2+1}\,dy = \frac{1}{\sqrt{1-x^2}}\,dx,$$

and then integrate to obtain

$$\tan^{-1} y = \sin^{-1} x + C.$$

Thus,

$$y = \tan(\sin^{-1} x + C),$$

where C is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of C.

$$0 = \tan\left(\sin^{-1} 0 + C\right) = \tan C \quad \text{so} \quad 0 = C.$$

Finally,

$$y = \tan\left(\sin^{-1} x\right).$$

41. $y' = \tan y, \quad y(\ln 2) = \dfrac{\pi}{2}$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = \tan y \qquad \text{as} \qquad \frac{dy}{\tan y} = dx,$$

and then integrate to obtain

$$\ln|\sin y| = x + C.$$

Thus,

$$y = \sin^{-1}(Ae^x),$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 2) = \frac{\pi}{2}$ allows us to determine the value of A.

$$\frac{\pi}{2} = \sin^{-1}(2A); \quad 1 = 2A \quad \text{so} \quad A = \frac{1}{2}.$$

Finally,

$$y = \sin^{-1}\left(\frac{1}{2}e^x\right).$$

42. $y' = y^2 \sin x, \quad y(\pi) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = y^2 \sin x \qquad \text{as} \qquad y^{-2}\,dy = \sin x\,dx,$$

and then integrate to obtain

$$-y^{-1} = -\cos x + C.$$

Thus,

$$y = \frac{1}{A + \cos x},$$

where $A = -C$ is an arbitrary constant. The initial condition $y(\pi) = 2$ allows us to determine the value of A.

$$2 = \frac{1}{A-1}; \quad A - 1 = \frac{1}{2} \quad \text{so} \quad A = \frac{1}{2} + 1 = \frac{3}{2}.$$

Finally,

$$y = \frac{1}{\cos x + (3/2)} = \frac{2}{3 + 2\cos x}.$$

43. Find all values of a such that $y = x^a$ is a solution of

$$y'' - 12x^{-2}y = 0$$

SOLUTION Let $y = x^a$. Then

$$y' = ax^{a-1} \quad \text{and} \quad y'' = a(a-1)x^{a-2}.$$

Substituting into the differential equation, we find

$$y'' - 12x^{-2}y = a(a-1)x^{a-2} - 12x^{a-2} = x^{a-2}(a^2 - a - 12).$$

Thus, $y'' - 12x^{-2}y = 0$ if and only if

$$a^2 - a - 12 = (a-4)(a+3) = 0.$$

Hence, $y = x^a$ is a solution of the differential equation $y'' - 12x^{-2}y = 0$ provided $a = 4$ or $a = -3$.

44. Find all values of a such that $y = e^{ax}$ is a solution of

$$y'' + 4y' - 12y = 0$$

SOLUTION Let $y = e^{ax}$. Then

$$y' = ae^{ax} \quad \text{and} \quad y'' = a^2e^{ax}.$$

Substituting into the differential equation, we find

$$y'' + 4y' - 12y = e^{ax}(a^2 + 4a - 12).$$

Because e^{ax} is never zero, $y'' + 4y' - 12y = 0$ if only if $a^2 + 4a - 12 = (a+6)(a-2) = 0$. Hence, $y = e^{ax}$ is a solution of the differential equation $y'' + 4y' - 12y = 0$ provided $a = -6$ or $a = 2$.

In Exercises 45 and 46, let $y(t)$ *be a solution of* $(\cos y + 1)\dfrac{dy}{dt} = 2t$ *such that* $y(2) = 0$.

45. Show that $\sin y + y = t^2 + C$. We cannot solve for y as a function of t, but, assuming that $y(2) = 0$, find the values of t at which $y(t) = \pi$.

SOLUTION Rewrite

$$(\cos y + 1)\frac{dy}{dt} = 2t \quad \text{as} \quad (\cos y + 1)\,dy = 2t\,dt$$

and integrate to obtain

$$\sin y + y = t^2 + C$$

where C is an arbitrary constant. Since $y(2) = 0$, we have $\sin 0 + 0 = 4 + C$ so that $C = -4$ and the particular solution we seek is $\sin y + y = t^2 - 4$. To find values of t at which $y(t) = \pi$, we must solve $\sin \pi + \pi = t^2 - 4$, or $t^2 - 4 = \pi$; thus $t = \pm\sqrt{\pi + 4}$.

46. Assuming that $y(6) = \pi/3$, find an equation of the tangent line to the graph of $y(t)$ at $(6, \pi/3)$.

SOLUTION At $(6, \pi/3)$, we have

$$\left(\cos\frac{\pi}{3} + 1\right)\frac{dy}{dt} = 2(6) = 12 \quad \Rightarrow \quad \frac{3}{2}y' = 12$$

and hence $y' = 8$. The tangent line has equation

$$(y - \pi/3) = 8(x - 6)$$

In Exercises 47–52, use Eq. (4) and Torricelli's Law [Eq. (5)].

47. Water leaks through a hole of area 0.002 m^2 at the bottom of a cylindrical tank that is filled with water and has height 3 m and a base of area 10 m^2. How long does it take (a) for half of the water to leak out and (b) for the tank to empty?

SOLUTION Because the tank has a constant cross-sectional area of 10 m^2 and the hole has an area of 0.002 m^2, the differential equation for the height of the water in the tank is

$$\frac{dy}{dt} = \frac{0.002v}{10} = 0.0002v.$$

By Torricelli's Law,

$$v = -\sqrt{2gy} = -\sqrt{19.6y},$$

using $g = 9.8\ \text{m/s}^2$. Thus,

$$\frac{dy}{dt} = -0.0002\sqrt{19.6y} = -0.0002\sqrt{19.6}\cdot\sqrt{y}.$$

Separating variables and then integrating yields

$$\begin{aligned} y^{-1/2}\,dy &= -0.0002\sqrt{19.6}\,dt \\ 2y^{1/2} &= -0.0002\sqrt{19.6}t + C \end{aligned}$$

Solving for y, we find

$$y(t) = \left(C - 0.0001\sqrt{19.6}t\right)^2.$$

Since the tank is originally full, we have the initial condition $y(0) = 10$, whence $\sqrt{10} = C$. Therefore,

$$y(t) = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2.$$

When half of the water is out of the tank, $y = 1.5$, so we solve:

$$1.5 = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2$$

for t, finding

$$t = \frac{1}{0.0002\sqrt{19.6}}(2\sqrt{10} - \sqrt{6}) \approx 4376.44 \text{ sec.}$$

When all of the water is out of the tank, $y = 0$, so

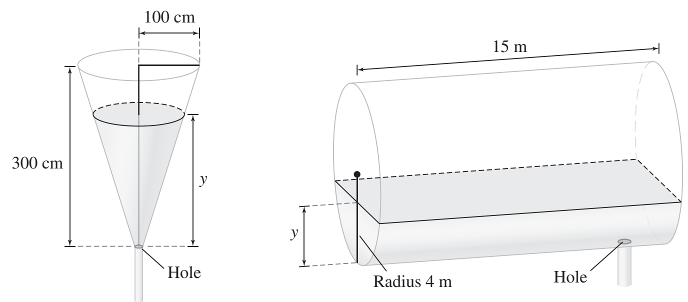
$$\sqrt{10} - 0.0001\sqrt{19.6}t = 0 \quad \text{and} \quad t = \frac{\sqrt{10}}{0.0001\sqrt{19.6}} \approx 7142.86 \text{ sec.}$$

48. At $t = 0$, a conical tank of height 300 cm and top radius 100 cm [Figure 7(A)] is filled with water. Water leaks through a hole in the bottom of area 3 cm^2. Let $y(t)$ be the water level at time t.

(a) Show that the tank's cross-sectional area at height y is $A(y) = \frac{\pi}{9}y^2$.

(b) Find and solve the differential equation satisfied by $y(t)$

(c) How long does it take for the tank to empty?



(A) Conical tank (B) Horizontal tank

FIGURE 7

SOLUTION

(a) By similar triangles, the radius r at height y satisfies

$$\frac{r}{y} = \frac{100}{300} = \frac{1}{3},$$

so $r = y/3$ and

$$A(y) = \pi r^2 = \frac{\pi}{9}y^2.$$

(b) The area of the hole is $B = 3$ cm^2, so the differential equation for the height of the water in the tank becomes:

$$\frac{dy}{dt} = -\frac{3\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{27\sqrt{19.6}}{\pi}y^{-3/2}.$$

Separating variables and integrating then yields

$$y^{3/2}\,dy = -\frac{27\sqrt{19.6}}{\pi}\,dt$$

$$\frac{2}{5}y^{5/2} = C - \frac{27\sqrt{19.6}}{\pi}t$$

When $t = 0$, $y = 300$, so we find $C = \frac{2}{5}(300)^{5/2}$. Therefore,

$$y(t) = \left(300^{5/2} - \frac{135\sqrt{19.6}}{2\pi}t\right)^{2/5}.$$

(c) The tank is empty when $y = 0$. Using the result from part (b), $y = 0$ when

$$t = \frac{4000\pi\sqrt{300}}{3\sqrt{19.6}} \approx 16{,}387.82 \text{ seconds.}$$

Thus, it takes roughly 4 hours, 33 minutes for the tank to empty.

49. The tank in Figure 7(B) is a cylinder of radius 4 m and height 15 m. Assume that the tank is half-filled with water and that water leaks through a hole in the bottom of area $B = 0.001$ m^2. Determine the water level $y(t)$ and the time t_e when the tank is empty.

SOLUTION When the water is at height y over the bottom, the top cross section is a rectangle with length 15 m, and with width x satisfying the equation:

$$(x/2)^2 + (y-4)^2 = 16.$$

Thus, $x = 2\sqrt{8y - y^2}$, and

$$A(y) = 15x = 30\sqrt{8y - y^2}.$$

With $B = 0.001$ m^2 and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.001\sqrt{19.6}\sqrt{y}}{30\sqrt{8y-y^2}} = -\frac{0.001\sqrt{19.6}}{30\sqrt{8-y}}.$$

Separating variables and integrating then yields:

$$\sqrt{8-y}\,dy = -\frac{0.001\sqrt{19.6}}{30}\,dt = -\frac{0.0001\sqrt{19.6}}{3}\,dt$$
$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t + C$$

When $t = 0$, $y = 4$, so $C = -\frac{2}{3}4^{3/2} = -\frac{16}{3}$, and

$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t - \frac{16}{3}$$
$$y(t) = 8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3}.$$

The tank is empty when $y = 0$. Thus, t_e satisfies the equation

$$8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3} = 0.$$

It follows that

$$t_e = \frac{2(8^{3/2} - 8)}{0.0001\sqrt{19.6}} \approx 66{,}079.9 \text{ seconds.}$$

50. A tank has the shape of the parabola $y = x^2$, revolved around the y-axis. Water leaks from a hole of area $B = 0.0005$ m^2 at the bottom of the tank. Let $y(t)$ be the water level at time t. How long does it take for the tank to empty if it is initially filled to height $y_0 = 1$ m.

SOLUTION When the water is at height y, the surface of the water is a circle with radius $\sqrt{y}$, so the cross-sectional area is $A(y) = \pi y$. With $B = 0.0005$ m and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{\pi y} = -\frac{0.0005\sqrt{19.6}}{\pi\sqrt{y}}$$

Separating variables and integrating yields

$$\pi y^{1/2}\,dy = -0.0005\sqrt{19.6}\,dt$$
$$\frac{2}{3}\pi y^{3/2} = -0.0005\sqrt{19.6}t + C$$
$$y^{3/2} = -\frac{0.00075\sqrt{19.6}}{\pi}t + C$$

Since $y(0) = 1$, we have $C = 1$, so that

$$y = \left(1 - \frac{0.00075\sqrt{19.6}}{\pi}t\right)^{2/3}$$

The tank is empty when $y = 0$, so when $1 - \frac{0.00075\sqrt{19.6}}{\pi}t = 0$ and thus

$$t = \frac{\pi}{0.00075\sqrt{19.6}} \approx 946.15 \text{ s}$$

51. A tank has the shape of the parabola $y = ax^2$ (where a is a constant) revolved around the y-axis. Water drains from a hole of area B m^2 at the bottom of the tank.

(a) Show that the water level at time t is

$$y(t) = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

where y_0 is the water level at time $t = 0$.

(b) Show that if the total volume of water in the tank has volume V at time $t = 0$, then $y_0 = \sqrt{2aV/\pi}$. *Hint:* Compute the volume of the tank as a volume of rotation.

(c) Show that the tank is empty at time

$$t_e = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

We see that for fixed initial water volume V, the time t_e is proportional to $a^{-1/4}$. A large value of a corresponds to a tall thin tank. Such a tank drains more quickly than a short wide tank of the same initial volume.

SOLUTION

(a) When the water is at height y, the surface of the water is a circle of radius $\sqrt{y/a}$, so that the cross-sectional area is $A(y) = \pi y/a$. With $v = -\sqrt{2gy} = -\sqrt{2g}\sqrt{y}$, we have

$$\frac{dy}{dt} = -\frac{B\sqrt{2g}\sqrt{y}}{A} = -\frac{aB\sqrt{2g}\sqrt{y}}{\pi y} = -\frac{aB\sqrt{2g}}{\pi}y^{-1/2}$$

Separating variables and integrating gives

$$\sqrt{y}\,dy = -\frac{aB\sqrt{2g}}{\pi}\,dt$$

$$\frac{2}{3}y^{3/2} = -\frac{aB\sqrt{2g}}{\pi}t + C_1$$

$$y^{3/2} = -\frac{3aB\sqrt{2g}}{2\pi}t + C$$

Since $y(0) = y_0$, we have $C = y_0^{3/2}$; solving for y gives

$$y = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

(b) The volume of the tank can be computed as a volume of rotation. Using the disk method and applying it to the function $x = \sqrt{y/a}$, we have

$$V = \int_0^{y_0} \pi\sqrt{\frac{y}{a}}^2\,dy = \frac{\pi}{a}\int_0^{y_0} y\,dy = \frac{\pi}{2a}y^2\Big|_0^{y_0} = \frac{\pi}{2a}y_0^2$$

Solving for y_0 gives

$$y_0 = \sqrt{2aV/\pi}$$

(c) The tank is empty when $y = 0$; this occurs when

$$y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t = 0$$

From part (b), we have

$$y_0^{3/2} = \sqrt{2aV/\pi}^{3/2} = ((2aV/\pi)^{1/2})^{3/2} = (2aV/\pi)^{3/4}$$

so that

$$t_e = \frac{2\pi y_0^{3/2}}{3aB\sqrt{2g}} = \frac{2\pi\sqrt[4]{8a^3V^3}}{3\pi^{3/4}B\sqrt[4]{a^4}\sqrt[4]{4}\sqrt{g}} = \frac{2\pi^{1/4}\sqrt[4]{2V^3a^{-1}}}{3B\sqrt{g}} = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

52. A cylindrical tank filled with water has height h and a base of area A. Water leaks through a hole in the bottom of area B.

(a) Show that the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
(b) Show that the emptying time is proportional to $Vh^{-1/2}$, where V is the volume of the tank.
(c) Two tanks have the same volume and a hole of the same size, but they have different heights and bases. Which tank empties first: the taller or the shorter tank?

SOLUTION Torricelli's law gives the differential equation for the height of the water in the tank as

$$\frac{dy}{dt} = -\sqrt{2g}\frac{B\sqrt{y}}{A}$$

Separating variables and integrating then yields:

$$y^{-1/2}\,dy = -\sqrt{2g}\frac{B}{A}\,dt$$

$$2y^{1/2} = -\sqrt{2g}\frac{Bt}{A} + C$$

$$y^{1/2} = -\sqrt{g/2}\frac{Bt}{A} + C$$

When $t = 0$, $y = h$, so $C = h^{1/2}$ and

$$y^{1/2} = \sqrt{h} - \sqrt{g/2}\frac{Bt}{A}.$$

(a) When the tank is empty, $y = 0$. Thus, the time required for the tank to empty, t_e, satisfies the equation

$$0 = \sqrt{h} - \sqrt{g/2}\frac{Bt_e}{A}.$$

It follows that

$$t_e = \frac{A}{B}\sqrt{2h/g} = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right);$$

that is, the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
(b) The volume of the tank is $V = Ah$; therefore

$$\frac{A\sqrt{h}}{B} = \frac{1}{B}\frac{V}{\sqrt{h}},$$

and

$$t_e = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right) = \frac{\sqrt{2/g}}{B}\left(\frac{V}{\sqrt{h}}\right);$$

that is, the time required for the tank to empty is proportional to $Vh^{-1/2}$.
(c) By part (b), with V and B held constant, the emptying time decreases with height. The taller tank therefore empties first.

53. Figure 8 shows a circuit consisting of a resistor of R ohms, a capacitor of C farads, and a battery of voltage V. When the circuit is completed, the amount of charge $q(t)$ (in coulombs) on the plates of the capacitor varies according to the differential equation (t in seconds)

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$

where R, C, and V are constants.

(a) Solve for $q(t)$, assuming that $q(0) = 0$.
(b) Show that $\lim_{t\to\infty} q(t) = CV$.
(c) Show that the capacitor charges to approximately 63% of its final value CV after a time period of length $\tau = RC$ (τ is called the time constant of the capacitor).

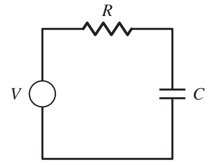


FIGURE 8 An RC circuit.

SOLUTION

(a) Upon rearranging the terms of the differential equation, we have

$$\frac{dq}{dt} = -\frac{q - CV}{RC}.$$

Separating the variables and integrating both sides, we obtain

$$\frac{dq}{q - CV} = -\frac{dt}{RC}$$

$$\int \frac{dq}{q - CV} = -\int \frac{dt}{RC}$$

and

$$\ln|q - CV| = -\frac{t}{RC} + k,$$

where k is an arbitrary constant. Solving for $q(t)$ yields

$$q(t) = CV + Ke^{-\frac{1}{RC}t},$$

where $K = \pm e^k$. We use the initial condition $q(0) = 0$ to solve for K:

$$0 = CV + K \quad \Rightarrow \quad K = -CV$$

so that the particular solution is

$$q(t) = CV(1 - e^{-\frac{1}{RC}t})$$

(b) Using the result from part (a), we calculate

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} CV(1 - e^{-\frac{1}{RC}t}) = CV(1 - \lim_{t\to\infty} 1 - e^{-\frac{1}{RC}t}) = CV.$$

(c) We have

$$q(\tau) = q(RC) = CV(1 - e^{-\frac{1}{RC}RC}) = CV(1 - e^{-1}) \approx 0.632CV.$$

54. Assume in the circuit of Figure 8 that $R = 200\ \Omega$, $C = 0.02$ F, and $V = 12$ V. How many seconds does it take for the charge on the capacitor plates to reach half of its limiting value?

SOLUTION From Exercise 53, we know that

$$q(t) = CV\left(1 - e^{-t/(RC)}\right) = 0.24(1 - e^{-t/4}),$$

and the limiting value of $q(t)$ is $CV = 0.24$. If the charge on the capacitor plates has reached half its limiting value, then

$$\begin{aligned} \frac{0.24}{2} &= 0.24(1 - e^{-t/4}) \\ 1 - e^{-t/4} &= 1/2 \\ e^{-t/4} &= 1/2 \\ t &= 4\ln 2 \end{aligned}$$

Therefore, the charge on the capacitor plates reaches half of its limiting value after $4\ln 2 \approx 2.773$ seconds.

55. According to one hypothesis, the growth rate dV/dt of a cell's volume V is proportional to its surface area A. Since V has cubic units such as cm^3 and A has square units such as cm^2, we may assume roughly that $A \propto V^{2/3}$, and hence $dV/dt = kV^{2/3}$ for some constant k. If this hypothesis is correct, which dependence of volume on time would we expect to see (again, roughly speaking) in the laboratory?

(a) Linear **(b)** Quadratic **(c)** Cubic

SOLUTION Rewrite

$$\frac{dV}{dt} = kV^{2/3} \qquad \text{as} \qquad V^{-2/3}\,dv = k\,dt,$$

and then integrate both sides to obtain

$$3V^{1/3} = kt + C$$
$$V = (kt/3 + C)^3.$$

Thus, we expect to see V increasing roughly like the cube of time.

56. We might also guess that the volume V of a melting snowball decreases at a rate proportional to its surface area. Argue as in Exercise 55 to find a differential equation satisfied by V. Suppose the snowball has volume 1000 cm^3 and that it loses half of its volume after 5 min. According to this model, when will the snowball disappear?

SOLUTION Since the volume is decreasing, we write (as in Exercise 55) $V' = -kV^{2/3}$ where k is positive, so $V(t) = (C - kt/3)^3$. $V(0) = 1000$ implies that $C = 10$ so $V(t) = (10 - kt/3)^3$. Since it loses half of its volume after 5 minutes, we have $V(5) = \frac{1}{2}V(0)$, so that

$$(10 - 5k/3)^3 = 500 \qquad \text{so that} \quad k = 6 - 3 \cdot 2^{2/3} \approx 1.2378$$

and finally the equation is

$$V(t) = \left(10 - \frac{1.2378t}{3}\right)^3$$

The snowball is melted when its volume is zero, so when

$$10 - \frac{1.2378t}{3} = 0 \quad \Rightarrow \quad t = \frac{30}{1.2378} \approx 24.24 \text{ minutes}$$

57. In general, $(fg)'$ is not equal to $f'g'$, but let $f(x) = e^{3x}$ and find a function $g(x)$ such that $(fg)' = f'g'$. Do the same for $f(x) = x$.

SOLUTION If $(fg)' = f'g'$, we have

$$f'(x)g(x) + g'(x)f(x) = f'(x)g'(x)$$
$$g'(x)(f(x) - f'(x)) = -g(x)f'(x)$$
$$\frac{g'(x)}{g(x)} = \frac{f'(x)}{f'(x) - f(x)}$$

Now, let $f(x) = e^{3x}$. Then $f'(x) = 3e^{3x}$ and

$$\frac{g'(x)}{g(x)} = \frac{3e^{3x}}{3e^{3x} - e^{3x}} = \frac{3}{2}.$$

Integrating and solving for $g(x)$, we find

$$\frac{dg}{g} = \frac{3}{2}\,dx$$
$$\ln|g| = \frac{3}{2}x + C$$
$$g(x) = Ae^{(3/2)x},$$

where $A = \pm e^C$ is an arbitrary constant.

If $f(x) = x$, then $f'(x) = 1$, and

$$\frac{g'(x)}{g(x)} = \frac{1}{1-x}.$$

Thus,

$$\frac{dg}{g} = \frac{1}{1-x}\,dx$$
$$\ln|g| = -\ln|1-x| + C$$
$$g(x) = \frac{A}{1-x},$$

where $A = \pm e^C$ is an arbitrary constant.

58. A boy standing at point B on a dock holds a rope of length ℓ attached to a boat at point A [Figure 9(A)]. As the boy walks along the dock, holding the rope taut, the boat moves along a curve called a **tractrix** (from the Latin *tractus*, meaning "to pull"). The segment from a point P on the curve to the x-axis along the tangent line has constant length ℓ. Let $y = f(x)$ be the equation of the tractrix.

(a) Show that $y^2 + (y/y')^2 = \ell^2$ and conclude $y' = -\frac{y}{\sqrt{\ell^2 - y^2}}$. Why must we choose the negative square root?

(b) Prove that the tractrix is the graph of

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}$$

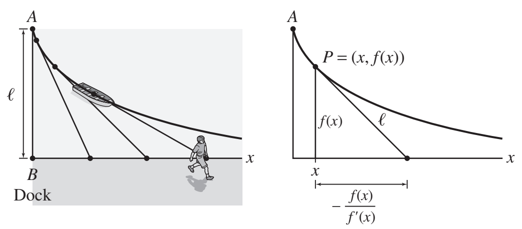


FIGURE 9

SOLUTION

(a) From the diagram on the right in Figure 9, we see that

$$f(x)^2 + \left(-\frac{f(x)}{f'(x)}\right)^2 = \ell^2.$$

If we let $y = f(x)$, this last equation reduces to $y^2 + (y/y')^2 = \ell^2$. Solving for y', we find

$$y' = -\frac{y}{\sqrt{\ell^2 - y^2}},$$

where we must choose the negative sign because y is a decreasing function of x.

(b) Rewrite

$$\frac{dy}{dx} = -\frac{y}{\sqrt{\ell^2 - y^2}} \qquad \text{as} \qquad \frac{\sqrt{\ell^2 - y^2}}{y}\, dy = -dx,$$

and then integrate both sides to obtain

$$-x + C = \int \frac{\sqrt{\ell^2 - y^2}}{y}\, dy.$$

For the remaining integral, we use the trigonometric substitution $y = \ell \sin\theta$, $dy = \ell \cos\theta\, d\theta$. Then

$$\int \frac{\sqrt{\ell^2 - y^2}}{y}\, dy = \ell \int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \ell \int \frac{1 - \sin^2\theta}{\sin\theta}\, d\theta = \ell \int (\csc\theta - \sin\theta)\, d\theta$$

$$= \ell\,[\ln|\csc\theta - \cot\theta| + \cos\theta] + C = \ell \ln\left(\frac{\ell}{y} - \frac{\sqrt{\ell^2 - y^2}}{y}\right) + \sqrt{\ell^2 - y^2} + C$$

Therefore,

$$x = -\ell \ln\left(\frac{\ell - \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C = \ell \ln\left(\frac{y}{\ell - \sqrt{\ell^2 - y^2}}\right) - \sqrt{\ell^2 - y^2} + C$$

$$= \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C$$

Now, when $x = 0$, $y = \ell$, so we find $C = 0$. Finally, the equation for the tractrix is

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}.$$

59. Show that the differential equations $y' = 3y/x$ and $y' = -x/3y$ define **orthogonal families** of curves; that is, the graphs of solutions to the first equation intersect the graphs of the solutions to the second equation in right angles (Figure 10). Find these curves explicitly.

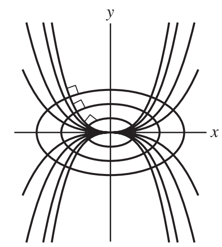


FIGURE 10 Two orthogonal families of curves.

SOLUTION Let y_1 be a solution to $y' = \frac{3y}{x}$ and let y_2 be a solution to $y' = -\frac{x}{3y}$. Suppose these two curves intersect at a point (x_0, y_0). The line tangent to the curve $y_1(x)$ at (x_0, y_0) has a slope of $\frac{3y_0}{x_0}$ and the line tangent to the curve $y_2(x)$ has a slope of $-\frac{x_0}{3y_0}$. The slopes are negative reciprocals of one another; hence the tangent lines are perpendicular.

Separation of variables and integration applied to $y' = \frac{3y}{x}$ gives

$$\frac{dy}{y} = 3\frac{dx}{x}$$
$$\ln|y| = 3\ln|x| + C$$
$$y = Ax^3$$

On the other hand, separation of variables and integration applied to $y' = -\frac{x}{3y}$ gives

$$3y\,dy = -x\,dx$$
$$3y^2/2 = -x^2/2 + C$$
$$y = \pm\sqrt{C - x^2/3}$$

60. Find the family of curves satisfying $y' = x/y$ and sketch several members of the family. Then find the differential equation for the orthogonal family (see Exercise 59), find its general solution, and add some members of this orthogonal family to your plot.

SOLUTION Separation of variables and integration applied to $y' = x/y$ gives

$$y\,dy = x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y = \pm\sqrt{x^2 + C}$$

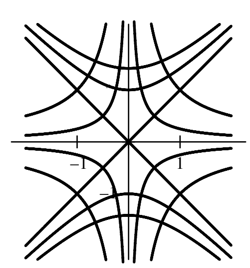
If $y(x)$ is a curve of the family orthogonal to these, it must have tangent lines of slope $-y/x$ at every point (x, y). This gives

$$y' = -y/x$$

Separation of variables and integration give

$$\frac{dy}{y} = -\frac{dx}{x}$$
$$\ln|y| = -\ln|x| + C$$
$$y = \frac{A}{x}$$

Several solution curves of both differential equations appear below:



61. A 50-kg model rocket lifts off by expelling fuel at a rate of $k = 4.75$ kg/s for 10 s. The fuel leaves the end of the rocket with an exhaust velocity of $b = -100$ m/s. Let $m(t)$ be the mass of the rocket at time t. From the law of conservation of momentum, we find the following differential equation for the rocket's velocity $v(t)$ (in meters per second):

$$m(t)v'(t) = -9.8m(t) + b\frac{dm}{dt}$$

(a) Show that $m(t) = 50 - 4.75t$ kg.
(b) Solve for $v(t)$ and compute the rocket's velocity at rocket burnout (after 10 s).

SOLUTION

(a) For $0 \le t \le 10$, the rocket is expelling fuel at a constant rate of 4.75 kg/s, giving $m'(t) = -4.75$. Hence, $m(t) = -4.75t + C$. Initially, the rocket has a mass of 50 kg, so $C = 50$. Therefore, $m(t) = 50 - 4.75t$.

(b) With $m(t) = 50 - 4.75t$ and $\dfrac{dm}{dt} = -4.75$, the equation for v becomes

$$\frac{dv}{dt} = -9.8 + \frac{b\frac{dm}{dt}}{50 - 4.75t} = -9.8 + \frac{(100)(-4.75)}{50 - 4.75t}$$

and therefore

$$v(t) = -9.8t + 100\int \frac{4.75\,dt}{50 - 4.75t} = -9.8t - 100\ln(50 - 4.75t) + C$$

Because $v(0) = 0$, we find $C = 100\ln 50$ and

$$v(t) = -9.8t - 100\ln(50 - 4.75t) + 100\ln(50).$$

After 10 seconds the velocity is:

$$v(10) = -98 - 100\ln(2.5) + 100\ln(50) \approx 201.573 \text{ m/s}.$$

62. Let $v(t)$ be the velocity of an object of mass m in free fall near the earth's surface. If we assume that air resistance is proportional to v^2, then v satisfies the differential equation $m\frac{dv}{dt} = -g + kv^2$ for some constant $k > 0$.

(a) Set $\alpha = (g/k)^{1/2}$ and rewrite the differential equation as

$$\frac{dv}{dt} = -\frac{k}{m}(\alpha^2 - v^2)$$

Then solve using separation of variables with initial condition $v(0) = 0$.
(b) Show that the terminal velocity $\lim_{t\to\infty} v(t)$ is equal to $-\alpha$.

SOLUTION

(a) Let $\alpha = (g/k)^{1/2}$. Then

$$\frac{dv}{dt} = -\frac{g}{m} + \frac{k}{m}v^2 = -\frac{k}{m}\left(\frac{g}{k} - v^2\right) = -\frac{k}{m}\left(\alpha^2 - v^2\right)$$

Separating variables and integrating yields

$$\int \frac{dv}{\alpha^2 - v^2} = -\frac{k}{m}\int dt = -\frac{k}{m}t + C$$

We now use partial fraction decomposition for the remaining integral to obtain

$$\int \frac{dv}{\alpha^2 - v^2} = \frac{1}{2\alpha}\int \left(\frac{1}{\alpha + v} + \frac{1}{\alpha - v}\right)dv = \frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right|$$

Therefore,

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right| = -\frac{k}{m}t + C.$$

The initial condition $v(0) = 0$ allows us to determine the value of C:

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + 0}{\alpha - 0}\right| = -\frac{k}{m}(0) + C$$

$$C = \frac{1}{2\alpha}\ln 1 = 0.$$

Finally, solving for v, we find

$$v(t) = -\alpha\left(\frac{1 - e^{-2(\sqrt{gk}/m)t}}{1 + e^{-2(\sqrt{gk}/m)t}}\right).$$

(b) As $t \to \infty$, $e^{-2(\sqrt{gk}/m)t} \to 0$, so

$$v(t) \to -\alpha\left(\frac{1-0}{1+0}\right) = -\alpha.$$

63. If a bucket of water spins about a vertical axis with constant angular velocity ω (in radians per second), the water climbs up the side of the bucket until it reaches an equilibrium position (Figure 11). Two forces act on a particle located at a distance x from the vertical axis: the gravitational force $-mg$ acting downward and the force of the bucket on the particle (transmitted indirectly through the liquid) in the direction perpendicular to the surface of the water. These two forces must combine to supply a centripetal force $m\omega^2 x$, and this occurs if the diagonal of the rectangle in Figure 11 is normal to the water's surface (that is, perpendicular to the tangent line). Prove that if $y = f(x)$ is the equation of the curve obtained by taking a vertical cross section through the axis, then $-1/y' = -g/(\omega^2 x)$. Show that $y = f(x)$ is a parabola.

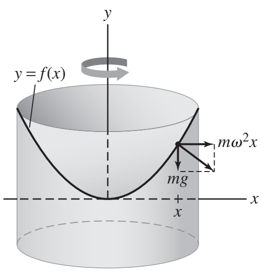


FIGURE 11

SOLUTION At any point along the surface of the water, the slope of the tangent line is given by the value of y' at that point; hence, the slope of the line perpendicular to the surface of the water is given by $-1/y'$. The slope of the resultant force generated by the gravitational force and the centrifugal force is

$$\frac{-mg}{m\omega^2 x} = -\frac{g}{\omega^2 x}.$$

Therefore, the curve obtained by taking a vertical cross-section of the water surface is determined by the equation

$$-\frac{1}{y'} = -\frac{g}{\omega^2 x} \qquad \text{or} \qquad y' = \frac{\omega^2}{g}x.$$

Performing one integration yields

$$y = f(x) = \frac{\omega^2}{2g}x^2 + C,$$

where C is a constant of integration. Thus, $y = f(x)$ is a parabola.

Further Insights and Challenges

64. In Section 6.2, we computed the volume V of a solid as the integral of cross-sectional area. Explain this formula in terms of differential equations. Let $V(y)$ be the volume of the solid up to height y, and let $A(y)$ be the cross-sectional area at height y as in Figure 12.

(a) Explain the following approximation for small Δy:

$$V(y + \Delta y) - V(y) \approx A(y)\,\Delta y \tag{8}$$

(b) Use Eq. (8) to justify the differential equation $dV/dy = A(y)$. Then derive the formula

$$V = \int_a^b A(y)\,dy$$

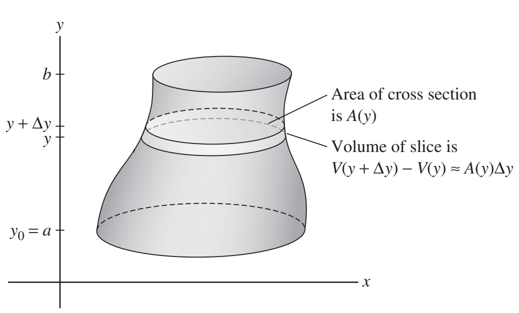


FIGURE 12

SOLUTION

(a) If Δy is very small, then the slice between y and $y + \Delta y$ is very similar to the *prism* formed by thickening the cross-sectional area $A(y)$ by a thickness of Δy. A prism with cross-sectional area A and height Δy has volume $A\Delta y$. This gives

$$V(y + \Delta y) - V(y) \approx A(y)\Delta y.$$

(b) Dividing Eq. (8) by Δy, we obtain

$$\frac{V(y + \Delta y) - V(y)}{\Delta y} \approx A(y).$$

In the limit as $\Delta y \to 0$, this becomes

$$\frac{dV}{dy} = A(y).$$

Integrating this last equation yields

$$V = \int_a^b A(y)\,dy.$$

65. A basic theorem states that a *linear* differential equation of order n has a general solution that depends on n arbitrary constants. There are, however, nonlinear exceptions.

(a) Show that $(y')^2 + y^2 = 0$ is a first-order equation with only one solution $y = 0$.

(b) Show that $(y')^2 + y^2 + 1 = 0$ is a first-order equation with no solutions.

SOLUTION

(a) $(y')^2 + y^2 \geq 0$ and equals zero if and only if $y' = 0$ and $y = 0$

(b) $(y')^2 + y^2 + 1 \geq 1 > 0$ for all y' and y, so $(y')^2 + y^2 + 1 = 0$ has no solution

66. Show that $y = Ce^{rx}$ is a solution of $y'' + ay' + by = 0$ if and only if r is a root of $P(r) = r^2 + ar + b$. Then verify directly that $y = C_1e^{3x} + C_2e^{-x}$ is a solution of $y'' - 2y' - 3y = 0$ for any constants C_1, C_2.

SOLUTION

(a) Let $y(x) = Ce^{rx}$. Then $y' = rCe^{rx}$, and $y'' = r^2Ce^{rx}$. Thus

$$y'' + ay' + by = r^2Ce^{rx} + arCe^{rx} + bCe^{rx} = Ce^{rx}\left(r^2 + ar + b\right) = Ce^{rx}P(r).$$

Hence, Ce^{rx} is a solution of the differential equation $y'' + ay' + by = 0$ if and only if $P(r) = 0$.

(b) Let $y(x) = C_1e^{3x} + C_2e^{-x}$. Then

$$y'(x) = 3C_1e^{3x} - C_2e^{-x}$$
$$y''(x) = 9C_1e^{3x} + C_2e^{-x}$$

and

$$\begin{aligned} y'' - 2y' - 3y &= 9C_1e^{3x} + C_2e^{-x} - 6C_1e^{3x} + 2C_2e^{-x} - 3C_1e^{3x} - 3C_2e^{-x} \\ &= (9 - 6 - 3)\,C_1e^{3x} + (1 + 2 - 3)\,C_2e^{-x} = 0. \end{aligned}$$

67. A spherical tank of radius R is half-filled with water. Suppose that water leaks through a hole in the bottom of area B. Let $y(t)$ be the water level at time t (seconds).

(a) Show that $\dfrac{dy}{dt} = \dfrac{-\sqrt{2g}B\sqrt{y}}{\pi(2Ry - y^2)}$.

(b) Show that for some constant C,

$$\frac{2\pi}{15B\sqrt{2g}}\left(10Ry^{3/2} - 3y^{5/2}\right) = C - t$$

(c) Use the initial condition $y(0) = R$ to compute C, and show that $C = t_e$, the time at which the tank is empty.

(d) Show that t_e is proportional to $R^{5/2}$ and inversely proportional to B.

SOLUTION

(a) At height y above the bottom of the tank, the cross section is a circle of radius

$$r = \sqrt{R^2 - (R - y)^2} = \sqrt{2Ry - y^2}.$$

The cross-sectional area function is then $A(y) = \pi(2Ry - y^2)$. The differential equation for the height of the water in the tank is then

$$\frac{dy}{dt} = -\frac{\sqrt{2g}B\sqrt{y}}{\pi(2Ry - y^2)}$$

by Torricelli's law.

(b) Rewrite the differential equation as

$$\frac{\pi}{\sqrt{2g}B}\left(2Ry^{1/2} - y^{3/2}\right)dy = -dt,$$

and then integrate both sides to obtain

$$\frac{2\pi}{\sqrt{2g}B}\left(\frac{2}{3}Ry^{3/2} - \frac{1}{5}y^{5/2}\right) = C - t,$$

where C is an arbitrary constant. Simplifying gives

$$\frac{2\pi}{15B\sqrt{2g}}(10Ry^{3/2} - 3y^{5/2}) = C - t \qquad (*)$$

(c) From Equation (*) we see that $y = 0$ when $t = C$. It follows that $C = t_e$, the time at which the tank is empty. Moreover, the initial condition $y(0) = R$ allows us to determine the value of C:

$$\frac{2\pi}{15B\sqrt{2g}}(10R^{5/2} - 3R^{5/2}) = \frac{14\pi}{15B\sqrt{2g}}R^{5/2} = C$$

(d) From part (c),

$$t_e = \frac{14\pi}{15\sqrt{2g}} \cdot \frac{R^{5/2}}{B},$$

from which it is clear that t_e is proportional to $R^{5/2}$ and inversely proportional to B.

9.2 Models Involving $y' = k(y - b)$

Preliminary Questions

1. Write down a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

SOLUTION The general solution is $y(t) = 5 + Ce^{4t}$ for any constant C; thus the solution tends to $-\infty$ as $t \to \infty$ whenever $C < 0$. One specific example is $y(t) = 5 - e^{4t}$.

2. Does $y' = -4(y - 5)$ have a solution that tends to ∞ as $t \to \infty$?

SOLUTION The general solution is $y(t) = 5 + Ce^{-4t}$ for any constant C. As $t \to \infty$, $y(t) \to 5$. Thus, there is no solution of $y' = -4(y - 5)$ that tends to ∞ as $t \to \infty$.

3. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

SOLUTION True. The general solution of $y' = -k(y - b)$ is $y(t) = b + Ce^{-kt}$ for any constant C. If $k > 0$, then $y(t) \to b$ as $t \to \infty$.

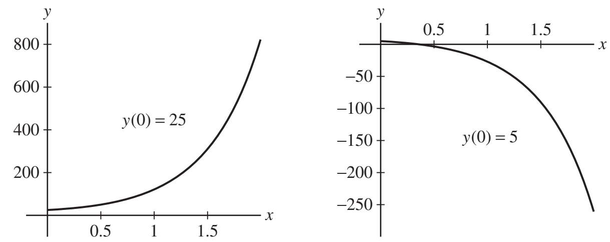
4. As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

SOLUTION Newton's Law of Cooling states that $y' = -k(y - T_0)$ where $y(t)$ is the temperature and T_0 is the ambient temperature. Thus as $y(t)$ gets closer to T_0, $y'(t)$, the rate of cooling, gets smaller and the rate of cooling slows.

Exercises

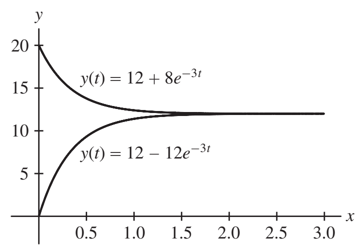
1. Find the general solution of $y' = 2(y - 10)$. Then find the two solutions satisfying $y(0) = 25$ and $y(0) = 5$, and sketch their graphs.

SOLUTION The general solution of $y' = 2(y - 10)$ is $y(t) = 10 + Ce^{2t}$ for any constant C. If $y(0) = 25$, then $10 + C = 25$, or $C = 15$; therefore, $y(t) = 10 + 15e^{2t}$. On the other hand, if $y(0) = 5$, then $10 + C = 5$, or $C = -5$; therefore, $y(t) = 10 - 5e^{2t}$. Graphs of these two functions are given below.



2. Verify directly that $y = 12 + Ce^{-3t}$ satisfies $y' = -3(y - 12)$ for all C. Then find the two solutions satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

SOLUTION The general solution of $y' = -3(y - 12)$ is $y(t) = 12 + Ce^{-3t}$ for any constant C. If $y(0) = 20$, then $12 + C = 20$, or $C = 8$; therefore, $y(t) = 12 + 8e^{-3t}$. On the other hand, if $y(0) = 0$, then $12 + C = 0$, or $C = -12$; therefore, $y(t) = 12 - 12e^{-3t}$. Graphs of these two functions are given below.



3. Solve $y' = 4y + 24$ subject to $y(0) = 5$.

SOLUTION Rewrite

$$y' = 4y + 24 \quad \text{as} \quad \frac{1}{4y + 24}\,dy = 1\,dt$$

Integrating gives

$$\begin{aligned} \frac{1}{4}\ln|4y + 24| &= t + C \\ \ln|4y + 24| &= 4t + C \\ 4y + 24 &= \pm e^{4t+C} \\ y &= Ae^{4t} - 6 \end{aligned}$$

where $A = \pm e^{C}/4$ is any constant. Since $y(0) = 5$ we have $5 = A - 6$ so that $A = 11$, and the solution is $y = 11e^{4t} - 6$.

4. Solve $y' + 6y = 12$ subject to $y(2) = 10$.

SOLUTION Rewrite

$$y' + 6y = 12 \quad \text{as} \quad \frac{dy}{dt} = 12 - 6y \quad \text{and then as} \quad \frac{1}{12 - 6y}\,dy = 1\,dt$$

Integrate both sides:

$$\begin{aligned} -\frac{1}{6}\ln|12 - 6y| &= t + C \\ \ln|12 - 6y| &= -6t + C \\ 12 - 6y &= \pm e^{-6t+C} \\ y &= Ae^{-6t} + 2 \end{aligned}$$

where $A = \pm e^{C}/6$ is any constant. Since $y(2) = 10$ we have $10 = Ae^{-12} + 2$ so that $A = 8e^{12}$, and the solution is $y = 8e^{12-6t} + 2$.

In Exercises 5–12, use Newton's Law of Cooling.

5. A hot anvil with cooling constant $k = 0.02\ \text{s}^{-1}$ is submerged in a large pool of water whose temperature is 10°C. Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?
(b) Find a formula for $y(t)$, assuming the object's initial temperature is 100°C.
(c) How long does it take the object to cool down to 20°?

SOLUTION

(a) By Newton's Law of Cooling, the differential equation is

$$y' = -0.02(y - 10)$$

(b) Separating variables gives

$$\frac{1}{y - 10}\,dy = -0.02\,dt$$

Integrate to get

$$\ln|y - 10| = -0.02t + C$$
$$y - 10 = \pm e^{-0.02t+C}$$
$$y = 10 + Ae^{-0.02t}$$

where $A = \pm e^C$ is a constant. Since the initial temperature is 100°C, we have $y(0) = 100 = 10 + A$ so that $A = 90$, and $y = 10 + 90e^{-0.02t}$.

(c) We must find the value of t such that $y(t) = 20$, so we need to solve $20 = 10 + 90e^{-0.02t}$. Thus

$$10 = 90e^{-0.02t} \quad \Rightarrow \quad \frac{1}{9} = e^{-0.02t} \quad \Rightarrow \quad -\ln 9 = -0.02t \quad \Rightarrow \quad t = 50\ln 9 \approx 109.86 \text{ s}$$

6. Frank's automobile engine runs at 100°C. On a day when the outside temperature is 21°C, he turns off the ignition and notes that five minutes later, the engine has cooled to 70°C.

(a) Determine the engine's cooling constant k.

(b) What is the formula for $y(t)$?

(c) When will the engine cool to 40°C?

SOLUTION

(a) The differential equation is

$$y' = -k(y - 21)$$

Rewriting gives $\frac{1}{y - 21}\,dy = -k\,dt$. Integrate to get

$$\ln|y - 21| = -kt + C$$
$$y - 21 = \pm e^{C-kt}$$
$$y = 21 + Ae^{-kt}$$

where $A = \pm e^C$ is a constant. The initial temperature is 100°C, so $y(0) = 100$. Thus $100 = 21 + A$ and $A = 79$, so that $y = 21 + 79e^{-kt}$. The second piece of information tells us that $y(5) = 70 = 21 + 79e^{-5k}$. Solving for k gives

$$k = -\frac{1}{5}\ln\frac{49}{79} \approx 0.0955$$

(b) From part (b), the equation is $y = 21 + 79e^{-0.0955t}$.

(c) The engine has cooled to 40°C when $y(t) = 40$; solving gives

$$40 = 21 + 79e^{-0.0955t} \quad \Rightarrow \quad e^{-0.0955t} = \frac{19}{79} \quad \Rightarrow \quad t = -\frac{1}{0.0955}\ln\frac{19}{79} \approx 14.92 \text{ m}$$

7. At 10:30 AM, detectives discover a dead body in a room and measure its temperature at 26°C. One hour later, the body's temperature had dropped to 24.8°C. Determine the time of death (when the body temperature was a normal 37°C), assuming that the temperature in the room was held constant at 20°C.

SOLUTION Let $t = 0$ be the time when the person died, and let t_0 denote 10:30AM. The differential equation satisfied by the body temperature, $y(t)$, is

$$y' = -k(y - 20)$$

by Newton's Law of Cooling. Separating variables gives $\frac{1}{y-20}\,dy = -k\,dt$. Integrate to get

$$\ln|y-20| = -kt + C$$
$$y - 20 = \pm e^{-kt+C}$$
$$y = 20 + Ae^{-kt}$$

where $A = \pm e^C$ is a constant. Since normal body temperature is 37°C, we have $y(0) = 37 = 20 + A$ so that $A = 17$. To determine k, note that

$$26 = 20 + 17e^{-kt_0} \qquad \text{and} \qquad 24.8 = 20 + 17e^{-k(t_0+1)}$$
$$kt_0 = -\ln\frac{6}{17} \qquad\qquad kt_0 + k = -\ln\frac{4.8}{17}$$

Subtracting these equations gives

$$k = \ln\frac{6}{17} - \ln\frac{4.8}{17} = \ln\frac{6}{4.8} \approx 0.223$$

We thus have

$$y = 20 + 17e^{-0.223t}$$

as the equation for the body temperature at time t. Since $y(t_0) = 26$, we have

$$26 = 20 + 17e^{-0.223t} \quad \Rightarrow \quad e^{-0.223t} = \frac{6}{17} \quad \Rightarrow \quad t = -\frac{1}{0.223}\ln\frac{6}{17} \approx 4.667 \text{ h}$$

so that the time of death was approximately 4 hours and 40 minutes ago.

8. A cup of coffee with cooling constant $k = 0.09$ min^{-1} is placed in a room at temperature 20°C.

(a) How fast is the coffee cooling (in degrees per minute) when its temperature is $T = 80$°C?

(b) Use the Linear Approximation to estimate the change in temperature over the next 6 s when $T = 80$°C.

(c) If the coffee is served at 90°C, how long will it take to reach an optimal drinking temperature of 65°C?

SOLUTION

(a) According to Newton's Law of Cooling, the coffee will cool at the rate $k(T - T_0)$, where k is the cooling constant of the coffee, T is the current temperature of the coffee and T_0 is the temperature of the surroundings. With $k = 0.09$ min^{-1}, $T = 80$°C and $T_0 = 20$°C, the coffee is cooling at the rate

$$0.09(80 - 20) = 5.4°\text{C/min}.$$

(b) Using the result from part (a) and the Linear Approximation, we estimate that the coffee will cool

$$(5.4°\text{C/min})(0.1 \text{ min}) = 0.54°\text{C}$$

over the next 6 seconds.

(c) With $T_0 = 20$°C and an initial temperature of 90°C, the temperature of the coffee at any time t is $T(t) = 20 + 70e^{-0.09t}$. Solving $20 + 70e^{-0.09t} = 65$ for t yields

$$t = -\frac{1}{0.09}\ln\left(\frac{45}{70}\right) \approx 4.91 \text{ minutes.}$$

9. A cold metal bar at −30°C is submerged in a pool maintained at a temperature of 40°C. Half a minute later, the temperature of the bar is 20°C. How long will it take for the bar to attain a temperature of 30°C?

SOLUTION With $T_0 = 40$°C, the temperature of the bar is given by $F(t) = 40 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 40 + C = -30$, so $C = -70$. After 30 seconds, $F(30) = 40 - 70e^{-30k} = 20$, so

$$k = -\frac{1}{30}\ln\left(\frac{20}{70}\right) \approx 0.0418 \text{ seconds}^{-1}.$$

To attain a temperature of 30°C we must solve $40 - 70e^{-0.0418t} = 30$ for t. This yields

$$t = \frac{\ln\left(\frac{10}{70}\right)}{-0.0418} \approx 46.55 \text{ seconds.}$$

10. When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100°C to 50°C in 150 s. In another bath, the same cooling occurs in 120 s. Find the temperature of the second bath.

SOLUTION With $T_0 = 25$°C, the temperature of the object is given by $F(t) = 25 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 25 + C = 100$, so $C = 75$. After 150 seconds, $F(150) = 25 + 75e^{-150k} = 50$, so

$$k = -\frac{1}{150}\ln\left(\frac{25}{75}\right) \approx 0.0073 \text{ seconds}^{-1}.$$

If we place the same object with a temperature of 100°C into a second bath whose temperature is T_0, then the temperature of the object is given by

$$F(t) = T_0 + (100 - T_0)e^{-0.0073t}.$$

To cool from 100°C to 50°C in 120 seconds, T_0 must satisfy

$$T_0 + (100 - T_0)e^{-0.0073(120)} = 50.$$

Thus, $T_0 = 14.32$°C.

11. $\boxed{\text{GU}}$ Objects A and B are placed in a warm bath at temperature $T_0 = 40$°C. Object A has initial temperature −20°C and cooling constant $k = 0.004\ \text{s}^{-1}$. Object B has initial temperature 0°C and cooling constant $k = 0.002\ \text{s}^{-1}$. Plot the temperatures of A and B for $0 \le t \le 1000$. After how many seconds will the objects have the same temperature?

SOLUTION With $T_0 = 40$°C, the temperature of A and B are given by

$$A(t) = 40 + C_A e^{-0.004t} \qquad B(t) = 40 + C_B e^{-0.002t}$$

Since $A(0) = -20$ and $B(0) = 0$, we have

$$A(t) = 40 - 60e^{-0.004t} \qquad B(t) = 40 - 40e^{-0.002t}$$

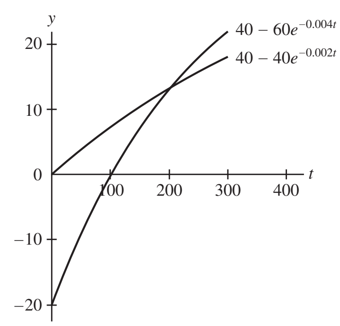
The two objects will have the same temperature whenever $A(t) = B(t)$, so we must solve

$$40 - 60e^{-0.004t} = 40 - 40e^{-0.002t} \quad \Rightarrow \quad 3e^{-0.004t} = 2e^{-0.002t}$$

Take logs to get

$$-0.004t + \ln 3 = -0.002t + \ln 2 \quad \Rightarrow \quad t = \frac{\ln 3 - \ln 2}{0.002} \approx 202.7 \text{ s}$$

or about 3 minutes 22 seconds.



12. In Newton's Law of Cooling, the constant $\tau = 1/k$ is called the "characteristic time." Show that τ is the time required for the temperature difference $(y - T_0)$ to decrease by the factor $e^{-1} \approx 0.37$. For example, if $y(0) = 100$°C and $T_0 = 0$°C, then the object cools to $100/e \approx 37$°C in time τ, to $100/e^2 \approx 13.5$°C in time 2τ, and so on.

SOLUTION If $y' = -k(y - T_0)$, then $y(t) = T_0 + Ce^{-kt}$. But then

$$\frac{y(t+\tau) - T_0}{y(t) - T_0} = \frac{Ce^{-k(t+\tau)}}{Ce^{-kt}} = e^{-k\tau} = e^{-k\cdot 1/k} = e^{-1}$$

Thus after time τ starting from any time t, the temperature difference will have decreased by a factor of e^{-1}.

In Exercises 13–16, use Eq. (3) as a model for free-fall with air resistance.

13. A 60-kg skydiver jumps out of an airplane. What is her terminal velocity, in meters per second, assuming that $k = 10$ kg/s for free-fall (no parachute)?

SOLUTION The free-fall terminal velocity is

$$\frac{-gm}{k} = \frac{-9.8(60)}{10} = -58.8 \text{ m/s.}$$

14. Find the terminal velocity of a skydiver of weight $w = 192$ lb if $k = 1.2$ lb-s/ft. How long does it take him to reach half of his terminal velocity if his initial velocity is zero? Mass and weight are related by $w = mg$, and Eq. (3) becomes $v' = -(kg/w)(v + w/k)$ with $g = 32$ ft/s^2.

SOLUTION The skydiver's velocity $v(t)$ satisfies the differential equation

$$v' = -\frac{kg}{w}\left(v + \frac{w}{k}\right),$$

where

$$\frac{kg}{w} = \frac{(1.2)(32)}{192} = 0.2 \text{ sec}^{-1} \quad \text{and} \quad \frac{w}{k} = \frac{192}{1.2} = 160 \text{ ft/sec.}$$

The general solution to this equation is $v(t) = -160 + Ce^{-0.2t}$, for some constant C. From the initial condition $v(0) = 0$, we find $0 = -160 + C$, or $C = 160$. Therefore,

$$v(t) = -160 + 160e^{-0.2t} = -160(1 - e^{-0.2t}).$$

Now, the terminal velocity of the skydiver is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -160(1 - e^{-0.2t}) = -160 \text{ ft/sec.}$$

To determine how long it takes for the skydiver to reach half this terminal velocity, we must solve the equation $v(t) = -80$ for t:

$$\begin{aligned} -160(1 - e^{-0.2t}) &= -80 \\ 1 - e^{-0.2t} &= \frac{1}{2} \\ e^{-0.2t} &= \frac{1}{2} \\ t &= -\frac{1}{0.2}\ln\frac{1}{2} \approx 3.47 \text{ sec.} \end{aligned}$$

15. A 80-kg skydiver jumps out of an airplane (with zero initial velocity). Assume that $k = 12$ kg/s with a closed parachute and $k = 70$ kg/s with an open parachute. What is the skydiver's velocity at $t = 25$ s if the parachute opens after 20 s of free fall?

SOLUTION We first compute the skydiver's velocity after 20 s of free fall, then use that as the initial velocity to calculate her velocity after an additional 5 s of restrained fall. We have $m = 80$ and $g = 9.8$; for free fall, $k = 12$, so

$$\frac{k}{m} = \frac{12}{80} = 0.15, \qquad \frac{-mg}{k} = \frac{-80 \cdot 9.8}{12} \approx -65.33$$

The general solution is thus $v(t) = -65.33 + Ce^{-0.15t}$. Since $v(0) = 0$, we have $C = 65.33$, so that

$$v(t) = -65.33(1 - e^{-0.15t})$$

After 20 s of free fall, the diver's velocity is thus

$$v(20) = -65.33(1 - e^{-0.15\cdot 20}) \approx -62.08 \text{ m/s}$$

Once the parachute opens, $k = 70$, so

$$\frac{k}{m} = \frac{70}{80} = 0.875, \qquad \frac{mg}{k} = \frac{80 \cdot 9.8}{70} = 11.2$$

so that the general solution for the restrained fall model is $v_r(t) = -11.2 + Ce^{-0.875t}$. Here $v_r(0) = -62.08$, so that $C = 11.2 - 62.08 = -50.88$ and $v_r(t) = -11.20 - 50.88e^{-0.875t}$. After 5 additional seconds, the diver's velocity is therefore

$$v_r(5) = -11.20 - 50.88e^{-0.875\cdot 5} \approx -11.84 \text{ m/s}$$

16. Does a heavier or a lighter skydiver reach terminal velocity faster?

SOLUTION The velocity of a skydiver is

$$v(t) = -\frac{gm}{k} + Ce^{-kt/m}.$$

As m decreases, the fraction $-k/m$ becomes more negative and $e^{-(k/m)t}$ approaches zero more rapidly. Thus, a lighter skydiver approaches terminal velocity faster.

17. A continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 5\%$ is funded by an initial deposit of $P_0 = \$50{,}000$.

(a) What is the balance in the annuity after 10 years?

(b) When will the annuity run out of funds?

SOLUTION

(a) From Equation , the value of the annuity is given by

$$P(t) = \frac{5000}{0.05} + Ce^{0.05t} = 100{,}000 + Ce^{0.05t}$$

for some constant C. Since $P(0) = 50{,}000$, we have $C = -50{,}000$ and $P(t) = 100{,}000 - 50{,}000e^{0.05t}$. After ten years, then, the balance in the annuity is

$$P(10) = 100{,}000 - 50{,}000e^{0.05\cdot 10} = 100{,}000 - 50{,}000e^{0.5} \approx \$17{,}563.94$$

(b) The annuity will run out of funds when $P(t) = 0$:

$$0 = 100{,}000 - 50{,}000e^{0.05t} \quad\Rightarrow\quad e^{0.05t} = 2 \quad\Rightarrow\quad t = \frac{\ln 2}{0.05} \approx 13.86$$

The annuity will run out of funds after approximately 13 years 10 months.

18. Show that a continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 8\%$, funded by an initial deposit of $P_0 = \$75{,}000$, never runs out of money.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{5000}{0.08} + Ce^{0.08t} = 62500 + Ce^{0.08t}$$

for some constant C. If $P_0 = 75{,}000$, then $75{,}000 = 62{,}500 + C$ and $C = 12{,}500$. Thus, $P(t) = 62{,}500 + 12{,}500e^{0.08t}$. As $t \to \infty$, $P(t) \to \infty$, so the annuity lives forever. Note the annuity will live forever for any $P_0 \geq \$62{,}500$.

19. Find the minimum initial deposit P_0 that will allow an annuity to pay out \$6000/year indefinitely if it earns interest at a rate of 5%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{6000}{0.05} + Ce^{0.05t} = 120{,}000 + Ce^{0.05t}$$

for some constant C. To fund the annuity indefinitely, we must have $C \geq 0$. If the initial deposit is P_0, then $P_0 = 120{,}000 + C$ and $C = P_0 - 120{,}000$. Thus, to fund the annuity indefinitely, we must have $P_0 \geq \$120{,}000$.

20. Find the minimum initial deposit P_0 necessary to fund an annuity for 20 years if withdrawals are made at a rate of \$10,000/year and interest is earned at a rate of 7%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{10{,}000}{0.07} + Ce^{0.07t} = 142{,}857.14 + Ce^{0.07t}$$

for some constant C. If the initial deposit is P_0, then $P_0 = 142{,}857.14 + C$ and $C = 142{,}857.14 - P_0$. To fund the annuity for 20 years, we need

$$P(20) = 142{,}857.14 + (P_0 - 142{,}857.14)e^{0.07(20)} \geq 0.$$

Hence,

$$P_0 \geq 142{,}857.14(1 - e^{-1.4}) = \$107{,}629.00.$$

21. An initial deposit of 100,000 euros are placed in an annuity with a French bank. What is the minimum interest rate the annuity must earn to allow withdrawals at a rate of 8000 euros/year to continue indefinitely?

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{8000}{r} + Ce^{rt}$$

for some constant C. To fund the annuity indefinitely, we need $C \geq 0$. If the initial deposit is 100,000 euros, then $100{,}000 = \frac{8000}{r} + C$ and $C = 100{,}000 - \frac{8000}{r}$. Thus, to fund the annuity indefinitely, we need $100{,}000 - \frac{8000}{r} \geq 0$, or $r \geq 0.08$. The bank must pay at least 8%.

22. Show that a continuous annuity never runs out of money if the initial balance is greater than or equal to N/r, where N is the withdrawal rate and r the interest rate.

SOLUTION With a withdrawal rate of N and an interest rate of r, the balance in the annuity is $P(t) = \frac{N}{r} + Ce^{rt}$ for some constant C. Let P_0 denote the initial balance. Then $P_0 = P(0) = \frac{N}{r} + C$ and $C = P_0 - \frac{N}{r}$. If $P_0 \geq \frac{N}{r}$, then $C \geq 0$ and the annuity lives forever.

23. Sam borrows \$10,000 from a bank at an interest rate of 9% and pays back the loan continuously at a rate of N dollars per year. Let $P(t)$ denote the amount still owed at time t.

(a) Explain why $P(t)$ satisfies the differential equation

$$y' = 0.09y - N$$

(b) How long will it take Sam to pay back the loan if $N = \$1200$?

(c) Will the loan ever be paid back if $N = \$800$?

SOLUTION

(a)

$$\begin{aligned}\text{Rate of Change of Loan} &= \text{(Amount still owed)(Interest rate)} - \text{(Payback rate)}\\ &= P(t)\cdot r - N = r\left(P - \frac{N}{r}\right).\end{aligned}$$

Therefore, if $y = P(t)$,

$$y' = r\left(y - \frac{N}{r}\right) = ry - N$$

(b) From the differential equation derived in part (a), we know that $P(t) = \frac{N}{r} + Ce^{rt} = 13{,}333.33 + Ce^{0.09t}$. Since \$10,000 was initially borrowed, $P(0) = 13{,}333.33 + C = 10{,}000$, and $C = -3333.33$. The loan is paid off when $P(t) = 13{,}333.33 - 3333.33e^{0.09t} = 0$. This yields

$$t = \frac{1}{0.09}\ln\left(\frac{13{,}333.33}{3333.33}\right) \approx 15.4 \text{ years.}$$

(c) If the annual rate of payment is \$800, then $P(t) = 800/0.09 + Ce^{0.09t} = 8888.89 + Ce^{0.09t}$. With $P(0) = 8888.89 + C = 10{,}000$, it follows that $C = 1111.11$. Since $C > 0$ and $e^{0.09t} \to \infty$ as $t \to \infty$, $P(t) \to \infty$, and the loan will never be paid back.

24. April borrows \$18,000 at an interest rate of 5% to purchase a new automobile. At what rate (in dollars per year) must she pay back the loan, if the loan must be paid off in 5 years? *Hint:* Set up the differential equation as in Exercise 23).

SOLUTION As in Exercise 23, the differential equation is

$$P(t)' = rP(t) - N = r\left(P(t) - \frac{N}{r}\right)$$

where r is the interest rate and N is the payment amount, so that here

$$P(t)' = 0.05\left(P(t) - \frac{N}{0.05}\right) \quad \Rightarrow \quad P(t) = \frac{N}{0.05} + Ce^{0.05t}$$

Since $P(0) = 18{,}000$, we have $C = 18{,}000 - \frac{N}{0.05}$, so that

$$P(t) = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right)e^{0.05t}$$

If the loan is to be paid back in 5 years, we must have

$$P(5) = 0 = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right)e^{0.05 \cdot 5}$$

Solving for N gives

$$N = \frac{900}{1 - e^{-0.25}} \approx 4068.73$$

so the payments must be at least \$4068.73 per year.

25. Let $N(t)$ be the fraction of the population who have heard a given piece of news t hours after its initial release. According to one model, the rate $N'(t)$ at which the news spreads is equal to k times the fraction of the population that has not yet heard the news, for some constant $k > 0$.

(a) Determine the differential equation satisfied by $N(t)$.

(b) Find the solution of this differential equation with the initial condition $N(0) = 0$ in terms of k.

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate k and estimate the percentage that will know about the earthquake 12 hours after it occurs.

SOLUTION

(a) $N'(t) = k(1 - N(t)) = -k(N(t) - 1)$.

(b) The general solution of the differential equation from part (a) is $N(t) = 1 + Ce^{-kt}$. The initial condition determines the value of C: $N(0) = 1 + C = 0$ so $C = -1$. Thus, $N(t) = 1 - e^{-kt}$.

(c) Knowing that $N(8) = 1 - e^{-8k} = \frac{1}{2}$, we find that

$$k = -\frac{1}{8}\ln\left(\frac{1}{2}\right) \approx 0.0866 \text{ hours}^{-1}.$$

With the value of k determined, we estimate that

$$N(12) = 1 - e^{-0.0866(12)} \approx 0.6463 = 64.63\%$$

of the population will know about the earthquake after 12 hours.

26. Current in a Circuit When the circuit in Figure 6 (which consists of a battery of V volts, a resistor of R ohms, and an inductor of L henries) is connected, the current $I(t)$ flowing in the circuit satisfies

$$L\frac{dI}{dt} + RI = V$$

with the initial condition $I(0) = 0$.

(a) Find a formula for $I(t)$ in terms of L, V, and R.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$.

(c) Show that $I(t)$ reaches approximately 63% of its maximum value at the "characteristic time" $\tau = L/R$.

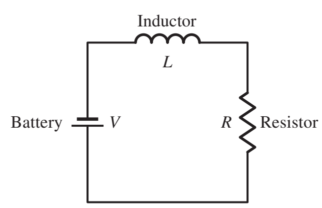


FIGURE 6 Current flow approaches the level $I_{\max} = V/R$.

SOLUTION

(a) Solve the differential equation for $\frac{dI}{dt}$:

$$\frac{dI}{dt} = -\frac{1}{L}(RI - V) = -\frac{R}{L}\left(I - \frac{V}{R}\right)$$

so that the general solution is

$$I(t) = \frac{V}{R} + Ce^{-(R/L)t}$$

The initial condition $I(0) = 0$ gives $C = -\frac{V}{R}$, so that

$$I(t) = \frac{V}{R}(1 - e^{-(R/L)t})$$

(b) As $t \to \infty$, $e^{-(R/L)t} \to 0$, so that $I(t) \to \frac{V}{R}$.
(c) When $t = \tau = L/R$,

$$I(\tau) = \frac{V}{R}(1 - e^{-(R/L)\tau}) = \frac{V}{R}(1 - e^{-(R/L)(L/R)}) = \frac{V}{R}(1 - e^{-1}) \approx 0.63\frac{V}{R}$$

which is 63% of the maximum value of V/R.

Further Insights and Challenges

27. Show that the cooling constant of an object can be determined from two temperature readings $y(t_1)$ and $y(t_2)$ at times $t_1 \neq t_2$ by the formula

$$k = \frac{1}{t_1 - t_2} \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right)$$

SOLUTION We know that $y(t_1) = T_0 + Ce^{-kt_1}$ and $y(t_2) = T_0 + Ce^{-kt_2}$. Thus, $y(t_1) - T_0 = Ce^{-kt_1}$ and $y(t_2) - T_0 = Ce^{-kt_2}$. Dividing the latter equation by the former yields

$$e^{-kt_2 + kt_1} = \frac{y(t_2) - T_0}{y(t_1) - T_0},$$

so that

$$k(t_1 - t_2) = \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right) \quad \text{and} \quad k = \frac{1}{t_1 - t_2} \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right).$$

28. Show that by Newton's Law of Cooling, the time required to cool an object from temperature A to temperature B is

$$t = \frac{1}{k} \ln\left(\frac{A - T_0}{B - T_0}\right)$$

where T_0 is the ambient temperature.

SOLUTION At any time t, the temperature of the object is $y(t) = T_0 + Ce^{-kt}$ for some constant C. Suppose the object is initially at temperature A and reaches temperature B at time t. Then $A = T_0 + C$, so $C = A - T_0$. Moreover,

$$B = T_0 + Ce^{-kt} = T_0 + (A - T_0)e^{-kt}.$$

Solving this last equation for t yields

$$t = \frac{1}{k} \ln\left(\frac{A - T_0}{B - T_0}\right).$$

29. Air Resistance A projectile of mass $m = 1$ travels straight up from ground level with initial velocity v_0. Suppose that the velocity v satisfies $v' = -g - kv$.
(a) Find a formula for $v(t)$.
(b) Show that the projectile's height $h(t)$ is given by

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t$$

where $C = k^{-2}(g + kv_0)$.
(c) Show that the projectile reaches its maximum height at time $t_{\max} = k^{-1}\ln(1 + kv_0/g)$.
(d) In the absence of air resistance, the maximum height is reached at time $t = v_0/g$. In view of this, explain why we should expect that

$$\lim_{k \to 0} \frac{\ln(1 + \frac{kv_0}{g})}{k} = \frac{v_0}{g} \qquad \boxed{8}$$

(e) Verify Eq. (8). *Hint:* Use Theorem 2 in Section 5.8 to show that $\lim_{k \to 0}\left(1 + \frac{kv_0}{g}\right)^{1/k} = e^{v_0/g}$ or use L'Hôpital's Rule.

SOLUTION

(a) Since $v' = -g - kv = -k\left(v - \frac{-g}{k}\right)$ it follows that $v(t) = \frac{-g}{k} + Be^{-kt}$ for some constant B. The initial condition $v(0) = v_0$ determines B: $v_0 = -\frac{g}{k} + B$, so $B = v_0 + \frac{g}{k}$. Thus,

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}.$$

(b) $v(t) = h'(t)$ so

$$h(t) = \int \left(-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}\right) dt = -\frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + D.$$

The initial condition $h(0) = 0$ determines

$$D = \frac{1}{k}\left(v_0 + \frac{g}{k}\right) = \frac{1}{k^2}(v_0 k + g).$$

Let $C = \frac{1}{k^2}(v_0 k + g)$. Then

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t.$$

(c) The projectile reaches its maximum height when $v(t) = 0$. This occurs when

$$-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt} = 0,$$

or

$$t = \frac{1}{-k}\ln\left(\frac{g}{kv_0 + g}\right) = \frac{1}{k}\ln\left(1 + \frac{kv_0}{g}\right).$$

(d) Recall that k is the proportionality constant for the force due to air resistance. Thus, as $k \to 0$, the effect of air resistance disappears. We should therefore expect that, as $k \to 0$, the time at which the maximum height is achieved from part (c) should approach v_0/g. In other words, we should expect

$$\lim_{k\to 0}\frac{1}{k}\ln\left(1 + \frac{kv_0}{g}\right) = \frac{v_0}{g}.$$

(e) Recall that

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

If we substitute $x = v_0/g$ and $k = 1/n$, we find

$$e^{v_0/g} = \lim_{k\to 0}\left(1 + \frac{v_0 k}{g}\right)^{1/k}.$$

Then

$$\lim_{k\to 0}\frac{1}{k}\ln\left(1 + \frac{kv_0}{g}\right) = \lim_{k\to 0}\ln\left(1 + \frac{v_0 k}{g}\right)^{1/k} = \ln\left(\lim_{k\to 0}\left(1 + \frac{v_0 k}{g}\right)^{1/k}\right) = \ln(e^{v_0/g}) = \frac{v_0}{g}.$$

9.3 Graphical and Numerical Methods

Preliminary Questions

1. What is the slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point (2, 3)?

SOLUTION The slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point (2, 3) is $(2)(3) + 1 = 7$.

2. What is the equation of the isocline of slope $c = 1$ for $\dot{y} = y^2 - t$?

SOLUTION The isocline of slope $c = 1$ has equation $y^2 - t = 1$, or $y = \pm\sqrt{1 + t}$.

3. For which of the following differential equations are the slopes at points on a vertical line $t = C$ all equal?

(a) $\dot{y} = \ln y$ **(b)** $\dot{y} = \ln t$

SOLUTION Only for the equation in part (b). The slope at a point is simply the value of $\dot{y}$ at that point, so for part (a), the slope depends on y, while for part (b), the slope depends only on t.

4. Let $y(t)$ be the solution to $\dot{y} = F(t, y)$ with $y(1) = 3$. How many iterations of Euler's Method are required to approximate $y(3)$ if the time step is $h = 0.1$?

SOLUTION The initial condition is specified at $t = 1$ and we want to obtain an approximation to the value of the solution at $t = 3$. With a time step of $h = 0.1$,

$$\frac{3 - 1}{0.1} = 20$$

iterations of Euler's method are required.

Exercises

1. Figure 8 shows the slope field for $\dot{y} = \sin y \sin t$. Sketch the graphs of the solutions with initial conditions $y(0) = 1$ and $y(0) = -1$. Show that $y(t) = 0$ is a solution and add its graph to the plot.

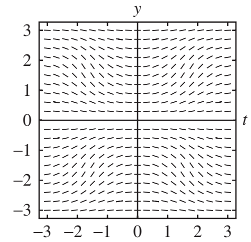
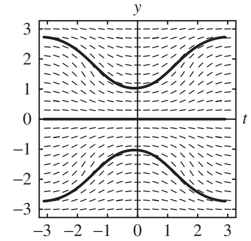


FIGURE 8 Slope field for $\dot{y} = \sin y \sin t$.

SOLUTION The sketches of the solutions appear below.



If $y(t) = 0$, then $y' = 0$; moreover, $\sin 0 \sin t = 0$. Thus, $y(t) = 0$ is a solution of $\dot{y} = \sin y \sin t$.

2. Figure 9 shows the slope field for $\dot{y} = y^2 - t^2$. Sketch the integral curve passing through the point $(0, -1)$, the curve through $(0, 0)$, and the curve through $(0, 2)$. Is $y(t) = 0$ a solution?

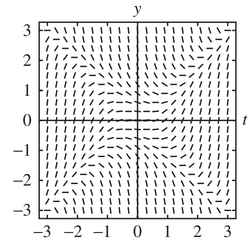
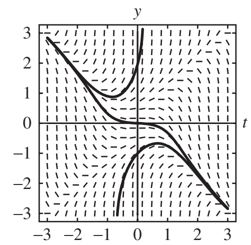


FIGURE 9 Slope field for $\dot{y} = y^2 - t^2$.

SOLUTION The sketches of the solutions appear below.



Let $y(t) = 0$. Because $\dot{y} = 0$ but $y^2 - t^2 = -t^2 \neq 0$, it follows that $y(t) = 0$ is not a solution of $\dot{y} = y^2 - t^2$.

3. Show that $f(t) = \frac{1}{2}\left(t - \frac{1}{2}\right)$ is a solution to $\dot{y} = t - 2y$. Sketch the four solutions with $y(0) = \pm 0.5, \pm 1$ on the slope field in Figure 10. The slope field suggests that every solution approaches $f(t)$ as $t \to \infty$. Confirm this by showing that $y = f(t) + Ce^{-2t}$ is the general solution.

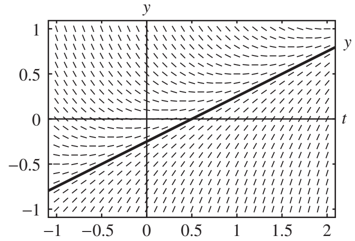
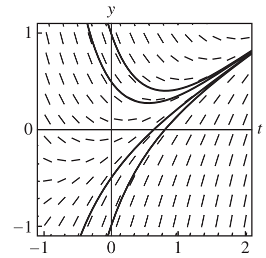


FIGURE 10 Slope field for $\dot{y} = t - 2y$.

SOLUTION Let $y = f(t) = \frac{1}{2}(t - \frac{1}{2})$. Then $\dot{y} = \frac{1}{2}$ and

$$\dot{y} + 2y = \frac{1}{2} + t - \frac{1}{2} = t,$$

so $f(t) = \frac{1}{2}(t - \frac{1}{2})$ is a solution to $\dot{y} = t - 2y$. The slope field with the four required solutions is shown below.



Now, let $y = f(t) + Ce^{-2t} = \frac{1}{2}(t - \frac{1}{2}) + Ce^{-2t}$. Then

$$\dot{y} = \frac{1}{2} - 2Ce^{-2t},$$

and

$$\dot{y} + 2y = \frac{1}{2} - 2Ce^{-2t} + \left(t - \frac{1}{2}\right) + 2Ce^{-2t} = t.$$

Thus, $y = f(t) + Ce^{-2t}$ is the general solution to the equation $\dot{y} = t - 2y$.

4. One of the slope fields in Figures 11(A) and (B) is the slope field for $\dot{y} = t^2$. The other is for $\dot{y} = y^2$. Identify which is which. In each case, sketch the solutions with initial conditions $y(0) = 1$, $y(0) = 0$, and $y(0) = -1$.

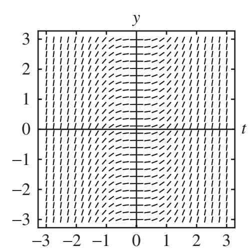


FIGURE 11(A)

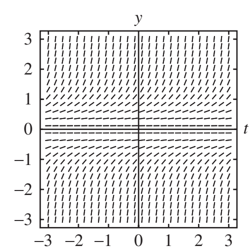
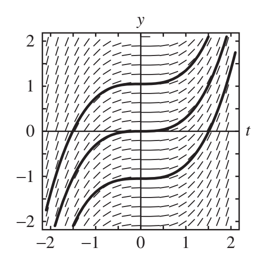
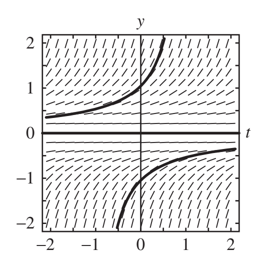


FIGURE 11(B)

SOLUTION For $y' = t^2$, y' only depends on t. The isoclines of any slope c will be the two vertical lines $t = \pm\sqrt{c}$. This indicates that the slope field will be the one given in Figure 11(A). The solutions are sketched below:



For $y' = y^2$, y' only depends on y. The isoclines of any slope c will be the two *horizontal* lines $y = \pm\sqrt{c}$. This indicates that the slope field will be the one given in Figure 11(B). The solutions are sketched below:



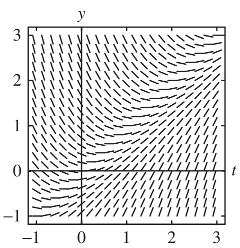
5. Consider the differential equation $\dot{y} = t - y$.

(a) Sketch the slope field of the differential equation $\dot{y} = t - y$ in the range $-1 \le t \le 3$, $-1 \le y \le 3$. As an aid, observe that the isocline of slope c is the line $t - y = c$, so the segments have slope c at points on the line $y = t - c$.

(b) Show that $y = t - 1 + Ce^{-t}$ is a solution for all C. Since $\lim_{t\to\infty} e^{-t} = 0$, these solutions approach the particular solution $y = t - 1$ as $t \to \infty$. Explain how this behavior is reflected in your slope field.

SOLUTION

(a) Here is a sketch of the slope field:



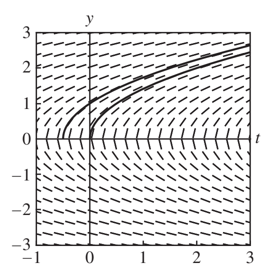
(b) Let $y = t - 1 + Ce^{-t}$. Then $\dot{y} = 1 - C^{-t}$, and

$$t - y = t - (t - 1 + Ce^{-t}) = 1 - Ce^{-t}.$$

Thus, $y = t - 1 + Ce^{-t}$ is a solution of $\dot{y} = t - y$. On the slope field, we can see that the isoclines of 1 all lie along the line $y = t - 1$. Whenever $y > t - 1$, $\dot{y} = t - y < 1$, so the solution curve will converge downward towards the line $y = t - 1$. On the other hand, if $y < t - 1$, $\dot{y} = t - y > 1$, so the solution curve will converge upward towards $y = t - 1$. In either case, the solution is approaching $t - 1$.

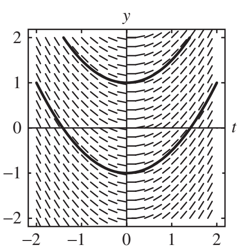
6. Show that the isoclines of $\dot{y} = 1/y$ are horizontal lines. Sketch the slope field for $-2 \le t \le 2$, $-2 \le y \le 2$ and plot the solutions with initial conditions $y(0) = 0$ and $y(0) = 1$.

SOLUTION The isocline of slope c is defined by $\frac{1}{y} = c$. This is equivalent to $y = \frac{1}{c}$, which is a horizontal line. The slope field and the solutions are shown below.



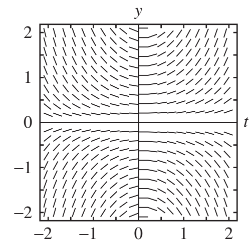
7. Show that the isoclines of $\dot{y} = t$ are vertical lines. Sketch the slope field for $-2 \le t \le 2$, $-2 \le y \le 2$ and plot the integral curves passing through $(0, -1)$ and $(0, 1)$.

SOLUTION The isocline of slope c for the differential equation $\dot{y} = t$ has equation $t = c$, which is the equation of a vertical line. The slope field and the required solution curves are shown below.



8. Sketch the slope field of $\dot{y} = ty$ for $-2 \le t \le 2$, $-2 \le y \le 2$. Based on the sketch, determine $\lim_{t\to\infty} y(t)$, where $y(t)$ is a solution with $y(0) > 0$. What is $\lim_{t\to\infty} y(t)$ if $y(0) < 0$?

SOLUTION The slope field for $\dot{y} = ty$ is shown below.



With $y(0) > 0$, the slope field indicates that y is an always increasing, always concave up function; consequently, $\lim_{t\to\infty} y = \infty$. On the other hand, when $y(0) < 0$, the slope field indicates that y is an always decreasing, always concave down function; consequently, $\lim_{t\to\infty} y = -\infty$.

9. Match each differential equation with its slope field in Figures 12(A)–(F).

(i) $\dot{y} = -1$

(ii) $\dot{y} = \dfrac{y}{t}$

(iii) $\dot{y} = t^2 y$

(iv) $\dot{y} = ty^2$

(v) $\dot{y} = t^2 + y^2$

(vi) $\dot{y} = t$

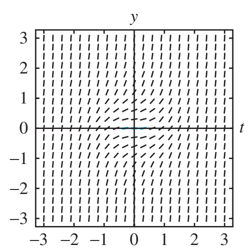


FIGURE 12(A)

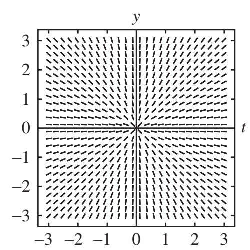


FIGURE 12(B)

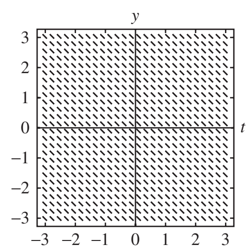


FIGURE 12(C)

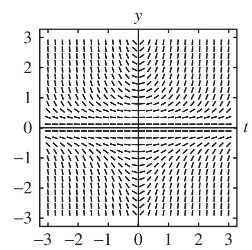


FIGURE 12(D)

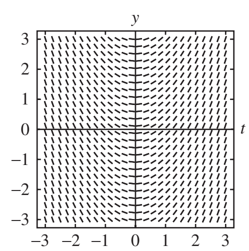


FIGURE 12(E)

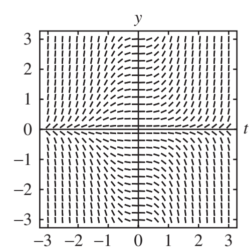


FIGURE 12(F)

SOLUTION

(i) Every segment in the slope field for $\dot{y} = -1$ will have slope -1; this matches Figure 12(C).

(ii) The segments in the slope field for $\dot{y} = \dfrac{y}{t}$ will have positive slope in the first and third quadrants and negative slopes in the second and fourth quadrant; this matches Figure 12(B).

(iii) The segments in the slope field for $\dot{y} = t^2 y$ will have positive slope in the upper half of the plane and negative slopes in the lower half of the plane; this matches Figure 12(F).
(iv) The segments in the slope field for $\dot{y} = ty^2$ will have positive slope on the right side of the plane and negative slopes on the left side of the plane; this matches Figure 12(D).
(v) Every segment in the slope field for $\dot{y} = t^2 + y^2$, except at the origin, will have positive slope; this matches Figure 12(A).
(vi) The isoclines for $\dot{y} = t$ are vertical lines; this matches Figure 12(E).

10. Sketch the solution of $\dot{y} = ty^2$ satisfying $y(0) = 1$ in the appropriate slope field of Figure 12(A)–(F). Then show, using separation of variables, that if $y(t)$ is a solution such that $y(0) > 0$, then $y(t)$ tends to infinity as $t \to \sqrt{2/y(0)}$.

SOLUTION Rewrite

$$\dot{y} = ty^2 \quad \text{as} \quad \frac{1}{y^2}\,dy = t\,dt$$

Integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int t\,dt$$

$$-y^{-1} = \frac{1}{2}t^2 + C_1$$

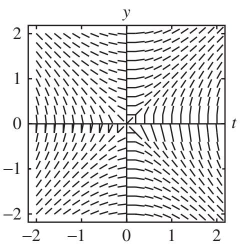
$$-y = \frac{2}{t^2 + C}$$

$$y = \frac{2}{C - t^2}$$

where $C = -C_1$ is an arbitrary constant. Then $y(0) = 2/C$ so that $C = 2/y(0)$, and then the denominator of y approaches 0 as $t \to \sqrt{2/y(0)}$, so that y tends to infinity.

11. **(a)** Sketch the slope field of $\dot{y} = t/y$ in the region $-2 \le t \le 2$, $-2 \le y \le 2$.
(b) Check that $y = \pm\sqrt{t^2 + C}$ is the general solution.
(c) Sketch the solutions on the slope field with initial conditions $y(0) = 1$ and $y(0) = -1$.

SOLUTION
(a) The slope field is shown below:



(b) Rewrite

$$\frac{dy}{dt} = \frac{t}{y} \qquad \text{as} \qquad y\,dy = t\,dt,$$

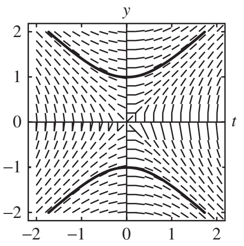
and then integrate both sides to obtain

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + C.$$

Solving for y, we find that the general solution is

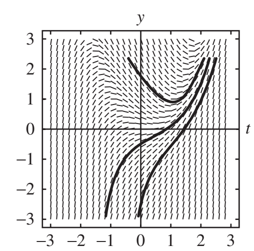
$$y = \pm\sqrt{t^2 + C}.$$

(c) The sketches of the two solutions are shown below:



12. Sketch the slope field of $\dot{y} = t^2 - y$ in the region $-3 \le t \le 3$, $-3 \le y \le 3$ and sketch the solutions satisfying $y(1) = 0$, $y(1) = 1$, and $y(1) = -1$.

SOLUTION The slope field for $\dot{y} = t^2 - y$, together with the required solution curves, is shown below.



13. Let $F(t, y) = t^2 - y$ and let $y(t)$ be the solution of $\dot{y} = F(t, y)$ satisfying $y(2) = 3$. Let $h = 0.1$ be the time step in Euler's Method, and set $y_0 = y(2) = 3$.

(a) Calculate $y_1 = y_0 + hF(2, 3)$.

(b) Calculate $y_2 = y_1 + hF(2.1, y_1)$.

(c) Calculate $y_3 = y_2 + hF(2.2, y_2)$ and continue computing y_4, y_5, and y_6.

(d) Find approximations to $y(2.2)$ and $y(2.5)$.

SOLUTION

(a) With $y_0 = 3$, $t_0 = 2$, $h = 0.1$, and $F(t, y) = t^2 - y$, we find

$$y_1 = y_0 + hF(t_0, y_0) = 3 + 0.1(1) = 3.1.$$

(b) With $y_1 = 3.1$, $t_1 = 2.1$, $h = 0.1$, and $F(t, y) = t^2 - y$, we find

$$y_2 = y_1 + hF(t_1, y_1) = 3.1 + 0.1(4.41 - 3.1) = 3.231.$$

(c) Continuing as in the previous two parts, we find

$$y_3 = y_2 + hF(t_2, y_2) = 3.3919;$$

$$y_4 = y_3 + hF(t_3, y_3) = 3.58171;$$

$$y_5 = y_4 + hF(t_4, y_4) = 3.799539;$$

$$y_6 = y_5 + hF(t_5, y_5) = 4.0445851.$$

(d) $y(2.2) \approx y_2 = 3.231$, and $y(2.5) \approx y_5 = 3.799539$.

14. Let $y(t)$ be the solution to $\dot{y} = te^{-y}$ satisfying $y(0) = 0$.

(a) Use Euler's Method with time step $h = 0.1$ to approximate $y(0.1), y(0.2), \ldots, y(0.5)$.

(b) Use separation of variables to find $y(t)$ exactly.

(c) Compute the errors in the approximations to $y(0.1)$ and $y(0.5)$.

SOLUTION

(a) With $y_0 = 0$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = te^{-y}$, we compute

n	t_n	y_n
0	0	0
1	0.1	$y_0 + hF(t_0, y_0) = 0$
2	0.2	$y_1 + hF(t_1, y_1) = 0.01$
3	0.3	$y_2 + hF(t_2, y_2) = 0.029801$
4	0.4	$y_3 + hF(t_3, y_3) = 0.058920$
5	0.5	$y_4 + hF(t_4, y_4) = 0.096631$
6	0.6	$y_5 + hF(t_5, y_5) = 0.142026$
7	0.7	$y_6 + hF(t_6, y_6) = 0.194082$
8	0.8	$y_7 + hF(t_7, y_7) = 0.251733$
9	0.9	$y_8 + hF(t_8, y_8) = 0.313929$
10	1.0	$y_9 + hF(t_9, y_9) = 0.379681$

(b) Rewrite

$$\frac{dy}{dt} = te^{-y} \qquad \text{as} \qquad e^y\,dy = t\,dt,$$

and then integrate both sides to obtain

$$e^y = \frac{1}{2}t^2 + C.$$

Thus,

$$y = \ln\left|\frac{1}{2}t^2 + C\right|.$$

Applying the initial condition $y(0) = 0$ yields $0 = \ln|C|$, so $C = 1$. The exact solution to the initial value problem is then $y = \ln\left(\frac{1}{2}t^2 + 1\right)$.

(c) The two errors requested are computed here:

$$|y(0.1) - y_1| = |0.00498754 - 0| = 0.00498754;$$
$$|y(0.5) - y_5| = |0.117783 - 0.0966314| = 0.021152$$

In Exercises 15–20, use Euler's Method to approximate the given value of $y(t)$ with the time step h indicated.

15. $y(0.5);\quad \dot{y} = y + t,\quad y(0) = 1,\quad h = 0.1$

SOLUTION With $y_0 = 1$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = y + t$, we compute

n	t_n	y_n
0	0	1
1	0.1	$y_0 + hF(t_0, y_0) = 1.1$
2	0.2	$y_1 + hF(t_1, y_1) = 1.22$
3	0.3	$y_2 + hF(t_2, y_2) = 1.362$
4	0.4	$y_3 + hF(t_3, y_3) = 1.5282$
5	0.5	$y_4 + hF(t_4, y_4) = 1.72102$

16. $y(0.7);\quad \dot{y} = 2y,\quad y(0) = 3,\quad h = 0.1$

SOLUTION With $y_0 = 3$, $t_0 = 0$, $h = 0.1$, and $F(t, y) = 2y$, we compute

n	t_n	y_n
0	0	3
1	0.1	$y_0 + hF(t_0, y_0) = 3.6$
2	0.2	$y_1 + hF(t_1, y_1) = 4.32$
3	0.3	$y_2 + hF(t_2, y_2) = 5.184$
4	0.4	$y_3 + hF(t_3, y_3) = 6.2208$
5	0.5	$y_4 + hF(t_4, y_4) = 7.464960$
6	0.6	$y_5 + hF(t_5, y_5) = 8.957952$
7	0.7	$y_6 + hF(t_6, y_6) = 10.749542$

17. $y(3.3)$; $\dot{y} = t^2 - y$, $y(3) = 1$, $h = 0.05$

SOLUTION With $y_0 = 1$, $t_0 = 3$, $h = 0.05$, and $F(t, y) = t^2 - y$, we compute

n	t_n	y_n
0	3	1
1	3.05	$y_0 + hF(t_0, y_0) = 1.4$
2	3.1	$y_1 + hF(t_1, y_1) = 1.795125$
3	3.15	$y_2 + hF(t_2, y_2) = 2.185869$
4	3.2	$y_3 + hF(t_3, y_3) = 2.572700$
5	3.25	$y_4 + hF(t_4, y_4) = 2.956065$
6	3.3	$y_5 + hF(t_5, y_5) = 3.336387$

18. $y(3)$; $\dot{y} = \sqrt{t + y}$, $y(2.7) = 5$, $h = 0.05$

SOLUTION With $y_0 = 5$, $t_0 = 2.7$, $h = 0.05$, and $F(t, y) = \sqrt{t + y}$, we compute

n	t_n	y_n
0	2.7	5
1	2.75	$y_0 + hF(t_0, y_0) = 5.138744$
2	2.8	$y_1 + hF(t_1, y_1) = 5.279179$
3	2.85	$y_2 + hF(t_2, y_2) = 5.421298$
4	2.9	$y_3 + hF(t_3, y_3) = 5.565098$
5	2.95	$y_4 + hF(t_4, y_4) = 5.710572$
6	3.0	$y_5 + hF(t_5, y_5) = 5.857716$

19. $y(2)$; $\dot{y} = t \sin y$, $y(1) = 2$, $h = 0.2$

SOLUTION Let $F(t, y) = t \sin y$. With $t_0 = 1$, $y_0 = 2$ and $h = 0.2$, we compute

n	t_n	y_n
0	1	2
1	1.2	$y_0 + hF(t_0, y_0) = 2.181859$
2	1.4	$y_1 + hF(t_1, y_1) = 2.378429$
3	1.6	$y_2 + hF(t_2, y_2) = 2.571968$
4	1.8	$y_3 + hF(t_3, y_3) = 2.744549$
5	2.0	$y_4 + hF(t_4, y_4) = 2.883759$

20. $y(5.2)$; $\dot{y} = t - \sec y$, $y(4) = -2$, $h = 0.2$

SOLUTION With $t_0 = 4$, $y_0 = -2$, $F(t, y) = t - \sec y$, and $h = 0.2$, we compute

n	t_n	y_n
0	4	-2
1	4.2	$y_0 + hF(t_0, y_0) = -0.7194$
2	4.4	$y_1 + hF(t_1, y_1) = -0.142587$
3	4.6	$y_2 + hF(t_2, y_2) = 0.532584$
4	4.8	$y_3 + hF(t_3, y_3) = 1.220430$
5	5.0	$y_4 + hF(t_4, y_4) = 1.597751$
6	5.2	$y_5 + hF(t_5, y_5) = 10.018619$

Note that $\sec y$ has a discontinuity at $y = \pi/2 \approx 1.57$ and at $y = 3\pi/2 \approx 4.71$, so this numerical solution should be regarded with some skepticism.

Further Insights and Challenges

21. If $f(t)$ is continuous on $[a, b]$, then the solution to $\dot{y} = f(t)$ with initial condition $y(a) = 0$ is $y(t) = \int_a^t f(u)\,du$. Show that Euler's Method with time step $h = (b-a)/N$ for N steps yields the Nth left-endpoint approximation to $y(b) = \int_a^b f(u)\,du$.

SOLUTION For a differential equation of the form $\dot{y} = f(t)$, the equation for Euler's method reduces to

$$y_k = y_{k-1} + hf(t_{k-1}).$$

With a step size of $h = (b-a)/N$, $y(b) =\approx y_N$. Starting from $y_0 = 0$, we compute

$$\begin{aligned}
y_1 &= y_0 + hf(t_0) = hf(t_0)\\
y_2 &= y_1 + hf(t_1) = h\,[f(t_0) + f(t_1)]\\
y_3 &= y_2 + hf(t_2) = h\,[f(t_0) + f(t_1) + f(t_2)]\\
&\;\;\vdots\\
y_N &= y_{N_1} + hf(t_{N-1}) = h\left[f(t_0) + f(t_1) + f(t_2) + \ldots + f(t_{N-1})\right] = h\sum_{k=0}^{N-1} f(t_k)
\end{aligned}$$

Observe this last expression is exactly the Nth left-endpoint approximation to $y(b) = \displaystyle\int_a^b f(u)\,du$.

*Exercises 22–27: **Euler's Midpoint Method** is a variation on Euler's Method that is significantly more accurate in general. For time step h and initial value $y_0 = y(t_0)$, the values y_k are defined successively by*

$$y_k = y_{k-1} + hm_{k-1}$$

where $m_{k-1} = F\left(t_{k-1} + \dfrac{h}{2}, y_{k-1} + \dfrac{h}{2}F(t_{k-1}, y_{k-1})\right)$.

22. Apply both Euler's Method and the Euler Midpoint Method with $h = 0.1$ to estimate $y(1.5)$, where $y(t)$ satisfies $\dot{y} = y$ with $y(0) = 1$. Find $y(t)$ exactly and compute the errors in these two approximations.

SOLUTION Let $F(t, y) = y$. With $t_0 = 0$, $y_0 = 1$, and $h = 0.1$, fifteen iterations of Euler's method yield

$$y(1.5) \approx y_{15} = 4.177248.$$

The Euler midpoint approximation with $F(t, y) = y$ is

$$m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}F(t_{k-1}, y_{k-1})\right) = y_{k-1} + \frac{h}{2}y_{k-1}$$

$$y_k = y_{k-1} + h\left(y_{k-1} + \frac{h}{2}y_{k-1}\right) = y_{k-1} + hy_{k-1} + \frac{h^2}{2}y_{k-1}$$

Fifteen iterations of Euler's midpoint method yield:

$$y(1.5) \approx y_{15} = 4.471304.$$

The exact solution to $y' = y$, $y(0) = 1$ is $y(t) = e^t$; therefore $y(1.5) = 4.481689$. The error from Euler's method is $|4.177248 - 4.481689| = 0.304441$, while the error from Euler's midpoint method is $|4.471304 - 4.481689| = 0.010385$.

In Exercises 23–26, use Euler's Midpoint Method with the time step indicated to approximate the given value of $y(t)$.

23. $y(0.5)$; $\dot{y} = y + t$, $y(0) = 1$, $h = 0.1$

SOLUTION With $t_0 = 0$, $y_0 = 1$, $F(t, y) = y + t$, and $h = 0.1$ we compute

n	t_n	y_n
0	0	1
1	0.1	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.11$
2	0.2	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.242050$
3	0.3	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.398465$
4	0.4	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 1.581804$
5	0.5	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 1.794894$

24. $y(2)$; $\quad \dot{y} = t^2 - y, \quad y(1) = 3, \quad h = 0.2$

SOLUTION With $t_0 = 1$, $y_0 = 3$, $F(t, y) = t^2 - y$, and $h = 0.2$ we compute

n	t_n	y_n
0	1	3
1	1.2	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 2.682$
2	1.4	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 2.50844$
3	1.6	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 2.467721$
4	1.8	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 2.550331$
5	2.0	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 2.748471$

25. $y(0.25)$; $\quad \dot{y} = \cos(y + t), \quad y(0) = 1, \quad h = 0.05$

SOLUTION With $t_0 = 0$, $y_0 = 1$, $F(t, y) = \cos(y + t)$, and $h = 0.05$ we compute

n	t_n	y_n
0	0	1
1	0.05	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.025375$
2	0.10	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.047507$
3	0.15	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.066425$
4	0.20	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 1.082186$
5	0.25	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 1.094871$

26. $y(2.3)$; $\quad \dot{y} = y + t^2, \quad y(2) = 1, \quad h = 0.05$

SOLUTION With $t_0 = 2$, $y_0 = 1$, $F(t, y) = y + t^2$, and $h = 0.05$ we compute

n	t_n	y_n
0	2.00	1
1	2.05	$y_0 + hF(t_0 + h/2, y_0 + (h/2)F(t_0, y_0)) = 1.261281$
2	2.10	$y_1 + hF(t_1 + h/2, y_1 + (h/2)F(t_1, y_1)) = 1.546456$
3	2.15	$y_2 + hF(t_2 + h/2, y_2 + (h/2)F(t_2, y_2)) = 1.857006$
4	2.20	$y_3 + hF(t_3 + h/2, y_3 + (h/2)F(t_3, y_3)) = 2.194487$
5	2.25	$y_4 + hF(t_4 + h/2, y_4 + (h/2)F(t_4, y_4)) = 2.560536$
6	2.30	$y_5 + hF(t_5 + h/2, y_5 + (h/2)F(t_5, y_5)) = 2.956872$

27. Assume that $f(t)$ is continuous on $[a, b]$. Show that Euler's Midpoint Method applied to $\dot{y} = f(t)$ with initial condition $y(a) = 0$ and time step $h = (b - a)/N$ for N steps yields the Nth midpoint approximation to

$$y(b) = \int_a^b f(u)\, du$$

SOLUTION For a differential equation of the form $\dot{y} = f(t)$, the equations for Euler's midpoint method reduce to

$$m_{k-1} = f\left(t_{k-1} + \frac{h}{2}\right) \qquad \text{and} \qquad y_k = y_{k-1} + hf\left(t_{k-1} + \frac{h}{2}\right).$$

With a step size of $h = (b - a)/N$, $y(b) =\approx y_N$. Starting from $y_0 = 0$, we compute

$$y_1 = y_0 + hf\left(t_0 + \frac{h}{2}\right) = hf\left(t_0 + \frac{h}{2}\right)$$

$$y_2 = y_1 + hf\left(t_1 + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right)\right]$$

$$y_3 = y_2 + hf\left(t_2 + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right) + f\left(t_2 + \frac{h}{2}\right)\right]$$

$$\vdots$$

$$y_N = y_{N_1} + hf\left(t_{N-1} + \frac{h}{2}\right) = h\left[f\left(t_0 + \frac{h}{2}\right) + f\left(t_1 + \frac{h}{2}\right) + f\left(t_2 + \frac{h}{2}\right) + \ldots + f\left(t_{N-1} + \frac{h}{2}\right)\right]$$

$$= h\sum_{k=0}^{N-1} f\left(t_k + \frac{h}{2}\right)$$

Observe this last expression is exactly the Nth midpoint approximation to $y(b) = \int_a^b f(u)\,du$.

9.4 The Logistic Equation

Preliminary Questions

1. Which of the following differential equations is a logistic differential equation?

(a) $\dot{y} = 2y(1 - y^2)$ **(b)** $\dot{y} = 2y\left(1 - \frac{y}{3}\right)$

(c) $\dot{y} = 2y\left(1 - \frac{t}{4}\right)$ **(d)** $\dot{y} = 2y(1 - 3y)$

SOLUTION The differential equations in **(b)** and **(d)** are logistic equations. The equation in **(a)** is not a logistic equation because of the y^2 term inside the parentheses on the right-hand side; the equation in **(c)** is not a logistic equation because of the presence of the independent variable on the right-hand side.

2. Is the logistic equation a linear differential equation?

SOLUTION No, the logistic equation is not linear.

$$\dot{y} = ky\left(1 - \frac{y}{A}\right) \quad \text{can be rewritten} \quad \dot{y} = ky - \frac{k}{A}y^2$$

and we see that a term involving y^2 occurs.

3. Is the logistic equation separable?

SOLUTION Yes, the logistic equation is a separable differential equation.

Exercises

1. Find the general solution of the logistic equation

$$\dot{y} = 3y\left(1 - \frac{y}{5}\right)$$

Then find the particular solution satisfying $y(0) = 2$.

SOLUTION $\dot{y} = 3y(1 - y/5)$ is a logistic equation with $k = 3$ and $A = 5$; therefore, the general solution is

$$y = \frac{5}{1 - e^{-3t}/C}.$$

The initial condition $y(0) = 2$ allows us to determine the value of C:

$$2 = \frac{5}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{5}{2}; \quad \text{so} \quad C = -\frac{2}{3}.$$

The particular solution is then

$$y = \frac{5}{1 + \frac{3}{2}e^{-3t}} = \frac{10}{2 + 3e^{-3t}}.$$

2. Find the solution of $\dot{y} = 2y(3 - y)$, $y(0) = 10$.

SOLUTION By rewriting

$$2y(3 - y) \qquad \text{as} \qquad 6y\left(1 - \frac{y}{3}\right),$$

we identify the given differential equation as a logistic equation with $k = 6$ and $A = 3$. The general solution is therefore

$$y = \frac{3}{1 - e^{-6t}/C}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = \frac{3}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{3}{10}; \quad \text{so} \quad C = \frac{10}{7}.$$

The particular solution is then

$$y = \frac{3}{1 - \frac{7}{10}e^{-6t}} = \frac{30}{10 - 7e^{-6t}}.$$

3. Let $y(t)$ be a solution of $\dot{y} = 0.5y(1 - 0.5y)$ such that $y(0) = 4$. Determine $\lim_{t\to\infty} y(t)$ without finding $y(t)$ explicitly.

SOLUTION This is a logistic equation with $k = \frac{1}{2}$ and $A = 2$, so the carrying capacity is 2. Thus the required limit is 2.

4. Let $y(t)$ be a solution of $\dot{y} = 5y(1 - y/5)$. State whether $y(t)$ is increasing, decreasing, or constant in the following cases:

(a) $y(0) = 2$ **(b)** $y(0) = 5$ **(c)** $y(0) = 8$

SOLUTION This is a logistic equation with $k = A = 5$.

(a) $0 < y(0) < A$, so $y(t)$ is increasing and approaches A asymptotically.

(b) $y(0) = A$; this represents a stable equilibrium and $y(t)$ is constant.

(c) $y(0) > A$, so $y(t)$ is decreasing and approaches A asymptotically.

5. A population of squirrels lives in a forest with a carrying capacity of 2000. Assume logistic growth with growth constant $k = 0.6\ \text{yr}^{-1}$.

(a) Find a formula for the squirrel population $P(t)$, assuming an initial population of 500 squirrels.

(b) How long will it take for the squirrel population to double?

SOLUTION

(a) Since $k = 0.6$ and the carrying capacity is $A = 2000$, the population $P(t)$ of the squirrels satisfies the differential equation

$$P'(t) = 0.6P(t)(1 - P(t)/2000),$$

with general solution

$$P(t) = \frac{2000}{1 - e^{-0.6t}/C}.$$

The initial condition $P(0) = 500$ allows us to determine the value of C:

$$500 = \frac{2000}{1 - 1/C}; \quad 1 - \frac{1}{C} = 4; \quad \text{so} \quad C = -\frac{1}{3}.$$

The formula for the population is then

$$P(t) = \frac{2000}{1 + 3e^{-0.6t}}.$$

(b) The squirrel population will have doubled at the time t where $P(t) = 1000$. This gives

$$1000 = \frac{2000}{1 + 3e^{-0.6t}}; \quad 1 + 3e^{-0.6t} = 2; \quad \text{so} \quad t = \frac{5}{3}\ln 3 \approx 1.83.$$

It therefore takes approximately 1.83 years for the squirrel population to double.

6. The population $P(t)$ of mosquito larvae growing in a tree hole increases according to the logistic equation with growth constant $k = 0.3\ \text{day}^{-1}$ and carrying capacity $A = 500$.

(a) Find a formula for the larvae population $P(t)$, assuming an initial population of $P_0 = 50$ larvae.

(b) After how many days will the larvae population reach 200?

SOLUTION

(a) Since $k = 0.3$ and $A = 500$, the population of the larvae satisfies the differential equation

$$P'(t) = 0.3P(t)(1 - P(t)/500),$$

with general solution

$$P(t) = \frac{500}{1 - e^{-0.3t}/C}.$$

The initial condition $P(0) = 50$ allows us to determine the value of C:

$$50 = \frac{500}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

The particular solution is then

$$P(t) = \frac{500}{1 + 9e^{-0.3t}}.$$

(b) The population will reach 200 after t days, where $P(t) = 200$. This gives

$$200 = \frac{500}{1 + 9e^{-0.3t}}; \quad 1 + 9e^{-0.3t} = 2.5; \quad \text{so} \quad t = \frac{10}{3}\ln 6 \approx 5.97.$$

It therefore takes approximately 5.97 days for the larvae to reach 200 in number.

7. Sunset Lake is stocked with 2000 rainbow trout, and after 1 year the population has grown to 4500. Assuming logistic growth with a carrying capacity of 20,000, find the growth constant k (specify the units) and determine when the population will increase to 10,000.

SOLUTION Since $A = 20{,}000$, the trout population $P(t)$ satisfies the logistic equation

$$P'(t) = kP(t)(1 - P(t)/20{,}000),$$

with general solution

$$P(t) = \frac{20{,}000}{1 - e^{-kt}/C}.$$

The initial condition $P(0) = 2000$ allows us to determine the value of C:

$$2000 = \frac{20{,}000}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

After one year, we know the population has grown to 4500. Let's measure time in years. Then

$$\begin{aligned}
4500 &= \frac{20{,}000}{1 + 9e^{-k}} \\
1 + 9e^{-k} &= \frac{40}{9} \\
e^{-k} &= \frac{31}{81} \\
k &= \ln\frac{81}{31} \approx 0.9605 \text{ years}^{-1}.
\end{aligned}$$

The population will increase to 10,000 at time t where $P(t) = 10{,}000$. This gives

$$\begin{aligned}
10{,}000 &= \frac{20{,}000}{1 + 9e^{-0.9605t}} \\
1 + 9e^{-0.9605t} &= 2 \\
e^{-0.9605t} &= \frac{1}{9} \\
t &= \frac{1}{0.9605}\ln 9 \approx 2.29 \text{ years.}
\end{aligned}$$

8. Spread of a Rumor A rumor spreads through a small town. Let $y(t)$ be the fraction of the population that has heard the rumor at time t and assume that the rate at which the rumor spreads is proportional to the product of the fraction y of the population that has heard the rumor and the fraction $1 - y$ that has not yet heard the rumor.

(a) Write down the differential equation satisfied by y in terms of a proportionality factor k.

(b) Find k (in units of day^{-1}), assuming that 10% of the population knows the rumor at $t = 0$ and 40% knows it at $t = 2$ days.

(c) Using the assumptions of part (b), determine when 75% of the population will know the rumor.

SOLUTION

(a) $y'(t)$ is the rate at which the rumor is spreading, in percentage of the population per day. By the description given, the rate satisfies:

$$y'(t) = ky(1-y),$$

where k is a constant of proportionality.

(b) The equation in part (a) is a logistic equation with constant k and capacity 1 (no more than 100% of the population can hear the rumor). Thus, y takes the form

$$y(t) = \frac{1}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = \frac{1}{10}$ allows us to determine the value of C:

$$\frac{1}{10} = \frac{1}{1 - 1/C}; \quad 1 - \frac{1}{C} = 10; \quad \text{so} \quad C = -\frac{1}{9}.$$

The condition $y(2) = \frac{2}{5}$ now allows us to determine the value of k:

$$\frac{2}{5} = \frac{1}{1 + 9e^{-2k}}; \quad 1 + 9e^{-2k} = \frac{5}{2}; \quad \text{so} \quad k = \frac{1}{2}\ln 6 \approx 0.896 \text{ days}^{-1}.$$

The particular solution of the differential equation for y is then

$$y(t) = \frac{1}{1 + 9e^{-0.896t}}.$$

(c) If 75% of the population knows the rumor at time t, we have

$$\begin{aligned}
\frac{3}{4} &= \frac{1}{1 + 9e^{-0.896t}} \\
1 + 9e^{-0.896t} &= \frac{4}{3} \\
t &= \frac{\ln 27}{0.896} \approx 3.67839
\end{aligned}$$

Thus, 75% of the population knows the rumor after approximately 3.67 days.

9. A rumor spreads through a school with 1000 students. At 8 AM, 80 students have heard the rumor, and by noon, half the school has heard it. Using the logistic model of Exercise 8, determine when 90% of the students will have heard the rumor.

SOLUTION Let $y(t)$ be the proportion of students that have heard the rumor at a time t hours after 8 AM. In the logistic model of Exercise 8, we have a capacity of $A = 1$ (100% of students) and an unknown growth factor of k. Hence,

$$y(t) = \frac{1}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = 0.08$ allows us to determine the value of C:

$$\frac{2}{25} = \frac{1}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{25}{2}; \quad \text{so} \quad C = -\frac{2}{23}.$$

so that

$$y(t) = \frac{2}{2 + 23e^{-kt}}.$$

The condition $y(4) = 0.5$ now allows us to determine the value of k:

$$\frac{1}{2} = \frac{2}{2+23e^{-4k}}; \quad 2+23e^{-4k} = 4; \quad \text{so} \quad k = \frac{1}{4}\ln\frac{23}{2} \approx 0.6106 \text{ hours}^{-1}.$$

90% of the students have heard the rumor when $y(t) = 0.9$. Thus

$$\frac{9}{10} = \frac{2}{2+23e^{-0.6106t}}$$
$$2+23e^{-0.6106t} = \frac{20}{9}$$
$$t = \frac{1}{0.6106}\ln\frac{207}{2} \approx 7.6 \text{ hours.}$$

Thus, 90% of the students have heard the rumor after 7.6 hours, or at 3:36 PM.

10. GU A simpler model for the spread of a rumor assumes that the rate at which the rumor spreads is proportional (with factor k) to the fraction of the population that has not yet heard the rumor.

(a) Compute the solutions to this model and the model of Exercise 8 with the values $k = 0.9$ and $y_0 = 0.1$.

(b) Graph the two solutions on the same axis.

(c) Which model seems more realistic? Why?

SOLUTION

(a) Let $y(t)$ denote the fraction of a population that has heard a rumor, and suppose the rumor spreads at a rate proportional to the fraction of the population that has not yet heard the rumor. Then

$$y' = k(1-y),$$

for some constant of proportionality k. Separating variables and integrating both sides yields

$$\frac{dy}{1-y} = k\,dt$$
$$-\ln|1-y| = kt + C.$$

Thus,

$$y(t) = 1 - Ae^{-kt},$$

where $A = \pm e^{-C}$ is an arbitrary constant. The initial condition $y(0) = 0.1$ allows us to determine the value of A:

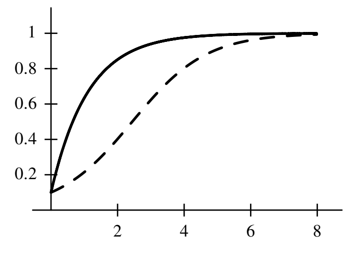
$$0.1 = 1 - A \qquad \text{so} \qquad A = 0.9.$$

With $k = 0.9$, we have $y(t) = 1 - 0.9e^{-0.9t}$.

Using the model from Exercise 8 with $k = 0.9$ and $y(0) = 0.1$, we find

$$y(t) = \frac{1}{1+9e^{-0.9t}}.$$

(b) The figure below shows the solutions from part (a): the solid curve corresponds to the model presented in this exercise while the dashed curve corresponds to the model from Exercise 8.



(c) The model from Exercise 8 seems more realistic because it predicts the rumor starts spreading slowly, picks up speed and then levels off as we near the time when the entire population has heard the rumor.

11. Let $k = 1$ and $A = 1$ in the logistic equation.

(a) Find the solutions satisfying $y_1(0) = 10$ and $y_2(0) = -1$.

(b) Find the time t when $y_1(t) = 5$.

(c) When does $y_2(t)$ become infinite?

SOLUTION The general solution of the logistic equation with $k = 1$ and $A = 1$ is

$$y(t) = \frac{1}{1 - e^{-t}/C}.$$

(a) Given $y_1(0) = 10$, we find $C = \frac{10}{9}$, and

$$y_1(t) = \frac{1}{1 - \frac{10}{9}e^{-t}} = \frac{10}{10 - 9e^{-t}}.$$

On the other hand, given $y_2(0) = -1$, we find $C = \frac{1}{2}$, and

$$y_2(t) = \frac{1}{1 - 2e^{-t}}.$$

(b) From part (a), we have

$$y_1(t) = \frac{10}{10 - 9e^{-t}}.$$

Thus, $y_1(t) = 5$ when

$$5 = \frac{10}{10 - 9e^{-t}}; \quad 10 - 9e^{-t} = 2; \quad \text{so} \quad t = \ln \frac{9}{8}.$$

(c) From part (a), we have

$$y_2(t) = \frac{1}{1 - 2e^{-t}}.$$

Thus, $y_2(t)$ becomes infinite when

$$1 - 2e^{-t} = 0 \qquad \text{or} \qquad t = \ln 2.$$

12. A tissue culture grows until it has a maximum area of M cm^2. The area $A(t)$ of the culture at time t may be modeled by the differential equation

$$\dot{A} = k\sqrt{A}\left(1 - \frac{A}{M}\right) \qquad \boxed{7}$$

where k is a growth constant.

(a) Show that if we set $A = u^2$, then

$$\dot{u} = \frac{1}{2}k\left(1 - \frac{u^2}{M}\right)$$

Then find the general solution using separation of variables.

(b) Show that the general solution to Eq. (7) is

$$A(t) = M\left(\frac{Ce^{(k/\sqrt{M})t} - 1}{Ce^{(k/\sqrt{M})t} + 1}\right)^2$$

SOLUTION

(a) Let $A = u^2$. This gives $\dot{A} = 2u\dot{u}$, so that Eq. (7) becomes:

$$2u\dot{u} = ku\left(1 - \frac{u^2}{M}\right)$$

$$\dot{u} = \frac{k}{2}\left(1 - \frac{u^2}{M}\right)$$

Now, rewrite

$$\frac{du}{dt} = \frac{k}{2}\left(1 - \frac{u^2}{M}\right) \qquad \text{as} \qquad \frac{du}{1 - u^2/M} = \frac{1}{2}k\,dt.$$

The partial fraction decomposition for the term on the left-hand side is

$$\frac{1}{1 - u^2/M} = \frac{\sqrt{M}}{2}\left(\frac{1}{\sqrt{M} + u} + \frac{1}{\sqrt{M} - u}\right),$$

so after integrating both sides, we obtain

$$\frac{\sqrt{M}}{2}\ln\left|\frac{\sqrt{M}+u}{\sqrt{M}-u}\right| = \tfrac{1}{2}kt + C.$$

Thus,

$$\frac{\sqrt{M}+u}{\sqrt{M}-u} = Ce^{(k/\sqrt{M})t}$$
$$u(Ce^{(k/\sqrt{M})t}+1) = \sqrt{M}(Ce^{(k/\sqrt{M})t}-1)$$

and

$$u = \sqrt{M}\frac{Ce^{(k/\sqrt{M})t}-1}{Ce^{(k/\sqrt{M})t}+1}.$$

(b) Recall $A = u^2$. Therefore,

$$A(t) = M\left(\frac{Ce^{(k/\sqrt{M})t}-1}{Ce^{(k/\sqrt{M})t}+1}\right)^2.$$

13. GU In the model of Exercise 12, let $A(t)$ be the area at time t (hours) of a growing tissue culture with initial size $A(0) = 1$ cm^2, assuming that the maximum area is $M = 16$ cm^2 and the growth constant is $k = 0.1$.

(a) Find a formula for $A(t)$. *Note:* The initial condition is satisfied for two values of the constant C. Choose the value of C for which $A(t)$ is increasing.

(b) Determine the area of the culture at $t = 10$ hours.

(c) GU Graph the solution using a graphing utility.

SOLUTION

(a) From the values for M and k we have

$$A(t) = 16\left(\frac{Ce^{t/40}-1}{Ce^{t/40}+1}\right)^2$$

and the initial condition then gives us

$$A(0) = 1 = 16\left(\frac{Ce^{0/40}-1}{Ce^{0/40}+1}\right)^2$$

so, simplifying,

$$1 = 16\left(\frac{C-1}{C+1}\right)^2 \quad\Rightarrow\quad C^2+2C+1 = 16C^2-32C+16 \quad\Rightarrow\quad 15C^2-34C+15 = 0$$

and thus $C = \dfrac{5}{3}$ or $C = \dfrac{3}{5}$. The derivative of $A(t)$ is

$$A'(t) = \frac{16Ce^{t/40}}{(Ce^{t/40}+1)^3}\cdot(Ce^{t/40}-1)$$

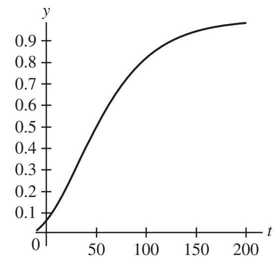
For $C = 3/5$, $A'(t)$ can be negative, while for $C = 5/3$, it is always positive. So let $C = 5/3$.

(b) From part (a), we have

$$A(t) = 16\left(\frac{\frac{5}{3}e^{t/40}-1}{\frac{5}{3}e^{t/40}+1}\right)^2$$

and $A(10) \approx 2.11$.

(c)



14. Show that if a tissue culture grows according to Eq. (7), then the growth rate reaches a maximum when $A = M/3$.

SOLUTION According to Eq. (7), the growth rate of the tissue culture is $k\sqrt{A}(1 - \frac{A}{M})$. Therefore

$$\frac{d}{dA}\left(k\sqrt{A}\left(1 - \frac{A}{M}\right)\right) = \frac{1}{2}kA^{-1/2} - \frac{3}{2}kA^{1/2}/M = \frac{1}{2}kA^{-1/2}\left(1 - \frac{3A}{M}\right) = 0$$

when $A = M/3$. Because the growth rate is zero for $A = 0$ and for $A = M$ and is positive for $0 < A < M$, it follows that the maximum growth rate occurs when $A = M/3$.

15. In 1751, Benjamin Franklin predicted that the U.S. population $P(t)$ would increase with growth constant $k = 0.028$ year^{-1}. According to the census, the U.S. population was 5 million in 1800 and 76 million in 1900. Assuming logistic growth with $k = 0.028$, find the predicted carrying capacity for the U.S. population. *Hint:* Use Eqs. (3) and (4) to show that

$$\frac{P(t)}{P(t) - A} = \frac{P_0}{P_0 - A}e^{kt}$$

SOLUTION Assuming the population grows according to the logistic equation,

$$\frac{P(t)}{P(t) - A} = Ce^{kt}.$$

But

$$C = \frac{P_0}{P_0 - A},$$

so

$$\frac{P(t)}{P(t) - A} = \frac{P_0}{P_0 - A}e^{kt}.$$

Now, let $t = 0$ correspond to the year 1800. Then the year 1900 corresponds to $t = 100$, and with $k = 0.028$, we have

$$\frac{76}{76 - A} = \frac{5}{5 - A}e^{(0.028)(100)}.$$

Solving for A, we find

$$A = \frac{5(e^{2.8} - 1)}{\frac{5}{76}e^{2.8} - 1} \approx 943.07.$$

Thus, the predicted carrying capacity for the U.S. population is approximately 943 million.

16. **Reverse Logistic Equation** Consider the following logistic equation (with $k, B > 0$):

$$\frac{dP}{dt} = -kP\left(1 - \frac{P}{B}\right) \qquad \boxed{8}$$

(a) Sketch the slope field of this equation.

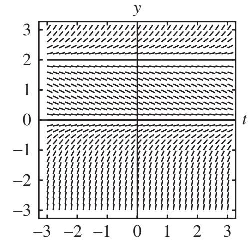
(b) The general solution is $P(t) = B/(1 - e^{kt}/C)$, where C is a nonzero constant. Show that $P(0) > B$ if $C > 1$ and $0 < P(0) < B$ if $C < 0$.

(c) Show that Eq. (8) models an "extinction–explosion" population. That is, $P(t)$ tends to zero if the initial population satisfies $0 < P(0) < B$, and it tends to ∞ after a finite amount of time if $P(0) > B$.

(d) Show that $P = 0$ is a stable equilibrium and $P = B$ an unstable equilibrium.

SOLUTION

(a) The slope field of this equation is shown below.



(b) Suppose that $C > 0$. Then $1 - \frac{1}{C} < 1$, $\left(1 - \frac{1}{C}\right)^{-1} > 1$, and

$$P(0) = \frac{B}{1 - \frac{1}{C}} > B.$$

On the other hand, if $C < 0$, then $1 - \frac{1}{C} > 1$, $0 < \left(1 - \frac{1}{C}\right)^{-1} < 1$, and

$$0 < P(0) = \frac{B}{1 - \frac{1}{C}} < B.$$

(c) From part (b), $0 < P(0) < B$ when $C < 0$. In this case, $1 - e^{kt}/C$ is never zero, but

$$1 - \frac{e^{kt}}{C} \to \infty$$

as $t \to \infty$. Thus, $P(t) \to 0$ as $t \to \infty$. On the other hand, $P(0) > B$ when $C > 0$. In this case $1 - e^{kt}/C = 0$ when $t = \frac{1}{k}\ln C$. Thus,

$$P(t) \to \infty \qquad \text{as} \qquad t \to \frac{1}{k}\ln C.$$

(d) Let

$$F(P) = -kP\left(1 - \frac{P}{B}\right).$$

Then, $F'(P) = -k + \frac{2kP}{B}$. Thus, $F'(0) = -k < 0$, and $F'(B) = -k + 2k = k > 0$, so $P = 0$ is a stable equilibrium and $P = B$ is an unstable equilibrium.

Further Insights and Challenges

In Exercises 17 and 18, let $y(t)$ be a solution of the logistic equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right) \qquad \boxed{9}$$

where $A > 0$ and $k > 0$.

17. **(a)** Differentiate Eq. (9) with respect to t and use the Chain Rule to show that

$$\frac{d^2y}{dt^2} = k^2y\left(1 - \frac{y}{A}\right)\left(1 - \frac{2y}{A}\right)$$

(b) Show that $y(t)$ is concave up if $0 < y < A/2$ and concave down if $A/2 < y < A$.

(c) Show that if $0 < y(0) < A/2$, then $y(t)$ has a point of inflection at $y = A/2$ (Figure 6).

(d) Assume that $0 < y(0) < A/2$. Find the time t when $y(t)$ reaches the inflection point.

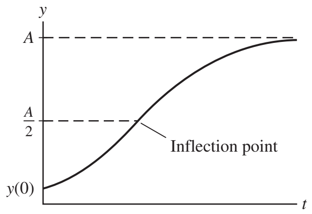


FIGURE 6 An inflection point occurs at $y = A/2$ in the logistic curve.

SOLUTION

(a) The derivative of Eq. (9) with respect to t is

$$y'' = ky' - \frac{2kyy'}{A} = ky'\left(1 - \frac{2y}{A}\right) = k\left(1 - \frac{y}{A}\right)ky\left(1 - \frac{2y}{A}\right) = k^2y\left(1 - \frac{y}{A}\right)\left(1 - \frac{2y}{A}\right).$$

(b) If $0 < y < A/2$, $1 - \frac{y}{A}$ and $1 - \frac{2y}{A}$ are both positive, so $y'' > 0$. Therefore, y is concave up. If $A/2 < y < A$, $1 - \frac{y}{A} > 0$, but $1 - \frac{2y}{A} < 0$, so $y'' < 0$, so y is concave down.

(c) If $y_0 < A$, y grows and $\lim_{t\to\infty} y(t) = A$. If $0 < y < A/2$, y is concave up at first. Once y passes $A/2$, y becomes concave down, so y has an inflection point at $y = A/2$.

(d) The general solution to Eq. (9) is

$$y = \frac{A}{1 - e^{-kt}/C};$$

thus, $y = A/2$ when

$$\frac{A}{2} = \frac{A}{1 - e^{-kt}/C}$$
$$1 - e^{-kt}/C = 2$$
$$t = -\frac{1}{k}\ln(-C)$$

Now, $C = y_0/(y_0 - A)$, so

$$t = -\frac{1}{k}\ln\frac{y_0}{A - y_0} = \frac{1}{k}\ln\frac{A - y_0}{y_0}.$$

18. Let $y = \dfrac{A}{1 - e^{-kt}/C}$ be the general nonequilibrium Eq. (9). If $y(t)$ has a vertical asymptote at $t = t_b$, that is, if $\lim_{t\to t_b-} y(t) = \pm\infty$, we say that the solution "blows up" at $t = t_b$.

(a) Show that if $0 < y(0) < A$, then y does not blow up at any time t_b.

(b) Show that if $y(0) > A$, then y blows up at a time t_b, which is negative (and hence does not correspond to a real time).

(c) Show that y blows up at some positive time t_b if and only if $y(0) < 0$ (and hence does not correspond to a real population).

SOLUTION

(a) Let $y(0) = y_0$. From the general solution, we find

$$y_0 = \frac{A}{1 - 1/C}; \quad 1 - \frac{1}{C} = \frac{A}{y_0}; \quad \text{so} \quad C = \frac{y_0}{y_0 - A}.$$

If $y_0 < A$, then $C < 0$, and the denominator in the general solution, $1 - e^{-kt}/C$, is always positive. Thus, when $0 < y(0) < A$, y does not blow up at any time.

(b) $1 - e^{-kt}/C = 0$ when $C = e^{-kt}$. Solving for t we find

$$t = -\frac{1}{k}\ln C.$$

Because $C = \frac{y_0}{y_0 - A}$ and $y_0 > A$, it follows that $C > 1$, and thus, $\ln C > 0$. Therefore, y blows up at a time which is negative.

(c) Suppose that y blows up at some $t_b > 0$. From part (b), we know that

$$t_b = -\frac{1}{k}\ln C.$$

Thus, in order for t_b to be positive, we must have $\ln C < 0$, which requires $C < 1$. Now,

$$C = \frac{y_0}{y_0 - A},$$

so $t_b > 0$ if and only if

$$\frac{y_0}{y_0 - A} < 1 \qquad \text{or equivalently} \qquad \frac{y_0 - A}{y_0} = 1 - \frac{A}{y_0} > 1.$$

This last inequality holds if and only if $y_0 = y(0) < 0$.

9.5 First-Order Linear Equations

Preliminary Questions

1. Which of the following are first-order linear equations?

(a) $y' + x^2 y = 1$ **(b)** $y' + xy^2 = 1$

(c) $x^5 y' + y = e^x$ **(d)** $x^5 y' + y = e^y$

SOLUTION The equations in **(a)** and **(c)** are first-order linear differential equations. The equation in **(b)** is not linear because of the y^2 factor in the second term on the left-hand side of the equation; the equation in **(d)** is not linear because of the e^y term on the right-hand side of the equation.

2. If $\alpha(x)$ is an integrating factor for $y' + A(x)y = B(x)$, then $\alpha'(x)$ is equal to (choose the correct answer):

(a) $B(x)$ **(b)** $\alpha(x)A(x)$

(c) $\alpha(x)A'(x)$ **(d)** $\alpha(x)B(x)$

SOLUTION The correct answer is **(b)**: $\alpha(x)A(x)$.

Exercises

1. Consider $y' + x^{-1}y = x^3$.

(a) Verify that $\alpha(x) = x$ is an integrating factor.

(b) Show that when multiplied by $\alpha(x)$, the differential equation can be written $(xy)' = x^4$.

(c) Conclude that xy is an antiderivative of x^4 and use this information to find the general solution.

(d) Find the particular solution satisfying $y(1) = 0$.

SOLUTION

(a) The equation is of the form

$$y' + A(x)y = B(x)$$

for $A(x) = x^{-1}$ and $B(x) = x^3$. By Theorem 1, $\alpha(x)$ is defined by

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

(b) When multiplied by $\alpha(x)$, the equation becomes:

$$xy' + y = x^4.$$

Now, $xy' + y = xy' + (x)'y = (xy)'$, so

$$(xy)' = x^4.$$

(c) Since $(xy)' = x^4$, $(xy) = \frac{x^5}{5} + C$ and

$$y = \frac{x^4}{5} + \frac{C}{x}$$

(d) If $y(1) = 0$, we find

$$0 = \frac{1}{5} + C \quad \text{so} \quad -\frac{1}{5} = C.$$

The solution, therefore, is

$$y = \frac{x^4}{5} - \frac{1}{5x}.$$

2. Consider $\dfrac{dy}{dt} + 2y = e^{-3t}$.

(a) Verify that $\alpha(t) = e^{2t}$ is an integrating factor.

(b) Use Eq. (4) to find the general solution.

(c) Find the particular solution with initial condition $y(0) = 1$.

SOLUTION

(a) The equation is of the form

$$y' + A(t)y = B(t)$$

for $A(t) = 2$ and $B(t) = e^{-3t}$. Thus,

$$\alpha(t) = e^{\int A(t)\,dt} = e^{2t}.$$

(b) According to Eq. (4),

$$y(t) = \frac{1}{\alpha(t)}\left(\int \alpha(t)B(t)\,dt + C\right).$$

With $\alpha(t) = e^{2t}$ and $B(t) = e^{-3t}$, this yields

$$y(t) = e^{-2t}\left(\int e^{-t}\,dt + C\right) = e^{-2t}\left(C - e^{-t}\right) = Ce^{-2t} - e^{-3t}.$$

(c) Using the initial condition $y(0) = 1$, we find

$$1 = -1 + C \quad \text{so} \quad 2 = C.$$

The particular solution is therefore

$$y = -e^{-3t} + 2e^{-2t}.$$

3. Let $\alpha(x) = e^{x^2}$. Verify the identity

$$(\alpha(x)y)' = \alpha(x)(y' + 2xy)$$

and explain how it is used to find the general solution of

$$y' + 2xy = x$$

SOLUTION Let $\alpha(x) = e^{x^2}$. Then

$$(\alpha(x)y)' = (e^{x^2}y)' = 2xe^{x^2}y + e^{x^2}y' = e^{x^2}\left(2xy + y'\right) = \alpha(x)\left(y' + 2xy\right).$$

If we now multiply both sides of the differential equation $y' + 2xy = x$ by $\alpha(x)$, we obtain

$$\alpha(x)(y' + 2xy) = x\alpha(x) = xe^{x^2}.$$

But $\alpha(x)(y' + 2xy) = (\alpha(x)y)'$, so by integration we find

$$\alpha(x)y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C.$$

Finally,

$$y(x) = \frac{1}{2} + Ce^{-x^2}.$$

4. Find the solution of $y' - y = e^{2x}$, $y(0) = 1$.

SOLUTION We first find the general solution of the differential equation $y' - y = e^{2x}$. This is of the standard linear form

$$y' + A(x)y = B(x)$$

with $A(x) = -1$, $B(x) = e^{2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^{-x}y' - e^{-x}y = e^x \qquad \text{or} \qquad (e^{-x}y)' = e^x.$$

Integration of both sides now yields

$$e^{-x}y = \int e^x\,dx = e^x + C.$$

Therefore,

$$y(x) = e^{2x} + Ce^x.$$

Using the initial condition $y(0) = 1$, we find

$$1 = 1 + C \quad \text{so} \quad 0 = C.$$

Therefore,

$$y = e^{2x}.$$

In Exercises 5–18, find the general solution of the first-order linear differential equation.

5. $xy' + y = x$

SOLUTION Rewrite the equation as

$$y' + \frac{1}{x}y = 1,$$

which is in standard linear form with $A(x) = \frac{1}{x}$ and $B(x) = 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$xy' + y = x \qquad \text{or} \qquad (xy)' = x.$$

Integration of both sides now yields

$$xy = \frac{1}{2}x^2 + C.$$

Finally,

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

6. $xy' - y = x^2 - x$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{x}y = x - 1,$$

which is in standard linear form with $A(x) = -\frac{1}{x}$ and $B(x) = x - 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-\ln x} = x^{-1}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1 - \frac{1}{x} \qquad \text{or} \qquad \left(\frac{y}{x}\right)' = 1 - \frac{1}{x}.$$

Integration of both sides now yields

$$\frac{y}{x} = x - \ln x + C.$$

Finally,

$$y(x) = x^2 - x\ln x + Cx.$$

7. $3xy' - y = x^{-1}$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{3x}y = \frac{1}{3x^2},$$

which is in standard form with $A(x) = -\frac{1}{3}x^{-1}$ and $B(x) = \frac{1}{3}x^{-2}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{-(1/3)\ln x} = x^{-1/3}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$x^{-1/3}y' - \frac{1}{3}x^{-4/3} = \frac{1}{3}x^{-7/3} \qquad \text{or} \qquad (x^{-1/3}y)' = \frac{1}{3}x^{-7/3}.$$

Integration of both sides now yields

$$x^{-1/3}y = -\frac{1}{4}x^{-4/3} + C.$$

Finally,

$$y(x) = -\frac{1}{4}x^{-1} + Cx^{1/3}.$$

8. $y' + xy = x$

SOLUTION This equation is in standard form with $A(x) = x$ and $B(x) = x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x\,dx} = e^{(1/2)x^2}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^{(1/2)x^2}y' + xe^{(1/2)x^2}y = xe^{(1/2)x^2} \qquad \text{or} \qquad (e^{(1/2)x^2}y)' = xe^{(1/2)x^2}.$$

Integration of both sides now yields

$$e^{(1/2)x^2}y = e^{(1/2)x^2} + C.$$

Finally,

$$y(x) = 1 + Ce^{-(1/2)x^2}.$$

9. $y' + 3x^{-1}y = x + x^{-1}$

SOLUTION This equation is in standard form with $A(x) = 3x^{-1}$ and $B(x) = x + x^{-1}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 3x^{-1}} = e^{3\ln x} = x^3.$$

When multiplied by the integrating factor, the original differential equation becomes

$$x^3y' + 3x^2y = x^4 + x^2 \qquad \text{or} \qquad (x^3y)' = x^4 + x^3.$$

Integration of both sides now yields

$$x^3y = \frac{1}{5}x^5 + \frac{1}{3}x^3 + C.$$

Finally,

$$y(x) = \frac{1}{5}x^2 + \frac{1}{3} + Cx^{-3}.$$

10. $y' + x^{-1}y = \cos(x^2)$

SOLUTION This equation is in standard form with $A(x) = x^{-1}$ and $B(x) = \cos(x^2)$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x^{-1}\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$xy' + y = x\cos(x^2) \qquad \text{or} \qquad (xy)' = x\cos(x^2).$$

Integration of both sides now yields

$$xy = \frac{1}{2}\sin(x^2) + C.$$

Finally,

$$y(x) = \frac{1}{2}x^{-1}\sin(x^2) + Cx^{-1}.$$

11. $xy' = y - x$

SOLUTION Rewrite the equation as

$$y' - \frac{1}{x}y = -1,$$

which is in standard form with $A(x) = -\frac{1}{x}$ and $B(x) = -1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -(1/x)\,dx} = e^{-\ln x} = x^{-1}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\frac{1}{x}y' - \frac{1}{x^2}y = -\frac{1}{x} \qquad \text{or} \qquad \left(\frac{1}{x}y\right)' = -\frac{1}{x}.$$

Integration on both sides now yields

$$\frac{1}{x}y = -\ln x + C.$$

Finally,

$$y(x) = -x\ln x + Cx.$$

12. $xy' = x^{-2} - \dfrac{3y}{x}$

SOLUTION Rewrite the equation is

$$y' + \frac{3}{x^2}y = \frac{1}{x^3}$$

which is in standard form with $A(x) = \frac{3}{x^2}$ and $B(x) = \frac{1}{x^3}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int (3/x^2)\,dx} = e^{-3x^{-1}}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$e^{-3/x}y' + \frac{3}{x^2}e^{-3/x}y = \frac{1}{x^3}e^{-3/x}$$

Integration on both sides now yields

$$e^{-3/x}y = \frac{x+3}{9x}e^{-3/x} + C \qquad \text{or} \qquad y = \frac{x+3}{9x} + Ce^{3/x}$$

13. $y' + y = e^x$

SOLUTION This equation is in standard form with $A(x) = 1$ and $B(x) = e^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^x y' + e^x y = e^{2x} \qquad \text{or} \qquad (e^x y)' = e^{2x}.$$

Integration on both sides now yields

$$e^x y = \frac{1}{2}e^{2x} + C.$$

Finally,

$$y(x) = \frac{1}{2}e^x + Ce^{-x}.$$

14. $y' + (\sec x)y = \cos x$

SOLUTION This equation is in standard form with $A(x) = \sec x$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \sec x\,dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(\sec x + \tan x)y' + (\sec^2 x + \sec x \tan x)y = 1 + \sin x$$

or

$$((\sec x + \tan x)y)' = 1 + \sin x.$$

Integration on both sides now yields

$$(\sec x + \tan x)y = x - \cos x + C.$$

Finally,

$$y(x) = \frac{x - \cos x + C}{\sec x + \tan x}.$$

15. $y' + (\tan x)y = \cos x$

SOLUTION This equation is in standard form with $A(x) = \tan x$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\sec x y' + \sec x \tan x y = 1 \qquad \text{or} \qquad (y \sec x)' = 1.$$

Integration on both sides now yields

$$y \sec x = x + C.$$

Finally,

$$y(x) = x \cos x + C \cos x.$$

16. $e^{2x} y' = 1 - e^x y$

SOLUTION Rewrite the equation as

$$y' + e^{-x} y = e^{-2x},$$

which is in standard form with $A(x) = e^{-x}$ and $B(x) = e^{-2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int e^{-x}\,dx} = e^{-e^{-x}}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$e^{-e^{-x}} y' + e^{-x-e^{-x}} y = e^{-2x} e^{-e^{-x}} \qquad \text{or} \qquad (e^{-e^{-x}} y)' = e^{-2x} e^{-e^{-x}}.$$

Integration on both sides now yields

$$(e^{-e^{-x}} y) = \int e^{-2x} e^{-e^{-x}}\,dx.$$

To handle the remaining integral, make the substitution $u = -e^{-x}$, $du = e^{-x}\,dx$. Then

$$\int e^{-2x} e^{-e^{-x}}\,dx = -\int u e^u\,du = -ue^u + e^u + C = e^{-x} e^{-e^{-x}} + e^{-e^{-x}} + C.$$

Finally,

$$y(x) = 1 + e^{-x} + Ce^{e^{-x}}.$$

17. $y' - (\ln x)y = x^x$

SOLUTION This equation is in standard form with $A(x) = -\ln x$ and $B(x) = x^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -\ln x\,dx} = e^{x - x\ln x} = \frac{e^x}{x^x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$x^{-x}e^x y' - (\ln x)x^{-x}e^x y = e^x \qquad \text{or} \qquad (x^{-x}e^x y)' = e^x.$$

Integration on both sides now yields

$$x^{-x}e^x y = e^x + C.$$

Finally,

$$y(x) = x^x + Cx^x e^{-x}.$$

18. $y' + y = \cos x$

SOLUTION This equation is in standard form with $A(x) = 1$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$e^x y' + e^x y = e^x \cos x \qquad \text{or} \qquad (e^x y)' = e^x \cos x.$$

Integration on both sides (integration by parts is needed on the right-hand side of the equation) now yields

$$e^x y = \frac{1}{2}e^x (\sin x + \cos x) + C.$$

Finally,

$$y(x) = \frac{1}{2}(\sin x + \cos x) + Ce^{-x}.$$

In Exercises 19–26, solve the initial value problem.

19. $y' + 3y = e^{2x}, \quad y(0) = -1$

SOLUTION First, we find the general solution of the differential equation. This linear equation is in standard form with $A(x) = 3$ and $B(x) = e^{2x}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{3x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{3x}y)' = e^{5x}.$$

Integration on both sides now yields

$$(e^{3x}y) = \frac{1}{5}e^{5x} + C;$$

hence,

$$y(x) = \frac{1}{5}e^{2x} + Ce^{-3x}.$$

The initial condition $y(0) = -1$ allows us to determine the value of C:

$$-1 = \frac{1}{5} + C \qquad \text{so} \qquad C = -\frac{6}{5}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{5}e^{2x} - \frac{6}{5}e^{-3x}.$$

20. $xy' + y = e^x, \quad y(1) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' + \frac{1}{x}y = \frac{1}{x}e^x,$$

which is in standard form with $A(x) = x^{-1}$ and $B(x) = x^{-1}e^x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int x^{-1}\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(xy)' = e^x.$$

Integration on both sides now yields

$$xy = e^x + C;$$

hence,

$$y(x) = \frac{1}{x}e^x + \frac{C}{x}.$$

The initial condition $y(1) = 3$ allows us to determine the value of C:

$$3 = e + \frac{C}{1} \qquad \text{so} \qquad C = 3 - e.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{x}e^x + \frac{3-e}{x}.$$

21. $y' + \dfrac{1}{x+1}y = x^{-2}, \quad y(1) = 2$

SOLUTION First, we find the general solution of the differential equation. This linear equation is in standard form with $A(x) = \frac{1}{x+1}$ and $B(x) = x^{-2}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1/(x+1)\,dx} = e^{\ln(x+1)} = x + 1.$$

When multiplied by the integrating factor, the original differential equation becomes

$$((x+1)y)' = x^{-1} + x^{-2}.$$

Integration on both sides now yields

$$(x+1)y = \ln x - x^{-1} + C;$$

hence,

$$y(x) = \frac{1}{x+1}\left(C + \ln x - \frac{1}{x}\right).$$

The initial condition $y(1) = 2$ allows us to determine the value of C:

$$2 = \frac{1}{2}(C - 1) \qquad \text{so} \qquad C = 5.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{x+1}\left(5 + \ln x - \frac{1}{x}\right).$$

22. $y' + y = \sin x, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = 1$ and $B(x) = \sin x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int 1\,dx} = e^x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^x y)' = e^x \sin x.$$

Integration on both sides (integration by parts is needed on the right-hand side of the equation) now yields

$$(e^x y) = \frac{1}{2}e^x(\sin x - \cos x) + C;$$

hence,

$$y(x) = \frac{1}{2}(\sin x - \cos x) + Ce^{-x}.$$

The initial condition $y(0) = 1$ allows us to determine the value of C:

$$1 = -\frac{1}{2} + C \qquad \text{so} \qquad C = \frac{3}{2}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{2}(\sin x - \cos x) + \frac{3}{2}e^{-x}.$$

23. $(\sin x)y' = (\cos x)y + 1, \quad y\left(\frac{\pi}{4}\right) = 0$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - (\cot x)y = \csc x,$$

which is in standard form with $A(x) = -\cot x$ and $B(x) = \csc x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int -\cot x\, dx} = e^{-\ln \sin x} = \csc x.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(\csc x y)' = \csc^2 x.$$

Integration on both sides now yields

$$(\csc x)y = -\cot x + C;$$

hence,

$$y(x) = -\cos x + C \sin x.$$

The initial condition $y(\pi/4) = 0$ allows us to determine the value of C:

$$0 = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2} \qquad \text{so} \qquad C = 1.$$

The solution to the initial value problem is therefore

$$y(x) = -\cos x + \sin x.$$

24. $y' + (\sec t)y = \sec t, \quad y\left(\frac{\pi}{4}\right) = 1$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(t) = \sec t$ and $B(t) = \sec t$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int \sec t\, dt} = e^{\ln(\sec t + \tan t)} = \sec t + \tan t.$$

When multiplied by the integrating factor, the original differential equation becomes

$$((\sec t + \tan t)y)' = \sec^2 t + \sec t \tan t.$$

Integration on both sides now yields

$$(\sec t + \tan t)y = \tan t + \sec t + C;$$

hence,

$$y(t) = 1 + \frac{C}{\sec t + \tan t}.$$

The initial condition $y(\pi/4) = 1$ allows us to determine the value of C:

$$1 = 1 + \frac{C}{\sqrt{2}+1} \qquad \text{so} \qquad C = 0.$$

The solution to the initial value problem is therefore

$$y(x) = 1.$$

25. $y' + (\tanh x)y = 1, \quad y(0) = 3$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = \tanh x$ and $B(x) = 1$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int \tanh x\, dx} = e^{\ln \cosh x} = \cosh x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(\cosh xy)' = \cosh x.$$

Integration on both sides now yields

$$(\cosh xy) = \sinh x + C;$$

hence,

$$y(x) = \tanh x + C \operatorname{sech} x.$$

The initial condition $y(0) = 3$ allows us to determine the value of C:

$$3 = C.$$

The solution to the initial value problem is therefore

$$y(x) = \tanh x + 3 \operatorname{sech} x.$$

26. $y' + \dfrac{x}{1+x^2}y = \dfrac{1}{(1+x^2)^{3/2}}, \quad y(1) = 0$

SOLUTION First, we find the general solution of the differential equation. This equation is in standard form with $A(x) = \frac{x}{1+x^2}$ and $B(x) = \frac{1}{(1+x^2)^{3/2}}$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int (x/(1+x^2))\,dx} = e^{(1/2)\ln(1+x^2)} = \sqrt{1+x^2}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\left(\sqrt{1+x^2}y\right)' = \frac{1}{1+x^2}.$$

Integration on both sides now yields

$$\sqrt{1+x^2}y = \tan^{-1} x + C;$$

hence,

$$y(x) = \frac{\tan^{-1} x}{\sqrt{1+x^2}} + \frac{C}{\sqrt{1+x^2}}.$$

The initial condition $y(1) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + C\right) \qquad \text{so} \qquad C = -\frac{\pi}{4}.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1} x - \frac{\pi}{4}\right).$$

27. Find the general solution of $y' + ny = e^{mx}$ for all m, n. *Note:* The case $m = -n$ must be treated separately.

SOLUTION For any m, n, Theorem 1 gives us the formula for $\alpha(x)$:

$$\alpha(x) = e^{\int n\,dx} = e^{nx}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{nx}y)' = e^{(m+n)x}.$$

If $m \neq -n$, integration on both sides yields

$$e^{nx}y = \frac{1}{m+n}e^{(m+n)x} + C,$$

so

$$y(x) = \frac{1}{m+n}e^{mx} + Ce^{-nx}.$$

However, if $m = -n$, then $m + n = 0$ and the equation reduces to

$$(e^{nx}y)' = 1,$$

so integration yields

$$e^{nx}y = x + C \qquad \text{or} \qquad y(x) = (x + C)e^{-nx}.$$

28. Find the general solution of $y' + ny = \cos x$ for all n.

SOLUTION This equation is in standard form with $A(x) = n$ and $B(x) = \cos x$. By Theorem 1, the integrating factor is

$$\alpha(x) = e^{\int n\,dx} = e^{nx}$$

When multiplied by the integrating factor, the differential equation becomes

$$e^{nx}y' + ne^{nx}y = e^{nx}\cos x$$

Integrating both sides gives

$$e^{nx}y = \frac{e^{nx}}{n^2+1}(\sin x + n\cos x) + C$$

(To integrate the right hand side, apply integration by parts twice with $u = e^{nx}$). Finally

$$y = Ce^{-nx} + \frac{\sin x + n\cos x}{n^2+1}$$

In Exercises 29–32, a 1000 L *tank contains* 500 L *of water with a salt concentration of* 10 g/L*. Water with a salt concentration of* 50 g/L *flows into the tank at a rate of* 80 L/min*. The fluid mixes instantaneously and is pumped out at a specified rate* R_{out}*. Let* $y(t)$ *denote the quantity of salt in the tank at time* t*.*

29. Assume that $R_{\text{out}} = 40$ L/min.

(a) Set up and solve the differential equation for $y(t)$.

(b) What is the salt concentration when the tank overflows?

SOLUTION Because water flows into the tank at the rate of 80 L/min but flows out at the rate of $R_{\text{out}} = 40$ L/min, there is a net inflow of 40 L/min. Therefore, at any time t, there are $500 + 40t$ liters of water in the tank.

(a) The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y\text{ g}}{500+40t\text{ L}}\right) = 4000 - 40\cdot\frac{y}{500+40t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{4}{50+4t}y = 4000,$$

so $A(t) = \frac{4}{50+4t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 4(50+4t)^{-1}\,dt} = e^{\ln(50+4t)} = 50 + 4t.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((50+4t)y)' = 4000(50+4t).$$

Integration on both sides now yields

$$(50+4t)y = 200{,}000t + 16{,}000t^2 + C;$$

hence,

$$y(t) = \frac{200{,}000t + 8000t^2 + C}{50+4t}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = \frac{C}{50} \qquad \text{so} \qquad C = 500.$$

The solution to the initial value problem is therefore

$$y(t) = \frac{200{,}000t + 8000t^2 + 500}{50+4t} = \frac{250 + 4000t^2 + 100{,}000t}{25+2t}.$$

(b) The tank overflows when $t = 25/2 = 12.5$. The amount of salt in the tank at that time is

$$y(12.5) = 37{,}505 \text{ g},$$

so the concentration of salt is

$$\frac{37{,}505 \text{ g}}{1000 \text{ L}} = 37.505 \text{ g/L}.$$

30. Find the salt concentration when the tank overflows, assuming that $R_{\text{out}} = 60$ L/min.

SOLUTION We work as in Exercise 29, but with $R_{\text{out}} = 60$. There is a net inflow of 20 L/min, so at time t, there are $500 + 20t$ liters of water in the tank. The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(60\frac{\text{L}}{\text{min}}\right)\left(\frac{y \text{ g}}{500 + 20t \text{ L}}\right) = 4000 - 6 \cdot \frac{y}{50 + 2t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{6}{50 + 2t}y = 4000,$$

so $A(t) = \frac{6}{50+2t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 6(50+2t)^{-1}\,dt} = e^{3\ln(50+2t)} = (50 + 2t)^3.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((50 + 2t)^3 y)' = 4000(50 + 2t)^3.$$

Integration on both sides now yields

$$(50 + 2t)^3 y = 500(50 + 2t)^4 + C;$$

hence,

$$y(t) = 25{,}000 + 1000t + \frac{C}{(50 + 2t)^3}.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = 25{,}000 + \frac{C}{50^3} \qquad \text{so} \qquad C = -3123.75 \times 10^6.$$

The solution to the initial value problem is therefore

$$y(t) = 25{,}000 + 1000t - \frac{390{,}468{,}750}{(25 + t)^2}.$$

The tank overflows when $t = 25$. The amount of salt in the tank at that time is

$$y(25) = 46{,}876.25 \text{ g},$$

so the concentration of salt is

$$\frac{46{,}876.25 \text{ g}}{1000 \text{ L}} \approx 46.876 \text{ g/L}.$$

31. Find the limiting salt concentration as $t \to \infty$ assuming that $R_{\text{out}} = 80$ L/min.

SOLUTION The total volume of water is now constant at 500 liters, so the net flow of salt at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(80\frac{\text{L}}{\text{min}}\right)\left(\frac{y \text{ g}}{500 \text{ L}}\right) = 4000 - \frac{8}{50}y$$

Rewriting this equation in standard form gives

$$\frac{dy}{dt} + \frac{8}{50}y = 4000$$

so that the integrating factor is

$$e^{\int (8/50)\,dt} = e^{0.16t}$$

Multiplying both sides by the integrating factor gives

$$(e^{0.16t}y)' = 4000e^{0.16t}$$

Integrate both sides to get

$$e^{0.16t}y = 25{,}000e^{0.16t} + C \quad \text{so that} \quad y = 25{,}000 + Ce^{-0.16t}$$

As $t \to \infty$, the exponential term tends to zero, so that the amount of salt tends to 25,000g, or 50 g/L. (Note that this is precisely what would be expected naïvely, since the salt concentration flowing in is also 50 g/L).

32. Assuming that $R_{\text{out}} = 120$ L/min. Find $y(t)$. Then calculate the tank volume and the salt concentration at $t = 10$ minutes.

SOLUTION We work as in Exercise 29, but with $R_{\text{out}} = 120$. There is a net outflow of 40 L/min, so at time t, there are $500 - 40t$ liters of water in the tank. Note that after ten minutes, the volume of water in the tank is 100 liters.

The net flow of salt into the tank at time t is

$$\frac{dy}{dt} = \text{salt rate in} - \text{salt rate out} = \left(80\frac{\text{L}}{\text{min}}\right)\left(50\frac{\text{g}}{\text{L}}\right) - \left(120\frac{\text{L}}{\text{min}}\right)\left(\frac{y\text{ g}}{500 - 40t\text{ L}}\right) = 4000 - 12 \cdot \frac{y}{50 - 4t}$$

Rewriting this linear equation in standard form, we have

$$\frac{dy}{dt} + \frac{6}{25 - 2t}y = 4000,$$

so $A(t) = \frac{6}{25-2t}$ and $B(t) = 4000$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int 6(25-2t)^{-1}\,dt} = e^{-3\ln(25-2t)} = (25 - 2t)^{-3}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$((25 - 2t)^{-3}y)' = 4000(25 - 2t)^{-3}.$$

Integration on both sides now yields

$$(25 - 2t)^{-3}y = 1000(25 - 2t)^{-2} + C;$$

hence,

$$y(t) = 25{,}000 - 2000t + C(25 - 2t)^3.$$

The initial condition $y(0) = 10$ allows us to determine the value of C:

$$10 = 25{,}000 + C \cdot 50^3 \qquad \text{so} \qquad C = -1.599.$$

The solution to the initial value problem is therefore

$$y(t) = 25{,}000 - 2000t - 1.599(25 - 2t)^3.$$

The amount of salt in the tank at time $t = 10$ is then

$$y(10) = 4800.08 \text{ g},$$

so the concentration of salt is

$$\frac{4800.08\text{ g}}{100\text{ L}} \approx 48 \text{ g/L}.$$

33. Water flows into a tank at the variable rate of $R_{\text{in}} = 20/(1 + t)$ gal/min and out at the constant rate $R_{\text{out}} = 5$ gal/min. Let $V(t)$ be the volume of water in the tank at time t.

(a) Set up a differential equation for $V(t)$ and solve it with the initial condition $V(0) = 100$.

(b) Find the maximum value of V.

(c) CAS Plot $V(t)$ and estimate the time t when the tank is empty.

SOLUTION

(a) The rate of change of the volume of water in the tank is given by

$$\frac{dV}{dt} = R_{\text{in}} - R_{\text{out}} = \frac{20}{1 + t} - 5.$$

Because the right-hand side depends only on the independent variable t, we integrate to obtain

$$V(t) = 20\ln(1+t) - 5t + C.$$

The initial condition $V(0) = 100$ allows us to determine the value of C:

$$100 = 20\ln 1 - 0 + C \qquad \text{so} \qquad C = 100.$$

Therefore

$$V(t) = 20\ln(1+t) - 5t + 100.$$

(b) Using the result from part (a),

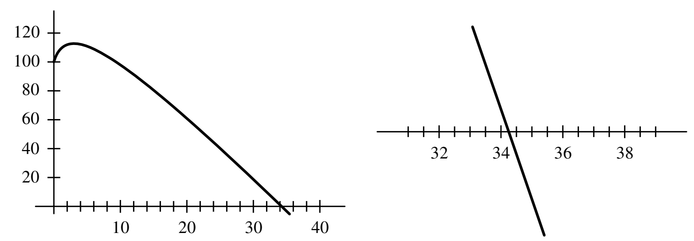
$$\frac{dV}{dt} = \frac{20}{1+t} - 5 = 0$$

when $t = 3$. Because $\frac{dV}{dt} > 0$ for $t < 3$ and $\frac{dV}{dt} < 0$ for $t > 3$, it follows that

$$V(3) = 20\ln 4 - 15 + 100 \approx 112.726 \text{ gal}$$

is the maximum volume.

(c) $V(t)$ is plotted in the figure below at the left. On the right, we zoom in near the location where the curve crosses the t-axis. From this graph, we estimate that the tank is empty after roughly 34.25 minutes.



34. A stream feeds into a lake at a rate of 1000 m^3/day. The stream is polluted with a toxin whose concentration is 5 g/m^3. Assume that the lake has volume 10^6 m^3 and that water flows out of the lake at the same rate of 1000 m^3/day.

(a) Set up a differential equation for the concentration $c(t)$ of toxin in the lake and solve for $c(t)$, assuming that $c(0) = 0$. *Hint:* Find the differential equation for the quantity of toxin $y(t)$, and observe that $c(t) = y(t)/10^6$.

(b) What is the limiting concentration for large t?

SOLUTION

(a) Let $M(t)$ denote the amount of toxin, in grams, in the lake at time t. The rate at which toxin enters the lake is given by

$$5\frac{\text{g}}{\text{m}^3} \cdot 1000\frac{\text{m}^3}{\text{day}} = 5000\frac{\text{g}}{\text{day}},$$

while the rate at which toxin exits the lake is given by

$$\frac{M(t)\ \text{g}}{10^6\ \text{m}^3} \cdot 1000\frac{\text{m}^3}{\text{day}} = \frac{M(t)}{1000}\frac{\text{g}}{\text{day}},$$

where we have assumed that any toxin in the lake is spread uniformly throughout the lake. The differential equation for $M(t)$ is then

$$\frac{dM}{dt} = 5000 - \frac{M}{1000}.$$

The concentration of the toxin in the lake is given by $c(t) = \dfrac{M(t)}{10^6}$, so $c'(t) = \frac{1}{10^6}M'(t)$, giving

$$\frac{dc}{dt} = \frac{1}{200} - \frac{1}{1000}c.$$

Rewriting this linear equation in standard form, we have

$$\frac{dc}{dt} + \frac{1}{1000}c = \frac{1}{200},$$

so $A(t) = \frac{1}{1000}$ and $B(t) = \frac{1}{200}$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{t/1000}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{t/1000}c)' = \frac{1}{200}e^{t/1000}.$$

Integration on both sides now yields

$$e^{t/1000}c = 5e^{t/1000} + A;$$

hence,

$$c(t) = 5 + Ae^{-t/1000}.$$

The initial condition $c(0) = 0$ allows us to determine the value of A:

$$0 = 5 + A \qquad \text{so} \qquad A = -5.$$

Therefore

$$c(t) = 5\left(1 - e^{-t/1000}\right) \text{ grams/m}^3.$$

(b) As $t \to \infty$, $c(t) \to 5$, so the limiting concentration of pollution is $5\ \dfrac{\text{grams}}{\text{m}^3}$.

In Exercises 35–38, consider a series circuit (Figure 4) consisting of a resistor of R ohms, an inductor of L henries, and a variable voltage source of $V(t)$ volts (time t in seconds). The current through the circuit $I(t)$ (in amperes) satisfies the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t) \qquad \boxed{10}$$

35. Find the solution to Eq. (10) with initial condition $I(0) = 0$, assuming that $R = 100\ \Omega$, $L = 5$ H, and $V(t)$ is constant with $V(t) = 10$ V.

SOLUTION If $R = 100$, $V(t) = 10$, and $L = 5$, the differential equation becomes

$$\frac{dI}{dt} + 20I = 2,$$

which is a linear equation in standard form with $A(t) = 20$ and $B(t) = 2$. The integrating factor is $\alpha(t) = e^{20t}$, and when multiplied by the integrating factor, the differential equation becomes

$$(e^{20t}I)' = 2e^{20t}.$$

Integration of both sides now yields

$$e^{20t}I = \frac{1}{10}e^{20t} + C;$$

hence,

$$I(t) = \frac{1}{10} + Ce^{-20t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{10} + C \qquad \text{so} \qquad C = -\frac{1}{10}.$$

Finally,

$$I(t) = \frac{1}{10}\left(1 - e^{-20t}\right).$$

36. Assume that $R = 110\ \Omega$, $L = 10$ H, and $V(t) = e^{-t}$.

(a) Solve Eq. (10) with initial condition $I(0) = 0$.

(b) Calculate t_m and $I(t_m)$, where t_m is the time at which $I(t)$ has a maximum value.

(c) GU Use a computer algebra system to sketch the graph of the solution for $0 \le t \le 3$.

SOLUTION

(a) If $R = 110$, $V(t) = e^{-t}$, and $L = 10$, the differential equation becomes

$$\frac{dI}{dt} + 11I = \frac{1}{10}e^{-t},$$

which is a linear equation in standard form with $A(t) = 11$ and $B(t) = \frac{1}{10}e^{-t}$. The integrating factor is $\alpha(t) = e^{11t}$, and when multiplied by the integrating factor, the differential equation becomes

$$(e^{11t}I)' = \frac{1}{10}e^{10t}.$$

Integration of both sides now yields

$$e^{11t}I = \frac{1}{100}e^{10t} + C;$$

hence,

$$I(t) = \frac{1}{100}e^{-t} + Ce^{-11t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{1}{100} + C \qquad \text{so} \qquad C = -\frac{1}{100}.$$

Finally,

$$I(t) = \frac{1}{100}\left(e^{-t} - e^{-11t}\right).$$

(b) Using the result from part (a),

$$\frac{dI}{dt} = \frac{1}{100}\left(-e^{-t} + 11e^{-11t}\right) = 0$$

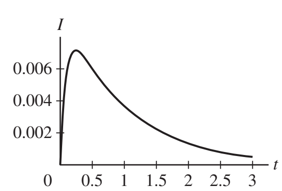
when

$$t = t_m = \frac{1}{10}\ln 11 \text{ seconds.}$$

Now,

$$I(t_m) = \frac{1}{100}\left(e^{-(1/10)\ln 11} - e^{-(11/10)\ln 11}\right) = \frac{1}{100}\left(11^{-1/10} - 11^{-11/10}\right) \approx 0.00715 \text{ amperes.}$$

(c) The graph of $I(t)$ is shown below.



37. Assume that $V(t) = V$ is constant and $I(0) = 0$.

(a) Solve for $I(t)$.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$ and that $I(t)$ reaches approximately 63% of its limiting value after L/R seconds.

(c) How long does it take for $I(t)$ to reach 90% of its limiting value if $R = 500\ \Omega$, $L = 4$ H, and $V = 20$ V?

SOLUTION

(a) The equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V$$

is a linear equation in standard form with $A(t) = \frac{R}{L}$ and $B(t) = \frac{1}{L}V(t)$. By Theorem 1, the integrating factor is

$$\alpha(t) = e^{\int (R/L)\,dt} = e^{(R/L)t}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{(R/L)t}I)' = e^{(R/L)t}\frac{V}{L}.$$

Integration on both sides now yields

$$(e^{(R/L)t}I) = \frac{V}{R}e^{(R/L)t} + C;$$

hence,

$$I(t) = \frac{V}{R} + Ce^{-(R/L)t}.$$

The initial condition $I(0) = 0$ allows us to determine the value of C:

$$0 = \frac{V}{R} + C \qquad \text{so} \qquad C = -\frac{V}{R}.$$

Therefore the current is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-(R/L)t}\right).$$

(b) As $t \to \infty$, $e^{-(R/L)t} \to 0$, so $I(t) \to \frac{V}{R}$. Moreover, when $t = (L/R)$ seconds, we have

$$I\left(\frac{L}{R}\right) = \frac{V}{R}\left(1 - e^{-(R/L)(L/R)}\right) = \frac{V}{R}\left(1 - e^{-1}\right) \approx 0.632\frac{V}{R}.$$

(c) Using the results from part (a) and part (b), $I(t)$ reaches 90% of its limiting value when

$$\frac{9}{10} = 1 - e^{-(R/L)t},$$

or when

$$t = \frac{L}{R}\ln 10.$$

With $L = 4$ and $R = 500$, this takes approximately 0.0184 seconds.

38. Solve for $I(t)$, assuming that $R = 500\ \Omega$, $L = 4$ H, and $V = 20\cos(80t)$ volts.

SOLUTION With $R = 500$, $L = 4$, and $V = 20\cos(80t)$, Eq. (10) becomes

$$\frac{dI}{dt} + 125I = 5\cos(80t)$$

which is a linear equation in standard form with $A(t) = 125$ and $B(t) = 5\cos(80t)$. The integrating factor is e^{125t}; when multiplied by the integrating factor, the differential equation becomes

$$(e^{125t}I)' = 5e^{125t}\cos(80t)$$

To integrate the right side, apply integration by parts twice and solve the resulting formula for the desired integral, giving

$$\int 5e^{125t}\cos(80t)\,dt = \frac{1}{881}e^{125t}(25\cos(80t) + 16\sin(80t) + C$$

so that the solution is

$$e^{125t}I = \frac{1}{881}e^{125t}(25\cos(80t) + 16\sin(80t) + C$$

Multiply through by e^{-125t} to get

$$I = \frac{1}{881}(25\cos(80t) + 16\sin(80t) + Ce^{-125t}$$

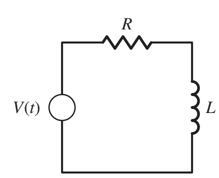


FIGURE 4 RL circuit.

39. Tank 1 in Figure 5 is filled with V_1 liters of water containing blue dye at an initial concentration of c_0 g/L. Water flows into the tank at a rate of R L/min, is mixed instantaneously with the dye solution, and flows out through the bottom at the same rate R. Let $c_1(t)$ be the dye concentration in the tank at time t.

(a) Explain why c_1 satisfies the differential equation $\dfrac{dc_1}{dt} = -\dfrac{R}{V_1}c_1$.

(b) Solve for $c_1(t)$ with $V_1 = 300$ L, $R = 50$, and $c_0 = 10$ g/L.

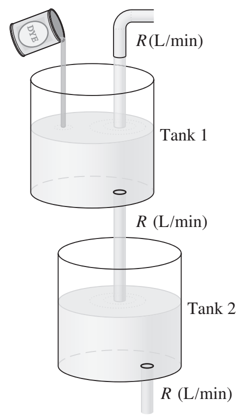


FIGURE 5

SOLUTION

(a) Let $g_1(t)$ be the number of grams of dye in the tank at time t. Then $g_1(t) = V_1c_1(t)$ and $g_1'(t) = V_1c_1'(t)$. Now,

$$g_1'(t) = \text{grams of dye in} - \text{grams of dye out} = 0 - \frac{g(t)}{V_1}\text{ g/L} \cdot R\text{ L/min} = -\frac{R}{V_1}g(t)$$

Substituting gives

$$V_1c_1'(t) = -\frac{R}{V_1}c_1(t)V_1 \quad \text{and simplifying yields} \quad c_1'(t) = -\frac{R}{V_1}c_1(t)$$

(b) In standard form, the equation is

$$c_1'(t) + \frac{R}{V_1}c_1(t) = 0$$

so that $A(t) = \dfrac{R}{V_1}$ and $B(t) = 0$. The integrating factor is $e^{(R/V_1)t}$; multiplying through gives

$$(e^{(R/V_1)t}c_1(t))' = 0 \quad \text{so, integrating,} \quad e^{(R/V_1)t}c_1(t) = C$$

and thus $c_1(t) = Ce^{-(R/V_1)t}$. With $R = 50$ and $V_1 = 300$ we have $c_1(t) = Ce^{-t/6}$; the initial condition $c_1(0) = c_0 = 10$ gives $C = 10$. Finally,

$$c_1(t) = 10e^{-t/6}$$

40. Continuing with the previous exercise, let Tank 2 be another tank filled with V_2 gal of water. Assume that the dye solution from Tank 1 empties into Tank 2 as in Figure 5, mixes instantaneously, and leaves Tank 2 at the same rate R. Let $c_2(t)$ be the dye concentration in Tank 2 at time t.

(a) Explain why c_2 satisfies the differential equation

$$\frac{dc_2}{dt} = \frac{R}{V_2}(c_1 - c_2)$$

(b) Use the solution to Exercise 39 to solve for $c_2(t)$ if $V_1 = 300$, $V_2 = 200$, $R = 50$, and $c_0 = 10$.

(c) Find the maximum concentration in Tank 2.

(d) GU Plot the solution.

SOLUTION

(a) Let $g_2(t)$ be the amount in grams of dye in Tank 2 at time t. At time t, the concentration of dye in Tank 1, and thus the concentration of dye coming into Tank 2, is $c_1(t)$. Thus

$$\begin{aligned} g_2'(t) &= \text{grams of dye in} - \text{grams of dye out} \\ &= c_1(t)\text{ g/L} \cdot R\text{ L/min} - c_2(t)\text{g/L} \cdot R\text{ L/min} = R(c_1(t) - c_2(t)) \end{aligned}$$

Since $g_2'(t) = V_2 c_2'(t)$, we get

$$c_2'(t) = \frac{R}{V_2}(c_1(t) - c_2(t))$$

(b) With $V_1 = 300$, $R = 50$, and $c_0 = 10$, part (a) tells us that

$$c_1(t) = 10e^{-t/6}$$

Since $V_2 = 200$, we have

$$c_2'(t) = \frac{1}{4}(10e^{-t/6} - c_2(t))$$

Putting this linear equation in standard form gives

$$c_2'(t) + \frac{1}{4}c_2(t) = \frac{5}{2}e^{-t/6}$$

The integrating factor is $e^{t/4}$; multiplying through gives

$$(e^{t/4}c_2(t))' = \frac{5}{2}e^{t/12}$$

Integrate to get

$$e^{t/4}c_2(t) = 30e^{t/12} + C \quad \text{so that} \quad c_2(t) = 30e^{-t/6} + Ce^{-t/4}$$

Since Tank 2 starts out filled entirely with water, we have $c_2(0) = 0$ so that $C = -30$ and

$$c_2(t) = 30(e^{-t/6} - e^{-t/4})$$

(c) The maximum concentration in Tank 2 occurs when $c_2'(t) = 0$.

$$c_2'(t) = 0 = -5e^{-t/6} + \frac{15}{2}e^{-t/4}$$

Solve this equation for t as follows:

$$\begin{aligned}
5e^{-t/6} &= \frac{15}{2}e^{-t/4} \\
2e^{-t/6} &= 3e^{-t/4} \\
-\frac{t}{6} + \ln 2 &= -\frac{t}{4} + \ln 3 \\
\frac{t}{12} &= \ln 3 - \ln 2 = \ln(3/2) \\
t &= 12\ln(3/2) \approx 4.866
\end{aligned}$$

When $t = 12\ln(3/2)$,

$$c_2(t) = 30(e^{-2\ln(3/2)} - e^{-3\ln(3/2)}) = 30\left(\frac{4}{9} - \frac{8}{27}\right) = \frac{40}{9}$$

41. Let a, b, r be constants. Show that

$$y = Ce^{-kt} + a + bk\left(\frac{k\sin rt - r\cos rt}{k^2 + r^2}\right)$$

is a general solution of

$$\frac{dy}{dt} = -k\Big(y - a - b\sin rt\Big)$$

SOLUTION This is a linear differential equation; in standard form, it is

$$\frac{dy}{dt} + ky = k(a + b\sin rt)$$

The integrating factor is then e^{kt}; multiplying through gives

$$(e^{kt}y)' = kae^{kt} + kbe^{kt}\sin rt \qquad (*)$$

The first term on the right-hand side has integral ae^{kt}. To integrate the second term, use integration by parts twice; this result in an equation of the form

$$\int kbe^{kt}\sin rt = F(t) + A\int kbe^{kt}\sin rt$$

for some function $F(t)$ and constant A. Solving for the integral gives

$$\int kbe^{kt}\sin rt = kbe^{kt}\frac{k\sin rt - r\cos rt}{k^2+r^2}$$

so that integrating equation (*) gives

$$e^{kt}y = ae^{kt} + kbe^{kt}\frac{k\sin rt - r\cos rt}{k^2+r^2} + C$$

Divide through by e^{kt} to get

$$y = a + bk\left(\frac{k\sin rt - r\cos rt}{k^2+r^2}\right) + Ce^{-kt}$$

42. Assume that the outside temperature varies as

$$T(t) = 15 + 5\sin(\pi t/12)$$

where $t = 0$ is 12 noon. A house is heated to 25°C at $t = 0$ and after that, its temperature $y(t)$ varies according to Newton's Law of Cooling (Figure 6):

$$\frac{dy}{dt} = -0.1\Big(y(t) - T(t)\Big)$$

Use Exercise 41 to solve for $y(t)$.

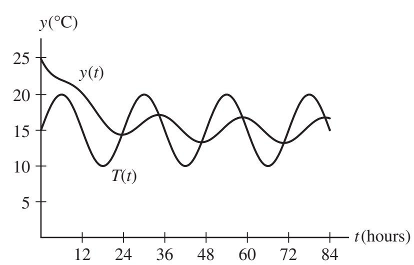


FIGURE 6 House temperature $y(t)$

SOLUTION The differential equation is

$$\frac{dy}{dt} = -0.1\Big(y(t) - 15 - 5\sin(\frac{\pi t}{12})\Big)$$

This differential equation is of the form considered in Exercise 41, with $a = 15$, $b = 5$, $r = \pi/12$, and $k = 0.1$. Thus the general solution is

$$y(t) = Ce^{-0.1t} + 15 + 0.5\left(\frac{0.1\sin(\pi t/12) - (\pi/12)\cos(\pi t/12)}{0.01 + \pi^2/144}\right)$$

Since $y(0) = 25$, we have

$$25 = C + 15 + 0.5\left(\frac{0 - \pi/12}{0.01 + \pi^2/144}\right) \approx C + 15 - 1.667$$

so that $C \approx 11.667$ and

$$y(t) = 11.667e^{-0.1t} + 15 + 0.5\left(\frac{0.1\sin(\pi t/12) - (\pi/12)\cos(\pi t/12)}{0.01 + \pi^2/144}\right)$$

Further Insights and Challenges

43. Let $\alpha(x)$ be an integrating factor for $y' + A(x)y = B(x)$. The differential equation $y' + A(x)y = 0$ is called the associated **homogeneous equation**.

(a) Show that $1/\alpha(x)$ is a solution of the associated homogeneous equation.

(b) Show that if $y = f(x)$ is a particular solution of $y' + A(x)y = B(x)$, then $f(x) + C/\alpha(x)$ is also a solution for any constant C.

SOLUTION

(a) Remember that $\alpha'(x) = A(x)\alpha(x)$. Now, let $y(x) = (\alpha(x))^{-1}$. Then

$$y' + A(x)y = -\frac{1}{(\alpha(x))^2}\alpha'(x) + \frac{A(x)}{\alpha(x)} = -\frac{1}{(\alpha(x))^2}A(x)\alpha(x) + \frac{A(x)}{\alpha(x)} = 0.$$

(b) Suppose $f(x)$ satisfies $f'(x) + A(x)f(x) = B(x)$. Now, let $y(x) = f(x) + C/\alpha(x)$, where C is an arbitrary constant. Then

$$\begin{aligned} y' + A(x)y &= f'(x) - \frac{C}{(\alpha(x))^2}\alpha'(x) + A(x)f(x) + \frac{CA(x)}{\alpha(x)} \\ &= \big(f'(x) + A(x)f(x)\big) + \frac{C}{\alpha(x)}\left(A(x) - \frac{\alpha'(x)}{\alpha(x)}\right) = B(x) + 0 = B(x). \end{aligned}$$

44. Use the Fundamental Theorem of Calculus and the Product Rule to verify directly that for any x_0, the function

$$f(x) = \alpha(x)^{-1}\int_{x_0}^{x} \alpha(t)B(t)\,dt$$

is a solution of the initial value problem

$$y' + A(x)y = B(x), \qquad y(x_0) = 0$$

where $\alpha(x)$ is an integrating factor [a solution to Eq. (3)].

SOLUTION Remember that $\alpha'(x) = A(x)\alpha(x)$. Now, let

$$y(x) = \frac{1}{\alpha(x)}\int_{x_0}^{x} \alpha(t)B(t)\,dt.$$

Then,

$$y(x_0) = \frac{1}{\alpha(x)}\int_{x_0}^{x_0} \alpha(t)B(t)\,dt = 0,$$

and

$$\begin{aligned} y' + A(x)y &= -\frac{\alpha'(x)}{(\alpha(x))^2}\int_{x_0}^{x} \alpha(t)B(t)\,dt + B(x) + \frac{A(x)}{\alpha(x)}\int_{x_0}^{x} \alpha(t)B(t)\,dt \\ &= B(x) + \left(-\frac{A(x)}{\alpha(x)} + \frac{A(x)}{\alpha(x)}\right)\int_{x_0}^{x} \alpha(t)B(t)\,dt = B(x). \end{aligned}$$

45. Transient Currents Suppose the circuit described by Eq. (10) is driven by a sinusoidal voltage source $V(t) = V\sin\omega t$ (where V and ω are constant).

(a) Show that

$$I(t) = \frac{V}{R^2 + L^2\omega^2}(R\sin\omega t - L\omega\cos\omega t) + Ce^{-(R/L)t}$$

(b) Let $Z = \sqrt{R^2 + L^2\omega^2}$. Choose θ so that $Z\cos\theta = R$ and $Z\sin\theta = L\omega$. Use the addition formula for the sine function to show that

$$I(t) = \frac{V}{Z}\sin(\omega t - \theta) + Ce^{-(R/L)t}$$

This shows that the current in the circuit varies sinusoidally apart from a DC term (called the **transient current** in electronics) that decreases exponentially.

SOLUTION
(a) With $V(t) = V \sin \omega t$, the equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t)$$

becomes

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}\sin \omega t.$$

This is a linear equation in standard form with $A(t) = \frac{R}{L}$ and $B(t) = \frac{V}{L}\sin \omega t$. By Theorem 1, the integrating factor is

$$\alpha(t) = \int e^{\int A(t)\,dt} = e^{(R/L)\,t}.$$

When multiplied by the integrating factor, the equation becomes

$$(e^{(R/L)\,t}I)' = \frac{V}{L}e^{(R/L)\,t}\sin \omega t.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side) now yields

$$(e^{(R/L)\,t}I) = \frac{V}{R^2 + L^2\omega^2}e^{(R/L)\,t}\,(R \sin \omega t - L\omega \cos \omega t) + C;$$

hence,

$$I(t) = \frac{V}{R^2 + L^2\omega^2}\,(R \sin \omega t - L\omega \cos \omega t) + Ce^{-(R/L)\,t}.$$

(b) Let $Z = \sqrt{R^2 + L^2\omega^2}$, and choose θ so that $Z \cos \theta = R$ and $Z \sin \theta = L\omega$. Then

$$\begin{aligned}\frac{V}{R^2 + L^2\omega^2}\,(R \sin \omega t - L\omega \cos \omega t) &= \frac{V}{Z^2}\,(Z \cos \theta \sin \omega t - Z \sin \theta \cos \omega t)\\ &= \frac{V}{Z}\,(\cos \theta \sin \omega t - \sin \theta \cos \omega t) = \frac{V}{Z}\sin(\omega t - \theta).\end{aligned}$$

Thus,

$$I(t) = \frac{V}{Z}\sin(\omega t - \theta) + Ce^{-(R/L)\,t}.$$

CHAPTER REVIEW EXERCISES

1. Which of the following differential equations are linear? Determine the order of each equation.

(a) $y' = y^5 - 3x^4 y$
(b) $y' = x^5 - 3x^4 y$
(c) $y = y''' - 3x\sqrt{y}$
(d) $\sin x \cdot y'' = y - 1$

SOLUTION

(a) y^5 is a nonlinear term involving the dependent variable, so this is not a linear equation; the highest order derivative that appears in the equation is a first derivative, so this is a first-order equation.

(b) This is linear equation; the highest order derivative that appears in the equation is a first derivative, so this is a first-order equation.

(c) $\sqrt{y}$ is a nonlinear term involving the dependent variable, so this is not a linear equation; the highest order derivative that appears in the equation is a third derivative, so this is a third-order equation.

(d) This is linear equation; the highest order derivative that appears in the equation is a second derivative, so this is a second-order equation.

2. Find a value of c such that $y = x - 2 + e^{cx}$ is a solution of $2y' + y = x$.

SOLUTION Let $y = x - 2 + e^{cx}$. Then

$$y' = 1 + ce^{cx},$$

and

$$2y' + y = 2\left(1 + ce^{cx}\right) + \left(x - 2 + e^{cx}\right) = 2 + 2ce^{cx} + x - 2 + e^{cx} = (2c + 1)e^{cx} + x.$$

For this to equal x, we must have $2c + 1 = 0$, or $c = -\frac{1}{2}$ (remember that e^{cx} is never zero).

In Exercises 3–6, solve using separation of variables.

3. $\dfrac{dy}{dt} = t^2 y^{-3}$

SOLUTION Rewrite the equation as

$$y^3\,dy = t^2\,dt.$$

Upon integrating both sides of this equation, we obtain:

$$\int y^3\,dy = \int t^2\,dt$$
$$\frac{y^4}{4} = \frac{t^3}{3} + C.$$

Thus,

$$y = \pm\left(\frac{4}{3}t^3 + C\right)^{1/4},$$

where C is an arbitrary constant.

4. $xyy' = 1 - x^2$

SOLUTION Rewrite the equation

$$xy\frac{dy}{dx} = 1 - x^2 \qquad \text{as} \qquad y\,dy = \left(\frac{1}{x} - x\right)dx.$$

Upon integrating both sides of this equation, we obtain

$$\int y\,dy = \int \left(\frac{1}{x} - x\right)dx$$
$$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C.$$

Thus,

$$y = \pm\sqrt{\ln x^2 + A - x^2},$$

where $A = 2C$ is an arbitrary constant.

5. $x\dfrac{dy}{dx} - y = 1$

SOLUTION Rewrite the equation as

$$\frac{dy}{1+y} = \frac{dx}{x}.$$

upon integrating both sides of this equation, we obtain

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$
$$\ln|1+y| = \ln|x| + C.$$

Thus,

$$y = -1 + Ax,$$

where $A = \pm e^C$ is an arbitrary constant.

6. $y' = \dfrac{xy^2}{x^2+1}$

SOLUTION Rewrite

$$\frac{dy}{dx} = \frac{xy^2}{x^2+1} \qquad \text{as} \qquad \frac{dy}{y^2} = \frac{x}{x^2+1}\,dx.$$

Upon integrating both sides of this equation, we obtain

$$\int \frac{dy}{y^2} = \int \frac{x}{x^2+1}\,dx$$
$$-\frac{1}{y} = \frac{1}{2}\ln\left(x^2+1\right) + C.$$

Thus,

$$y = -\frac{1}{\frac{1}{2}\ln\left(x^2+1\right)+C},$$

where C is an arbitrary constant.

In Exercises 7–10, solve the initial value problem using separation of variables.

7. $y' = \cos^2 x, \quad y(0) = \frac{\pi}{4}$

SOLUTION First, we find the general solution of the differential equation. Because the variables are already separated, we integrate both sides to obtain

$$y = \int \cos^2 x\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

The initial condition $y(0) = \frac{\pi}{4}$ allows us to determine $C = \frac{\pi}{4}$. Thus, the solution is:

$$y(x) = \frac{x}{2} + \frac{\sin 2x}{4} + \frac{\pi}{4}.$$

8. $y' = \cos^2 y, \quad y(0) = \frac{\pi}{4}$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = \cos^2 y \qquad \text{as} \qquad \frac{dy}{\cos^2 y} = dx.$$

Upon integrating both sides of this equation, we find

$$\tan y = x + C;$$

thus,

$$y = \tan^{-1}(x+C).$$

The initial condition $y(0) = \frac{\pi}{4}$ allows us to determine the value of C:

$$\frac{\pi}{4} = \tan^{-1} C \qquad \text{so} \qquad C = 1.$$

Hence, the solution is $y = \tan^{-1}(x+1)$.

9. $y' = xy^2, \quad y(1) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = xy^2 \qquad \text{as} \qquad \frac{dy}{y^2} = x\,dx.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dy}{y^2} = \int x\,dx$$
$$-\frac{1}{y} = \frac{1}{2}x^2 + C.$$

Thus,

$$y = -\frac{1}{\frac{1}{2}x^2 + C}.$$

The initial condition $y(1) = 2$ allows us to determine the value of C:

$$2 = -\frac{1}{\frac{1}{2}\cdot 1^2 + C} = -\frac{2}{1+2C}$$
$$1 + 2C = -1$$
$$C = -1$$

Hence, the solution to the initial value problem is

$$y = -\frac{1}{\frac{1}{2}x^2 - 1} = -\frac{2}{x^2 - 2}.$$

10. $xyy' = 1, \quad y(3) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite

$$xy\frac{dy}{dx} = 1 \qquad \text{as} \qquad y\,dy = \frac{dx}{x}.$$

Next we integrate both sides of the equation to obtain

$$\int y\,dy = \int \frac{dx}{x}$$
$$\frac{1}{2}y^2 = \ln|x| + C.$$

Thus,

$$y = \pm\sqrt{2(\ln|x| + C)}.$$

To satisfy the initial condition $y(3) = 2$ we must choose the positive square root; moreover,

$$2 = \sqrt{2(\ln 3 + C)} \qquad \text{so} \qquad C = 2 - \ln 3.$$

Hence, the solution to the initial value problem is

$$y = \sqrt{2(\ln|x| + 2 - \ln 3)} = \sqrt{\ln\left(\frac{x^2}{9}\right) + 4}.$$

11. Figure 1 shows the slope field for $\dot{y} = \sin y + ty$. Sketch the graphs of the solutions with the initial conditions $y(0) = 1$, $y(0) = 0$, and $y(0) = -1$.

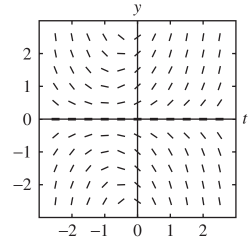
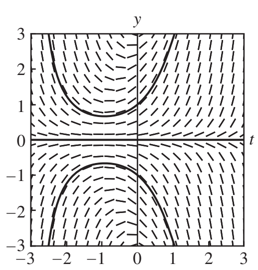


FIGURE 1

SOLUTION



12. Which of the equations (i)–(iii) corresponds to the slope field in Figure 2?

(i) $\dot{y} = 1 - y^2$
(ii) $\dot{y} = 1 + y^2$
(iii) $\dot{y} = y^2$

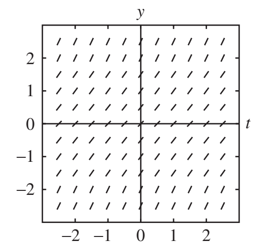
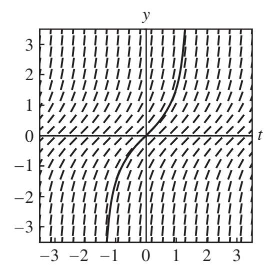


FIGURE 2

SOLUTION From the figure we see that the the slope is positive even for $y > 1$, thus, the slope field does not correspond to the equation $\dot{y} = 1 - y^2$. Moreover, the slope at $y = 0$ is positive, so the slope field also does not correspond to the equation $\dot{y} = y^2$. The slope field must therefore correspond to **(ii)**: $\dot{y} = 1 + y^2$.

13. Let $y(t)$ be the solution to the differential equation with slope field as shown in Figure 2, satisfying $y(0) = 0$. Sketch the graph of $y(t)$. Then use your answer to Exercise 12 to solve for $y(t)$.

SOLUTION As explained in the previous exercise, the slope field in Figure 2 corresponds to the equation $\dot{y} = 1 + y^2$. The graph of the solution satisfying $y(0) = 0$ is:



To solve the initial value problem $\dot{y} = 1 + y^2$, $y(0) = 0$, we first find the general solution of the differential equation. Separating variables yields:

$$\frac{dy}{1 + y^2} = dt.$$

Upon integrating both sides of this equation, we find

$$\tan^{-1} y = t + C \qquad \text{or} \qquad y = \tan(t + C).$$

The initial condition gives $C = 0$, so the solution is $y = \tan x$.

14. Let $y(t)$ be the solution of $4\dot{y} = y^2 + t$ satisfying $y(2) = 1$. Carry out Euler's Method with time step $h = 0.05$ for $n = 6$ steps.

SOLUTION Rewrite the differential equation as $\dot{y} = \frac{1}{4}(y^2 + t)$ to identify $F(t, y) = \frac{1}{4}\left(y^2 + t\right)$. With $t_0 = 2$, $y_0 = 1$, and $h = 0.05$, we calculate

$$\begin{aligned}
y_1 &= y_0 + hF(t_0, y_0) = 1.0375 \\
y_2 &= y_1 + hF(t_1, y_1) = 1.076580 \\
y_3 &= y_2 + hF(t_2, y_2) = 1.117318 \\
y_4 &= y_3 + hF(t_3, y_3) = 1.159798 \\
y_5 &= y_4 + hF(t_4, y_4) = 1.204112 \\
y_6 &= y_5 + hF(t_5, y_5) = 1.250361
\end{aligned}$$

15. Let $y(t)$ be the solution of $(x^3 + 1)\dot{y} = y$ satisfying $y(0) = 1$. Compute approximations to $y(0.1)$, $y(0.2)$, and $y(0.3)$ using Euler's Method with time step $h = 0.1$.

SOLUTION Rewriting the equation as $\dot{y} = \frac{y}{x^3+1}$ we have $F(x, y) = \frac{y}{x^3+1}$. Using Euler's Method with $x_0 = 0$, $y_0 = 1$ and $h = 0.1$, we calculate

$$\begin{aligned}
y(0.1) &\approx y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1 \cdot \frac{1}{0^3 + 1} = 1.1 \\
y(0.2) &\approx y_2 = y_1 + hF(x_1, y_1) = 1.209890 \\
y(0.3) &\approx y_3 = y_2 + hF(x_2, y_2) = 1.329919
\end{aligned}$$

In Exercises 16–19, solve using the method of integrating factors.

16. $\dfrac{dy}{dt} = y + t^2, \quad y(0) = 4$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - y = t^2,$$

which is in standard form with $A(t) = -1$ and $B(t) = t^2$. The integrating factor is

$$\alpha(t) = e^{\int -1\,dt} = e^{-t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-t}y)' = t^2 e^{-t}.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side of the equation) now yields

$$e^{-t}y = -e^{-t}(t^2 + 2t + 2) + C;$$

hence,

$$y(t) = Ce^t - t^2 - 2t - 2.$$

The initial condition $y(0) = 4$ allows us to determine the value of C:

$$4 = -2 + C \qquad \text{so} \qquad C = 6.$$

The solution to the initial value problem is then

$$y = 6e^t - t^2 - 2t - 2.$$

17. $\dfrac{dy}{dx} = \dfrac{y}{x} + x, \quad y(1) = 3$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - \frac{1}{x}y = x,$$

which is in standard form with $A(x) = -\frac{1}{x}$ and $B(x) = x$. The integrating factor is

$$\alpha(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\left(\frac{1}{x}y\right)' = 1.$$

Integration on both sides now yields

$$\frac{1}{x}y = x + C;$$

hence,

$$y(x) = x^2 + Cx.$$

The initial condition $y(1) = 3$ allows us to determine the value of C:

$$3 = 1 + C \qquad \text{so} \qquad C = 2.$$

The solution to the initial value problem is then

$$y = x^2 + 2x.$$

18. $\dfrac{dy}{dt} = y - 3t, \quad y(-1) = 2$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' - y = -3t,$$

which is in standard form with $A(t) = -1$ and $B(t) = -3t$. The integrating factor is

$$\alpha(t) = e^{\int A(t)\,dt} = e^{\int -dt} = e^{-t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-t}y)' = -3te^{-t}.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side of the equation) now yields

$$e^{-t}y = (3t+3)e^{-t} + C;$$

hence,

$$y(t) = 3t + 3 + Ce^{t}.$$

The initial condition $y(-1) = 2$ allows us to determine the value of C;

$$2 = Ce^{-1} + 3(-1) + 3 \qquad \text{so} \qquad C = 2e.$$

The solution to the initial value problem is then

$$y = 2e \cdot e^{t} + 3t + 3 = 2e^{t+1} + 3t + 3.$$

19. $y' + 2y = 1 + e^{-x}, \quad y(0) = -4$

SOLUTION The equation is already in standard form with $A(x) = 2$ and $B(x) = 1 + e^{-x}$. The integrating factor is

$$\alpha(x) = e^{\int 2\,dx} = e^{2x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{2x}y)' = e^{2x} + e^{x}.$$

Integration on both sides now yields

$$e^{2x}y = \frac{1}{2}e^{2x} + e^{x} + C;$$

hence,

$$y(x) = \frac{1}{2} + e^{-x} + Ce^{-2x}.$$

The initial condition $y(0) = -4$ allows us to determine the value of C:

$$-4 = \frac{1}{2} + 1 + C \qquad \text{so} \qquad C = -\frac{11}{2}.$$

The solution to the initial value problem is then

$$y(x) = \frac{1}{2} + e^{-x} - \frac{11}{2}e^{-2x}.$$

In Exercises 20–27, solve using the appropriate method.

20. $x^2y' = x^2 + 1, \quad y(1) = 10$

SOLUTION First, we find the general solution of the differential equation. Rewrite the equation as

$$y' = 1 + \frac{1}{x^2}.$$

Because the variables have already been separated, we integrate both sides to obtain

$$y = \int \left(1 + \frac{1}{x^2}\right) dx = x - \frac{1}{x} + C.$$

The initial condition $y(1) = 10$ allows us to determine the value of C:

$$10 = 1 - 1 + C \qquad \text{so} \qquad C = 10.$$

The solution to the initial value problem is then

$$y = x - \frac{1}{x} + 10.$$

21. $y' + (\tan x)y = \cos^2 x, \quad y(\pi) = 2$

SOLUTION First, we find the general solution of the differential equation. As this is a first order linear equation with $A(x) = \tan x$ and $B(x) = \cos^2 x$, we compute the integrating factor

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\int \tan x\,dx} = e^{-\ln\cos x} = \frac{1}{\cos x}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$\left(\frac{1}{\cos x}y\right)' = \cos x.$$

Integration on both sides now yields

$$\frac{1}{\cos x}y = \sin x + C;$$

hence,

$$y(x) = \sin x\cos x + C\cos x = \frac{1}{2}\sin 2x + C\cos x.$$

The initial condition $y(\pi) = 2$ allows us to determine the value of C:

$$2 = 0 + C(-1) \qquad \text{so} \qquad C = -2.$$

The solution to the initial value problem is then

$$y = \frac{1}{2}\sin 2x - 2\cos x.$$

22. $xy' = 2y + x - 1, \quad y\left(\frac{3}{2}\right) = 9$

SOLUTION First, we find the general solution of the differential equation. This is a linear equation which can be rewritten as

$$y' - \frac{2}{x}y = 1 - \frac{1}{x}.$$

Thus, $A(x) = -\frac{2}{x}$, $B(x) = 1 - \frac{1}{x}$ and the integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = \frac{1}{x^2}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$\left(\frac{1}{x^2}y\right)' = \frac{1}{x^2} - \frac{1}{x^3}.$$

Integration on both sides now yields

$$\frac{1}{x^2}y = -\frac{1}{x} + \frac{1}{2x^2} + C;$$

hence,

$$y(x) = -x + \frac{1}{2} + Cx^2.$$

The initial condition $y\left(\frac{3}{2}\right) = 9$ allows us to determine the value of C:

$$9 = -\frac{3}{2} + \frac{1}{2} + \frac{9}{4}C \qquad \text{so} \qquad C = \frac{40}{9}.$$

The solution to the initial value problem is then

$$y = \frac{40}{9}x^2 - x + \frac{1}{2}.$$

23. $(y-1)y' = t, \quad y(1) = -3$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$(y-1)\,dy = t\,dt.$$

Upon integrating both sides of this equation, we find

$$\int (y-1)\,dy = \int t\,dt$$
$$\frac{y^2}{2} - y = \frac{1}{2}t^2 + C$$
$$y^2 - 2y + 1 = t^2 + C$$
$$(y-1)^2 = t^2 + C$$
$$y(t) = \pm\sqrt{t^2+C} + 1$$

To satisfy the initial condition $y(1) = -3$ we must choose the negative square root; moreover,

$$-3 = -\sqrt{1+C} + 1 \qquad \text{so} \qquad C = 15.$$

The solution to the initial value problem is then

$$y(t) = -\sqrt{t^2+15} + 1$$

24. $\left(\sqrt{y}+1\right)y' = yte^{t^2}, \quad y(0) = 1$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$\left(\frac{1}{\sqrt{y}} + \frac{1}{y}\right) dy = te^{t^2}\,dt.$$

Upon integrating both sides of this equation, we find

$$\int \left(\frac{1}{\sqrt{y}} + \frac{1}{y}\right) dy = \int te^{t^2}\,dt$$
$$2\sqrt{y} + \ln y = \frac{1}{2}e^{t^2} + C.$$

Note that we cannot solve explicitly for $y(t)$. The initial condition $y(0) = 1$ still allows us to determine the value of C:

$$2(1) + \ln 1 = \frac{1}{2} + C \qquad \text{so} \qquad C = \frac{3}{2}.$$

Hence, the general solution is given implicitly by the equation

$$2\sqrt{y} + \ln y = \frac{1}{2}e^{x^2} + \frac{3}{2}.$$

25. $\dfrac{dw}{dx} = k\dfrac{1+w^2}{x}, \quad w(1) = 1$

SOLUTION First, we find the general solution of the differential equation. This is a separable equation that we rewrite as

$$\frac{dw}{1+w^2} = \frac{k}{x}\,dx.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dw}{1+w^2} = \int \frac{k}{x}\,dx$$
$$\tan^{-1} w = k\ln x + C$$
$$w(x) = \tan(k\ln x + C).$$

Because the initial condition is specified at $x = 1$, we are interested in the solution for $x > 0$; we can therefore omit the absolute value within the natural logarithm function. The initial condition $w(1) = 1$ allows us to determine the value of C:

$$1 = \tan(k\ln 1 + C) \qquad \text{so} \qquad C = \tan^{-1} 1 = \frac{\pi}{4}.$$

The solution to the initial value problem is then

$$w = \tan\left(k \ln x + \frac{\pi}{4}\right).$$

26. $y' + \dfrac{3y-1}{t} = t + 2$

SOLUTION We rewrite this first order linear equation in standard form:

$$y' + \frac{3}{t}y = t + 2 + \frac{1}{t}.$$

Thus, $A(t) = \frac{3}{t}$, $B(t) = t + 2 + \frac{1}{t}$, and the integrating factor is

$$\alpha(t) = e^{\int A(t)\,dt} = e^{3\ln t} = t^3.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(t^3 y)' = t^4 + 2t^3 + t^2.$$

Integration on both sides now yields

$$t^3 y = \frac{1}{5}t^5 + \frac{1}{2}t^4 + \frac{1}{3}t^3 + C;$$

hence,

$$y(t) = \frac{1}{5}t^2 + \frac{1}{2}t + \frac{1}{3} + \frac{C}{t^3}.$$

27. $y' + \dfrac{y}{x} = \sin x$

SOLUTION This is a first order linear equation with $A(x) = \frac{1}{x}$ and $B(x) = \sin x$. The integrating factor is

$$\alpha(x) = e^{\int A(x)\,dx} = e^{\ln x} = x.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(xy)' = x \sin x.$$

Integration on both sides (integration by parts is needed for the integral on the right-hand side) now yields

$$xy = -x\cos x + \sin x + C;$$

hence,

$$y(x) = -\cos x + \frac{\sin x}{x} + \frac{C}{x}.$$

28. Find the solutions to $y' = 4(y - 12)$ satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

SOLUTION The general solution of the differential equation $y' = 4(y - 12)$ is

$$y(t) = 12 + Ce^{4t},$$

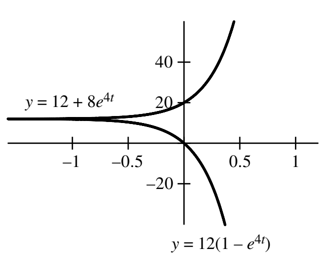
for some constant C. If $y(0) = 20$, then

$$20 = 12 + Ce^0 \quad \text{and} \quad C = 8.$$

Thus, $y(t) = 12 + 8e^{4t}$. If $y(0) = 0$, then

$$0 = 12 + Ce^0 \quad \text{and} \quad C = -12;$$

hence, $y(t) = 12(1 - e^{4t})$. The graphs of the two solutions are shown below.



29. Find the solutions to $y' = -2y + 8$ satisfying $y(0) = 3$ and $y(0) = 4$, and sketch their graphs.

SOLUTION First, rewrite the differential equation as $y' = -2(y - 4)$; from here we see that the general solution is

$$y(t) = 4 + Ce^{-2t},$$

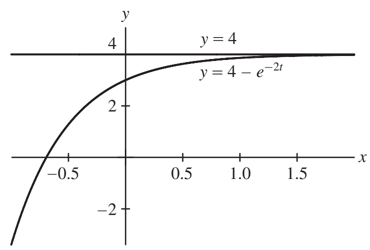
for some constant C. If $y(0) = 3$, then

$$3 = 4 + Ce^0 \quad \text{and} \quad C = -1.$$

Thus, $y(t) = 4 - e^{-2t}$. If $y(0) = 4$, then

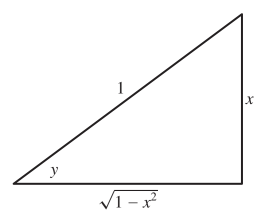
$$4 = 4 + Ce^0 \quad \text{and} \quad C = 0;$$

hence, $y(t) = 4$. The graphs of the two solutions are shown below.



30. Show that $y = \sin^{-1} x$ satisfies the differential equation $y' = \sec y$ with initial condition $y(0) = 0$.

SOLUTION Let $y = \sin^{-1} x$. Then $x = \sin y$ and we construct the right triangle shown below.



Thus,

$$\sec y = \frac{1}{\sqrt{1 - x^2}} = \frac{d}{dx} \sin^{-1} x = y'.$$

Moreover, $y(0) = \sin^{-1} 0 = 0$. Consequently, $y = \sin^{-1} x$ satisfies the differential equation $y' = \sec y$ with initial condition $y(0) = 0$.

31. What is the limit $\lim\limits_{t\to\infty} y(t)$ if $y(t)$ is a solution of:

(a) $\dfrac{dy}{dt} = -4(y - 12)$?

(b) $\dfrac{dy}{dt} = 4(y - 12)$?

(c) $\dfrac{dy}{dt} = -4y - 12$?

SOLUTION

(a) The general solution of $\dfrac{dy}{dt} = -4(y - 12)$ is $y(t) = 12 + Ce^{-4t}$, where C is an arbitrary constant. Regardless of the value of C,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (12 + Ce^{-4t}) = 12.$$

(b) The general solution of $\dfrac{dy}{dt} = 4(y - 12)$ is $y(t) = 12 + Ce^{4t}$, where C is an arbitrary constant. Here, the limit depends on the value of C. Specifically,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (12 + Ce^{4t}) = \begin{cases} \infty, & C > 0 \\ 12, & C = 0 \\ -\infty, & C < 0 \end{cases}$$

(c) The general solution of $\frac{dy}{dt} = -4y - 12 = -4(y+3)$ is $y(t) = -3 + Ce^{-4t}$, where C is an arbitrary constant. Regardless of the value of C,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (-3 + Ce^{-4t}) = -3.$$

In Exercises 32–35, let $P(t)$ denote the balance at time t (years) of an annuity that earns 5% interest continuously compounded and pays out \$20,000*/year continuously.*

32. Find the differential equation satisfied by $P(t)$.

SOLUTION Since money is withdrawn continuously at a rate of \$20,000 a year and the growth due to interest is $0.05P$, the rate of change of the balance is

$$P'(t) = 0.05P - 20{,}000.$$

Thus, the differential equation satisfied by $P(t)$ is

$$P'(t) = 0.05(P - 400{,}000).$$

33. Determine $P(5)$ if $P(0) = \$200{,}000$.

SOLUTION In the previous exercise we concluded that $P(t)$ satisfies the equation $P' = 0.05(P - 400{,}000)$. The general solution of this differential equation is

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

Given $P(0) = 200{,}000$, it follows that

$$200{,}000 = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = -200{,}000.$$

Thus,

$$P(t) = 400{,}000 - 200{,}000e^{0.05t},$$

and

$$P(5) = 400{,}000 - 200{,}000e^{0.05(5)} \approx \$143{,}194.90.$$

34. When does the annuity run out of money if $P(0) = \$300{,}000$?

SOLUTION We found that

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

If $P(0) = 300{,}000$, then

$$300{,}000 = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = -100{,}000.$$

Thus,

$$P(t) = 400{,}000 - 100{,}000e^{0.05t}.$$

The annuity runs out of money when $P(t) = 0$; that is, when

$$400{,}000 - 100{,}000e^{0.05t} = 0.$$

Solving for t yields

$$t = \frac{1}{0.05}\ln 4 = 20 \ln 4 \approx 27.73.$$

The money runs out after roughly 27.73 years.

35. What is the minimum initial balance that will allow the annuity to make payments indefinitely?

SOLUTION In Exercise 33, we found that the balance at time t is

$$P(t) = 400{,}000 + Ce^{0.05t}.$$

If initial balance is P_0 then

$$P_0 = P(0) = 400{,}000 + Ce^{0.05\cdot 0} = 400{,}000 + C$$

or

$$C = P_0 - 400{,}000.$$

Thus,

$$P(t) = 400{,}000 + (P_0 - 400{,}000)\,e^{0.05t}.$$

If $P_0 \geq 400{,}000$, then $P(t)$ is always positive. Therefore, the minimum initial balance that allows the annuity to make payments indefinitely is $P_0 = \$400{,}000$.

36. State whether the differential equation can be solved using separation of variables, the method of integrating factors, both, or neither.

(a) $y' = y + x^2$

(b) $xy' = y + 1$

(c) $y' = y^2 + x^2$

(d) $xy' = y^2$

SOLUTION

(a) The equation $y' = y + x^2$ is a first order linear equation; hence, it can be solved by the method of integration factors. However, it cannot be written in the form $y' = f(x)g(y)$; therefore, separation of variables cannot be used.

(b) The equation $xy' = y + 1$ is a first order linear equation; hence, it can be solved using the method of integration factors. We can rewrite this equation as $y' = \frac{1}{x}(y+1)$; therefore, it can also be solved by separating the variables.

(c) The equation $y' = y^2 + x^2$ cannot be written in the form $y' = f(x)g(y)$; hence, separation of variables cannot be used. This equation is also not linear; hence, the method of integrating factors cannot be used.

(d) The equation $xy' = y^2$ can be rewritten as $y' = \frac{1}{x}y^2$; therefore, it can be solved by separating the variables. Since it is not a linear equation, the method of integrating factors cannot be used.

37. Let A and B be constants. Prove that if $A > 0$, then all solutions of $\frac{dy}{dt} + Ay = B$ approach the same limit as $t \to \infty$.

SOLUTION This is a linear first-order equation in standard form with integrating factor

$$\alpha(t) = e^{\int A\,dt} = e^{At}.$$

When multiplied by the integrating factor, the original differential equation becomes

$$(e^{At}y)' = Be^{At}.$$

Integration on both sides now yields

$$e^{At}y = \frac{B}{A}e^{At} + C;$$

hence,

$$y(t) = \frac{B}{A} + Ce^{-At}.$$

Because $A > 0$,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{B}{A} + Ce^{-At}\right) = \frac{B}{A}.$$

We conclude that if $A > 0$, all solutions approach the limit $\frac{B}{A}$ as $t \to \infty$.

38. At time $t = 0$, a tank of height 5 m in the shape of an inverted pyramid whose cross section at the top is a square of side 2 m is filled with water. Water flows through a hole at the bottom of area 0.002 m^2. Use Torricelli's Law to determine the time required for the tank to empty.

SOLUTION $y(t)$, the height of the water at time t, obeys the differential equation:

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)}$$

where $v(y)$ is the velocity of the water flowing through the hole when the height of the water is y, B is the area of the hole, and $A(y)$ is the cross-sectional area of the surface of the water when it is at height y. By Torricelli's Law, $v(y) = -\sqrt{19.6}\sqrt{y} = -14\sqrt{y}/\sqrt{10}$ m/s. The area of the hole is $B = 0.002$. To determine $A(y)$, note that the ratio of the length of a side of the square forming the surface of the water to the height of the water is $2/5$ (using similar triangles). Thus when the water is at height y, the area is $A(y) = \left(\frac{2}{5}y\right)^2 = \frac{4y^2}{25}$. Thus

$$\frac{dy}{dt} = \frac{-0.002 \cdot 14\sqrt{y} \cdot 25}{4y^2\sqrt{10}} = \frac{-0.175}{\sqrt{10}}y^{-3/2}$$

Separating variables gives

$$y^{3/2}\,dy = \frac{-0.175}{\sqrt{10}}\,dt$$

Integrating both sides gives

$$\frac{2}{5}y^{5/2} = \frac{-0.175}{\sqrt{10}}t + C \quad \text{so that} \quad y = \left(\frac{-0.4375}{\sqrt{10}}t + \frac{5}{2}C\right)^{2/5}$$

At $t = 0$, $y(t) = 5$, so that

$$5 = \left(\frac{5}{2}C\right)^{2/5} \quad \text{and} \quad C \approx 22.36$$

so that

$$y(t) \approx (-0.138t + 55.9)^{2/5}$$

The tank is empty when $y(t) = 0$, so when $t = 55.9/0.138 \approx 405.07$. The tank is empty after approximately 405 seconds, or 6 minutes 45 seconds.

39. The trough in Figure 3 (dimensions in centimeters) is filled with water. At time $t = 0$ (in seconds), water begins leaking through a hole at the bottom of area 4 cm^2. Let $y(t)$ be the water height at time t. Find a differential equation for $y(t)$ and solve it to determine when the water level decreases to 60 cm.

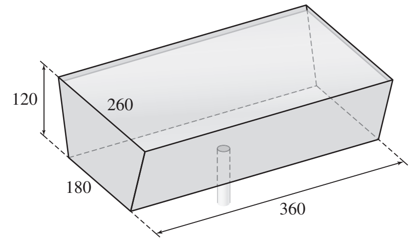


FIGURE 3

SOLUTION $y(t)$ obeys the differential equation:

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)},$$

where $v(y)$ denotes the velocity of the water flowing through the hole when the trough is filled to height y, B denotes the area of the hole and $A(y)$ denotes the area of the horizontal cross section of the trough at height y. Since measurements are all in centimeters, we will work in centimeters. We have

$$g = 9.8 \text{ m/s}^2 = 980 \text{ cm/s}^2$$

By Torricelli's Law, $v(y) = -\sqrt{2 \cdot 980}\sqrt{y} = -14\sqrt{10}\sqrt{y}$ m/s. The area of the hole is $B = 4\ \text{cm}^2$. The horizontal cross section of the trough at height y is a rectangle of length 360 and width $w(y)$. As $w(y)$ varies linearly from 180 when $y = 0$ to 260 when $y = 120$, it follows that

$$w(y) = 180 + \frac{80y}{120} = 180 + \frac{2}{3}y$$

so that the area of the horizontal cross-section at height y is

$$A(y) = 360w(y) = 64800 + 240y = 240(y + 270)$$

The differential equation for $y(t)$ then becomes

$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)} = \frac{-4 \cdot 14\sqrt{10}\sqrt{y}}{240(y+270)} = \frac{-7\sqrt{10}}{30} \cdot \frac{\sqrt{y}}{y+270}$$

This equation is separable, so

$$\begin{aligned}
\frac{y+270}{\sqrt{y}}\,dy &= \frac{-7\sqrt{10}}{30}\,dt \\
(y^{1/2} + 270y^{-1/2})\,dy &= \frac{-7\sqrt{10}}{30}\,dt \\
\int (y^{1/2} + 270y^{-1/2})\,dy &= \frac{-7\sqrt{10}}{30}\int 1\,dt \\
\frac{2}{3}y^{3/2} + 540y^{1/2} &= -\frac{7\sqrt{10}}{30}t + C \\
y^{3/2} + 810y^{1/2} &= -\frac{7\sqrt{10}}{20}t + C
\end{aligned}$$

The initial condition $y(0) = 120$ allows us to determine the value of C:

$$120^{3/2} + 810 \cdot 120^{1/2} = 0 + C \quad \text{so} \quad C = 930\sqrt{120} = 1860\sqrt{30}$$

Thus the height of the water is given implicitly by the equation

$$y^{3/2} + 810y^{1/2} = -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30}$$

We want to find t such that $y(t) = 60$:

$$\begin{aligned}
60^{3/2} + 810 \cdot 60^{1/2} &= -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30} \\
1740\sqrt{15} &= -\frac{7\sqrt{10}}{20}t + 1860\sqrt{30} \\
t &= \frac{120}{7}\sqrt{10}(31\sqrt{30} - 29\sqrt{15}) \approx 3115.88 \text{ s}
\end{aligned}$$

The height of the water in the tank is 60 cm after approximately 3116 seconds, or 51 minutes 56 seconds.

40. Find the solution of the logistic equation $\dot{y} = 0.4y(4 - y)$ satisfying $y(0) = 8$.

SOLUTION We can write the given equation as

$$\dot{y} = 1.6y\left(1 - \frac{y}{4}\right).$$

This is a logistic equation with $k = 1.6$ and $A = 4$. Therefore,

$$y(t) = \frac{A}{1 - e^{-kt}/C} = \frac{4}{1 - e^{-1.6t}/C}.$$

The initial condition $y(0) = 8$ allows us to determine the value of C:

$$8 = \frac{4}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = \frac{1}{2}; \quad \text{so} \quad C = 2.$$

Thus,

$$y(t) = \frac{4}{1 - e^{-1.6t}/2} = \frac{8}{2 - e^{-1.6t}}.$$

41. Let $y(t)$ be the solution of $\dot{y} = 0.3y(2 - y)$ with $y(0) = 1$. Determine $\lim_{t\to\infty} y(t)$ without solving for y explicitly.

SOLUTION We write the given equation in the form

$$\dot{y} = 0.6y\left(1 - \frac{y}{2}\right).$$

This is a logistic equation with $A = 2$ and $k = 0.6$. Because the initial condition $y(0) = y_0 = 1$ satisfies $0 < y_0 < A$, the solution is increasing and approaches A as $t \to \infty$. That is, $\lim_{t\to\infty} y(t) = 2$.

42. Suppose that $y' = ky(1 - y/8)$ has a solution satisfying $y(0) = 12$ and $y(10) = 24$. Find k.

SOLUTION The given differential equation is a logistic equation with $A = 8$. Thus,

$$y(t) = \frac{8}{1 - e^{-kt}/C}.$$

The initial condition $y(0) = 12$ allows us to determine the value of C:

$$12 = \frac{8}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = \frac{2}{3}; \quad \text{so} \quad C = 3.$$

Hence,

$$y(t) = \frac{8}{1 - e^{-kt}/3} = \frac{24}{3 - e^{-kt}}.$$

Now, the condition $y(10) = 24$ allows us to determine the value of k:

$$\begin{aligned} 24 &= \frac{24}{3 - e^{-10k}} \\ 3 - e^{-10k} &= 1 \\ k &= -\frac{\ln 2}{10} \approx -0.0693. \end{aligned}$$

43. A lake has a carrying capacity of 1000 fish. Assume that the fish population grows logistically with growth constant $k = 0.2$ day^{-1}. How many days will it take for the population to reach 900 fish if the initial population is 20 fish?

SOLUTION Let $y(t)$ represent the fish population. Because the population grows logistically with $k = 0.2$ and $A = 1000$,

$$y(t) = \frac{1000}{1 - e^{-0.2t}/C}.$$

The initial condition $y(0) = 20$ allows us to determine the value of C:

$$20 = \frac{1000}{1 - \frac{1}{C}}; \quad 1 - \frac{1}{C} = 50; \quad \text{so} \quad C = -\frac{1}{49}.$$

Hence,

$$y(t) = \frac{1000}{1 + 49e^{-0.2t}}.$$

The population will reach 900 fish when

$$\frac{1000}{1 + 49e^{-0.2t}} = 900.$$

Solving for t, we find

$$t = 5 \ln 441 \approx 30.44 \text{ days}.$$

44. A rabbit population on an island increases exponentially with growth rate $k = 0.12$ months^{-1}. When the population reaches 300 rabbits (say, at time $t = 0$), wolves begin eating the rabbits at a rate of r rabbits per month.

(a) Find a differential equation satisfied by the rabbit population $P(t)$.

(b) How large can r be without the rabbit population becoming extinct?

SOLUTION

(a) The rabbit population $P(t)$ obeys the differential equation

$$\frac{dP}{dt} = 0.12P - r,$$

where the term $0.12P$ accounts for the exponential growth of the population and the term $-r$ accounts for the rate of decline in the rabbit population due to their being food for wolves.

(b) Rewrite the linear differential equation from part (a) as

$$\frac{dP}{dt} - 0.12P = -r,$$

which is in standard form with $A = -0.12$ and $B = -r$. The integrating factor is

$$\alpha(t) = e^{\int A\,dt} = e^{\int -0.12\,dt} = e^{-0.12t}.$$

When multiplied by the integrating factor, the rewritten differential equation becomes

$$(e^{-0.12t}P)' = -re^{-0.12t}.$$

Integration on both sides now yields

$$e^{-0.12t}P = \frac{r}{0.12}e^{-0.12t} + C;$$

hence,

$$P(t) = \frac{r}{0.12} + Ce^{0.12t}.$$

The initial condition $P(0) = 300$ allows us to determine the value of C:

$$300 = \frac{r}{0.12} + C \qquad \text{so} \qquad C = 300 - \frac{r}{0.12}.$$

The solution to the initial value problem is then

$$P(t) = \left(300 - \frac{r}{0.12}\right)e^{0.12t} + \frac{r}{0.12}.$$

Now, if $300 - \frac{r}{0.12} < 0$, then $\lim_{t\to\infty} P(t) = -\infty$, and the population becomes extinct. Therefore, in order for the population to survive, we must have

$$300 - \frac{r}{0.12} \ge 0 \qquad \text{or} \qquad r \le 36.$$

We conclude that the maximum rate at which the wolves can eat the rabbits without driving the rabbits to extinction is $r = 36$ rabbits per month.

45. Show that $y = \sin(\tan^{-1} x + C)$ is the general solution of $y' = \sqrt{1-y^2}/\left(1+x^2\right)$. Then use the addition formula for the sine function to show that the general solution may be written

$$y = \frac{(\cos C)x + \sin C}{\sqrt{1+x^2}}$$

SOLUTION Rewrite

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2} \qquad \text{as} \qquad \frac{dy}{\sqrt{1-y^2}} = \frac{dx}{1+x^2}.$$

Upon integrating both sides of this equation, we find

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2}$$
$$\sin^{-1} y = \tan^{-1} x + C$$

Thus,

$$y(x) = \sin\left(\tan^{-1} x + C\right).$$

To express the solution in the required form, we use the addition formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

This yields

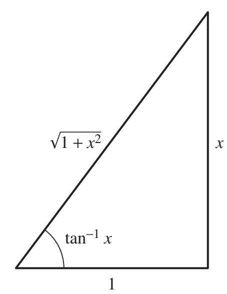
$$y(x) = \sin\left(\tan^{-1}x\right)\cos C + \sin C\cos\left(\tan^{-1}x\right).$$

Using the figure below, we see that

$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}}; \text{ and}$$
$$\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}.$$

Finally,

$$y = \frac{x\cos C}{\sqrt{1+x^2}} + \frac{\sin C}{\sqrt{1+x^2}} = \frac{(\cos C)x + \sin C}{\sqrt{1+x^2}}.$$



46. A tank is filled with 300 liters of contaminated water containing 3 kg of toxin. Pure water is pumped in at a rate of 40 L/min, mixes instantaneously, and is then pumped out at the same rate. Let $y(t)$ be the quantity of toxin present in the tank at time t.

(a) Find a differential equation satisfied by $y(t)$.

(b) Solve for $y(t)$.

(c) Find the time at which there is 0.01 kg of toxin present.

SOLUTION

(a) The net flow of toxin into or out of the tank at time t is

$$\frac{dy}{dt} = \text{toxin rate in} - \text{toxin rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(0\frac{\text{kg}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y(t)}{300}\frac{\text{kg}}{\text{L}}\right)$$
$$= -\frac{2}{15}y(t)$$

(b) This is a linear differential equation. Putting it in standard form gives

$$\frac{dy}{dt} + \frac{2}{15}y = 0$$

The integrating factor is

$$\alpha(t) = e^{\int (2/15)\,dt} = e^{2t/15}$$

When multiplied by the integrating factor, the differential equation becomes

$$(e^{2t/15}y)' = 0$$

Integrate both sides and multiply through by $e^{-2t/15}$ to get

$$y = Ce^{-2t/15}$$

Since there are initially 3 kg of toxin present, $y(0) = 3$ so that $C = 3$. Finally, we have

$$y = 3e^{-2t/15}$$

(c) We solve for t:

$$0.01 = 3e^{-2t/15} \quad \Rightarrow \quad t = -\frac{5}{2}\ln 0.01 \approx 11.51$$

There is 0.01 kg of toxin in the tank after about 11 and a half minutes.

47. At $t = 0$, a tank of volume 300 L is filled with 100 L of water containing salt at a concentration of 8 g/L. Fresh water flows in at a rate of 40 L/min, mixes instantaneously, and exits at the same rate. Let $c_1(t)$ be the salt concentration at time t.

(a) Find a differential equation satisfied by $c_1(t)$ *Hint:* Find the differential equation for the quantity of salt $y(t)$, and observe that $c_1(t) = y(t)/100$.

(b) Find the salt concentration $c_1(t)$ in the tank as a function of time.

SOLUTION

(a) Let $y(t)$ be the amount of salt in the tank at time t; then $c_1(t) = y(t)/100$. The rate of change of the amount of salt in the tank is

$$\begin{aligned}\frac{dy}{dt} &= \text{salt rate in} - \text{salt rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(0\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y}{100}\cdot\frac{\text{g}}{\text{L}}\right)\\ &= -\frac{2}{5}y\end{aligned}$$

Now, $c_1'(t) = y'(t)/100$ and $c(t) = y(t)/100$, so that c_1 satisfies the same differential equation:

$$\frac{dc_1}{dt} = -\frac{2}{5}c_1$$

(b) This is a linear differential equation. Putting it in standard form gives

$$\frac{dc_1}{dt} + \frac{2}{5}c_1 = 0$$

The integrating factor is $e^{2t/5}$; multiplying both sides by the integrating factor gives

$$(e^{2t/5}c_1)' = 0$$

Integrate and multiply through by $e^{-2t/5}$ to get

$$c_1(t) = Ce^{-2t/5}$$

The initial condition tells us that $y(0) = Ce^{-2\cdot 0/5} = C = 8$, so that finally,

$$c_1(t) = 8e^{-2t/5}$$

48. The outflow of the tank in Exercise 47 is directed into a second tank containing V liters of fresh water where it mixes instantaneously and exits at the same rate of 40 L/min. Determine the salt concentration $c_2(t)$ in the second tank as a function of time in the following two cases:

(a) $V = 200$ **(b)** $V = 300$

In each case, determine the maximum concentration.

SOLUTION Let $y_2(t)$ be the amount of salt in the second tank at time t; then $y_2(t) = c_2(t)V$ and $y_2'(t) = c_2'(t)V$. The rate of change in the amount of salt in the second tank is

$$\begin{aligned}\frac{dy_2}{dt} &= \text{salt rate in} - \text{salt rate out} = \left(40\frac{\text{L}}{\text{min}}\right)\left(c_1\frac{\text{g}}{\text{L}}\right) - \left(40\frac{\text{L}}{\text{min}}\right)\left(\frac{y_2}{V}\cdot\frac{\text{g}}{\text{L}}\right)\\ &= 40c_1 - \frac{40}{V}y_2\end{aligned}$$

Substituting for $y_2(t)$ and $y_2'(t)$ gives

$$Vc_2'(t) = 40c_1(t) - \frac{40}{V}Vc_2(t) \quad \text{so} \quad c_2'(t) = \frac{40}{V}(c_1(t) - c_2(t))$$

This is a linear differential equation; in standard form, it is

$$c_2'(t) + \frac{40}{V}c_2(t) = \frac{40}{V}c_1(t)$$

From the previous problem, we know that $c_1(t) = 8e^{-2t/5}$; substituting gives

$$c_2'(t) + \frac{40}{V}c_2(t) = \frac{320}{V}e^{-2t/5}$$

The integrating factor is $e^{40t/V}$; multiplying through by this factor gives

$$(e^{40t/V}c_2)' = \frac{320}{V}e^{(40t/V)-(2t/5)} = \frac{320}{V}e^{(200-2V)t/5V}$$

Integrate both sides to get

$$e^{40t/V}c_2 = \frac{320}{V} \cdot \frac{5V}{200-2V}e^{(200-2V)t/5V} + C = \frac{800}{100-V}e^{(200-2V)t/5V} + C$$

Multiply through by $e^{-40t/V}$ to get

$$c_2(t) = \frac{800}{100-V}e^{-2t/5} + Ce^{-40t/V}$$

Since tank 2 initially contains fresh water, $c_2(0) = 0$, so that $C = -\frac{800}{100-V}$ and

$$c_2(t) = \frac{800}{100-V}(e^{-2t/5} - e^{-40t/V})$$

(a) If $V = 200$, we have

$$c_2(t) = -8(e^{-2t/5} - e^{-t/5}) = 8(e^{-t/5} - e^{-2t/5})$$

The concentration of salt is at a maximum when $c_2'(t) = 0$:

$$\begin{aligned} 0 = c_2'(t) &= \frac{16}{5}e^{-2t/5} - \frac{8}{5}e^{-t/5} \\ e^{-t/5} &= 2e^{-2t/5} \\ -\frac{t}{5} &= -\frac{2t}{5} + \ln 2 \\ t &= 5\ln 2 \approx 3.47 \end{aligned}$$

so that the concentration of salt is at a maximum after about 3 and a half minutes.
(b) If $V = 300$, we have

$$c_2(t) = -4(e^{-2t/5} - e^{-2t/15}) = 4(e^{-2t/15} - e^{-2t/5})$$

The concentration of salt is at a maximum when $c_2'(t) = 0$:

$$\begin{aligned} 0 = c_2'(t) &= \frac{8}{5}e^{-2t/5} - \frac{8}{15}e^{-2t/15} \\ e^{-2t/15} &= 3e^{-2t/5} \\ -\frac{2}{15}t &= -\frac{2}{5}t + \ln 3 \\ t &= \frac{15}{4}\ln 3 \approx 4.12 \end{aligned}$$

so that the concentration of salt is at a maximum after about 4 minutes 7 seconds.

10 INFINITE SERIES

10.1 Sequences

Preliminary Questions

1. What is a_4 for the sequence $a_n = n^2 - n$?

SOLUTION Substituting $n = 4$ in the expression for a_n gives

$$a_4 = 4^2 - 4 = 12.$$

2. Which of the following sequences converge to zero?

(a) $\dfrac{n^2}{n^2+1}$ **(b)** 2^n **(c)** $\left(\dfrac{-1}{2}\right)^n$

SOLUTION

(a) This sequence does not converge to zero:

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1.$$

(b) This sequence does not converge to zero: this is a geometric sequence with $r = 2 > 1$; hence, the sequence diverges to ∞.

(c) Recall that if $|a_n|$ converges to 0, then a_n must also converge to zero. Here,

$$\left|\left(-\frac{1}{2}\right)^n\right| = \left(\frac{1}{2}\right)^n,$$

which is a geometric sequence with $0 < r < 1$; hence, $(\frac{1}{2})^n$ converges to zero. It therefore follows that $(-\frac{1}{2})^n$ converges to zero.

3. Let a_n be the nth decimal approximation to $\sqrt{2}$. That is, $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, etc. What is $\lim_{n\to\infty} a_n$?

SOLUTION $\lim_{n\to\infty} a_n = \sqrt{2}$.

4. Which of the following sequences is defined recursively?

(a) $a_n = \sqrt{4+n}$ **(b)** $b_n = \sqrt{4+b_{n-1}}$

SOLUTION

(a) a_n can be computed directly, since it depends on n only and not on preceding terms. Therefore a_n is defined explicitly and not recursively.

(b) b_n is computed in terms of the preceding term b_{n-1}, hence the sequence $\{b_n\}$ is defined recursively.

5. Theorem 5 says that every convergent sequence is bounded. Determine if the following statements are true or false and if false, give a counterexample.

(a) If $\{a_n\}$ is bounded, then it converges.

(b) If $\{a_n\}$ is not bounded, then it diverges.

(c) If $\{a_n\}$ diverges, then it is not bounded.

SOLUTION

(a) This statement is false. The sequence $a_n = \cos \pi n$ is bounded since $-1 \le \cos \pi n \le 1$ for all n, but it does not converge: since $a_n = \cos n\pi = (-1)^n$, the terms assume the two values 1 and -1 alternately, hence they do not approach one value.

(b) By Theorem 5, a converging sequence must be bounded. Therefore, if a sequence is not bounded, it certainly does not converge.

(c) The statement is false. The sequence $a_n = (-1)^n$ is bounded, but it does not approach one limit.

Exercises

1. Match each sequence with its general term:

$a_1, a_2, a_3, a_4, \ldots$	**General term**
(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$	(i) $\cos \pi n$
(b) $-1, 1, -1, 1, \ldots$	(ii) $\dfrac{n!}{2^n}$
(c) $1, -1, 1, -1, \ldots$	(iii) $(-1)^{n+1}$
(d) $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16} \ldots$	(iv) $\dfrac{n}{n+1}$

SOLUTION

(a) The numerator of each term is the same as the index of the term, and the denominator is one more than the numerator; hence $a_n = \frac{n}{n+1}$, $n = 1, 2, 3, \ldots$.

(b) The terms of this sequence are alternating between -1 and 1 so that the positive terms are in the even places. Since $\cos \pi n = 1$ for even n and $\cos \pi n = -1$ for odd n, we have $a_n = \cos \pi n$, $n = 1, 2, \ldots$.

(c) The terms a_n are 1 for odd n and -1 for even n. Hence, $a_n = (-1)^{n+1}$, $n = 1, 2, \ldots$

(d) The numerator of each term is $n!$, and the denominator is 2^n; hence, $a_n = \frac{n!}{2^n}$, $n = 1, 2, 3, \ldots$.

2. Let $a_n = \dfrac{1}{2n-1}$ for $n = 1, 2, 3, \ldots$. Write out the first three terms of the following sequences.

(a) $b_n = a_{n+1}$

(b) $c_n = a_{n+3}$

(c) $d_n = a_n^2$

(d) $e_n = 2a_n - a_{n+1}$

SOLUTION

(a) The first three terms of $\{b_n\}$ are:

$$b_1 = a_2 = \frac{1}{2 \cdot 2 - 1} = \frac{1}{3}, \quad b_2 = a_3 = \frac{1}{2 \cdot 3 - 1} = \frac{1}{5}, \quad b_3 = a_4 = \frac{1}{2 \cdot 4 - 1} = \frac{1}{7}.$$

(b) The first three terms of $\{c_n\}$ are:

$$c_1 = a_4 = \frac{1}{2 \cdot 4 - 1} = \frac{1}{7}, \quad c_2 = a_5 = \frac{1}{2 \cdot 5 - 1} = \frac{1}{9}, \quad c_3 = a_6 = \frac{1}{2 \cdot 6 - 1} = \frac{1}{11}.$$

(c) Note

$$a_1 = \frac{1}{2 \cdot 1 - 1} = 1, \quad a_2 = \frac{1}{2 \cdot 2 - 1} = \frac{1}{3}, \quad a_3 = \frac{1}{2 \cdot 3 - 1} = \frac{1}{5}.$$

Thus,

$$d_1 = a_1^2 = 1^2 = 1, \quad d_2 = a_2^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}, \quad d_3 = a_3^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}.$$

(d) The first three terms of $\{e_n\}$ are:

$$e_1 = 2a_1 - a_2, \quad e_2 = 2a_2 - a_3, \quad e_3 = 2a_3 - a_4.$$

Thus, we must compute a_1, a_2, a_3 and a_4. We set $n = 1, 2, 3$ and 4 in the formula for a_n to obtain:

$$a_1 = \frac{1}{2 \cdot 1 - 1} = 1, \quad a_2 = \frac{1}{2 \cdot 2 - 1} = \frac{1}{3}, \quad a_3 = \frac{1}{2 \cdot 3 - 1} = \frac{1}{5}, \quad a_4 = \frac{1}{2 \cdot 4 - 1} = \frac{1}{7}.$$

Therefore,

$$e_1 = 2 \cdot 1 - \frac{1}{3} = \frac{5}{3}, \quad e_2 = 2 \cdot \frac{1}{3} - \frac{1}{5} = \frac{7}{15}, \quad e_3 = 2 \cdot \frac{1}{5} - \frac{1}{7} = \frac{9}{35}.$$

In Exercises 3–12, calculate the first four terms of the sequence, starting with $n = 1$.

3. $c_n = \dfrac{3^n}{n!}$

SOLUTION Setting $n = 1, 2, 3, 4$ in the formula for c_n gives

$$c_1 = \frac{3^1}{1!} = \frac{3}{1} = 3, \qquad c_2 = \frac{3^2}{2!} = \frac{9}{2},$$
$$c_3 = \frac{3^3}{3!} = \frac{27}{6} = \frac{9}{2}, \qquad c_4 = \frac{3^4}{4!} = \frac{81}{24} = \frac{27}{8}.$$

4. $b_n = \dfrac{(2n-1)!}{n!}$

SOLUTION Setting $n = 1, 2, 3, 4$ in the formula for b_n gives

$$b_1 = \frac{(2\cdot 1-1)!}{1!} = \frac{1}{1} = 1, \qquad b_2 = \frac{(2\cdot 2-1)}{2!} = \frac{6}{2} = 3,$$
$$b_3 = \frac{(2\cdot 3-1)!}{3!} = \frac{120}{6} = 20, \qquad b_4 = \frac{(2\cdot 4-1)}{4!} = \frac{5040}{24} = 210.$$

5. $a_1 = 2, \quad a_{n+1} = 2a_n^2 - 3$

SOLUTION For $n = 1, 2, 3$ we have:

$$a_2 = a_{1+1} = 2a_1^2 - 3 = 2\cdot 4 - 3 = 5;$$
$$a_3 = a_{2+1} = 2a_2^2 - 3 = 2\cdot 25 - 3 = 47;$$
$$a_4 = a_{3+1} = 2a_3^2 - 3 = 2\cdot 2209 - 3 = 4415.$$

The first four terms of $\{a_n\}$ are 2, 5, 47, 4415.

6. $b_1 = 1, \quad b_n = b_{n-1} + \dfrac{1}{b_{n-1}}$

SOLUTION For $n = 2, 3, 4$ we have

$$b_2 = b_1 + \frac{1}{b_1} = 1 + \frac{1}{1} = 2;$$
$$b_3 = b_2 + \frac{1}{b_2} = 2 + \frac{1}{2} = \frac{5}{2};$$
$$b_4 = b_3 + \frac{1}{b_2} = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}.$$

The first four terms of $\{b_n\}$ are $1, 2, \frac{5}{2}, \frac{29}{10}$.

7. $b_n = 5 + \cos \pi n$

SOLUTION For $n = 1, 2, 3, 4$ we have

$$b_1 = 5 + \cos \pi = 4;$$
$$b_2 = 5 + \cos 2\pi = 6;$$
$$b_3 = 5 + \cos 3\pi = 4;$$
$$b_4 = 5 + \cos 4\pi = 6.$$

The first four terms of $\{b_n\}$ are 4, 6, 4, 6.

8. $c_n = (-1)^{2n+1}$

SOLUTION for $n = 1, 2, 3, 4$ we have

$$c_1 = (-1)^{2\cdot 1+1} = (-1)^3 = -1;$$
$$c_2 = (-1)^{2\cdot 2+1} = (-1)^5 = -1;$$
$$c_3 = (-1)^{2\cdot 3+1} = (-1)^7 = -1;$$
$$c_4 = (-1)^{2\cdot 4+1} = (-1)^9 = -1.$$

The first four terms of $\{c_n\}$ are $-1, -1, -1, -1$.

9. $c_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$

SOLUTION

$$c_1 = 1;$$
$$c_2 = 1 + \frac{1}{2} = \frac{3}{2};$$
$$c_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6};$$
$$c_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{11}{6} + \frac{1}{4} = \frac{25}{12}.$$

10. $a_n = n + (n+1) + (n+2) + \cdots + (2n)$

SOLUTION The general term a_n is the sum of $n+1$ successive numbers, where the first one is n and the last one is $2n$. Thus,

$$a_1 = 1 + 2 = 3;$$
$$a_2 = 2 + 3 + 4 = 9;$$
$$a_3 = 3 + 4 + 5 + 6 = 18;$$
$$a_4 = 4 + 5 + 6 + 7 + 8 = 30.$$

11. $b_1 = 2, \quad b_2 = 3, \quad b_n = 2b_{n-1} + b_{n-2}$

SOLUTION We need to find b_3 and b_4. Setting $n = 3$ and $n = 4$ and using the given values for b_1 and b_2 we obtain:

$$b_3 = 2b_{3-1} + b_{3-2} = 2b_2 + b_1 = 2 \cdot 3 + 2 = 8;$$
$$b_4 = 2b_{4-1} + b_{4-2} = 2b_3 + b_2 = 2 \cdot 8 + 3 = 19.$$

The first four terms of the sequence $\{b_n\}$ are 2, 3, 8, 19.

12. $c_n = n$-place decimal approximation to e

SOLUTION Using a calculator we find that $e = 2.718281828\ldots$. Thus, the four first terms of $\{c_n\}$ are

$$c_1 = 2.7; \quad c_2 = 2.72; \quad c_3 = 2.718; \quad c_4 = 2.7183.$$

13. Find a formula for the nth term of each sequence.

(a) $\frac{1}{1}, \frac{-1}{8}, \frac{1}{27}, \ldots$

(b) $\frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \ldots$

SOLUTION

(a) The denominators are the third powers of the positive integers starting with $n = 1$. Also, the sign of the terms is alternating with the sign of the first term being positive. Thus,

$$a_1 = \frac{1}{1^3} = \frac{(-1)^{1+1}}{1^3}; \quad a_2 = -\frac{1}{2^3} = \frac{(-1)^{2+1}}{2^3}; \quad a_3 = \frac{1}{3^3} = \frac{(-1)^{3+1}}{3^3}.$$

This rule leads to the following formula for the nth term:

$$a_n = \frac{(-1)^{n+1}}{n^3}.$$

(b) Assuming a starting index of $n = 1$, we see that each numerator is one more than the index and the denominator is four more than the numerator. Thus, the general term a_n is

$$a_n = \frac{n+1}{n+5}.$$

14. Suppose that $\lim_{n\to\infty} a_n = 4$ and $\lim_{n\to\infty} b_n = 7$. Determine:

(a) $\lim_{n\to\infty} (a_n + b_n)$

(b) $\lim_{n\to\infty} a_n^3$

(c) $\lim_{n\to\infty} \cos(\pi b_n)$

(d) $\lim_{n\to\infty} (a_n^2 - 2a_n b_n)$

SOLUTION

(a) By the Limit Laws for Sequences, we find

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 4 + 7 = 11.$$

(b) By the Limit Laws for Sequences, we find

$$\lim_{n\to\infty} a_n^3 = \lim_{n\to\infty} (a_n \cdot a_n \cdot a_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} a_n\right) = \left(\lim_{n\to\infty} a_n\right)^3 = 4^3 = 64.$$

(c) By Theorem 4, we can "bring the limit inside the function":

$$\lim_{n\to\infty} \cos(\pi b_n) = \cos\left(\lim_{n\to\infty} \pi b_n\right) = \cos\left(\pi \lim_{n\to\infty} b_n\right) = \cos(7\pi) = -1.$$

(d) By the Limit Laws of Sequences, we find

$$\lim_{n\to\infty}\left(a_n^2 - 2a_nb_n\right) = \lim_{n\to\infty} a_n^2 - \lim_{n\to\infty} 2a_nb_n = \left(\lim_{n\to\infty} a_n\right)^2 - 2\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = 4^2 - 2\cdot 4\cdot 7 = -40.$$

In Exercises 15–26, use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.

15. $a_n = 12$

SOLUTION We have $a_n = f(n)$ where $f(x) = 12$; thus,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} 12 = 12.$$

16. $a_n = 20 - \dfrac{4}{n^2}$

SOLUTION We have $a_n = f(n)$ where $f(x) = 20 - \dfrac{4}{x^2}$; thus,

$$\lim_{n\to\infty}\left(20 - \frac{4}{n^2}\right) = \lim_{x\to\infty}\left(20 - \frac{4}{x^2}\right) = 20 - 0 = 20.$$

17. $b_n = \dfrac{5n-1}{12n+9}$

SOLUTION We have $b_n = f(n)$ where $f(x) = \dfrac{5x-1}{12x+9}$; thus,

$$\lim_{n\to\infty}\frac{5n-1}{12n+9} = \lim_{x\to\infty}\frac{5x-1}{12x+9} = \frac{5}{12}.$$

18. $a_n = \dfrac{4+n-3n^2}{4n^2+1}$

SOLUTION We have $a_n = f(n)$ where $f(x) = \dfrac{4+x-3x^2}{4x^2+1}$; thus,

$$\lim_{n\to\infty}\frac{4+n-3n^2}{4n^2+1} = \lim_{x\to\infty}\frac{4+x-3x^2}{4x^2+1} = -\frac{3}{4}$$

19. $c_n = -2^{-n}$

SOLUTION We have $c_n = f(n)$ where $f(x) = -2^{-x}$; thus,

$$\lim_{n\to\infty}\left(-2^{-n}\right) = \lim_{x\to\infty} -2^{-x} = \lim_{x\to\infty} -\frac{1}{2^x} = 0.$$

20. $z_n = \left(\dfrac{1}{3}\right)^n$

SOLUTION We have $z_n = f(n)$ where $f(x) = \left(\dfrac{1}{3}\right)^x$; thus,

$$\lim_{n\to\infty}\left(\frac{1}{3}\right)^n = \lim_{x\to\infty}\left(\frac{1}{3}\right)^x = 0.$$

21. $c_n = 9^n$

SOLUTION We have $c_n = f(n)$ where $f(x) = 9^x$; thus,

$$\lim_{n\to\infty} 9^n = \lim_{x\to\infty} 9^x = \infty$$

Thus, the sequence 9^n diverges.

22. $z_n = 10^{-1/n}$

SOLUTION We have $z_n = f(n)$ where $f(x) = (0.1)^{-1/x}$; thus

$$\lim_{n\to\infty}(0.1)^{-1/n} = \lim_{x\to\infty}(0.1)^{-1/x} = (0.1)^{\lim_{x\to\infty}(-1/x)} = (0.1)^0 = 1.$$

23. $a_n = \dfrac{n}{\sqrt{n^2+1}}$

SOLUTION We have $a_n = f(n)$ where $f(x) = \dfrac{x}{\sqrt{x^2+1}}$; thus,

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{\sqrt{x^2+1}}{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+1}{x^2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1.$$

24. $a_n = \dfrac{n}{\sqrt{n^3+1}}$

SOLUTION We have $a_n = f(n)$ where $f(x) = \dfrac{x}{\sqrt{x^3+1}}$; thus,

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^3+1}} = \lim_{x\to\infty} \frac{\frac{x}{x^{3/2}}}{\frac{\sqrt{x^3+1}}{x^{3/2}}} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\sqrt{1+\frac{1}{x^3}}} = \frac{0}{\sqrt{1+0}} = \frac{0}{1} = 0.$$

25. $a_n = \ln\left(\dfrac{12n+2}{-9+4n}\right)$

SOLUTION We have $a_n = f(n)$ where $f(x) = \ln\left(\dfrac{12x+2}{-9+4x}\right)$; thus,

$$\lim_{n\to\infty} \ln\left(\frac{12n+2}{-9+4n}\right) = \lim_{x\to\infty} \ln\left(\frac{12x+2}{-9+4x}\right) = \ln \lim_{x\to\infty}\left(\frac{12x+2}{-9+4x}\right) = \ln 3$$

26. $r_n = \ln n - \ln(n^2+1)$

SOLUTION We have $r_n = f(n)$ where $f(x) = \ln x - \ln(x^2+1)$; thus,

$$\lim_{n\to\infty} (\ln n - \ln(n^2+1)) = \lim_{x\to\infty} (\ln x - \ln(x^2+1)) = \lim_{x\to\infty} \ln \frac{x}{x^2+1}$$

But this function diverges as $x \to \infty$, so that r_n diverges as well.

In Exercises 27–30, use Theorem 4 to determine the limit of the sequence.

27. $a_n = \sqrt{4 + \dfrac{1}{n}}$

SOLUTION We have

$$\lim_{n\to\infty} 4 + \frac{1}{n} = \lim_{x\to\infty} 4 + \frac{1}{x} = 4$$

Since $\sqrt{x}$ is a continuous function for $x > 0$, Theorem 4 tells us that

$$\lim_{n\to\infty} \sqrt{4 + \frac{1}{n}} = \sqrt{\lim_{n\to\infty} 4 + \frac{1}{n}} = \sqrt{4} = 2$$

28. $a_n = e^{4n/(3n+9)}$

SOLUTION We have

$$\lim_{n\to\infty} \frac{4n}{3n+9} = \frac{4}{3}$$

Since e^x is continuous for all x, Theorem 4 tells us that

$$\lim_{n\to\infty} e^{4n/(3n+9)} = e^{\lim_{n\to\infty} 4n/(3n+9)} = e^{4/3}$$

29. $a_n = \cos^{-1}\left(\dfrac{n^3}{2n^3+1}\right)$

SOLUTION We have

$$\lim_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2}$$

Since $\cos^{-1}(x)$ is continuous for all x, Theorem 4 tells us that

$$\lim_{n\to\infty} \cos^{-1}\left(\frac{n^3}{2n^3+1}\right) = \cos^{-1}\left(\lim_{n\to\infty} \frac{n^3}{2n^3+1}\right) = \cos^{-1}(1/2) = \frac{\pi}{3}$$

30. $a_n = \tan^{-1}(e^{-n})$

SOLUTION We have

$$\lim_{n\to\infty} = e^{-n} \lim_{x\to\infty} e^{-x} = 0$$

Since $\tan^{-1}(x)$ is continuous for all x, Theorem 4 tells us that

$$\lim_{n\to\infty} \tan^{-1}(e^{-n}) = \tan^{-1}\left(\lim_{n\to\infty} e^{-n}\right) = \tan^{-1}(0) = 0$$

31. Let $a_n = \dfrac{n}{n+1}$. Find a number M such that:

(a) $|a_n - 1| \le 0.001$ for $n \ge M$.

(b) $|a_n - 1| \le 0.00001$ for $n \ge M$.

Then use the limit definition to prove that $\lim_{n\to\infty} a_n = 1$.

SOLUTION

(a) We have

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}.$$

Therefore $|a_n - 1| \le 0.001$ provided $\frac{1}{n+1} \le 0.001$, that is, $n \ge 999$. It follows that we can take $M = 999$.

(b) By part (a), $|a_n - 1| \le 0.00001$ provided $\frac{1}{n+1} \le 0.00001$, that is, $n \ge 99999$. It follows that we can take $M = 99999$.

We now prove formally that $\lim_{n\to\infty} a_n = 1$. Using part (a), we know that

$$|a_n - 1| = \frac{1}{n+1} < \epsilon,$$

provided $n > \frac{1}{\epsilon} - 1$. Thus, Let $\epsilon > 0$ and take $M = \frac{1}{\epsilon} - 1$. Then, for $n > M$, we have

$$|a_n - 1| = \frac{1}{n+1} < \frac{1}{M+1} = \epsilon.$$

32. Let $b_n = \left(\frac{1}{3}\right)^n$.

(a) Find a value of M such that $|b_n| \le 10^{-5}$ for $n \ge M$.

(b) Use the limit definition to prove that $\lim_{n\to\infty} b_n = 0$.

SOLUTION

(a) Solving $\left(\frac{1}{3}\right)^n \le 10^{-5}$ for n, we find

$$n \ge 5\log_3 10 = 5\frac{\ln 10}{\ln 3} \approx 10.48.$$

It follows that we can take $M = 10.5$.

(b) We see that

$$\left|\left(\frac{1}{3}\right)^n - 0\right| = \frac{1}{3^n} < \epsilon$$

provided

$$n > \log_3 \frac{1}{\epsilon}.$$

Thus, let $\epsilon > 0$ and take $M = \log_3 \frac{1}{\epsilon}$. Then, for $n > M$, we have

$$\left|\left(\frac{1}{3}\right)^n - 0\right| = \frac{1}{3^n} < \frac{1}{3^M} = \epsilon.$$

33. Use the limit definition to prove that $\lim_{n\to\infty} n^{-2} = 0$.

SOLUTION We see that

$$|n^{-2} - 0| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \epsilon$$

provided

$$n > \frac{1}{\sqrt{\epsilon}}.$$

Thus, let $\epsilon > 0$ and take $M = \frac{1}{\sqrt{\epsilon}}$. Then, for $n > M$, we have

$$|n^{-2} - 0| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{M^2} = \epsilon.$$

34. Use the limit definition to prove that $\lim_{n\to\infty} \frac{n}{n+n^{-1}} = 1$.

SOLUTION Since

$$\frac{n}{n+n^{-1}} = \frac{n^2}{n(n+n^{-1})} = \frac{n^2}{n^2+1}$$

we see that

$$\left|\frac{n^2}{n^2+1} - 1\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{n^2+1} < \epsilon$$

provided

$$n > \sqrt{\frac{1}{\epsilon} - 1}$$

So choose $\epsilon > 0$, and let $M = \sqrt{\frac{1}{\epsilon} - 1}$. Then, for $n > M$, we have

$$\left|\frac{n}{n+n^{-1}} - 1\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{(\frac{1}{\epsilon} - 1) + 1} = \epsilon$$

In Exercises 35–62, use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

35. $a_n = 10 + \left(-\frac{1}{9}\right)^n$

SOLUTION By the Limit Laws for Sequences we have:

$$\lim_{n\to\infty}\left(10 + \left(-\frac{1}{9}\right)^n\right) = \lim_{n\to\infty} 10 + \lim_{n\to\infty}\left(-\frac{1}{9}\right)^n = 10 + \lim_{n\to\infty}\left(-\frac{1}{9}\right)^n.$$

Now,

$$-\left(\frac{1}{9}\right)^n \le \left(-\frac{1}{9}\right)^n \le \left(\frac{1}{9}\right)^n.$$

Because

$$\lim_{n\to\infty}\left(\frac{1}{9}\right)^n = 0,$$

by the Limit Laws for Sequences,

$$\lim_{n\to\infty} -\left(\frac{1}{9}\right)^n = -\lim_{n\to\infty}\left(\frac{1}{9}\right)^n = 0.$$

Thus, we have

$$\lim_{n\to\infty}\left(-\frac{1}{9}\right)^n = 0,$$

and

$$\lim_{n\to\infty}\left(10 + \left(-\frac{1}{9}\right)^n\right) = 10 + 0 = 10.$$

36. $d_n = \sqrt{n+3} - \sqrt{n}$

SOLUTION We multiply and divide d_n by the conjugate expression $\sqrt{n+3} + \sqrt{n}$ and use the identity $(a-b)(a+b) = a^2 - b^2$ to obtain:

$$d_n = \frac{\left(\sqrt{n+3} - \sqrt{n}\right)\left(\sqrt{n+3} + \sqrt{n}\right)}{\sqrt{n+3} + \sqrt{n}} = \frac{(n+3) - n}{\sqrt{n+3} + \sqrt{n}} = \frac{3}{\sqrt{n+3} + \sqrt{n}}.$$

Thus,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{3}{\sqrt{n+3} + \sqrt{n}} = \lim_{x\to\infty} \frac{3}{\sqrt{x+3} + \sqrt{x}} = 0.$$

37. $c_n = 1.01^n$

SOLUTION Since $c_n = f(n)$ where $f(x) = 1.01^x$, we have

$$\lim_{n\to\infty} 1.01^n = \lim_{x\to\infty} 1.01^x = \infty$$

so that the sequence diverges.

38. $b_n = e^{1-n^2}$

SOLUTION Since $b_n = f(n)$ where $f(x) = e^{1-x^2}$, we have

$$\lim_{n\to\infty} e^{1-n^2} = \lim_{x\to\infty} e^{1-x^2} = \lim_{x\to\infty} \frac{e}{e^{x^2}} = 0$$

39. $a_n = 2^{1/n}$

SOLUTION Because 2^x is a continuous function,

$$\lim_{n\to\infty} 2^{1/n} = \lim_{x\to\infty} 2^{1/x} = 2^{\lim_{x\to\infty}(1/x)} = 2^0 = 1.$$

40. $b_n = n^{1/n}$

SOLUTION Let $b_n = n^{1/n}$. Take the natural logarithm of both sides of this expression to obtain

$$\ln b_n = \ln n^{1/n} = \frac{\ln n}{n}.$$

Thus,

$$\lim_{n\to\infty} (\ln b_n) = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{\ln b_n} = e^{\lim_{n\to\infty}(\ln b_n)} = e^0 = 1.$$

That is,

$$\lim_{n\to\infty} n^{1/n} = 1.$$

41. $c_n = \dfrac{9^n}{n!}$

SOLUTION For $n \ge 9$, write

$$c_n = \frac{9^n}{n!} = \underbrace{\frac{9}{1} \cdot \frac{9}{2} \cdots \frac{9}{9}}_{\text{call this } C} \cdot \underbrace{\frac{9}{10} \cdot \frac{9}{11} \cdots \frac{9}{n-1} \cdot \frac{9}{n}}_{\text{Each factor is less than 1}}$$

Then clearly

$$0 \le \frac{9^n}{n!} \le C\frac{9}{n}$$

since each factor after the first nine is < 1. The squeeze theorem tells us that

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{9^n}{n!} \le \lim_{n\to\infty} C\frac{9}{n} = C \lim_{n\to\infty} \frac{9}{n} = C \cdot 0 = 0$$

so that $\lim_{n\to\infty} c_n = 0$ as well.

42. $a_n = \dfrac{8^{2n}}{n!}$

SOLUTION Note that

$$a_n = \frac{8^{2n}}{n!} = \frac{64^n}{n!}$$

Now apply the same method as in the Exercise 41. For $n \ge 64$, write

$$c_n = \frac{64^n}{n!} = \underbrace{\frac{64}{1} \cdot \frac{64}{2} \cdots \frac{64}{64}}_{\text{call this } C} \cdot \underbrace{\frac{64}{65} \cdot \frac{64}{66} \cdots \frac{64}{n-1} \cdot \frac{64}{n}}_{\text{Each factor is less than 1}}$$

Then clearly

$$0 \le \frac{64^n}{n!} \le C\frac{64}{n}$$

since each factor after the first 64 is < 1. The squeeze theorem tells us that

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{64^n}{n!} \le \lim_{n\to\infty} C\frac{64}{n} = C \lim_{n\to\infty} \frac{64}{n} = C \cdot 0 = 0$$

so that $\lim_{n\to\infty} a_n = 0$ as well.

43. $a_n = \dfrac{3n^2+n+2}{2n^2-3}$

SOLUTION

$$\lim_{n\to\infty} \frac{3n^2+n+2}{2n^2-3} = \lim_{x\to\infty} \frac{3x^2+x+2}{2x^2-3} = \frac{3}{2}.$$

44. $a_n = \dfrac{\sqrt{n}}{\sqrt{n}+4}$

SOLUTION

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x}}{\sqrt{x}}+\frac{4}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{1+\frac{4}{\sqrt{x}}} = \frac{1}{1+0} = 1.$$

45. $a_n = \dfrac{\cos n}{n}$

SOLUTION Since $-1 \le \cos n \le 1$ the following holds:

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}.$$

We now apply the Squeeze Theorem for Sequences and the limits

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

to conclude that $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

46. $c_n = \dfrac{(-1)^n}{\sqrt{n}}$

SOLUTION Clearly

$$-\frac{1}{\sqrt{n}} \le \frac{(-1)^n}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

Since

$$\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

the Squeeze Theorem tells us that $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$.

47. $d_n = \ln 5^n - \ln n!$

SOLUTION Note that

$$d_n = \ln \frac{5^n}{n!}$$

so that

$$e^{d_n} = \frac{5^n}{n!} \quad \text{so} \quad \lim_{n\to\infty} e^{d_n} = \lim_{n\to\infty} \frac{5^n}{n!} = 0$$

by the method of Exercise 41. If d_n converged, we could, since $f(x) = e^x$ is continuous, then write

$$\lim_{n\to\infty} e^{d_n} = e^{\lim_{n\to\infty} d_n} = 0$$

which is impossible. Thus $\{d_n\}$ diverges.

48. $d_n = \ln(n^2+4) - \ln(n^2-1)$

SOLUTION Note that

$$d_n = \ln \frac{n^2+4}{n^2-1}$$

so exponentiating both sides of this expression gives

$$e^{d_n} = \frac{n^2+4}{n^2-1} = \frac{1+(4/n^2)}{1-(1/n^2)}$$

Thus,

$$\lim_{n\to\infty} e^{d_n} = \lim_{n\to\infty} \frac{1+(4/n^2)}{1-(1/n^2)} = 1$$

Because $f(x) = \ln x$ is continuous for $x > 0$, it follows that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \ln(e^{d_n}) = \ln(\lim_{n\to\infty} e^{d_n}) = \ln 1 = 0$$

49. $a_n = \left(2 + \frac{4}{n^2}\right)^{1/3}$

SOLUTION Let $a_n = \left(2+\frac{4}{n^2}\right)^{1/3}$. Taking the natural logarithm of both sides of this expression yields

$$\ln a_n = \ln\left(2+\frac{4}{n^2}\right)^{1/3} = \frac{1}{3}\ln\left(2+\frac{4}{n^2}\right).$$

Thus,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{3}\ln\left(2+\frac{4}{n^2}\right)^{1/3} = \frac{1}{3}\lim_{x\to\infty}\ln\left(2+\frac{4}{x^2}\right) = \frac{1}{3}\ln\left(\lim_{x\to\infty}\left(2+\frac{4}{x^2}\right)\right)$$
$$= \frac{1}{3}\ln(2+0) = \frac{1}{3}\ln 2 = \ln 2^{1/3}.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty}(\ln a_n)} = e^{\ln 2^{1/3}} = 2^{1/3}.$$

50. $b_n = \tan^{-1}\left(1 - \frac{2}{n}\right)$

SOLUTION Because $f(x) = \tan^{-1} x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \tan^{-1}\left(1-\frac{2}{x}\right) = \tan^{-1}\left(\lim_{x\to\infty}\left(1-\frac{2}{x}\right)\right) = \tan^{-1} 1 = \frac{\pi}{4}.$$

51. $c_n = \ln\left(\frac{2n+1}{3n+4}\right)$

SOLUTION Because $f(x) = \ln x$ is a continuous function, it follows that

$$\lim_{n\to\infty} c_n = \lim_{x\to\infty} \ln\left(\frac{2x+1}{3x+4}\right) = \ln\left(\lim_{x\to\infty}\frac{2x+1}{3x+4}\right) = \ln\frac{2}{3}.$$

52. $c_n = \frac{n}{n+n^{1/n}}$

SOLUTION We rewrite $\frac{n}{n+n^{1/n}}$ as follows:

$$\frac{n}{n+n^{1/n}} = \frac{\frac{n}{n}}{\frac{n}{n}+\frac{n^{1/n}}{n}} = \frac{1}{1+\frac{n^{1/n}}{n}}.$$

Now,

$$\frac{n^{1/n}}{n} = n^{\frac{1}{n}-1} = \frac{1}{n^{1-1/n}},$$

and

$$\lim_{n\to\infty}\frac{n^{1/n}}{n}=\lim_{n\to\infty}\frac{1}{n^{1-1/n}}=\lim_{x\to\infty}\frac{1}{x^{1-1/x}}=0.$$

Thus,

$$\lim_{n\to\infty}\frac{n}{n+n^{1/n}}=\lim_{x\to\infty}\frac{1}{1+\frac{x^{1/x}}{x}}=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\left(1+\frac{x^{1/x}}{x}\right)}=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{x^{1/x}}{x}}=\frac{1}{1+0}=1.$$

53. $y_n=\dfrac{e^n}{2^n}$

SOLUTION $\frac{e^n}{2^n}=\left(\frac{e}{2}\right)^n$ and $\frac{e}{2}>1$. By the Limit of Geometric Sequences,we conclude that $\lim_{n\to\infty}\left(\frac{e}{2}\right)^n=\infty$. Thus, the given sequence diverges.

54. $a_n=\dfrac{n}{2^n}$

SOLUTION

$$\lim_{n\to\infty}\frac{n}{2^n}=\lim_{x\to\infty}\frac{x}{2^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(2^x)}=\lim_{x\to\infty}\frac{1}{(\ln 2)\,2^x}=\frac{1}{\ln 2}\lim_{x\to\infty}\frac{1}{2^x}=\frac{1}{\ln 2}\cdot 0=0.$$

55. $y_n=\dfrac{e^n+(-3)^n}{5^n}$

SOLUTION

$$\lim_{n\to\infty}\frac{e^n+(-3)^n}{5^n}=\lim_{n\to\infty}\left(\frac{e}{5}\right)^n+\lim_{n\to\infty}\left(\frac{-3}{5}\right)^n$$

assuming both limits on the right-hand side exist. But by the Limit of Geometric Sequences, since

$$-1<\frac{-3}{5}<0<\frac{e}{5}<1$$

both limits on the right-hand side are 0, so that y_n converges to 0.

56. $b_n=\dfrac{(-1)^n n^3+2^{-n}}{3n^3+4^{-n}}$

SOLUTION Assuming both limits on the right-hand side exist, we have

$$\lim_{n\to\infty}\frac{(-1)^n n^3+2^{-n}}{3n^3+4^{-n}}=\lim_{n\to\infty}\frac{(-1)^n n^3}{3n^3+4^{-n}}+\lim_{n\to\infty}\frac{2^{-n}}{3n^3+4^{-n}}$$

For the first limit, let us consider instead the limit of its reciprocal:

$$\begin{aligned}\lim_{n\to\infty}(-1)^n\frac{3n^3+4^{-n}}{n^3}&=\lim_{n\to\infty}(-1)^n\frac{3n^3}{n^3}+\lim_{n\to\infty}(-1)^n\frac{4^{-n}}{n^3}\\&=\lim_{n\to\infty}(-1)^n\cdot 3+\lim_{n\to\infty}(-1)^n\frac{1}{4^n n^3}\\&=\lim_{n\to\infty}\left((-1)^n\cdot 3\right)+0\end{aligned}$$

so that one limit on the right-hand side exists and the other does not; thus the left-hand side diverges as well.

57. $a_n=n\sin\dfrac{\pi}{n}$

SOLUTION By the Theorem on Sequences Defined by a Function, we have

$$\lim_{n\to\infty}n\sin\frac{\pi}{n}=\lim_{x\to\infty}x\sin\frac{\pi}{x}.$$

Now,

$$\begin{aligned}\lim_{x\to\infty}x\sin\frac{\pi}{x}&=\lim_{x\to\infty}\frac{\sin\frac{\pi}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\left(\cos\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\left(\pi\cos\frac{\pi}{x}\right)\\&=\pi\lim_{x\to\infty}\cos\frac{\pi}{x}=\pi\cos 0=\pi\cdot 1=\pi.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} n \sin\frac{\pi}{n} = \pi.$$

58. $b_n = \dfrac{n!}{\pi^n}$

SOLUTION By the method of Exercise 41, we can see that $\lim_{n\to\infty} \dfrac{4^n}{n!} = 0$ so that $c_n = \dfrac{n!}{4^n}$ diverges. But $\pi < 4$ so that $c_n < b_n$ and thus b_n diverges as well.

59. $b_n = \dfrac{3-4^n}{2+7\cdot 4^n}$

SOLUTION Divide the numerator and denominator by 4^n to obtain

$$a_n = \frac{3-4^n}{2+7\cdot 4^n} = \frac{\frac{3}{4^n} - \frac{4^n}{4^n}}{\frac{2}{4^n} + \frac{7\cdot 4^n}{4^n}} = \frac{\frac{3}{4^n} - 1}{\frac{2}{4^n} + 7}.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\frac{3}{4^x} - 1}{\frac{2}{4^x} + 7} = \frac{\lim_{x\to\infty}\left(\frac{3}{4^x} - 1\right)}{\lim_{x\to\infty}\left(\frac{2}{4^x} + 7\right)} = \frac{3\lim_{x\to\infty}\frac{1}{4^x} - \lim_{x\to\infty} 1}{2\lim_{x\to\infty}\frac{1}{4^x} - \lim_{x\to\infty} 7} = \frac{3\cdot 0 - 1}{2\cdot 0 + 7} = -\frac{1}{7}.$$

60. $a_n = \dfrac{3-4^n}{2+7\cdot 3^n}$

SOLUTION Divide the numerator and denominator by 3^n to obtain

$$a_n = \frac{3-4^n}{2+7\cdot 3^n} = \frac{\frac{3}{3^n} - \frac{4^n}{3^n}}{\frac{2}{3^n} + \frac{7\cdot 3^n}{3^n}} = \frac{\frac{3}{3^n} - \left(\frac{4}{3}\right)^n}{\frac{2}{3^n} + 7}.$$

We examine the limits of the numerator and the denominator:

$$\lim_{n\to\infty}\left(\frac{3}{3^n} - \left(\frac{4}{3}\right)^n\right) = 3\lim_{n\to\infty}\left(\frac{1}{3}\right)^n - 3\lim_{n\to\infty}\left(\frac{4}{3}\right)^n = 3\cdot 0 - \infty = -\infty,$$

whereas

$$\lim_{n\to\infty}\left(\frac{2}{3^n} + 7\right) = \lim_{n\to\infty}\frac{2}{3^n} + \lim_{n\to\infty} 7 = 2\lim_{n\to\infty}\left(\frac{1}{3}\right)^n + \lim_{n\to\infty} 7 = 2\cdot 0 + 7 = 7.$$

Thus, $\lim\limits_{n\to\infty} a_n = -\infty$; that is, the sequence diverges.

61. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

SOLUTION Taking the natural logarithm of both sides of this expression yields

$$\ln a_n = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}.$$

Thus,

$$\lim_{n\to\infty}(\ln a_n) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty}(\ln a_n)} = e^1 = e.$$

62. $a_n = \left(1 + \frac{1}{n^2}\right)^n$

SOLUTION Taking the natural logarithm of both sides of this expression yields

$$\ln a_n = \ln\left(1 + \frac{1}{n^2}\right)^n = n \ln\left(1 + \frac{1}{n^2}\right) = \frac{\ln\left(1 + \frac{1}{n^2}\right)}{\frac{1}{n}}.$$

Thus,

$$\lim_{n\to\infty}(\ln a_n) = \lim_{x\to\infty}\frac{\ln(1 + x^{-2})}{x^{-1}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln(1 + x^{-2})\right)}{\frac{d}{dx}(x^{-1})}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+x^{-2}}(-2x^{-3})}{-x^{-2}} = \lim_{x\to\infty}\frac{2x^{-1}}{1 + x^{-2}} = \lim_{x\to\infty}\frac{\frac{2}{x}}{1 + \frac{1}{x^2}} = \frac{0}{1+0} = 0.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty}(\ln a_n)} = e^0 = 1.$$

In Exercises 63–66, find the limit of the sequence using L'Hôpital's Rule.

63. $a_n = \frac{(\ln n)^2}{n}$

SOLUTION

$$\lim_{n\to\infty}\frac{(\ln n)^2}{n} = \lim_{x\to\infty}\frac{(\ln x)^2}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)^2}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{\frac{2\ln x}{x}}{1} = \lim_{x\to\infty}\frac{2\ln x}{x}$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}2\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{\frac{2}{x}}{1} = \lim_{x\to\infty}\frac{2}{x} = 0$$

64. $b_n = \sqrt{n}\ln\left(1 + \frac{1}{n}\right)$

SOLUTION

$$\lim_{n\to\infty}\sqrt{n}\ln\left(1 + \frac{1}{n}\right) = \lim_{x\to\infty}\sqrt{x}\ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1 + \frac{1}{x}\right)}{x^{-1/2}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx}x^{-1/2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\frac{-1}{x^2}\right)}{\frac{-1}{2}x^{-3/2}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}\left(1 + \frac{1}{x}\right)} = 0$$

65. $c_n = n\left(\sqrt{n^2+1} - n\right)$

SOLUTION

$$\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right) = \lim_{x\to\infty} x\left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty}\frac{x\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1} + x} = \lim_{x\to\infty}\frac{\frac{d}{dx}x}{\frac{d}{dx}\sqrt{x^2+1} + x} = \lim_{x\to\infty}\frac{1}{1 + \frac{x}{\sqrt{x^2+1}}}$$

$$= \lim_{x\to\infty}\frac{1}{1 + \sqrt{\frac{x^2}{x^2+1}}} = \lim_{x\to\infty}\frac{1}{1 + \sqrt{\frac{1}{1+(1/x^2)}}} = \frac{1}{2}$$

66. $d_n = n^2\left(\sqrt[3]{n^3+1} - n\right)$

SOLUTION We rewrite d_n as follows:

$$d_n = n^2\left(\sqrt[3]{n^3+1} - n\right) = n^2\left(\sqrt[3]{n^3\left(1 + \frac{1}{n^3}\right)} - n\right) = n^2\left(n\sqrt[3]{1 + \frac{1}{n^3}} - n\right)$$

$$= n^3\left(\sqrt[3]{1 + \frac{1}{n^3}} - 1\right) = \frac{\left((1 + n^{-3})^{1/3} - 1\right)}{n^{-3}}.$$

Thus,

$$\lim_{n\to\infty} d_n = \lim_{x\to\infty} \frac{(1+x^{-3})^{1/3}-1}{x^{-3}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left[(1+x^{-3})^{1/3}-1\right]}{\frac{d}{dx}[x^{-3}]}$$
$$= \lim_{x\to\infty} \frac{\frac{1}{3}(1+x^{-3})^{-2/3}(-3x^{-4})}{-3x^{-4}} = \lim_{x\to\infty} \frac{1}{3}(1+x^{-3})^{-2/3} = \lim_{x\to\infty} \frac{1}{3\left(1+\frac{1}{x^3}\right)^{2/3}} = \frac{1}{3}.$$

In Exercises 67–70, use the Squeeze Theorem to evaluate $\lim_{n\to\infty} a_n$ by verifying the given inequality.

67. $a_n = \dfrac{1}{\sqrt{n^4+n^8}}, \quad \dfrac{1}{\sqrt{2n^4}} \le a_n \le \dfrac{1}{\sqrt{2n^2}}$

SOLUTION For all $n > 1$ we have $n^4 < n^8$, so the quotient $\frac{1}{\sqrt{n^4+n^8}}$ is smaller than $\frac{1}{\sqrt{n^4+n^4}}$ and larger than $\frac{1}{\sqrt{n^8+n^8}}$. That is,

$$a_n < \frac{1}{\sqrt{n^4+n^4}} = \frac{1}{\sqrt{n^4\cdot 2}} = \frac{1}{\sqrt{2n^2}}; \text{ and}$$
$$a_n > \frac{1}{\sqrt{n^8+n^8}} = \frac{1}{\sqrt{2n^8}} = \frac{1}{\sqrt{2n^4}}.$$

Now, since $\lim_{n\to\infty} \frac{1}{\sqrt{2n^4}} = \lim_{n\to\infty} \frac{1}{\sqrt{2n^2}} = 0$, the Squeeze Theorem for Sequences implies that $\lim_{n\to\infty} a_n = 0$.

68. $c_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}$,
$\dfrac{n}{\sqrt{n^2+n}} \le c_n \le \dfrac{n}{\sqrt{n^2+1}}$

SOLUTION Since each of the n terms in the sum defining c_n is not smaller than $\frac{1}{\sqrt{n^2+n}}$ and not larger than $\frac{1}{\sqrt{n^2+1}}$ we obtain the following inequalities:

$$c_n \ge \underbrace{\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}}}_{n \text{ terms}} = n\cdot\frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}};$$
$$c_n \le \underbrace{\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}}_{n \text{ terms}} = n\cdot\frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}.$$

Thus,

$$\frac{n}{\sqrt{n^2+n}} \le c_n \le \frac{n}{\sqrt{n^2+1}}.$$

We now compute the limits of the two sequences:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+1}}{n}} = \lim_{n\to\infty} \frac{1}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1;$$
$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+n}}{n}} = \lim_{n\to\infty} \frac{1}{\frac{\sqrt{n^2+n}}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1.$$

By the Squeeze Theorem we conclude that:

$$\lim_{n\to\infty} c_n = 1.$$

69. $a_n = (2^n+3^n)^{1/n}, \quad 3 \le a_n \le (2\cdot 3^n)^{1/n} = 2^{1/n}\cdot 3$

SOLUTION Clearly $2^n + 3^n \ge 3^n$ for all $n \ge 1$. Therefore:

$$(2^n+3^n)^{1/n} \ge (3^n)^{1/n} = 3.$$

Also $2^n + 3^n \le 3^n + 3^n = 2 \cdot 3^n$, so

$$(2^n + 3^n)^{1/n} \le (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3.$$

Thus,

$$3 \le (2^n + 3^n)^{1/n} \le 2^{1/n} \cdot 3.$$

Because

$$\lim_{n\to\infty} 2^{1/n} \cdot 3 = 3 \lim_{n\to\infty} 2^{1/n} = 3 \cdot 1 = 3$$

and $\lim_{n\to\infty} 3 = 3$, the Squeeze Theorem for Sequences guarantees

$$\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3.$$

70. $a_n = (n + 10^n)^{1/n}, \quad 10 \le a_n \le (2 \cdot 10^n)^{1/n}$

SOLUTION Clearly

$$10^n \le n + 10^n \le 10^n + 10^n = 2 \cdot 10^n$$

for all $n \ge 0$. Thus

$$10 \le (n + 10^n)^{1/n} \le (2 \cdot 10^n)^{1/n}$$

Now,

$$\lim_{n\to\infty} (2 \cdot 10^n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot 10 = 10 \lim_{n\to\infty} 2^{1/n} = 10 \cdot 1 = 10$$

and $\lim_{n\to\infty} 10 = 10$, so that the Squeeze Theorem for Sequences tells us that

$$\lim_{n\to\infty} (n + 10^n)^{1/n} = 10$$

71. Which of the following statements is equivalent to the assertion $\lim\limits_{n\to\infty} a_n = L$? Explain.

(a) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains at least one element of the sequence $\{a_n\}$.

(b) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains all but at most finitely many elements of the sequence $\{a_n\}$.

SOLUTION Statement (b) is equivalent to Definition 1 of the limit, since the assertion "$|a_n - L| < \epsilon$ for all $n > M$" means that $L - \epsilon < a_n < L + \epsilon$ for all $n > M$; that is, the interval $(L - \epsilon, L + \epsilon)$ contains all the elements a_n except (maybe) the finite number of elements $a_1, a_2, \ldots, a_M$.

Statement (a) is not equivalent to the assertion $\lim\limits_{n\to\infty} a_n = L$. We show this, by considering the following sequence:

$$a_n = \begin{cases} \dfrac{1}{n} & \text{for odd } n \\ 1 + \dfrac{1}{n} & \text{for even } n \end{cases}$$

Clearly for every $\epsilon > 0$, the interval $(-\epsilon, \epsilon) = (L - \epsilon, L + \epsilon)$ for $L = 0$ contains at least one element of $\{a_n\}$, but the sequence diverges (rather than converges to $L = 0$). Since the terms in the odd places converge to 0 and the terms in the even places converge to 1. Hence, a_n does not approach one limit.

72. Show that $a_n = \dfrac{1}{2n+1}$ is decreasing.

SOLUTION Let $f(x) = \frac{1}{2x+1}$. Then

$$f'(x) = -\frac{1}{(2x+1)^2} \cdot 2 = \frac{-2}{(2x+1)^2} < 0 \qquad \text{for } x \neq -\frac{1}{2}.$$

Since $f'(x) < 0$ for $x \neq -\frac{1}{2}$, f is decreasing on the interval $x > -\frac{1}{2}$. It follows that $a_n = f(n)$ is also decreasing.

73. Show that $a_n = \dfrac{3n^2}{n^2+2}$ is increasing. Find an upper bound.

SOLUTION Let $f(x) = \frac{3x^2}{x^2+2}$. Then

$$f'(x) = \frac{6x(x^2+2) - 3x^2 \cdot 2x}{(x^2+2)^2} = \frac{12x}{(x^2+2)^2}.$$

$f'(x) > 0$ for $x > 0$, hence f is increasing on this interval. It follows that $a_n = f(n)$ is also increasing. We now show that $M = 3$ is an upper bound for a_n, by writing:

$$a_n = \frac{3n^2}{n^2+2} \le \frac{3n^2+6}{n^2+2} = \frac{3(n^2+2)}{n^2+2} = 3.$$

That is, $a_n \le 3$ for all n.

74. Show that $a_n = \sqrt[3]{n+1} - n$ is decreasing.

SOLUTION Let $f(x) = \sqrt[3]{x+1} - x$. Then

$$f'(x) = \frac{d}{dx}\left((x+1)^{1/3} - x\right) = \frac{1}{3}(x+1)^{-2/3} - 1.$$

For $x \ge 1$,

$$\frac{1}{3}(x+1)^{-2/3} - 1 \le \frac{1}{3}2^{-2/3} - 1 < 0.$$

We conclude that f is decreasing on the interval $x \ge 1$. It follows that $a_n = f(n)$ is also decreasing.

75. Give an example of a divergent sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} |a_n|$ converges.

SOLUTION Let $a_n = (-1)^n$. The sequence $\{a_n\}$ diverges because the terms alternate between $+1$ and -1; however, the sequence $\{|a_n|\}$ converges because it is a constant sequence, all of whose terms are equal to 1.

76. Give an example of *divergent* sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ converges.

SOLUTION Let $a_n = 2^n$ and $b_n = -2^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent geometric sequences. However, since $a_n + b_n = 2^n - 2^n = 0$, the sequence $\{a_n + b_n\}$ is the constant sequence with all the terms equal zero, so it converges to zero.

77. Using the limit definition, prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

SOLUTION We will prove this result by contradiction. Suppose $\lim_{n\to\infty} a_n = L_1$ and that $\{a_n + b_n\}$ converges to a limit L_2. Now, let $\epsilon > 0$. Because $\{a_n\}$ converges to L_1 and $\{a_n + b_n\}$ converges to L_2, it follows that there exist numbers M_1 and M_2 such that:

$$|a_n - L_1| < \frac{\epsilon}{2} \qquad \text{for all } n > M_1,$$
$$|(a_n + b_n) - L_2| < \frac{\epsilon}{2} \qquad \text{for all } n > M_2.$$

Thus, for $n > M = \max\{M_1, M_2\}$,

$$|a_n - L_1| < \frac{\epsilon}{2} \quad \text{and} \quad |(a_n + b_n) - L_2| < \frac{\epsilon}{2}.$$

By the triangle inequality,

$$\begin{aligned} |b_n - (L_2 - L_1)| &= |a_n + b_n - a_n - (L_2 - L_1)| = |(-a_n + L_1) + (a_n + b_n - L_2)| \\ &\le |L_1 - a_n| + |a_n + b_n - L_2|. \end{aligned}$$

Thus, for $n > M$,

$$|b_n - (L_2 - L_1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

that is, $\{b_n\}$ converges to $L_2 - L_1$, in contradiction to the given data. Thus, $\{a_n + b_n\}$ must diverge.

78. Use the limit definition to prove that if $\{a_n\}$ is a convergent sequence of integers with limit L, then there exists a number M such that $a_n = L$ for all $n \geq M$.

SOLUTION Suppose $\{a_n\}$ converges to L, and let $\epsilon = \frac{1}{2}$. Then, there exists a number M such that

$$|a_n - L| < \frac{1}{2}$$

for all $n \geq M$. In other words, for all $n \geq M$,

$$L - \frac{1}{2} < a_n < L + \frac{1}{2}.$$

However, we are given that $\{a_n\}$ is a sequence of integers. Thus, it must be that $a_n = L$ for all $n \geq M$.

79. Theorem 1 states that if $\lim_{x\to\infty} f(x) = L$, then the sequence $a_n = f(n)$ converges and $\lim_{n\to\infty} a_n = L$. Show that the *converse* is false. In other words, find a function $f(x)$ such that $a_n = f(n)$ converges but $\lim_{x\to\infty} f(x)$ does not exist.

SOLUTION Let $f(x) = \sin \pi x$ and $a_n = \sin \pi n$. Then $a_n = f(n)$. Since $\sin \pi x$ is oscillating between -1 and 1 the limit $\lim_{x\to\infty} f(x)$ does not exist. However, the sequence $\{a_n\}$ is the constant sequence in which $a_n = \sin \pi n = 0$ for all n, hence it converges to zero.

80. Use the limit definition to prove that the limit does not change if a finite number of terms are added or removed from a convergent sequence.

SOLUTION Suppose that $\{a_n\}$ is a sequence such that $\lim_{n\to\infty} a_n = L$. For every $\epsilon > 0$, there is a number M such that $|a_n - L| < \epsilon$ for all $n > M$. That is, the inequality $|a_n - L| < \epsilon$ holds for all the terms of $\{a_n\}$ except possibly a finite number of terms. If we add a finite number of terms, these terms may not satisfy the inequality $|a_n - L| < \epsilon$, but there are still only a finite number of terms that do not satisfy this inequality. By removing terms from the sequence, the number of terms in the new sequence that do not satisfy $|a_n - L| < \epsilon$ are no more than in the original sequence. Hence the new sequence also converges to L.

81. Let $b_n = a_{n+1}$. Use the limit definition to prove that if $\{a_n\}$ converges, then $\{b_n\}$ also converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

SOLUTION Suppose $\{a_n\}$ converges to L. Let $b_n = a_{n+1}$, and let $\epsilon > 0$. Because $\{a_n\}$ converges to L, there exists an M' such that $|a_n - L| < \epsilon$ for $n > M'$. Now, let $M = M' - 1$. Then, whenever $n > M$, $n + 1 > M + 1 = M'$. Thus, for $n > M$,

$$|b_n - L| = |a_{n+1} - L| < \epsilon.$$

Hence, $\{b_n\}$ converges to L.

82. Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} |a_n|$ exists and is nonzero. Show that $\lim_{n\to\infty} a_n$ exists if and only if there exists an integer M such that the sign of a_n does not change for $n > M$.

SOLUTION Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} |a_n|$ exists and is nonzero. Suppose $\lim_{n\to\infty} a_n$ exists and let $L = \lim_{n\to\infty} a_n$. Note that L cannot be zero for then $\lim_{n\to\infty} |a_n|$ would also be zero. Now, choose $\epsilon < |L|$. Then there exists an integer M such that $|a_n - L| < \epsilon$, or $L - \epsilon < a_n < L + \epsilon$, for all $n > M$. If $L < 0$, then $-2L < a_n < 0$, whereas if $L > 0$, then $0 < a_n < 2L$; that is, a_n does not change for $n > M$.

Now suppose that there exists an integer M such that a_n does not change for $n > M$. If $a_n > 0$ for $n > M$, then $a_n = |a_n|$ for $n > M$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} |a_n|.$$

On the other hand, if $a_n < 0$ for $n > M$, then $a_n = -|a_n|$ for $n > M$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n|.$$

In either case, $\lim_{n\to\infty} a_n$ exists. Thus, $\lim_{n\to\infty} a_n$ exists if and only if there exists an integer M such that the sign of a_n does not change for $n > M$.

83. Proceed as in Example 12 to show that the sequence $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \dots$ is increasing and bounded above by $M = 3$. Then prove that the limit exists and find its value.

SOLUTION This sequence is defined recursively by the formula:

$$a_{n+1} = \sqrt{3a_n}, \qquad a_1 = \sqrt{3}.$$

Consider the following inequalities:

$$a_2 = \sqrt{3a_1} = \sqrt{3\sqrt{3}} > \sqrt{3} = a_1 \quad \Rightarrow \quad a_2 > a_1;$$
$$a_3 = \sqrt{3a_2} > \sqrt{3a_1} = a_2 \quad \Rightarrow \quad a_3 > a_2;$$
$$a_4 = \sqrt{3a_3} > \sqrt{3a_2} = a_3 \quad \Rightarrow \quad a_4 > a_3.$$

In general, if we assume that $a_k > a_{k-1}$, then

$$a_{k+1} = \sqrt{3a_k} > \sqrt{3a_{k-1}} = a_k.$$

Hence, by mathematical induction, $a_{n+1} > a_n$ for all n; that is, the sequence $\{a_n\}$ is increasing.

Because $a_{n+1} = \sqrt{3a_n}$, it follows that $a_n \geq 0$ for all n. Now, $a_1 = \sqrt{3} < 3$. If $a_k \leq 3$, then

$$a_{k+1} = \sqrt{3a_k} \leq \sqrt{3 \cdot 3} = 3.$$

Thus, by mathematical induction, $a_n \leq 3$ for all n.

Since $\{a_n\}$ is increasing and bounded, it follows by the Theorem on Bounded Monotonic Sequences that this sequence is converging. Denote the limit by $L = \lim_{n\to\infty} a_n$. Using Exercise 81, it follows that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n} = \sqrt{3 \lim_{n\to\infty} a_n} = \sqrt{3L}.$$

Thus, $L^2 = 3L$, so $L = 0$ or $L = 3$. Because the sequence is increasing, we have $a_n \geq a_1 = \sqrt{3}$ for all n. Hence, the limit also satisfies $L \geq \sqrt{3}$. We conclude that the appropriate solution is $L = 3$; that is, $\lim_{n\to\infty} a_n = 3$.

84. Let $\{a_n\}$ be the sequence defined recursively by

$$a_0 = 0, \qquad a_{n+1} = \sqrt{2 + a_n}$$

Thus, $a_1 = \sqrt{2}, \quad a_2 = \sqrt{2+\sqrt{2}}, \quad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots.$

(a) Show that if $a_n < 2$, then $a_{n+1} < 2$. Conclude by induction that $a_n < 2$ for all n.

(b) Show that if $a_n < 2$, then $a_n \leq a_{n+1}$. Conclude by induction that $\{a_n\}$ is increasing.

(c) Use (a) and (b) to conclude that $L = \lim_{n\to\infty} a_n$ exists. Then compute L by showing that $L = \sqrt{2 + L}$.

SOLUTION

(a) Assume $a_n < 2$. Then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2$$

so that $a_{n+1} < 2$. So by induction, $a_n < 2$ for all n and $\{a_n\}$ is bounded above by 2.

(b) Assume $a_n < 2$. Then

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{a_n + a_n} = \sqrt{2a_n} > \sqrt{a_n^2} = a_n$$

so that $a_n < a_{n+1}$. It follows by induction that $\{a_n\}$ is increasing.

(c) Since $\{a_n\}$ is increasing and bounded above, the Theorem on Bounded Monotone Sequences tells us that $L = \lim_{n\to\infty} a_n$ exists. We have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2 + L}$$

by Exercise 81. It follows that $L = \sqrt{2 + L}$, so that $L^2 - L - 2 = 0$. Thus $L = 2$ or $L = -1$. But all terms of $\{a_n\}$ are positive, so we must have $L = 2$.

Further Insights and Challenges

85. Show that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. *Hint:* Verify that $n! \geq (n/2)^{n/2}$ by observing that half of the factors of $n!$ are greater than or equal to $n/2$.

SOLUTION We show that $n! \geq \left(\frac{n}{2}\right)^{n/2}$. For $n \geq 4$ even, we have:

$$n! = \underbrace{1 \cdot \cdots \cdot \frac{n}{2}}_{\frac{n}{2}\text{ factors}} \cdot \underbrace{\left(\frac{n}{2} + 1\right) \cdot \cdots \cdot n}_{\frac{n}{2}\text{ factors}} \geq \underbrace{\left(\frac{n}{2} + 1\right) \cdot \cdots \cdot n}_{\frac{n}{2}\text{ factors}}.$$

Since each one of the $\frac{n}{2}$ factors is greater than $\frac{n}{2}$, we have:

$$n! \geq \underbrace{\left(\frac{n}{2}+1\right)\cdot\cdots\cdot n}_{\frac{n}{2}\text{ factors}} \geq \underbrace{\frac{n}{2}\cdot\cdots\cdot\frac{n}{2}}_{\frac{n}{2}\text{ factors}} = \left(\frac{n}{2}\right)^{n/2}.$$

For $n \geq 3$ odd, we have:

$$n! = \underbrace{1\cdot\cdots\cdot\frac{n-1}{2}}_{\frac{n-1}{2}\text{ factors}}\cdot\underbrace{\frac{n+1}{2}\cdot\cdots\cdot n}_{\frac{n+1}{2}\text{ factors}} \geq \underbrace{\frac{n+1}{2}\cdot\cdots\cdot n}_{\frac{n+1}{2}\text{ factors}}.$$

Since each one of the $\frac{n+1}{2}$ factors is greater than $\frac{n}{2}$, we have:

$$n! \geq \underbrace{\frac{n+1}{2}\cdot\cdots\cdot n}_{\frac{n+1}{2}\text{ factors}} \geq \underbrace{\frac{n}{2}\cdot\cdots\cdot\frac{n}{2}}_{\frac{n+1}{2}\text{ factors}} = \left(\frac{n}{2}\right)^{(n+1)/2} = \left(\frac{n}{2}\right)^{n/2}\sqrt{\frac{n}{2}} \geq \left(\frac{n}{2}\right)^{n/2}.$$

In either case we have $n! \geq \left(\frac{n}{2}\right)^{n/2}$. Thus,

$$\sqrt[n]{n!} \geq \sqrt{\frac{n}{2}}.$$

Since $\lim_{n\to\infty}\sqrt{\frac{n}{2}} = \infty$, it follows that $\lim_{n\to\infty}\sqrt[n]{n!} = \infty$. Thus, the sequence $a_n = \sqrt[n]{n!}$ diverges.

86. Let $b_n = \dfrac{\sqrt[n]{n!}}{n}$.

(a) Show that $\ln b_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\ln\frac{k}{n}$.

(b) Show that $\ln b_n$ converges to $\displaystyle\int_0^1 \ln x\,dx$, and conclude that $b_n \to e^{-1}$.

SOLUTION

(a) Let $b_n = \frac{(n!)^{1/n}}{n}$. Then

$$\ln b_n = \ln(n!)^{1/n} - \ln n = \frac{1}{n}\ln(n!) - \ln n = \frac{\ln(n!) - n\ln n}{n} = \frac{1}{n}\left[\ln(n!) - \ln n^n\right] = \frac{1}{n}\ln\frac{n!}{n^n}$$

$$= \frac{1}{n}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\cdots\cdot\frac{n}{n}\right) = \frac{1}{n}\left(\ln\frac{1}{n} + \ln\frac{2}{n} + \ln\frac{3}{n} + \cdots + \ln\frac{n}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}.$$

(b) By part (a) we have,

$$\lim_{n\to\infty}(\ln b_n) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}.$$

Notice that $\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}$ is the nth right-endpoint approximation of the integral of $\ln x$ over the interval $[0, 1]$. Hence,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n} = \int_0^1 \ln x\,dx.$$

We compute the improper integral using integration by parts, with $u = \ln x$ and $v' = 1$. Then $u' = \frac{1}{x}$, $v = x$ and

$$\int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - \int_0^1 \frac{1}{x}x\,dx = 1\cdot\ln 1 - \lim_{x\to 0+}(x\ln x) - \int_0^1 dx$$

$$= 0 - \lim_{x\to 0+}(x\ln x) - x\Big|_0^1 = -1 - \lim_{x\to 0+}(x\ln x).$$

We compute the remaining limit using L'Hôpital's Rule. This gives:

$$\lim_{x\to 0+}(x\cdot\ln x) = \lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+}(-x) = 0.$$

Thus,

$$\lim_{n\to\infty} \ln b_n = \int_0^1 \ln x\, dx = -1,$$

and

$$\lim_{n\to\infty} b_n = e^{-1}.$$

87. Given positive numbers $a_1 < b_1$, define two sequences recursively by

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}$$

(a) Show that $a_n \le b_n$ for all n (Figure 13).

(b) Show that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing.

(c) Show that $b_{n+1} - a_{n+1} \le \dfrac{b_n - a_n}{2}$.

(d) Prove that both $\{a_n\}$ and $\{b_n\}$ converge and have the same limit. This limit, denoted $\text{AGM}(a_1, b_1)$, is called the **arithmetic-geometric mean** of a_1 and b_1.

(e) Estimate $\text{AGM}(1, \sqrt{2})$ to three decimal places.

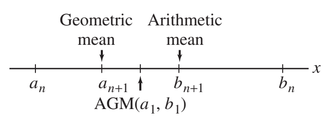


FIGURE 13

SOLUTION

(a) Examine the following:

$$b_{n+1} - a_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{a_n + b_n - 2\sqrt{a_n b_n}}{2} = \frac{\left(\sqrt{a_n}\right)^2 - 2\sqrt{a_n}\sqrt{b_n} + \left(\sqrt{b_n}\right)^2}{2}$$
$$= \frac{\left(\sqrt{a_n} - \sqrt{b_n}\right)^2}{2} \ge 0.$$

We conclude that $b_{n+1} \ge a_{n+1}$ for all $n > 1$. By the given information $b_1 > a_1$; hence, $b_n \ge a_n$ for all n.

(b) By part (a), $b_n \ge a_n$ for all n, so

$$a_{n+1} = \sqrt{a_n b_n} \ge \sqrt{a_n \cdot a_n} = \sqrt{a_n^2} = a_n$$

for all n. Hence, the sequence $\{a_n\}$ is increasing. Moreover, since $a_n \le b_n$ for all n,

$$b_{n+1} = \frac{a_n + b_n}{2} \le \frac{b_n + b_n}{2} = \frac{2b_n}{2} = b_n$$

for all n; that is, the sequence $\{b_n\}$ is decreasing.

(c) Since $\{a_n\}$ is increasing, $a_{n+1} \ge a_n$. Thus,

$$b_{n+1} - a_{n+1} \le b_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n = \frac{a_n + b_n - 2a_n}{2} = \frac{b_n - a_n}{2}.$$

Now, by part (a), $a_n \le b_n$ for all n. By part (b), $\{b_n\}$ is decreasing. Hence $b_n \le b_1$ for all n. Combining the two inequalities we conclude that $a_n \le b_1$ for all n. That is, the sequence $\{a_n\}$ is increasing and bounded ($0 \le a_n \le b_1$). By the Theorem on Bounded Monotonic Sequences we conclude that $\{a_n\}$ converges. Similarly, since $\{a_n\}$ is increasing, $a_n \ge a_1$ for all n. We combine this inequality with $b_n \ge a_n$ to conclude that $b_n \ge a_1$ for all n. Thus, $\{b_n\}$ is decreasing and bounded ($a_1 \le b_n \le b_1$); hence this sequence converges.

To show that $\{a_n\}$ and $\{b_n\}$ converge to the same limit, note that

$$b_n - a_n \le \frac{b_{n-1} - a_{n-1}}{2} \le \frac{b_{n-2} - a_{n-2}}{2^2} \le \cdots \le \frac{b_1 - a_1}{2^{n-1}}.$$

Thus,

$$\lim_{n\to\infty} (b_n - a_n) = (b_1 - a_1) \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0.$$

(d) We have

$$a_{n+1} = \sqrt{a_n b_n}, \quad a_1 = 1; \quad b_{n+1} = \frac{a_n + b_n}{2}, \quad b_1 = \sqrt{2}$$

Computing the values of a_n and b_n until the first three decimal digits are equal in successive terms, we obtain:

$$\begin{aligned}
a_2 &= \sqrt{a_1 b_1} = \sqrt{1 \cdot \sqrt{2}} = 1.1892 \\
b_2 &= \frac{a_1 + b_1}{2} = \frac{1 + \sqrt{2}}{2} = 1.2071 \\
a_3 &= \sqrt{a_2 b_2} = \sqrt{1.1892 \cdot 1.2071} = 1.1981 \\
b_3 &= \frac{a_2 + b_2}{2} = \frac{1.1892 \cdot 1.2071}{2} = 1.1981 \\
a_4 &= \sqrt{a_3 b_3} = 1.1981 \\
b_4 &= \frac{a_3 + b_3}{2} = 1.1981
\end{aligned}$$

Thus,

$$AGM\left(1, \sqrt{2}\right) \approx 1.198.$$

88. Let $c_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

(a) Calculate c_1, c_2, c_3, c_4.

(b) Use a comparison of rectangles with the area under $y = x^{-1}$ over the interval $[n, 2n]$ to prove that

$$\int_n^{2n} \frac{dx}{x} + \frac{1}{2n} \le c_n \le \int_n^{2n} \frac{dx}{x} + \frac{1}{n}$$

(c) Use the Squeeze Theorem to determine $\lim_{n\to\infty} c_n$.

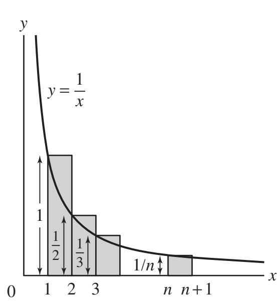
SOLUTION

(a)

$$\begin{aligned}
c_1 &= 1 + \frac{1}{2} = \frac{3}{2}; \\
c_2 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}; \\
c_3 &= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}; \\
c_4 &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{743}{840};
\end{aligned}$$

(b) We consider the left endpoint approximation to the integral of $y = \frac{1}{x}$ over the interval $[n, 2n]$. Since the function $y = \frac{1}{x}$ is decreasing, the left endpoint approximation is greater than $\int_n^{2n} \frac{dx}{x}$; that is,

$$\int_n^{2n} \frac{dx}{x} \le \frac{1}{n} \cdot 1 + \frac{1}{n+1} \cdot 1 + \frac{1}{n+2} \cdot 1 + \cdots + \frac{1}{2n-1} \cdot 1.$$

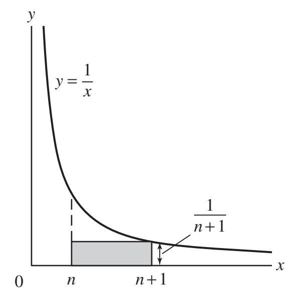


We express the right hand-side of the inequality in terms of c_n, obtaining:

$$\int_n^{2n} \frac{dx}{x} \le c_n - \frac{1}{2n}.$$

We now consider the right endpoint approximation to the integral $\int_n^{2n} \frac{dx}{x}$; that is,

$$\frac{1}{n+1} \cdot 1 + \frac{1}{n+2} \cdot 1 + \cdots + \frac{1}{2n} \cdot 1 \le \int_n^{2n} \frac{dx}{x}.$$



We express the left hand-side of the inequality in terms of c_n, obtaining:

$$c_n - \frac{1}{n} \le \int_n^{2n} \frac{dx}{x}.$$

Thus,

$$\int_n^{2n} \frac{dx}{x} + \frac{1}{2n} \le c_n \le \int_n^{2n} \frac{dx}{x} + \frac{1}{n}.$$

(c) With

$$\int_n^{2n} \frac{dx}{x} = \ln x|_n^{2n} = \ln 2n - \ln n = \ln \frac{2n}{n} = \ln 2,$$

the result from part (b) becomes

$$\ln 2 + \frac{1}{2n} \le c_n \le \ln 2 + \frac{1}{n}.$$

Because

$$\lim_{n\to\infty} \frac{1}{2n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} c_n = \ln 2.$$

89. Let $a_n = H_n - \ln n$, where H_n is the nth harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(a) Show that $a_n \ge 0$ for $n \ge 1$. *Hint:* Show that $H_n \ge \int_1^{n+1} \frac{dx}{x}$.

(b) Show that $\{a_n\}$ is decreasing by interpreting $a_n - a_{n+1}$ as an area.

(c) Prove that $\lim_{n\to\infty} a_n$ exists.

This limit, denoted γ, is known as *Euler's Constant*. It appears in many areas of mathematics, including analysis and number theory, and has been calculated to more than 100 million decimal places, but it is still not known whether γ is an irrational number. The first 10 digits are $\gamma \approx 0.5772156649$.

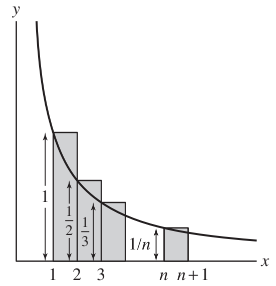
SOLUTION

(a) Since the function $y = \frac{1}{x}$ is decreasing, the left endpoint approximation to the integral $\int_1^{n+1} \frac{dx}{x}$ is greater than this integral; that is,

$$1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{n} \cdot 1 \ge \int_1^{n+1} \frac{dx}{x}$$

or

$$H_n \ge \int_1^{n+1} \frac{dx}{x}.$$



Moreover, since the function $y = \frac{1}{x}$ is positive for $x > 0$, we have:

$$\int_1^{n+1} \frac{dx}{x} \geq \int_1^n \frac{dx}{x}.$$

Thus,

$$H_n \geq \int_1^n \frac{dx}{x} = \ln x \Big|_1^n = \ln n - \ln 1 = \ln n,$$

and

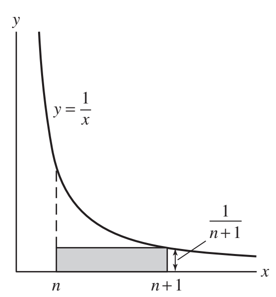
$$a_n = H_n - \ln n \geq 0 \qquad \text{for all } n \geq 1.$$

(b) To show that $\{a_n\}$ is decreasing, we consider the difference $a_n - a_{n+1}$:

$$\begin{aligned} a_n - a_{n+1} &= H_n - \ln n - \big(H_{n+1} - \ln(n+1)\big) = H_n - H_{n+1} + \ln(n+1) - \ln n \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) + \ln(n+1) - \ln n \\ &= -\frac{1}{n+1} + \ln(n+1) - \ln n. \end{aligned}$$

Now, $\ln(n+1) - \ln n = \int_n^{n+1} \frac{dx}{x}$, whereas $\frac{1}{n+1}$ is the right endpoint approximation to the integral $\int_n^{n+1} \frac{dx}{x}$. Recalling $y = \frac{1}{x}$ is decreasing, it follows that

$$\int_n^{n+1} \frac{dx}{x} \geq \frac{1}{n+1}$$



so

$$a_n - a_{n+1} \geq 0.$$

(c) By parts (a) and (b), $\{a_n\}$ is decreasing and 0 is a lower bound for this sequence. Hence $0 \leq a_n \leq a_1$ for all n. A monotonic and bounded sequence is convergent, so $\lim_{n\to\infty} a_n$ exists.

10.2 Summing an Infinite Series

Preliminary Questions

1. What role do partial sums play in defining the sum of an infinite series?

SOLUTION The sum of an infinite series is defined as the limit of the sequence of partial sums. If the limit of this sequence does not exist, the series is said to diverge.

2. What is the sum of the following infinite series?

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$$

SOLUTION This is a geometric series with $c=\frac{1}{4}$ and $r=\frac{1}{2}$. The sum of the series is therefore

$$\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}.$$

3. What happens if you apply the formula for the sum of a geometric series to the following series? Is the formula valid?

$$1+3+3^2+3^3+3^4+\cdots$$

SOLUTION This is a geometric series with $c=1$ and $r=3$. Applying the formula for the sum of a geometric series then gives

$$\sum_{n=0}^{\infty}3^n=\frac{1}{1-3}=-\frac{1}{2}.$$

Clearly, this is not valid: a series with all positive terms cannot have a negative sum. The formula is not valid in this case because a geometric series with $r=3$ diverges.

4. Arvind asserts that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=0$ because $\dfrac{1}{n^2}$ tends to zero. Is this valid reasoning?

SOLUTION Arvind's reasoning is not valid. Though the terms in the series do tend to zero, the general term in the sequence of partial sums,

$$S_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2},$$

is clearly larger than 1. The sum of the series therefore cannot be zero.

5. Colleen claims that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ converges because

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$$

Is this valid reasoning?

SOLUTION Colleen's reasoning is not valid. Although the general term of a convergent series must tend to zero, a series whose general term tends to zero need not converge. In the case of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, the series diverges even though its general term tends to zero.

6. Find an N such that $S_N>25$ for the series $\displaystyle\sum_{n=1}^{\infty}2$.

SOLUTION The Nth partial sum of the series is:

$$S_N=\sum_{n=1}^{N}2=\underbrace{2+\cdots+2}_{N}=2N.$$

7. Does there exist an N such that $S_N>25$ for the series $\displaystyle\sum_{n=1}^{\infty}2^{-n}$? Explain.

SOLUTION The series $\displaystyle\sum_{n=1}^{\infty}2^{-n}$ is a convergent geometric series with the common ratio $r=\dfrac{1}{2}$. The sum of the series is:

$$S=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1.$$

Notice that the sequence of partial sums $\{S_N\}$ is increasing and converges to 1; therefore $S_N\le 1$ for all N. Thus, there does not exist an N such that $S_N>25$.

8. Give an example of a divergent infinite series whose general term tends to zero.

SOLUTION Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{10}}}$. The general term tends to zero, since $\lim_{n\to\infty} \frac{1}{n^{\frac{9}{10}}} = 0$. However, the Nth partial sum satisfies the following inequality:

$$S_N = \frac{1}{1^{\frac{9}{10}}} + \frac{1}{2^{\frac{9}{10}}} + \cdots + \frac{1}{N^{\frac{9}{10}}} \geq \frac{N}{N^{\frac{9}{10}}} = N^{1-\frac{9}{10}} = N^{\frac{1}{10}}.$$

That is, $S_N \geq N^{\frac{1}{10}}$ for all N. Since $\lim_{N\to\infty} N^{\frac{1}{10}} = \infty$, the sequence of partial sums S_n diverges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{10}}}$ diverges.

Exercises

1. Find a formula for the general term a_n (not the partial sum) of the infinite series.

(a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

(b) $\frac{1}{1} + \frac{5}{2} + \frac{25}{4} + \frac{125}{8} + \cdots$

(c) $\frac{1}{1} - \frac{2^2}{2\cdot 1} + \frac{3^3}{3\cdot 2\cdot 1} - \frac{4^4}{4\cdot 3\cdot 2\cdot 1} + \cdots$

(d) $\frac{2}{1^2+1} + \frac{1}{2^2+1} + \frac{2}{3^2+1} + \frac{1}{4^2+1} + \cdots$

SOLUTION

(a) The denominators of the terms are powers of 3, starting with the first power. Hence, the general term is:

$$a_n = \frac{1}{3^n}.$$

(b) The numerators are powers of 5, and the denominators are the same powers of 2. The first term is $a_1 = 1$ so,

$$a_n = \left(\frac{5}{2}\right)^{n-1}.$$

(c) The general term of this series is,

$$a_n = (-1)^{n+1}\frac{n^n}{n!}.$$

(d) Notice that the numerators of a_n equal 2 for odd values of n and 1 for even values of n. Thus,

$$a_n = \begin{cases} \dfrac{2}{n^2+1} & \text{odd } n \\ \dfrac{1}{n^2+1} & \text{even } n \end{cases}$$

The formula can also be rewritten as follows:

$$a_n = \frac{1 + \frac{(-1)^{n+1}+1}{2}}{n^2+1}.$$

2. Write in summation notation:

(a) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

(b) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$

(c) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(d) $\frac{125}{9} + \frac{625}{16} + \frac{3125}{25} + \frac{15{,}625}{36} + \cdots$

SOLUTION

(a) The general term is $a_n = \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$; hence, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(b) The general term is $a_n = \dfrac{1}{n^2}$, $n = 3, 4, 5, \ldots$ or $a_n = \dfrac{1}{(n+2)^2}$, $n = 1, 2, 3, \ldots$; hence, the series is $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$.

(c) The general term is $a_n = \dfrac{(-1)^{n+1}}{2n-1}$, $n = 1, 2, 3, \ldots$; hence, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$.

(d) The general term is $a_n = \dfrac{5^n}{n^2}$, $n = 3, 4, 5, \ldots$ or $a_n = \dfrac{5^{n+2}}{(n+2)^2}$, $n = 1, 2, 3, \ldots$; hence, the series is $\displaystyle\sum_{n=3}^{\infty} \frac{5^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^{n+2}}{(n+2)^2}$.

In Exercises 3–6, compute the partial sums S_2, S_4, and S_6.

3. $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

SOLUTION

$$S_2 = 1 + \frac{1}{2^2} = \frac{5}{4};$$
$$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} = \frac{205}{144};$$
$$S_6 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = \frac{5369}{3600}.$$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^k k^{-1}$

SOLUTION

$$S_2 = (-1)^1 \cdot 1^{-1} + (-1)^2 \cdot 2^{-1} = -1 + \frac{1}{2} = -\frac{1}{2};$$
$$S_4 = (-1)^1 \cdot 1^{-1} + (-1)^2 \cdot 2^{-1} + (-1)^3 \cdot 3^{-1} + (-1)^4 \cdot 4^{-1} = S_2 - \frac{1}{3} + \frac{1}{4} = -\frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12};$$
$$S_6 = -\frac{7}{12} + (-1)^5 \cdot 5^{-1} + (-1)^6 \cdot 6^{-1} = -\frac{7}{12} - \frac{1}{5} + \frac{1}{6} = -\frac{37}{60}.$$

5. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots$

SOLUTION

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3};$$
$$S_4 = S_2 + a_3 + a_4 = \frac{2}{3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{2}{3} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5};$$
$$S_6 = S_4 + a_5 + a_6 = \frac{4}{5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} = \frac{4}{5} + \frac{1}{30} + \frac{1}{42} = \frac{6}{7}.$$

6. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j!}$

SOLUTION

$$S_2 = \frac{1}{1!} + \frac{1}{2!} = 1 + \frac{1}{2} = \frac{3}{2};$$
$$S_4 = S_2 + \frac{1}{3!} + \frac{1}{4!} = \frac{3}{2} + \frac{1}{6} + \frac{1}{24} = \frac{41}{24};$$
$$S_6 = S_4 + \frac{1}{5!} + \frac{1}{6!} = \frac{41}{24} + \frac{1}{120} + \frac{1}{720} = \frac{1237}{720}.$$

7. The series $S = 1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \cdots$ converges to $\frac{5}{4}$. Calculate S_N for $N = 1, 2, \ldots$ until you find an S_N that approximates $\frac{5}{4}$ with an error less than 0.0001.

SOLUTION

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{5} = \frac{6}{5} = 1.2$$

$$S_3 = 1 + \frac{1}{5} + \frac{1}{25} = \frac{31}{25} = 1.24$$

$$S_3 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = \frac{156}{125} = 1.248$$

$$S_4 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} = \frac{781}{625} = 1.2496$$

$$S_5 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} = \frac{3906}{3125} = 1.24992$$

Note that

$$1.25 - S_5 = 1.25 - 1.24992 = 0.00008 < 0.0001$$

8. The series $S = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$ is known to converge to e^{-1} (recall that $0! = 1$). Calculate S_N for $N = 1, 2, \ldots$ until you find an S_N that approximates e^{-1} with an error less than 0.001.

SOLUTION The general term of the series is

$$a_n = \frac{(-1)^{n-1}}{(n-1)!};$$

thus, the Nth partial sum of the series is

$$S_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{N-1}}{(N-1)!}.$$

Using a calculator we find $e^{-1} = 0.367879$. Working sequentially, we find

$$S_1 = \frac{1}{0!} = 1$$

$$S_2 = S_1 + a_2 = 1 - \frac{1}{1!} = 0$$

$$S_3 = S_2 + a_3 = 0 + \frac{1}{2!} = \frac{1}{2} = 0.5$$

$$S_4 = S_3 + a_4 = 0.5 - \frac{1}{3!} = 0.333333$$

$$S_5 = S_4 + a_5 = 0.333333 + \frac{1}{4!} = 0.375$$

$$S_6 = S_5 + a_6 = 0.375 - \frac{1}{5!} = 0.366667$$

$$S_7 = S_6 + a_7 = 0.366667 + \frac{1}{6!} = 0.368056$$

Note that

$$|S_7 - e^{-1}| = 1.76 \times 10^{-4} < 10^{-3}.$$

In Exercises 9 and 10, use a computer algebra system to compute S_{10}, S_{100}, S_{500}, and S_{1000} for the series. Do these values suggest convergence to the given value?

9. CAS

$$\frac{\pi - 3}{4} = \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \frac{1}{8 \cdot 9 \cdot 10} + \cdots$$

SOLUTION Write

$$a_n = \frac{(-1)^{n+1}}{2n \cdot (2n+1) \cdot (2n+2)}$$

Then

$$S_N = \sum_{i=1}^{N} a_n$$

Computing, we find

$$\begin{aligned}
\frac{\pi - 3}{4} &\approx 0.0353981635 \\
S_{10} &\approx 0.03535167962 \\
S_{100} &\approx 0.03539810274 \\
S_{500} &\approx 0.03539816290 \\
S_{1000} &\approx 0.03539816334
\end{aligned}$$

It appears that $S_N \to \frac{\pi-3}{4}$.

10. CAS

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

SOLUTION Write

$$S_N = \sum_{i=1}^{N} \frac{1}{i^4}$$

Computing, we find

$$\begin{aligned}
\frac{\pi^4}{90} &\approx 1.082323234 \\
S(10) &\approx 1.082036583 \\
S(100) &\approx 1.082322905 \\
S(500) &\approx 1.082323231 \\
S(1000) &\approx 1.082323233
\end{aligned}$$

It appears that $S_N \to \frac{\pi^4}{90}$.

11. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

SOLUTION

$$\begin{aligned}
S_3 &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}; \\
S_4 &= S_3 + \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}; \\
S_5 &= S_4 + \left(\frac{1}{6} - \frac{1}{7}\right) = \frac{1}{2} - \frac{1}{7} = \frac{5}{14}.
\end{aligned}$$

The general term in the sequence of partial sums is

$$S_N = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{1}{2} - \frac{1}{N+2};$$

thus,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{1}{2}.$$

The sum of the telescoping series is therefore $\frac{1}{2}$.

12. Write $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$ as a telescoping series and find its sum.

SOLUTION By partial fraction decomposition

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n},$$

so

$$\sum_{n=3}^{\infty} \frac{1}{n(n-1)} = \sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right).$$

The general term in the sequence of partial sums for this series is

$$S_N = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) = \frac{1}{2} - \frac{1}{N};$$

thus,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N}\right) = \frac{1}{2}.$$

13. Calculate S_3, S_4, and S_5 and then find the sum $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ using the identity

$$\frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

SOLUTION

$$S_3 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{2}\left(1 - \frac{1}{7}\right) = \frac{3}{7};$$

$$S_4 = S_3 + \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right) = \frac{1}{2}\left(1 - \frac{1}{9}\right) = \frac{4}{9};$$

$$S_5 = S_4 + \frac{1}{2}\left(\frac{1}{9} - \frac{1}{11}\right) = \frac{1}{2}\left(1 - \frac{1}{11}\right) = \frac{5}{11}.$$

The general term in the sequence of partial sums is

$$S_N = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2N-1} - \frac{1}{2N+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2N+1}\right);$$

thus,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{1}{2}.$$

14. Use partial fractions to rewrite $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ as a telescoping series and find its sum.

SOLUTION By partial fraction decomposition

$$\frac{1}{n\,(n+3)} = \frac{A}{n} + \frac{B}{n+3};$$

clearing denominators gives

$$1 = A\,(n+3) + Bn.$$

Setting $n = 0$ yields $A = \frac{1}{3}$, while setting $n = -3$ yields $B = -\frac{1}{3}$. Thus,

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right).$$

The general term in the sequence of partial sums for the series on the right-hand side is

$$\begin{aligned} S_N &= \frac{1}{3}\left(1-\frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{6}-\frac{1}{9}\right) \\ &\quad + \cdots + \frac{1}{3}\left(\frac{1}{N-2}-\frac{1}{N+1}\right) + \frac{1}{3}\left(\frac{1}{N-1}-\frac{1}{N+2}\right) + \frac{1}{3}\left(\frac{1}{N}-\frac{1}{N+3}\right) \\ &= \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right) = \frac{11}{18} - \frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right). \end{aligned}$$

Thus,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left[\frac{11}{18} - \frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right)\right] = \frac{11}{18},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18}.$$

15. Find the sum of $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$.

SOLUTION We may write this sum as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

The general term in the sequence of partial sums is

$$S_N = \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2N-1}-\frac{1}{2N+1}\right) = \frac{1}{2}\left(1-\frac{1}{2N+1}\right);$$

thus,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2}\left(1-\frac{1}{2N+1}\right) = \frac{1}{2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

16. Find a formula for the partial sum S_N of $\sum_{n=1}^{\infty} (-1)^{n-1}$ and show that the series diverges.

SOLUTION The partial sums of the series are:

$$\begin{aligned} S_1 &= (-1)^{1-1} = 1; \\ S_2 &= (-1)^0 + (-1)^1 = 1 - 1 = 0; \\ S_3 &= (-1)^0 + (-1)^1 + (-1)^2 = 1; \\ S_4 &= (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 0; \cdots \end{aligned}$$

In general,

$$S_N = \begin{cases} 1 & \text{if } N \text{ odd} \\ 0 & \text{if } N \text{ even} \end{cases}$$

Because the values of S_N alternate between 0 and 1, the sequence of partial sums diverges; this, in turn, implies that the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

In Exercises 17–22, use Theorem 3 to prove that the following series diverge.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n+12}$

SOLUTION The general term, $\dfrac{n}{10n+12}$, has limit

$$\lim_{n\to\infty} \frac{n}{10n+12} = \lim_{n\to\infty} \frac{1}{10+(12/n)} = \frac{1}{10}$$

Since the general term does not tend to zero, the series diverges.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

SOLUTION The general term, $\dfrac{n}{\sqrt{n^2+1}}$, has limit

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+(1/n^2)}} = 1$$

Since the general term does not tend to zero, the series diverges.

19. $\dfrac{0}{1} - \dfrac{1}{2} + \dfrac{2}{3} - \dfrac{3}{4} + \cdots$

SOLUTION The general term $a_n = (-1)^{n-1}\frac{n-1}{n}$ does not tend to zero. In fact, because $\lim_{n\to\infty} \frac{n-1}{n} = 1$, $\lim_{n\to\infty} a_n$ does not exist. By Theorem 3, we conclude that the given series diverges.

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2$

SOLUTION The general term $a_n = (-1)^n n^2$ does not tend to zero. In fact, because $\lim_{n\to\infty} n^2 = \infty$, $\lim_{n\to\infty} a_n$ does not exist. By Theorem 3, we conclude that the given series diverges.

21. $\cos\dfrac{1}{2} + \cos\dfrac{1}{3} + \cos\dfrac{1}{4} + \cdots$

SOLUTION The general term $a_n = \cos\frac{1}{n+1}$ tends to 1, not zero. By Theorem 3, we conclude that the given series diverges.

22. $\displaystyle\sum_{n=0}^{\infty} \left(\sqrt{4n^2+1} - n\right)$

SOLUTION The general term of the series satisfies

$$\sqrt{4n^2+1} - n > \sqrt{4n^2} - n = n$$

Thus the general term tends to infinity. The series diverges by Theorem 2.

In Exercises 23–36, use the formula for the sum of a geometric series to find the sum or state that the series diverges.

23. $\dfrac{1}{1} + \dfrac{1}{8} + \dfrac{1}{8^2} + \cdots$

SOLUTION This is a geometric series with $c = 1$ and $r = \frac{1}{8}$, so its sum is

$$\frac{1}{1-\frac{1}{8}} = \frac{1}{7/8} = \frac{8}{7}$$

24. $\dfrac{4^3}{5^3} + \dfrac{4^4}{5^4} + \dfrac{4^5}{5^5} + \cdots$

SOLUTION This is a geometric series with

$$c = \frac{4^3}{5^3} \quad \text{and} \quad r = \frac{4}{5}$$

so its sum is

$$\frac{c}{1-r} = \frac{4^3/5^3}{1-\frac{4}{5}} = \frac{4^3}{5^3 - 4\cdot 5^2} = \frac{64}{25}$$

25. $\displaystyle\sum_{n=3}^{\infty}\left(\frac{3}{11}\right)^{-n}$

SOLUTION Rewrite this series as

$$\sum_{n=3}^{\infty}\left(\frac{11}{3}\right)^{n}$$

This is a geometric series with $r = \dfrac{11}{3} > 1$, so it is divergent.

26. $\displaystyle\sum_{n=2}^{\infty}\frac{7\cdot(-3)^n}{5^n}$

SOLUTION This is a geometric series with $c = 7$ and $r = -\dfrac{3}{5}$, starting at $n = 2$. Its sum is thus

$$\frac{cr^2}{1-r} = \frac{7\cdot(9/25)}{1-\frac{3}{5}} = \frac{63}{25}\cdot\frac{5}{8} = \frac{63}{40}$$

27. $\displaystyle\sum_{n=-4}^{\infty}\left(-\frac{4}{9}\right)^{n}$

SOLUTION This is a geometric series with $c = 1$ and $r = -\dfrac{4}{9}$, starting at $n = -4$. Its sum is thus

$$\frac{cr^{-4}}{1-r} = \frac{c}{r^4-r^5} = \frac{1}{\frac{4^4}{9^4}+\frac{4^5}{9^5}} = \frac{9^5}{9\cdot 4^4+4^5} = \frac{59{,}049}{3328}$$

28. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\pi}{e}\right)^{n}$

SOLUTION Since $\pi > e$, this is a geometric series with $r > 1$, so it diverges.

29. $\displaystyle\sum_{n=1}^{\infty}e^{-n}$

SOLUTION Rewrite the series as

$$\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{n}$$

to recognize it as a geometric series with $c = \frac{1}{e}$ and $r = \frac{1}{e}$. Thus,

$$\sum_{n=1}^{\infty}e^{-n} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}.$$

30. $\displaystyle\sum_{n=2}^{\infty}e^{3-2n}$

SOLUTION Rewrite the series as

$$\sum_{n=2}^{\infty}e^3e^{-2n} = \sum_{n=2}^{\infty}e^3\left(\frac{1}{e^2}\right)^{n}$$

to recognize it as a geometric series with $c = e^3\left(\frac{1}{e^2}\right)^2 = \frac{1}{e}$ and $r = \frac{1}{e^2}$. Thus,

$$\sum_{n=2}^{\infty}e^{3-2n} = \frac{\frac{1}{e}}{1-\frac{1}{e^2}} = \frac{e}{e^2-1}.$$

31. $\displaystyle\sum_{n=0}^{\infty}\frac{8+2^n}{5^n}$

SOLUTION Rewrite the series as

$$\sum_{n=0}^{\infty}\frac{8}{5^n}+\sum_{n=0}^{\infty}\frac{2^n}{5^n} = \sum_{n=0}^{\infty}8\cdot\left(\frac{1}{5}\right)^{n}+\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^{n},$$

which is a sum of two geometric series. The first series has $c = 8\left(\frac{1}{5}\right)^0 = 8$ and $r = \frac{1}{5}$; the second has $c = \left(\frac{2}{5}\right)^0 = 1$ and $r = \frac{2}{5}$. Thus,

$$\sum_{n=0}^{\infty} 8 \cdot \left(\frac{1}{5}\right)^n = \frac{8}{1-\frac{1}{5}} = \frac{8}{\frac{4}{5}} = 10,$$

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3},$$

and

$$\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} = 10 + \frac{5}{3} = \frac{35}{3}.$$

32. $\displaystyle\sum_{n=0}^{\infty} \frac{3(-2)^n - 5^n}{8^n}$

SOLUTION Rewrite the series as

$$\sum_{n=0}^{\infty} \frac{3(-2)^n - 5^n}{8^n} = \sum_{n=0}^{\infty} \frac{3(-2)^n}{8^n} - \sum_{n=0}^{\infty} \frac{5^n}{8^n}$$

which is a difference of two geometric series. The first has $c = 3$ and $r = -\frac{1}{4}$; the second has $c = 1$ and $r = \frac{5}{8}$. Thus

$$\sum_{n=0}^{\infty} \frac{3(-2)^n}{8^n} = \frac{3}{1+\frac{1}{4}} = \frac{12}{5}$$

$$\sum_{n=0}^{\infty} \frac{5^n}{8^n} = \frac{1}{1-\frac{5}{8}} = \frac{8}{3}$$

so that

$$\sum_{n=0}^{\infty} \frac{3(-2)^n - 5^n}{8^n} = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15}$$

33. $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots$

SOLUTION This is a geometric series with $c = 5$ and $r = -\frac{1}{4}$. Thus,

$$\sum_{n=0}^{\infty} 5 \cdot \left(-\frac{1}{4}\right)^n = \frac{5}{1-\left(-\frac{1}{4}\right)} = \frac{5}{1+\frac{1}{4}} = \frac{5}{\frac{5}{4}} = 4.$$

34. $\frac{2^3}{7} + \frac{2^4}{7^2} + \frac{2^5}{7^3} + \frac{2^6}{7^4} + \cdots$

SOLUTION This is a geometric series with $c = \frac{8}{7}$ and $r = \frac{2}{7}$. Thus,

$$\sum_{n=0}^{\infty} \frac{8}{7} \cdot \left(\frac{2}{7}\right)^n = \frac{\frac{8}{7}}{1-\frac{2}{7}} = \frac{\frac{8}{7}}{\frac{5}{7}} = \frac{8}{5}.$$

35. $\frac{7}{8} - \frac{49}{64} + \frac{343}{512} - \frac{2401}{4096} + \cdots$

SOLUTION This is a geometric series with $c = \frac{7}{8}$ and $r = -\frac{7}{8}$. Thus,

$$\sum_{n=0}^{\infty} \frac{7}{8} \cdot \left(-\frac{7}{8}\right)^n = \frac{\frac{7}{8}}{1-\left(-\frac{7}{8}\right)} = \frac{\frac{7}{8}}{\frac{15}{8}} = \frac{7}{15}.$$

36. $\frac{25}{9}+\frac{5}{3}+1+\frac{3}{5}+\frac{9}{25}+\frac{27}{125}+\cdots$

SOLUTION This appears to be a geometric series with

$$c=\frac{25}{9}\quad\text{and}\quad r=\frac{3}{5}$$

so its sum is

$$\frac{c}{1-r}=\frac{25/9}{1-\frac{3}{5}}=\frac{25}{9}\cdot\frac{5}{2}=\frac{125}{18}$$

37. Which of the following are *not* geometric series?

(a) $\sum_{n=0}^{\infty}\frac{7^n}{29^n}$

(b) $\sum_{n=3}^{\infty}\frac{1}{n^4}$

(c) $\sum_{n=0}^{\infty}\frac{n^2}{2^n}$

(d) $\sum_{n=5}^{\infty}\pi^{-n}$

SOLUTION

(a) $\sum_{n=0}^{\infty}\frac{7^n}{29^n}=\sum_{n=0}^{\infty}\left(\frac{7}{29}\right)^n$: this is a geometric series with common ratio $r=\frac{7}{29}$.

(b) The ratio between two successive terms is

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{(n+1)^4}}{\frac{1}{n^4}}=\frac{n^4}{(n+1)^4}=\left(\frac{n}{n+1}\right)^4.$$

This ratio is not constant since it depends on n. Hence, the series $\sum_{n=3}^{\infty}\frac{1}{n^4}$ is not a geometric series.

(c) The ratio between two successive terms is

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}=\frac{(n+1)^2}{n^2}\cdot\frac{2^n}{2^{n+1}}=\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2}.$$

This ratio is not constant since it depends on n. Hence, the series $\sum_{n=0}^{\infty}\frac{n^2}{2^n}$ is not a geometric series.

(d) $\sum_{n=5}^{\infty}\pi^{-n}=\sum_{n=5}^{\infty}\left(\frac{1}{\pi}\right)^n$: this is a geometric series with common ratio $r=\frac{1}{\pi}$.

38. Use the method of Example 8 to show that $\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ diverges.

SOLUTION Each term in the Nth partial sum is greater than or equal to $\frac{1}{N^{\frac{1}{3}}}$, hence:

$$S_N=\frac{1}{1^{1/3}}+\frac{1}{2^{1/3}}+\frac{3}{3^{1/3}}+\cdots+\frac{1}{N^{1/3}}\geq\frac{1}{N^{1/3}}+\frac{1}{N^{1/3}}+\frac{1}{N^{1/3}}+\cdots+\frac{1}{N^{1/3}}=N\cdot\frac{1}{N^{1/3}}=N^{2/3}.$$

Since $\lim_{N\to\infty}N^{2/3}=\infty$, it follows that

$$\lim_{N\to\infty}S_N=\infty.$$

Thus, the series $\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ diverges.

39. Prove that if $\sum_{n=1}^{\infty}a_n$ converges and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n+b_n)$ diverges. *Hint:* If not, derive a contradiction by writing

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}(a_n+b_n)-\sum_{n=1}^{\infty}a_n$$

SOLUTION Suppose to the contrary that $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ diverges, but $\sum_{n=1}^{\infty}(a_n + b_n)$ converges. Then by the Linearity of Infinite Series, we have

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n + b_n) - \sum_{n=1}^{\infty} a_n$$

so that $\sum_{n=1}^{\infty} b_n$ converges, a contradiction.

40. Prove the divergence of $\displaystyle\sum_{n=0}^{\infty} \frac{9^n + 2^n}{5^n}$.

SOLUTION Note that this is the sum of two infinite series:

$$\sum_{n=0}^{\infty} \frac{9^n + 2^n}{5^n} = \sum_{n=0}^{\infty} \frac{9^n}{5^n} + \sum_{n=0}^{\infty} \frac{2^n}{5^n}$$

The first of these is a geometric series with $r = \frac{9}{5} > 1$, so diverges, while the second is a geometric series with $r = \frac{2}{5} < 1$, so converges. By the previous exercise, the sum of the two also diverges.

41. Give a counterexample to show that each of the following statements is false.

(a) If the general term a_n tends to zero, then $\displaystyle\sum_{n=1}^{\infty} a_n = 0$.

(b) The Nth partial sum of the infinite series defined by $\{a_n\}$ is a_N.

(c) If a_n tends to zero, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(d) If a_n tends to L, then $\displaystyle\sum_{n=1}^{\infty} a_n = L$.

SOLUTION

(a) Let $a_n = 2^{-n}$. Then $\lim_{n\to\infty} a_n = 0$, but a_n is a geometric series with $c = 2^0 = 1$ and $r = 1/2$, so its sum is $\dfrac{1}{1 - (1/2)} = 2$.

(b) Let $a_n = 1$. Then the n^{th} partial sum is $a_1 + a_2 + \cdots + a_n = n$ while $a_n = 1$.

(c) Let $a_n = \dfrac{1}{\sqrt{n}}$. An example in the text shows that while a_n tends to zero, the sum $\sum_{n=1}^{\infty} a_n$ does not converge.

(d) Let $a_n = 1$. Then clearly a_n tends to $L = 1$, while the series $\sum_{n=1}^{\infty} a_n$ obviously diverges.

42. Suppose that $S = \displaystyle\sum_{n=1}^{\infty} a_n$ is an infinite series with partial sum $S_N = 5 - \dfrac{2}{N^2}$.

(a) What are the values of $\displaystyle\sum_{n=1}^{10} a_n$ and $\displaystyle\sum_{n=5}^{16} a_n$?

(b) What is the value of a_3?

(c) Find a general formula for a_n.

(d) Find the sum $\displaystyle\sum_{n=1}^{\infty} a_n$.

SOLUTION

(a)

$$\sum_{n=1}^{10} a_n = S_{10} = 5 - \frac{2}{10^2} = \frac{249}{50};$$

$$\sum_{n=5}^{16} a_n = (a_1 + \cdots + a_{16}) - (a_1 + a_2 + a_3 + a_4) = S_{16} - S_4 = \left(5 - \frac{2}{16^2}\right) - \left(5 - \frac{2}{4^2}\right) = \frac{2}{16} - \frac{2}{256} = \frac{15}{128}.$$

(b)

$$a_3 = (a_1 + a_2 + a_3) - (a_1 + a_2) = S_3 - S_2 = \left(5 - \frac{2}{3^2}\right) - \left(5 - \frac{2}{2^2}\right) = \frac{1}{2} - \frac{2}{9} = \frac{5}{18}.$$

(c) Since $a_n = S_n - S_{n-1}$, we have:

$$a_n = S_n - S_{n-1} = \left(5 - \frac{2}{n^2}\right) - \left(5 - \frac{2}{(n-1)^2}\right) = \frac{2}{(n-1)^2} - \frac{2}{n^2}$$
$$= \frac{2\left(n^2 - (n-1)^2\right)}{(n(n-1))^2} = \frac{2\left(n^2 - n^2 + 2n - 1\right)}{(n(n-1))^2} = \frac{2(2n-1)}{n^2(n-1)^2}.$$

(d) The sum $\sum_{n=1}^{\infty} a_n$ is the limit of the sequence of partial sums $\{S_N\}$. Hence:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(5 - \frac{2}{N^2}\right) = 5.$$

43. Compute the total area of the (infinitely many) triangles in Figure 4.

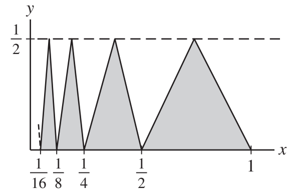


FIGURE 4

SOLUTION The area of a triangle with base B and height H is $A = \frac{1}{2}BH$. Because all of the triangles in Figure 4 have height $\frac{1}{2}$, the area of each triangle equals one-quarter of the base. Now, for $n \geq 0$, the nth triangle has a base which extends from $x = \frac{1}{2^{n+1}}$ to $x = \frac{1}{2^n}$. Thus,

$$B = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \quad \text{and} \quad A = \frac{1}{4}B = \frac{1}{2^{n+3}}.$$

The total area of the triangles is then given by the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+3}} = \sum_{n=0}^{\infty} \frac{1}{8}\left(\frac{1}{2}\right)^n = \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \frac{1}{4}.$$

44. The winner of a lottery receives m dollars at the end of each year for N years. The present value (PV) of this prize in today's dollars is $\text{PV} = \sum_{i=1}^{N} m(1+r)^{-i}$, where r is the interest rate. Calculate PV if $m = \$50{,}000$, $r = 0.06$, and $N = 20$. What is PV if $N = \infty$?

SOLUTION For the given values r, m and N, we have

$$PV = \sum_{i=1}^{20} 50{,}000(1+0.06)^{-i} = \sum_{i=1}^{20} 50{,}000\left(\frac{50}{53}\right)^i = 50{,}000\frac{1 - \left(\frac{50}{53}\right)^{21}}{1 - \frac{50}{53}} = \$623{,}496.06.$$

If we extend the payments forever, then $N = \infty$ and

$$PV = \sum_{i=1}^{\infty} 50{,}000(1+0.06)^{-i} = \sum_{i=1}^{\infty} 50{,}000\left(\frac{50}{53}\right)^i = \frac{50{,}000\left(\frac{50}{53}\right)}{1 - \frac{50}{53}} = \$833{,}333.33.$$

45. Find the total length of the infinite zigzag path in Figure 5 (each zag occurs at an angle of $\frac{\pi}{4}$).

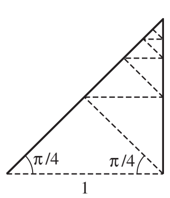


FIGURE 5

SOLUTION Because the angle at the lower left in Figure 5 has measure $\frac{\pi}{4}$ and each zag in the path occurs at an angle of $\frac{\pi}{4}$, every triangle in the figure is an isosceles right triangle. Accordingly, the length of each new segment in the path is $\frac{1}{\sqrt{2}}$ times the length of the previous segment. Since the first segment has length 1, the total length of the path is

$$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2+\sqrt{2}.$$

46. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$. *Hint:* Find constants A, B, and C such that

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

SOLUTION By partial fraction decomposition

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2};$$

clearing denominators then gives

$$1 = A\,(n+1)\,(n+2) + Bn\,(n+2) + Cn\,(n+1)\,.$$

Setting $n=0$ now yields $A=\frac{1}{2}$, while setting $n=-1$ yields $B=-1$ and setting $n=-2$ yields $C=\frac{1}{2}$. Thus,

$$\frac{1}{n\,(n+1)\,(n+2)} = \frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right).$$

The general term of the sequence of partial sums for the series on the right-hand side is

$$\begin{aligned} S_N &= \frac{1}{2}\left(1-\frac{2}{2}+\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4}+\frac{2}{5}+\frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{5}-\frac{2}{6}+\frac{1}{7}\right) \\ &\quad + \cdots + \frac{1}{2}\left(\frac{1}{N-2}-\frac{2}{N-1}+\frac{1}{N}\right) + \frac{1}{2}\left(\frac{1}{N-1}-\frac{2}{N}+\frac{1}{N+1}\right) + \frac{1}{2}\left(\frac{1}{N}-\frac{2}{N+1}+\frac{1}{N+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{N+1}+\frac{1}{N+2}\right). \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N+1}+\frac{1}{N+2}\right) = \frac{1}{4}.$$

47. Show that if a is a positive integer, then

$$\sum_{n=1}^{\infty}\frac{1}{n(n+a)} = \frac{1}{a}\left(1+\frac{1}{2}+\cdots+\frac{1}{a}\right)$$

SOLUTION By partial fraction decomposition

$$\frac{1}{n\,(n+a)} = \frac{A}{n} + \frac{B}{n+a};$$

clearing the denominators gives

$$1 = A(n+a) + Bn.$$

Setting $n=0$ then yields $A=\frac{1}{a}$, while setting $n=-a$ yields $B=-\frac{1}{a}$. Thus,

$$\frac{1}{n\,(n+a)} = \frac{\frac{1}{a}}{n} - \frac{\frac{1}{a}}{n+a} = \frac{1}{a}\left(\frac{1}{n}-\frac{1}{n+a}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \sum_{n=1}^{\infty} \frac{1}{a}\left(\frac{1}{n} - \frac{1}{n+a}\right).$$

For $N > a$, the Nth partial sum is

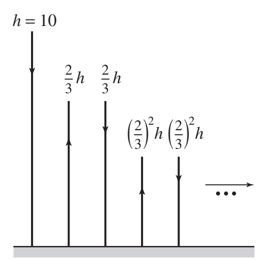
$$S_N = \frac{1}{a}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a}\right) - \frac{1}{a}\left(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{N+a}\right).$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \lim_{N\to\infty} S_N = \frac{1}{a}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a}\right).$$

48. A ball dropped from a height of 10 ft begins to bounce. Each time it strikes the ground, it returns to two-thirds of its previous height. What is the total distance traveled by the ball if it bounces infinitely many times?

SOLUTION The distance traveled by the ball is shown in the accompanying figure:



The total distance d traveled by the ball is given by the following infinite sum:

$$d = h + 2\cdot\frac{2}{3}h + 2\cdot\left(\frac{2}{3}\right)^2 h + 2\cdot\left(\frac{2}{3}\right)^3 h + \cdots = h + 2h\left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right) = h + 2h\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n.$$

We use the formula for the sum of a geometric series to compute the sum of the resulting series:

$$d = h + 2h \cdot \frac{\left(\frac{2}{3}\right)^1}{1 - \frac{2}{3}} = h + 2h(2) = 5h.$$

With $h = 10$ feet, it follows that the total distance traveled by the ball is 50 feet.

49. Let $\{b_n\}$ be a sequence and let $a_n = b_n - b_{n-1}$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

SOLUTION Let $a_n = b_n - b_{n-1}$. The general term in the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is then

$$S_N = (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_N - b_{N-1}) = b_N - b_0.$$

Now, if $\lim_{N\to\infty} b_N$ exists, then so does $\lim_{N\to\infty} S_N$ and $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} S_N$ exists, which implies that $\lim_{N\to\infty} b_N$ also exists. Thus, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

50. Assumptions Matter Show, by giving counterexamples, that the assertions of Theorem 1 are not valid if the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are not convergent.

SOLUTION Let $a_n = 2^{-n} - 2^n$ and $b_n = 2^n$. Then, both

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n$$

diverge, so the sum

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

is not defined. However,

$$\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty}((2^{-n} - 2^n) + 2^n) = \sum_{n=0}^{\infty} 2^{-n} = 1.$$

Further Insights and Challenges

Exercises 51–53 use the formula

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1 - r^N}{1 - r} \qquad \boxed{7}$$

51. Professor George Andrews of Pennsylvania State University observed that we can use Eq. (7) to calculate the derivative of $f(x) = x^N$ (for $N \geq 0$). Assume that $a \neq 0$ and let $x = ra$. Show that

$$f'(a) = \lim_{x \to a} \frac{x^N - a^N}{x - a} = a^{N-1} \lim_{r \to 1} \frac{r^N - 1}{r - 1}$$

and evaluate the limit.

SOLUTION According to the definition of derivative of $f(x)$ at $x = a$

$$f'(a) = \lim_{x \to a} \frac{x^N - a^N}{x - a}.$$

Now, let $x = ra$. Then $x \to a$ if and only if $r \to 1$, and

$$f'(a) = \lim_{x \to a} \frac{x^N - a^N}{x - a} = \lim_{r \to 1} \frac{(ra)^N - a^N}{ra - a} = \lim_{r \to 1} \frac{a^N\left(r^N - 1\right)}{a(r - 1)} = a^{N-1} \lim_{r \to 1} \frac{r^N - 1}{r - 1}.$$

By Eq. (7) for a geometric sum,

$$\frac{1 - r^N}{1 - r} = \frac{r^N - 1}{r - 1} = 1 + r + r^2 + \cdots + r^{N-1},$$

so

$$\lim_{r \to 1} \frac{r^N - 1}{r - 1} = \lim_{r \to 1}\left(1 + r + r^2 + \cdots + r^{N-1}\right) = 1 + 1 + 1^2 + \cdots + 1^{N-1} = N.$$

Therefore, $f'(a) = a^{N-1} \cdot N = Na^{N-1}$

52. Pierre de Fermat used geometric series to compute the area under the graph of $f(x) = x^N$ over $[0, A]$. For $0 < r < 1$, let $F(r)$ be the sum of the areas of the infinitely many right-endpoint rectangles with endpoints Ar^n, as in Figure 6. As r tends to 1, the rectangles become narrower and $F(r)$ tends to the area under the graph.

(a) Show that $F(r) = A^{N+1}\dfrac{1 - r}{1 - r^{N+1}}$.

(b) Use Eq. (7) to evaluate $\displaystyle\int_0^A x^N\,dx = \lim_{r \to 1} F(r)$.

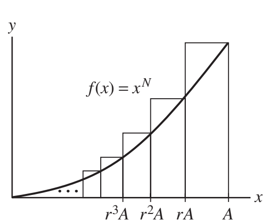


FIGURE 6

SOLUTION

(a) The area of the rectangle whose base extends from $x = r^n A$ to $x = r^{n-1}A$ is

$$(r^{n-1}A)^N(r^{n-1}A - r^n A).$$

Hence, $F(r)$ is the sum

$$F(r) = \sum_{n=1}^{\infty} \left(r^{n-1}A\right)^N \left(r^{n-1}A - r^n A\right) = \sum_{n=1}^{\infty} r^{(n-1)N} r^{n-1}(1-r)A^{N+1} = A^{N+1}(1-r)\sum_{n=1}^{\infty} r^{nN-N+n-1}$$

$$= \frac{A^{N+1}(1-r)}{r^{N+1}} \sum_{n=1}^{\infty} \left(r^{N+1}\right)^n = \frac{A^{N+1}(1-r)}{r^{N+1}} \cdot \frac{r^{N+1}}{1-r^{N+1}} = A^{N+1}\frac{1-r}{1-r^{N+1}}.$$

(b) Using the result from part (a) and Eq. (7) from Exercise 51,

$$\int_0^A x^N\,dx = \lim_{r\to 1} F(r) = A^{N+1} \lim_{r\to 1} \frac{1-r}{1-r^{N+1}} = A^{N+1} \lim_{r\to 1} \frac{1}{1+r+r^2+\cdots+r^N} = A^{N+1}\cdot\frac{1}{N+1} = \frac{A^{N+1}}{N+1}.$$

53. Verify the Gregory–Leibniz formula as follows.

(a) Set $r = -x^2$ in Eq. (7) and rearrange to show that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}$$

(b) Show, by integrating over [0, 1], that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{N-1}}{2N-1} + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$

(c) Use the Comparison Theorem for integrals to prove that

$$0 \le \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \le \frac{1}{2N+1}$$

Hint: Observe that the integrand is $\le x^{2N}$.

(d) Prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Hint: Use (b) and (c) to show that the partial sums S_N of satisfy $\left|S_N - \frac{\pi}{4}\right| \le \frac{1}{2N+1}$, and thereby conclude that $\lim_{N\to\infty} S_N = \frac{\pi}{4}$.

SOLUTION

(a) Start with Eq. (7), and substitute $-x^2$ for r:

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1-r^N}{1-r}$$

$$1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} = \frac{1-(-1)^N x^{2N}}{1-(-x^2)}$$

$$1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} = \frac{1}{1+x^2} - \frac{(-1)^N x^{2N}}{1+x^2}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}$$

(b) The integrals of both sides must be equal. Now,

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

while

$$\int_0^1 \left(1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}\right) dx$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^{N-1}\frac{1}{2N-1}x^{2N-1}\right) + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} + \cdots + (-1)^{N-1}\frac{1}{2N-1} + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$

(c) Note that for $x \in [0, 1]$, we have $1 + x^2 \geq 1$, so that

$$0 \leq \frac{x^{2N}}{1+x^2} \leq x^{2N}$$

By the Comparison Theorem for integrals, we then see that

$$0 \leq \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \leq \int_0^1 x^{2N}\,dx = \frac{1}{2N+1}x^{2N+1}\Big|_0^1 = \frac{1}{2N+1}$$

(d) Write

$$a_n = (-1)^n \frac{1}{2n-1}, \quad n \geq 1$$

and let S_N be the partial sums. Then

$$\left|S_N - \frac{\pi}{4}\right| = \left|(-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}\right| = \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \leq \frac{1}{2N+1}$$

Thus $\lim_{N\to\infty} S_N = \frac{\pi}{4}$ so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

54. Cantor's Disappearing Table (following Larry Knop of Hamilton College) Take a table of length L (Figure 7). At stage 1, remove the section of length $L/4$ centered at the midpoint. Two sections remain, each with length less than $L/2$. At stage 2, remove sections of length $L/4^2$ from each of these two sections (this stage removes $L/8$ of the table). Now four sections remain, each of length less than $L/4$. At stage 3, remove the four central sections of length $L/4^3$, etc.

(a) Show that at the Nth stage, each remaining section has length less than $L/2^N$ and that the total amount of table removed is

$$L\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{N+1}}\right)$$

(b) Show that in the limit as $N \to \infty$, precisely one-half of the table remains.

This result is curious, because there are no nonzero intervals of table left (at each stage, the remaining sections have a length less than $L/2^N$). So the table has "disappeared." However, we can place any object longer than $L/4$ on the table. It will not fall through because it will not fit through any of the removed sections.

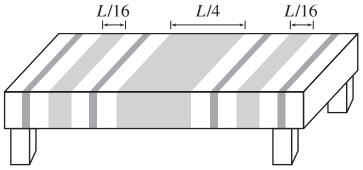


FIGURE 7

SOLUTION

(a) After the Nth stage, the total amount of table that has been removed is

$$\frac{L}{4} + \frac{2L}{4^2} + \frac{4L}{4^3} + \cdots + \frac{2^{N-1}L}{4^N} = L\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{2^{N-1}}{2^{2N}}\right) = L\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{N+1}}\right)$$

At the first stage ($N = 1$), there are two remaining sections each of length

$$\frac{L - \frac{L}{4}}{2} = \frac{3L}{8} < \frac{L}{2}.$$

Suppose that at the Kth stage, each of the 2^K remaining sections has length less than $\frac{L}{2^K}$. The $(K+1)$st stage is obtained by removing the section of length $\frac{L}{4^{K+1}}$ centered at the midpoint of each segment in the Kth stage. Let a_k and a_{K+1}, respectively, denote the length of each segment in the Kth and $(K+1)$st stage. Then,

$$a_{K+1} = \frac{a_K - \frac{L}{4^{K+1}}}{2} < \frac{\frac{L}{2^K} - \frac{L}{4^{K+1}}}{2} = \frac{L}{2^K}\left(\frac{1 - \frac{1}{2^{K+2}}}{2}\right) < \frac{L}{2^K}\cdot\frac{1}{2} = \frac{L}{2^{K+1}}.$$

Thus, by mathematical induction, each remaining section at the Nth stage has length less than $\frac{L}{2^N}$.

(b) From part (a), we know that after N stages, the amount of the table that has been removed is

$$L\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^{N+1}}\right)=\sum_{n=1}^{N}\frac{1}{2^{n+1}}.$$

As $N\to\infty$, the amount of the table that has been removed becomes a geometric series whose sum is

$$L\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n=L\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{1}{2}L.$$

Thus, the amount of table that remains is $L-\frac{1}{2}L=\frac{1}{2}L$.

55. The **Koch snowflake** (described in 1904 by Swedish mathematician Helge von Koch) is an infinitely jagged "fractal" curve obtained as a limit of polygonal curves (it is continuous but has no tangent line at any point). Begin with an equilateral triangle (stage 0) and produce stage 1 by replacing each edge with four edges of one-third the length, arranged as in Figure 8. Continue the process: At the nth stage, replace each edge with four edges of one-third the length.

(a) Show that the perimeter P_n of the polygon at the nth stage satisfies $P_n=\frac{4}{3}P_{n-1}$. Prove that $\lim_{n\to\infty}P_n=\infty$. The snowflake has infinite length.

(b) Let A_0 be the area of the original equilateral triangle. Show that $(3)4^{n-1}$ new triangles are added at the nth stage, each with area $A_0/9^n$ (for $n\geq 1$). Show that the total area of the Koch snowflake is $\frac{8}{5}A_0$.

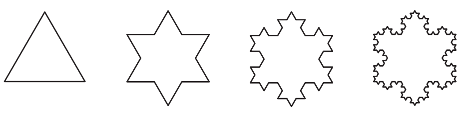


FIGURE 8

SOLUTION

(a) Each edge of the polygon at the $(n-1)$st stage is replaced by four edges of one-third the length; hence the perimeter of the polygon at the nth stage is $\frac{4}{3}$ times the perimeter of the polygon at the $(n-1)$th stage. That is, $P_n=\frac{4}{3}P_{n-1}$. Thus,

$$P_1=\frac{4}{3}P_0;\quad P_2=\frac{4}{3}P_1=\left(\frac{4}{3}\right)^2P_0,\quad P_3=\frac{4}{3}P_2=\left(\frac{4}{3}\right)^3P_0,$$

and, in general, $P_n=\left(\frac{4}{3}\right)^nP_0$. As $n\to\infty$, it follows that

$$\lim_{n\to\infty}P_n=P_0\lim_{n\to\infty}\left(\frac{4}{3}\right)^n=\infty.$$

(b) When each edge is replaced by four edges of one-third the length, one new triangle is created. At the $(n-1)$st stage, there are $3\cdot 4^{n-1}$ edges in the snowflake, so $3\cdot 4^{n-1}$ new triangles are generated at the nth stage. Because the area of an equilateral triangle is proportional to the square of its side length and the side length for each new triangle is one-third the side length of triangles from the previous stage, it follows that the area of the triangles added at each stage is reduced by a factor of $\frac{1}{9}$ from the area of the triangles added at the previous stage. Thus, each triangle added at the nth stage has an area of $A_0/9^n$. This means that the nth stage contributes

$$3\cdot 4^{n-1}\cdot\frac{A_0}{9^n}=\frac{3}{4}A_0\left(\frac{4}{9}\right)^n$$

to the area of the snowflake. The total area is therefore

$$A=A_0+\frac{3}{4}A_0\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n=A_0+\frac{3}{4}A_0\frac{\frac{4}{9}}{1-\frac{4}{9}}=A_0+\frac{3}{4}A_0\cdot\frac{4}{5}=\frac{8}{5}A_0.$$

10.3 Convergence of Series with Positive Terms

Preliminary Questions

1. Let $S=\sum_{n=1}^{\infty}a_n$. If the partial sums S_N are increasing, then (choose the correct conclusion):

(a) $\{a_n\}$ is an increasing sequence.

(b) $\{a_n\}$ is a positive sequence.

SOLUTION The correct response is **(b)**. Recall that $S_N = a_1 + a_2 + a_3 + \cdots + a_N$; thus, $S_N - S_{N-1} = a_N$. If S_N is increasing, then $S_N - S_{N-1} \ge 0$. It then follows that $a_N \ge 0$; that is, $\{a_n\}$ is a positive sequence.

2. What are the hypotheses of the Integral Test?

SOLUTION The hypotheses for the Integral Test are: A function $f(x)$ such that $a_n = f(n)$ must be positive, decreasing, and continuous for $x \ge 1$.

3. Which test would you use to determine whether $\sum_{n=1}^{\infty} n^{-3.2}$ converges?

SOLUTION Because $n^{-3.2} = \frac{1}{n^{3.2}}$, we see that the indicated series is a p-series with $p = 3.2 > 1$. Therefore, the series converges.

4. Which test would you use to determine whether $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$ converges?

SOLUTION Because

$$\frac{1}{2^n + \sqrt{n}} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent geometric series, the comparison test would be an appropriate choice to establish that the given series converges.

5. Ralph hopes to investigate the convergence of $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$ by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n}$. Is Ralph on the right track?

SOLUTION No, Ralph is not on the right track. For $n \ge 1$,

$$\frac{e^{-n}}{n} < \frac{1}{n};$$

however, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series. The Comparison Test therefore does not allow us to draw a conclusion about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$.

Exercises

In Exercises 1–14, use the Integral Test to determine whether the infinite series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

SOLUTION Let $f(x) = \frac{1}{x^4}$. This function is continuous, positive and decreasing on the interval $x \ge 1$, so the Integral Test applies. Moreover,

$$\int_1^{\infty} \frac{dx}{x^4} = \lim_{R\to\infty} \int_1^R x^{-4}\,dx = -\frac{1}{3} \lim_{R\to\infty} \left(\frac{1}{R^3} - 1\right) = \frac{1}{3}.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ also converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

SOLUTION Let $f(x) = \frac{1}{x+3}$. This function is continuous, positive and decreasing on the interval $x \ge 1$, so the Integral Test applies. Moreover,

$$\int_1^{\infty} \frac{dx}{x+3} = \lim_{R\to\infty} \int_1^R \frac{dx}{x+3} = \lim_{R\to\infty} (\ln(R+3) - \ln 4) = \infty.$$

The integral diverges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n+3}$ also diverges.

3. $\displaystyle\sum_{n=1}^{\infty} n^{-1/3}$

SOLUTION Let $f(x) = x^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{x}}$. This function is continuous, positive and decreasing on the interval $x \ge 1$, so the Integral Test applies. Moreover,

$$\int_1^{\infty} x^{-1/3}\,dx = \lim_{R\to\infty}\int_1^R x^{-1/3}\,dx = \frac{3}{2}\lim_{R\to\infty}\left(R^{2/3}-1\right) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=1}^{\infty} n^{-1/3}$ also diverges.

4. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-4}}$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{x-4}}$. This function is continuous, positive and decreasing on the interval $x \ge 5$, so the Integral Test applies. Moreover,

$$\int_5^{\infty} \frac{dx}{\sqrt{x-4}} = \lim_{R\to\infty}\int_5^R \frac{dx}{\sqrt{x-4}} = 2\lim_{R\to\infty}\left(\sqrt{R-4}-1\right) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-4}}$ also diverges.

5. $\displaystyle\sum_{n=25}^{\infty} \frac{n^2}{(n^3+9)^{5/2}}$

SOLUTION Let $f(x) = \dfrac{x^2}{\left(x^3+9\right)^{5/2}}$. This function is positive and continuous for $x \ge 25$. Moreover, because

$$f'(x) = \frac{2x(x^3+9)^{5/2} - x^2\cdot\frac{5}{2}(x^3+9)^{3/2}\cdot 3x^2}{(x^3+9)^5} = \frac{x(36-11x^3)}{2(x^3+9)^{7/2}},$$

we see that $f'(x) < 0$ for $x \ge 25$, so f is decreasing on the interval $x \ge 25$. The Integral Test therefore applies. To evaluate the improper integral, we use the substitution $u = x^3+9$, $du = 3x^2dx$. We then find

$$\int_{25}^{\infty} \frac{x^2}{(x^3+9)^{5/2}}\,dx = \lim_{R\to\infty}\int_{25}^R \frac{x^2}{(x^3+9)^{5/2}}\,dx = \frac{1}{3}\lim_{R\to\infty}\int_{15634}^{R^3+9}\frac{du}{u^{5/2}}$$
$$= -\frac{2}{9}\lim_{R\to\infty}\left(\frac{1}{(R^3+9)^{3/2}} - \frac{1}{15634^{3/2}}\right) = \frac{2}{9\cdot 15634^{3/2}}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=25}^{\infty} \frac{n^2}{\left(n^3+9\right)^{5/2}}$ also converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/5}}$

SOLUTION Let $f(x) = \dfrac{x}{(x^2+1)^{3/5}}$. Because

$$f'(x) = \frac{(x^2+1)^{3/5} - x\cdot\frac{6}{5}x(x^2+1)^{-2/5}}{(x^2+1)^{6/5}} = \frac{1-\frac{1}{5}x^2}{(x^2+1)^{8/5}},$$

we see that $f'(x) < 0$ for $x > \sqrt{5} \approx 2.236$. We conclude that f is decreasing on the interval $x \ge 3$. Since f is also positive and continuous on this interval, the Integral Test can be applied. To evaluate the improper integral, we make the substitution $u = x^2+1$, $du = 2x\,dx$. This gives

$$\int_3^{\infty} \frac{x}{(x^2+1)^{3/5}}\,dx = \lim_{R\to\infty}\int_3^R \frac{x}{(x^2+1)^{3/5}}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{10}^{R^2+1}\frac{du}{u^{3/5}} = \frac{5}{4}\lim_{R\to\infty}\left((R^2+1)^{2/5} - 10^{2/5}\right) = \infty.$$

The integral diverges; therefore, the series $\sum_{n=3}^{\infty} \frac{n}{(n^2+1)^{3/5}}$ also diverges. Since the divergence of the series is not affected by adding the finite sum $\sum_{n=1}^{2} \frac{n}{(n^2+1)^{3/5}}$, the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/5}}$ also diverges.

7. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

SOLUTION Let $f(x) = \frac{1}{x^2+1}$. This function is positive, decreasing and continuous on the interval $x \geq 1$, hence the Integral Test applies. Moreover,

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2+1} = \lim_{R\to\infty} \left(\tan^{-1} R - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

8. $\sum_{n=4}^{\infty} \frac{1}{n^2-1}$

SOLUTION Let $f(x) = \frac{1}{x^2-1}$.This function is continuous, positive and decreasing on the interval $x \geq 4$; therefore, the Integral Test applies. We compute the improper integral using partial fractions:

$$\int_4^{\infty} \frac{dx}{x^2-1} = \lim_{R\to\infty} \int_4^R \left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right) dx = \frac{1}{2} \lim_{R\to\infty} \ln \frac{x-1}{x+1}\bigg|_4^R = \frac{1}{2} \lim_{R\to\infty} \left(\ln \frac{R-1}{R+1} - \ln \frac{3}{5}\right)$$

$$= \frac{1}{2}\left(\ln 1 - \ln \frac{3}{5}\right) = -\frac{1}{2} \ln \frac{3}{5}.$$

The integral converges; hence, the series $\sum_{n=4}^{\infty} \frac{1}{n^2-1}$ also converges.

9. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

SOLUTION Let $f(x) = \frac{1}{x(x+1)}$. This function is positive, continuous and decreasing on the interval $x \geq 1$, so the Integral Test applies. We compute the improper integral using partial fractions:

$$\int_1^{\infty} \frac{dx}{x(x+1)} = \lim_{R\to\infty} \int_1^R \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \lim_{R\to\infty} \ln \frac{x}{x+1}\bigg|_1^R = \lim_{R\to\infty} \left(\ln \frac{R}{R+1} - \ln \frac{1}{2}\right) = \ln 1 - \ln \frac{1}{2} = \ln 2.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

10. $\sum_{n=1}^{\infty} ne^{-n^2}$

SOLUTION Let $f(x) = xe^{-x^2}$. This function is continuous and positive on the interval $x \geq 1$. Moreover, because

$$f'(x) = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = e^{-x^2}\left(1 - 2x^2\right),$$

we see that $f'(x) < 0$ for $x \geq 1$, so f is decreasing on this interval. To compute the improper integral we make the substitution $u = x^2$, $du = 2x\,dx$. Then, we find

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty} \int_1^R xe^{-x^2}\,dx = \frac{1}{2}\int_1^{R^2} e^{-u}\,du = -\frac{1}{2} \lim_{R\to\infty} \left(e^{-R^2} - e^{-1}\right) = \frac{1}{2e}.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x(\ln x)^2}$. This function is positive and continuous for $x \geq 2$. Moreover,

$$f'(x) = -\frac{1}{x^2(\ln x)^4}\left(1 \cdot (\ln x)^2 + x \cdot 2(\ln x) \cdot \frac{1}{x}\right) = -\frac{1}{x^2(\ln x)^4}\left((\ln x)^2 + 2\ln x\right).$$

Since $\ln x > 0$ for $x > 1$, $f'(x)$ is negative for $x > 1$; hence, f is decreasing for $x \geq 2$. To compute the improper integral, we make the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$. We obtain:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^2}$$
$$= -\lim_{R\to\infty}\left(\frac{1}{\ln R} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

SOLUTION Let $f(x) = \dfrac{\ln x}{x^2}$. Because

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - 2x\ln x}{x^4} = \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3},$$

we see that $f'(x) < 0$ for $x > \sqrt{e} \approx 1.65$. We conclude that f is decreasing on the interval $x \geq 2$. Since f is also positive and continuous on this interval, the Integral Test can be applied. By Integration by Parts, we find

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C;$$

therefore,

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\left(\frac{1}{2} + \frac{\ln 2}{2} - \frac{1}{R} - \frac{\ln R}{R}\right) = \frac{1 + \ln 2}{2} - \lim_{R\to\infty} \frac{\ln R}{R}.$$

We compute the resulting limit using L'Hôpital's Rule:

$$\lim_{R\to\infty} \frac{\ln R}{R} = \lim_{R\to\infty} \frac{1/R}{1} = 0.$$

Hence,

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \frac{1 + \ln 2}{2}.$$

The integral converges; therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ also converges. Since the convergence of the series is not affected by adding the finite sum $\displaystyle\sum_{n=1}^{1} \frac{\ln n}{n^2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ also converges.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

SOLUTION Note that

$$2^{\ln n} = (e^{\ln 2})^{\ln n} = (e^{\ln n})^{\ln 2} = n^{\ln 2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln 2}}.$$

Now, let $f(x) = \dfrac{1}{x^{\ln 2}}$. This function is positive, continuous and decreasing on the interval $x \geq 1$; therefore, the Integral Test applies. Moreover,

$$\int_1^\infty \frac{dx}{x^{\ln 2}} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^{\ln 2}} = \frac{1}{1-\ln 2} \lim_{R\to\infty} (R^{1-\ln 2} - 1) = \infty,$$

because $1 - \ln 2 > 0$. The integral diverges; hence, the series $\sum_{n=1}^{\infty} \dfrac{1}{2^{\ln n}}$ also diverges.

14. $\sum_{n=1}^{\infty} \dfrac{1}{3^{\ln n}}$

SOLUTION Note that

$$3^{\ln n} = (e^{\ln 3})^{\ln n} = (e^{\ln n})^{\ln 3} = n^{\ln 3}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}}.$$

Now, let $f(x) = \dfrac{1}{x^{\ln 3}}$. This function is positive, continuous and decreasing on the interval $x \geq 1$; therefore, the Integral Test applies. Moreover,

$$\int_1^\infty \frac{dx}{x^{\ln 3}} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^{\ln 3}} = \frac{1}{1-\ln 3} \lim_{R\to\infty} (R^{1-\ln 3} - 1) = -\frac{1}{1-\ln 3},$$

because $1 - \ln 3 < 0$. The integral converges; hence, the series $\sum_{n=1}^{\infty} \dfrac{1}{3^{\ln n}}$ also converges.

15. Show that $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + 8n}$ converges by using the Comparison Test with $\sum_{n=1}^{\infty} n^{-3}$.

SOLUTION We compare the series with the p-series $\sum_{n=1}^{\infty} n^{-3}$. For $n \geq 1$,

$$\frac{1}{n^3 + 8n} \leq \frac{1}{n^3}.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ converges (it is a p-series with $p = 3 > 1$), the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + 8n}$ also converges by the Comparison Test.

16. Show that $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n^2 - 3}}$ diverges by comparing with $\sum_{n=2}^{\infty} n^{-1}$.

SOLUTION For $n \geq 2$,

$$\frac{1}{\sqrt{n^2 - 3}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}.$$

The harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, and it still diverges if we drop the first term. Thus, the series $\sum_{n=2}^{\infty} \dfrac{1}{n}$ also diverges. The Comparison Test now lets us conclude that the larger series $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n^2 - 3}}$ also diverges.

17. Let $S = \sum_{n=1}^{\infty} \dfrac{1}{n + \sqrt{n}}$. Verify that for $n \geq 1$,

$$\frac{1}{n + \sqrt{n}} \leq \frac{1}{n}, \qquad \frac{1}{n + \sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Can either inequality be used to show that S diverges? Show that $\dfrac{1}{n + \sqrt{n}} \geq \dfrac{1}{2n}$ and conclude that S diverges.

SOLUTION For $n \ge 1$, $n + \sqrt{n} \ge n$ and $n + \sqrt{n} \ge \sqrt{n}$. Taking the reciprocal of each of these inequalities yields

$$\frac{1}{n+\sqrt{n}} \le \frac{1}{n} \quad \text{and} \quad \frac{1}{n+\sqrt{n}} \le \frac{1}{\sqrt{n}}.$$

These inequalities indicate that the series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ is smaller than both $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$; however, $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ both diverge so neither inequality allows us to show that S diverges.

On the other hand, for $n \ge 1$, $n \ge \sqrt{n}$, so $2n \ge n + \sqrt{n}$ and

$$\frac{1}{n+\sqrt{n}} \ge \frac{1}{2n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{2n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since the harmonic series diverges. The Comparison Test then lets us conclude that the larger series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ also diverges.

18. Which of the following inequalities can be used to study the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$? Explain.

$$\frac{1}{n^2+\sqrt{n}} \le \frac{1}{\sqrt{n}}, \qquad \frac{1}{n^2+\sqrt{n}} \le \frac{1}{n^2}$$

SOLUTION The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, hence the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. The first inequality given above therefore establishes that $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$ is smaller than a divergent series, which does not allow us to conclude whether $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$ converges or diverges.

However, the second inequality given above establishes that $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$ is smaller than the convergent p-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. By the Comparison Test, we therefore conclude that $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$ also converges.

In Exercises 19–30, use the Comparison Test to determine whether the infinite series is convergent.

19. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

SOLUTION We compare with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. For $n \ge 1$,

$$\frac{1}{n2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges (it is a geometric series with $r = \frac{1}{2}$), we conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ also converges.

20. $\sum_{n=1}^{\infty} \frac{n^3}{n^5+4n+1}$

SOLUTION For $n \ge 1$,

$$\frac{n^3}{n^5+4n+1} \le \frac{n^3}{n^5} = \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4n + 1}$ also converges.

21. $\sum_{n=1}^{\infty} \frac{1}{n^{1/3} + 2^n}$

SOLUTION For $n \geq 1$,

$$\frac{1}{n^{1/3} + 2^n} \leq \frac{1}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2}$, so it converges. By the Comparison test, so does $\sum_{n=1}^{\infty} \frac{1}{n^{1/3} + 2^n}$.

22. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n - 1}}$

SOLUTION For $n \geq 1$, we have $2n - 1 \geq 0$ so that

$$\frac{1}{\sqrt{n^3 + 2n - 1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}.$$

This latter series is a p-series with $p = \frac{3}{2} > 1$, so it converges. By the Comparison Test, so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n - 1}}$.

23. $\sum_{m=1}^{\infty} \frac{4}{m! + 4^m}$

SOLUTION For $m \geq 1$,

$$\frac{4}{m! + 4^m} \leq \frac{4}{4^m} = \left(\frac{1}{4}\right)^{m-1}.$$

The series $\sum_{m=1}^{\infty} \left(\frac{1}{4}\right)^{m-1}$ is a geometric series with $r = \frac{1}{4}$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{m=1}^{\infty} \frac{4}{m! + 4^m}$ also converges.

24. $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n - 3}$

SOLUTION For $n \geq 4$,

$$\frac{\sqrt{n}}{n - 3} \geq \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} < 1$, so it diverges, and it continues to diverge if we drop the terms $n = 1, 2, 3$; that is, $\sum_{n=4}^{\infty} \frac{1}{n^{1/2}}$ also diverges. By the Comparison Test we can therefore conclude that series $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n - 3}$ diverges.

25. $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

SOLUTION For $k \geq 1$, $0 \leq \sin^2 k \leq 1$, so

$$0 \leq \frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}.$$

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$ also converges.

26. $\displaystyle\sum_{k=2}^{\infty} \frac{k^{1/3}}{k^{5/4} - k}$

SOLUTION For $k \geq 2$, $k^{5/4} - k < k^{5/4}$ so that

$$\frac{k^{1/3}}{k^{5/4} - k} \geq \frac{k^{1/3}}{k^{5/4}} = \frac{1}{k^{11/12}}$$

The series $\sum_{k=2}^{\infty} \frac{1}{k^{11/12}}$ is a p-series with $p = \frac{11}{12} < 1$, so it diverges. By the Comparison Test, so does $\displaystyle\sum_{k=2}^{\infty} \frac{k^{1/3}}{k^{5/4} - k}$.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$

SOLUTION Since $3^{-n} > 0$ for all n,

$$\frac{2}{3^n + 3^{-n}} \leq \frac{2}{3^n} = 2\left(\frac{1}{3}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3}$, so it converges. By the Comparison Theorem we can therefore conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$ also converges.

28. $\displaystyle\sum_{k=1}^{\infty} 2^{-k^2}$

SOLUTION For $k \geq 1$, $k^2 \geq k$ and

$$\frac{1}{2^{k^2}} \leq \frac{1}{2^k} = \left(\frac{1}{2}\right)^k.$$

The series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ is a geometric series with $r = \frac{1}{2}$, so it converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^{k^2}} = \sum_{k=1}^{\infty} 2^{-k^2}$ also converges.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$

SOLUTION Note that for $n \geq 2$,

$$(n+1)! = 1 \cdot \underbrace{2 \cdot 3 \cdots n \cdot (n+1)}_{n \text{ factors}} \leq 2^n$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = 1 + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{2^n}$$

But $\sum_{n=2}^{\infty} \frac{1}{2^n}$ is a geometric series with ratio $r = \frac{1}{2}$, so it converges. By the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ converges as well.

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^3}$

SOLUTION Note that for $n \geq 4$, we have $(n-1)(n-2) > n$ [to see this, solve the equation $(n-1)(n-2) = n$ for n; the positive root is $2 + \sqrt{2} \approx 3.4$]. Thus

$$\sum_{n=4}^{\infty} \frac{n!}{n^3} = \sum_{n=4}^{\infty} \frac{n(n-1)(n-2)(n-3)!}{n^3} \geq \sum_{n=4}^{\infty} \frac{(n-3)!}{n} \geq \sum_{n=4}^{\infty} \frac{1}{n}$$

But $\sum_{n=4}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges, so that $\sum_{n=4}^{\infty} \frac{n!}{n^3}$ also diverges. Adding back in the terms for $n = 1$, 2, and 3 does not affect this result. Thus the original series diverges.

Exercise 31–36: For all $a > 0$ and $b > 1$, the inequalities

$$\ln n \le n^a, \qquad n^a < b^n$$

are true for n sufficiently large (this can be proved using L'Hopital's Rule). Use this, together with the Comparison Theorem, to determine whether the series converges or diverges.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

SOLUTION For n sufficiently large (say $n = k$, although in this case $n = 1$ suffices), we have $\ln n \le n$, so that

$$\sum_{n=k}^{\infty} \frac{\ln n}{n^3} \le \sum_{n=k}^{\infty} \frac{n}{n^3} = \sum_{n=k}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, so it converges. Thus $\sum_{n=k}^{\infty} \frac{\ln n}{n^3}$ also converges; adding back in the finite number of terms for $1 \le n \le k$ does not affect this result.

32. $\displaystyle\sum_{m=2}^{\infty} \frac{1}{\ln m}$

SOLUTION For $m > 1$ sufficiently large (say $m = k$, although in this case $m = 2$ suffices), we have $\ln m \le m$, so that

$$\sum_{m=k}^{\infty} \frac{1}{\ln m} \ge \sum_{m=k}^{\infty} \frac{1}{m}$$

This is the harmonic series, which diverges (the absence of the finite number of terms for $m = 1, \dots, k-1$ does not affect convergence). By the comparison theorem, $\displaystyle\sum_{m=2}^{\infty} \frac{1}{\ln m}$ also diverges (again, ignoring the finite number of terms for $m = 1, \dots, k-1$ does not affect convergence).

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$

SOLUTION Choose N so that $\ln n \le n^{0.0005}$ for $n \ge N$. Then also for $n > N$, $(\ln n)^{100} \le (n^{0.0005})^{100} = n^{0.05}$. Then

$$\sum_{n=N}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}} \le \sum_{n=N}^{\infty} \frac{n^{0.05}}{n^{1.1}} = \sum_{n=N}^{\infty} \frac{1}{n^{1.05}}$$

But $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n^{1.05}}$ is a p-series with $p = 1.05 > 1$, so is convergent. It follows that $\sum_{n=N}^{\infty} \frac{(\ln n)^100}{n^{1.1}}$ is also convergent; adding back in the finite number of terms for $n = 1, 2, \dots, N-1$ shows that $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$ converges as well.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{10}}$

SOLUTION Choose N such that $\ln n \le n^{0.1}$ for $n \ge N$; then also $(\ln n)^{10} \le n$ for $n \ge N$. So we have

$$\sum_{n=N}^{\infty} \frac{1}{(\ln n)^{10}} \ge \sum_{n=N}^{\infty} \frac{1}{n}$$

The latter sum is the harmonic series, which diverges, so the series on the left diverges as well. Adding back in the finite number of terms for $n < N$ shows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{10}}$ diverges.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

SOLUTION Choose N such that $n \le 2^n$ for $n \ge N$. Then

$$\sum_{n=N}^{\infty} \frac{n}{3^n} \le \sum_{n=N}^{\infty} \left(\frac{2}{3}\right)^n$$

The latter sum is a geometric series with $r = \frac{2}{3} < 1$, so it converges. Thus the series on the left converges as well. Adding back in the finite number of terms for $n < N$ shows that $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

36. $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

SOLUTION Choose N such that $n^5 \le 1.5^n$ for $n \ge N$. Then

$$\sum_{n=N}^{\infty} \frac{n^5}{2^n} \le \sum_{n=N}^{\infty} \left(\frac{1.5}{2}\right)^n$$

The latter sum is a geometric series with $r = \frac{1.5}{2} < 1$, so it converges. Thus the series on the left converges as well. Adding back in the finite number of terms for $n < N$ shows that $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$ converges.

37. Show that $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ converges. *Hint:* Use the inequality $\sin x \le x$ for $x \ge 0$.

SOLUTION For $n \ge 1$,

$$0 \le \frac{1}{n^2} \le 1 < \pi;$$

therefore, $\sin \frac{1}{n^2} > 0$ for $n \ge 1$. Moreover, for $n \ge 1$,

$$\sin \frac{1}{n^2} \le \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ also converges.

38. Does $\sum_{n=2}^{\infty} \frac{\sin(1/n)}{\ln n}$ converge?

SOLUTION No, it diverges. Either the Comparison Theorem or the Limit Comparison Theorem may be used. Using the Comparison Theorem, recall that

$$\frac{\sin x}{x} > \cos x \quad \text{for } x > 0$$

so that $\sin x > x \cos x$. Substituting $1/n$ for x gives

$$\sin\left(\frac{1}{n}\right) > \frac{1}{n} \cos\left(\frac{1}{n}\right) = \frac{\cos(1/n)}{n} \ge \frac{1}{2n}$$

since $\cos\left(\frac{1}{n}\right) \ge \frac{1}{2}$ for $n \ge 2$. Thus

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\ln n} > \sum_{n=1}^{\infty} \frac{1}{2n \ln n}$$

Apply the Integral Test to the latter expression, making the substitution $u = \ln x$:

$$\int_1^{\infty} \frac{1}{2x \ln x}\, dx = \frac{1}{2} \int_0^{\infty} \frac{1}{u}\, du = \frac{1}{2} \ln u\Big|_0^{\infty}$$

and the integral diverges. Thus

$$\sum_{n=1}^{\infty} \frac{1}{2n \ln n} \quad \text{diverges, and thus} \quad \sum_{n=1}^{\infty} \frac{\sin(1/n)}{\ln n} \quad \text{diverges as well.}$$

Applying the Limit Comparison Test is similar but perhaps simpler: Recall that

$$\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

so apply the Limit Comparison Test with $b_n=\dfrac{1/x}{\ln x}$:

$$L=\lim_{x\to\infty}\frac{\sin(1/x)}{\ln x}\cdot\frac{\ln x}{1/x}=\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=1$$

so that either both series converge or both diverge. But by the Integral Test as above,

$$\sum_{n=1}^{\infty}\frac{(1/x)}{\ln x}=\sum_{n=1}^{\infty}\frac{1}{x\ln x}$$

diverges.

In Exercises 39–48, use the Limit Comparison Test to prove convergence or divergence of the infinite series.

39. $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{n^4-1}$

SOLUTION Let $a_n=\dfrac{n^2}{n^4-1}$. For large n, $\dfrac{n^2}{n^4-1}\approx\dfrac{n^2}{n^4}=\dfrac{1}{n^2}$, so we apply the Limit Comparison Test with $b_n=\dfrac{1}{n^2}$. We find

$$L=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^4}{n^4-1}=1.$$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges; hence, $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{n^4-1}$ converges.

40. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-\sqrt{n}}$

SOLUTION Let $a_n=\dfrac{1}{n^2-\sqrt{n}}$. For large n, $\dfrac{1}{n^2-\sqrt{n}}\approx\dfrac{1}{n^2}$, so we apply the Limit Comparison Test with $b_n=\dfrac{1}{n^2}$. We find

$$L=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{n^2-\sqrt{n}}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n^2-\sqrt{n}}=1.$$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges; hence, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-\sqrt{n}}$ converges.

41. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\sqrt{n^3+1}}$

SOLUTION Let $a_n=\dfrac{n}{\sqrt{n^3+1}}$. For large n, $\dfrac{n}{\sqrt{n^3+1}}\approx\dfrac{n}{\sqrt{n^3}}=\dfrac{1}{\sqrt{n}}$, so we apply the Limit Comparison test with $b_n=\dfrac{1}{\sqrt{n}}$. We find

$$L=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n}{\sqrt{n^3+1}}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{n^3+1}}=1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$, so it diverges; hence, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$ diverges.

42. $\sum_{n=2}^{\infty} \frac{n^3}{\sqrt{n^7 + 2n^2 + 1}}$

SOLUTION Let a_n be the general term of our series. Observe that

$$a_n = \frac{n^3}{\sqrt{n^7+2n^2+1}} = \frac{n^{-3} \cdot n^3}{n^{-3} \cdot \sqrt{n^7+2n^2+1}} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}}$$

This suggests that we apply the Limit Comparison Test, comparing our series with

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$$

The ratio of the terms is

$$\frac{a_n}{b_n} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}} \cdot \frac{\sqrt{n}}{1} = \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}}$$

Hence

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}} = 1$$

The p-series $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges since $p = 1/2 < 1$. Therefore, our original series diverges.

43. $\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$

SOLUTION Let $a_n = \frac{3n+5}{n(n-1)(n-2)}$. For large n, $\frac{3n+5}{n(n-1)(n-2)} \approx \frac{3n}{n^3} = \frac{3}{n^2}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n^2}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n+5}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+5n^2}{n(n+1)(n+2)} = 3.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges; hence, the series $\sum_{n=3}^{\infty} \frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$ converges.

44. $\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$

SOLUTION Let

$$a_n = \frac{e^n+n}{e^{2n}-n^2} = \frac{e^n+n}{(e^n-n)(e^n+n)} = \frac{1}{e^n-n}.$$

For large n,

$$\frac{1}{e^n-n} \approx \frac{1}{e^n} = e^{-n},$$

so we apply the Limit Comparison Test with $b_n = e^{-n}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{e^n-n}}{e^{-n}} = \lim_{n\to\infty} \frac{e^n}{e^n-n} = 1.$$

The series $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with $r = \frac{1}{e} < 1$, so it converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{e^n + n}{e^{2n} - n^2}$ also converges.

45. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \ln n}$

SOLUTION Let

$$a_n = \frac{1}{\sqrt{n} + \ln n}$$

For large n, $\sqrt{n} + \ln n \approx \sqrt{n}$, so apply the Comparison Test with $b_n = \frac{1}{\sqrt{n}}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \ln n} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{1}{1 + \frac{\ln n}{\sqrt{n}}} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$, so it diverges. Because L exists, the Limit Comparison Test tells us the the original series also diverges.

46. $\sum_{n=1}^{\infty} \frac{\ln(n+4)}{n^{5/2}}$

SOLUTION Let

$$a_n = \frac{\ln(n+4)}{n^{5/2}}$$

For large n, $a_n \approx \frac{\ln n}{n^{5/2}}$, so apply the Comparison Test with $b_n = \frac{\ln n}{n^{5/2}}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+4)}{n^{5/2}} \cdot \frac{n^{5/2}}{\ln n} = \lim_{n\to\infty} \frac{\ln(n+4)}{\ln n}$$

Applying L'Hôpital's rule gives

$$L = \lim_{n\to\infty} \frac{\ln(n+4)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+4)}{1/n} = \lim_{n\to\infty} \frac{n}{n+4} = \lim_{n\to\infty} \frac{1}{1+4/n} = 1$$

To see that $\sum_{n=1}^{\infty} b_n$ converges, choose N so that $\ln n < n$ for $n \geq N$; then

$$\sum_{n=N}^{\infty} \frac{\ln n}{n^{5/2}} \leq \sum_{n=N}^{\infty} \frac{n}{n^{5/2}} = \sum_{n=N}^{\infty} \frac{1}{n^{3/2}}$$

which is a p-series with $p = \frac{3}{2} > 1$, so it converges. Adding back in the finite number of terms for $n < N$ shows that $\sum b_n$ converges as well. Since L exists and $\sum b_n$ converges, the Limit Comparison Test tells us that $\sum_{n=1}^{\infty} a_n$ converges.

47. $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ *Hint:* Compare with $\sum_{n=1}^{\infty} n^{-2}$.

SOLUTION Let $a_n = 1 - \cos \frac{1}{n}$, and apply the Limit Comparison Test with $b_n = \frac{1}{n^2}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 - \cos \frac{1}{n}}{\frac{1}{n^2}} = \lim_{x\to\infty} \frac{1 - \cos \frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2} \sin \frac{1}{x}}{-\frac{2}{x^3}} = \frac{1}{2} \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}}.$$

As $x \to \infty$, $u = \frac{1}{x} \to 0$, so

$$L = \frac{1}{2} \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \frac{1}{2} \lim_{u\to 0} \frac{\sin u}{u} = \frac{1}{2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ also converges.

48. $\sum_{n=1}^{\infty} (1 - 2^{-1/n})$ *Hint:* Compare with the harmonic series.

SOLUTION Let $a_n = 1 - 2^{-1/n}$, and apply the Limit Comparison Test with $b_n = \frac{1}{n}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 - 2^{-1/n}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{1 - 2^{-1/x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}(\ln 2)2^{-1/x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \left(2^{-1/x} \ln 2\right) = \ln 2.$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; because $L > 0$, we can conclude by the Limit Comparison Test that the series $\sum_{n=1}^{\infty} (1 - 2^{-1/n})$ also diverges.

In Exercises 49–74, determine convergence or divergence using any method covered so far.

49. $\sum_{n=4}^{\infty} \frac{1}{n^2 - 9}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{1}{n^2 - 9}$ and $b_n = \frac{1}{n^2}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-9}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2 - 9} = 1.$$

Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=4}^{\infty} \frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=4}^{\infty} \frac{1}{n^2 - 9}$ converges.

50. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$

SOLUTION For all $n \geq 1$, $0 \leq \cos^2 n \leq 1$, so

$$0 \leq \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series; hence, by the Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ also converges.

51. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n + 9}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{\sqrt{n}}{4n + 9}$ and $b_n = \frac{1}{\sqrt{n}}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{4n+9}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{4n + 9} = \frac{1}{4}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n + 9}$ also diverges.

52. $\displaystyle\sum_{n=1}^{\infty} \frac{n - \cos n}{n^3}$

SOLUTION Apply the Limit Comparison Test with $a_n = \dfrac{n - \cos n}{n^3}$ and $b_n = \dfrac{1}{n^2}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-\cos n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(1 - \frac{\cos n}{n}\right) = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n - \cos n}{n^3}$ also converges.

53. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{n^5 + n}$

SOLUTION First rewrite $a_n = \dfrac{n^2 - n}{n^5 + n} = \dfrac{n(n-1)}{n\left(n^4+1\right)} = \dfrac{n-1}{n^4+1}$ and observe

$$\frac{n-1}{n^4+1} < \frac{n}{n^4} = \frac{1}{n^3}$$

for $n \ge 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series, so by the Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{n^5 + n}$ also converges.

54. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \dfrac{1}{n^2 + \sin n}$ and $b_n = \dfrac{1}{n^2}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+\sin n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1 + \frac{\sin n}{n^2}} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$ also converges.

55. $\displaystyle\sum_{n=5}^{\infty} (4/5)^{-n}$

SOLUTION

$$\sum_{n=5}^{\infty} \left(\frac{4}{5}\right)^{-n} = \sum_{n=5}^{\infty} \left(\frac{5}{4}\right)^{n}$$

which is a geometric series starting at $n = 5$ with ratio $r = \dfrac{5}{4} > 1$. Thus the series diverges.

56. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n^2}}$

SOLUTION Because $n^2 \ge n$ for $n \ge 1$, $3^{n^2} \ge 3^n$ and

$$\frac{1}{3^{n^2}} \le \frac{1}{3^n} = \left(\frac{1}{3}\right)^n.$$

The series $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3}$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=1}^{\infty}\frac{1}{3^{n^2}}$ also converges.

57. $\sum_{n=2}^{\infty}\frac{1}{n^{3/2}\ln n}$

SOLUTION For $n \geq 3$, $\ln n > 1$, so $n^{3/2}\ln n > n^{3/2}$ and

$$\frac{1}{n^{3/2}\ln n} < \frac{1}{n^{3/2}}.$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\sum_{n=3}^{\infty}\frac{1}{n^{3/2}}$ also converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=3}^{\infty}\frac{1}{n^{3/2}\ln n}$ converges. Hence, the series $\sum_{n=2}^{\infty}\frac{1}{n^{3/2}\ln n}$ also converges.

58. $\sum_{n=2}^{\infty}\frac{(\ln n)^{12}}{n^{9/8}}$

SOLUTION By the comment preceding Exercise 31, we can choose N so that for $n \geq N$, we have $\ln n < n^{1/192}$. Then also for $n \geq N$ we have $(\ln n)^{12} < n^{12/192} = n^{1/16}$. Then

$$\sum_{n=N}^{\infty}\frac{(\ln n)^{12}}{n^{9/8}} \leq \sum_{n=N}^{\infty}\frac{n^{1/16}}{n^{9/8}} = \sum_{n=N}^{\infty}\frac{1}{n^{17/16}}$$

which is a convergent p-series. Thus the series on the left converges as well; adding back in the finite number of terms for $n \leq N$ shows that $\sum_{n=2}^{\infty}\frac{(\ln n)^{12}}{n^{9/8}}$ converges.

59. $\sum_{k=1}^{\infty}4^{1/k}$

SOLUTION

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} 4^{1/k} = 4^0 = 1 \neq 0;$$

therefore, the series $\sum_{k=1}^{\infty}4^{1/k}$ diverges by the Divergence Test.

60. $\sum_{n=1}^{\infty}\frac{4^n}{5^n - 2n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{4^n}{5^n - 2n}$ and $b_n = \frac{4^n}{5^n}$:

$$L = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{4^n}{5^n-2n}}{\frac{4^n}{5^n}} = \lim_{n\to\infty}\frac{1}{1-\frac{2n}{5^n}}.$$

Now,

$$\lim_{n\to\infty}\frac{2n}{5^n} = \lim_{x\to\infty}\frac{2x}{5^x} = \lim_{x\to\infty}\frac{2}{5^x\ln 5} = 0,$$

so

$$L = \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{1-0} = 1.$$

The series $\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is a convergent geometric series. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty}\frac{4^n}{5^n - 2n}$ also converges.

61. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$

SOLUTION By the comment preceding Exercise 31, we can choose N so that for $n \ge N$, we have $\ln n < n^{1/8}$, so that $(\ln n)^4 < n^{1/2}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{(\ln n)^4} > \sum_{n=N}^{\infty} \frac{1}{n^{1/2}}$$

which is a divergent p-series. Thus the series on the left diverges as well, and adding back in the finite number of terms for $n < N$ does not affect the result. Thus $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$ diverges.

62. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \dfrac{2^n}{3^n - n}$ and $b_n = \dfrac{2^n}{3^n}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n - n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{1}{1 - \frac{n}{3^n}}.$$

Now,

$$\lim_{n\to\infty} \frac{n}{3^n} = \lim_{x\to\infty} \frac{x}{3^x} = \lim_{x\to\infty} \frac{1}{3^x \ln 3} = 0,$$

so

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{1-0} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - n}$ also converges.

63. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n - n}$

SOLUTION For $n \ge 2$, $n \ln n - n \le n \ln n$; therefore,

$$\frac{1}{n \ln n - n} \ge \frac{1}{n \ln n}.$$

Now, let $f(x) = \dfrac{1}{x \ln x}$. For $x \ge 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^R \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n - n}$ diverges.

64. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2 - n}$

SOLUTION Use the Integral Test. Note that $x(\ln x)^2 - x$ has a zero at $x = e$, so restrict the integral to $[4, \infty)$:

$$\int_4^{\infty} \frac{1}{x(\ln x)^2 - x}\,dx$$

Substitute $u = \ln x$ so that $du = \frac{1}{x}\,dx$ to get

$$\int_{\ln 4}^{\infty} \frac{1}{u^2-1}\,du = \lim_{R\to\infty}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_4^R\right) = \frac{1}{2}\lim_{R\to\infty}\left(\ln\left(\frac{R-1}{R+1}\right) - \ln\left(\frac{3}{5}\right)\right)$$
$$= \frac{1}{2}\left(\ln\lim_{R\to\infty}\left(\frac{R-1}{R+1}\right) - \ln\left(\frac{3}{5}\right)\right) = \frac{1}{2}\left(\ln 1 - \ln\left(\frac{3}{5}\right)\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) < \infty$$

Since the integral converges, the series does as well starting at $n = 4$, using the Integral Test. Adding in the terms for $n = 1, 2, 3$ does not affect this result.

65. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION For $n \geq 2$, $n^n \geq 2^n$; therefore,

$$\frac{1}{n^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a convergent geometric series, so $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$ also converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^n}$ converges. Hence, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^n}$ converges.

66. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 4n^{3/2}}{n^3}$

SOLUTION Let $a_n = \frac{1}{n}$ and $b_n = -\frac{4}{n^{3/2}}$. Then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\frac{n^2 - 4n^{3/2}}{n^3}$$

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges since it is the harmonic series}$$

$$\sum_{n=1}^{\infty} b_n \quad \text{is a } p\text{-series with } p = \frac{3}{2} > 1\text{, so converges}$$

Since $\sum a_n$ diverges and $\sum b_n$ converges, it follows that $\sum(a_n + b_n)$ diverges.

67. $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}$

SOLUTION Let

$$a_n = \frac{1+(-1)^n}{n}$$

Then

$$a_n = \begin{cases} 0 & n \text{ odd} \\ \frac{2}{2k} = \frac{1}{k} & n = 2k \text{ even} \end{cases}$$

Therefore, $\{a_n\}$ consists of 0s in the odd places and the harmonic series in the even places, so $\sum_{i=1}^{\infty} a_n$ is just the sum of the harmonic series, which diverges. Thus $\sum_{i=1}^{\infty} a_n$ diverges as well.

68. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^{3/2}}$

SOLUTION For $n \geq 1$

$$0 < \frac{2+(-1)^n}{n^{3/2}} \leq \frac{2+1}{n^{3/2}} = \frac{3}{n^{3/2}}.$$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series; hence, the series $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^{3/2}}$ also converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^{3/2}}$ converges.

69. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \sin \frac{1}{n}$ and $b_n = \frac{1}{n}$:

$$L = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{u\to 0} \frac{\sin u}{u} = 1,$$

where $u = \frac{1}{n}$. The harmonic series diverges. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$ also diverges.

70. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

SOLUTION Apply the Limit Comparison Test with $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1/n}{\sqrt{n}}$:

$$L = \lim_{n\to\infty} \frac{\sin(1/n)}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1/n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

so that $\sum a_n$ and $\sum b_n$ either both converge or both diverge. But

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1/n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a convergent p-series. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges as well.

71. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{4^n}$

SOLUTION For $n \ge 3$, $2n + 1 < 2^n$, so

$$\frac{2n+1}{4^n} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, so $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$ also converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=3}^{\infty} \frac{2n+1}{4^n}$ converges. Finally, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{4^n}$ converges.

72. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$

SOLUTION Apply the integral test, making the substitution $z = \sqrt{x}$ so that $z^2 = x$ and $2z\,dz = dx$:

$$\int_3^{\infty} \frac{1}{e^{\sqrt{x}}}\,dx = \int_3^{\infty} e^{-x^{1/2}}\,dx = \int_{\sqrt{3}}^{\infty} 2ze^{-z}\,dz$$

Evaluate this integral using integration by parts with $u = 2z$, $dv = e^{-z}\,dz$:

$$\begin{aligned}\int_{\sqrt{3}}^{\infty} 2ze^{-z}\,dz &= uv\Big|_{\sqrt{3}}^{\infty} - \int_{\sqrt{3}}^{\infty} v\,du = (-2ze^{-z})\Big|_{\sqrt{3}}^{\infty} - \int_{\sqrt{3}}^{\infty}(-2e^{-z})\,dz = 2\sqrt{3}e^{-\sqrt{3}} - (2e^{-z})\Big|_{\sqrt{3}}^{\infty} \\ &= 2\sqrt{3}e^{-\sqrt{3}} + 2e^{-\sqrt{3}} < \infty\end{aligned}$$

Since the integral converges, so does the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$.

73. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n^2 - 3n}$

SOLUTION By the comment preceding Exercise 31, we can choose $N \geq 4$ so that for $n \geq N$, $\ln n < n^{1/2}$. Then

$$\sum_{n=N}^{\infty} \frac{\ln n}{n^2 - 3n} \leq \sum_{n=N}^{\infty} \frac{n^{1/2}}{n^2 - 3n} = \sum_{n=N}^{\infty} \frac{1}{n^{3/2} - 3n^{1/2}}$$

To evaluate convergence of the latter series, let $a_n = \dfrac{1}{n^{3/2} - 3n^{1/2}}$ and $b_n = \dfrac{1}{n^{3/2}}$, and apply the Limit Comparison Test:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{3/2} - 3n^{1/2}} \cdot n^{3/2} = \lim_{n\to\infty} \frac{1}{1 - 3n^{-1}} = 0$$

Thus $\sum a_n$ converges if $\sum b_n$ does. But $\sum b_n$ is a convergent p-series. Thus $\sum a_n$ converges and, by the comparison test, so does the original series. Adding back in the finite number of terms for $n < N$ does not affect convergence.

74. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$

SOLUTION Note that

$$3^{\ln n} = (e^{\ln 3})^{\ln n} = (e^{\ln n})^{\ln 3} = n^{\ln 3}.$$

Thus the sum is a p-series with $p = \ln 3 > 1$, so is convergent.

75. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2} \ln n}$

SOLUTION By the comment preceding Exercise 31, we can choose $N \geq 2$ so that for $n \geq N$, $\ln n < n^{1/4}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{n^{1/2} \ln n} > \sum_{n=N}^{\infty} \frac{1}{n^{3/4}}$$

which is a divergent p-series. Thus the original series diverges as well - as usual, adding back in the finite number of terms for $n < N$ does not affect convergence.

76. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2} - \ln^4 n}$

SOLUTION Let

$$a_n = \frac{1}{n^{3/2} - \ln^4 n}, \qquad b_n = \frac{1}{n^{3/2}},$$

and apply the Limit Comparison Test:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2} - \ln^4 n} = \lim_{n\to\infty} \frac{1}{1 - \frac{\ln^4 n}{n^{3/2}}}$$

But by the comment preceding Exercise 31, $\ln n$, and thus $\ln^4 n$, are eventually smaller than any positive power of n, so for n sufficiently large, $\dfrac{\ln^4 n}{n^{3/2}}$ is arbitrarily small. Thus $L = 1$ and $\sum a_n$ converges if and only if $\sum b_n$ does. But $\sum b_n$ is a convergent p-series, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2} - \ln^4 n}$ converges.

77. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}$

SOLUTION Apply the Limit Comparison Test with

$$a_n = \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}, \qquad b_n = \frac{4n^2}{3n^4} = \frac{4}{3n^2}$$

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17} \cdot \frac{3n^2}{4} = \lim_{n\to\infty} \frac{12n^4 + 45n^3}{12n^4 - 20n^2 - 68} = \lim_{n\to\infty} \frac{12 + 45/n}{12 - 20/n^2 - 68/n^4} = 1$$

Now, $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 2 > 1$, so converges. Since $L = 1$, we see that $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}$ converges as well.

78. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^{-n} + 5^{-n}}$

SOLUTION Note that

$$\lim_{n\to\infty} \frac{n}{4^{-n} + 5^{-n}} = \lim_{n\to\infty} \frac{n4^n}{1 + \left(\frac{4}{5}\right)^n}$$

This limit approaches $\infty/1 = \infty$, so the terms of the sequence do not tend to zero. Thus the series is divergent.

79. For which a does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a}$ converge?

SOLUTION First consider the case $a > 0$ but $a \neq 1$. Let $f(x) = \dfrac{1}{x(\ln x)^a}$. This function is continuous, positive and decreasing for $x \geq 2$, so the Integral Test applies. Now,

$$\int_2^{\infty} \frac{dx}{x(\ln x)^a} = \lim_{R\to\infty} \int_2^R \frac{dx}{x(\ln x)^a} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^a} = \frac{1}{1-a} \lim_{R\to\infty} \left(\frac{1}{(\ln R)^{a-1}} - \frac{1}{(\ln 2)^{a-1}} \right).$$

Because

$$\lim_{R\to\infty} \frac{1}{(\ln R)^{a-1}} = \begin{cases} \infty, & 0 < a < 1 \\ 0, & a > 1 \end{cases}$$

we conclude the integral diverges when $0 < a < 1$ and converges when $a > 1$. Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ converges for } a > 1 \text{ and diverges for } 0 < a < 1.$$

Next, consider the case $a = 1$. The series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Let $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^R \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series also diverges.

Finally, consider the case $a < 0$. Let $b = -a > 0$ so the series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^b}{n}$. Since $\ln n > 1$ for all $n \geq 3$, it follows that

$$(\ln n)^b > 1 \quad \text{so} \quad \frac{(\ln n)^b}{n} > \frac{1}{n}.$$

The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so by the Comparison Test we can conclude that $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln n)^b}{n}$ also diverges. Consequently, $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^b}{n}$ diverges. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ diverges for } a < 0.$$

To summarize:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ converges if } a > 1 \text{ and diverges if } a \leq 1.$$

80. For which a does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ converge?

SOLUTION First consider the case $a > 1$. For $n \geq 3$, $\ln n > 1$ and

$$\frac{1}{n^a \ln n} < \frac{1}{n^a}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^a}$ is a p-series with $p = a > 1$, so it converges; hence, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^a}$ also converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^a \ln n}$ converges, which implies the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ also converges.

For $a \leq 1$, $n^a \leq n$ so

$$\frac{1}{n^a \ln n} \geq \frac{1}{n \ln n}$$

for $n \geq 2$. Let $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^R \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ diverges.

To summarize,

$$\sum_{n=2}^{\infty} \frac{1}{n^a \ln n} \text{ converges for } a > 1 \text{ and diverges for } a \leq 1.$$

Approximating Infinite Sums *In Exercises 81–83, let $a_n = f(n)$, where $f(x)$ is a continuous, decreasing function such that $f(x) \geq 0$ and $\int_1^{\infty} f(x)\,dx$ converges.*

81. Show that

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx \qquad \boxed{3}$$

SOLUTION From the proof of the Integral Test, we know that

$$a_2 + a_3 + a_4 + \cdots + a_N \leq \int_1^N f(x)\,dx \leq \int_1^{\infty} f(x)\,dx;$$

that is,

$$S_N - a_1 \leq \int_1^{\infty} f(x)\,dx \quad \text{or} \quad S_N \leq a_1 + \int_1^{\infty} f(x)\,dx.$$

Also from the proof of the Integral test, we know that

$$\int_1^N f(x)\,dx \leq a_1 + a_2 + a_3 + \cdots + a_{N-1} = S_N - a_N \leq S_N.$$

Thus,

$$\int_1^N f(x)\,dx \leq S_N \leq a_1 + \int_1^{\infty} f(x)\,dx.$$

Taking the limit as $N \to \infty$ yields Eq. (3), as desired.

82. CAS Using Eq. (3), show that

$$5 \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \leq 6$$

This series converges slowly. Use a computer algebra system to verify that $S_N < 5$ for $N \leq 43{,}128$ and $S_{43,129} \approx 5.00000021$.

SOLUTION By Eq. (3), we have

$$\int_1^{\infty} \frac{dx}{x^{1.2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \leq 1 + \int_1^{\infty} \frac{dx}{x^{1.2}}.$$

Since

$$\int_1^{\infty} \frac{dx}{x^{1.2}} = \lim_{R\to\infty} \int_1^{R} \frac{dx}{x^{1.2}} = \lim_{R\to\infty} \left(\frac{1}{0.2} - \frac{R^{-0.2}}{0.2} \right) = 5,$$

it follows that

$$5 \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \leq 6.$$

Because $a_n = n^{-1.2} \geq 0$ for all N, S_N is increasing and it suffices to show that $S_N < 5$ for $N = 43{,}128$ to conclude that $S_N < 5$ for all $N \leq 43{,}128$. Using a computer algebra system, we obtain:

$$S_{43,128} = \sum_{n=1}^{43,128} \frac{1}{n^{1.2}} = 4.9999974685$$

and

$$S_{43,129} = \sum_{n=1}^{43,129} \frac{1}{n^{1.2}} = 5.0000002118.$$

83. Let $S = \sum_{n=1}^{\infty} a_n$. Arguing as in Exercise 81, show that

$$\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x)\,dx \leq S \leq \sum_{n=1}^{M+1} a_n + \int_{M+1}^{\infty} f(x)\,dx \qquad \boxed{4}$$

Conclude that

$$0 \leq S - \left(\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x)\,dx \right) \leq a_{M+1} \qquad \boxed{5}$$

This provides a method for approximating S with an error of at most a_{M+1}.

SOLUTION Following the proof of the Integral Test and the argument in Exercise 81, but starting with $n = M + 1$ rather than $n = 1$, we obtain

$$\int_{M+1}^{\infty} f(x)\,dx \leq \sum_{n=M+1}^{\infty} a_n \leq a_{M+1} + \int_{M+1}^{\infty} f(x)\,dx.$$

Adding $\sum_{n=1}^{M} a_n$ to each part of this inequality yields

$$\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n = S \leq \sum_{n=1}^{M+1} a_n + \int_{M+1}^{\infty} f(x)\,dx.$$

Subtracting $\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x)\,dx$ from each part of this last inequality then gives us

$$0 \leq S - \left(\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x)\,dx \right) \leq a_{M+1}.$$

84. CAS Use Eq. (4) with $M = 43{,}129$ to prove that

$$5.5915810 \le \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \le 5.5915839$$

SOLUTION Using Eq. (4) with $f(x) = \frac{1}{x^{1.2}}$, $a_n = \frac{1}{n^{1.2}}$ and $M = 43129$, we find

$$S_{43129} + \int_{43130}^{\infty} \frac{dx}{x^{1.2}} \le \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \le S_{43130} + \int_{43130}^{\infty} \frac{dx}{x^{1.2}}.$$

Now,

$$S_{43129} = 5.0000002118;$$

$$S_{43130} = S_{43129} + \frac{1}{43130^{1.2}} = 5.0000029551;$$

and

$$\int_{43130}^{\infty} \frac{dx}{x^{1.2}} = \lim_{R\to\infty} \int_{43130}^{R} \frac{dx}{x^{1.2}} = -5 \lim_{R\to\infty} \left(\frac{1}{R^{0.2}} - \frac{1}{43130^{0.2}} \right) = \frac{5}{43130^{0.2}} = 0.5915808577.$$

Thus,

$$5.0000002118 + 0.5915808577 \le \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \le 5.0000029551 + 0.5915808577,$$

or

$$5.5915810695 \le \sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \le 5.5915838128.$$

85. CAS Apply Eq. (4) with $M = 40{,}000$ to show that

$$1.644934066 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.644934068$$

Is this consistent with Euler's result, according to which this infinite series has sum $\pi^2/6$?

SOLUTION Using Eq. (4) with $f(x) = \frac{1}{x^2}$, $a_n = \frac{1}{n^2}$ and $M = 40{,}000$, we find

$$S_{40,000} + \int_{40,001}^{\infty} \frac{dx}{x^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le S_{40,001} + \int_{40,001}^{\infty} \frac{dx}{x^2}.$$

Now,

$$S_{40,000} = 1.6449090672;$$

$$S_{40,001} = S_{40,000} + \frac{1}{40{,}001} = 1.6449090678;$$

and

$$\int_{40,001}^{\infty} \frac{dx}{x^2} = \lim_{R\to\infty} \int_{40,001}^{R} \frac{dx}{x^2} = -\lim_{R\to\infty} \left(\frac{1}{R} - \frac{1}{40{,}001} \right) = \frac{1}{40{,}001} = 0.0000249994.$$

Thus,

$$1.6449090672 + 0.0000249994 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.6449090678 + 0.0000249994,$$

or

$$1.6449340665 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.6449340672.$$

Since $\frac{\pi^2}{6} \approx 1.6449340668$, our approximation is consistent with Euler's result.

86. CAS Using a CAS and Eq. (5), determine the value of $\sum_{n=1}^{\infty} n^{-6}$ to within an error less than 10^{-4}. Check that your result is consistent with that of Euler, who proved that the sum is equal to $\pi^6/945$.

SOLUTION According to Eq. (5), if we choose M so that $(M+1)^{-6} < 10^{-4}$, we can then approximate the sum to within 10^{-4}. Solving $(M+1)^{-6} = 10^{-4}$ gives $M+1 = 10^{-2/3} \approx 4.641$, so the smallest such integral M is $M = 4$. Denote by S the sum of the series. Then

$$0 \le S - \left(\sum_{n=1}^{4} n^{-6} + \int_5^{\infty} x^{-6}\,dx\right) \le (M+1)^{-6} < 10^{-4}$$

We have

$$\sum_{n=1}^{4} n^{-6} = \frac{1}{1} + \frac{1}{64} + \frac{1}{729} + \frac{1}{4096} \approx 1.017240883$$

$$\int_5^{\infty} x^{-6}\,dx = -\frac{1}{5}x^{-5}\Big|_5^{\infty} = \frac{1}{5^6} \approx 0.000064$$

The sum of these two is $\approx$ 1.017304883, while $\dfrac{\pi^6}{945} \approx 1.017343063$. These two values differ by approximately $0.000038180 < 10^{-4}$, so the result is consistent with Euler's calculation.

87. CAS Using a CAS and Eq. (5), determine the value of $\sum_{n=1}^{\infty} n^{-5}$ to within an error less than 10^{-4}.

SOLUTION Using Eq. (5) with $f(x) = x^{-5}$ and $a_n = n^{-5}$, we have

$$0 \le \sum_{n=1}^{\infty} n^{-5} - \left(\sum_{n=1}^{M+1} n^{-5} + \int_{M+1}^{\infty} x^{-5}\,dx\right) \le (M+1)^{-5}.$$

To guarantee an error less than 10^{-4}, we need $(M+1)^{-5} \le 10^{-4}$. This yields $M \ge 10^{4/5} - 1 \approx 5.3$, so we choose $M = 6$. Now,

$$\sum_{n=1}^{7} n^{-5} = 1.0368498887,$$

and

$$\int_7^{\infty} x^{-5}\,dx = \lim_{R\to\infty} \int_7^{R} x^{-5}\,dx = -\frac{1}{4}\lim_{R\to\infty}\left(R^{-4} - 7^{-4}\right) = \frac{1}{4\cdot 7^4} = 0.0001041233.$$

Thus,

$$\sum_{n=1}^{\infty} n^{-5} \approx \sum_{n=1}^{7} n^{-5} + \int_7^{\infty} x^{-5}\,dx = 1.0368498887 + 0.0001041233 = 1.0369540120.$$

88. How far can a stack of identical books (of mass m and unit length) extend without tipping over? The stack will not tip over if the $(n+1)$st book is placed at the bottom of the stack with its right edge located at the center of mass of the first n books (Figure 5). Let c_n be the center of mass of the first n books, measured along the x-axis, where we take the positive x-axis to the left of the origin as in Figure 6. Recall that if an object of mass m_1 has center of mass at x_1 and a second object of m_2 has center of mass x_2, then the center of mass of the system has x-coordinate

$$\frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

(a) Show that if the $(n+1)$st book is placed with its right edge at c_n, then its center of mass is located at $c_n + \frac{1}{2}$.

(b) Consider the first n books as a single object of mass nm with center of mass at c_n and the $(n+1)$st book as a second object of mass m. Show that if the $(n+1)$st book is placed with its right edge at c_n, then $c_{n+1} = c_n + \dfrac{1}{2(n+1)}$.

(c) Prove that $\lim_{n\to\infty} c_n = \infty$. Thus, by using enough books, the stack can be extended as far as desired without tipping over.

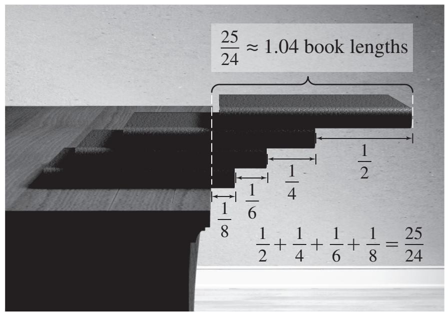


FIGURE 5

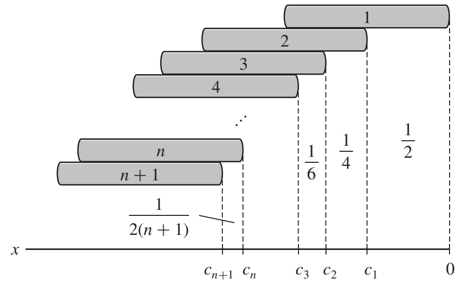


FIGURE 6

SOLUTION Let $f(x) = \frac{1}{x}$. This function is continuous, positive and decreasing, so following the argument of Exercise 81, we know that

$$\int_1^N f(x)\,dx \le S_N \le a_1 + \int_1^N f(x)\,dx,$$

or

$$\ln N \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \le 1 + \ln N.$$

Using the inequality on the right-hand side, we know that

$$S_{8100} \le 1 + \ln 8100 = 9.999619 < 10;$$

using the inequality on the left-hand side, we can guarantee $S_N \ge 100$ by making $\ln N \ge 100$. Thus, we can take

$$N \ge e^{100} \approx 2.688 \times 10^{43}.$$

89. The following argument proves the divergence of the harmonic series $S = \sum_{n=1}^{\infty} 1/n$ without using the Integral Test. Let

$$S_1 = 1 + \frac{1}{3} + \frac{1}{5} + \cdots, \qquad S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

Show that if S converges, then

(a) S_1 and S_2 also converge and $S = S_1 + S_2$.

(b) $S_1 > S_2$ and $S_2 = \frac{1}{2}S$.

Observe that (b) contradicts (a), and conclude that S diverges.

SOLUTION Assume throughout that S converges; we will derive a contradiction. Write

$$a_n = \frac{1}{n}, \quad b_n = \frac{1}{2n-1}, \quad c_n = \frac{1}{2n}$$

for the n^{th} terms in the series S, S_1, and S_2. Since $2n - 1 \ge n$ for $n \ge 1$, we have $b_n < a_n$. Since $S = \sum a_n$ converges, so does $S_1 = \sum b_n$ by the Comparison Test. Also, $c_n = \dfrac{1}{2}a_n$, so again by the Comparison Test, the convergence of S implies the convergence of $S_2 = \sum c_n$. Now, define two sequences

$$b'_n = \begin{cases} b_{(n+1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$c'_n = \begin{cases} 0 & n \text{ odd} \\ c_{n/2} & n \text{ even} \end{cases}$$

That is, b'_n and c'_n look like b_n and c_n, but have zeros inserted in the "missing" places compared to a_n. Then $a_n = b'_n + c'_n$; also $S_1 = \sum b_n = \sum b'_n$ and $S_2 = \sum c_n = \sum c'_n$. Finally, since S, S_1, and S_2 all converge, we have

$$S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b'_n + c'_n) = \sum_{n=1}^{\infty} b'_n + \sum_{n=1}^{\infty} c'_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n = S_1 + S_2$$

Now, $b_n > c_n$ for every n, so that $S_1 > S_2$. Also, we showed above that $c_n = \dfrac{1}{2}a_n$, so that $2S_2 = S$. Putting all this together gives

$$S = S_1 + S_2 > S_2 + S_2 = 2S_2 = S$$

so that $S > S$, a contradiction. Thus S must diverge.

Further Insights and Challenges

90. Let $S = \sum_{n=2}^{\infty} a_n$, where $a_n = (\ln(\ln n))^{-\ln n}$.

(a) Show, by taking logarithms, that $a_n = n^{-\ln(\ln(\ln n))}$.

(b) Show that $\ln(\ln(\ln n)) \geq 2$ if $n > C$, where $C = e^{e^{e^2}}$.

(c) Show that S converges.

SOLUTION

(a) Let $a_n = (\ln(\ln n))^{-\ln n}$. Then

$$\ln a_n = (-\ln n)\ln(\ln(\ln n)),$$

and

$$a_n = e^{(-\ln n)\ln(\ln(\ln n))} = \left(e^{\ln n}\right)^{-\ln(\ln(\ln n))} = n^{-\ln(\ln(\ln n))}.$$

(b) Suppose $n > e^{e^{e^2}}$. Then

$$\ln n > \ln e^{e^{e^2}} = e^{e^2};$$

$$\ln(\ln n) > \ln e^{e^2} = e^2; \text{ and}$$

$$\ln(\ln(\ln n)) > \ln e^2 = 2.$$

(c) Combining the results from parts (a) and (b), we have

$$a_n = \frac{1}{n^{\ln(\ln(\ln n))}} \leq \frac{1}{n^2}$$

for $n > C = e^{e^{e^2}}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so $\sum_{n=C+1}^{\infty} \frac{1}{n^2}$ also converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=C+1}^{\infty} a_n$ converges, which means that the series $\sum_{n=2}^{\infty} a_n$ converges.

91. Kummer's Acceleration Method Suppose we wish to approximate $S = \sum_{n=1}^{\infty} 1/n^2$. There is a similar telescoping series whose value can be computed exactly (Example 1 in Section 10.2):

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(a) Verify that

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right)$$

Thus for M large,

$$S \approx 1 + \sum_{n=1}^{M} \frac{1}{n^2(n+1)} \qquad \boxed{6}$$

(b) Explain what has been gained. Why is Eq. (6) a better approximation to S than is $\sum_{n=1}^{M} 1/n^2$?

(c) CAS Compute

$$\sum_{n=1}^{1000} \frac{1}{n^2}, \qquad 1 + \sum_{n=1}^{100} \frac{1}{n^2(n+1)}$$

Which is a better approximation to S, whose exact value is $\pi^2/6$?

SOLUTION

(a) Because the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ both converge,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} = S.$$

Now,

$$\frac{1}{n^2} - \frac{1}{n(n+1)} = \frac{n+1}{n^2(n+1)} - \frac{n}{n^2(n+1)} = \frac{1}{n^2(n+1)},$$

so, for M large,

$$S \approx 1 + \sum_{n=1}^{M} \frac{1}{n^2(n+1)}.$$

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ converges more rapidly than $\sum_{n=1}^{\infty} \frac{1}{n^2}$ since the degree of n in the denominator is larger.

(c) Using a computer algebra system, we find

$$\sum_{n=1}^{1000} \frac{1}{n^2} = 1.6439345667 \quad \text{and} \quad 1 + \sum_{n=1}^{100} \frac{1}{n^2(n+1)} = 1.6448848903.$$

The second sum is more accurate because it is closer to the exact solution $\frac{\pi^2}{6} \approx 1.6449340668$.

92. CAS The series $S = \sum_{k=1}^{\infty} k^{-3}$ has been computed to more than 100 million digits. The first 30 digits are

$$S = 1.202056903159594285399738161511$$

Approximate S using the Acceleration Method of Exercise 91 with $M = 100$ and auxiliary series

$$R = \sum_{n=1}^{\infty} (n(n+1)(n+2))^{-1}.$$

According to Exercise 46 in Section 10.2, R is a telescoping series with the sum $R = \frac{1}{4}$.

SOLUTION We compute the difference between the general term of the given series and the general term of the auxiliary series:

$$\frac{1}{k^3} - \frac{1}{k(k+1)(k+2)} = \frac{(k+1)(k+2) - k^2}{k^3(k+1)(k+2)} = \frac{k^2 + 3k + 2 - k^2}{k^3(k+1)(k+2)} = \frac{3k+2}{k^3(k+1)(k+2)}$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} + \sum_{k=1}^{\infty} \frac{3k+2}{k^3(k+1)(k+2)} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{3k+2}{k^3(k+1)(k+2)}$$

With $M = 100$ and using a computer algebra system, we find

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \approx \frac{1}{4} + \sum_{k=1}^{100} \frac{3k+2}{k^3(k+1)(k+2)} = 1.2020559349.$$

10.4 Absolute and Conditional Convergence

Preliminary Questions

1. Give an example of a series such that $\sum a_n$ converges but $\sum |a_n|$ diverges.

SOLUTION The series $\sum \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the Leibniz Test, but the positive series $\sum \frac{1}{\sqrt[3]{n}}$ is a divergent p-series.

2. Which of the following statements is equivalent to Theorem 1?

(a) If $\sum_{n=0}^{\infty} |a_n|$ diverges, then $\sum_{n=0}^{\infty} a_n$ also diverges.

(b) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} |a_n|$ also converges.

SOLUTION The correct answer is **(b)**: If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges. Take $a_n = (-1)^n \frac{1}{n}$ to see that statements **(a)** and **(c)** are not true in general.

3. Lathika argues that $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series and therefore converges. Is Lathika right?

SOLUTION No. Although $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series, the terms $a_n = \sqrt{n}$ do not form a decreasing sequence that tends to zero. In fact, $a_n = \sqrt{n}$ is an increasing sequence that tends to ∞, so $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ diverges by the Divergence Test.

4. Suppose that a_n is positive, decreasing, and tends to 0, and let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$. What can we say about $|S - S_{100}|$ if $a_{101} = 10^{-3}$? Is S larger or smaller than S_{100}?

SOLUTION From the text, we know that $|S - S_{100}| < a_{101} = 10^{-3}$. Also, the Leibniz test tells us that $S_{2N} < S < S_{2N+1}$ for any $N \geq 1$, so that $S_{100} < S$.

Exercises

1. Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely.

SOLUTION The positive series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2}$. Thus, the positive series converges, and the given series converges absolutely.

2. Show that the following series converges conditionally:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2/3}} = \frac{1}{1^{2/3}} - \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} - \frac{1}{4^{2/3}} + \cdots$$

SOLUTION Let $a_n = \frac{1}{n^{2/3}}$. Then a_n forms a decreasing sequence that tends to zero; hence, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2/3}}$ converges by the Leibniz Test. However, the positive series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series, so the original series converges conditionally.

In Exercises 3–10, determine whether the series converges absolutely, conditionally, or not at all.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

SOLUTION The sequence $a_n = \frac{1}{n^{1/3}}$ is positive, decreasing, and tends to zero; hence, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$ converges by the Leibniz Test. However, the positive series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a divergent p-series, so the original series converges conditionally.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$

SOLUTION Because

$$\lim_{n\to\infty} \frac{n^4}{n^3+1} = \infty,$$

the general term $\frac{(-1)^n n^4}{n^3+1}$ of the series does not tend to zero; hence, this series diverges by the Divergence Test.

5. $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$

SOLUTION The positive series $\sum_{n=0}^{\infty} \left(\frac{1}{1.1}\right)^n$ is a convergent geometric series; thus, the original series converges absolutely.

6. $\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{4})}{n^2}$

SOLUTION Because

$$\left| \frac{\sin\left(\frac{\pi n}{4}\right)}{n^2} \right| = \frac{\left|\sin\left(\frac{\pi n}{4}\right)\right|}{n^2} \le \frac{1}{n^2}$$

the positive series forms a convergent p-series; thus, the original series converges absolutely.

7. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

SOLUTION Let $a_n = \frac{1}{n \ln n}$. Then a_n forms a decreasing sequence (note that n and $\ln n$ are both increasing functions of n) that tends to zero; hence, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Leibniz Test. However, the positive series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, so the original series converges conditionally.

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}}$

SOLUTION Because

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1,$$

the general term $\frac{(-1)^n}{1+\frac{1}{n}}$ of the series does not tend to zero; hence, the series diverges by the Divergent Test.

9. $\sum_{n=2}^{\infty} \frac{\cos n\pi}{(\ln n)^2}$

SOLUTION Since $\cos n\pi$ alternates between $+1$ and -1,

$$\sum_{n=2}^{\infty} \frac{\cos n\pi}{(lnn)^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(lnn)^2}$$

This is an alternating series whose general term decreases to zero, so it converges. The associated positive series,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$$

is a divergent series, so the original series converges conditionally.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

SOLUTION The associated positive series is

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n}$$

which is a convergent geometric series. Thus the associated positive series converges, so the original series converges absolutely.

11. Let $S = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$.

(a) Calculate S_n for $1 \le n \le 10$.

(b) Use Eq. (2) to show that $0.9 \le S \le 0.902$.

SOLUTION

(a)

$$S_1 = 1 \qquad S_6 = S_5 - \frac{1}{6^3} = 0.899782407$$

$$S_2 = 1 - \frac{1}{2^3} = \frac{7}{8} = 0.875 \qquad S_7 = S_6 + \frac{1}{7^3} = 0.902697859$$

$$S_3 = S_2 + \frac{1}{3^3} = 0.912037037 \qquad S_8 = S_7 - \frac{1}{8^3} = 0.900744734$$

$$S_4 = S_3 - \frac{1}{4^3} = 0.896412037 \qquad S_9 = S_8 + \frac{1}{9^3} = 0.902116476$$

$$S_5 = S_4 + \frac{1}{5^3} = 0.904412037 \qquad S_{10} = S_9 - \frac{1}{10^3} = 0.901116476$$

(b) By Eq. (2),

$$|S_{10} - S| \le a_{11} = \frac{1}{11^3},$$

so

$$S_{10} - \frac{1}{11^3} \le S \le S_{10} + \frac{1}{11^3},$$

or

$$0.900365161 \le S \le 0.901867791.$$

12. Use Eq. (2) to approximate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$$

to four decimal places.

SOLUTION Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$, so that $a_n = \dfrac{1}{n!}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{1}{(N+1)!}.$$

To guarantee accuracy to four decimal places, we must choose N so that

$$\frac{1}{(N+1)!} < 5 \times 10^{-5} \quad \text{or} \quad (N+1)! > 20{,}000.$$

Because $7! = 5040$ and $8! = 40,320$, the smallest value that satisfies the required inequality is $N = 7$. Thus,

$$S \approx S_7 = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} = 0.632142857.$$

13. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to three decimal places.

SOLUTION Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$, so that $a_n = \frac{1}{n^4}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{1}{(N+1)^4}.$$

To guarantee accuracy to three decimal places, we must choose N so that

$$\frac{1}{(N+1)^4} < 5 \times 10^{-4} \quad \text{or} \quad N > \sqrt[4]{2000} - 1 \approx 5.7.$$

The smallest value that satisfies the required inequality is then $N = 6$. Thus,

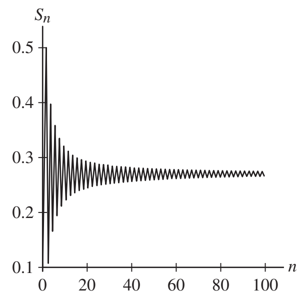
$$S \approx S_6 = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} = 0.946767824.$$

14. CAS Let

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

Use a computer algebra system to calculate and plot the partial sums S_n for $1 \le n \le 100$. Observe that the partial sums zigzag above and below the limit.

SOLUTION The partial sums associated with the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ are plotted below. As expected, the partial sums alternate between overestimating and underestimating the sum.



In Exercises 15 and 16, find a value of N such that S_N approximates the series with an error of at most 10^{-5}. If you have a CAS, compute this value of S_N.

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$

SOLUTION Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,(n+2)\,(n+3)}$, so that $a_n = \frac{1}{n\,(n+2)\,(n+3)}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{1}{(N+1)(N+3)(N+4)}.$$

We must choose N so that

$$\frac{1}{(N+1)(N+3)(N+4)} \le 10^{-5} \quad \text{or} \quad (N+1)(N+3)(N+4) \ge 10^5.$$

For $N = 43$, the product on the left hand side is 95,128, while for $N = 44$ the product is 101,520; hence, the smallest value of N which satisfies the required inequality is $N = 44$. Thus,

$$S \approx S_{44} = \sum_{n=1}^{44} \frac{(-1)^{n+1}}{n(n+2)(n+3)} = 0.0656746.$$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n!}$

SOLUTION Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n!}$, so that $a_n = \dfrac{\ln n}{n!}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{\ln(N+1)}{(N+1)!}.$$

To make the error at most 10^{-5}, we must choose N so that

$$\frac{\ln(N+1)}{(N+1)!} \le 10^{-5}.$$

For $N = 7$, the left-hand side of the above inequality is 5.157×10^{-5}, while for $N = 8$, the left-hand side is 6.055×10^{-6}; hence, the smallest value for N which satisfies the required inequality is $N = 8$. Thus,

$$S \approx S_8 = \sum_{n=1}^{8} \frac{(-1)^{n+1} \ln n}{n!} = -0.209975859.$$

In Exercises 17–32, determine convergence or divergence by any method.

17. $\displaystyle\sum_{n=0}^{\infty} 7^{-n}$

SOLUTION This is a (positive) geometric series with $r = \dfrac{1}{7} < 1$, so it converges.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{7.5}}$

SOLUTION This is a p-series with $p = 7.5 > 1$, so it converges.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 3^n}$

SOLUTION Use the Limit Comparison Test with $\dfrac{1}{5^n}$:

$$L = \lim_{n\to\infty} \frac{1/(5^n - 3^n)}{1/5^n} = \lim_{n\to\infty} \frac{5^n}{5^n - 3^n} = \lim_{n\to\infty} \frac{1}{1 - (3/5)^n} = 1$$

But $\sum_{n=1}^{\infty} \dfrac{1}{5^n}$ is a convergent geometric series. Since $L = 1$, the Limit Comparison Test tells us that the original series converges as well.

20. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2 - n}$

SOLUTION Apply the Limit Comparison Test and compare with the divergent harmonic series:

$$L = \lim_{n\to\infty} \frac{\frac{n}{n^2-n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 - n} = 1.$$

Because $L > 0$, we conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2 - n}$ diverges.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4 + 12n}$

SOLUTION Use the Limit Comparison Test with $\dfrac{1}{3n^4}$:

$$L = \lim_{n\to\infty} \frac{(1/(3n^4+12n)}{1/3n^4} = \lim_{n\to\infty} \frac{3n^4}{3n^4+12n} = \lim_{n\to\infty} \frac{1}{1+4n^{-3}} = 1$$

But $\sum_{n=1}^{\infty} \dfrac{1}{3n^4} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series. Since $L = 1$, the Limit Comparison Test tells us that the original series converges as well.

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

SOLUTION This is an alternating series with $a_n = \dfrac{1}{\sqrt{n^2+1}}$. Because a_n is a decreasing sequence that converges to zero, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges by the Leibniz Test.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

SOLUTION Apply the Limit Comparison Test and compare the series with the divergent harmonic series:

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1.$$

Because $L > 0$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}}$

SOLUTION This series diverges, since the general term of the associated positive series tends to 1, not to 0:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+n^{-2}}} = 1$$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{5^n}$

SOLUTION The series

$$\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$$

is a convergent geometric series, as is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^n}{5^n} = \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{3^n + (-1)^n 2^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n$$

also converges.

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

SOLUTION This is an alternating series with $a_n = \dfrac{1}{(2n+1)!}$. Because a_n is a decreasing sequence which converges to zero, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$ converges by the Leibniz Test.

27. $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$

SOLUTION Consider the associated positive series $\sum_{n=1}^{\infty} n^2 e^{-n^3/3}$. This series can be seen to converge by the Integral Test:

$$\int_1^{\infty} x^2 e^{-x^3/3}\,dx = \lim_{R\to\infty} \int_1^R x^2 e^{-x^3/3}\,dx = -\lim_{R\to\infty} e^{-x^3/3}\Big|_1^R = e^{-1/3} + \lim_{R\to\infty} e^{-R^3/3} = e^{-1/3}.$$

The integral converges, so the original series converges absolutely.

28. $\sum_{n=1}^{\infty} n e^{-n^3/3}$

SOLUTION This is a positive series, and by the Comparison Test with the associated positive series in the previous exercise,

$$\sum_{n=1}^{\infty} n e^{-n^3/3} \le \sum_{n=1}^{\infty} n^2 e^{-n^3/3}$$

Since the series on the right converges, so does the original series.

29. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2}(\ln n)^2}$

SOLUTION This is an alternating series with $a_n = \frac{1}{n^{1/2}(\ln n)^2}$. Because a_n is a decreasing sequence which converges to zero, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2}(\ln n)^2}$ converges by the Leibniz Test. (Note that the series converges only conditionally, not absolutely; the associated positive series is eventually greater than $\frac{1}{n^{3/4}}$, which is a divergent p-series).

30. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/4}}$

SOLUTION Use the Integral Test, with the substitution $u = \ln x$:

$$\int_2^{\infty} \frac{1}{x \ln^{1/4} x}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x \ln^{1/4} x}\,dx = \lim_{R\to\infty} \int_{\ln 2}^R u^{-1/4}\,du = \lim_{R\to\infty} \frac{4}{3} u^{3/4}\Big|_{\ln 2}^R$$
$$= -\frac{4}{3}\left((\ln 2)^{3/4} + \lim_{R\to\infty} R^{3/4}\right)$$

The integral diverges, so the original series diverges as well.

31. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.05}}$

SOLUTION Choose N so that for $n \ge N$ we have $\ln n \le n^{0.01}$. Then

$$\sum_{n=N}^{\infty} \frac{\ln n}{n^{1.05}} \le \sum_{n=N}^{\infty} \frac{n^{0.01}}{n^{1.05}} = \sum_{n=N}^{\infty} \frac{1}{n^{1.04}}$$

This is a convergent p-series, so by the Comparison Test, the original series converges as well.

32. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

SOLUTION Choose N so that for $n \ge N$ we have $\ln n < n^{0.25}$ so that $\ln^2 n < n^{0.5}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{(\ln n)^2} > \sum_{n=N}^{\infty} \frac{1}{n^{0.5}}$$

This is a divergent p-series, so by the Comparison Test, the original series diverges as well.

33. Show that

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$$

converges by computing the partial sums. Does it converge absolutely?

SOLUTION The sequence of partial sums is

$$S_1 = \frac{1}{2}$$

$$S_2 = S_1 - \frac{1}{2} = 0$$

$$S_3 = S_2 + \frac{1}{3} = \frac{1}{3}$$

$$S_4 = S_3 - \frac{1}{3} = 0$$

and, in general,

$$S_N = \begin{cases} \frac{1}{N}, & \text{for odd } N \\ 0, & \text{for even } N \end{cases}$$

Thus, $\lim_{N\to\infty} S_N = 0$, and the series converges to 0. The positive series is

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \cdots = 2\sum_{n=2}^{\infty} \frac{1}{n};$$

which diverges. Therefore, the original series converges conditionally, not absolutely.

34. The Leibniz Test cannot be applied to

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$$

Why not? Show that it converges by another method.

SOLUTION The sequence of terms $\{a_n\}$ for this alternating series is

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \dots, \frac{1}{2^n}, \frac{1}{3^n}, \frac{1}{2^{n+1}}, \frac{1}{3^{n+1}}, \dots$$

Now,

$$\frac{1}{3^2} = \frac{1}{9} < \frac{1}{8} = \frac{1}{2^3}.$$

Moreover, if we assume that

$$\frac{1}{3^k} < \frac{1}{2^{k+1}}$$

for some k, then

$$\frac{1}{3^{k+1}} = \frac{1}{3} \cdot \frac{1}{3^k} < \frac{1}{3}\frac{1}{2^{k+1}} < \frac{1}{2}\frac{1}{2^{k+1}} = \frac{1}{2^{k+2}}.$$

Thus, by mathematical induction,

$$\frac{1}{3^n} < \frac{1}{2^{n+1}}$$

for all $n \geq 2$. The sequence $\{a_n\}$ is therefore not decreasing, and the Leibniz Test does not apply.

We may express the given series as

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right).$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

are both convergent geometric series, it follows that this series converges, and

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}-\frac{\frac{1}{3}}{1-\frac{1}{3}}=1-\frac{1}{2}=\frac{1}{2}.$$

35. **Assumptions Matter** Show by counterexample that the Leibniz Test does not remain true if the sequence a_n tends to zero but is not assumed nonincreasing. *Hint:* Consider

$$R=\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+\cdots+\left(\frac{1}{n}-\frac{1}{2^n}\right)+\cdots$$

SOLUTION Let

$$R=\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+\cdots+\left(\frac{1}{n+1}-\frac{1}{2^{n+1}}\right)+\cdots$$

This is an alternating series with

$$a_n=\begin{cases}\dfrac{1}{k+1}, & n=2k-1\\[2mm] \dfrac{1}{2^{k+1}}, & n=2k\end{cases}$$

Note that $a_n \to 0$ as $n \to \infty$, but the sequence $\{a_n\}$ is not decreasing. We will now establish that R diverges.

For sake of contradiction, suppose that R converges. The geometric series

$$\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}$$

converges, so the sum of R and this geometric series must also converge; however,

$$R+\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}=\sum_{n=2}^{\infty}\frac{1}{n},$$

which diverges because the harmonic series diverges. Thus, the series R must diverge.

36. Determine whether the following series converges conditionally:

$$1-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}+\frac{1}{4}-\frac{1}{9}+\frac{1}{5}-\frac{1}{11}+\cdots$$

SOLUTION Although the signs alternate, the terms a_n are not decreasing, so we cannot apply the Leibniz Test. However, we may express the series as

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{2n+1}\right)=\sum_{n=1}^{\infty}\frac{n+1}{n(2n+1)}.$$

Using the Limit Comparison Test and comparing with the harmonic series, we find

$$L=\lim_{n\to\infty}\frac{\frac{n+1}{n(2n+1)}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n+1}{2n+1}=\frac{1}{2}.$$

Because $L>0$, we conclude that the series

$$1-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}+\frac{1}{4}-\frac{1}{9}+\frac{1}{5}-\frac{1}{11}+\cdots$$

diverges.

37. Prove that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges. Then give an example where $\sum a_n$ is only conditionally convergent and $\sum a_n^2$ diverges.

SOLUTION Suppose the series $\sum a_n$ converges absolutely. Because $\sum |a_n|$ converges, we know that

$$\lim_{n\to\infty}|a_n|=0.$$

Therefore, there exists a positive integer N such that $|a_n|<1$ for all $n\ge N$. It then follows that for $n\ge N$,

$$0\le a_n^2=|a_n|^2=|a_n|\cdot|a_n|<|a_n|\cdot 1=|a_n|.$$

By the Comparison Test we can then conclude that $\sum a_n^2$ also converges.

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This series converges by the Leibniz Test, but the corresponding positive series is a divergent p-series; that is, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent. Now, $\sum_{n=1}^{\infty} a_n^2$ is the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Thus, $\sum a_n^2$ need not converge if $\sum a_n$ is only conditionally convergent.

Further Insights and Challenges

38. Prove the following variant of the Leibniz Test: If $\{a_n\}$ is a positive, decreasing sequence with $\lim_{n\to\infty} a_n = 0$, then the series

$$a_1 + a_2 - 2a_3 + a_4 + a_5 - 2a_6 + \cdots$$

converges. *Hint:* Show that S_{3N} is increasing and bounded by $a_1 + a_2$, and continue as in the proof of the Leibniz Test.

SOLUTION Following the hint, we first examine the sequence $\{S_{3N}\}$. Now,

$$S_{3N+3} = S_{3(N+1)} = S_{3N} + a_{3N+1} + a_{3N+2} - 2a_{3N+3} = S_{3N} + \left(a_{3N+1} - a_{3N+3}\right) + \left(a_{3N+2} - a_{3N+3}\right) \ge S_{3N}$$

because $\{a_n\}$ is a decreasing sequence. Moreover,

$$\begin{aligned} S_{3N} &= a_1 + a_2 - \sum_{k=1}^{N-1} \left(2a_{3k} - a_{3k+1} - a_{3k+2}\right) - 2a_{3N} \\ &= a_1 + a_2 - \sum_{k=1}^{N-1} \left[\left(a_{3k} - a_{3k+1}\right) + \left(a_{3k} - a_{3k+2}\right) - 2a_{3N}\right] \le a_1 + a_2 \end{aligned}$$

again because $\{a_n\}$ is a decreasing sequence. Thus, $\{S_{3N}\}$ is an increasing sequence with an upper bound; hence, $\{S_{3N}\}$ converges. Next,

$$S_{3N+1} = S_{3N} + a_{3N+1} \quad \text{and} \quad S_{3N+2} = S_{3N} + a_{3N+1} + a_{3N+2}.$$

Given that $\lim_{n\to\infty} a_n = 0$, it follows that

$$\lim_{N\to\infty} S_{3N+1} = \lim_{N\to\infty} S_{3N+2} = \lim_{N\to\infty} S_{3N}.$$

Having just established that $\lim_{N\to\infty} S_{3N}$ exists, it follows that the sequences $\{S_{3N+1}\}$ and $\{S_{3N+2}\}$ converge to the same limit. Finally, we can conclude that the sequence of partial sums $\{S_N\}$ converges, so the given series converges.

39. Use Exercise 38 to show that the following series converges:

$$S = \frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{2}{\ln 4} + \frac{1}{\ln 5} + \frac{1}{\ln 6} - \frac{2}{\ln 7} + \cdots$$

SOLUTION The given series has the structure of the generic series from Exercise 38 with $a_n = \frac{1}{\ln(n+1)}$. Because a_n is a positive, decreasing sequence with $\lim_{n\to\infty} a_n = 0$, we can conclude from Exercise 38 that the given series converges.

40. Prove the conditional convergence of

$$R = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots$$

SOLUTION Using Exercise 38 as a template, we first examine the sequence $\{R_{4N}\}$. Now,

$$\begin{aligned} R_{4N+4} = R_{4(N+1)} &= R_{4N} + \frac{1}{4N+1} + \frac{1}{4N+2} + \frac{1}{4N+3} - \frac{3}{4N+4} \\ &= R_N + \left(\frac{1}{4N+1} - \frac{1}{4N+4}\right) + \left(\frac{1}{4N+2} - \frac{1}{4N+4}\right) + \left(\frac{1}{4N+3} - \frac{1}{4N+4}\right) \ge R_{4N}. \end{aligned}$$

Moreover,

$$R_{4N} = 1 + \frac{1}{2} + \frac{1}{3} - \sum_{k=1}^{N-1} \left(\frac{3}{4k} - \frac{1}{4k+1} - \frac{1}{4k+2} - \frac{1}{4k+3}\right) - \frac{3}{4N} \le 1 + \frac{1}{2} + \frac{1}{3}.$$

Thus, $\{R_{4N}\}$ is an increasing sequence with an upper bound; hence, $\{R_{4N}\}$ converges. Next,

$$R_{4N+1} = R_{4N} + \frac{1}{4N+1};$$

$$R_{4N+2} = R_{4N} + \frac{1}{4N+1} + \frac{1}{4N+2}; \text{ and}$$

$$R_{4N+3} = R_{4N} + \frac{1}{4N+1} + \frac{1}{4N+2} + \frac{1}{4N+3},$$

so

$$\lim_{n\to\infty} R_{4N+1} = \lim_{N\to\infty} R_{4N+2} = \lim_{N\to\infty} R_{4N+3} = \lim_{N\to\infty} R_{4N}.$$

Having just established that $\lim_{N\to\infty} R_{4N}$ exists, it follows that the sequences $\{R_{4N+1}\}$, $\{R_{4N+2}\}$ and $\{R_{4N+3}\}$ converge to the same limit. Finally, we can conclude that the sequence of partial sums $\{R_N\}$ converges, so the series R converges.

Now, consider the positive series

$$R^+ = 1 + \frac{1}{2} + \frac{1}{3} + \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{3}{8} + \cdots$$

Because the terms in this series are greater than or equal to the corresponding terms in the divergent harmonic series, it follows from the Comparison Test that R^+ diverges. Thus, by definition, R converges conditionally.

41. Show that the following series diverges:

$$S = 1 + \frac{1}{2} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \cdots$$

Hint: Use the result of Exercise 40 to write S as the sum of a convergent series and a divergent series.

SOLUTION Let

$$R = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots$$

and

$$S = 1 + \frac{1}{2} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \cdots$$

For sake of contradiction, suppose the series S converges. From Exercise 40, we know that the series R converges. Thus, the series $S - R$ must converge; however,

$$S - R = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \cdots = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges because the harmonic series diverges. Thus, the series S must diverge.

42. Prove that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\ln n)^a}{n}$$

converges for all exponents a. *Hint:* Show that $f(x) = (\ln x)^a/x$ is decreasing for x sufficiently large.

SOLUTION This is an alternating series with $a_n = \frac{(\ln n)^a}{n}$. Following the hint, consider the function $f(x) = \dfrac{(\ln x)^a}{x}$. Now,

$$f'(x) = \frac{a(\ln x)^{a-1} - (\ln x)^a}{x^2} = \frac{(\ln x)^{a-1}}{x^2}(a - \ln x),$$

so $f'(x) < 0$ and f is decreasing for $x > e^a$. If $a \le 0$, then it is clear that

$$\lim_{x\to\infty} \frac{(\ln x)^a}{x} = 0;$$

if $a > 0$, then repeated use of L'Hôpital's Rule leads to the same conclusion. Let N be any integer greater than e^a; then, $\{a_n\}$ is a decreasing sequence for $n \ge N$ which converges to zero and the series $\displaystyle\sum_{n=N}^{\infty} (-1)^{n+1} \frac{(\ln n)^a}{n}$ converges by the Leibniz Test. Finally, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\ln n)^a}{n}$ also converges.

43. We say that $\{b_n\}$ is a rearrangement of $\{a_n\}$ if $\{b_n\}$ has the same terms as $\{a_n\}$ but occurring in a different order. Show that if $\{b_n\}$ is a rearrangement of $\{a_n\}$ and $S = \sum_{n=1}^{\infty} a_n$ converges absolutely, then $T = \sum_{n=1}^{\infty} b_n$ also converges absolutely. (This result does not hold if S is only conditionally convergent.) *Hint:* Prove that the partial sums $\sum_{n=1}^{N} |b_n|$ are bounded. It can be shown further that $S = T$.

SOLUTION Suppose the series $S = \sum_{n=1}^{\infty} a_n$ converges absolutely and denote the corresponding positive series by

$$S^+ = \sum_{n=1}^{\infty} |a_n|.$$

Further, let $T_N = \sum_{n=1}^{N} |b_n|$ denote the Nth partial sum of the series $\sum_{n=1}^{\infty} |b_n|$. Because $\{b_n\}$ is a rearrangement of $\{a_n\}$, we know that

$$0 \le T_N \le \sum_{n=1}^{\infty} |a_n| = S^+;$$

that is, the sequence $\{T_N\}$ is bounded. Moreover,

$$T_{N+1} = \sum_{n=1}^{N+1} |b_n| = T_N + |b_{N+1}| \ge T_N;$$

that is, $\{T_N\}$ is increasing. It follows that $\{T_N\}$ converges, so the series $\sum_{n=1}^{\infty} |b_n|$ converges, which means the series $\sum_{n=1}^{\infty} b_n$ converges absolutely.

44. Assumptions Matter In 1829, Lejeune Dirichlet pointed out that the great French mathematician Augustin Louis Cauchy made a mistake in a published paper by improperly assuming the Limit Comparison Test to be valid for nonpositive series. Here are Dirichlet's two series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

Explain how they provide a counterexample to the Limit Comparison Test when the series are not assumed to be positive.

SOLUTION Let

$$R = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{and} \quad S = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

R is an alternating series that converges by the Leibniz Test; however, we cannot apply the Leibniz Test to S because the absolute value of the terms in S is not decreasing. Because

$$L = \lim_{n \to \infty} \frac{\frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)}{\frac{(-1)^n}{\sqrt{n}}} = \lim_{n \to \infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = 1,$$

if the Limit Comparison Test were valid for nonpositive series, we would conclude that S converges. However, if we assume that S converges, then the series $S - R$ would also converge. But

$$S - R = \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} - \frac{(-1)^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the divergent harmonic series. Thus, S diverges, and the Limit Comparison Test is not valid for nonpositive series.

10.5 The Ratio and Root Tests

Preliminary Questions

1. In the Ratio Test, is ρ equal to $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$?

SOLUTION In the Ratio Test ρ is the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

2. Is the Ratio Test conclusive for $\sum_{n=1}^{\infty}\frac{1}{2^n}$? Is it conclusive for $\sum_{n=1}^{\infty}\frac{1}{n}$?

SOLUTION The general term of $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is $a_n = \frac{1}{2^n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}}\cdot\frac{2^n}{1} = \frac{1}{2},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1.$$

Consequently, the Ratio Test guarantees that the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges.

The general term of $\sum_{n=1}^{\infty}\frac{1}{n}$ is $a_n = \frac{1}{n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1}\cdot\frac{n}{1} = \frac{n}{n+1},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

The Ratio Test is therefore inconclusive for the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

3. Can the Ratio Test be used to show convergence if the series is only conditionally convergent?

SOLUTION No. The Ratio Test can only establish absolute convergence and divergence, not conditional convergence.

Exercises

In Exercises 1–20, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

1. $\sum_{n=1}^{\infty}\frac{1}{5^n}$

SOLUTION With $a_n = \frac{1}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5^{n+1}}\cdot\frac{5^n}{1} = \frac{1}{5} \quad\text{and}\quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{1}{5^n}$ converges by the Ratio Test.

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{5^n}$

SOLUTION With $a_n = \frac{(-1)^{n-1}n}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n} = \frac{n+1}{5n} \quad\text{and}\quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{5^n}$ converges by the Ratio Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^{n+1}} \cdot \frac{n^n}{1} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot \frac{1}{e} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Ratio Test.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$

SOLUTION With $a_n = \frac{3n+2}{5n^3+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3(n+1)+2}{5(n+1)^3+1} \cdot \frac{5n^3+1}{3n+2} = \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1 = 1.$$

Therefore, for the series $\displaystyle\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$, the Ratio Test is inconclusive.

We can show that this series converges by using the Limit Comparison Test and comparing with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

SOLUTION With $a_n = \frac{n}{n^2+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1 = 1.$$

Therefore, for the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, the Ratio Test is inconclusive.

We can show that this series diverges by using the Limit Comparison Test and comparing with the divergent harmonic series.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$

SOLUTION With $a_n = \frac{2^n}{n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = \frac{2n}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2 > 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$ diverges by the Ratio Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

SOLUTION With $a_n = \frac{2^n}{n^{100}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^{100}} \cdot \frac{n^{100}}{2^n} = 2\left(\frac{n}{n+1}\right)^{100} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot 1^{100} = 2 > 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$ diverges by the Ratio Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^{n^2}}$

SOLUTION With $a_n = \frac{n^3}{3^{n^2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{(n+1)^2}} \cdot \frac{3^{n^2}}{n^3} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3^{2n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1^3 \cdot 0 = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^{n^2}}$ converges by the Ratio Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

SOLUTION With $a_n = \frac{10^n}{2^{n^2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{10^n} = 10 \cdot \frac{1}{2^{2n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 10 \cdot 0 = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ converges by the Ratio Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

SOLUTION With $a_n = \frac{e^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \frac{e}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$ converges by the Ratio Test.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

SOLUTION With $a_n = \frac{e^n}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{e^n} = \frac{e}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{e}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot \frac{1}{e} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges by the Ratio Test.

12. $\sum_{n=1}^{\infty} \frac{n^{40}}{n!}$

SOLUTION With $a_n = \frac{n^{40}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{40}}{(n+1)!} \cdot \frac{n!}{n^{40}} = \frac{1}{n+1}\left(\frac{n+1}{n}\right)^{40} = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{40},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot 1 = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^{40}}{n!}$ converges by the Ratio Test.

13. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

SOLUTION With $a_n = \frac{n!}{6^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{6^{n+1}} \cdot \frac{6^n}{n!} = \frac{n+1}{6} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{n!}{6^n}$ diverges by the Ratio Test.

14. $\sum_{n=1}^{\infty} \frac{n!}{n^9}$

SOLUTION With $a_n = \frac{n!}{n^9}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^9} \cdot \frac{n^9}{n!} = \frac{n^9}{(n+1)^8} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{n^9}$ diverges by the Ratio Test.

15. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

SOLUTION With $a_n = \frac{1}{n \ln n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{1} = \frac{n}{n+1} \frac{\ln n}{\ln(n+1)},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}.$$

Now,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Thus, $\rho = 1$, and the Ratio Test is inconclusive for the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Using the Integral Test, we can show that the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

16. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

SOLUTION With $a_n = \frac{1}{(2n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(2n+2)!} \cdot \frac{(2n)!}{1} = \frac{1}{(2n+2)(2n+1)} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges by the Ratio Test.

17. $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

SOLUTION With $a_n = \frac{n^2}{(2n+1)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{n^2} = \left(\frac{n+1}{n}\right)^2 \frac{1}{(2n+3)(2n+2)},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1^2 \cdot 0 = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$ converges by the Ratio Test.

18. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

SOLUTION With $a_n = \frac{(n!)^3}{(3n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{n^3+3n^2+3n+1}{27n^3+54n^2+33n+6}$$
$$= \frac{1+3n^{-1}+3n^{-2}+1n^{-3}}{27+54n^{-1}+33n^{-2}+6n^{-3}}$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{27} < 1$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ converges by the Ratio Test.

19. $\sum_{n=2}^{\infty} \frac{1}{2^n+1}$

SOLUTION With $a_n = \frac{1}{2^n+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}+1} \cdot \frac{2^n+1}{1} = \frac{1+2^{-n}}{2+2^{-n}}$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$$

Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{2^n+1}$ converges by the Ratio Test.

20. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

SOLUTION With $a_n = \dfrac{1}{\ln n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\ln n} \cdot \frac{\ln(n+1)}{1} = \frac{\ln(n+1)}{\ln n}$$

and (using L'Hôpital's rule)

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}\ln x} = \lim_{x\to\infty}\frac{x}{x+1} = 1$$

Therefore, the Ratio Test is inconclusive for $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$. This series can be shown to diverge using the Comparison Test with the harmonic series since $\ln n < n$ for $n \geq 2$.

21. Show that $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k.

SOLUTION With $a_n = n^k 3^{-n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^k 3^{-(n+1)}}{n^k 3^{-n}} = \frac{1}{3}\left(1+\frac{1}{n}\right)^k,$$

and, for all k,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k by the Ratio Test.

22. Show that $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$ converges if $|x| < 1$.

SOLUTION With $a_n = n^2 x^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2|x|^{n+1}}{n^2|x|^n} = \left(1+\frac{1}{n}\right)^2|x| \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1\cdot|x| = |x|.$$

Therefore, by the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$ converges provided $|x| < 1$.

23. Show that $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges if $|x| < \frac{1}{2}$.

SOLUTION With $a_n = 2^n x^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x|^{n+1}}{2^n|x|^n} = 2|x| \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x|.$$

Therefore, $\rho < 1$ and the series $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges by the Ratio Test provided $|x| < \frac{1}{2}$.

24. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{n!}$ converges for all r.

SOLUTION With $a_n = \frac{r^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{(n+1)!}\cdot\frac{n!}{|r|^n} = \frac{|r|}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0\cdot|r| = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{r^n}{n!}$ converges by the Ratio Test for all r.

25. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$.

SOLUTION With $a_n = \frac{r^n}{n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{n+1} \cdot \frac{n}{|r|^n} = |r|\frac{n}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot |r| = |r|.$$

Therefore, by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges provided $|r| < 1$.

26. Is there any value of k such that $\sum_{n=1}^{\infty} \frac{2^n}{n^k}$ converges?

SOLUTION With $a_n = \frac{2^n}{n^k}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^k} \cdot \frac{n^k}{2^n} = 2\left(\frac{n}{n+1}\right)^k,$$

and, for all k,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot 1^k = 2 > 1.$$

Therefore, by the Ratio Test, there is no value for k such that the series $\sum_{n=1}^{\infty} \frac{2^n}{n^k}$ converges.

27. Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. *Hint:* Use $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

SOLUTION With $a_n = \frac{n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Ratio Test.

In Exercises 28–33, assume that $|a_{n+1}/a_n|$ converges to $\rho = \frac{1}{3}$. What can you say about the convergence of the given series?

28. $\sum_{n=1}^{\infty} n a_n$

SOLUTION Let $b_n = n a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{n+1}{n}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} n a_n$ converges by the Ratio Test.

29. $\sum_{n=1}^{\infty} n^3 a_n$

SOLUTION Let $b_n = n^3 a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\left|\frac{a_{n+1}}{a_n}\right| = 1^3 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} n^3 a_n$ converges by the Ratio Test.

30. $\sum_{n=1}^{\infty} 2^n a_n$

SOLUTION Let $b_n = 2^n a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\left|\frac{a_{n+1}}{a_n}\right| = 2\cdot\frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} 2^n a_n$ converges by the Ratio Test.

31. $\sum_{n=1}^{\infty} 3^n a_n$

SOLUTION Let $b_n = 3^n a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\left|\frac{a_{n+1}}{a_n}\right| = 3\cdot\frac{1}{3} = 1.$$

Therefore, the Ratio Test is inconclusive for the series $\sum_{n=1}^{\infty} 3^n a_n$.

32. $\sum_{n=1}^{\infty} 4^n a_n$

SOLUTION Let $b_n = 4^n a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{4^{n+1}}{4^n}\left|\frac{a_{n+1}}{a_n}\right| = 4\cdot\frac{1}{3} = \frac{4}{3} > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} 4^n a_n$ diverges by the Ratio Test.

33. $\sum_{n=1}^{\infty} a_n^2$

SOLUTION Let $b_n = a_n^2$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n^2$ converges by the Ratio Test.

34. Assume that $|a_{n+1}/a_n|$ converges to $\rho = 4$. Does $\sum_{n=1}^{\infty} a_n^{-1}$ converge (assume that $a_n \neq 0$ for all n)?

SOLUTION Let $b_n = a_n^{-1}$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{4} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n^{-1}$ converges by the Ratio Test.

35. Is the Ratio Test conclusive for the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$?

SOLUTION With $a_n = \frac{1}{n^p}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^p}\cdot\frac{n^p}{1} = \left(\frac{n}{n+1}\right)^p \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1^p = 1.$$

Therefore, the Ratio Test is inconclusive for the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$.

In Exercises 36–41, use the Root Test to determine convergence or divergence (or state that the test is inconclusive).

36. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^n}$

SOLUTION With $a_n = \frac{1}{10^n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{10^n}} = \frac{1}{10} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{10} < 1.$$

Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^n}$ converges by the Root Test.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Root Test.

38. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$

SOLUTION With $a_k = \left(\frac{k}{k+10}\right)^k$,

$$\sqrt[k]{a_k} = \sqrt[k]{\left(\frac{k}{k+10}\right)^k} = \frac{k}{k+10} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{a_k} = 1.$$

Therefore, the Root Test is inconclusive for the series $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$. Because

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1 + \frac{10}{k}\right)^{-k} = \lim_{k\to\infty} \left[\left(1 + \frac{10}{k}\right)^{k/10}\right]^{-10} = e^{-10} \neq 0,$$

this series diverges by the Divergence Test.

39. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

SOLUTION With $a_k = \left(\frac{k}{3k+1}\right)^k$,

$$\sqrt[k]{a_k} = \sqrt[k]{\left(\frac{k}{3k+1}\right)^k} = \frac{k}{3k+1} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{a_k} = \frac{1}{3} < 1.$$

Therefore, the series $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$ converges by the Root Test.

40. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n}$

SOLUTION With $a_k = \left(1 + \frac{1}{n}\right)^{-n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1 + \frac{1}{n}\right)^{-n}} = \left(1 + \frac{1}{n}\right)^{-1} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = 1^{-1} = 1.$$

Therefore, the Root Test is inconclusive for the series $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n}$.

Because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} = e^{-1} \neq 0,$$

this series diverges by the Divergence Test.

41. $\displaystyle\sum_{n=4}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

SOLUTION With $a_k = \left(1+\frac{1}{n}\right)^{-n^2}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}} = \left(1+\frac{1}{n}\right)^{-n} \quad \text{and} \quad \lim_{n\to\infty}\sqrt[n]{a_n} = e^{-1} < 1.$$

Therefore, the series $\displaystyle\sum_{n=4}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ converges by the Root Test.

42. Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ diverges. *Hint:* Use $2^{n^2} = (2^n)^n$ and $n! \le n^n$.

SOLUTION Because $n! \le n^n$,

$$\frac{2^{n^2}}{n!} \ge \frac{2^{n^2}}{n^n}.$$

Now, let $a_n = \frac{2^{n^2}}{n^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^{n^2}}{n^n}} = \frac{2^n}{n},$$

and

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{2^n}{n} = \lim_{x\to\infty}\frac{2^x}{x} = \lim_{x\to\infty}\frac{2^x\ln 2}{1} = \infty > 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n^n}$ diverges by the Root Test. By the Comparison Test, we can then conclude that the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ also diverges.

In Exercises 43–56, determine convergence or divergence using any method covered in the text so far.

43. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+4^n}{7^n}$

SOLUTION Because the series

$$\sum_{n=1}^{\infty}\frac{2^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{4^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{4}{7}\right)^n$$

are both convergent geometric series, it follows that

$$\sum_{n=1}^{\infty}\frac{2^n+4^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n + \sum_{n=1}^{\infty}\left(\frac{4}{7}\right)^n$$

also converges.

44. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{n!}$

SOLUTION The presence of the factorial suggests applying the Ratio Test. With $a_n = \frac{n^3}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{(n+1)!}\cdot\frac{n!}{n^3} = \frac{(n+1)^2}{n^3} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{n!}$ converges by the Ratio Test.

45. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

SOLUTION The presence of the exponential term suggests applying the Ratio Test. With $a_n = \frac{n^3}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^3 \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} \cdot 1^3 = \frac{1}{5} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by the Ratio Test.

46. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

SOLUTION The general term in this series suggests applying the Integral Test. Let $f(x) = \frac{1}{x(\ln x)^3}$. This function is continuous, positive and decreasing for $x \geq 2$, so the Integral Test does apply. Now

$$\int_2^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{R\to\infty} \int_2^R \frac{dx}{x(\ln x)^3} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^3} = -\frac{1}{2} \lim_{R\to\infty}\left(\frac{1}{(\ln R)^2} - \frac{1}{(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ also converges.

47. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

SOLUTION This series is similar to a p-series; because

$$\frac{1}{\sqrt{n^3 - n^2}} \approx \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

for large n, we will apply the Limit Comparison Test comparing with the p-series with $p = \frac{3}{2}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3 - n^2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3 - n^2}} = 1.$$

The p-series with $p = \frac{3}{2}$ converges and L exists; therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$ also converges.

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

SOLUTION This series is similar to a p-series; because

$$\frac{n^2 + 4n}{3n^4 + 9} \approx \frac{n^2}{\sqrt{3n^4}} = \frac{1}{3n^2}$$

for large n, we will apply the Limit Comparison Test comparing with the p-series with $p = 2$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{n^2 + 4n}{3n^4 + 9}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4 + 4n^3}{3n^4 + 9} = \frac{1}{3}.$$

The p-series with $p = 2$ converges and L exists; therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$ also converges.

49. $\displaystyle\sum_{n=1}^{\infty} n^{-0.8}$

SOLUTION

$$\sum_{n=1}^{\infty} n^{-0.8} = \sum_{n=1}^{\infty} \frac{1}{n^{0.8}}$$

so that this is a divergent p-series.

50. $\displaystyle\sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8}$

SOLUTION

$$\sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8} = \sum_{n=1}^{\infty}(0.8^{-1})^{n}n^{-0.8} = \sum_{n=1}^{\infty}\frac{1.25^n}{n^{0.8}}$$

With $a_n = \frac{1.25^n}{n^{0.8}}$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1.25^{n+1}}{(n+1)^{0.8}}\cdot\frac{n^{0.8}}{1.25^n} = 1.25\left(\frac{n}{n+1}\right)^{0.8}$$

so that

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.25 > 1$$

Thus the original series diverges, by the Ratio Test.

51. $\displaystyle\sum_{n=1}^{\infty}4^{-2n+1}$

SOLUTION Observe

$$\sum_{n=1}^{\infty}4^{-2n+1} = \sum_{n=1}^{\infty}4\cdot(4^{-2})^n = \sum_{n=1}^{\infty}4\left(\frac{1}{16}\right)^n$$

is a geometric series with $r = \frac{1}{16}$; therefore, this series converges.

52. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

SOLUTION This is an alternating series with $a_n = \frac{1}{\sqrt{n}}$. Because a_n forms a decreasing sequence which converges to zero, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Leibniz Test.

53. $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$

SOLUTION Here, we will apply the Limit Comparison Test, comparing with the p-series with $p = 2$. Now,

$$L = \lim_{n\to\infty}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{u\to 0}\frac{\sin u}{u} = 1,$$

where $u = \frac{1}{n^2}$. The p-series with $p = 2$ converges and L exists; therefore, the series $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$ also converges.

54. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos\frac{1}{n}$

SOLUTION Because

$$\lim_{n\to\infty}\cos\frac{1}{n} = \cos 0 = 1 \neq 0,$$

the general term in the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos\frac{1}{n}$ does not tend toward zero; therefore, the series diverges by the Divergence Test.

55. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}$

SOLUTION Because

$$\lim_{n\to\infty}\frac{2^n}{\sqrt{n}} = \lim_{x\to\infty}\frac{2^x}{\sqrt{x}} = \lim_{x\to\infty}\frac{2^x\ln 2}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}2^{x+1}\sqrt{x}\ln 2 = \infty \neq 0,$$

the general term in the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}$ does not tend toward zero; therefore, the series diverges by the Divergence Test.

56. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

SOLUTION Because the general term has the form of a function of n raised to the nth power, we might be tempted to use the Root Test; however, the Root Test is inconclusive for this series. Instead, note

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{12}{n}\right)^{-n} = \lim_{n\to\infty} \left[\left(1+\frac{12}{n}\right)^{n/12}\right]^{-12} = e^{-12} \neq 0.$$

Therefore, the series diverges by the Divergence Test.

Further Insights and Challenges

57. **Proof of the Root Test** Let $S = \displaystyle\sum_{n=0}^{\infty} a_n$ be a positive series, and assume that $L = \displaystyle\lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

(a) Show that S converges if $L < 1$. *Hint:* Choose R with $L < R < 1$ and show that $a_n \le R^n$ for n sufficiently large. Then compare with the geometric series $\sum R^n$.

(b) Show that S diverges if $L > 1$.

SOLUTION Suppose $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists.

(a) If $L < 1$, let $\epsilon = \dfrac{1-L}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \le \sqrt[n]{a_n} - L \le \epsilon$$

for $n \ge N$. From this, we conclude that

$$0 \le \sqrt[n]{a_n} \le L + \epsilon$$

for $n \ge N$. Now, let $R = L + \epsilon$. Then

$$R = L + \frac{1-L}{2} = \frac{L+1}{2} < \frac{1+1}{2} = 1,$$

and

$$0 \le \sqrt[n]{a_n} \le R \quad \text{or} \quad 0 \le a_n \le R^n$$

for $n \ge N$. Because $0 \le R < 1$, the series $\displaystyle\sum_{n=N}^{\infty} R^n$ is a convergent geometric series, so the series $\displaystyle\sum_{n=N}^{\infty} a_n$ converges by the Comparison Test. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ also converges.

(b) If $L > 1$, let $\epsilon = \dfrac{L-1}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \le \sqrt[n]{a_n} - L \le \epsilon$$

for $n \ge N$. From this, we conclude that

$$L - \epsilon \le \sqrt[n]{a_n}$$

for $n \ge N$. Now, let $R = L - \epsilon$. Then

$$R = L - \frac{L-1}{2} = \frac{L+1}{2} > \frac{1+1}{2} = 1,$$

and

$$R \le \sqrt[n]{a_n} \quad \text{or} \quad R^n \le a_n$$

for $n \ge N$. Because $R > 1$, the series $\displaystyle\sum_{n=N}^{\infty} R^n$ is a divergent geometric series, so the series $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges by the Comparison Test. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ also diverges.

58. Show that the Ratio Test does not apply, but verify convergence using the Comparison Test for the series

$$\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\cdots$$

SOLUTION The general term of the series is:

$$a_n=\begin{cases}\dfrac{1}{2^n} & n \text{ odd}\\[2mm] \dfrac{1}{3^n} & n \text{ even}\end{cases}$$

First use the Ratio Test. If n is even,

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}}=\frac{3^n}{2^{n+1}}=\frac{1}{2}\cdot\left(\frac{3}{2}\right)^n$$

whereas, if n is odd,

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{3^{n+1}}}{\frac{1}{2^n}}=\frac{2^n}{3^{n+1}}=\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n$$

Since $\lim_{n\to\infty}\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n=0$ and $\lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{3}{2}\right)^n=\infty$, the sequence $\frac{a_{n+1}}{a_n}$ does not converge, and the Ratio Test is inconclusive.

However, we have $0\le a_n\le\frac{1}{2^n}$ for all n, so the series converges by comparison with the convergent geometric series

$$\sum_{n=1}^{\infty}\frac{1}{2^n}$$

59. Let $S=\sum_{n=1}^{\infty}\frac{c^n n!}{n^n}$, where c is a constant.

(a) Prove that S converges absolutely if $|c|<e$ and diverges if $|c|>e$.

(b) It is known that $\lim_{n\to\infty}\frac{e^n n!}{n^{n+1/2}}=\sqrt{2\pi}$. Verify this numerically.

(c) Use the Limit Comparison Test to prove that S diverges for $c=e$.

SOLUTION

(a) With $a_n=\frac{c^n n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{|c|^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{|c|^n n!}=|c|\left(\frac{n}{n+1}\right)^n=|c|\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|c|e^{-1}.$$

Thus, by the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{c^n n!}{n^n}$ converges when $|c|e^{-1}<1$, or when $|c|<e$. The series diverges when $|c|>e$.

(b) The table below lists the value of $\frac{e^n n!}{n^{n+1/2}}$ for several increasing values of n. Since $\sqrt{2\pi}=2.506628275$, the numerical evidence verifies that

$$\lim_{n\to\infty}\frac{e^n n!}{n^{n+1/2}}=\sqrt{2\pi}.$$

n	100	1000	10000	100000
$\frac{e^n n!}{n^{n+1/2}}$	2.508717995	2.506837169	2.506649163	2.506630363

(c) With $c = e$, the series S becomes $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$. Using the result from part (b),

$$L = \lim_{n\to\infty} \frac{\frac{e^n n!}{n^n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}.$$

Because the series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by the Divergence Test and $L > 0$, we conclude that $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ diverges by the Limit Comparison Test.

10.6 Power Series

Preliminary Questions

1. Suppose that $\sum a_n x^n$ converges for $x = 5$. Must it also converge for $x = 4$? What about $x = -3$?

SOLUTION The power series $\sum a_n x^n$ is centered at $x = 0$. Because the series converges for $x = 5$, the radius of convergence must be at least 5 and the series converges absolutely at least for the interval $|x| < 5$. Both $x = 4$ and $x = -3$ are inside this interval, so the series converges for $x = 4$ and for $x = -3$.

2. Suppose that $\sum a_n (x-6)^n$ converges for $x = 10$. At which of the points (a)–(d) must it also converge?

(a) $x = 8$ **(b)** $x = 11$ **(c)** $x = 3$ **(d)** $x = 0$

SOLUTION The given power series is centered at $x = 6$. Because the series converges for $x = 10$, the radius of convergence must be at least $|10 - 6| = 4$ and the series converges absolutely at least for the interval $|x - 6| < 4$, or $2 < x < 10$.

(a) $x = 8$ is inside the interval $2 < x < 10$, so the series converges for $x = 8$.

(b) $x = 11$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 11$.

(c) $x = 3$ is inside the interval $2 < x < 10$, so the series converges for $x = 2$.

(d) $x = 0$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 0$.

3. What is the radius of convergence of $F(3x)$ if $F(x)$ is a power series with radius of convergence $R = 12$?

SOLUTION If the power series $F(x)$ has radius of convergence $R = 12$, then the power series $F(3x)$ has radius of convergence $R = \frac{12}{3} = 4$.

4. The power series $F(x) = \sum_{n=1}^{\infty} n x^n$ has radius of convergence $R = 1$. What is the power series expansion of $F'(x)$ and what is its radius of convergence?

SOLUTION We obtain the power series expansion for $F'(x)$ by differentiating the power series expansion for $F(x)$ term-by-term. Thus,

$$F'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}.$$

The radius of convergence for this series is $R = 1$, the same as the radius of convergence for the series expansion for $F(x)$.

Exercises

1. Use the Ratio Test to determine the radius of convergence R of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. Does it converge at the endpoints $x = \pm R$?

SOLUTION With $a_n = \frac{x^n}{2^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{|x|^n} = \frac{|x|}{2} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2}.$$

By the Ratio Test, the series converges when $\rho = \frac{|x|}{2} < 1$, or $|x| < 2$, and diverges when $\rho = \frac{|x|}{2} > 1$, or $|x| > 2$. The radius of convergence is therefore $R = 2$. For $x = -2$, the left endpoint, the series becomes $\sum_{n=0}^{\infty} (-1)^n$, which is divergent. For $x = 2$, the right endpoint, the series becomes $\sum_{n=0}^{\infty} 1$, which is also divergent. Thus the series diverges at both endpoints.

2. Use the Ratio Test to show that $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}2^n}$ has radius of convergence $R = 2$. Then determine whether it converges at the endpoints $R = \pm 2$.

SOLUTION With $a_n = \frac{x^n}{\sqrt{n}2^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{\sqrt{n+1} \cdot 2^{n+1}} \cdot \frac{\sqrt{n} \cdot 2^n}{|x|^n} = \frac{|x|}{2} \cdot \sqrt{\frac{n}{n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2} \cdot 1 = \frac{|x|}{2}.$$

By the Ratio Test, the series converges when $\rho = \frac{|x|}{2} < 1$, or $|x| < 2$, and diverges when $\rho = \frac{|x|}{2} > 1$, or $|x| > 2$. The radius of convergence is therefore $R = 2$.

For the endpoint $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n} \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a divergent p-series. For the endpoint $x = -2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n} \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This alternating series converges by the Leibniz Test, but its associated positive series is a divergent p-series. Thus, the series for $x = -2$ is conditionally convergent.

3. Show that the power series (a)–(c) have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ **(c)** $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

SOLUTION

(a) With $a_n = \frac{x^n}{3^n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

Then $\rho < 1$ if $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1,$$

which diverges by the Divergence Test. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n,$$

which also diverges by the Divergence Test.

(b) With $a_n = \frac{x^n}{n3^n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{3}\left(\frac{n}{n+1}\right)\right| = \left|\frac{x}{3}\right|.$$

Then $\rho < 1$ when $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the divergent harmonic series. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Leibniz Test.

(c) With $a_n = \frac{x^n}{n^2 3^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\left(\frac{n}{n+1}\right)^2\right| = \left|\frac{x}{3}\right|$$

Then $\rho < 1$ when $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Leibniz Test.

4. Repeat Exercise 3 for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n9^n}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 9^n}$

SOLUTION

(a) With $a_n = \frac{(x-5)^n}{9^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{9^{n+1}} \cdot \frac{9^n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{x-5}{9}\right| = \left|\frac{x-5}{9}\right|$$

Then $\rho < 1$ when $|x-5| < 9$, so that the radius of convergence is $R = 9$. Because the series is centered at $x = 5$, the series converges absolutely on the interval $|x-5| < 9$, or $-4 < x < 14$. For the endpoint $x = 14$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(14-5)^n}{9^n} = \sum_{n=1}^{\infty} 1,$$

which diverges by the Divergence Test. For the endpoint $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n,$$

which also diverges by the Divergence Test.

(b) With $a_n = \frac{(x-5)^n}{n9^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)9^{n+1}} \cdot \frac{n9^n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{x-5}{9}\frac{n}{n+1}\right| = \left|\frac{x-5}{9}\right|.$$

Then $\rho < 1$ when $|x-5| < 9$, so that the radius of convergence is $R = 9$. Because the series is centered at $x = 5$, the series converges absolutely on the interval $|x-5| < 9$, or $-4 < x < 14$. For the endpoint $x = 14$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(14-5)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the divergent harmonic series. For the endpoint $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-4-5)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Leibniz Test.

(c) With $a_n = \frac{(x-5)^n}{n^2 9^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{x-5}{9}\left(\frac{n}{n+1}\right)^2\right| = \left|\frac{x-5}{9}\right|.$$

Then $\rho < 1$ when $|x-5| < 9$, so that the radius of convergence is $R = 9$. Because the series is centered at $x = 5$, the series converges absolutely on the interval $|x-5| < 9$, or $-4 < x < 14$. For the endpoint $x = 14$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(14-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series. For the endpoint $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-4-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Leibniz Test.

5. Show that $\displaystyle\sum_{n=0}^{\infty} n^n x^n$ diverges for all $x \neq 0$.

SOLUTION With $a_n = n^n x^n$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1} x^{n+1}}{n^n x^n}\right| = \lim_{n\to\infty} \left| x \left(1 + \frac{1}{n}\right)^n (n+1) \right| = \infty$$

$\rho < 1$ only if $x = 0$, so that the radius of convergence is therefore $R = 0$. In other words, the power series converges only for $x = 0$.

6. For which values of x does $\displaystyle\sum_{n=0}^{\infty} n! x^n$ converge?

SOLUTION With $a_n = n! x^n$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = \lim_{n\to\infty} |(n+1)x| = \infty$$

$\rho < 1$ only if $x = 0$, so that the radius of convergence is $R = 0$. In other words, the power series converges only for $x = 0$.

7. Use the Ratio Test to show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n}$ has radius of convergence $R = \sqrt{3}$.

SOLUTION With $a_n = \dfrac{x^{2n}}{3^n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{3}\right| = \left|\frac{x^2}{3}\right|$$

Then $\rho < 1$ when $|x^2| < 3$, or $x = \sqrt{3}$, so the radius of convergence is $R = \sqrt{3}$.

8. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{64^n}$ has radius of convergence $R = 4$.

SOLUTION With $a_n = \dfrac{x^{3n+1}}{64^n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{3(n+1)+1}}{64^{n+1}} \cdot \frac{64^n}{x^{3n+1}}\right| = \lim_{n\to\infty} \left|\frac{x^3}{64}\right| = \left|\frac{x^3}{64}\right|$$

Then $\rho < 1$ when $|x|^3 < 64$ or $|x| = 4$, so the radius of convergence is $R = 4$.

In Exercises 9–34, find the interval of convergence.

9. $\displaystyle\sum_{n=0}^{\infty} n x^n$

SOLUTION With $a_n = n x^n$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \lim_{n\to\infty} \left| x \frac{n+1}{n} \right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so that the radius of convergence is $R = 1$, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\sum_{n=0}^{\infty} n$, which diverges by the Divergence Test. For the endpoint $x = -1$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n$, which also diverges by the Divergence Test. Thus, the series $\sum_{n=0}^{\infty} nx^n$ converges for $-1 < x < 1$ and diverges elsewhere.

10. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

SOLUTION With $a_n = \frac{2^n}{n} x^n$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{2^n x^n} \right| = \lim_{n\to\infty} \left| 2x \frac{n}{n+1} \right| = |2x|$$

$\rho < 1$ when $|x| < \frac{1}{2}$, so the radius of convergence is $R = \frac{1}{2}$, and the series converges absolutely on the interval $|x| < \frac{1}{2}$, or $-\frac{1}{2} < x < \frac{1}{2}$. For the endpoint $x = \frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent harmonic series. For the endpoint $x = -\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=1}^{\infty} \frac{x^n}{n} x^n$ converges for $-\frac{1}{2} \le x < \frac{1}{2}$ and diverges elsewhere.

11. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$

SOLUTION With $a_n = (-1)^n \frac{x^{2n+1}}{2^n n}$,

$$\rho = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2} \cdot \frac{n}{n+1} \right| = \left| \frac{x^2}{2} \right|$$

Then $\rho < 1$ when $|x| < \sqrt{2}$, so the radius of convergence is $R = \sqrt{2}$, and the series converges absolutely on the interval $-\sqrt{2} < x < \sqrt{2}$. For the endpoint $x = -\sqrt{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{-\sqrt{2}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{2}}{n}$, which converges by the Leibniz test. For the endpoint $x = \sqrt{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{2}}{n}$ which also converges by the Leibniz test. Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$ converges for $-\sqrt{2} \le x \le \sqrt{2}$ and diverges elsewhere.

12. $\sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} x^{2n}$

SOLUTION With $a_n = (-1)^n \frac{n}{4^n} x^{2n}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{2(n+1)}}{4^{n+1}} \cdot \frac{4^n}{nx^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{4} \cdot \frac{n+1}{n} \right| = \left| \frac{x^2}{4} \right|$$

Then $\rho < 1$ when $|x^2| < 4$, or $|x| < 2$, so the radius of convergence is $R = 2$, and the series converges absolutely for $-2 < x < 2$. At both endpoints $x = \pm 2$, the series becomes $\sum_{n=0}^{\infty} (-1)^n n$, which diverges by the Divergence Test. Thus, the series $\sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} x^{2n}$ converges for $-2 < x < 2$ and diverges elsewhere.

13. $\displaystyle\sum_{n=4}^{\infty} \frac{x^n}{n^5}$

SOLUTION With $a_n = \frac{x^n}{n^5}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^5} \cdot \frac{n^5}{x^n}\right| = \lim_{n\to\infty} \left|x\left(\frac{n}{n+1}\right)^5\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$, which is a convergent p-series. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Leibniz Test. Thus, the series $\displaystyle\sum_{n=4}^{\infty} \frac{x^n}{n^5}$ converges for $-1 \le x \le 1$ and diverges elsewhere.

14. $\displaystyle\sum_{n=8}^{\infty} n^7 x^n$

SOLUTION With $a_n = n^7 x^n$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^7 x^{n+1}}{n^7 x^n}\right| = \lim_{n\to\infty} \left|x\left(\frac{n+1}{n}\right)^7\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so that the radius of convergence is $R = 1$, and the series converges absolutely on the intervale $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=8}^{\infty} n^7$, which diverges by the Divergence test; for the endpoints $x = -1$, the series becomes $\displaystyle\sum_{n=8}^{\infty} (-1)^n n^7$, which also diverges by the Divergence test. Thus the series $\displaystyle\sum_{n=8}^{\infty} n^7 x^n$ converges for $-1 < x < 1$ and diverges elsewhere.

15. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$

SOLUTION With $a_n = \frac{x^n}{(n!)^2}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{x^n}\right| = \lim_{n\to\infty} \left|x\left(\frac{1}{n+1}\right)^2\right| = 0$$

$\rho < 1$ for all x, so the radius of convergence is $R = \infty$, and the series converges absolutely for all x.

16. $\displaystyle\sum_{n=0}^{\infty} \frac{8^n}{n!} x^n$

SOLUTION With $a_n = \frac{8^n x^n}{n!}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{8^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{8^n x^n}\right| = \lim_{n\to\infty} \left|8x \cdot \frac{1}{n+1}\right| = 0$$

$\rho < 1$ for all x, so the radius of convergence is $R = \infty$, and the series converges absolutely for all x.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3} x^n$

SOLUTION With $a_n = \frac{(2n)! x^n}{(n!)^3}$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2(n+1))! x^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)! x^n}\right| = \lim_{n\to\infty} \left|x \frac{(2n+2)(2n+1)}{(n+1)^3}\right|$$

$$= \lim_{n\to\infty} \left|x \frac{4n^2 + 6n + 2}{n^3 + 3n^2 + 3n + 1}\right| = \lim_{n\to\infty} \left|x \frac{4n^{-1} + 6n^{-1} + 2n^{-3}}{1 + 3n^{-1} + 3n^{-2} + n^{-3}}\right| = 0$$

Then $\rho < 1$ for all x, so the radius of convergence is $R = \infty$, and the series converges absolutely for all x.

18. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n-1}$

SOLUTION With $a_n = \frac{4^n x^{2n-1}}{(2n+1)!}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{4^{n+1} x^{2n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{4^n x^{2n-1}} \right| = \lim_{n\to\infty} \left| \frac{4x^2}{(2n+3)(2n+2)} \right| = 0$$

Then ρ is always less than 1, and the series converges absolutely for all x.

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$

SOLUTION With $a_n = \frac{(-1)^n x^n}{\sqrt{n^2+1}}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt{n^2+2n+2}} \cdot \frac{\sqrt{n^2+1}}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| x \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}} \right| = \lim_{n\to\infty} \left| x \sqrt{\frac{n^2+1}{n^2+2n+2}} \right| = \lim_{n\to\infty} \left| x \sqrt{\frac{1+1/n^2}{1+2/n+2/n^2}} \right|$$

$$= |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely on the interval $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$, which converges by the Leibniz Test. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$, which diverges by the Limit Comparison Test comparing with the divergent harmonic series. Thus, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$ converges for $-1 < x \le 1$ and diverges elsewhere.

20. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$

SOLUTION With $a_n = \frac{x^n}{n^4+2}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^4+2} \cdot \frac{n^4+2}{x^n} \right| = \lim_{n\to\infty} \left| x \frac{n^4+2}{n^4+4n^3+6n^2+4n+3} \right| = |x|$$

$\rho < 1$ when $|x| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+2}$. Because $\frac{1}{n^4+2} < \frac{1}{n^4}$ and the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^4}$ is a convergent p-series, the endpoint series converges by the Comparison Test. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+2}$, which converges by the Leibniz Test. Thus, the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$ converges for $-1 \le x \le 1$ and diverges elsewhere.

21. $\displaystyle\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$

SOLUTION With $a_n = \dfrac{x^{2n+1}}{3n+1}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{3n+4} \cdot \frac{3n+1}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| x^2 \frac{3n+1}{3n+4} \right| = |x^2|$$

Then $\rho < 1$ when $|x^2| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely for $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=15}^{\infty} \frac{1}{3n+1}$, which diverges by the Limit Comparison Test comparing with the divergent harmonic series. For the endpoint $x = -1$, the series becomes $\sum_{n=15}^{\infty} \frac{-1}{3n+1}$, which also diverges by the Limit Comparison Test comparing with the divergent harmonic series. Thus, the series $\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$ converges for $-1 < x < 1$ and diverges elsewhere.

22. $\sum_{n=1}^{\infty} \frac{x^n}{n - 4\ln n}$

SOLUTION With $a_n = \frac{x^n}{n-4\ln n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1) - 4\ln(n+1)} \cdot \frac{n - 4\ln n}{x^n}\right| = \lim_{n\to\infty} \left|x \frac{n - 4\ln n}{(n+1) - 4\ln(n+1)}\right|$$

$$= \lim_{n\to\infty} \left|x \frac{1 - 4(\ln n)/n}{1 + n^{-1} - 4(\ln(n+1))/n}\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n - 4\ln n}$. Because $\frac{1}{n-4\ln n} > \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the endpoint series diverges by the Comparison Test. For the endpoint $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - 4\ln n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=1}^{\infty} \frac{x^n}{n - 4\ln n}$ converges for $-1 \le x < 1$ and diverges elsewhere.

23. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

SOLUTION With $a_n = \frac{x^n}{\ln n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n}\right| = \lim_{n\to\infty} \left|x \frac{\ln(n+1)}{\ln n}\right| = \lim_{n\to\infty} \left|x \frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty} \left|x \frac{n}{n+1}\right| = |x|$$

using L'Hôpital's rule. Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Because $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the endpoint series diverges by the Comparison Test. For the endpoint $x = -1$, the series becomes $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ converges for $-1 \le x < 1$ and diverges elsewhere.

24. $\sum_{n=2}^{\infty} \frac{x^{3n+2}}{\ln n}$

SOLUTION With $a_n = \frac{x^{3n+2}}{\ln n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{3n+5}}{\ln(n+1)} \cdot \frac{\ln n}{x^{3n+2}}\right| = \lim_{n\to\infty} \left|x^3 \cdot \frac{\ln(n+1)}{\ln n}\right| = \lim_{n\to\infty} \left|x^3 \cdot \frac{1/(n+1)}{1/n}\right|$$

$$= \lim_{n\to\infty} \left|x^3 \cdot \frac{n}{n+1}\right| = |x^3|$$

using L'Hôpital's rule. Thus $\rho < 1$ when $|x^3| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Because $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the endpoint series diverges by the Comparison Test. For the endpoint $x = -1$, the series becomes $\sum_{n=2}^{\infty} \frac{(-1)^{3n+2}}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=2}^{\infty} \frac{x^{3n+2}}{\ln n}$ converges for $-1 \le x < 1$ and diverges elsewhere.

25. $\displaystyle\sum_{n=1}^{\infty} n(x-3)^n$

SOLUTION With $a_n = n(x-3)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{n+1}}{n(x-3)^n}\right| = \lim_{n\to\infty}\left|(x-3)\cdot\frac{n+1}{n}\right| = |x-3|$$

Then $\rho < 1$ when $|x-3| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x-3| < 1$, or $2 < x < 4$. For the endpoint $x = 4$, the series becomes $\displaystyle\sum_{n=1}^{\infty} n$, which diverges by the Divergence Test. For the endpoint $x = 2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n n$, which also diverges by the Divergence Test. Thus, the series $\displaystyle\sum_{n=1}^{\infty} n(x-3)^n$ converges for $2 < x < 4$ and diverges elsewhere.

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n(x-3)^n}{n^2}$

SOLUTION With $a_n = \frac{(-5)^n(x-3)^n}{n^2}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{n+1}(x-3)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-5)^n(x-3)^n}\right| = \lim_{n\to\infty}\left|5(x-3)\cdot\frac{n^2}{n^2+2n+1}\right|$$
$$= \lim_{n\to\infty}\left|5(x-3)\cdot\frac{1}{1+2n^{-1}+n^{-2}}\right| = |5(x-3)|$$

Then $\rho < 1$ when $|5(x-3)| < 1$, or $|x-3| < \frac{1}{5}$. Thus the radius of convergence is 5, and the series converges absolutely on the interval $|x-3| < \frac{1}{5}$, or $\frac{14}{5} < x < \frac{16}{5}$. For the endpoint $x = \frac{16}{5}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges by the Leibniz Test. For the endpoint $x = \frac{14}{5}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series. Thus, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^n(x-3)^n}{n^2}$ converges for $\frac{14}{5} \le x \le \frac{16}{5}$ and diverges elsewhere.

27. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^5(x-7)^n$

SOLUTION With $a_n = (-1)^n n^5(x-7)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^5(x-7)^{n+1}}{(-1)^n n^5(x-7)^n}\right| = \lim_{n\to\infty}\left|(x-7)\cdot\frac{(n+1)^5}{n^5}\right|$$
$$= \lim_{n\to\infty}\left|(x-7)\cdot\frac{n^5+\dots}{n^5}\right| = |x-7|$$

Then $\rho < 1$ when $|x-7| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x-7| < 1$, or $6 < x < 8$. For the endpoint $x = 6$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^{2n}n^5 = \sum_{n=1}^{\infty}n^5$, which diverges by the Divergence Test. For the endpoint $x = 8$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^5$, which also diverges by the Divergence Test. Thus, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^5(x-7)^n$ converges for $6 < x < 8$ and diverges elsewhere.

28. $\sum_{n=0}^{\infty} 27^n (x-1)^{3n+2}$

SOLUTION With $a_n = 27^n (x-1)^{3n+2}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{27^{n+1}(x-1)^{3n+5}}{27^n (x-1)^{3n+2}}\right| = \lim_{n\to\infty} \left|27(x-1)^3\right| = |27(x-1)^3|$$

Then $\rho < 1$ when $|27(x-1)^3| < 1$, or when $|(x-1)^3| < \frac{1}{27}$, so when $|x-1| < \frac{1}{3}$. Thus the radius of convergence is $\frac{1}{3}$, and the series converges absolutely when $\frac{2}{3} < x < \frac{4}{3}$. For the endpoint $x = \frac{2}{3}$, the series becomes $\sum_{n=0}^{\infty} 27^n \left(\frac{-1}{3}\right)^{3n+2} = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n$ which diverges by the Divergence test. For the endpoint $x = \frac{4}{3}$, the series becomes $\sum_{n=0}^{\infty} 27^n \left(\frac{1}{3}\right)^{3n+2} = \frac{1}{9}\sum_{n=0}^{\infty} 1$, which also diverges by the Divergence Test. Thus the series $\sum_{n=0}^{\infty} 27^n (x-1)^{3n+2}$ converges for $\frac{2}{3} < x < \frac{4}{3}$ and diverges elsewhere.

29. $\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$

SOLUTION With $a_n = \frac{2^n (x+3)^n}{3n}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(x+3)^{n+1}}{3(n+1)} \cdot \frac{3n}{2^n (x+3)^n}\right| = \lim_{n\to\infty} \left|2(x+3) \cdot \frac{3n}{3n+3}\right|$$
$$= \lim_{n\to\infty} \left|2(x+3) \cdot \frac{1}{1+1/n}\right| = |2(x+3)|$$

Then $\rho < 1$ when $|2(x+3)| < 1$, so when $|x+3| < \frac{1}{2}$. Thus the radius of convergence is $\frac{1}{2}$, and the series converges absolutely on the interval $|x+3| < \frac{1}{2}$, or $-\frac{7}{2} < x < -\frac{5}{2}$. For the endpoint $x = -\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{3n}$, which diverges because it is a multiple of the divergent harmonic series. For the endpoint $x = -\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$ converges for $-\frac{7}{2} \le x < -\frac{5}{2}$ and diverges elsewhere.

30. $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$

SOLUTION With $a_n = \frac{(x-4)^n}{n!}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-4)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-4)^n}\right| = \lim_{n\to\infty} \left|(x-4)\frac{1}{n}\right| = 0$$

Thus $\rho < 1$ for all x, so the radius of convergence is infinite, and $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$ converges for all x.

31. $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}(x+10)^n$

SOLUTION With $a_n = \frac{(-5)^n}{n!}(x+10)^n$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-5)^{n+1}(x+10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-5)^n (x+10)^n}\right| = \lim_{n\to\infty} \left|5(x+10)\frac{1}{n}\right| = 0$$

Thus $\rho < 1$ for all x, so the radius of convergence is infinite, and $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}(x+10)^n$ converges for all x.

32. $\displaystyle\sum_{n=10}^{\infty} n!\,(x+5)^n$

SOLUTION With $a_n = n!(x+5)^n$,, and assuming $x + 5 \neq 0$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+5)^{n+1}}{n!(x+5)^n}\right| = \lim_{n\to\infty}|(n+1)(x+5)| = \infty$$

Thus $\rho < 1$ only if $x + 5 = 0$, so the radius of convergence is zero, and $\displaystyle\sum_{n=10}^{\infty} n!\,(x+5)^n$ converges only for $x = -5$.

33. $\displaystyle\sum_{n=12}^{\infty} e^n (x-2)^n$

SOLUTION With $a_n = e^n (x-2)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}(x-2)^{n+1}}{e^n(x-2)^n}\right| = \lim_{n\to\infty}|e(x-2)| = |e(x-2)|$$

Thus $\rho < 1$ when $|e(x-2)| < 1$, so when $|x-2| < e^{-1}$. Thus the radius of convergence is e^{-1}, and the series converges absolutely on the interval $|x-2| < e^{-1}$, or $2 - e^{-1} < x < 2 + e^{-1}$. For the endpoint $x = 2 + e^{-1}$, the series becomes $\displaystyle\sum_{n=1}^{\infty} 1$, which diverges by the Divergence Test. For the endpoint $x = 2 - e^{-1}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n$, which also diverges by the Divergence Test. Thus, the series $\displaystyle\sum_{n=12}^{\infty} e^n (x-2)^n$ converges for $2 - e^{-1} < x < 2 + e^{-1}$ and diverges elsewhere.

34. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+4)^n}{(n\ln n)^2}$

SOLUTION With $a_n = \frac{(x+4)^n}{(n\ln n)^2}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{((n+1)\ln(n+1))^2}\cdot\frac{(n\ln n)^2}{(x+4)^n}\right| = \lim_{n\to\infty}\left|(x+4)\cdot\left(\frac{n}{n+1}\cdot\frac{\ln n}{\ln(n+1)}\right)^2\right| = |x+4|$$

applying L'Hôpital's rule to evaluate the second term in the product. Thus $\rho < 1$ when $|x+4| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x+4| < 1$, or $-5 < x < -3$. For the endpoint $x = -3$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n\ln n)^2}$, which converges by the Limit Comparison Test comparing with the convergent p-series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$. For the endpoint $x = -5$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\ln n)^2}$, which converges by the Leibniz Test. Thus, the series $\displaystyle\sum_{n=2}^{\infty}\frac{(x+4)^n}{(n\ln n)^2}$ converges for $-5 \le x \le -3$ and diverges elsewhere.

In Exercises 35–40, use Eq. (2) to expand the function in a power series with center $c = 0$ and determine the interval of convergence.

35. $f(x) = \dfrac{1}{1-3x}$

SOLUTION Substituting $3x$ for x in Eq. (2), we obtain

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 3^n x^n.$$

This series is valid for $|3x| < 1$, or $|x| < \frac{1}{3}$.

36. $f(x) = \dfrac{1}{1+3x}$

SOLUTION Substituting $-3x$ for x in Eq. (2), we obtain

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty}(-3x)^n = \sum_{n=0}^{\infty}(-3)^n x^n.$$

This series is valid for $|-3x| < 1$, or $|x| < \frac{1}{3}$.

37. $f(x) = \dfrac{1}{3-x}$

SOLUTION First write

$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}}.$$

Substituting $\frac{x}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^n};$$

Thus,

$$\frac{1}{3-x} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}.$$

This series is valid for $|x/3| < 1$, or $|x| < 3$.

38. $f(x) = \dfrac{1}{4+3x}$

SOLUTION First write

$$\frac{1}{4+3x} = \frac{1}{4} \cdot \frac{1}{1+\frac{3x}{4}}.$$

Substituting $-\frac{3x}{4}$ for x in Eq. (2), we obtain

$$\frac{1}{1+\frac{3x}{4}} = \sum_{n=0}^{\infty}\left(-\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^n};$$

Thus,

$$\frac{1}{4+3x} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^n} = \sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^{n+1}}.$$

This series is valid for $|-3x/4| < 1$, or $|x| < \frac{4}{3}$.

39. $f(x) = \dfrac{1}{1+x^2}$

SOLUTION Substituting $-x^2$ for x in Eq. (2), we obtain

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

This series is valid for $|x| < 1$.

40. $f(x) = \dfrac{1}{16+2x^3}$

SOLUTION First rewrite

$$\frac{1}{16+2x^3} = \frac{1}{16} \cdot \frac{1}{1+\frac{x^3}{8}}$$

Now substitute $-\frac{x^3}{8}$ for x in Eq. (2) to obtain

$$\frac{1}{1+\frac{x^3}{8}} = \sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n}}{8}$$

Thus,

$$\frac{1}{16+2x^3} = \frac{1}{16} \cdot \frac{1}{1+\frac{x^3}{8}} = \frac{1}{16} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8}$$

This series is valid for $|x^3| < 8$, or $|x| < 2$.

41. Use the equalities

$$\frac{1}{1-x} = \frac{1}{-3-(x-4)} = \frac{-\frac{1}{3}}{1+\left(\frac{x-4}{3}\right)}$$

to show that for $|x-4| < 3$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}$$

SOLUTION Substituting $-\frac{x-4}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1+\left(\frac{x-4}{3}\right)} = \sum_{n=0}^{\infty} \left(-\frac{x-4}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{3^n}.$$

Thus,

$$\frac{1}{1-x} = -\frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}.$$

This series is valid for $|-\frac{x-4}{3}| < 1$, or $|x-4| < 3$.

42. Use the method of Exercise 41 to expand $1/(1-x)$ in power series with centers $c = 2$ and $c = -2$. Determine the interval of convergence.

SOLUTION For $c = 2$, write

$$\frac{1}{1-x} = \frac{1}{-1-(x-2)} = -\frac{1}{1+(x-2)}.$$

Substituting $-(x-2)$ for x in Eq. (2), we obtain

$$\frac{1}{1+(x-2)} = \sum_{n=0}^{\infty} (-(x-2))^n = \sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Thus,

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n.$$

This series is valid for $|-(x-2)| < 1$, or $|x-2| < 1$.

For $c = -2$, write

$$\frac{1}{1-x} = \frac{1}{3-(x+2)} = \frac{1}{3} \cdot \frac{1}{1-\frac{x+2}{3}}.$$

Substituting $\frac{x+2}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1-\frac{x+2}{3}} = \sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}.$$

Thus,

$$\frac{1}{1-x} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}.$$

This series is valid for $|\frac{x+2}{3}| < 1$, or $|x+2| < 3$.

43. Use the method of Exercise 41 to expand $1/(4-x)$ in a power series with center $c = 5$. Determine the interval of convergence.

SOLUTION First write

$$\frac{1}{4-x} = \frac{1}{-1-(x-5)} = -\frac{1}{1+(x-5)}.$$

Substituting $-(x-5)$ for x in Eq. (2), we obtain

$$\frac{1}{1+(x-5)} = \sum_{n=0}^{\infty}(-(x-5))^n = \sum_{n=0}^{\infty}(-1)^n(x-5)^n.$$

Thus,

$$\frac{1}{4-x} = -\sum_{n=0}^{\infty}(-1)^n(x-5)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(x-5)^n.$$

This series is valid for $|-(x-5)| < 1$, or $|x-5| < 1$.

44. Find a power series that converges only for x in $[2, 6)$.

SOLUTION The power series must be centered at $c = \frac{6+2}{2} = 4$, with radius of convergence $R = 2$. Consider the following series:

$$\sum_{n=1}^{\infty}\frac{(x-4)^n}{n2^n}.$$

With $a_n = \frac{1}{n2^n}$,

$$r = \lim_{n\to\infty}\frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{2}.$$

The radius of convergence is therefore $R = r^{-1} = 2$, and the series converges absolutely for $|x-4| < 2$, or $2 < x < 6$. For the endpoint $x = 6$, the series becomes $\sum_{n=1}^{\infty}\frac{(6-4)^n}{n{\cdot}2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series. For the endpoint $x = 2$, the series becomes $\sum_{n=1}^{\infty}\frac{(2-4)^n}{n{\cdot}2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges by the Leibniz Test. Therefore, the series converges for $2 \le x < 6$, as desired.

45. Apply integration to the expansion

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

to prove that for $-1 < x < 1$,

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

SOLUTION To obtain the first expansion, substitute $-x$ for x in Eq. (2):

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

This expansion is valid for $|-x| < 1$, or $-1 < x < 1$.

Upon integrating both sides of the above equation, we find

$$\ln(1+x) = \int\frac{dx}{1+x} = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx.$$

Integrating the series term-by-term then yields

$$\ln(1+x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

To determine the constant C, set $x = 0$. Then $0 = \ln(1+0) = C$. Finally,

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

46. Use the result of Exercise 45 to prove that

$$\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

Use your knowledge of alternating series to find an N such that the partial sum S_N approximates $\ln\frac{3}{2}$ to within an error of at most 10^{-3}. Confirm using a calculator to compute both S_N and $\ln\frac{3}{2}$.

SOLUTION In the previous exercise we found that

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

Setting $x = \frac{1}{2}$ yields:

$$\ln\frac{3}{2} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

Note that the series for $\ln\frac{3}{2}$ is an alternating series with $a_n = \frac{1}{n2^n}$. The error in approximating $\ln\frac{3}{2}$ by the partial sum S_N is therefore bounded by

$$\left|\ln\frac{3}{2} - S_N\right| < a_{N+1} = \frac{1}{(N+1)2^{N+1}}.$$

To obtain an error of at most 10^{-3}, we must find an N such that

$$\frac{1}{(N+1)2^{N+1}} < 10^{-3} \quad \text{or} \quad (N+1)2^{N+1} > 1000.$$

For $N = 6$, $(N+1)2^{N+1} = 7\cdot 2^7 = 896 < 1000$, but for $N = 7$, $(N+1)2^{N+1} = 8\cdot 2^8 = 2048 > 1000$; hence, the smallest value for N is $N = 7$. The corresponding approximation is

$$S_7 = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \frac{1}{6\cdot 2^6} + \frac{1}{7\cdot 2^7} = 0.405803571.$$

Now, $\ln\frac{3}{2} = 0.405465108$, so

$$\left|\ln\frac{3}{2} - S_7\right| = 3.385\times 10^{-4} < 10^{-3}.$$

47. Let $F(x) = (x+1)\ln(1+x) - x$.

(a) Apply integration to the result of Exercise 45 to prove that for $-1 < x < 1$,

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n(n+1)}$$

(b) Evaluate at $x = \frac{1}{2}$ to prove

$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} = \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} + \cdots$$

(c) Use a calculator to verify that the partial sum S_4 approximates the left-hand side with an error no greater than the term a_5 of the series.

SOLUTION

(a) Note that

$$\int \ln(x+1)\,dx = (x+1)\ln(x+1) - x + C$$

Then integrating both sides of the result of Exercise 45 gives

$$(x+1)\ln(x+1) - x = \int \ln(x+1)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}\,dx$$

For $-1 < x < 1$, which is the interval of convergence of the series in Exercise 45, therefore, we can integrate term by term to get

$$(x+1)\ln(x+1) - x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int x^n\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)} + C$$

(noting that $(-1)^{n-1} = (-1)^{n+1}$). To determine C, evaluate both sides at $x = 0$ to get

$$0 = \ln 1 - 0 = 0 + C$$

so that $C = 0$ and we get finally

$$(x+1)\ln(x+1) - x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)}$$

(b) Evaluating the result of part(a) at $x = \frac{1}{2}$ gives

$$\begin{aligned}\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)2^{n+1}} \\ &= \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} + \dots\end{aligned}$$

(c)

$$S_4 = \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} = 0.1078125$$

$$a_5 = \frac{1}{5\cdot 6\cdot 2^6} \approx 0.0005208$$

$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} \approx 0.10819766$$

and

$$\left| S_4 - \frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} \right| \approx 0.0003852 < a_5$$

48. Prove that for $|x| < 1$,

$$\int \frac{dx}{x^4+1} = x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots$$

Use the first two terms to approximate $\int_0^{1/2} dx/(x^4+1)$ numerically. Use the fact that you have an alternating series to show that the error in this approximation is at most 0.00022.

SOLUTION Substitute $-x^4$ for x in Eq. (2) to get

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

This is valid for $|x| < 1$, so for x in that range we can integrate the right-hand side term by term to get

$$\begin{aligned}\int \frac{1}{1+x^4}\,dx &= \sum_{n=0}^{\infty} \int (-1)^n x^{4n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} + C \\ &= x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots + C\end{aligned}$$

Using the first two terms, we have

$$\int_0^{1/2} \frac{1}{1+x^4}\,dx \approx \frac{1}{2} - \frac{1}{2^5 \cdot 5} = \frac{79}{160} = 0.49375$$

Since this is an alternating series, the error in the approximation is bounded by the first unused term, so by

$$\frac{1}{2^9 \cdot 9} = \frac{1}{4608} \approx 0.000217 < 0.00022$$

49. Use the result of Example 7 to show that

$$F(x) = \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots$$

is an antiderivative of $f(x) = \tan^{-1} x$ satisfying $F(0) = 0$. What is the radius of convergence of this power series?

SOLUTION For $-1 < x < 1$, which is the interval of convergence for the power series for arctangent, we can integrate term-by-term, so integrate that power series to get

$$\begin{aligned} F(x) &= \int \tan^{-1} x\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n+1}}{2n+1}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} \\ &= \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots + C \end{aligned}$$

If we assume $F(0) = 0$, then we have $C = 0$. The radius of convergence of this power series is the same as that of the original power series, which is 1.

50. Verify that function $F(x) = x\tan^{-1} x - \frac{1}{2}\log(x^2+1)$ is an antiderivative of $f(x) = \tan^{-1} x$ satisfying $F(0) = 0$. Then use the result of Exercise 49 with $x = \frac{\pi}{6}$ to show that

$$\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3} = \frac{1}{1\cdot 2(3)} - \frac{1}{3\cdot 4(3^2)} + \frac{1}{5\cdot 6(3^3)} - \frac{1}{7\cdot 8(3^4)} + \cdots$$

Use a calculator to compare the value of the left-hand side with the partial sum S_4 of the series on the right.

SOLUTION We have

$$F'(x) = \tan^{-1} x + \frac{x}{1+x^2} - \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x = \tan^{-1} x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \tan^{-1} x$$

so that $F(x)$ is an antiderivative of $\tan^{-1} x$, and clearly $F(0) = 0$. So applying Exercise 49 for this F, and setting $x = \frac{1}{\sqrt{3}}$, gives

$$\begin{aligned} \frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - \frac{1}{2}\ln\left(\frac{1}{3}+1\right) &= \frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3} \\ &= \frac{(1/\sqrt{3})^2}{1\cdot 2} - \frac{(1/\sqrt{3})^4}{3\cdot 4} + \frac{(1/\sqrt{3})^6}{5\cdot 6} - \frac{(1/\sqrt{3})^8}{7\cdot 8} + \dots \\ &= \frac{1}{1\cdot 2(3)} - \frac{1}{3\cdot 4(3^2)} + \frac{1}{5\cdot 6(3^3)} - \frac{1}{7\cdot 8(3^4)} + \dots \end{aligned}$$

Now, we have

$$S_4 = \frac{1}{1\cdot 2(3)} - \frac{1}{3\cdot 4(3^2)} + \frac{1}{5\cdot 6(3^3)} - \frac{1}{7\cdot 8(3^4)} = \frac{3593}{22680} \approx 0.1548215$$

$$\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3} \approx 0.158459$$

so the two differ by less than 0.00004.

51. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. *Hint:* Use differentiation to show that

$$(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1} \quad (\text{for } |x| < 1)$$

SOLUTION Differentiate both sides of Eq. (2) to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Setting $x = \frac{1}{2}$ then yields

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

Divide this equation by 2 to obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

52. Use the power series for $(1+x^2)^{-1}$ and differentiation to prove that for $|x| < 1$,

$$\frac{2x}{(x^2+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1}(2n)x^{2n-1}$$

SOLUTION From Exercise 39, we know that for $-1 < x < 1$,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Thus for x in this range, we can differentiate both sides, and differentiate the right-hand side term by term, to get

$$\frac{d}{dx}\frac{1}{1+x^2} = \frac{-2x}{(x^2+1)^2} = \sum_{n=1}^{\infty} (-1)^n 2nx^{2n-1}$$

(Note the change in the lower limit of summation, since the $n = 0$ term is a constant, whose derivative is zero). Cancelling the minus sign on the left gives

$$\frac{2x}{(x^2+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1}(2n)x^{2n-1}$$

53. Show that the following series converges absolutely for $|x| < 1$ and compute its sum:

$$F(x) = 1 - x - x^2 + x^3 - x^4 - x^5 + x^6 - x^7 - x^8 + \cdots$$

Hint: Write $F(x)$ as a sum of three geometric series with common ratio x^3.

SOLUTION Because the coefficients in the power series are all ± 1, we find

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

The radius of convergence is therefore $R = r^{-1} = 1$, and the series converges absolutely for $|x| < 1$.

By Exercise 43 of Section 11.4, any rearrangement of the terms of an absolutely convergent series yields another absolutely convergent series with the same sum as the original series. Following the hint, we now rearrange the terms of $F(x)$ as the sum of three geometric series:

$$\begin{aligned} F(x) &= \left(1 + x^3 + x^6 + \cdots\right) - \left(x + x^4 + x^7 + \cdots\right) - \left(x^2 + x^5 + x^8 + \cdots\right) \\ &= \sum_{n=0}^{\infty} (x^3)^n - \sum_{n=0}^{\infty} x(x^3)^n - \sum_{n=0}^{\infty} x^2(x^3)^n = \frac{1}{1-x^3} - \frac{x}{1-x^3} - \frac{x^2}{1-x^3} = \frac{1-x-x^2}{1-x^3}. \end{aligned}$$

54. Show that for $|x| < 1$,

$$\frac{1+2x}{1+x+x^2} = 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \cdots$$

Hint: Use the hint from Exercise 53.

SOLUTION The terms in the series on the right-hand side are either of the form x^n or $-2x^n$ for some n. Because

$$\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} \sqrt[n]{1} = 1,$$

it follows that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|.$$

Hence, by the Root Test, the series converges absolutely for $|x| < 1$.

By Exercise 43 of Section 11.4, any rearrangement of the terms of an absolutely convergent series yields another absolutely convergent series with the same sum as the original series. If we let S denote the sum of the series, then

$$\begin{aligned} S &= \left(1 + x^3 + x^6 + \cdots\right) + \left(x + x^4 + x^7 + \cdots\right) - 2\left(x^2 + x^5 + x^8 + \cdots\right) \\ &= \frac{1}{1-x^3} + \frac{x}{1-x^3} - \frac{2x^2}{1-x^3} = \frac{1+x-2x^2}{1-x^3} = \frac{(1-x)(2x+1)}{(1-x)(1+x+x^2)} = \frac{2x+1}{1+x+x^2}. \end{aligned}$$

55. Find all values of x such that $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges.

SOLUTION With $a_n = \frac{x^{n^2}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{|x|^{n^2}} = \frac{|x|^{2n+1}}{n+1}.$$

if $|x| \le 1$, then

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{n+1} = 0,$$

and the series converges absolutely. On the other hand, if $|x| > 1$, then

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{n+1} = \infty,$$

and the series diverges. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges for $-1 \le x \le 1$ and diverges elsewhere.

56. Find all values of x such that the following series converges:

$$F(x) = 1 + 3x + x^2 + 27x^3 + x^4 + 243x^5 + \cdots$$

SOLUTION Observe that $F(x)$ can be written as the sum of two geometric series:

$$F(x) = \left(1 + x^2 + x^4 + \cdots\right) + \left(3x + 27x^3 + 243x^5 + \cdots\right) = \sum_{n=0}^{\infty} (x^2)^n + \sum_{n=0}^{\infty} 3x(9x^2)^n$$

The first geometric series converges for $|x^2| < 1$, or $|x| < 1$; the second geometric series converges for $|9x^2| < 1$, or $|x| < \frac{1}{3}$. Since both geometric series must converge for $F(x)$ to converge, we find that $F(x)$ converges for $|x| < \frac{1}{3}$, the intersection of the intervals of convergence for the two geometric series.

57. Find a power series $P(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ satisfying the differential equation $y' = -y$ with initial condition $y(0) = 1$. Then use Theorem 1 of Section 5.8 to conclude that $P(x) = e^{-x}$.

SOLUTION Let $P(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and note that $P(0) = a_0$; thus, to satisfy the initial condition $P(0) = 1$, we must take $a_0 = 1$. Now,

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

so

$$P'(x) + P(x) = \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + a_n\right] x^n.$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus

$$(n+1)a_{n+1} + a_n = 0 \quad \text{or} \quad a_{n+1} = -\frac{a_n}{n+1}.$$

Starting from $a_0 = 1$, we then calculate

$$a_1 = -\frac{a_0}{1} = -1;$$

$$a_2 = -\frac{a_1}{2} = \frac{1}{2};$$

$$a_3 = -\frac{a_2}{3} = -\frac{1}{6} = -\frac{1}{3!};$$

and, in general,

$$a_n = (-1)^n \frac{1}{n!}.$$

Hence,

$$P(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

The solution to the initial value problem $y' = -y$, $y(0) = 1$ is $y = e^{-x}$. Because this solution is unique, it follows that

$$P(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = e^{-x}.$$

58. Let $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.
(a) Show that $C(x)$ has an infinite radius of convergence.
(b) Prove that $C(x)$ and $f(x) = \cos x$ are both solutions of $y'' = -y$ with initial conditions $y(0) = 1$, $y'(0) = 0$. This initial value problem has a unique solution, so we have $C(x) = \cos x$ for all x.

SOLUTION
(a) Consider the series

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

With $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} = \frac{|x|^2}{(2n+2)(2n+1)},$$

and

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.$$

The radius of convergence for $C(x)$ is therefore $R = r^{-1} = \infty$.
(b) Differentiating the series defining $C(x)$ term-by-term, we find

$$C'(x) = \sum_{n=1}^{\infty} (-1)^n (2n) \frac{x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$$

and

$$C''(x) = \sum_{n=1}^{\infty} (-1)^n (2n-1) \frac{x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n-2)!}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = -C(x).$$

Moreover, $C(0) = 1$ and $C'(0) = 0$.

59. Use the power series for $y = e^x$ to show that

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

Use your knowledge of alternating series to find an N such that the partial sum S_N approximates e^{-1} to within an error of at most 10^{-3}. Confirm this using a calculator to compute both S_N and e^{-1}.

SOLUTION Recall that the series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Setting $x = -1$ yields

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots.$$

This is an alternating series with $a_n = \frac{1}{(n+1)!}$. The error in approximating e^{-1} with the partial sum S_N is therefore bounded by

$$|S_N - e^{-1}| \le a_{N+1} = \frac{1}{(N+2)!}.$$

To make the error at most 10^{-3}, we must choose N such that

$$\frac{1}{(N+2)!} \le 10^{-3} \quad \text{or} \quad (N+2)! \ge 1000.$$

For $N = 4$, $(N+2)! = 6! = 720 < 1000$, but for $N = 5$, $(N+2)! = 7! = 5040$; hence, $N = 5$ is the smallest value that satisfies the error bound. The corresponding approximation is

$$S_5 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = 0.368055555$$

Now, $e^{-1} = 0.367879441$, so

$$|S_5 - e^{-1}| = 1.761 \times 10^{-4} < 10^{-3}.$$

60. Let $P(x) = \sum_{n=0} a_n x^n$ be a power series solution to $y' = 2xy$ with initial condition $y(0) = 1$.

(a) Show that the odd coefficients a_{2k+1} are all zero.

(b) Prove that $a_{2k} = a_{2k-2}/k$ and use this result to determine the coefficients a_{2k}.

SOLUTION Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$ and note that $P(0) = a_0$; thus, to satisfy the initial condition $P(0) = 1$, we must take $a_0 = 1$. Now,

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

so

$$P'(x) - 2xP(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1}$$

$$= a_1 + \sum_{n=2}^{\infty} \left[n a_n - 2a_{n-2} \right] x^{n-1}.$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus, $a_1 = 0$ and

$$n a_n - 2a_{n-2} = 0 \quad \text{or} \quad a_n = \frac{2a_{n-2}}{n}.$$

(a) We know that $a_1 = 0$ and

$$a_n = \frac{2a_{n-2}}{n}.$$

Thus,

$$a_3 = \frac{2a_1}{3} = 0;$$
$$a_5 = \frac{2a_3}{5} = 0;$$
$$a_7 = \frac{2a_5}{7} = 0;$$

and, in general, $a_{2k+1} = 0$ for all k.

(b) Replace n by $2k$ in the equation

$$a_n = \frac{2a_{n-2}}{n} \quad \text{to obtain} \quad a_{2k} = \frac{2a_{2k-2}}{2k} = \frac{a_{2k-2}}{k}.$$

Starting from $a_0 = 1$, we then calculate

$$a_2 = \frac{a_0}{1} = 1 = \frac{1}{1!};$$
$$a_4 = \frac{a_2}{2} = \frac{1}{2} = \frac{1}{2!};$$
$$a_6 = \frac{a_4}{3} = \frac{1}{6} = \frac{1}{3!};$$

and, in general, $a_{2k} = \frac{1}{k!}$.

61. Find a power series $P(x)$ satisfying the differential equation

$$y'' - xy' + y = 0$$ 9

with initial condition $y(0) = 1$, $y'(0) = 0$. What is the radius of convergence of the power series?

SOLUTION Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad P''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Note that $P(0) = a_0$ and $P'(0) = a_1$; in order to satisfy the initial conditions $P(0) = 1$, $P'(0) = 0$, we must have $a_0 = 1$ and $a_1 = 0$. Now,

$$\begin{aligned} P''(x) - xP'(x) + P(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - n a_n + a_n\right] x^n. \end{aligned}$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus, $2a_2 + a_0 = 0$ and $(n+2)(n+1)a_{n+2} - (n-1)a_n = 0$, or

$$a_2 = -\frac{1}{2}a_0 \quad \text{and} \quad a_{n+2} = \frac{n-1}{(n+2)(n+1)} a_n.$$

Starting from $a_1 = 0$, we calculate

$$a_3 = \frac{1-1}{(3)(2)} a_1 = 0;$$
$$a_5 = \frac{2}{(5)(4)} a_3 = 0;$$
$$a_7 = \frac{4}{(7)(6)} a_5 = 0;$$

and, in general, all of the odd coefficients are zero. As for the even coefficients, we have $a_0 = 1$, $a_2 = -\frac{1}{2}$,

$$a_4 = \frac{1}{(4)(3)}a_2 = -\frac{1}{4!};$$
$$a_6 = \frac{3}{(6)(5)}a_4 = -\frac{3}{6!};$$
$$a_8 = \frac{5}{(8)(7)}a_6 = -\frac{15}{8!}$$

and so on. Thus,

$$P(x) = 1 - \frac{1}{2}x^2 - \frac{1}{4!}x^4 - \frac{3}{6!}x^6 - \frac{15}{8!}x^8 - \cdots$$

To determine the radius of convergence, treat this as a series in the variable x^2, and observe that

$$r = \lim_{k\to\infty}\left|\frac{a_{2k+2}}{a_{2k}}\right| = \lim_{k\to\infty}\frac{2k-1}{(2k+2)(2k+1)} = 0.$$

Thus, the radius of convergence is $R = r^{-1} = \infty$.

62. Find a power series satisfying Eq. (9) with initial condition $y(0) = 0$, $y'(0) = 1$.

SOLUTION Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution to Eq. (9). From the previous exercise, we know that

$$a_2 = -\frac{1}{2}a_0 \quad \text{and} \quad a_{n+2} = \frac{n-1}{(n+2)(n+1)}a_n.$$

To satisfy the initial condition $P(0) = 0$, $P'(0) = 1$, we must have $a_0 = 0$ and $a_1 = 1$. Then

$$a_2 = -\frac{1}{2}a_0 = 0;$$
$$a_4 = \frac{1}{(4)(3)}a_2 = 0;$$
$$a_6 = \frac{3}{(6)(5)}a_4 = 0;$$

and, in general, all of the even coefficients are zero. As in the previous exercise, all of the odd coefficients past a_1 are also equal to zero. Thus,

$$P(x) = x.$$

63. Prove that

$$J_2(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+2}\,k!\,(k+3)!}x^{2k+2}$$

is a solution of the Bessel differential equation of order 2:

$$x^2y'' + xy' + (x^2-4)y = 0$$

SOLUTION Let $J_2(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+2}\,k!\,(k+2)!}x^{2k+2}$. Then

$$J_2'(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{2^{2k+1}\,k!\,(k+2)!}x^{2k+1}$$
$$J_2''(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{2^{2k+1}\,k!\,(k+2)!}x^{2k}$$

and

$$x^2J_2''(x) + xJ_2'(x) + (x^2-4)J_2(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{2^{2k+1}\,k!\,(k+2)!}x^{2k+2} + \sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{2^{2k+1}\,k!\,(k+2)!}x^{2k+2}$$
$$- \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+2}\,k!\,(k+2)!}x^{2k+4} - \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\,k!\,(k+2)!}x^{2k+2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k(k+2)}{2^{2k} k!(k+2)!} x^{2k+2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k}\,(k-1)!\,(k+1)!} x^{2k+2}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k-1)!(k+1)!} x^{2k+2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k-1)!(k+1)!} x^{2k+2} = 0.$$

64. Why is it impossible to expand $f(x) = |x|$ as a power series that converges in an interval around $x = 0$? Explain using Theorem 2.

SOLUTION Suppose that there exists a $c > 0$ such that f can be represented by a power series on the interval $(-c, c)$; that is,

$$|x| = \sum_{n=0}^{\infty} a_n x^n$$

for $|x| < c$. Then it follows by Theorem 2 that $|x|$ is differentiable on $(-c, c)$. This contradicts the well known property that $f(x) = |x|$ is not differentiable at the point $x = 0$.

Further Insights and Challenges

65. Suppose that the coefficients of $F(x) = \sum_{n=0}^{\infty} a_n x^n$ are *periodic*; that is, for some whole number $M > 0$, we have $a_{M+n} = a_n$. Prove that $F(x)$ converges absolutely for $|x| < 1$ and that

$$F(x) = \frac{a_0 + a_1 x + \cdots + a_{M-1} x^{M-1}}{1 - x^M}$$

Hint: Use the hint for Exercise 53.

SOLUTION Suppose the coefficients of $F(x)$ are periodic, with $a_{M+n} = a_n$ for some whole number M and all n. The $F(x)$ can be written as the sum of M geometric series:

$$F(x) = a_0\left(1 + x^M + x^{2M} + \cdots\right) + a_1\left(x + x^{M+1} + x^{2M+1} + \cdots\right) +$$

$$= a_2\left(x^2 + x^{M+2} + x^{2M+2} + \cdots\right) + \cdots + a_{M-1}\left(x^{M-1} + x^{2M-1} + x^{3M-1} + \cdots\right)$$

$$= \frac{a_0}{1 - x^M} + \frac{a_1 x}{1 - x^M} + \frac{a_2 x^2}{1 - x^M} + \cdots + \frac{a_{M-1} x^{M-1}}{1 - x^M} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{M-1} x^{M-1}}{1 - x^M}.$$

As each geometric series converges absolutely for $|x| < 1$, it follows that $F(x)$ also converges absolutely for $|x| < 1$.

66. Continuity of Power Series Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.

(a) Prove the inequality

$$|x^n - y^n| \le n|x - y|(|x|^{n-1} + |y|^{n-1}) \qquad \boxed{10}$$

Hint: $x^n - y^n = (x - y)(x^{n-1} + x^{n-2} y + \cdots + y^{n-1})$.

(b) Choose R_1 with $0 < R_1 < R$. Show that the infinite series $M = \sum_{n=0}^{\infty} 2n|a_n| R_1^n$ converges. *Hint:* Show that $n|a_n| R_1^n < |a_n| x^n$ for all n sufficiently large if $R_1 < x < R$.

(c) Use Eq. (10) to show that if $|x| < R_1$ and $|y| < R_1$, then $|F(x) - F(y)| \le M|x - y|$.

(d) Prove that if $|x| < R$, then $F(x)$ is continuous at x. *Hint:* Choose R_1 such that $|x| < R_1 < R$. Show that if $\epsilon > 0$ is given, then $|F(x) - F(y)| \le \epsilon$ for all y such that $|x - y| < \delta$, where δ is any positive number that is less than ϵ/M and $R_1 - |x|$ (see Figure 6).

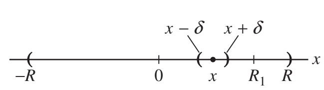


FIGURE 6 If $x > 0$, choose $\delta > 0$ less than ϵ/M and $R_1 - x$.

SOLUTION

(a) Take the absolute value of both sides of the identity

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2} y + \cdots + y^{n-1}),$$

and then apply the triangle inequality to obtain

$$|x^n - y^n| \le |x - y|\left(|x|^{n-1} + |x|^{n-2}|y| + |x|^{n-3}|y|^2 + \cdots + |x||y|^{n-2} + |y|^{n-1}\right).$$

Now, if $|x| \le |y|$ then $|x|^{n-k}|y|^{k-1} \le |y|^{n-k}|y|^{k-1} = |y|^{n-1}$, and if $|y| \le |x|$ then $|x|^{n-k}|y|^{k-1} \le |x|^{n-k}|x|^{k-1} = |x|^{n-1}$. In either case, $|x|^{n-k}|y|^{k-1} \le |x|^{n-1} + |y|^{n-1}$. Thus,

$$|x^n - y^n| \le |x - y|\left(|x|^{n-1} + (n-2)(|x|^{n-1} + |y|^{n-1}) + |y|^{n-1}\right)$$

$$= (n-1)|x-y|\left(|x|^{n-1} + |y|^{n-1}\right) \le n|x-y|\left(|x|^{n-1} + |y|^{n-1}\right).$$

(b) Let $0 < R_1 < R$. Then,

$$\rho = \lim_{n\to\infty} \frac{2(n+1)|a_{n+1}|R_1^{n+1}}{2n|a_n|R_1^n} = R_1 \lim_{n\to\infty} \frac{n+1}{n} \cdot \left|\frac{a_{n+1}}{a_n}\right| = R_1 \cdot 1 \cdot \frac{1}{R} = \frac{R_1}{R} < 1.$$

Thus, the series $M = \sum_{n=0}^{\infty} 2n|a_n|R_1^n$ converges by the Ratio Test.

(c) Suppose $|x| < R_1$ and $|y| < R_1$. Then

$$|F(x) - F(y)| = \left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n y^n\right| \le \sum_{n=0}^{\infty} |a_n||x^n - y^n| \le \sum_{n=0}^{\infty} n|a_n||x-y|\left(|x|^{n-1} + |y|^{n-1}\right)$$

$$\le |x-y| \sum_{n=0}^{\infty} n|a_n|\left(R_1^{n-1} + R_1^{n-1}\right) = M|x-y|$$

(d) Let $|x| < R$, and let R_1 be a number such that $|x| < R_1 < R$. Then by part (b), $M = \sum_{n=0}^{\infty} 2n|a_n|R_1^n$ is finite and by part (c)

$$|F(x) - F(y)| \le M|x-y|$$

for $|y| < R_1$. Now, let $\epsilon > 0$, and choose $\delta > 0$ so that $\delta < \frac{\epsilon}{M}$ and $\delta < R_1 - |x|$. Then, whenever $|y - x| < \delta$,

$$|y| = |(y-x) + x| \le |y-x| + |x| < \delta + |x| < R_1,$$

so

$$|F(x) - F(y)| < M|x-y| < M\delta < M \cdot \frac{\epsilon}{M} = \epsilon.$$

Thus, F is continuous at x.

10.7 Taylor Series

Preliminary Questions

1. Determine $f(0)$ and $f'''(0)$ for a function $f(x)$ with Maclaurin series

$$T(x) = 3 + 2x + 12x^2 + 5x^3 + \cdots$$

SOLUTION The Maclaurin series for a function f has the form

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Matching this general expression with the given series, we find $f(0) = 3$ and $\frac{f'''(0)}{3!} = 5$. From this latter equation, it follows that $f'''(0) = 30$.

2. Determine $f(-2)$ and $f^{(4)}(-2)$ for a function with Taylor series

$$T(x) = 3(x+2) + (x+2)^2 - 4(x+2)^3 + 2(x+2)^4 + \cdots$$

SOLUTION The Taylor series for a function f centered at $x = -2$ has the form

$$f(-2) + \frac{f'(-2)}{1!}(x+2) + \frac{f''(-2)}{2!}(x+2)^2 + \frac{f'''(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \cdots$$

Matching this general expression with the given series, we find $f(-2) = 0$ and $\frac{f^{(4)}(-2)}{4!} = 2$. From this latter equation, it follows that $f^{(4)}(-2) = 48$.

3. What is the easiest way to find the Maclaurin series for the function $f(x) = \sin(x^2)$?

SOLUTION The easiest way to find the Maclaurin series for $\sin\left(x^2\right)$ is to substitute x^2 for x in the Maclaurin series for $\sin x$.

4. Find the Taylor series for $f(x)$ centered at $c = 3$ if $f(3) = 4$ and $f'(x)$ has a Taylor expansion

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

SOLUTION Integrating the series for $f'(x)$ term-by-term gives

$$f(x) = C + \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{n(n+1)}.$$

Substituting $x = 3$ then yields

$$f(3) = C = 4;$$

so

$$f(x) = 4 + \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{n(n+1)}.$$

5. Let $T(x)$ be the Maclaurin series of $f(x)$. Which of the following guarantees that $f(2) = T(2)$?

(a) $T(x)$ converges for $x = 2$.
(b) The remainder $R_k(2)$ approaches a limit as $k \to \infty$.
(c) The remainder $R_k(2)$ approaches zero as $k \to \infty$.

SOLUTION The correct response is **(c)**: $f(2) = T(2)$ if and only if the remainder $R_k(2)$ approaches zero as $k \to \infty$.

Exercises

1. Write out the first four terms of the Maclaurin series of $f(x)$ if

$$f(0) = 2, \quad f'(0) = 3, \quad f''(0) = 4, \quad f'''(0) = 12$$

SOLUTION The first four terms of the Maclaurin series of $f(x)$ are

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 2 + 3x + \frac{4}{2}x^2 + \frac{12}{6}x^3 = 2 + 3x + 2x^2 + 2x^3.$$

2. Write out the first four terms of the Taylor series of $f(x)$ centered at $c = 3$ if

$$f(3) = 1, \quad f'(3) = 2, \quad f''(3) = 12, \quad f'''(3) = 3$$

SOLUTION The first four terms of the Taylor series centered at $c = 3$ are:

$$\begin{aligned} f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 &= 1 + 2(x-3) + \frac{12}{2}(x-3)^2 + \frac{3}{6}(x-3)^3 \\ &= 1 + 2(x-3) + 6(x-3)^2 + \frac{1}{2}(x-3)^3. \end{aligned}$$

In Exercises 3–18, find the Maclaurin series and find the interval on which the expansion is valid.

3. $f(x) = \dfrac{1}{1-2x}$

SOLUTION Substituting $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n.$$

This series is valid for $|2x| < 1$, or $|x| < \frac{1}{2}$.

4. $f(x) = \dfrac{x}{1-x^4}$

SOLUTION Substituting x^4 for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1-x^4} = \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n}.$$

Therefore

$$\frac{x}{1-x^4} = x\sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty} x^{4n+1}.$$

This series is valid for $|x^4| < 1$, or $|x| < 1$.

5. $f(x) = \cos 3x$

SOLUTION Substituting $3x$ for x in the Maclaurin series for $\cos x$ gives

$$\cos 3x = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{9^n x^{2n}}{(2n)!}.$$

This series is valid for all x.

6. $f(x) = \sin(2x)$

SOLUTION Substituting $2x$ for x in the Maclaurin series for $\sin x$ gives

$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}.$$

This series is valid for all x.

7. $f(x) = \sin(x^2)$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\sin x$ gives

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

This series is valid for all x.

8. $f(x) = e^{4x}$

SOLUTION Substituting $4x$ for x in the Maclaurin series for e^x gives

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=o}^{\infty} \frac{4^n x^n}{n!}.$$

This series is valid for all x.

9. $f(x) = \ln(1-x^2)$

SOLUTION Substituting $-x^2$ for x in the Maclaurin series for $\ln(1+x)$ gives

$$\ln(1-x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}x^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}.$$

This series is valid for $|x| < 1$.

10. $f(x) = (1-x)^{-1/2}$

SOLUTION Substituting $-x$ for x and using $a = -\frac{1}{2}$ in the Binomial series gives

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^n.$$

This series is valid for $|x| < 1$.

11. $f(x) = \tan^{-1}(x^2)$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\tan^{-1} x$ gives

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}.$$

This series is valid for $|x| \le 1$.

12. $f(x) = x^2 e^{x^2}$

SOLUTION First substitute x^2 for x in the Maclaurin series for e^x to obtain

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Now, multiply by x^2 to obtain

$$x^2 e^{x^2} = x^2 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}.$$

This series is valid for all x.

13. $f(x) = e^{x-2}$

SOLUTION $e^{x-2} = e^{-2}e^x$; thus,

$$e^{x-2} = e^{-2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{e^2 n!}.$$

This series is valid for all x.

14. $f(x) = \dfrac{1 - \cos x}{x}$

SOLUTION $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$, so

$$\frac{1 - \cos x}{x} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!}$$

This series is valid for all x.

15. $f(x) = \ln(1 - 5x)$

SOLUTION Substituting $-5x$ for x in the Maclaurin series for $\ln(1 + x)$ gives

$$\ln(1 - 5x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-5x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} 5^n x^n}{n} = -\sum_{n=1}^{\infty} \frac{5^n x^n}{n}.$$

This series is valid for $|5x| < 1$, or $|x| < \frac{1}{5}$, and for $x = -\frac{1}{5}$.

16. $f(x) = (x^2 + 2x)e^x$

SOLUTION Using the Maclaurin series for e^x, we find

$$(x^2 + 2x)e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} + 2x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} + \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} = 2x + \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)!} + \frac{2}{n!} \right) x^{n+1}$$

$$= 2x + \sum_{n=1}^{\infty} \frac{n+2}{n!} x^{n+1} = \sum_{n=0}^{\infty} \frac{n+2}{n!} x^{n+1}.$$

This series is valid for all x.

17. $f(x) = \sinh x$

SOLUTION Recall that

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

Therefore,

$$\sinh x = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^n}{2(n!)} \left(1 - (-1)^n\right).$$

Now,

$$1-(-1)^n = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

so

$$\sinh x = \sum_{k=0}^{\infty} 2\frac{x^{2k+1}}{2(2k+1)!} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

This series is valid for all x.

18. $f(x) = \cosh x$

SOLUTION Recall that

$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Therefore,

$$\cosh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^n}{2(n!)}\left(1+(-1)^n\right).$$

Now,

$$1+(-1)^n = \begin{cases} 0, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}$$

so

$$\cosh x = \sum_{k=0}^{\infty} 2\frac{x^{2k}}{2(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

This series is valid for all x.

In Exercises 19–28, find the terms through degree four of the Maclaurin series of $f(x)$. Use multiplication and substitution as necessary.

19. $f(x) = e^x \sin x$

SOLUTION Multiply the fourth-order Taylor Polynomials for e^x and $\sin x$:

$$\begin{aligned}
&\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\left(x - \frac{x^3}{6}\right) \\
&\quad = x + x^2 - \frac{x^3}{6} + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^4}{6} + \text{ higher-order terms} \\
&\quad = x + x^2 + \frac{x^3}{3} + \text{ higher-order terms.}
\end{aligned}$$

The terms through degree four in the Maclaurin series for $f(x) = e^x \sin x$ are therefore

$$x + x^2 + \frac{x^3}{3}.$$

20. $f(x) = e^x \ln(1-x)$

SOLUTION Multiply the fourth order Taylor Polynomials for e^x and $\ln(1-x)$:

$$\begin{aligned}
&\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right) \\
&\quad = -x - \frac{x^2}{2} - x^2 - \frac{x^3}{3} - \frac{x^3}{2} - \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^4}{3} - \frac{x^4}{4} - \frac{x^4}{6} + \text{ higher-order terms} \\
&\quad = -x - \frac{3x^2}{2} - \frac{4x^3}{3} - x^4 + \text{ higher-order terms.}
\end{aligned}$$

The first four terms of the Maclaurin series for $f(x) = e^x \ln(1 - x)$ are therefore

$$-x - \frac{3x^2}{2} - \frac{4x^3}{3} - x^4.$$

21. $f(x) = \dfrac{\sin x}{1 - x}$

SOLUTION Multiply the fourth order Taylor Polynomials for $\sin x$ and $\dfrac{1}{1-x}$:

$$\begin{aligned}&\left(x - \frac{x^3}{6}\right)\left(1 + x + x^2 + x^3 + x^4\right)\\ &\qquad = x + x^2 - \frac{x^3}{6} + x^3 + x^4 - \frac{x^4}{6} + \text{ higher-order terms}\\ &\qquad = x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6} + \text{ higher-order terms.}\end{aligned}$$

The terms through order four of the Maclaurin series for $f(x) = \dfrac{\sin x}{1 - x}$ are therefore

$$x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6}.$$

22. $f(x) = \dfrac{1}{1 + \sin x}$

SOLUTION Substituting $\sin x$ for x in the Maclaurin series for $\frac{1}{1-x}$ and then using the Maclaurin series for $\sin x$ gives

$$\begin{aligned}\frac{1}{1+\sin x} &= 1 - \sin x + \sin^2 x - \sin^3 x + \sin^4 x - \dots\\ &= 1 - \left(x - \frac{x^3}{6} + \cdots\right) + \left(x - \frac{x^3}{6} + \cdots\right)^2 - \left(x - \frac{x^3}{6} + \dots\right)^3 + \left(x - \frac{x^3}{6} + \dots\right)^4 \cdots\\ &= 1 - x + \frac{x^3}{6} + x^2 - \frac{x^4}{3} - x^3 + x^4 = 1 - x + x^2 - \frac{5x^3}{6} + \frac{2x^4}{3}\end{aligned}$$

Therefore, the terms of the Maclaurin series for $f(x) = \dfrac{1}{1 + \sin x}$ through order four are

$$1 - x + x^2 - \frac{5x^3}{6} + \frac{2x^4}{3}.$$

23. $f(x) = (1 + x)^{1/4}$

SOLUTION The first five generalized binomial coefficients for $a = \frac{1}{4}$ are

$$1, \quad \frac{1}{4}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)}{2!} = -\frac{3}{32}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{3!} = \frac{7}{128}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-11}{4}\right)}{4!} = \frac{-77}{2048}$$

Therefore, the first four terms in the binomial series for $(1 + x)^{1/4}$ are

$$1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4$$

24. $f(x) = (1 + x)^{-3/2}$

SOLUTION The first five generalized binomial coefficients for $a = -\frac{3}{2}$ are

$$1, \quad -\frac{3}{2}, \quad \frac{-\frac{3}{2}(-\frac{5}{2})}{2!} = \frac{15}{8}, \quad \frac{-\frac{3}{2}(-\frac{5}{2})(-\frac{7}{2})}{3!} = -\frac{35}{16}, \quad \frac{-\frac{3}{2}(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})}{4!} = \frac{315}{128}.$$

Therefore, the first five terms in the binomial series for $f(x) = (1 + x)^{-3/2}$ are

$$1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \frac{315}{128}x^4.$$

25. $f(x) = e^x \tan^{-1} x$

SOLUTION Using the Maclaurin series for e^x and $\tan^{-1} x$, we find

$$e^x \tan^{-1} x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{3} + \cdots\right) = x + x^2 - \frac{x^3}{3} + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^4}{3} + \cdots$$
$$= x + x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \cdots.$$

26. $f(x) = \sin(x^3 - x)$

SOLUTION Substitute $x^3 - x$ into the first two terms of the Maclaurin series for $\sin x$:

$$(x^3 - x) - \frac{(x^3 - x)^3}{3!} = x^3 - x - \frac{x^9 - 3x^7 + 3x^5 - x^3}{3!}$$

so that the terms of the Maclaurin series for $\sin(x^3 - x)$ through degree four are

$$-x + \frac{7}{6}x^3$$

27. $f(x) = e^{\sin x}$

SOLUTION Substituting $\sin x$ for x in the Maclaurin series for e^x and then using the Maclaurin series for $\sin x$, we find

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + \cdots$$
$$= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{6}(x - \cdots)^3 + \frac{1}{24}(x - \cdots)^4$$
$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^4 + \cdots$$
$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots.$$

28. $f(x) = e^{(e^x)}$

SOLUTION With $f(x) = e^{(e^x)}$, we find

$$f'(x) = e^{(e^x)} \cdot e^x$$
$$f''(x) = e^{(e^x)} \cdot e^x + e^{(e^x)} \cdot e^{2x} = e^{(e^x)}\left(e^{2x} + e^x\right)$$
$$f'''(x) = e^{(e^x)}\left(2e^{2x} + e^x\right) + e^{(e^x)}\left(e^{2x} + e^x\right)e^x$$
$$= e^{(e^x)}\left(e^{3x} + 3e^{2x} + e^x\right)$$
$$f^{(4)}(x) = e^{(e^x)}(3e^{3x} + 6e^{2x} + e^x) + e^{(e^x)}(e^{3x} + 3e^{2x} + e^x)e^x$$
$$= e^{(e^x)}(e^{4x} + 6e^{3x} + 7e^{2x} + e^x)$$

and

$$f(0) = e, \quad f'(0) = e, \quad f''(0) = 2e, \quad f'''(0) = 5e, \quad f^{(4)}(0) = 15e.$$

Therefore, the first four terms of the Maclaurin for $f(x) = e^{(e^x)}$ are

$$e + ex + ex^2 + \frac{5e}{6}x^3 + \frac{5e}{8}x^4.$$

In Exercises 29–38, find the Taylor series centered at c and find the interval on which the expansion is valid.

29. $f(x) = \dfrac{1}{x}, \quad c = 1$

SOLUTION Write

$$\frac{1}{x} = \frac{1}{1 + (x-1)},$$

and then substitute $-(x-1)$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{x} = \sum_{n=0}^{\infty} [-(x-1)]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

This series is valid for $|x-1| < 1$.

30. $f(x) = e^{3x}, \quad c = -1$

SOLUTION Write

$$e^{3x} = e^{3(x+1)-3} = e^{-3}e^{3(x+1)}.$$

Now, substitute $3(x+1)$ for x in the Maclaurin series for e^x to obtain

$$e^{3(x+1)} = \sum_{n=0}^{\infty} \frac{(3(x+1))^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!}(x+1)^n.$$

Thus,

$$e^{3x} = e^{-3} \sum_{n=0}^{\infty} \frac{3^n}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{3^n e^{-3}}{n!}(x+1)^n,$$

This series is valid for all x.

31. $f(x) = \dfrac{1}{1-x}, \quad c = 5$

SOLUTION Write

$$\frac{1}{1-x} = \frac{1}{-4-(x-5)} = -\frac{1}{4} \cdot \frac{1}{1 + \frac{x-5}{4}}.$$

Substituting $-\frac{x-5}{4}$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1 + \frac{x-5}{4}} = \sum_{n=0}^{\infty} \left(-\frac{x-5}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{4^n}.$$

Thus,

$$\frac{1}{1-x} = -\frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-5)^n}{4^{n+1}}.$$

This series is valid for $\left|\frac{x-5}{4}\right| < 1$, or $|x-5| < 4$.

32. $f(x) = \sin x, \quad c = \dfrac{\pi}{2}$

SOLUTION Note that the odd derivatives of $\sin x$ are zero at $\frac{\pi}{2}$, and the even derivatives alternate between $+1$ and -1. Thus the Taylor series centered at $\frac{\pi}{2}$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$$

33. $f(x) = x^4 + 3x - 1, \quad c = 2$

SOLUTION To determine the Taylor series with center $c = 2$, we compute

$$f'(x) = 4x^3 + 3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x,$$

and $f^{(4)}(x) = 24$. All derivatives of order five and higher are zero. Now,

$$f(2) = 21, \quad f'(2) = 35, \quad f''(2) = 48, \quad f'''(2) = 48,$$

and $f^{(4)}(2) = 24$. Therefore, the Taylor series is

$$21 + 35(x-2) + \frac{48}{2}(x-2)^2 + \frac{48}{6}(x-2)^3 + \frac{24}{24}(x-2)^4,$$

or

$$21 + 35(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4.$$

34. $f(x) = x^4 + 3x - 1, \quad c = 0$

SOLUTION The function $x^4 + 3x - 1$ is a polynomial in x, hence it is already in the form of a Maclaurin series.

35. $f(x) = \dfrac{1}{x^2}, \quad c = 4$

SOLUTION We will first find the Taylor series for $\frac{1}{x}$ and then differentiate to obtain the series for $\frac{1}{x^2}$. Write

$$\frac{1}{x} = \frac{1}{4 + (x-4)} = \frac{1}{4} \cdot \frac{1}{1 + \frac{x-4}{4}}.$$

Now substitute $-\frac{x-4}{4}$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{x} = \frac{1}{4}\sum_{n=}^{\infty}\left(-\frac{x-4}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{(x-4)^n}{4^{n+1}}.$$

Differentiating term-by-term yields

$$-\frac{1}{x^2} = \sum_{n=1}^{\infty}(-1)^n n\frac{(x-4)^{n-1}}{4^{n+1}},$$

so that

$$\frac{1}{x^2} = \sum_{n=1}^{\infty}(-1)^{n-1} n\frac{(x-4)^{n-1}}{4^{n+1}} = \sum_{n=0}^{\infty}(-1)^n(n+1)\frac{(x-4)^n}{4^{n+2}}.$$

This series is valid for $\left|\frac{x-4}{4}\right| < 1$, or $|x-4| < 4$.

36. $f(x) = \sqrt{x}, \quad c = 4$

SOLUTION Write

$$\sqrt{x} = \sqrt{4 + (x-4)} = 2\sqrt{1 + \frac{x-4}{4}}.$$

Substituting $\frac{x-4}{4}$ for x in the binomial series with $a = \frac{1}{2}$ yields

$$\sqrt{x} = 2\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{x-4}{4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{2n-1}}\binom{\frac{1}{2}}{n}(x-4)^n.$$

This series is valid for $\left|\frac{x-4}{4}\right| < 1$, or $|x-4| < 4$.

37. $f(x) = \dfrac{1}{1-x^2}, \quad c = 3$

SOLUTION By partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x},$$

so

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{-2-(x-3)} + \frac{\frac{1}{2}}{4+(x-3)} = -\frac{1}{4}\cdot\frac{1}{1+\frac{x-3}{2}} + \frac{1}{8}\cdot\frac{1}{1+\frac{x-3}{4}}.$$

Substituting $-\frac{x-3}{2}$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1+\frac{x-3}{2}} = \sum_{n=0}^{\infty}\left(-\frac{x-3}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(x-3)^n,$$

while substituting $-\frac{x-3}{4}$ for x in the same series gives

$$\frac{1}{1+\frac{x-3}{4}} = \sum_{n=0}^{\infty}\left(-\frac{x-3}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(x-3)^n.$$

Thus,

$$\begin{aligned}\frac{1}{1-x^2} &= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(x-3)^n + \frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(x-3)^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(x-3)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+3}}(x-3)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}}{2^{n+2}} + \frac{(-1)^n}{2^{2n+3}}\right)(x-3)^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2^{n+1}-1)}{2^{2n+3}}(x-3)^n.\end{aligned}$$

This series is valid for $|x-3| < 2$.

38. $f(x) = \dfrac{1}{3x-2}, \quad c = -1$

SOLUTION Write

$$\frac{1}{3x-2} = \frac{1}{-5+3(x+1)} = -\frac{1}{5}\frac{1}{1-\frac{3(x+1)}{5}},$$

and then substitute $\frac{3(x+1)}{5}$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{1-\frac{3(x+1)}{5}} = \sum_{n=0}^{\infty}\left(\frac{3(x+1)}{5}\right)^n = \sum_{n=0}^{\infty}\frac{3^n}{5^n}(x+1)^n.$$

Thus,

$$\frac{1}{3x-2} = -\sum_{n=0}^{\infty}\frac{3^n}{5^{n+1}}(x+1)^n.$$

This series is valid for $\left|\frac{3(x+1)}{5}\right| < 1$, or $|x+1| < \frac{5}{3}$.

39. Use the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to find the Maclaurin series for $\cos^2 x$.

SOLUTION The Maclaurin series for $\cos 2x$ is

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!}$$

so the Maclaurin series for $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ is

$$\frac{1+\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!}\right)}{2} = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}$$

40. Show that for $|x| < 1$,

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Hint: Recall that $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$.

SOLUTION Because

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty}x^{2n},$$

we have

$$\tanh^{-1} x = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = C + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

Now, $\tanh^{-1} 0 = 0$, so it follows that $C = 0$, and

$$\tanh^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

41. Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to show that

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

for $|x| < 1$. What can you conclude by comparing this result with that of Exercise 40?

SOLUTION Using the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$, we have for $|x| < 1$

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-x)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1 + (-1)^{n-1}}{n} x^n.$$

Since $1 + (-1)^{n-1} = 0$ for even n and $1 + (-1)^{n-1} = 2$ for odd n,

$$\ln(1+x) - \ln(1-x) = \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1}.$$

Thus,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$

Observe that this is the same series we found in Exercise 40; therefore,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x.$$

42. Differentiate the Maclaurin series for $\frac{1}{1-x}$ twice to find the Maclaurin series of $\frac{1}{(1-x)^3}$.

SOLUTION Differentiating the Maclaurin series for $\frac{1}{1-x}$ term-by-term, we obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Differentiating again then yields

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2},$$

so that

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n.$$

43. Show, by integrating the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$, that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}$$

SOLUTION From Example 10, we know that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^{2n},$$

so, for $|x| < 1$,

$$\sin^{-1} x = \int \frac{dx}{\sqrt{1-x^2}} = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

Since $\sin^{-1} 0 = 0$, we find that $C = 0$. Thus,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

44. Use the first five terms of the Maclaurin series in Exercise 43 to approximate $\sin^{-1} \frac{1}{2}$. Compare the result with the calculator value.

SOLUTION From Exercise 43 we know that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

The first five terms of the series are:

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\frac{x^9}{9} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152}$$

Setting $x = \dfrac{1}{2}$, we obtain the following approximation:

$$\sin^{-1}\frac{1}{2} \approx \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{3 \cdot \left(\frac{1}{2}\right)^5}{40} + \frac{5 \cdot \left(\frac{1}{2}\right)^7}{112} + \frac{35 \cdot \left(\frac{1}{2}\right)^9}{1152} \approx 0.52358519539.$$

The calculator value is $\sin^{-1}\frac{1}{2} \approx 0.5235988775$.

45. How many terms of the Maclaurin series of $f(x) = \ln(1+x)$ are needed to compute $\ln 1.2$ to within an error of at most 0.0001? Make the computation and compare the result with the calculator value.

SOLUTION Substitute $x = 0.2$ into the Maclaurin series for $\ln(1+x)$ to obtain:

$$\ln 1.2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(0.2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{5^n n}.$$

This is an alternating series with $a_n = \dfrac{1}{n \cdot 5^n}$. Using the error bound for alternating series

$$|\ln 1.2 - S_N| \le a_{N+1} = \frac{1}{(N+1)5^{N+1}},$$

so we must choose N so that

$$\frac{1}{(N+1)5^{N+1}} < 0.0001 \quad \text{or} \quad (N+1)5^{N+1} > 10{,}000.$$

For $N = 3$, $(N+1)5^{N+1} = 4 \cdot 5^4 = 2500 < 10,000$, and for $N = 4$, $(N+1)5^{N+1} = 5 \cdot 5^5 = 15,625 > 10,000$; thus, the smallest acceptable value for N is $N = 4$. The corresponding approximation is:

$$S_4 = \sum_{n=1}^{4} \frac{(-1)^{n-1}}{5^n \cdot n} = \frac{1}{5} - \frac{1}{5^2 \cdot 2} + \frac{1}{5^3 \cdot 3} - \frac{1}{5^4 \cdot 4} = 0.182266666.$$

Now, $\ln 1.2 = 0.182321556$, so

$$|\ln 1.2 - S_4| = 5.489 \times 10^{-5} < 0.0001.$$

46. Show that

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

converges to zero. How many terms must be computed to get within 0.01 of zero?

SOLUTION Set $x = \pi$ in the Maclaurin series for $\sin x$ to obtain:

$$0 = \sin \pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots .$$

Using the error bound for an alternating series, we have

$$\left| 0 - \sum_{n=0}^{N} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \right| \le \frac{\pi^{2N+3}}{(2N+3)!}.$$

$N = 4$ is the smallest value for which the error bound is less than 0.01, so five terms are needed.

47. Use the Maclaurin expansion for e^{-t^2} to express the function $F(x) = \int_0^x e^{-t^2}\, dt$ as an alternating power series in x (Figure 4).

(a) How many terms of the Maclaurin series are needed to approximate the integral for $x = 1$ to within an error of at most 0.001?

(b) CAS Carry out the computation and check your answer using a computer algebra system.

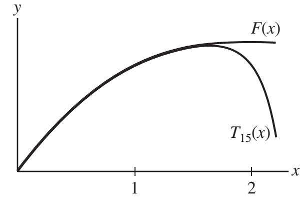


FIGURE 4 The Maclaurin polynomial $T_{15}(x)$ for $F(t) = \int_0^x e^{-t^2}\, dt$.

SOLUTION Substituting $-t^2$ for t in the Maclaurin series for e^t yields

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!};$$

thus,

$$\int_0^x e^{-t^2}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

(a) For $x = 1$,

$$\int_0^1 e^{-t^2}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(2n+1)}.$$

This is an alternating series with $a_n = \frac{1}{n!(2n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left| \int_0^1 e^{-t^2}\, dt - S_N \right| \le a_{N+1} = \frac{1}{(N+1)!(2N+3)}.$$

To guarantee the error is at most 0.001, we must choose N so that

$$\frac{1}{(N+1)!(2N+3)} < 0.001 \quad \text{or} \quad (N+1)!(2N+3) > 1000.$$

For $N = 3$, $(N+1)!(2N+3) = 4! \cdot 9 = 216 < 1000$ and for $N = 4$, $(N+1)!(2N+3) = 5! \cdot 11 = 1320 > 1000$; thus, the smallest acceptable value for N is $N = 4$. The corresponding approximation is

$$S_4 = \sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+1)} = 1 - \frac{1}{3} + \frac{1}{2! \cdot 5} - \frac{1}{3! \cdot 7} + \frac{1}{4! \cdot 9} = 0.747486772.$$

(b) Using a computer algebra system, we find

$$\int_0^1 e^{-t^2}\,dt = 0.746824133;$$

therefore

$$\left|\int_0^1 e^{-t^2}\,dt - S_4\right| = 6.626 \times 10^{-4} < 10^{-3}.$$

48. Let $F(x) = \int_0^x \frac{\sin t\,dt}{t}$. Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

Evaluate $F(1)$ to three decimal places.

SOLUTION Divide the Maclaurin series for $\sin t$ by t to obtain

$$\frac{\sin t}{t} = \frac{1}{t}\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n+1)!}.$$

Integrating both sides of this equation and using term-by-term integration, we find

$$F(x) = \int_0^x \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

For $x = 1$,

$$F(1) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!(2n+1)}.$$

This is an alternating series with $a_n = \frac{1}{(2n+1)!(2n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 \frac{\sin t}{t}\,dt - S_N\right| \le a_{N+1} = \frac{1}{(2N+3)!(2N+3)}.$$

To guarantee the error is at most 0.0005, we must choose N so that

$$\frac{1}{(2N+3)!(2N+3)} < 0.0005 \quad \text{or} \quad (2N+3)!(2N+3) > 2000.$$

For $N = 1$, $(2N+3)!(2N+3) = 5! \cdot 5 = 600 < 2000$ and for $N = 2$, $(2N+3)!(2N+3) = 7! \cdot 7 = 35{,}280 > 2000$; thus, the smallest acceptable value for N is $N = 2$. The corresponding approximation is

$$S_2 = \sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!(2n+1)} = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} = 0.946111111.$$

In Exercises 49–52, express the definite integral as an infinite series and find its value to within an error of at most 10^{-4}.

49. $\int_0^1 \cos(x^2)\,dx$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\cos x$ yields

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!};$$

therefore,

$$\int_0^1 \cos(x^2)\,dx = \sum_{n=0}^{\infty}(-1)^n \left.\frac{x^{4n+1}}{(2n)!(4n+1)}\right|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}.$$

This is an alternating series with $a_n = \frac{1}{(2n)!(4n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 \cos(x^2)\,dx - S_N\right| \le a_{N+1} = \frac{1}{(2N+2)!(4N+5)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(2N+2)!(4N+5)} < 0.0001 \quad \text{or} \quad (2N+2)!(4N+5) > 10{,}000.$$

For $N = 2$, $(2N+2)!(4N+5) = 6! \cdot 13 = 9360 < 10{,}000$ and for $N = 3$, $(2N+2)!(4N+5) = 8! \cdot 17 = 685{,}440 > 10{,}000$; thus, the smallest acceptable value for N is $N = 3$. The corresponding approximation is

$$S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{(2n)!(4n+1)} = 1 - \frac{1}{5 \cdot 2!} + \frac{1}{9 \cdot 4!} - \frac{1}{13 \cdot 6!} = 0.904522792.$$

50. $\displaystyle\int_0^1 \tan^{-1}(x^2)\,dx$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\tan^{-1} x$ yields

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1};$$

therefore,

$$\int_0^1 \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty} (-1)^n \left.\frac{x^{4n+3}}{(2n+1)(4n+3)}\right|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)}.$$

This is an alternating series with $a_n = \frac{1}{(2n+1)(4n+3)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 \tan^{-1}(x^2)\,dx - S_N\right| \le a_{N+1} = \frac{1}{(2N+3)(4N+7)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(2N+3)(4N+7)} < 0.0001 \quad \text{or} \quad (2N+3)(4N+7) > 10{,}000.$$

For $N = 33$, $(2N+3)(4N+7) = (69)(139) = 9591 < 10{,}000$ and for $N = 34$, $(2N+3)(4N+7) = (71)(143) = 10{,}153 > 10{,}000$; thus, the smallest acceptable value for N is $N = 34$. The corresponding approximation is

$$S_{34} = \sum_{n=0}^{34} \frac{(-1)^n}{(2n)!(4n+1)} = 0.297953297.$$

51. $\displaystyle\int_0^1 e^{-x^3}\,dx$

SOLUTION Substituting $-x^3$ for x in the Maclaurin series for e^x yields

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!};$$

therefore,

$$\int_0^1 e^{-x^3}\,dx = \sum_{n=0}^{\infty} (-1)^n \left.\frac{x^{3n+1}}{n!(3n+1)}\right|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n+1)}.$$

This is an alternating series with $a_n = \frac{1}{n!(3n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 e^{-x^3}\,dx - S_N\right| \le a_{N+1} = \frac{1}{(N+1)!(3N+4)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(N+1)!(3N+4)} < 0.0001 \quad \text{or} \quad (N+1)!(3N+4) > 10{,}000.$$

For $N = 4$, $(N+1)!(3N+4) = 5! \cdot 16 = 1920 < 10{,}000$ and for $N = 5$, $(N+1)!(3N+4) = 6! \cdot 19 = 13{,}680 > 10{,}000$; thus, the smallest acceptable value for N is $N = 5$. The corresponding approximation is

$$S_5 = \sum_{n=0}^{5} \frac{(-1)^n}{n!(3n+1)} = 0.807446200.$$

52. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^4+1}}$

SOLUTION From Example 10, we know that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^{2n};$$

therefore,

$$\frac{1}{\sqrt{x^4+1}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} (-x^2)^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^{4n},$$

and

$$\int_0^1 \frac{dx}{\sqrt{x^4+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \left.\frac{x^{4n+1}}{4n+1}\right|_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(4n+1)(n!)^2}.$$

This is an alternating series with

$$a_n = \frac{(2n)!}{2^{2n}(4n+1)(n!)^2};$$

therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left| \int_0^1 \frac{dx}{\sqrt{x^4+1}} - S_N \right| \le a_{N+1} = \frac{(2N+2)!}{2^{2N+2}(4N+5)((N+1)!)^2}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{(2N+2)!}{2^{2N+2}(4N+5)((N+1)!)^2} < 0.0001.$$

For $N = 124$,

$$\frac{(2N+2)!}{2^{2N+2}(4N+5)((N+1)!)^2} = 0.0001006 > 0.0001,$$

and for $N = 125$,

$$\frac{(2N+2)!}{2^{2N+2}(4N+5)((N+1)!)^2} = 0.00009943 < 0.0001,$$

thus, the smallest acceptable value for N is $N = 125$. The corresponding approximation is

$$S_{125} = \sum_{n=0}^{125} (-1)^n \frac{(2n)!}{2^{2n}(4n+1)(n!)^2} = 0.926987328.$$

In Exercises 53–56, express the integral as an infinite series.

53. $\displaystyle\int_0^x \frac{1-\cos(t)}{t}\,dt$, for all x

SOLUTION The Maclaurin series for $\cos t$ is

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!},$$

so

$$1 - \cos t = -\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!},$$

and

$$\frac{1 - \cos t}{t} = \frac{1}{t} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{(2n)!}.$$

Thus,

$$\int_0^x \frac{1 - \cos(t)}{t}\, dt = \sum_{n=1}^{\infty} (-1)^{n+1} \left. \frac{t^{2n}}{(2n)!2n} \right|_0^x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!2n}.$$

54. $\displaystyle\int_0^x \frac{t - \sin t}{t}\, dt$, for all x

SOLUTION The Maclaurin series for $\sin t$ is

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!},$$

so

$$t - \sin t = -\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n+1}}{(2n+1)!},$$

and

$$\frac{t - \sin t}{t} = \frac{1}{t} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n+1)!}.$$

Thus,

$$\int_0^x \frac{t - \sin(t)}{t}\, dt = \sum_{n=1}^{\infty} (-1)^{n+1} \left. \frac{t^{2n+1}}{(2n+1)!(2n+1)} \right|_0^x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!(2n+1)}.$$

55. $\displaystyle\int_0^x \ln(1 + t^2)\, dt$, for $|x| < 1$

SOLUTION Substituting t^2 for t in the Maclaurin series for $\ln(1 + t)$ yields

$$\ln(1 + t^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(t^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{n}.$$

Thus,

$$\int_0^x \ln(1 + t^2)\, dt = \sum_{n=1}^{\infty} (-1)^n \left. \frac{t^{2n+1}}{n(2n+1)} \right|_0^x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n(2n+1)}.$$

56. $\displaystyle\int_0^x \frac{dt}{\sqrt{1 - t^4}}$, for $|x| < 1$

SOLUTION From Example 10, we know that for $|t| < 1$

$$\frac{1}{\sqrt{1 - t^2}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} t^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} t^{2n};$$

therefore,

$$\frac{1}{\sqrt{1 - t^4}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} (t^2)^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} t^{4n},$$

and

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \left.\frac{t^{4n+1}}{4n+1}\right|_0^x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{4n+1}}{4n+1}.$$

57. Which function has Maclaurin series $\sum_{n=0}^{\infty}(-1)^n 2^n x^n$?

SOLUTION We recognize that

$$\sum_{n=0}^{\infty}(-1)^n 2^n x^n = \sum_{n=0}^{\infty}(-2x)^n$$

is the Maclaurin series for $\frac{1}{1-x}$ with x replaced by $-2x$. Therefore,

$$\sum_{n=0}^{\infty}(-1)^n 2^n x^n = \frac{1}{1-(-2x)} = \frac{1}{1+2x}.$$

58. Which function has Maclaurin series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(x-3)^k?$$

For which values of x is the expansion valid?

SOLUTION Write the series as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(x-3)^k = \frac{1}{3}\sum_{k=0}^{\infty}\left(-\frac{x-3}{3}\right)^k,$$

which we recognize as $\frac{1}{3}$ times the Maclaurin series for $\frac{1}{1-x}$ with x replaced by $-\frac{x-3}{3}$. Therefore,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(x-3)^k = \frac{1}{3}\cdot\frac{1}{1+\frac{x-3}{3}} = \frac{1}{3+x-3} = \frac{1}{x}.$$

The series is valid for $\left|\frac{x-3}{3}\right| < 1$, or $|x-3| < 3$.

In Exercises 59–62, use Theorem 2 to prove that the $f(x)$ is represented by its Maclaurin series on the interval I.

59. $f(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$,

SOLUTION All derivatives of $f(x)$ consist of sin or cos applied to each of $x/2$ and $x/3$ and added together, so each summand is bounded by 1. Thus $\left|f^{(n)}(x)\right| \le 2$ for all n and x. By Theorem 2, $f(x)$ is represented by its Taylor series for every x.

60. $f(x) = e^{-x}$,

SOLUTION For any c, choose any $R > 0$ and consider the interval $(c-R, c+R)$. For $f(x) = e^{-x}$, we have

$$\left|f^{(n)}(x)\right| = \left|(-1)^n e^{-x}\right| = e^{-x}$$

and on $(c-R, c+R)$, e^{-x} is bounded above by $e^{-(c-R)} = e^{R-c}$. Thus all derivatives of $f(x)$ are bounded by e^{R-c} for any $x \in (c-R, c+R)$, so by Theorem 2, $f(x)$ is represented by its Taylor series centered at c.

61. $f(x) = \sinh x$,

SOLUTION By definition, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, so if both e^x and e^{-x} are represented by their Taylor series centered at c, then so is $\sinh x$. But the previous exercise shows that e^{-x} is so represented, and the text shows that e^x is.

62. $f(x) = (1+x)^{100}$

SOLUTION $f(x)$ is a polynomial, so it is equal to its Taylor series and thus is obviously represented by its Taylor series.

In Exercises 63–66, find the functions with the following Maclaurin series (refer to Table 1 on page 599).

63. $1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$

SOLUTION We recognize

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

as the Maclaurin series for e^x with x replaced by x^3. Therefore,

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots = e^{x^3}.$$

64. $1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 - 4^5x^5 + \cdots$

SOLUTION We recognize

$$1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 - 4^5x^5 + \cdots = \sum_{n=0}^{\infty} (-4x)^n$$

as the Maclaurin series for $\frac{1}{1-x}$ with x replaced by $-4x$. Therefore,

$$1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 - 4^5x^5 + \cdots = \frac{1}{1-(-4x)} = \frac{1}{1+4x}.$$

65. $1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots$

SOLUTION Note

$$\begin{aligned} 1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots &= 1 - 5x + \left(5x - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots\right) \\ &= 1 - 5x + \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{(2n+1)!}. \end{aligned}$$

The series is the Maclaurin series for $\sin x$ with x replaced by $5x$, so

$$1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots = 1 - 5x + \sin(5x).$$

66. $x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots$

SOLUTION We recognize

$$x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{2n+1}$$

as the Maclaurin series for $\tan^{-1} x$ with x replaced by x^4. Therefore,

$$x^4 - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \frac{x^{28}}{7} + \cdots = \tan^{-1}(x^4).$$

In Exercises 67 and 68, let

$$f(x) = \frac{1}{(1-x)(1-2x)}$$

67. Find the Maclaurin series of $f(x)$ using the identity

$$f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

SOLUTION Substituting $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}2^n x^n$$

which is valid for $|2x| < 1$, or $|x| < \frac{1}{2}$. Because the Maclaurin series for $\frac{1}{1-x}$ is valid for $|x| < 1$, the two series together are valid for $|x| < \frac{1}{2}$. Thus, for $|x| < \frac{1}{2}$,

$$\begin{aligned}\frac{1}{(1-2x)(1-x)} &= \frac{2}{1-2x} - \frac{1}{1-x} = 2\sum_{n=0}^{\infty}2^n x^n - \sum_{n=0}^{\infty}x^n\\ &= \sum_{n=0}^{\infty}2^{n+1}x^n - \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\left(2^{n+1}-1\right)x^n.\end{aligned}$$

68. Find the Taylor series for $f(x)$ at $c = 2$. *Hint:* Rewrite the identity of Exercise 67 as

$$f(x) = \frac{2}{-3-2(x-2)} - \frac{1}{-1-(x-2)}$$

SOLUTION Using the given identity,

$$f(x) = \frac{2}{-3-2(x-2)} - \frac{1}{-1-(x-2)} = -\frac{2}{3}\frac{1}{1+\frac{2}{3}(x-2)} + \frac{1}{1+(x-2)}.$$

Substituting $-\frac{2}{3}(x-2)$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1+\frac{2}{3}(x-2)} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n(x-2)^n,$$

and substituting $-(x-2)$ for x in the same Maclaurin series yields

$$\frac{1}{1+(x-2)} = \sum_{n=0}^{\infty}(-1)^n(x-2)^n.$$

The first series is valid for $\left|-\frac{2}{3}(x-2)\right| < 1$, or $|x-2| < \frac{3}{2}$, and the second series is valid for $|x-2| < 1$; therefore, the two series together are valid for $|x-2| < 1$. Finally, for $|x-2| < 1$,

$$f(x) = -\frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n(x-2)^n + \sum_{n=0}^{\infty}(-1)^n(x-2)^n = \sum_{n=0}^{\infty}(-1)^n\left[1-\left(\frac{2}{3}\right)^{n+1}\right](x-2)^n.$$

69. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is

$$I(t) = \left(\frac{V}{R}\right)\left(1-e^{-Rt/L}\right)$$

Expand $I(t)$ in a Maclaurin series. Show that $I(t) \approx \frac{Vt}{L}$ for small t.

SOLUTION Substituting $-\frac{Rt}{L}$ for t in the Maclaurin series for e^t gives

$$e^{-Rt/L} = \sum_{n=0}^{\infty}\frac{\left(-\frac{Rt}{L}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n$$

Thus,

$$1-e^{-Rt/L} = 1-\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n,$$

and

$$I(t) = \frac{V}{R}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n = \frac{Vt}{L} + \frac{V}{R}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n.$$

If t is small, then we can approximate $I(t)$ by the first (linear) term, and ignore terms with higher powers of t; then we find

$$V(t) \approx \frac{Vt}{L}.$$

70. Use the result of Exercise 69 and your knowledge of alternating series to show that

$$\frac{Vt}{L}\left(1 - \frac{R}{2L}t\right) \le I(t) \le \frac{Vt}{L} \qquad \text{(for all } t\text{)}$$

SOLUTION Since the series for $I(t)$ is an alternating series, we know that the true value lies between any two successive partial sums. Since the term for $n = 2$ is negative, we have

$$S_2 \le I(t) \le S_1 \qquad \text{for all } t$$

Clearly $S_1 = \frac{Vt}{L}$, and

$$S_2 = \frac{Vt}{L} + \frac{V}{R}\left(\frac{-1}{n!} \cdot \frac{R^2t^2}{L^2}\right) = \frac{Vt}{L} - \frac{VR^2t^2}{2RL^2} = \frac{Vt}{L}\left(1 - \frac{R}{2L}t\right)$$

71. Find the Maclaurin series for $f(x) = \cos(x^3)$ and use it to determine $f^{(6)}(0)$.

SOLUTION The Maclaurin series for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Substituting x^3 for x gives

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

Now, the coefficient of x^6 in this series is

$$-\frac{1}{2!} = -\frac{1}{2} = \frac{f^{(6)}(0)}{6!}$$

so

$$f^{(6)}(0) = -\frac{6!}{2} = -360$$

72. Find $f^{(7)}(0)$ and $f^{(8)}(0)$ for $f(x) = \tan^{-1} x$ using the Maclaurin series.

SOLUTION The Maclaurin series for $f(x) = \tan^{-1} x$ is:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

The coefficient of x^7 in this series is

$$\frac{(-1)^3}{7} = -\frac{1}{7} = \frac{f^{(7)}(0)}{7!},$$

so

$$f^{(7)}(0) = -\frac{7!}{7} = -6! = -720.$$

The coefficient of x^8 is 0, so $f^{(8)}(0) = 0$.

73. Use substitution to find the first three terms of the Maclaurin series for $f(x) = e^{x^{20}}$. How does the result show that $f^{(k)}(0) = 0$ for $1 \le k \le 19$?

SOLUTION Substituting x^{20} for x in the Maclaurin series for e^x yields

$$e^{x^{20}} = \sum_{n=0}^{\infty} \frac{(x^{20})^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{20n}}{n!};$$

the first three terms in the series are then

$$1 + x^{20} + \frac{1}{2}x^{40}.$$

Recall that the coefficient of x^k in the Maclaurin series for f is $\frac{f^{(k)}(0)}{k!}$. For $1 \le k \le 19$, the coefficient of x^k in the Maclaurin series for $f(x) = e^{x^{20}}$ is zero; it therefore follows that

$$\frac{f^{(k)}(0)}{k!} = 0 \quad \text{or} \quad f^{(k)}(0) = 0$$

for $1 \le k \le 19$.

74. Use the binomial series to find $f^{(8)}(0)$ for $f(x) = \sqrt{1 - x^2}$.

SOLUTION We obtain the Maclaurin series for $f(x) = \sqrt{1 - x^2}$ by substituting $-x^2$ for x in the binomial series with $a = \frac{1}{2}$. This gives

$$\sqrt{1 - x^2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} x^{2n}.$$

The coefficient of x^8 is

$$(-1)^4 \binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 3\right)}{4!} = -\frac{15}{16 \cdot 4!} = \frac{f^{(8)}(0)}{8!},$$

so

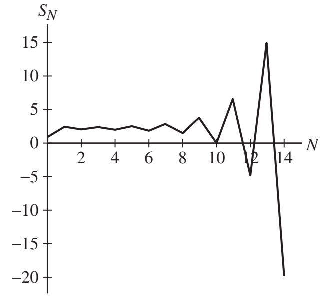
$$f^{(8)}(0) = \frac{-15 \cdot 8!}{16 \cdot 4!} = -1575.$$

75. Does the Maclaurin series for $f(x) = (1 + x)^{3/4}$ converge to $f(x)$ at $x = 2$? Give numerical evidence to support your answer.

SOLUTION The Taylor series for $f(x) = (1 + x)^{3/4}$ converges to $f(x)$ for $|x| < 1$; because $x = 2$ is not contained on this interval, the series does not converge to $f(x)$ at $x = 2$. The graph below displays

$$S_N = \sum_{n=0}^{N} \binom{\frac{3}{4}}{n} 2^n$$

for $0 \le N \le 14$. The divergent nature of the sequence of partial sums is clear.



76. Explain the steps required to verify that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$ for all x.

SOLUTION To show that the Maclaurin series for e^x converges to e^x for all x, we show that for any real number c, the Maclaurin series converges to e^x on an interval containing c. To do this, it suffices to show that for any interval $I = (-R, R)$, the Maclaurin series for e^x converges to e^x on I, since each real number is contained in some such interval. By Theorem 2, it suffices to show that there is a number K that bounds all derivatives of e^x for all numbers in the interval $(-R, R)$. But each derivative of e^x is also e^x, so it suffices to show that there is a number K that bounds e^x for all $x \in (-R, R)$. But e^x is an increasing function, so that $e^x < e^R$ for all $x \in (-R, R)$. Thus $K = e^R$ is the bound we want. Theorem 2 then assures us that the Maclaurin series for e^x converges to e^x on I.

77. GU Let $f(x) = \sqrt{1 + x}$.

(a) Use a graphing calculator to compare the graph of f with the graphs of the first five Taylor polynomials for f. What do they suggest about the interval of convergence of the Taylor series?

(b) Investigate numerically whether or not the Taylor expansion for f is valid for $x = 1$ and $x = -1$.

SOLUTION

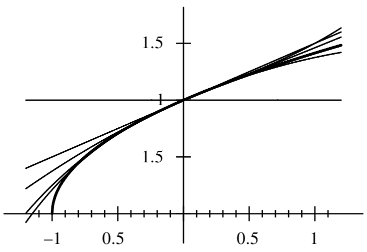
(a) The five first terms of the Binomial series with $a = \frac{1}{2}$ are

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}x^4 + \cdots$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{9}{4}x^3 - \frac{45}{2}x^4 + \cdots$$

Therefore, the first five Taylor polynomials are

$$T_0(x) = 1;$$
$$T_1(x) = 1 + \frac{1}{2}x;$$
$$T_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2;$$
$$T_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3;$$
$$T_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3 - \frac{5}{128}x^4.$$

The figure displays the graphs of these Taylor polynomials, along with the graph of the function $f(x) = \sqrt{1+x}$, which is shown in red.



The graphs suggest that the interval of convergence for the Taylor series is $-1 < x < 1$.

(b) Using a computer algebra system to calculate $S_N = \sum_{n=0}^{N} \binom{\frac{1}{2}}{n} x^n$ for $x = 1$ we find

$$S_{10} = 1.409931183, \quad S_{100} = 1.414073048, \quad S_{1000} = 1.414209104,$$

which appears to be converging to $\sqrt{2}$ as expected. At $x = -1$ we calculate $S_N = \sum_{n=0}^{N} \binom{\frac{1}{2}}{n} \cdot (-1)^n$, and find

$$S_{10} = 0.176197052, \quad S_{100} = 0.056348479, \quad S_{1000} = 0.017839011,$$

which appears to be converging to zero, though slowly.

78. Use the first five terms of the Maclaurin series for the elliptic function $E(k)$ to estimate the period T of a 1-meter pendulum released at an angle $\theta = \frac{\pi}{4}$ (see Example 11).

SOLUTION The period T of a pendulum of length L released from an angle θ is

$$T = 4\sqrt{\frac{L}{g}}E(k),$$

where $g \approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity, $E(k)$ is the elliptic function of the first kind and $k = \sin\frac{\theta}{2}$. From Example 11, we know that

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2 k^{2n}.$$

With $\theta = \frac{\pi}{4}$,

$$k = \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2},$$

and using the first five terms of the series for $E(k)$, we find

$$E\left(\sin\frac{\pi}{8}\right) \approx \frac{\pi}{2}\left(1+\left(\frac{1}{2}\right)^2\sin^2\frac{\pi}{8}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\frac{\pi}{8}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\sin^6\frac{\pi}{8}+\left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\sin^8\frac{\pi}{8}\right)$$
$$= 1.633578996$$

Therefore,

$$T \approx 4\sqrt{\frac{1}{9.8}}\cdot 1.633578996 = 2.09 \text{ seconds.}$$

79. Use Example 11 and the approximation $\sin x \approx x$ to show that the period T of a pendulum released at an angle θ has the following second-order approximation:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{\theta^2}{16}\right)$$

SOLUTION The period T of a pendulum of length L released from an angle θ is

$$T = 4\sqrt{\frac{L}{g}}E(k),$$

where $g \approx 9.8$ m/s^2 is the acceleration due to gravity, $E(k)$ is the elliptic function of the first kind and $k = \sin\frac{\theta}{2}$. From Example 11, we know that

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2 k^{2n}.$$

Using the approximation $\sin x \approx x$, we have

$$k = \sin\frac{\theta}{2} \approx \frac{\theta}{2};$$

moreover, using the first two terms of the series for $E(k)$, we find

$$E(k) \approx \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{\theta}{2}\right)^2\right] = \frac{\pi}{2}\left(1+\frac{\theta^2}{16}\right).$$

Therefore,

$$T = 4\sqrt{\frac{L}{g}}E(k) \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{\theta^2}{16}\right).$$

In Exercises 80–83, find the Maclaurin series of the function and use it to calculate the limit.

80. $\displaystyle\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

SOLUTION Using the Maclaurin series for $\cos x$, we find

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

Thus,

$$\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{24} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

and

$$\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{24} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n)!}$$

Note that the radius of convergence for this series is infinite, and recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x \to 0$ it suffices to evaluate it at $x = 0$:

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \left(\frac{1}{24} + \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n-4}}{(2n)!} \right) = \frac{1}{24} + 0 = \frac{1}{24}.$$

81. $\lim_{x\to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$

SOLUTION Using the Maclaurin series for $\sin x$, we find

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Thus,

$$\sin x - x + \frac{x^3}{6} = \frac{x^5}{120} + \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and

$$\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120} + \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n-4}}{(2n+1)!}$$

Note that the radius of convergence for this series is infinite, and recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x \to 0$ it suffices to evaluate it at $x = 0$:

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \left(\frac{1}{120} + \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n-4}}{(2n+1)!} \right) = \frac{1}{120} + 0 = \frac{1}{120}$$

82. $\lim_{x\to 0} \dfrac{\tan^{-1} x - x\cos x - \frac{1}{6}x^3}{x^5}$

SOLUTION Start with the Maclaurin series for $\tan^{-1} x$ and $\cos x$:

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Then

$$x\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

so that

$$\begin{aligned} \tan^{-1} x - x\cos x &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1} - \frac{1}{(2n)!} \right) x^{2n+1} \\ &= \frac{1}{6}x^3 + \frac{19}{120}x^5 + \sum_{n=3}^{\infty} (-1)^n \left(\frac{1}{2n+1} - \frac{1}{(2n)!} \right) x^{2n+1} \end{aligned}$$

and

$$\frac{\tan^{-1} x - x\cos x - \frac{1}{6}x^3}{x^5} = \frac{19}{120} + \sum_{n=3}^{\infty} (-1)^n \left(\frac{1}{2n+1} - \frac{1}{(2n)!} \right) x^{2n-4}$$

Since the radius of convergence of the series for $\tan^{-1} x$ is 1 and that of $\cos x$ is infinite, the radius of convergence of this series is 1. Recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x \to 0$ it suffices to evaluate it at $x = 0$:

$$\lim_{x\to 0} \frac{\tan^{-1} x - x\cos x - \frac{1}{6}x^3}{x^5} = \lim_{x\to 0}\left(\frac{19}{120} + \sum_{n=3}^{\infty}(-1)^n\left(\frac{1}{2n+1} - \frac{1}{(2n)!}\right)x^{2n-4}\right) = \frac{19}{120} + 0 = \frac{19}{120}$$

83. $\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^4} - \frac{\cos x}{x^2}\right)$

SOLUTION We start with

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

so that

$$\frac{\sin(x^2)}{x^4} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!x^4} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n-2}}{(2n+1)!}$$

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n-2}}{(2n)!}$$

Expanding the first few terms gives

$$\frac{\sin(x^2)}{x^4} = \frac{1}{x^2} - \sum_{n=1}^{\infty}(-1)^n \frac{x^{4n-2}}{(2n+1)!}$$

$$\frac{\cos x}{x^2} = \frac{1}{x^2} - \frac{1}{2} + \sum_{n=2}^{\infty}(-1)^n \frac{x^{2n-2}}{(2n)!}$$

so that

$$\frac{\sin(x^2)}{x^4} - \frac{\cos x}{x^2} = \frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n \frac{x^{4n-2}}{(2n+1)!} - \sum_{n=2}^{\infty}(-1)^n \frac{x^{2n-2}}{(2n)!}$$

Note that all terms under the summation signs have positive powers of x. Now, the radius of convergence of the series for both sin and cos is infinite, so the radius of convergence of this series is infinite. Recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x \to 0$ it suffices to evaluate it at $x = 0$:

$$\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^4} - \frac{\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n \frac{x^{4n-2}}{(2n+1)!} - \sum_{n=2}^{\infty}(-1)^n \frac{x^{2n-2}}{(2n)!}\right) = \frac{1}{2} + 0 = \frac{1}{2}$$

Further Insights and Challenges

84. In this exercise we show that the Maclaurin expansion of $f(x) = \ln(1+x)$ is valid for $x = 1$.

(a) Show that for all $x \neq -1$,

$$\frac{1}{1+x} = \sum_{n=0}^{N}(-1)^n x^n + \frac{(-1)^{N+1}x^{N+1}}{1+x}$$

(b) Integrate from 0 to 1 to obtain

$$\ln 2 = \sum_{n=1}^{N}\frac{(-1)^{n-1}}{n} + (-1)^{N+1}\int_0^1 \frac{x^{N+1}\,dx}{1+x}$$

(c) Verify that the integral on the right tends to zero as $N \to \infty$ by showing that it is smaller than $\int_0^1 x^{N+1}dx$.

(d) Prove the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

SOLUTION

(a) Substituting $-x$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Now, rewrite the series as

$$\sum_{n=0}^{N} (-1)^n x^n + \sum_{n=N+1}^{\infty} (-1)^n x^n,$$

and use the formula for the sum of a geometric series on the second term to obtain

$$\frac{1}{1+x} = \sum_{n=0}^{N} (-1)^n x^n + \frac{(-1)^{N+1} x^{N+1}}{1+x}.$$

(b) Integrate the equation derived in part (a) from 0 to 1 to obtain

$$\ln(1+x)\Big|_0^1 = \sum_{n=0}^{N} (-1)^n \frac{x^{n+1}}{n+1}\Big|_0^1 + (-1)^{N+1} \int_0^1 \frac{x^{N+1}}{1+x}\,dx,$$

or

$$\ln 2 = \sum_{n=0}^{N} \frac{(-1)^n}{n+1} + (-1)^{N+1} \int_0^1 \frac{x^{N+1}}{1+x}\,dx = \sum_{n=1}^{N+1} \frac{(-1)^{n-1}}{n} + (-1)^{N+1} \int_0^1 \frac{x^{N+1}}{1+x}\,dx.$$

(c) For $0 < x < 1$,

$$0 \le \frac{x^{N+1}}{1+x} \le x^{N+1} \quad \text{so} \quad 0 \le \int_0^1 \frac{x^{N+1}}{1+x}\,dx \le \int_0^1 x^{N+1}\,dx.$$

Now,

$$\int_0^1 x^{N+1}\,dx = \frac{x^{N+2}}{N+2}\Big|_0^1 = \frac{1}{N+2} \to 0 \text{ as } N \to \infty.$$

Thus, by the Squeeze Theorem,

$$\lim_{N\to\infty} \int_0^1 \frac{x^{N+1}}{1+x}\,dx = 0.$$

(d) Taking the limit as $N \to \infty$ of the equation derived in part (b) and using the result from part (c), we find

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

85. Let $g(t) = \dfrac{1}{1+t^2} - \dfrac{t}{1+t^2}$.

(a) Show that $\displaystyle\int_0^1 g(t)\,dt = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

(b) Show that $g(t) = 1 - t - t^2 + t^3 - t^4 - t^5 + \cdots$.

(c) Evaluate $S = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \cdots$.

SOLUTION

(a)

$$\int_0^1 g(t)\,dt = \left(\tan^{-1} t - \frac{1}{2}\ln(t^2+1)\right)\Big|_0^1 = \tan^{-1} 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(b) Start with the Taylor series for $\frac{1}{1+t}$:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

and substitute t^2 for t to get

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n} = 1 - t^2 + t^4 - t^6 + \dots$$

so that

$$\frac{t}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n+1} = t - t^3 + t^5 - t^7 + \dots$$

Finally,

$$g(t) = \frac{1}{1+t^2} - \frac{t}{1+t^2} = 1 - t - t^2 + t^3 + t^4 - t^5 - t^6 + t^7 + \dots$$

(c) We have

$$\int g(t)\,dt = \int (1 - t - t^2 + t^3 + t^4 - t^5 - \dots)\,dt = t - \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{1}{6}t^6 - \dots + C$$

The radius of convergence of the series for $g(t)$ is 1, so the radius of convergence of this series is also 1. However, this series converges at the right endpoint, $t = 1$, since

$$\left(1 - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) - \dots$$

is an alternating series with general term decreasing to zero. Thus by part (a),

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \dots = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

In Exercises 86 and 87, we investigate the convergence of the binomial series

$$T_a(x) = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

86. Prove that $T_a(x)$ has radius of convergence $R = 1$ if a is not a whole number. What is the radius of convergence if a is a whole number?

SOLUTION Suppose that a is not a whole number. Then

$$\binom{a}{n} = \frac{a\,(a-1)\cdots(a-n+1)}{n!}$$

is never zero. Moreover,

$$\left|\frac{\binom{a}{n+1}}{\binom{a}{n}}\right| = \left|\frac{a(a-1)\cdots(a-n+1)(a-n)}{(n+1)!} \cdot \frac{n!}{a(a-1)\cdots(a-n+1)}\right| = \left|\frac{a-n}{n+1}\right|,$$

so, by the formula for the radius of convergence

$$r = \lim_{n\to\infty}\left|\frac{a-n}{n+1}\right| = 1.$$

The radius of convergence of $T_a(x)$ is therefore $R = r^{-1} = 1$.

If a is a whole number, then $\binom{a}{n} = 0$ for all $n > a$. The infinite series then reduces to a polynomial of degree a, so it converges for all x (i.e. $R = \infty$).

87. By Exercise 86, $T_a(x)$ converges for $|x| < 1$, but we do not yet know whether $T_a(x) = (1+x)^a$.

(a) Verify the identity

$$a\binom{a}{n} = n\binom{a}{n} + (n+1)\binom{a}{n+1}$$

(b) Use (a) to show that $y = T_a(x)$ satisfies the differential equation $(1+x)y' = ay$ with initial condition $y(0) = 1$.

(c) Prove that $T_a(x) = (1+x)^a$ for $|x| < 1$ by showing that the derivative of the ratio $\dfrac{T_a(x)}{(1+x)^a}$ is zero.

SOLUTION

(a)

$$n\binom{a}{n}+(n+1)\binom{a}{n+1}=n\cdot\frac{a\,(a-1)\cdots(a-n+1)}{n!}+(n+1)\cdot\frac{a\,(a-1)\cdots(a-n+1)\,(a-n)}{(n+1)!}$$
$$=\frac{a\,(a-1)\cdots(a-n+1)}{(n-1)!}+\frac{a\,(a-1)\cdots(a-n+1)\,(a-n)}{n!}$$
$$=\frac{a\,(a-1)\cdots(a-n+1)\,(n+(a-n))}{n!}=a\cdot\binom{a}{n}$$

(b) Differentiating $T_a(x)$ term-by-term yields

$$T_a'(x)=\sum_{n=1}^{\infty}n\binom{a}{n}x^{n-1}.$$

Thus,

$$(1+x)T_a'(x)=\sum_{n=1}^{\infty}n\binom{a}{n}x^{n-1}+\sum_{n=1}^{\infty}n\binom{a}{n}x^{n}=\sum_{n=0}^{\infty}(n+1)\binom{a}{n+1}x^{n}+\sum_{n=0}^{\infty}n\binom{a}{n}x^{n}$$
$$=\sum_{n=0}^{\infty}\left[(n+1)\binom{a}{n+1}+n\binom{a}{n}\right]x^n=a\sum_{n=0}^{\infty}\binom{a}{n}x^n=aT_a(x).$$

Moreover,

$$T_a(0)=\binom{a}{0}=1.$$

(c)

$$\frac{d}{dx}\left(\frac{T_a(x)}{(1+x)^a}\right)=\frac{(1+x)^aT_a'(x)-a(1+x)^{a-1}T_a(x)}{(1+x)^{2a}}=\frac{(1+x)T_a'(x)-aT_a(x)}{(1+x)^{a+1}}=0.$$

Thus,

$$\frac{T_a(x)}{(1+x)^a}=C,$$

for some constant C. For $x=0$,

$$\frac{T_a(0)}{(1+0)^a}=\frac{1}{1}=1,\ \text{so } C=1.$$

Finally, $T_a(x)=(1+x)^a$.

88. The function $G(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt$ is called an **elliptic function of the second kind**. Prove that for $|k|<1$,

$$G(k)=\frac{\pi}{2}-\frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdots 4\cdot(2n)}\right)^2\frac{k^{2n}}{2n-1}$$

SOLUTION For $|k|<1$, $|k^2\sin^2 t|<1$ for all t. Substituting $-k^2\sin^2 t$ for t in the binomial series for $a=\frac{1}{2}$, we find

$$\sqrt{1-k^2\sin^2 t}=1+\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\left(-k^2\sin^2 t\right)^n$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}k^{2n}\sin^{2n}t$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\frac{1(1-2)(1-4)\cdots(1-2(n-1))}{2^n n!}k^{2n}\sin^{2n}t$$
$$=1+\sum_{n=1}^{\infty}(-1)^n(-1)^{n-1}\frac{(2-1)(4-1)\cdots(2n-3)}{2^n n!}k^{2n}\sin^{2n}t$$
$$=1-\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n)}k^{2n}\sin^{2n}t.$$

Integrating from 0 to $\frac{\pi}{2}$ term-by-term, we obtain

$$G(k) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n)} k^{2n} \int_0^{\pi/2} \sin^{2n} t\, dt.$$

Finally, using the formula

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2},$$

we arrive at

$$G(k) = \frac{\pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^2 (2n-1) k^{2n} = \frac{\pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^2 \frac{k^{2n}}{2n-1}.$$

89. Assume that $a < b$ and let L be the arc length (circumference) of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ shown in Figure 5. There is no explicit formula for L, but it is known that $L = 4bG(k)$, with $G(k)$ as in Exercise 88 and $k = \sqrt{1 - a^2/b^2}$. Use the first three terms of the expansion of Exercise 88 to estimate L when $a = 4$ and $b = 5$.

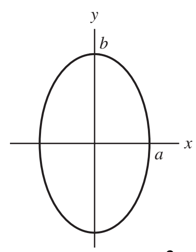


FIGURE 5 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION With $a = 4$ and $b = 5$,

$$k = \sqrt{1 - \frac{4^2}{5^2}} = \frac{3}{5},$$

and the arc length of the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ is

$$L = 20G\left(\frac{3}{5}\right) = 20\left(\frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 \frac{\left(\frac{3}{5}\right)^{2n}}{2n-1}\right).$$

Using the first three terms in the series for $G(k)$ gives

$$L \approx 10\pi - 10\pi\left(\left(\frac{1}{2}\right)^2 \cdot \frac{(3/5)^2}{1} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot \frac{(3/5)^4}{3}\right) = 10\pi\left(1 - \frac{9}{100} - \frac{243}{40{,}000}\right) = \frac{36{,}157\pi}{4000} \approx 28.398.$$

90. Use Exercise 88 to prove that if $a < b$ and a/b is near 1 (a nearly circular ellipse), then

$$L \approx \frac{\pi}{2}\left(3b + \frac{a^2}{b}\right)$$

Hint: Use the first two terms of the series for $G(k)$.

SOLUTION From the previous exercise, we know that

$$L = 4bG(k), \quad \text{where} \quad k = \sqrt{1 - \frac{a^2}{b^2}}.$$

Following the hint and using only the first two terms of the series expansion for $G(k)$ from Exercise 88, we find

$$L \approx 4b\left(\frac{\pi}{2} - \frac{\pi}{2}\left(\frac{1}{2}\right)^2 k^2\right) = \frac{\pi}{2}\left(4b - b\left(1 - \frac{a^2}{b^2}\right)\right) = \frac{\pi}{2}\left(3b + \frac{a^2}{b}\right).$$

91. Irrationality of e Prove that e is an irrational number using the following argument by contradiction. Suppose that $e = M/N$, where M, N are nonzero integers.

(a) Show that $M!\,e^{-1}$ is a whole number.

(b) Use the power series for e^x at $x = -1$ to show that there is an integer B such that $M!\,e^{-1}$ equals

$$B + (-1)^{M+1}\left(\frac{1}{M+1} - \frac{1}{(M+1)(M+2)} + \cdots\right)$$

(c) Use your knowledge of alternating series with decreasing terms to conclude that $0 < |M!\,e^{-1} - B| < 1$ and observe that this contradicts (a). Hence, e is not equal to M/N.

SOLUTION Suppose that $e = M/N$, where M, N are nonzero integers.

(a) With $e = M/N$,

$$M!e^{-1} = M!\frac{N}{M} = (M-1)!N,$$

which is a whole number.

(b) Substituting $x = -1$ into the Maclaurin series for e^x and multiplying the resulting series by $M!$ yields

$$M!e^{-1} = M!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots\right).$$

For all $k \le M$, $\dfrac{M!}{k!}$ is a whole number, so

$$M!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{M!}\right)$$

is an integer. Denote this integer by B. Thus,

$$M!\,e^{-1} = B + M!\left(\frac{(-1)^{M+1}}{(M+1)!} + \frac{(-1)^{M+2}}{(M+2)!} + \cdots\right) = B + (-1)^{M+1}\left(\frac{1}{M+1} - \frac{1}{(M+1)(M+2)} + \cdots\right).$$

(c) The series for $M!\,e^{-1}$ obtained in part (b) is an alternating series with $a_n = \frac{M!}{n!}$. Using the error bound for an alternating series and noting that $B = S_M$, we have

$$\left|M!\,e^{-1} - B\right| \le a_{M+1} = \frac{1}{M+1} < 1.$$

This inequality implies that $M!\,e^{-1} - B$ is not a whole number; however, B is a whole number so $M!\,e^{-1}$ cannot be a whole number. We get a contradiction to the result in part (a), which proves that the original assumption that e is a rational number is false.

92. Use the result of Exercise 73 in Section 4.5 to show that the Maclaurin series of the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is $T(x) = 0$. This provides an example of a function $f(x)$ whose Maclaurin series converges but does not converge to $f(x)$ (except at $x = 0$).

SOLUTION By the referenced exercise, $f(x)$ has continuous derivatives of all orders at 0, and $f^{(n)}(0) = 0$ for all $n > 0$. But then the Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0$$

CHAPTER REVIEW EXERCISES

1. Let $a_n = \dfrac{n-3}{n!}$ and $b_n = a_{n+3}$. Calculate the first three terms in each sequence.

(a) a_n^2

(b) b_n

(c) $a_n b_n$

(d) $2a_{n+1} - 3a_n$

SOLUTION

(a)

$$a_1^2 = \left(\frac{1-3}{1!}\right)^2 = (-2)^2 = 4;$$
$$a_2^2 = \left(\frac{2-3}{2!}\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4};$$
$$a_3^2 = \left(\frac{3-3}{3!}\right)^2 = 0.$$

(b)

$$b_1 = a_4 = \frac{4-3}{4!} = \frac{1}{24};$$
$$b_2 = a_5 = \frac{5-3}{5!} = \frac{1}{60};$$
$$b_3 = a_6 = \frac{6-3}{6!} = \frac{1}{240}.$$

(c) Using the formula for a_n and the values in (b) we obtain:

$$a_1b_1 = \frac{1-3}{1!}\cdot\frac{1}{24} = -\frac{1}{12};$$
$$a_2b_2 = \frac{2-3}{2!}\cdot\frac{1}{60} = -\frac{1}{120};$$
$$a_3b_3 = \frac{3-3}{3!}\cdot\frac{1}{240} = 0.$$

(d)

$$2a_2 - 3a_1 = 2\left(-\frac{1}{2}\right) - 3(-2) = 5;$$
$$2a_3 - 3a_2 = 2\cdot 0 - 3\left(-\frac{1}{2}\right) = \frac{3}{2};$$
$$2a_4 - 3a_3 = 2\cdot\frac{1}{24} - 3\cdot 0 = \frac{1}{12}.$$

2. Prove that $\lim_{n\to\infty}\frac{2n-1}{3n+2} = \frac{2}{3}$ using the limit definition.

SOLUTION Note

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \left|\frac{6n-3-2(3n+2)}{3(3n+2)}\right| = \left|-\frac{7}{3(3n+2)}\right| = \frac{7}{3(3n+2)} < \frac{7}{9n}.$$

Therefore, to have $\left|a_n - \frac{2}{3}\right| < \epsilon$, we need

$$\frac{7}{9n} < \epsilon \quad \text{or} \quad n > \frac{7}{9\epsilon}.$$

Thus, let $\epsilon > 0$ and take $M = \frac{7}{9\epsilon}$. Then, whenever $n > M$,

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \frac{7}{3(3n+2)} < \frac{7}{9n} < \frac{7}{9}\cdot\frac{9\epsilon}{7} = \epsilon.$$

In Exercises 3–8, compute the limit (or state that it does not exist) assuming that $\lim_{n\to\infty} a_n = 2$.

3. $\lim_{n\to\infty}(5a_n - 2a_n^2)$

SOLUTION

$$\lim_{n\to\infty}\left(5a_n - 2a_n^2\right) = 5\lim_{n\to\infty} a_n - 2\lim_{n\to\infty} a_n^2 = 5\lim_{n\to\infty} a_n - 2\left(\lim_{n\to\infty} a_n\right)^2 = 5\cdot 2 - 2\cdot 2^2 = 2.$$

4. $\displaystyle\lim_{n\to\infty} \frac{1}{a_n}$

SOLUTION $\displaystyle\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n} = \frac{1}{2}.$

5. $\displaystyle\lim_{n\to\infty} e^{a_n}$

SOLUTION The function $f(x) = e^x$ is continuous, hence:

$$\lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^2.$$

6. $\displaystyle\lim_{n\to\infty} \cos(\pi a_n)$

SOLUTION The function $f(x) = \cos(\pi x)$ is continuous, hence:

$$\lim_{n\to\infty} \cos(\pi a_n) = \cos\left(\pi \lim_{n\to\infty} a_n\right) = \cos(2\pi) = 1.$$

7. $\displaystyle\lim_{n\to\infty} (-1)^n a_n$

SOLUTION Because $\displaystyle\lim_{n\to\infty} a_n \neq 0$, it follows that $\displaystyle\lim_{n\to\infty} (-1)^n a_n$ does not exist.

8. $\displaystyle\lim_{n\to\infty} \frac{a_n + n}{a_n + n^2}$

SOLUTION Because the sequence $\{a_n\}$ converges, $\{a_n\}$ is bounded and

$$\lim_{n\to\infty} \frac{a_n}{n^2} = 0.$$

Thus,

$$\lim_{n\to\infty} \frac{a_n + n}{a_n + n^2} = \lim_{n\to\infty} \frac{\frac{a_n}{n^2} + \frac{1}{n}}{\frac{a_n}{n^2} + 1} = \frac{0+0}{0+1} = 0.$$

In Exercises 9–22, determine the limit of the sequence or show that the sequence diverges.

9. $a_n = \sqrt{n+5} - \sqrt{n+2}$

SOLUTION First rewrite a_n as follows:

$$a_n = \frac{\left(\sqrt{n+5} - \sqrt{n+2}\right)\left(\sqrt{n+5} + \sqrt{n+2}\right)}{\sqrt{n+5} + \sqrt{n+2}} = \frac{(n+5) - (n+2)}{\sqrt{n+5} + \sqrt{n+2}} = \frac{3}{\sqrt{n+5} + \sqrt{n+2}}.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{\sqrt{n+5} + \sqrt{n+2}} = 0.$$

10. $a_n = \dfrac{3n^3 - n}{1 - 2n^3}$

SOLUTION $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^3 - n}{1 - 2n^3} = -\frac{3}{2}.$

11. $a_n = 2^{1/n^2}$

SOLUTION The function $f(x) = 2^x$ is continuous, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1/n^2} = 2^{\lim_{n\to\infty}(1/n^2)} = 2^0 = 1.$$

12. $a_n = \dfrac{10^n}{n!}$

SOLUTION For $n > 10$, write a_n as

$$0 \le a_n = \underbrace{\left(\frac{10}{1} \cdot \frac{10}{2} \cdot \cdots \cdot \frac{10}{10}\right)}_{\text{equals } \frac{10^{10}}{10!}} \underbrace{\left(\frac{10}{11}\right) \cdot \left(\frac{10}{12}\right) \cdot \cdots \cdot \left(\frac{10}{n}\right)}_{\text{each factor is less than 1}} < \frac{10^{10}}{10!} \cdot \frac{10}{n} = \frac{10^{10}}{9!n};$$

Thus, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} a_n = 0.$

13. $b_m = 1 + (-1)^m$

SOLUTION Because $1 + (-1)^m$ is equal to 0 for m odd and is equal to 2 for m even, the sequence $\{b_m\}$ does not approach one limit; hence this sequence diverges.

14. $b_m = \dfrac{1 + (-1)^m}{m}$

SOLUTION The numerator is equal to zero for m odd and is equal to 2 for m even. Therefore,

$$0 \le \frac{1 + (-1)^m}{m} \le \frac{2}{m},$$

and by the Squeeze Theorem, $\lim_{m\to\infty} b_m = 0$.

15. $b_n = \tan^{-1}\left(\dfrac{n+2}{n+5}\right)$

SOLUTION The function $\tan^{-1}x$ is continuous, so

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \tan^{-1}\left(\frac{n+2}{n+5}\right) = \tan^{-1}\left(\lim_{n\to\infty}\frac{n+2}{n+5}\right) = \tan^{-1} 1 = \frac{\pi}{4}.$$

16. $a_n = \dfrac{100^n}{n!} - \dfrac{3+\pi^n}{5^n}$

SOLUTION For $n > 100$,

$$0 \le \frac{100^n}{n!} = \left(\frac{100}{1}\cdot\frac{100}{2}\cdots\frac{100}{100}\right)\frac{100}{101}\cdot\frac{100}{102}\cdot\frac{100}{n} < \frac{100^{100}}{99!n};$$

therefore,

$$\lim_{n\to\infty}\frac{100^n}{n!} = 0$$

by the Squeeze Theorem. Moreover,

$$\lim_{n\to\infty}\left(\frac{3+\pi^n}{5^n}\right) = \lim_{n\to\infty}\frac{3}{5^n} + \lim_{n\to\infty}\left(\frac{\pi}{5}\right)^n = 0 + 0 = 0.$$

Thus,

$$\lim_{n\to\infty} a_n = 0 + 0 = 0.$$

17. $b_n = \sqrt{n^2+n} - \sqrt{n^2+1}$

SOLUTION Rewrite b_n as

$$b_n = \frac{\left(\sqrt{n^2+n} - \sqrt{n^2+1}\right)\left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}{\sqrt{n^2+n} + \sqrt{n^2+1}} = \frac{\left(n^2+n\right) - \left(n^2+1\right)}{\sqrt{n^2+n} + \sqrt{n^2+1}} = \frac{n-1}{\sqrt{n^2+n} + \sqrt{n^2+1}}.$$

Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\frac{n}{n} - \frac{1}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{n}{n^2}} + \sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}} = \lim_{n\to\infty}\frac{1 - \frac{1}{n}}{\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{1}{n^2}}} = \frac{1-0}{\sqrt{1+0} + \sqrt{1+0}} = \frac{1}{2}.$$

18. $c_n = \sqrt{n^2+n} - \sqrt{n^2-n}$

SOLUTION Rewrite c_n as

$$c_n = \frac{\left(\sqrt{n^2+n} - \sqrt{n^2-n}\right)\left(\sqrt{n^2+n} + \sqrt{n^2-n}\right)}{\sqrt{n^2+n} + \sqrt{n^2-n}} = \frac{\left(n^2+n\right) - \left(n^2-n\right)}{\sqrt{n^2+n} + \sqrt{n^2-n}} = \frac{2n}{\sqrt{n^2+n} + \sqrt{n^2-n}}.$$

Then

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\frac{\frac{2n}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{n}{n^2}} + \sqrt{\frac{n^2}{n^2} - \frac{n}{n^2}}} = \lim_{n\to\infty}\frac{2}{\sqrt{1 + \frac{1}{n}} + \sqrt{1 - \frac{1}{n}}} = \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = 1.$$

19. $b_m = \left(1 + \frac{1}{m}\right)^{3m}$

SOLUTION $\lim_{m\to\infty} b_m = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e.$

20. $c_n = \left(1 + \frac{3}{n}\right)^n$

SOLUTION Write

$$c_n = \left(1 + \frac{1}{n/3}\right)^n = \left[\left(1 + \frac{1}{n/3}\right)^{n/3}\right]^3.$$

Then, because x^3 is a continuous function,

$$\lim_{n\to\infty} c_n = \left[\lim_{n\to\infty}\left(1 + \frac{1}{n/3}\right)^{n/3}\right]^3 = e^3.$$

21. $b_n = n\big(\ln(n+1) - \ln n\big)$

SOLUTION Write

$$b_n = n\ln\left(\frac{n+1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}.$$

Using L'Hôpital's Rule, we find

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{-1}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-1} = 1.$$

22. $c_n = \frac{\ln(n^2+1)}{\ln(n^3+1)}$

SOLUTION Using L'Hôpital's Rule, we find

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\frac{\ln(n^2+1)}{\ln(n^3+1)} = \lim_{n\to\infty}\frac{2n/(n^2+1)}{3n^2/(n^3+1)} = \lim_{n\to\infty}\frac{2n^4+2n}{3n^4+3n^2} = \lim_{n\to\infty}\frac{2+2n^{-3}}{3+3n^{-2}} = \frac{2}{3}$$

23. Use the Squeeze Theorem to show that $\lim_{n\to\infty}\frac{\arctan(n^2)}{\sqrt{n}} = 0.$

SOLUTION For all x,

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2},$$

so

$$-\frac{\pi/2}{\sqrt{n}} < \frac{\arctan(n^2)}{\sqrt{n}} < \frac{\pi/2}{\sqrt{n}},$$

for all n. Because

$$\lim_{n\to\infty}\left(-\frac{\pi/2}{\sqrt{n}}\right) = \lim_{n\to\infty}\frac{\pi/2}{\sqrt{n}} = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{n\to\infty}\frac{\arctan(n^2)}{\sqrt{n}} = 0.$$

24. Give an example of a divergent sequence $\{a_n\}$ such that $\{\sin a_n\}$ is convergent.

SOLUTION Let $a_n = (-1)^n\pi$. This is an alternating series, which does not approach 0, hence it diverges. However, a_n is a multiple of π for every n, and thus, $\sin a_n = 0$. Since $\{\sin a_n\}$ is a constant sequence, it converges.

25. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, where $a_n = \frac{1}{2}3^n - \frac{1}{3}2^n$.

SOLUTION Because

$$\frac{1}{2}3^n - \frac{1}{3}2^n \geq \frac{1}{2}3^n - \frac{1}{3}3^n = \frac{3^n}{6}$$

and

$$\lim_{n\to\infty} \frac{3^n}{6} = \infty,$$

we conclude that $\lim_{n\to\infty} a_n = \infty$, so L'Hôpital's rule may be used:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{2}3^{n+1} - \frac{1}{3}2^{n+1}}{\frac{1}{2}3^n - \frac{1}{3}2^n} = \lim_{n\to\infty} \frac{3^{n+2} - 2^{n+2}}{3^{n+1} - 2^{n+1}} = \lim_{n\to\infty} \frac{3 - 2\left(\frac{2}{3}\right)^{n+1}}{1 - \left(\frac{2}{3}\right)^{n+1}} = \frac{3-0}{1-0} = 3.$$

26. Define $a_{n+1} = \sqrt{a_n + 6}$ with $a_1 = 2$.

(a) Compute a_n for $n = 2, 3, 4, 5$.

(b) Show that $\{a_n\}$ is increasing and is bounded by 3.

(c) Prove that $\lim_{n\to\infty} a_n$ exists and find its value.

SOLUTION

(a) We compute the first four values of a_n recursively:

$$a_2 = \sqrt{a_1 + 6} = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2} \approx 2.828427;$$

$$a_3 = \sqrt{a_2 + 6} = \sqrt{2\sqrt{2} + 6} \approx 2.971267;$$

$$a_4 = \sqrt{a_3 + 6} = \sqrt{\sqrt{2\sqrt{2} + 6} + 6} \approx 2.995207;$$

$$a_5 = \sqrt{a_4 + 6} = \sqrt{\sqrt{\sqrt{2\sqrt{2} + 6} + 6} + 6} \approx 2.999201.$$

(b) By part (a) and the given data, $a_2 \approx 2.8$ and $a_1 = 2$, so $a_2 > a_1$. Now, suppose that $a_k > a_{k-1}$; then

$$a_{k+1} = \sqrt{a_k + 6} > \sqrt{a_{k-1} + 6} = a_k.$$

Thus, by mathematical induction, $a_{n+1} > a_n$ for all n and $\{a_n\}$ is increasing.

Next, note that $a_1 = 2 < 3$. Suppose $a_k < 3$, then

$$a_{k+1} = \sqrt{a_k + 6} < \sqrt{3+6} = 3.$$

Thus, by mathematical induction, $a_n < 3$ for all n.

(c) Since $\{a_n\}$ is increasing and has an upper bound, $\{a_n\}$ converges. Let

$$L = \lim_{n\to\infty} a_n.$$

Then,

$$\begin{aligned} L &= \sqrt{L+6} \\ L^2 &= L + 6 \\ L^2 - L - 6 &= 0 \\ (L-3)(L+2) &= 0 \end{aligned}$$

so $L = 3$ or $L = -2$; however, the sequence is increasing and its first term is positive, so -2 cannot be the limit. Therefore,

$$\lim_{n\to\infty} a_n = 3.$$

27. Calculate the partial sums S_4 and S_7 of the series $\sum_{n=1}^{\infty} \frac{n-2}{n^2+2n}$.

SOLUTION

$$S_4 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} = -\frac{11}{60} = -0.183333;$$

$$S_7 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} + \frac{3}{35} + \frac{4}{48} + \frac{5}{63} = \frac{287}{4410} = 0.065079.$$

28. Find the sum $1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots$.

SOLUTION This is a geometric series with $r = -\frac{1}{4}$. Therefore,

$$1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots = \frac{1}{1-(-\frac{1}{4})} = \frac{4}{5}.$$

29. Find the sum $\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$.

SOLUTION This is a geometric series with common ratio $r = \frac{2}{3}$. Therefore,

$$\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots = \frac{\frac{4}{9}}{1-\frac{2}{3}} = \frac{4}{3}.$$

30. Find the sum $\sum_{n=2}^{\infty} \left(\frac{2}{e}\right)^n$.

SOLUTION This is a geometric series with common ratio $r = \frac{2}{e}$. Therefore,

$$\sum_{n=2}^{\infty} \left(\frac{2}{e}\right)^n = \frac{\left(\frac{2}{e}\right)^2}{1-\frac{2}{e}} = \frac{\frac{4}{e^2}}{1-\frac{2}{e}} = \frac{4}{e^2-2e}.$$

31. Find the sum $\sum_{n=-1}^{\infty} \frac{2^{n+3}}{3^n}$.

SOLUTION Note

$$\sum_{n=-1}^{\infty} \frac{2^{n+3}}{3^n} = 2^3 \sum_{n=-1}^{\infty} \frac{2^n}{3^n} = 8 \sum_{n=-1}^{\infty} \left(\frac{2}{3}\right)^n;$$

therefore,

$$\sum_{n=-1}^{\infty} \frac{2^{n+3}}{3^n} = 8 \cdot \frac{3}{2} \cdot \frac{1}{1-\frac{2}{3}} = 36.$$

32. Show that $\sum_{n=1}^{\infty} \left(b - \tan^{-1} n^2\right)$ diverges if $b \neq \frac{\pi}{2}$.

SOLUTION Note

$$\lim_{n\to\infty} \left(b - \tan^{-1} n^2\right) = b - \lim_{n\to\infty} \tan^{-1} n^2 = b - \frac{\pi}{2}.$$

If $b \neq \frac{\pi}{2}$, then the limit of the terms in the series is not 0; hence, the series diverges by the Divergence Test.

33. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} (a_n + b_n) = 1$.

SOLUTION Let $a_n = \left(\frac{1}{2}\right)^n + 1$, $b_n = -1$. The corresponding series diverge by the Divergence Test; however,

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

34. Let $S = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$. Compute S_N for $N = 1, 2, 3, 4$. Find S by showing that

$$S_N = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

SOLUTION

$$S_1 = 1 - \frac{1}{3} = \frac{2}{3};$$
$$S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2} - \frac{7}{12} = \frac{11}{12};$$
$$S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{3}{2} - \frac{9}{20} = \frac{21}{20};$$
$$S_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{3}{2} - \frac{11}{30} = \frac{17}{15}.$$

The general term in the sequence of partial sums is

$$\begin{aligned} S_N &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} = \frac{3}{2} - \left(\frac{1}{N+1} + \frac{1}{N+2}\right). \end{aligned}$$

Finally,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left[\frac{3}{2} - \left(\frac{1}{N+1} + \frac{1}{N+2}\right)\right] = \frac{3}{2}.$$

35. Evaluate $S = \sum_{n=3}^{\infty} \frac{1}{n(n+3)}$.

SOLUTION Note that

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

so that

$$\begin{aligned} \sum_{n=3}^{N} \frac{1}{n(n+3)} &= \frac{1}{3} \sum_{n=3}^{N} \left(\frac{1}{n} - \frac{1}{n+3}\right) \\ &= \frac{1}{3}\left(\left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right)\right. \\ &\qquad \left.\left(\frac{1}{6} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N+2}\right) + \left(\frac{1}{N} - \frac{1}{N+3}\right)\right) \\ &= \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3}\right) \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{n(n+3)} &= \frac{1}{3} \lim_{N\to\infty} \sum_{n=3}^{N} \left(\frac{1}{n} - \frac{1}{n+3}\right) \\ &= \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3}\right) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{47}{180} \end{aligned}$$

36. Find the total area of the infinitely many circles on the interval [0, 1] in Figure 1.

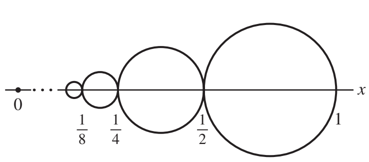


FIGURE 1

SOLUTION The diameter of the largest circle is $\frac{1}{2}$, and the diameter of each smaller circle is $\frac{1}{2}$ the diameter of the previous circle; thus, the diameter of the nth circle (for $n \geq 1$) is $\frac{1}{2^n}$ and the area is

$$\pi\left(\frac{1}{2^{n+1}}\right)^2 = \frac{\pi}{4^{n+1}}.$$

The total area of the circles is

$$\sum_{n=1}^{\infty} \frac{\pi}{4^{n+1}} = \frac{\pi}{4}\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{\pi}{4}\cdot\frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\pi}{12}.$$

In Exercises 37–40, use the Integral Test to determine whether the infinite series converges.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

SOLUTION Let $f(x) = \frac{x^2}{x^3+1}$. This function is continuous and positive for $x \geq 1$. Because

$$f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2},$$

we see that $f'(x) < 0$ and f is decreasing on the interval $x \geq 2$. Therefore, the Integral Test applies on the interval $x \geq 2$. Now,

$$\int_2^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{R\to\infty}\int_2^R \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\lim_{R\to\infty}\Big(\ln(R^3+1) - \ln 9\Big) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{n^3+1}$ diverges, as does the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$.

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n^3+1)^{1.01}}$

SOLUTION Let $f(x) = \frac{x^2}{(x^3+1)^{1.01}}$. This function is continuous and positive for $x \geq 1$. Because

$$f'(x) = \frac{(x^3+1)^{1.01}(2x) - x^2\cdot 1.01(x^3+1)^{0.01}(3x^2)}{(x^3+1)^{2.02}} = \frac{x(x^3+1)^{0.01}(2-1.03x^3)}{(x^3+1)^{2.02}},$$

we see that $f'(x) < 0$ and f is decreasing on the interval $x \geq 2$. Therefore, the Integral Test applies on the interval $x \geq 2$. Now,

$$\int_2^{\infty}\frac{x^2}{(x^3+1)^{1.01}}\,dx = \lim_{R\to\infty}\int_2^R \frac{x^2}{(x^3+1)^{1.01}}\,dx = -\frac{1}{0.03}\lim_{R\to\infty}\left(\frac{1}{(R^3+1)^{0.01}} - \frac{1}{9^{0.01}}\right) = \frac{1}{0.03\cdot 9^{0.01}}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{(n^3+1)^{1.01}}$ converges, as does the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{(n^3+1)^{1.01}}$.

39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(\ln(n+2))^3}$

SOLUTION Let $f(x) = \frac{1}{(x+2)\ln^3(x+2)}$. Using the substitution $u = \ln(x+2)$, so that $du = \frac{1}{x+2}\,dx$, we have

$$\int_0^{\infty} f(x)\,dx = \int_{\ln 2}^{\infty}\frac{1}{u^3}\,du = \lim_{R\to\infty}\int_{\ln 2}^{\infty}\frac{1}{u^3}\,du = \lim_{R\to\infty}\left(-\frac{1}{2u^2}\Big|_{\ln 2}^{R}\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln R)^2}\right) = \frac{1}{2(\ln 2)^2}$$

Since the integral of $f(x)$ converges, so does the series.

40. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^{n^4}}$

SOLUTION Let $f(x) = x^3 e^{-x^4}$. This function is continuous and positive for $x \ge 1$. Because

$$f'(x) = x^3 \left(-4x^3 e^{-x^4}\right) + 3x^2 e^{-x^4} = x^2 e^{-x^4}\left(3 - 4x^4\right),$$

we see that $f'(x) < 0$ and f is decreasing on the interval $x \ge 1$. Therefore, the Integral Test applies on the interval $x \ge 1$. Now,

$$\int_1^{\infty} x^3 e^{-x^4}\,dx = \lim_{R\to\infty} \int_1^R x^3 e^{-x^4}\,dx = -\frac{1}{4}\lim_{R\to\infty}\left(e^{-R^4} - e^{-1}\right) = \frac{1}{4e}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^{n^4}}$ also converges.

In Exercises 41–48, use the Comparison or Limit Comparison Test to determine whether the infinite series converges.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

SOLUTION For all $n \ge 1$,

$$0 < \frac{1}{n+1} < \frac{1}{n} \quad \text{so} \quad \frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by the Comparison Test.

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{1}{\sqrt{n}+n}$ and $b_n = \frac{1}{n}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}+n} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1.$$

Because $L > 0$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, we conclude by the Limit Comparison Test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$ also diverges.

43. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^{3.5}-2}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{n^2+1}{n^{3.5}-2}$ and $b_n = \frac{1}{n^{1.5}}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{n^2+1}{n^{3.5}-2}}{\frac{1}{n^{1.5}}} = \lim_{n\to\infty} \frac{n^{3.5}+n^{1.5}}{n^{3.5}-2} = 1.$$

Because L exists and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series, we conclude by the Limit Comparison Test that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^{3.5}-2}$ also converges.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$

SOLUTION Since $0 \le \ln n \le n$ for all $n \ge 1$, we have $0 \le n - \ln n \le n$ and

$$\frac{1}{n} \le \frac{1}{n-\ln n}$$

The harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so we conclude by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$ also diverges.

45. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5+5}}$

SOLUTION For all $n \geq 2$,

$$\frac{n}{\sqrt{n^5+5}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}.$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5+5}}$ converges by the Comparison Test.

46. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{1}{3^n-2^n}$ and $b_n = \frac{1}{3^n}$. Then,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n}{3^n - 2^n} = \lim_{n\to\infty} \frac{1}{1 - \left(\frac{2}{3}\right)^n} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; because L exists, we may therefore conclude by the Limit Comparison Test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$ also converges.

47. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10} + 10^n}{n^{11} + 11^n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{n^{10}+10^n}{n^{11}+11^n}$ and $b_n = \left(\frac{10}{11}\right)^n$. Then,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^{10}+10^n}{n^{11}+11^n}}{\left(\frac{10}{11}\right)^n} = \lim_{n\to\infty} \frac{\frac{n^{10}+10^n}{10^n}}{\frac{n^{11}+11^n}{11^n}} = \lim_{n\to\infty} \frac{\frac{n^{10}}{10^n}+1}{\frac{n^{11}}{11^n}+1} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{10}{11}\right)^n$ is a convergent geometric series; because L exists, we may therefore conclude by the Limit Comparison Test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10} + 10^n}{n^{11} + 11^n}$ also converges.

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{20} + 21^n}{n^{21} + 20^n}$

SOLUTION Apply the Limit Comparison Theorem with $a_n = \frac{n^{20}+21^n}{n^{21}+20^n}$ and $b_n = \left(\frac{21}{20}\right)^n$. Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^{20}+21^n}{n^{21}+20^n}}{\left(\frac{21}{20}\right)^n} = \lim_{n\to\infty} \frac{\frac{n^{20}+21^n}{21^n}}{\frac{n^{21}+20^n}{20^n}} = \lim_{n\to\infty} \frac{\frac{n^{20}}{21^n}+1}{\frac{n^{21}}{20^n}+1} = 1$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{21}{20}\right)^n$ is a divergent geometric series. Since $L = 1$, the two series either both converge or both diverge; thus, we may conclude from the Limit Comparison Test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{20} + 21^n}{n^{21} + 20^n}$ diverges.

49. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + n}{3^n - 2}$ using the Limit Comparison Test with $b_n = \left(\frac{2}{3}\right)^n$.

SOLUTION With $a_n = \frac{2^n+n}{3^n-2}$, we have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n + n}{3^n - 2} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{6^n + n3^n}{6^n - 2^{n+1}} = \lim_{n\to\infty} \frac{1 + n\left(\frac{1}{2}\right)^n}{1 - 2\left(\frac{1}{3}\right)^n} = 1$$

Since $L = 1$, the two series either both converge or both diverge. Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series, the Limit Comparison Test tells us that $\sum_{n=1}^{\infty} \frac{2^n + n}{3^n - 2}$ also converges.

50. Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln n}{1.5^n}$ using the Limit Comparison Test with $b_n = \frac{1}{1.4^n}$.

SOLUTION With $a_n = \frac{\ln n}{1.5^n}$, and using L'Hôpital's Rule,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{1.5^n}}{\frac{1}{1.4^n}} = \lim_{n\to\infty} \frac{\ln n}{\left(\frac{1.5}{1.4}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1/n}{\ln(1.5/1.4)\left(\frac{1.5}{1.4}\right)^n} = \frac{1}{\ln(1.5/1.4)} \lim_{n\to\infty} \frac{\left(\frac{1.4}{1.5}\right)^n}{n} = 0$$

Since $L < \infty$ and $\sum_{n=1}^{\infty} b_n$ is a convergent geometric series, it follows from the Limit Comparison Test that $\sum_{n=1}^{\infty} \frac{\ln n}{1.5^n}$ also converges.

51. Let $a_n = 1 - \sqrt{1 - \frac{1}{n}}$. Show that $\lim_{n\to\infty} a_n = 0$ and that $\sum_{n=1}^{\infty} a_n$ diverges. *Hint:* Show that $a_n \geq \frac{1}{2n}$.

SOLUTION

$$1 - \sqrt{1 - \frac{1}{n}} = 1 - \sqrt{\frac{n-1}{n}} = \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}} = \frac{n - (n-1)}{\sqrt{n}(\sqrt{n} + \sqrt{n-1})} = \frac{1}{n + \sqrt{n^2 - n}}$$

$$\geq \frac{1}{n + \sqrt{n^2}} = \frac{1}{2n}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{2n}$ diverges, so the series $\sum_{n=2}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n}}\right)$ also diverges by the Comparison Test.

52. Determine whether $\sum_{n=2}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n^2}}\right)$ converges.

SOLUTION

$$1 - \sqrt{1 - \frac{1}{n^2}} = 1 - \sqrt{\frac{n^2 - 1}{n^2}} = \frac{n - \sqrt{n^2 - 1}}{n} = \frac{n^2 - \left(n^2 - 1\right)}{n\left(n + \sqrt{n^2 - 1}\right)}$$

$$= \frac{1}{n\left(n + \sqrt{n^2 - 1}\right)} = \frac{1}{n^2 + n\sqrt{n^2 - 1}} \leq \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so the series $\sum_{n=2}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n^2}}\right)$ also converges by the Comparison Test.

53. Let $S = \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$.

(a) Show that S converges.

(b) CAS Use Eq. (4) in Exercise 83 of Section 10.3 with $M = 99$ to approximate S. What is the maximum size of the error?

SOLUTION

(a) For $n \geq 1$,

$$\frac{n}{(n^2 + 1)^2} < \frac{n}{(n^2)^2} = \frac{1}{n^3}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series, so the series $\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$ also converges by the Comparison Test.

(b) With $a_n = \frac{n}{(n^2+1)^2}$, $f(x) = \frac{x}{(x^2+1)^2}$ and $M = 99$, Eq. (4) in Exercise 83 of Section 10.3 becomes

$$\sum_{n=1}^{99} \frac{n}{(n^2+1)^2} + \int_{100}^{\infty} \frac{x}{(x^2+1)^2}\,dx \le S \le \sum_{n=1}^{100} \frac{n}{(n^2+1)^2} + \int_{100}^{\infty} \frac{x}{(x^2+1)^2}\,dx,$$

or

$$0 \le S - \left(\sum_{n=1}^{99} \frac{n}{(n^2+1)^2} + \int_{100}^{\infty} \frac{x}{(x^2+1)^2}\,dx \right) \le \frac{100}{(100^2+1)^2}.$$

Now,

$$\sum_{n=1}^{99} \frac{n}{(n^2+1)^2} = 0.397066274; \text{ and}$$

$$\int_{100}^{\infty} \frac{x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_{100}^{R} \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2} \lim_{R\to\infty} \left(-\frac{1}{R^2+1} + \frac{1}{100^2+1} \right)$$
$$= \frac{1}{20002} = 0.000049995;$$

thus,

$$S \approx 0.397066274 + 0.000049995 = 0.397116269.$$

The bound on the error in this approximation is

$$\frac{100}{(100^2+1)^2} = 9.998 \times 10^{-7}.$$

In Exercises 54–57, determine whether the series converges absolutely. If it does not, determine whether it converges conditionally.

54. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n} + 2n}$

SOLUTION Both $\sqrt[3]{n}$ and $2n$ are increasing functions, so $\sqrt[3]{n} + 2n$ is also increasing. Therefore, $\frac{1}{\sqrt[3]{n}+2n}$ is decreasing. Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n} + 2n} = 0,$$

so the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n} + 2n}$ converges by the Leibniz Test.

The corresponding positive series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + 2n}$. Because

$$\frac{1}{\sqrt[3]{n} + 2n} > \frac{1}{n + 2n} = \frac{1}{3} \cdot \frac{1}{n}$$

and the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + 2n}$ also diverges by the Comparison Test. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}+2n}$ converges conditionally.

55. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$

SOLUTION Consider the corresponding positive series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}\ln(n+1)}$. Because

$$\frac{1}{n^{1.1}\ln(n+1)} < \frac{1}{n^{1.1}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is a convergent p-series, we can conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$ also converges. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$ converges absolutely.

56. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+\pi n\right)}{\sqrt{n}}$

SOLUTION Note

$$\cos\left(\frac{\pi}{4}+\pi n\right) = \cos\frac{\pi}{4}\cos n\pi - \sin\frac{\pi}{4}\sin n\pi = (-1)^n\frac{\sqrt{2}}{2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+\pi n\right)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

Now, the sequence $\{\frac{1}{\sqrt{n}}\}$ is decreasing and converges to 0 as $n \to \infty$. Therefore, $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+\pi n\right)}{\sqrt{n}}$ converges by the Leibniz Test. However, the corresponding positive series is a divergent p-series ($p = \frac{1}{2}$), so the original series converges conditionally.

57. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}}$

SOLUTION $\cos\left(\frac{\pi}{4}+2\pi n\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}} = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a divergent p-series, so the series $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}}$ diverges.

58. CAS Use a computer algebra system to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+\sqrt{n}}$ to within an error of at most 10^{-5}.

SOLUTION The sequence $\{\frac{1}{n^3+\sqrt{n}}\}$ is decreasing and converges to 0, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+\sqrt{n}}$ converges by the Leibniz Test. Using the error bound for an alternating series,

$$\left| S_N - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+\sqrt{n}} \right| \le a_{N+1} = \frac{1}{(N+1)^3+\sqrt{N+1}}.$$

If we want an approximation with an error of at most 10^{-5}, we must choose N such that

$$\frac{1}{(N+1)^3+\sqrt{N+1}} < 10^{-5} \quad \text{or} \quad (N+1)^3+\sqrt{N+1} > 10^5.$$

For $N = 45$, $(N+1)^3+\sqrt{N+1} = 97{,}342.8 < 10^5$, and for $N = 46$, $(N+1)^3+\sqrt{N+1} = 103{,}829.9 > 10^5$. The smallest acceptable value for N is therefore $N = 46$. Using a computer algebra system, we find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+\sqrt{n}} \approx S_{46} = -0.418452236.$$

59. Catalan's constant is defined by $K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$.

(a) How many terms of the series are needed to calculate K with an error of less than 10^{-6}?

(b) CAS Carry out the calculation.

SOLUTION Using the error bound for an alternating series, we have

$$|S_N - K| \le \frac{1}{(2(N+1)+1)^2} = \frac{1}{(2N+3)^2}.$$

For accuracy to three decimal places, we must choose N so that

$$\frac{1}{(2N+3)^2} < 5 \times 10^{-3} \quad \text{or} \quad (2N+3)^2 > 2000.$$

Solving for N yields

$$N > \frac{1}{2}\left(\sqrt{2000} - 3\right) \approx 20.9.$$

Thus,

$$K \approx \sum_{k=0}^{21} \frac{(-1)^k}{(2k+1)^2} = 0.915707728.$$

60. Give an example of conditionally convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges absolutely.

SOLUTION Let $a_n = \frac{(-1)^n}{n}$ and $b_n = \frac{(-1)^{n+1}}{n}$. The corresponding alternating series converge by the Leibniz Test; however, the corresponding positive series are the divergent harmonic series. Thus, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge conditionally. On the other hand, the series

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n} + \frac{(-1)^{n+1}}{n}\right) = \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n} + \frac{-1}{n}\right) = \sum_{n=1}^{\infty} 0$$

converges absolutely.

61. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Determine whether the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2}\right)$

(b) $\sum_{n=1}^{\infty}(-1)^n a_n$

(c) $\sum_{n=1}^{\infty}\frac{1}{1+a_n^2}$

(d) $\sum_{n=1}^{\infty}\frac{|a_n|}{n}$

SOLUTION Because $\sum_{n=1}^{\infty} a_n$ converges absolutely, we know that $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty} |a_n|$ converges.

(a) Because we know that $\sum_{n=1}^{\infty} a_n$ converges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the sum of these two series, $\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2}\right)$ also converges.

(b) We have,

$$\sum_{n=1}^{\infty} \left|(-1)^n a_n\right| = \sum_{n=1}^{\infty} |a_n|$$

Because $\sum_{n=1}^{\infty} |a_n|$ converges, it follows that $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely, which implies that $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(c) Because $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{1+a_n^2} = \frac{1}{1+0^2} = 1 \ne 0,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n^2}$ diverges by the Divergence Test.

(d) $\frac{|a_n|}{n} \le |a_n|$ and the series $\sum_{n=1}^{\infty} |a_n|$ converges, so the series $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ also converges by the Comparison Test.

62. Let $\{a_n\}$ be a positive sequence such that $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$. Determine whether the following series converge or diverge:

(a) $\sum_{n=1}^{\infty} 2a_n$ **(b)** $\sum_{n=1}^{\infty} 3^n a_n$ **(c)** $\sum_{n=1}^{\infty} \sqrt{a_n}$

SOLUTION
(a)

$$L = \lim_{n\to\infty} \sqrt[n]{2a_n} = \lim_{n\to\infty} \sqrt[n]{2}\sqrt[n]{a_n} = 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

Because $L < 1$, the series converges by the Root Test.
(b)

$$L = \lim_{n\to\infty} \sqrt[n]{3^n a_n} = \lim_{n\to\infty} 3\sqrt[n]{a_n} = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Because $L > 1$, the series diverges by the Root Test.
(c)

$$L = \lim_{n\to\infty} \sqrt[n]{\sqrt{a_n}} = \lim_{n\to\infty} \sqrt{\sqrt[n]{a_n}} = \sqrt{\frac{1}{2}}.$$

Because $L < 1$, the series converges by the Root Test.

In Exercises 63–70, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

63. $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

SOLUTION With $a_n = \frac{n^5}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^5}{5^{n+1}} \cdot \frac{5^n}{n^5} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^5,$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^5 = \frac{1}{5} \cdot 1 = \frac{1}{5}.$$

Because $\rho < 1$, the series converges by the Ratio Test.

64. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^8}$

SOLUTION With $a_n = \frac{\sqrt{n+1}}{n^8}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n+2}}{(n+1)^8} \cdot \frac{n^8}{\sqrt{n+1}} = \sqrt{\frac{n+2}{n+1}}\left(\frac{n}{n+1}\right)^8,$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1^8 = 1.$$

Because $\rho = 1$, the Ratio Test is inconclusive.

65. $\sum_{n=1}^{\infty} \frac{1}{n2^n + n^3}$

SOLUTION With $a_n = \frac{1}{n2^n+n^3}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n2^n + n^3}{(n+1)2^{n+1} + (n+1)^3} = \frac{n2^n\left(1 + \frac{n^2}{2^n}\right)}{(n+1)2^{n+1}\left(1 + \frac{(n+1)^2}{2^{n+1}}\right)} = \frac{1}{2} \cdot \frac{n}{n+1} \cdot \frac{1 + \frac{n^2}{2^n}}{1 + \frac{(n+1)^2}{2^{n+1}}},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$$

Because $\rho < 1$, the series converges by the Ratio Test.

66. $\displaystyle\sum_{n=1}^{\infty}\frac{n^4}{n!}$

SOLUTION With $a_n = \frac{n^4}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^4}{(n+1)!}\cdot\frac{n!}{n^4} = \frac{(n+1)^3}{n^4} \quad \text{and} \quad \rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0.$$

Because $\rho < 1$, the series converges by the Ratio Test.

67. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$

SOLUTION With $a_n = \frac{2^{n^2}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{(n+1)^2}}{(n+1)!}\cdot\frac{n!}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty.$$

Because $\rho > 1$, the series diverges by the Ratio Test.

68. $\displaystyle\sum_{n=4}^{\infty}\frac{\ln n}{n^{3/2}}$

SOLUTION With $a_n = \frac{\ln n}{n^{3/2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln(n+1)}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{\ln n} = \left(\frac{n}{n+1}\right)^{3/2}\frac{\ln(n+1)}{\ln n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1^{3/2}\cdot 1 = 1.$$

Because $\rho = 1$, the Ratio Test is inconclusive.

69. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2}\right)^n\frac{1}{n!}$

SOLUTION With $a_n = \left(\frac{n}{2}\right)^n\frac{1}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{2}\right)^{n+1}\frac{1}{(n+1)!}\cdot\left(\frac{2}{n}\right)^n n! = \frac{1}{2}\left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1+\frac{1}{n}\right)^n,$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}e.$$

Because $\rho = \frac{e}{2} > 1$, the series diverges by the Ratio Test.

70. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{4}\right)^n\frac{1}{n!}$

SOLUTION With $a_n = \left(\frac{n}{4}\right)^n\frac{1}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{4}\right)^{n+1}\frac{1}{(n+1)!}\cdot\left(\frac{4}{n}\right)^n n! = \frac{1}{4}\left(\frac{n+1}{n}\right)^n = \frac{1}{4}\left(1+\frac{1}{n}\right)^n,$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}e.$$

Because $\rho = \frac{e}{4} < 1$, the series converges by the Ratio Test.

In Exercises 71–74, apply the Root Test to determine convergence or divergence, or state that the Root Test is inconclusive.

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$

SOLUTION With $a_n = \frac{1}{4^n}$,

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{4^n}} = \frac{1}{4}.$$

Because $L < 1$, the series converges by the Root Test.

72. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$

SOLUTION With $a_n = \left(\frac{2}{n}\right)^n$,

$$L = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Because $L < 1$, the series converges by the Root Test.

73. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4n}\right)^n$

SOLUTION With $a_n = \left(\frac{3}{4n}\right)^n$,

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{4n}\right)^n} = \lim_{n\to\infty} \frac{3}{4n} = 0.$$

Because $L < 1$, the series converges by the Root Test.

74. $\displaystyle\sum_{n=1}^{\infty} \left(\cos\frac{1}{n}\right)^{n^3}$

SOLUTION With $a_n = \left(\cos\frac{1}{n}\right)^{n^3}$,

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\cos\left(\frac{1}{n}\right)^{n^3}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)^{n^2} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)^{x^2}.$$

Now,

$$\ln L = \lim_{x\to\infty} x^2 \ln\cos\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\cos\left(\frac{1}{x}\right)}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{\cos\left(\frac{1}{x}\right)}\left(-\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}$$

$$= -\frac{1}{2}\lim_{x\to\infty} \frac{1}{\cos\left(\frac{1}{x}\right)} \cdot \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = -\frac{1}{2}\cdot 1 \cdot 1 = -\frac{1}{2}.$$

Therefore, $L = e^{-1/2}$. Because $L < 1$, the series converges by the Root Test.

In Exercises 75–92, determine convergence or divergence using any method covered in the text.

75. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

SOLUTION This is a geometric series with ratio $r = \frac{2}{3} < 1$; hence, the series converges.

76. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^{7n}}{e^{8n}}$

SOLUTION This is a geometric series with ratio $r = \frac{\pi^7}{e^8} \approx 1.013$, so it diverges.

77. $\displaystyle\sum_{n=1}^{\infty} e^{-0.02n}$

SOLUTION This is a geometric series with common ratio $r = \frac{1}{e^{0.02}} \approx 0.98 < 1$; hence, the series converges.

78. $\displaystyle\sum_{n=1}^{\infty} ne^{-0.02n}$

SOLUTION With $a_n = ne^{-0.02n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)e^{-0.02(n+1)}}{ne^{-0.02n}} = \frac{n+1}{n}e^{-0.02},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot e^{-0.02} = e^{-0.02}.$$

Because $\rho < 1$, the series converges by the Ratio Test.

79. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+\sqrt{n+1}}$

SOLUTION In this alternating series, $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$. The sequence $\{a_n\}$ is decreasing, and

$$\lim_{n\to\infty} a_n = 0;$$

therefore the series converges by the Leibniz Test.

80. $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

SOLUTION Let $f(x) = \frac{1}{x(\ln x)^{3/2}}$. This function is continuous, positive and decreasing for $x > e^{-3/2}$ and thus for $x \ge 10$; therefore, the Integral Test applies. Now,

$$\int_{10}^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{R\to\infty}\int_{10}^{R} \frac{dx}{x(\ln x)^{3/2}} = \lim_{R\to\infty}\int_{\ln 10}^{\ln R} \frac{1}{u^{3/2}}\,du$$
$$= \lim_{R\to\infty}\left(\left.\frac{-2}{\sqrt{u}}\right|_{\ln 10}^{\ln R}\right) = 2\lim_{R\to\infty}\left(\frac{1}{\sqrt{\ln 10}} - \frac{1}{\sqrt{\ln R}}\right) = 2.$$

The integral converges; hence, the series converges as well.

81. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

SOLUTION The sequence $a_n = \frac{1}{\ln n}$ is decreasing for $n \ge 10$ and

$$\lim_{n\to\infty} a_n = 0;$$

therefore, the series converges by the Leibniz Test.

82. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

SOLUTION With $a_n = \frac{e^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n} = \frac{e}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.$$

Because $\rho < 1$, the series converges by the Ratio Test.

83. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+\ln n}}$

SOLUTION For $n \geq 1$,

$$\frac{1}{n\sqrt{n+\ln n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+\ln n}}$ converges by the Comparison Test.

84. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}(1+\sqrt{n})}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{1}{\sqrt[3]{n}(1+\sqrt{n})}$ and $b_n = \frac{1}{n^{5/6}}$. Then,

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}(1+\sqrt{n})}}{\frac{1}{n^{5/6}}} = \lim_{n\to\infty} \frac{n^{5/6}}{\sqrt[3]{n} + n^{5/6}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is a divergent p-series. Because $L > 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}(1+\sqrt{n})}$ also diverges by the Limit Comparison Test.

85. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

SOLUTION This series telescopes:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \dots$$

so that the n^{th} partial sum S_n is

$$S_n = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \dots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$$

and then

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1$$

86. $\displaystyle\sum_{n=1}^{\infty} \big(\ln n - \ln(n+1)\big)$

SOLUTION This series telescopes:

$$\sum_{n=1}^{\infty} \big(\ln n - \ln(n+1)\big) = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \dots$$

so that the n^{th} partial sum S_n is

$$\begin{aligned} S_n &= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \dots + (\ln n - \ln(n+1)) \\ &= \ln 1 - \ln(n+1) = -\ln(n+1) \end{aligned}$$

and then

$$\sum_{n=1}^{\infty} \big(\ln n - \ln(n+1)\big) = \lim_{n\to\infty} S_n = -\lim_{n\to\infty} \ln(n+1) = \infty$$

so the series diverges.

87. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

SOLUTION For $n \geq 1$, $\sqrt{n} \leq n$, so that

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2n}$$

which diverges since it is a constant multiple of the harmonic series. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges as well, by the Comparison Test.

88. $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$

SOLUTION $\cos(\pi n) = (-1)^n$, so

$$\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n^{2/3}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{2/3}}.$$

The sequence $a_n = \frac{1}{n^{2/3}}$ is decreasing and

$$\lim_{n\to\infty} a_n = 0;$$

therefore, the series converges by the Leibniz Test.

89. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\ln n}}$

SOLUTION For $n \geq N$ large enough, $\ln n \geq 2$ so that

$$\sum_{n=N}^{\infty} \frac{1}{n^{\ln n}} \leq \sum_{n=N}^{\infty} \frac{1}{n^2}$$

which is a convergent p-series. Thus by the Comparison Test, $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n^{\ln n}}$ also converges; adding back in the terms for $n < N$ does not affect convergence.

90. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln^3 n}$

SOLUTION For N large enough, $\ln n \leq n^{1/4}$ when $n \geq N$ so that

$$\sum_{n=N}^{\infty} \frac{1}{\ln^3 n} > \sum_{n=N}^{\infty} \frac{1}{n^{3/4}}$$

which is a divergent p-series. Thus by the Comparison Test, $\displaystyle\sum_{n=N}^{\infty} \frac{1}{\ln^3 n}$ diverges; adding back in the terms for $n < N$ does not affect this result.

91. $\displaystyle\sum_{n=1}^{\infty} \sin^2 \frac{\pi}{n}$

SOLUTION For all $x > 0$, $\sin x < x$. Therefore, $\sin^2 x < x^2$, and for $x = \frac{\pi}{n}$,

$$\sin^2 \frac{\pi}{n} < \frac{\pi^2}{n^2} = \pi^2 \cdot \frac{1}{n^2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so the series $\displaystyle\sum_{n=1}^{\infty} \sin^2 \frac{\pi}{n}$ also converges by the Comparison Test.

92. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}$

SOLUTION With $a_n = \frac{2^{2n}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{2^{2n}} = \frac{4}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.$$

Because $\rho < 1$, the series converges by the Ratio Test.

In Exercises 93–98, find the interval of convergence of the power series.

93. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

SOLUTION With $a_n = \frac{2^n x^n}{n!}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n}\right| = \lim_{n\to\infty} \left|x \cdot \frac{2}{n}\right| = 0$$

Then $\rho < 1$ for all x, so that the radius of convergence is $R = \infty$, and the series converges for all x.

94. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$

SOLUTION With $a_n = \frac{x^n}{n+1}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right| = \lim_{n\to\infty} \left|x \cdot \frac{n+1}{n+2}\right| = \lim_{n\to\infty} \left|x \cdot \frac{1+1/n}{1+2/n}\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is 1, and the series converges absolutely for $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent harmonic series. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, which converges by the Leibniz Test. The series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ therefore converges for $-1 \le x < 1$.

95. $\displaystyle\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}(x-3)^n$

SOLUTION With $a_n = \frac{n^6(x-3)^n}{n^8+1}$,

$$\begin{aligned}
\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)^6(x-3)^{n+1}}{(n+1)^8-1} \cdot \frac{n^8+1}{n^6(x-3)^n}\right| \\
&= \lim_{n\to\infty} \left|(x-3) \cdot \frac{(n+1)^6(n^8+1)}{n^6((n+1)^8+1)}\right| \\
&= \lim_{n\to\infty} \left|(x-3) \cdot \frac{n^{14} + \text{terms of lower degree}}{n^{14} + \text{terms of lower degree}}\right| = |x-3|
\end{aligned}$$

Then $\rho < 1$ when $|x-3| < 1$, so the radius of convergence is 1, and the series converges absolutely for $|x-3| < 1$, or $2 < x < 4$. For the endpoint $x = 4$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}$, which converges by the Comparison Test comparing with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. For the endpoint $x = 2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{n^6(-1)^n}{n^8+1}$, which converges by the Leibniz Test. The series $\displaystyle\sum_{n=0}^{\infty} \frac{n^6(x-3)^n}{n^8+1}$ therefore converges for $2 \le x \le 4$.

96. $\displaystyle\sum_{n=0}^{\infty} nx^n$

SOLUTION With $a_n = nx^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \lim_{n\to\infty}\left|x\cdot\frac{n+1}{n}\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is 1, and the series converges for $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=0}^{\infty} n$, which diverges by the Divergence Test. For the endpoint $x = -1$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n$, which also diverges by the Divergence Test. The series $\displaystyle\sum_{n=0}^{\infty} nx^n$ therefore converges for $-1 < x < 1$.

97. $\displaystyle\sum_{n=0}^{\infty} (nx)^n$

SOLUTION With $a_n = n^n x^n$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| = \lim_{n\to\infty}\left|x(n+1)\cdot\left(\frac{n+1}{n}\right)^n\right| = \infty$$

since $\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$ converges to e and the $(n+1)$ term diverges to ∞. Thus $\rho < 1$ only when $x = 0$, so the series converges only for $x = 0$.

98. $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n\ln n}$

SOLUTION With $a_n = \frac{(2x-3)^n}{n\ln n}$, and using L'Hôpital's Rule,

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(n+1)\ln(n+1)}\cdot\frac{n\ln n}{(2x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|(2x-3)\frac{n\ln n}{(n+1)\ln(n+1)}\right| = \lim_{n\to\infty}\left|(2x-3)\frac{1+\ln n}{1+\ln(n+1)}\right| \\ &= \lim_{n\to\infty}\left|(2x-3)\frac{1/n}{1/(n+1)}\right| = \lim_{n\to\infty}\left|(2x-3)\frac{n+1}{n}\right| = |2x-3|\end{aligned}$$

Then $\rho < 1$ when $|2x-3| < 1$, so the radius of convergence is 1, and the series converges absolutely for $|2x-3| < 1$, or $1 < x < 2$. For the endpoint $x = 2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n\ln n}$, which diverges by the Integral Test. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n\ln n}$, which converges by the Leibniz Test. The series $\displaystyle\sum_{n=0}^{\infty}\frac{(2x-3)^n}{n\ln n}$ therefore converges for $1 \le x < 2$.

99. Expand $f(x) = \dfrac{2}{4-3x}$ as a power series centered at $c = 0$. Determine the values of x for which the series converges.

SOLUTION Write

$$\frac{2}{4-3x} = \frac{1}{2}\frac{1}{1-\frac{3}{4}x}.$$

Substituting $\frac{3}{4}x$ for x in the Maclaurin series for $\frac{1}{1-x}$, we obtain

$$\frac{1}{1-\frac{3}{4}x} = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n x^n.$$

This series converges for $\left|\frac{3}{4}x\right| < 1$, or $|x| < \frac{4}{3}$. Hence, for $|x| < \frac{4}{3}$,

$$\frac{2}{4-3x} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n x^n.$$

100. Prove that

$$\sum_{n=0}^{\infty} ne^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2}$$

Hint: Express the left-hand side as the derivative of a geometric series.

SOLUTION For $x > 0$, $\sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} (e^{-x})^n$ is a convergent geometric series with ratio $r = e^{-x}$; hence,

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}.$$

Differentiating term-by-term then yields

$$\sum_{n=0}^{\infty} \left(-ne^{-nx}\right) = -\frac{e^{-x}}{(1-e^{-x})^2}.$$

Therefore, for $x > 0$,

$$\sum_{n=0}^{\infty} ne^{-nx} = \frac{e^{-x}}{\left(1-e^{-x}\right)^2}.$$

101. Let $F(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!}$.

(a) Show that $F(x)$ has infinite radius of convergence.

(b) Show that $y = F(x)$ is a solution of

$$y'' = xy' + y, \qquad y(0) = 1, \qquad y'(0) = 0$$

(c) CAS Plot the partial sums S_N for $N = 1, 3, 5, 7$ on the same set of axes.

SOLUTION

(a) With $a_k = \frac{x^{2k}}{2^k \cdot k!}$,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|^{2k+2}}{2^{k+1} \cdot (k+1)!} \cdot \frac{2^k \cdot k!}{|x|^{2k}} = \frac{x^2}{2(k+1)},$$

and

$$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = x^2 \cdot 0 = 0.$$

Because $\rho < 1$ for all x, we conclude that the series converges for all x; that is, $R = \infty$.

(b) Let

$$y = F(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!}.$$

Then

$$y' = \sum_{k=1}^{\infty} \frac{2kx^{2k-1}}{2^k k!} = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2^{k-1}(k-1)!},$$

$$y'' = \sum_{k=1}^{\infty} \frac{(2k-1)x^{2k-2}}{2^{k-1}(k-1)!},$$

and

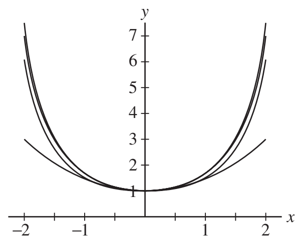
$$\begin{aligned} xy' + y &= x\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2^{k-1}(k-1)!} + \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \sum_{k=1}^{\infty} \frac{x^{2k}}{2^{k-1}(k-1)!} + 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{2^k k!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(2k+1)x^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{(2k+1)x^{2k}}{2^k k!} = \sum_{k=1}^{\infty} \frac{(2k-1)x^{2k-2}}{2^{k-1}(k-1)!} = y''. \end{aligned}$$

Moreover,

$$y(0) = 1 + \sum_{k=1}^{\infty} \frac{0^{2k}}{2^k k!} = 1 \quad \text{and} \quad y'(0) = \sum_{k=1}^{\infty} \frac{0^{2k-1}}{2^{k-1}(k-1)!} = 0.$$

Thus, $\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$ is the solution to the equation $y'' = xy' + y$ satisfying $y(0) = 1$, $y'(0) = 0$.

(c) The partial sums S_1, S_3, S_5 and S_7 are plotted in the figure below.



102. Find a power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$ that satisfies the Laguerre differential equation

$$xy'' + (1 - x)y' - y = 0$$

with initial condition satisfying $P(0) = 1$.

SOLUTION Let

$$y = P(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and

$$\begin{aligned} xy'' + (1-x)y' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= (a_1 - a_0) + \sum_{n=1}^{\infty} \left[(n+1)^2 a_{n+1} - (n+1) a_n \right] x^n. \end{aligned}$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus

$$a_1 = a_0 \quad \text{and} \quad a_{n+1} = \frac{a_n}{n+1}.$$

Now, $y(0) = P(0) = a_0$, so to satisfy the initial condition $P(0) = 1$, we must set $a_0 = 1$. Then,

$$\begin{aligned} a_1 &= a_0 = 1; \\ a_2 &= \frac{a_1}{2} = \frac{1}{2}; \\ a_3 &= \frac{a_2}{3} = \frac{1}{6} = \frac{1}{3!}; \\ a_4 &= \frac{a_3}{4} = \frac{1}{4!}; \end{aligned}$$

and, in general, $a_n = \frac{1}{n!}$. Thus,

$$P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

In Exercises 103–112, find the Taylor series centered at c.

103. $f(x) = e^{4x}, \quad c = 0$

SOLUTION Substituting $4x$ for x in the Maclaurin series for e^x yields

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n.$$

104. $f(x) = e^{2x}, \quad c = -1$

SOLUTION Write:

$$e^{2x} = e^{2(x+1)-2} = e^{-2} e^{2(x+1)}.$$

Substituting $2(x+1)$ for x in the Maclaurin series for e^x yields

$$e^{2(x+1)} = \sum_{n=0}^{\infty} \frac{(2(x+1))^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!}(x+1)^n;$$

hence,

$$e^{2x} = e^{-2} \sum_{n=0}^{\infty} \frac{2^n (x+1)^n}{n!}.$$

105. $f(x) = x^4, \quad c = 2$

SOLUTION We have

$$f'(x) = 4x^3 \quad f''(x) = 12x^2 \quad f'''(x) = 24x \quad f^{(4)}(x) = 24$$

and all higher derivatives are zero, so that

$$f(2) = 2^4 = 16 \quad f'(2) = 4 \cdot 2^3 = 32 \quad f''(2) = 12 \cdot 2^2 = 48 \quad f'''(2) = 24 \cdot 2 = 48 \quad f^{(4)}(2) = 24$$

Thus the Taylor series centered at $c = 2$ is

$$\begin{aligned}\sum_{n=0}^{4} \frac{f^{(n)}(2)}{n!}(x-2)^n &= 16 + \frac{32}{1!}(x-2) + \frac{48}{2!}(x-2)^2 + \frac{48}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4 \\ &= 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4\end{aligned}$$

106. $f(x) = x^3 - x, \quad c = -2$

SOLUTION We have

$$f'(x) = 3x^2 - 1 \quad f''(x) = 6x \quad f'''(x) = 6$$

and all higher derivatives are zero, so that

$$f(-2) = -8 + 2 = -6 \quad f'(-2) = 3(-2)^2 - 1 = 11 \quad f''(-2) = 6(-2) = -12 \quad f'''(-2) = 6$$

Thus the Taylor series centered at $c = -2$ is

$$\begin{aligned}\sum_{n=0}^{3} \frac{f^{(n)}(-2)}{n!}(x+2)^n &= -6 + \frac{11}{1!}(x+2) + \frac{-12}{2!}(x+2)^2 + \frac{6}{3!}(x+2)^3 \\ &= -6 + 11(x+2) - 6(x+2)^2 + (x+2)^3\end{aligned}$$

107. $f(x) = \sin x, \quad c = \pi$

SOLUTION We have

$$f^{(4n)}(x) = \sin x \quad f^{(4n+1)}(x) = \cos x \quad f^{(4n+2)}(x) = -\sin x \quad f^{(4n+3)}(x) = -\cos x$$

so that

$$f^{(4n)}(\pi) = \sin \pi = 0 \quad f^{(4n+1)}(\pi) = \cos \pi = -1 \quad f^{(4n+2)}(\pi) = -\sin \pi = 0 \quad f^{(4n+3)}(\pi) = -\cos \pi = 1$$

Then the Taylor series centered at $c = \pi$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n = \frac{-1}{1!}(x-\pi) + \frac{1}{3!}(x-\pi)^3 + \frac{-1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \dots$$

$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \frac{1}{5040}(x-\pi)^7 - \dots$$

108. $f(x) = e^{x-1}, \quad c = -1$

SOLUTION Write

$$e^{x-1} = e^{x+1-1-1} = e^{-2}e^{x+1}.$$

Substituting $x+1$ for x in the Maclaurin series for e^x yields

$$e^{x+1} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!};$$

hence,

$$e^{x-1} = e^{-2}\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!e^2}.$$

109. $f(x) = \dfrac{1}{1-2x}, \quad c = -2$

SOLUTION Write

$$\frac{1}{1-2x} = \frac{1}{5-2(x+2)} = \frac{1}{5}\frac{1}{1-\frac{2}{5}(x+2)}.$$

Substituting $\frac{2}{5}(x+2)$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1-\frac{2}{5}(x+2)} = \sum_{n=0}^{\infty} \frac{2^n}{5^n}(x+2)^n;$$

hence,

$$\frac{1}{1-2x} = \frac{1}{5}\sum_{n=0}^{\infty} \frac{2^n}{5^n}(x+2)^n = \sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+2)^n.$$

110. $f(x) = \dfrac{1}{(1-2x)^2}, \quad c = -2$

SOLUTION Note that

$$\frac{d}{dx}\frac{1}{1-2x} = \frac{2}{1-2x}$$

so that we can derive the Taylor series for $f(x)$ by differentiating the Taylor series for $\frac{1}{1-2x}$, computed in the previous exercise, and dividing by 2. Thus

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\cdot\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}}(x+2)^n\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n2^n}{5^{n+1}}(x+2)^{n-1} = \frac{2}{50}\sum_{n=1}^{\infty} \frac{n2^{n-1}}{5^{n-1}}(x+2)^{n-1}$$

$$= \frac{1}{25}\sum_{k=0}^{\infty} \frac{(k+1)2^k}{5^k}(x+2)^k$$

111. $f(x) = \ln \dfrac{x}{2}, \quad c = 2$

SOLUTION Write

$$\ln \frac{x}{2} = \ln\left(\frac{(x-2)+2}{2}\right) = \ln\left(1 + \frac{x-2}{2}\right).$$

Substituting $\frac{x-2}{2}$ for x in the Maclaurin series for $\ln(1+x)$ yields

$$\ln \frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{x-2}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n \cdot 2^n}.$$

This series is valid for $|x-2| < 2$.

112. $f(x) = x \ln\left(1 + \dfrac{x}{2}\right), \quad c = 0$

SOLUTION Substituting $\frac{x}{2}$ for x in the Maclaurin series for $\ln(1+x)$ yields

$$\ln\left(1 + \frac{x}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{x}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n2^n}.$$

Thus,

$$x \ln\left(1 + \frac{x}{2}\right) = x\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+1}}{n2^n}.$$

In Exercises 113–116, find the first three terms of the Maclaurin series of $f(x)$ and use it to calculate $f^{(3)}(0)$.

113. $f(x) = (x^2 - x)e^{x^2}$

SOLUTION Substitute x^2 for x in the Maclaurin series for e^x to get

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots$$

so that the Maclaurin series for $f(x)$ is

$$(x^2 - x)e^{x^2} = x^2 + x^4 + \frac{1}{2}x^6 + \cdots - x - x^3 - \frac{1}{2}x^5 - \cdots = -x + x^2 - x^3 + x^4 + \dots$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = -1$$

so that $f'''(0) = -6$.

114. $f(x) = \tan^{-1}(x^2 - x)$

SOLUTION Substitute $x^2 - x$ for x in the Maclaurin series for $\tan^{-1} x$ to get

$$\tan^{-1}(x^2 - x) = (x^2 - x) - \frac{1}{3}(x^2 - x)^3 + \cdots = -x + x^2 + \frac{1}{3}x^3 + \dots$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = \frac{1}{3}$$

so that $f'''(0) = 3!\frac{1}{3} = 2$.

115. $f(x) = \dfrac{1}{1 + \tan x}$

SOLUTION Substitute $-\tan x$ in the Maclaurin series for $\frac{1}{1-x}$ to get

$$\frac{1}{1 + \tan x} = 1 - \tan x + (\tan x)^2 - (\tan x)^3 + \dots$$

We have not yet encountered the Maclaurin series for $\tan x$. We need only the terms up through x^3, so compute

$$\tan'(x) = \sec^2 x \quad \tan''(x) = 2(\tan x)\sec^2 x \quad \tan'''(x) = 2(1 + \tan^2 x)\sec^2 x + 4(\tan^2 x)\sec^2 x$$

so that

$$\tan'(0) = 1 \quad \tan''(0) = 0 \quad \tan'''(0) = 2$$

Then the Maclaurin series for $\tan x$ is

$$\tan x = \tan 0 + \frac{\tan'(0)}{1!}x + \frac{\tan''(0)}{2!}x^2 + \frac{\tan'''(0)}{3!}x^3 + \cdots = x + \frac{1}{3}x^3 + \ldots$$

Substitute these into the series above to get

$$\begin{aligned}\frac{1}{1+\tan x} &= 1 - \left(x + \frac{1}{3}x^3\right) + \left(x + \frac{1}{3}x^3\right)^2 - \left(x + \frac{1}{3}x^3\right)^3 + \ldots \\ &= 1 - x - \frac{1}{3}x^3 + x^2 - x^3 + \text{higher degree terms} \\ &= 1 - x + x^2 - \frac{4}{3}x^3 + \text{higher degree terms}\end{aligned}$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = -\frac{4}{3}$$

so that

$$f'''(0) = -6 \cdot \frac{4}{3} = -8$$

116. $f(x) = (\sin x)\sqrt{1+x}$

SOLUTION The binomial series for $\sqrt{1+x}$ is

$$\begin{aligned}\sqrt{1+x} = (1+x)^{1/2} &= \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \ldots \\ &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\end{aligned}$$

So, multiply the first few terms of the two Maclaurin series together:

$$\begin{aligned}(\sin x)\sqrt{1+x} &= \left(x - \frac{x^3}{6}\right)\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3\right) \\ &= x + \frac{1}{2}x^2 - \frac{1}{8}x^3 - \frac{1}{6}x^3 + \text{higher degree terms} \\ &= x + \frac{1}{2}x^2 - \frac{7}{24}x^3 + \text{higher degree terms}\end{aligned}$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = -\frac{7}{24}$$

so that

$$f'''(0) = -6 \cdot \frac{7}{24} = -\frac{7}{4}$$

117. Calculate $\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots$.

SOLUTION We recognize that

$$\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/2)^{2n+1}}{(2n+1)!}$$

is the Maclaurin series for $\sin x$ with x replaced by $\pi/2$. Therefore,

$$\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots = \sin\frac{\pi}{2} = 1.$$

118. Find the Maclaurin series of the function $F(x) = \int_0^x \frac{e^t - 1}{t} dt$.

SOLUTION Subtracting 1 from the Maclaurin series for e^t yields

$$e^t - 1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} - 1 = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}.$$

Thus,

$$\frac{e^t - 1}{t} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}.$$

Finally, integrating term-by-term yields

$$\int_0^x \frac{e^t - 1}{t} dt = \int_0^x \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} dt = \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n!} dt = \sum_{n=1}^{\infty} \frac{x^n}{n!\,n}.$$

11 PARAMETRIC EQUATIONS, POLAR COORDINATES, AND CONIC SECTIONS

11.1 Parametric Equations

Preliminary Questions

1. Describe the shape of the curve $x = 3\cos t$, $y = 3\sin t$.

SOLUTION For all t,

$$x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9 \cdot 1 = 9,$$

therefore the curve is on the circle $x^2 + y^2 = 9$. Also, each point on the circle $x^2 + y^2 = 9$ can be represented in the form $(3\cos t, 3\sin t)$ for some value of t. We conclude that the curve $x = 3\cos t$, $y = 3\sin t$ is the circle of radius 3 centered at the origin.

2. How does $x = 4 + 3\cos t$, $y = 5 + 3\sin t$ differ from the curve in the previous question?

SOLUTION In this case we have

$$(x-4)^2 + (y-5)^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9 \cdot 1 = 9$$

Therefore, the given equations parametrize the circle of radius 3 centered at the point $(4, 5)$.

3. What is the maximum height of a particle whose path has parametric equations $x = t^9$, $y = 4 - t^2$?

SOLUTION The particle's height is $y = 4 - t^2$. To find the maximum height we set the derivative equal to zero and solve:

$$\frac{dy}{dt} = \frac{d}{dt}(4 - t^2) = -2t = 0 \quad \text{or} \quad t = 0$$

The maximum height is $y(0) = 4 - 0^2 = 4$.

4. Can the parametric curve $(t, \sin t)$ be represented as a graph $y = f(x)$? What about $(\sin t, t)$?

SOLUTION In the parametric curve $(t, \sin t)$ we have $x = t$ and $y = \sin t$, therefore, $y = \sin x$. That is, the curve can be represented as a graph of a function. In the parametric curve $(\sin t, t)$ we have $x = \sin t$, $y = t$, therefore $x = \sin y$. This equation does not define y as a function of x, therefore the parametric curve $(\sin t, t)$ cannot be represented as a graph of a function $y = f(x)$.

5. Match the derivatives with a verbal description:

(a) $\dfrac{dx}{dt}$ **(b)** $\dfrac{dy}{dt}$ **(c)** $\dfrac{dy}{dx}$

(i) Slope of the tangent line to the curve

(ii) Vertical rate of change with respect to time

(iii) Horizontal rate of change with respect to time

SOLUTION

(a) The derivative $\dfrac{dx}{dt}$ is the horizontal rate of change with respect to time.

(b) The derivative $\dfrac{dy}{dt}$ is the vertical rate of change with respect to time.

(c) The derivative $\dfrac{dy}{dx}$ is the slope of the tangent line to the curve.

Hence, (a) $\leftrightarrow$ (iii), (b) $\leftrightarrow$ (ii), (c) $\leftrightarrow$ (i)

Exercises

1. Find the coordinates at times $t = 0, 2, 4$ of a particle following the path $x = 1 + t^3$, $y = 9 - 3t^2$.

SOLUTION Substituting $t = 0$, $t = 2$, and $t = 4$ into $x = 1 + t^3$, $y = 9 - 3t^2$ gives the coordinates of the particle at these times respectively. That is,

$$\begin{aligned}
&(t = 0) \quad x = 1 + 0^3 = 1,\ y = 9 - 3 \cdot 0^2 = 9 &&\Rightarrow (1, 9)\\
&(t = 2) \quad x = 1 + 2^3 = 9,\ y = 9 - 3 \cdot 2^2 = -3 &&\Rightarrow (9, -3)\\
&(t = 4) \quad x = 1 + 4^3 = 65,\ y = 9 - 3 \cdot 4^2 = -39 &&\Rightarrow (65, -39).
\end{aligned}$$

2. Find the coordinates at $t = 0, \frac{\pi}{4}, \pi$ of a particle moving along the path $c(t) = (\cos 2t, \sin^2 t)$.

SOLUTION Setting $t = 0$, $t = \frac{\pi}{4}$, and $t = \pi$ in $c(t) = (\cos 2t, \sin^2 t)$ we obtain the following coordinates of the particle:

$$\begin{aligned}
t = 0: &\quad (\cos 2 \cdot 0, \sin^2 0) = (1, 0)\\
t = \tfrac{\pi}{4}: &\quad (\cos \tfrac{2\pi}{4}, \sin^2 \tfrac{\pi}{4}) = (0, \tfrac{1}{2})\\
t = \pi: &\quad (\cos 2\pi, \sin^2 \pi) = (1, 0)
\end{aligned}$$

3. Show that the path traced by the bullet in Example 3 is a parabola by eliminating the parameter.

SOLUTION The path traced by the bullet is given by the following parametric equations:

$$x = 200t,\ \ y = 400t - 16t^2$$

We eliminate the parameter. Since $x = 200t$, we have $t = \dfrac{x}{200}$. Substituting into the equation for y we obtain:

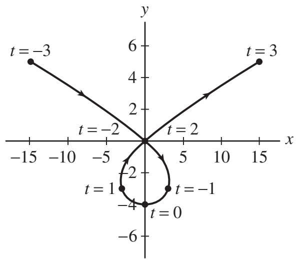
$$y = 400t - 16t^2 = 400 \cdot \frac{x}{200} - 16\left(\frac{x}{200}\right)^2 = 2x - \frac{x^2}{2500}$$

The equation $y = -\dfrac{x^2}{2500} + 2x$ is the equation of a parabola.

4. Use the table of values to sketch the parametric curve $(x(t), y(t))$, indicating the direction of motion.

t	-3	-2	-1	0	1	2	3
x	-15	0	3	0	-3	0	15
y	5	0	-3	-4	-3	0	5

SOLUTION We mark the given points on the xy-plane and connect the points corresponding to successive values of t in the direction of increasing t. We get the following trajectory (there are other correct answers):



5. Graph the parametric curves. Include arrows indicating the direction of motion.

(a) (t, t), $\ -\infty < t < \infty$

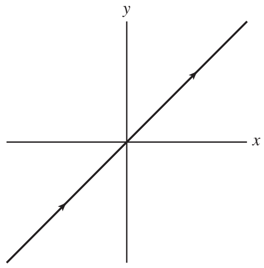
(b) $(\sin t, \sin t)$, $\ 0 \le t \le 2\pi$

(c) (e^t, e^t), $\ -\infty < t < \infty$

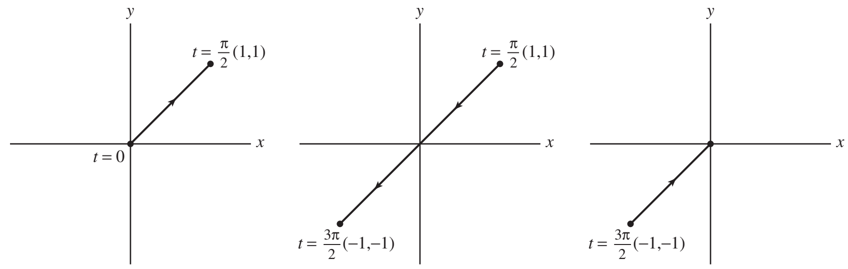
(d) (t^3, t^3), $\ -1 \le t \le 1$

SOLUTION

(a) For the trajectory $c(t) = (t, t)$, $-\infty < t < \infty$ we have $y = x$. Also the two coordinates tend to ∞ and $-\infty$ as $t \to \infty$ and $t \to -\infty$ respectively. The graph is shown next:

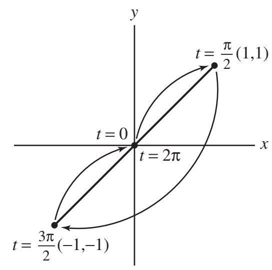


(b) For the curve $c(t) = (\sin t, \sin t)$, $0 \le t \le 2\pi$, we have $y = x$. $\sin t$ is increasing for $0 \le t \le \frac{\pi}{2}$, decreasing for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$ and increasing again for $\frac{3\pi}{2} \le t \le 2\pi$. Hence the particle moves from $c(0) = (0,0)$ to $c(\frac{\pi}{2}) = (1,1)$, then moves back to $c(\frac{3\pi}{2}) = (-1,-1)$ and then returns to $c(2\pi) = (0,0)$. We obtain the following trajectory:

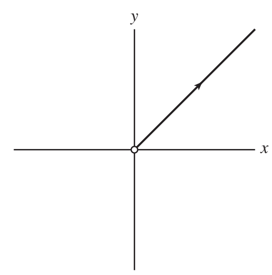


$0 < t \le \frac{\pi}{2}$ $\qquad$ $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$ $\qquad$ $\frac{3\pi}{2} \le t < 2\pi$

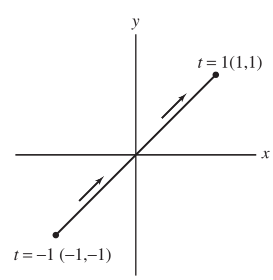
These three parts of the trajectory are shown together in the next figure:



(c) For the trajectory $c(t) = (e^t, e^t)$, $-\infty < t < \infty$, we have $y = x$. However since $\lim_{t\to-\infty} e^t = 0$ and $\lim_{t\to\infty} e^t = \infty$, the trajectory is the part of the line $y = x$, $0 < x$.



(d) For the trajectory $c(t) = (t^3, t^3)$, $-1 \le t \le 1$, we have again $y = x$. Since the function t^3 is increasing the particle moves in one direction starting at $((-1)^3, (-1)^3) = (-1,-1)$ and ending at $(1^3, 1^3) = (1,1)$. The trajectory is shown next:



6. Give two different parametrizations of the line through $(4, 1)$ with slope 2.

SOLUTION The equation of the line through $(4, 1)$ with slope 2 is $y - 1 = 2(x - 4)$ or $y = 2x - 7$. One parametrization is obtained by choosing the x coordinate as the parameter. That is, $x = t$. Hence $y = 2t - 7$ and we get $x = t$, $y = 2t - 7$, $-\infty < t < \infty$. Another parametrization is given by $x = \frac{t}{2}$, $y = t - 7$, $-\infty < t < \infty$.

In Exercises 7–14, express in the form $y = f(x)$ by eliminating the parameter.

7. $x = t + 3, \quad y = 4t$

SOLUTION We eliminate the parameter. Since $x = t + 3$, we have $t = x - 3$. Substituting into $y = 4t$ we obtain

$$y = 4t = 4(x - 3) \Rightarrow y = 4x - 12$$

8. $x = t^{-1}, \quad y = t^{-2}$

SOLUTION From $x = t^{-1}$, we have $t = x^{-1}$. Substituting in $y = t^{-2}$ we obtain

$$y = t^{-2} = (x^{-1})^{-2} = x^2 \Rightarrow y = x^2, \quad x \neq 0.$$

9. $x = t, \quad y = \tan^{-1}(t^3 + e^t)$

SOLUTION Replacing t by x in the equation for y we obtain $y = \tan^{-1}(x^3 + e^x)$.

10. $x = t^2, \quad y = t^3 + 1$

SOLUTION From $x = t^2$ we get $t = \pm\sqrt{x}$. Substituting into $y = t^3 + 1$ we obtain

$$y = t^3 + 1 = (\pm\sqrt{x})^3 + 1 = \pm\sqrt{x^3} + 1, \quad x \geq 0.$$

Since we must have y a function of x, we should probably choose either the positive or negative root.

11. $x = e^{-2t}, \quad y = 6e^{4t}$

SOLUTION We eliminate the parameter. Since $x = e^{-2t}$, we have $-2t = \ln x$ or $t = -\frac{1}{2}\ln x$. Substituting in $y = 6e^{4t}$ we get

$$y = 6e^{4t} = 6e^{4\cdot(-\frac{1}{2}\ln x)} = 6e^{-2\ln x} = 6e^{\ln x^{-2}} = 6x^{-2} \Rightarrow y = \frac{6}{x^2}, \quad x > 0.$$

12. $x = 1 + t^{-1}, \quad y = t^2$

SOLUTION From $x = 1 + t^{-1}$, we get $t^{-1} = x - 1$ or $t = \frac{1}{x-1}$. We now substitute $t = \frac{1}{x-1}$ in $y = t^2$ to obtain

$$y = t^2 = \left(\frac{1}{x-1}\right)^2 \Rightarrow y = \frac{1}{(x-1)^2}, \quad x \neq 1.$$

13. $x = \ln t, \quad y = 2 - t$

SOLUTION Since $x = \ln t$ we have $t = e^x$. Substituting in $y = 2 - t$ we obtain $y = 2 - e^x$.

14. $x = \cos t, \quad y = \tan t$

SOLUTION We use the trigonometric identity $\sin t = \pm\sqrt{1 - \cos^2 t}$ to write

$$y = \tan t = \frac{\sin t}{\cos t} = \pm\frac{\sqrt{1 - \cos^2 t}}{\cos t}.$$

We now express y in terms of x:

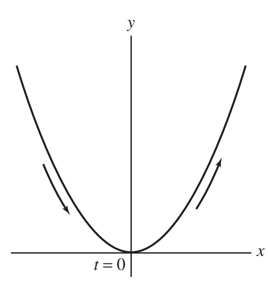
$$y = \tan t = \pm\frac{\sqrt{1 - x^2}}{x} \Rightarrow y = \pm\frac{\sqrt{1 - x^2}}{x}, \quad x \neq 0.$$

Since we must have y a function of x, we should probably choose either the positive or negative root.

In Exercises 15–18, graph the curve and draw an arrow specifying the direction corresponding to motion.

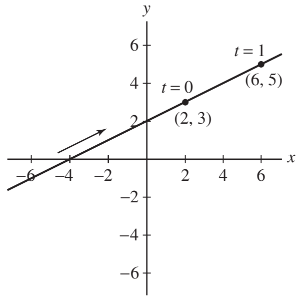
15. $x = \frac{1}{2}t, \quad y = 2t^2$

SOLUTION Let $c(t) = (x(t), y(t)) = (\frac{1}{2}t, 2t^2)$. Then $c(-t) = (-x(t), y(t))$ so the curve is symmetric with respect to the y-axis. Also, the function $\frac{1}{2}t$ is increasing. Hence there is only one direction of motion on the curve. The corresponding function is the parabola $y = 2 \cdot (2x)^2 = 8x^2$. We obtain the following trajectory:



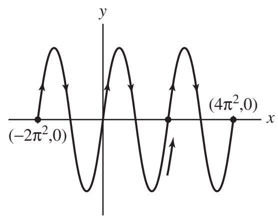
16. $x = 2 + 4t, \quad y = 3 + 2t$

SOLUTION We find the function by eliminating the parameter. Since $x = 2 + 4t$ we have $t = \frac{x-2}{4}$, hence $y = 3 + 2(\frac{x-2}{4})$ or $y = \frac{x}{2} + 2$. Also, since $2 + 4t$ and $3 + 2t$ are increasing functions, the direction of motion is the direction of increasing t. We obtain the following curve:



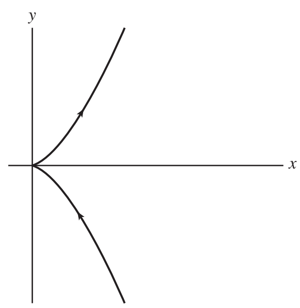
17. $x = \pi t, \quad y = \sin t$

SOLUTION We find the function by eliminating t. Since $x = \pi t$, we have $t = \frac{x}{\pi}$. Substituting $t = \frac{x}{\pi}$ into $y = \sin t$ we get $y = \sin \frac{x}{\pi}$. We obtain the following curve:



18. $x = t^2, \quad y = t^3$

SOLUTION From $x = t^2$ we have $t = \pm x^{1/2}$. Hence, $y = \pm x^{3/2}$. Since the functions t^2 and t^3 are increasing, there is only one direction of motion, which is the direction of increasing t. Notice that for $c(t) = (t^2, t^3)$ we have $c(-t) = (t^2, -t^3) = (x(t), -y(t))$. Hence the curve is symmetric with respect to the x axis. We obtain the following curve:



19. Match the parametrizations (a)–(d) below with their plots in Figure 14, and draw an arrow indicating the direction of motion.

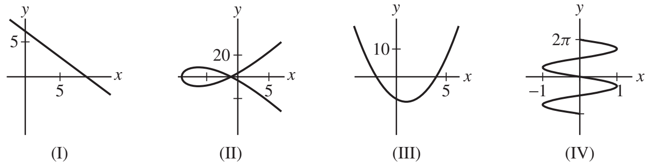


FIGURE 14

(a) $c(t) = (\sin t, -t)$

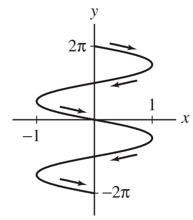
(b) $c(t) = (t^2 - 9, 8t - t^3)$

(c) $c(t) = (1 - t, t^2 - 9)$

(d) $c(t) = (4t + 2, 5 - 3t)$

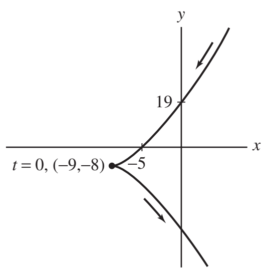
SOLUTION

(a) In the curve $c(t) = (\sin t, -t)$ the x-coordinate is varying between -1 and 1 so this curve corresponds to plot IV. As t increases, the y-coordinate $y = -t$ is decreasing so the direction of motion is downward.



(IV) $c(t) = (\sin t, -t)$

(b) The curve $c(t) = (t^2 - 9, -t^3 - 8)$ intersects the x-axis where $y = -t^3 - 8 = 0$, or $t = -2$. The x-intercept is $(-5, 0)$. The y-intercepts are obtained where $x = t^2 - 9 = 0$, or $t = \pm 3$. The y-intercepts are $(0, -35)$ and $(0, 19)$. As t increases from $-\infty$ to 0, x and y decrease, and as t increases from 0 to ∞, x increases and y decreases. We obtain the following trajectory:

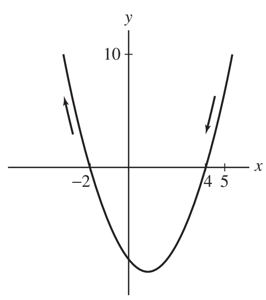


(II)

(c) The curve $c(t) = (1 - t, t^2 - 9)$ intersects the y-axis where $x = 1 - t = 0$, or $t = 1$. The y-intercept is $(0, -8)$. The x-intercepts are obtained where $t^2 - 9 = 0$ or $t = \pm 3$. These are the points $(-2, 0)$ and $(4, 0)$. Setting $t = 1 - x$ we get

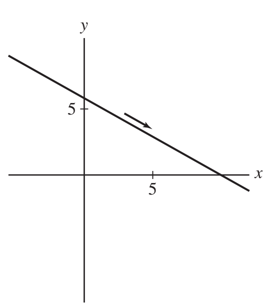
$$y = t^2 - 9 = (1 - x)^2 - 9 = x^2 - 2x - 8.$$

As t increases the x coordinate decreases and we obtain the following trajectory:



(III)

(d) The curve $c(t) = (4t + 2, 5 - 3t)$ is a straight line, since eliminating t in $x = 4t + 2$ and substituting in $y = 5 - 3t$ gives $y = 5 - 3 \cdot \frac{x-2}{4} = -\frac{3}{4}x + \frac{13}{2}$ which is the equation of a line. As t increases, the x coordinate $x = 4t + 2$ increases and the y-coordinate $y = 5 - 3t$ decreases. We obtain the following trajectory:



(I)

20. A particle follows the trajectory

$$x(t) = \frac{1}{4}t^3 + 2t, \qquad y(t) = 20t - t^2$$

with t in seconds and distance in centimeters.

(a) What is the particle's maximum height?

(b) When does the particle hit the ground and how far from the origin does it land?

SOLUTION

(a) To find the maximum height $y(t)$, we set the derivative of $y(t)$ equal to zero and solve:

$$\frac{dy}{dt} = \frac{d}{dt}(20t - t^2) = 20 - 2t = 0 \Rightarrow t = 10.$$

The maximum height is $y(10) = 20 \cdot 10 - 10^2 = 100$ cm.

(b) The object hits the ground when its height is zero. That is, when $y(t) = 0$. Solving for t we get

$$20t - t^2 = t(20 - t) = 0 \Rightarrow t = 0, t = 20.$$

$t = 0$ is the initial time, so the solution is $t = 20$. At that time, the object's x coordinate is $x(20) = \frac{1}{4} \cdot 20^3 + 2 \cdot 20 = 2040$. Thus, when it hits the ground, the object is 2040 cm away from the origin.

21. Find an interval of t-values such that $c(t) = (\cos t, \sin t)$ traces the lower half of the unit circle.

SOLUTION For $t = \pi$, we have $c(\pi) = (-1, 0)$. As t increases from π to 2π, the x-coordinate of $c(t)$ increases from -1 to 1, and the y-coordinate decreases from 0 to -1 (at $t = 3\pi/2$) and then returns to 0. Thus, for t in $[\pi, 2\pi]$, the equation traces the lower part of the circle.

22. Find an interval of t-values such that $c(t) = (2t + 1, 4t - 5)$ parametrizes the segment from $(0, -7)$ to $(7, 7)$.

SOLUTION Note that $2t + 1 = 0$ at $t = -1/2$, and $2t + 1 = 7$ at $t = 3$. Also, $4t - 5$ takes on the values of -7 and 7 at $t = -1/2$ and $t = 3$. Thus, the interval is $[-1/2, 3]$.

In Exercises 23–38, find parametric equations for the given curve.

23. $y = 9 - 4x$

SOLUTION This is a line through $P = (0, 9)$ with slope $m = -4$. Using the parametric representation of a line, as given in Example 3, we obtain $c(t) = (t, 9 - 4t)$.

24. $y = 8x^2 - 3x$

SOLUTION Letting $t = x$ yields the parametric representation $c(t) = (t, 8t^2 - 3t)$.

25. $4x - y^2 = 5$

SOLUTION We define the parameter $t = y$. Then, $x = \dfrac{5 + y^2}{4} = \dfrac{5 + t^2}{4}$, giving us the parametrization $c(t) = \left(\dfrac{5 + t^2}{4}, t\right)$.

26. $x^2 + y^2 = 49$

SOLUTION The curve $x^2 + y^2 = 49$ is a circle of radius 7 centered at the origin. We use the parametric representation of a circle to obtain the representation $c(t) = (7\cos t, 7\sin t)$.

27. $(x + 9)^2 + (y - 4)^2 = 49$

SOLUTION This is a circle of radius 7 centered at $(-9, 4)$. Using the parametric representation of a circle we get $c(t) = (-9 + 7\cos t, 4 + 7\sin t)$.

28. $\left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{12}\right)^2 = 1$

SOLUTION This is an ellipse centered at the origin with $a = 5$ and $b = 12$. Using the parametric representation of an ellipse we get $c(t) = (5\cos t, 12\sin t)$ for $-\pi \le t \le \pi$.

29. Line of slope 8 through $(-4, 9)$

SOLUTION Using the parametric representation of a line given in Example 3, we get the parametrization $c(t) = (-4 + t, 9 + 8t)$.

30. Line through $(2, 5)$ perpendicular to $y = 3x$

SOLUTION The line perpendicular to $y = 3x$ has slope $m = -\frac{1}{3}$. We use the parametric representation of a line given in Example 3 to obtain the parametrization $c(t) = (2 + t, 5 - \frac{1}{3}t)$.

31. Line through $(3, 1)$ and $(-5, 4)$

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (3, 1)$ and $Q = (c, d) = (-5, 4)$. Then $c(t) = (3 - 8t, 1 + 3t)$ for $-\infty < t < \infty$.

32. Line through $\left(\frac{1}{3}, \frac{1}{6}\right)$ and $\left(-\frac{7}{6}, \frac{5}{3}\right)$

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = \left(\frac{1}{3}, \frac{1}{6}\right)$ and $Q = (c, d) = \left(-\frac{7}{6}, \frac{5}{3}\right)$. Then

$$c(t) = \left(\frac{1}{3} - \frac{3}{2}t, \frac{1}{6} + \frac{3}{2}t\right)$$

for $-\infty < t < \infty$.

33. Segment joining $(1, 1)$ and $(2, 3)$

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (1, 1)$ and $Q = (c, d) = (2, 3)$. Then $c(t) = (1 + t, 1 + 2t)$; since we want only the segment joining the two points, we want $0 \le t \le 1$.

34. Segment joining $(-3, 0)$ and $(0, 4)$

SOLUTION We use the two-point parametrization of a line with $P = (a, b) = (-3, 0)$ and $Q = (c, d) = (0, 4)$. Then $c(t) = (-3 + 3t, 4t)$; since we want only the segment joining the two points, we want $0 \le t \le 1$.

35. Circle of radius 4 with center $(3, 9)$

SOLUTION Substituting $(a, b) = (3, 9)$ and $R = 4$ in the parametric equation of the circle we get $c(t) = (3 + 4\cos t, 9 + 4\sin t)$.

36. Ellipse of Exercise 28, with its center translated to $(7, 4)$

SOLUTION Since the center is translated by $(7, 4)$, so is every point. Thus the original parametrization becomes $c(t) = (7 + 5\cos t, 4 + 12\sin t)$ for $-\pi \le t \le \pi$.

37. $y = x^2$, translated so that the minimum occurs at $(-4, -8)$

SOLUTION We may parametrize $y = x^2$ by (t, t^2) for $-\infty < t < \infty$. The minimum of $y = x^2$ occurs at $(0, 0)$, so the desired curve is translated by $(-4, -8)$ from $y = x^2$. Thus a parametrization of the desired curve is $c(t) = (-4 + t, -8 + t^2)$.

38. $y = \cos x$ translated so that a maximum occurs at $(3, 5)$

SOLUTION A maximum value 1 of $y = \cos x$ occurs at $x = 0$. Hence, the curve $y - 4 = \cos(x - 3)$, or $y = 4 + \cos(x - 3)$ has a maximum at the point $(3, 5)$. We let $t = x - 3$, then $x = t + 3$ and $y = 4 + \cos t$. We obtain the representation $c(t) = (t + 3, 4 + \cos t)$.

In Exercises 39–42, find a parametrization $c(t)$ of the curve satisfying the given condition.

39. $y = 3x - 4, \quad c(0) = (2, 2)$

SOLUTION Let $x(t) = t + a$ and $y(t) = 3x - 4 = 3(t + a) - 4$. We want $x(0) = 2$, thus we must use $a = 2$. Our line is $c(t) = (x(t), y(t)) = (t + 2, 3(t + 2) - 4) = (t + 2, 3t + 2)$.

40. $y = 3x - 4, \quad c(3) = (2, 2)$

SOLUTION Let $x(t) = t + a$; since $x(3) = 2$ we have $2 = 3 + a$ so that $a = -1$. Then $y = 3x - 4 = 3(t - 1) - 4 = 3t - 7$, so that the line is $c(t) = (t - 1, 3t - 7)$ for $-\infty < t < \infty$.

41. $y = x^2, \quad c(0) = (3, 9)$

SOLUTION Let $x(t) = t + a$ and $y(t) = x^2 = (t + a)^2$. We want $x(0) = 3$, thus we must use $a = 3$. Our curve is $c(t) = (x(t), y(t)) = (t + 3, (t + 3)^2) = (t + 3, t^2 + 6t + 9)$.

42. $x^2 + y^2 = 4, \quad c(0) = (1, \sqrt{3})$

SOLUTION This is a circle of radius 2 centered at the origin, so we are looking for a parametrization of that circle that starts at a different point. Thus instead of the standard parametrization $(2\cos\theta, 2\sin\theta)$, $\theta = 0$ must correspond to some other angle ω. We choose the parametrization $(2\cos(\theta + \omega), 2\sin(\theta + \omega))$ and must determine the value of ω. Now,

$$x(0) = 1, \quad \text{so} \quad 1 = 2\cos(0 + \omega) = 2\cos\omega \quad \text{and} \quad \omega = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

Since

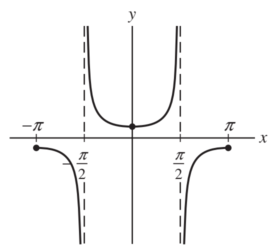
$$y(0) = \sqrt{3}, \quad \text{we have} \quad \sqrt{3} = 2\sin(0 + \omega) = 2\sin\omega \quad \text{and} \quad \omega = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

Comparing these results we see that we must have $\omega = \dfrac{\pi}{3}$ so that the parametrization is

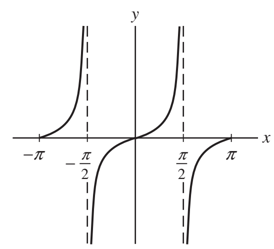
$$c(t) = \left(2\cos\left(\theta + \frac{\pi}{3}\right), 2\sin\left(\theta + \frac{\pi}{3}\right)\right)$$

43. Describe $c(t) = (\sec t, \tan t)$ for $0 \le t < \frac{\pi}{2}$ in the form $y = f(x)$. Specify the domain of x.

SOLUTION The function $x = \sec t$ has period 2π and $y = \tan t$ has period π. The graphs of these functions in the interval $-\pi \le t \le \pi$, are shown below:



$x = \sec t$

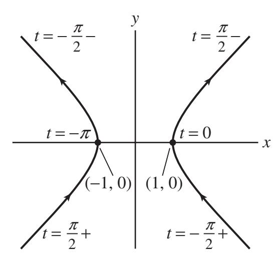


$y = \tan t$

$$x = \sec t \Rightarrow x^2 = \sec^2 t$$

$$y = \tan t \Rightarrow y^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{1 - \cos^2 t}{\cos^2 t} = \sec^2 t - 1 = x^2 - 1$$

Hence the graph of the curve is the hyperbola $x^2 - y^2 = 1$. The function $x = \sec t$ is an even function while $y = \tan t$ is odd. Also x has period 2π and y has period π. It follows that the intervals $-\pi \le t < -\frac{\pi}{2}$, $\frac{-\pi}{2} < t < \frac{\pi}{2}$ and $\frac{\pi}{2} < t < \pi$ trace the curve exactly once. The corresponding curve is shown next:



$c(t) = (\sec t, \tan t)$

44. Find a parametrization of the right branch ($x > 0$) of the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

using the functions $\cosh t$ and $\sinh t$. How can you parametrize the branch $x < 0$?

SOLUTION We show first that $x = \cosh t$, $y = \sinh t$ parametrizes the hyperbola when $a = b = 1$: then

$$x^2 - y^2 = (\cosh t)^2 - (\sinh t)^2 = 1.$$

using the identity $\cosh^2 - \sinh^2 = 1$. Generalize this parametrization to get a parametrization for the general hyperbola $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$:

$$x = a \cosh t, \ \ y = b \sinh t.$$

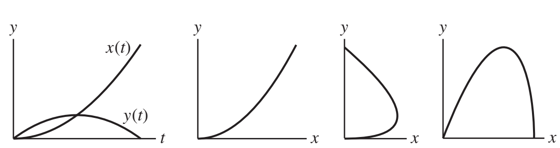
We must of course check that this parametrization indeed parametrizes the curve, i.e. that $x = a \cosh t$ and $y = b \sin t$ satisfy the equation $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \left(\frac{a \cosh t}{a}\right)^2 - \left(\frac{b \sinh t}{b}\right)^2 = (\cosh t)^2 - (\sinh t)^2 = 1.$$

The left branch of the hyperbola is the reflection of the right branch around the line $x = 0$, so it clearly has the parametrization

$$x = -a \cosh t, \ \ y = b \sinh t.$$

45. The graphs of $x(t)$ and $y(t)$ as functions of t are shown in Figure 15(A). Which of (I)–(III) is the plot of $c(t) = (x(t), y(t))$? Explain.



(A) (I) (II) (III)

FIGURE 15

SOLUTION As seen in Figure 15(A), the x-coordinate is an increasing function of t, while $y(t)$ is first increasing and then decreasing. In Figure I, x and y are both increasing or both decreasing (depending on the direction on the curve). In Figure II, x does not maintain one tendency, rather, it is decreasing and increasing for certain values of t. The plot $c(t) = (x(t), y(t))$ is plot III.

46. Which graph, (I) or (II), is the graph of $x(t)$ and which is the graph of $y(t)$ for the parametric curve in Figure 16(A)?

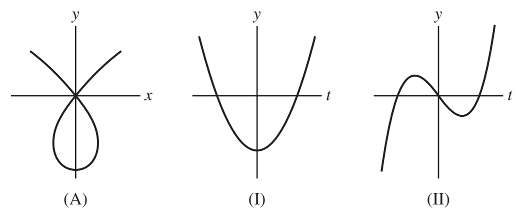
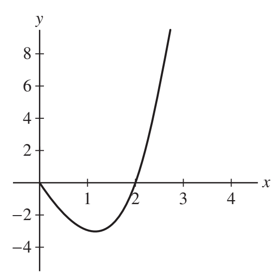


FIGURE 16

SOLUTION As indicated by Figure 16(A), the y-coordinate is decreasing and then increasing, so plot I is the graph of y. Figure 16(A) also shows that the x-coordinate is increasing, decreasing and then increasing, so plot II is the graph for x.

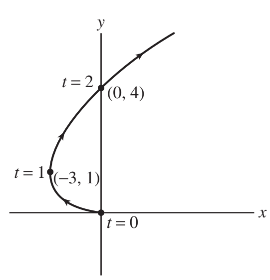
47. Sketch $c(t) = (t^3 - 4t, t^2)$ following the steps in Example 7.

SOLUTION We note that $x(t) = t^3 - 4t$ is odd and $y(t) = t^2$ is even, hence $c(-t) = (x(-t), y(-t)) = (-x(t), y(t))$. It follows that $c(-t)$ is the reflection of $c(t)$ across y-axis. That is, $c(-t)$ and $c(t)$ are symmetric with respect to the y-axis; thus, it suffices to graph the curve for $t \geq 0$. For $t = 0$, we have $c(0) = (0, 0)$ and the y-coordinate $y(t) = t^2$ tends to ∞ as $t \to \infty$. To analyze the x-coordinate, we graph $x(t) = t^3 - 4t$ for $t \geq 0$:

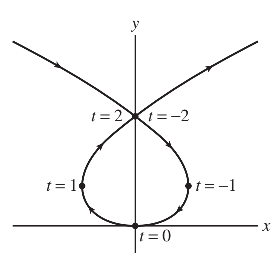


$x = t^3 - 4t$

We see that $x(t) < 0$ and decreasing for $0 < t < 2/\sqrt{3}$, $x(t) < 0$ and increasing for $2/\sqrt{3} < t < 2$ and $x(t) > 0$ and increasing for $t > 2$. Also $x(t)$ tends to ∞ as $t \to \infty$. Therefore, starting at the origin, the curve first directs to the left of the y-axis, then at $t = 2/\sqrt{3}$ it turns to the right, always keeping an upward direction. The part of the path for $t \leq 0$ is obtained by reflecting across the y-axis. We also use the points $c(0) = (0, 0)$, $c(1) = (-3, 1)$, $c(2) = (0, 4)$ to obtain the following graph for $c(t)$:



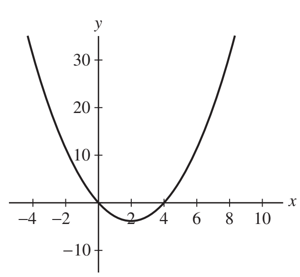
Graph of $c(t)$ for $t \geq 0$.



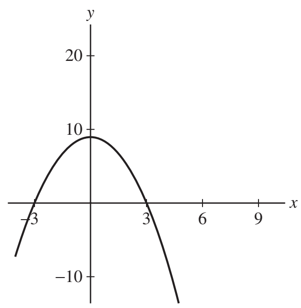
Graph of $c(t)$ for all t.

48. Sketch $c(t) = (t^2 - 4t, 9 - t^2)$ for $-4 \leq t \leq 10$.

SOLUTION The graphs of $x(t) = t^2 - 4t$ and $y(t) = 9 - t^2$ for $-4 \leq t \leq 10$ are shown in the following figures:



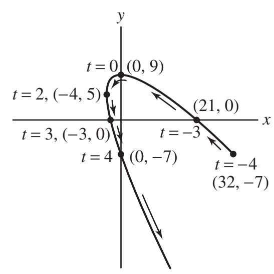
$x(t) = t^2 - 4t$



$y(t) = 9 - t^2$

The curve starts at $c(-4) = (32, -7)$. For $-4 < t < 0$, $x(t)$ is decreasing and $y(t)$ is increasing, so the graph turns to the left and upwards to $c(0) = (0, 9)$. Then for $0 < t < 2$, $x(t)$ is decreasing and so is $y(t)$, hence the graph turns to the left and downwards towards $c(2) = (-4, 5)$.

For $2 < t < 10$, $x(t)$ is increasing and $y(t)$ is decreasing, hence the graph turns to the right and downwards, ending at $c(10) = (60, -91)$. The intercept are the points where $t^2 - 4t = t(t-4) = 0$ or $9 - t^2 = 0$, that is $t = 0, 4, \pm 3$. These are the points $c(0) = (0, 9)$, $c(4) = (0, -7)$, $c(3) = (-3, 0)$, $c(-3) = (21, 0)$. These properties lead to the following path:



In Exercises 49–52, use Eq. (7) to find dy/dx at the given point.

49. $(t^3, t^2 - 1), \quad t = -4$

SOLUTION By Eq. (7) we have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(t^2-1)'}{(t^3)'} = \frac{2t}{3t^2} = \frac{2}{3t}$$

Substituting $t = -4$ we get

$$\frac{dy}{dx} = \frac{2}{3t}\bigg|_{t=-4} = \frac{2}{3 \cdot (-4)} = -\frac{1}{6}.$$

50. $(2t + 9, 7t - 9), \quad t = 1$

SOLUTION We find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{(7t-9)'}{(2t+9)'} = \frac{7}{2} \Rightarrow \frac{dy}{dx}\bigg|_{t=1} = \frac{7}{2}.$$

51. $(s^{-1} - 3s, s^3), \quad s = -1$

SOLUTION Using Eq. (7) we get

$$\frac{dy}{dx} = \frac{y'(s)}{x'(s)} = \frac{(s^3)'}{(s^{-1} - 3s)'} = \frac{3s^2}{-s^{-2} - 3} = \frac{3s^4}{-1 - 3s^2}$$

Substituting $s = -1$ we obtain

$$\frac{dy}{dx} = \frac{3s^4}{-1 - 3s^2}\bigg|_{s=-1} = \frac{3 \cdot (-1)^4}{-1 - 3 \cdot (-1)^2} = -\frac{3}{4}.$$

52. $(\sin 2\theta, \cos 3\theta), \quad \theta = \frac{\pi}{6}$

SOLUTION Using Eq. (7) we get

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-3\sin 3\theta}{2\cos 2\theta}$$

Substituting $\theta = \dfrac{\pi}{6}$ we get

$$\frac{dy}{dx} = \left.\frac{-3\sin 3\theta}{2\cos 2\theta}\right|_{\theta=\pi/6} = \frac{-3\sin\frac{\pi}{2}}{2\cos\frac{\pi}{3}} = \frac{-3}{2\cdot\frac{1}{2}} = -3$$

In Exercises 53–56, find an equation $y = f(x)$ for the parametric curve and compute dy/dx in two ways: using Eq. (7) and by differentiating $f(x)$.

53. $c(t) = (2t+1, 1-9t)$

SOLUTION Since $x = 2t+1$, we have $t = \dfrac{x-1}{2}$. Substituting in $y = 1 - 9t$ we have

$$y = 1 - 9\left(\frac{x-1}{2}\right) = -\frac{9}{2}x + \frac{11}{2}$$

Differentiating $y = -\dfrac{9}{2}x + \dfrac{11}{2}$ gives $\dfrac{dy}{dx} = -\dfrac{9}{2}$. We now find $\dfrac{dy}{dx}$ using Eq. (7):

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(1-9t)'}{(2t+1)'} = -\frac{9}{2}$$

54. $c(t) = \left(\frac{1}{2}t, \frac{1}{4}t^2 - t\right)$

SOLUTION Since $x = \frac{1}{2}t$ we have $t = 2x$. Substituting in $y = \frac{1}{4}t^2 - t$ yields

$$y = \frac{1}{4}(2x)^2 - 2x = x^2 - 2x.$$

We differentiate $y = x^2 - 2x$:

$$\frac{dy}{dx} = 2x - 2$$

Now, we find $\frac{dy}{dx}$ using Eq. (7). Thus,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\frac{1}{4}t^2 - t\right)'}{\left(\frac{1}{2}t\right)'} = \frac{\frac{1}{2}t - 1}{\frac{1}{2}} = t - 2.$$

Since $t = 2x$, then this $t - 2$ is the same as $2x - 2$.

55. $x = s^3, \quad y = s^6 + s^{-3}$

SOLUTION We find y as a function of x:

$$y = s^6 + s^{-3} = \left(s^3\right)^2 + \left(s^3\right)^{-1} = x^2 + x^{-1}.$$

We now differentiate $y = x^2 + x^{-1}$. This gives

$$\frac{dy}{dx} = 2x - x^{-2}.$$

Alternatively, we can use Eq. (7) to obtain the following derivative:

$$\frac{dy}{dx} = \frac{y'(s)}{x'(s)} = \frac{\left(s^6 + s^{-3}\right)'}{\left(s^3\right)'} = \frac{6s^5 - 3s^{-4}}{3s^2} = 2s^3 - s^{-6}.$$

Hence, since $x = s^3$,

$$\frac{dy}{dx} = 2x - x^{-2}.$$

56. $x = \cos\theta, \quad y = \cos\theta + \sin^2\theta$

SOLUTION To find y as a function of x, we first use the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$ to write

$$y = \cos\theta + 1 - \cos^2\theta.$$

We substitute $x = \cos\theta$ to obtain $y = x + 1 - x^2$. Differentiating this function yields

$$\frac{dy}{dx} = 1 - 2x.$$

Alternatively, we can compute $\frac{dy}{dx}$ using Eq. (7). That is,

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{\left(\cos\theta + \sin^2\theta\right)'}{(\cos\theta)'} = \frac{-\sin\theta + 2\sin\theta\cos\theta}{-\sin\theta} = 1 - 2\cos\theta.$$

Hence, since $x = \cos\theta$,

$$\frac{dy}{dx} = 1 - 2x.$$

57. Find the points on the curve $c(t) = (3t^2 - 2t, t^3 - 6t)$ where the tangent line has slope 3.

SOLUTION We solve

$$\frac{dy}{dx} = \frac{3t^2 - 6}{6t - 2} = 3$$

or $3t^2 - 6 = 18t - 6$, or $t^2 - 6t = 0$, so the slope is 3 at $t = 0, 6$ and the points are $(0, 0)$ and $(96, 180)$

58. Find the equation of the tangent line to the cycloid generated by a circle of radius 4 at $t = \frac{\pi}{2}$.

SOLUTION The cycloid generated by a circle of radius 4 can be parametrized by

$$c(t) = (4t - 4\sin t, 4 - 4\cos t)$$

Then we compute

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \left.\frac{4\sin t}{4 - 4\cos t}\right|_{t=\pi/2} = \frac{4}{4} = 1$$

so that the slope of the tangent line is 1 and the equation of the tangent line is

$$y - \left(4 - 4\cos\frac{\pi}{2}\right) = 1 \cdot \left(x - \left(4 \cdot \frac{\pi}{2} - 4\sin\frac{\pi}{2}\right)\right) \quad \text{or} \quad y = x + 8 - 2\pi$$

In Exercises 59–62, let $c(t) = (t^2 - 9, t^2 - 8t)$ (see Figure 17).

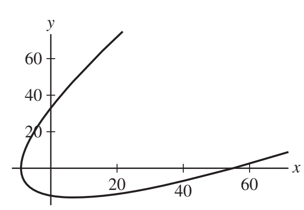
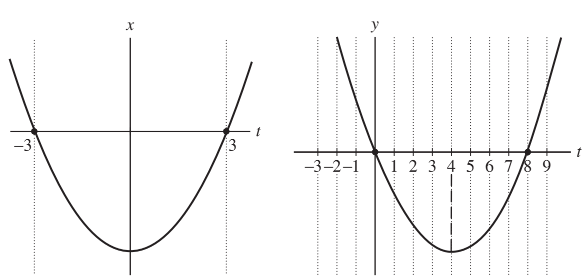


FIGURE 17 Plot of $c(t) = (t^2 - 9, t^2 - 8t)$.

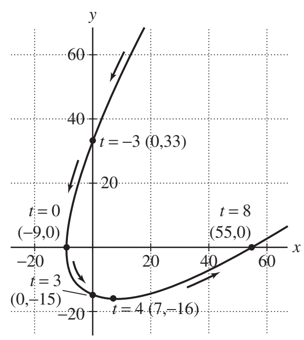
59. Draw an arrow indicating the direction of motion, and determine the interval of t-values corresponding to the portion of the curve in each of the four quadrants.

SOLUTION We plot the functions $x(t) = t^2 - 9$ and $y(t) = t^2 - 8t$:



$x = t^2 - 9$ $\qquad$ $y = t^2 - 8t$

We note carefully where each of these graphs are positive or negative, increasing or decreasing. In particular, $x(t)$ is decreasing for $t < 0$, increasing for $t > 0$, positive for $|t| > 3$, and negative for $|t| < 3$. Likewise, $y(t)$ is decreasing for $t < 4$, increasing for $t > 4$, positive for $t > 8$ or $t < 0$, and negative for $0 < t < 8$. We now draw arrows on the path following the decreasing/increasing behavior of the coordinates as indicated above. We obtain:



This plot also shows that:

- The graph is in the first quadrant for $t < -3$ or $t > 8$.
- The graph is in the second quadrant for $-3 < t < 0$.
- The graph is in the third quadrant for $0 < t < 3$.
- The graph is in the fourth quadrant for $3 < t < 8$.

60. Find the equation of the tangent line at $t = 4$.

SOLUTION Using the formula for the slope m of the tangent line we have:

$$m = \frac{dy}{dx}\bigg|_{t=4} = \frac{\left(t^2 - 8t\right)'}{\left(t^2 - 9\right)'}\bigg|_{t=4} = \frac{2t - 8}{2t}|_{t=4} = 1 - \frac{4}{t}|_{t=4} = 0.$$

Since the slope is zero, the tangent line is horizontal. The y-coordinate corresponding to $t = 4$ is $y = 4^2 - 8 \cdot 4 = -16$. Hence the equation of the tangent line is $y = -16$.

61. Find the points where the tangent has slope $\frac{1}{2}$.

SOLUTION The slope of the tangent at t is

$$\frac{dy}{dx} = \frac{\left(t^2 - 8t\right)'}{\left(t^2 - 9\right)'} = \frac{2t - 8}{2t} = 1 - \frac{4}{t}$$

The point where the tangent has slope $\frac{1}{2}$ corresponds to the value of t that satisfies

$$\frac{dy}{dx} = 1 - \frac{4}{t} = \frac{1}{2} \Rightarrow \frac{4}{t} = \frac{1}{2} \Rightarrow t = 8.$$

We substitute $t = 8$ in $x(t) = t^2 - 9$ and $y(t) = t^2 - 8t$ to obtain the following point:

$$\begin{aligned} x(8) &= 8^2 - 9 = 55 \\ y(8) &= 8^2 - 8 \cdot 8 = 0 \end{aligned} \quad \Rightarrow \quad (55, 0)$$

62. Find the points where the tangent is horizontal or vertical.

SOLUTION In Exercise 61 we found that the slope of the tangent at t is

$$\frac{dy}{dx} = 1 - \frac{4}{t} = \frac{t - 4}{t}$$

The tangent is horizontal where its slope is zero. We set the slope equal to zero and solve for t. This gives

$$\frac{t - 4}{t} = 0 \Rightarrow t = 4.$$

The corresponding point is

$$(x(4), y(4)) = (4^2 - 9, 4^2 - 8 \cdot 4) = (7, -16).$$

The tangent is vertical where it has infinite slope; that is, at $t = 0$. The corresponding point is

$$(x(0), y(0)) = (0^2 - 9, 0^2 - 8 \cdot 0) = (-9, 0).$$

63. Let A and B be the points where the ray of angle θ intersects the two concentric circles of radii $r < R$ centered at the origin (Figure 18). Let P be the point of intersection of the horizontal line through A and the vertical line through B. Express the coordinates of P as a function of θ and describe the curve traced by P for $0 \le \theta \le 2\pi$.

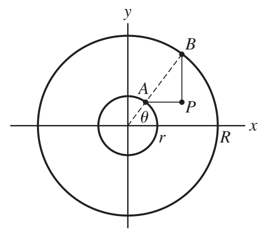


FIGURE 18

SOLUTION We use the parametric representation of a circle to determine the coordinates of the points A and B. That is,

$$A = (r\cos\theta, r\sin\theta), \; B = (R\cos\theta, R\sin\theta)$$

The coordinates of P are therefore

$$P = (R\cos\theta, r\sin\theta)$$

In order to identify the curve traced by P, we notice that the x and y coordinates of P satisfy $\frac{x}{R} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Hence

$$\left(\frac{x}{R}\right)^2 + \left(\frac{y}{r}\right)^2 = \cos^2\theta + \sin^2\theta = 1.$$

The equation

$$\left(\frac{x}{R}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

is the equation of ellipse. Hence, the coordinates of P, $(R\cos\theta, r\sin\theta)$ describe an ellipse for $0 \le \theta \le 2\pi$.

64. A 10-ft ladder slides down a wall as its bottom B is pulled away from the wall (Figure 19). Using the angle θ as parameter, find the parametric equations for the path followed by (a) the top of the ladder A, (b) the bottom of the ladder B, and (c) the point P located 4 ft from the top of the ladder. Show that P describes an ellipse.

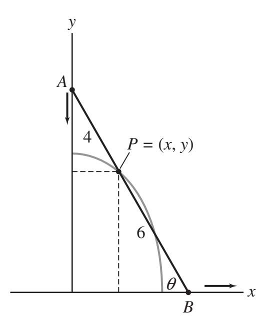
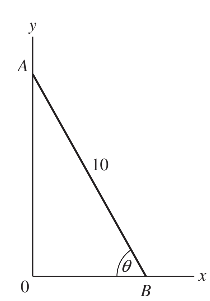


FIGURE 19

SOLUTION

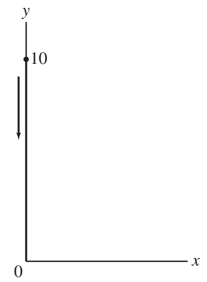
(a) We define the xy-coordinate system as shown in the figure:



As the ladder slides down the wall, the x-coordinate of A is always zero and the y-coordinate is $y = 10\sin\theta$. The parametric equations for the path followed by A are thus

$$x = 0, \; y = 10\sin\theta, \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$

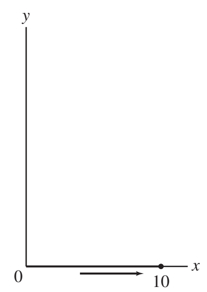
The path described by A is the segment [0, 10] on the y-axis.



(b) As the ladder slides down the wall, the y-coordinate of B is always zero and the x-coordinate is $x = 10\cos\theta$. The parametric equations for the path followed by B are therefore

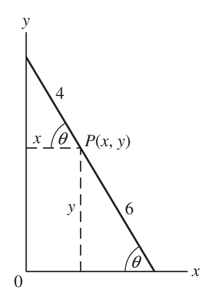
$$x = 10\cos\theta, \; y = 0, \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$

The path is the segment [0, 10] on the x-axis.

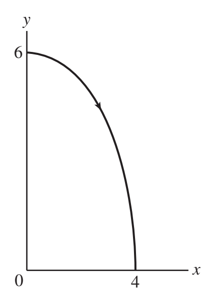


(c) The x and y coordinates of P are $x = 4\cos\theta$, $y = 6\sin\theta$. The path followed by P has the following parametrization:

$$c(\theta) = (4\cos\theta, 6\sin\theta), \quad \theta \text{ is between } \tfrac{\pi}{2} \text{ and } 0.$$



As shown in Example 4, the corresponding path is a part of an ellipse. Since θ is varying between $\frac{\pi}{2}$ and 0, we obtain the part of the ellipse in the first quadrant.



In Exercises 65–68, refer to the Bézier curve defined by Eqs. (8) and (9).

65. Show that the Bézier curve with control points

$$P_0 = (1, 4), \quad P_1 = (3, 12), \quad P_2 = (6, 15), \quad P_3 = (7, 4)$$

has parametrization

$$c(t) = (1 + 6t + 3t^2 - 3t^3, 4 + 24t - 15t^2 - 9t^3)$$

Verify that the slope at $t = 0$ is equal to the slope of the segment $\overline{P_0 P_1}$.

SOLUTION For the given Bézier curve we have $a_0 = 1, a_1 = 3, a_2 = 6, a_3 = 7$, and $b_0 = 4, b_1 = 12, b_2 = 15, b_3 = 4$. Substituting these values in Eq. (8)–(9) and simplifying gives

$$\begin{aligned} x(t) &= (1-t)^3 + 9t(1-t)^2 + 18t^2(1-t) + 7t^3 \\ &= 1 - 3t + 3t^2 - t^3 + 9t(1 - 2t + t^2) + 18t^2 - 18t^3 + 7t^3 \\ &= 1 - 3t + 3t^2 - t^3 + 9t - 18t^2 + 9t^3 + 18t^2 - 18t^3 + 7t^3 \\ &= -3t^3 + 3t^2 + 6t + 1 \\ y(t) &= 4(1-t)^3 + 36t(1-t)^2 + 45t^2(1-t) + 4t^3 \\ &= 4(1 - 3t + 3t^2 - t^3) + 36t(1 - 2t + t^2) + 45t^2 - 45t^3 + 4t^3 \\ &= 4 - 12t + 12t^2 - 4t^3 + 36t - 72t^2 + 36t^3 + 45t^2 - 45t^3 + 4t^3 \\ &= 4 + 24t - 15t^2 - 9t^3 \end{aligned}$$

Then

$$c(t) = (1 + 6t + 3t^2 - 3t^3, 4 + 24t - 15t^2 - 9t^3), \quad 0 \le t \le 1.$$

We find the slope at $t = 0$. Using the formula for slope of the tangent line we get

$$\frac{dy}{dx} = \frac{(4 + 24t - 15t^2 - 9t^3)'}{(1 + 6t + 3t^2 - 3t^3)'} = \frac{24 - 30t - 27t^2}{6 + 6t - 9t^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{24}{6} = 4.$$

The slope of the segment $\overline{P_0 P_1}$ is the slope of the line determined by the points $P_0 = (1, 4)$ and $P_1 = (3, 12)$. That is, $\frac{12-4}{3-1} = \frac{8}{2} = 4$. We see that the slope of the tangent line at $t = 0$ is equal to the slope of the segment $\overline{P_0 P_1}$, as expected.

66. Find an equation of the tangent line to the Bézier curve in Exercise 65 at $t = \frac{1}{3}$.

SOLUTION We have

$$\frac{dy}{dx} = \frac{y(t)'}{x(t)'} = \frac{24 - 30t - 27t^2}{6_6t - 9t^2}$$

so that at $t = \frac{1}{3}$,

$$\left.\frac{dy}{dx}\right|_{t=1/3} = \left.\frac{24 - 30t - 27t^2}{6 + 6t - 9t^2}\right|_{t=1/3} = \frac{11}{7}$$

and

$$x\left(\frac{1}{3}\right) = \frac{29}{9}, \qquad y\left(\frac{1}{3}\right) = 10$$

Thus the tangent line is

$$y - 10 = \frac{11}{7}\left(x - \frac{29}{9}\right) \qquad \text{or} \qquad y = \frac{11}{7}x + \frac{311}{63}$$

67. CAS Find and plot the Bézier curve $c(t)$ passing through the control points

$$P_0 = (3, 2), \quad P_1 = (0, 2), \quad P_2 = (5, 4), \quad P_3 = (2, 4)$$

SOLUTION Setting $a_0 = 3, a_1 = 0, a_2 = 5, a_3 = 2$, and $b_0 = 2, b_1 = 2, b_2 = 4, b_3 = 4$ into Eq. (8)–(9) and simplifying gives

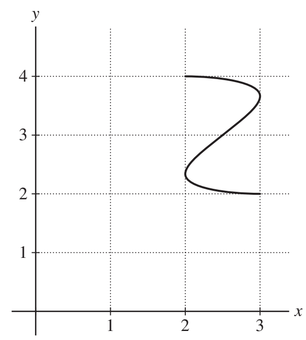
$$\begin{aligned} x(t) &= 3(1-t)^3 + 0 + 15t^2(1-t) + 2t^3 \\ &= 3(1 - 3t + 3t^2 - t^3) + 15t^2 - 15t^3 + 2t^3 = 3 - 9t + 24t^2 - 16t^3 \end{aligned}$$

$$\begin{aligned} y(t) &= 2(1-t)^3 + 6t(1-t)^2 + 12t^2(1-t) + 4t^3 \\ &= 2(1-3t+3t^2-t^3) + 6t(1-2t+t^2) + 12t^2 - 12t^3 + 4t^3 \\ &= 2 - 6t + 6t^2 - 2t^3 + 6t - 12t^2 + 6t^3 + 12t^2 - 12t^3 + 4t^3 = 2 + 6t^2 - 4t^3 \end{aligned}$$

We obtain the following equation

$$c(t) = (3 - 9t + 24t^2 - 16t^3, 2 + 6t^2 - 4t^3), \quad 0 \le t \le 1.$$

The graph of the Bézier curve is shown in the following figure:



68. Show that a cubic Bézier curve is tangent to the segment $\overline{P_2P_3}$ at P_3.

SOLUTION The equations of the cubic Bézier curve are

$$\begin{aligned} x(t) &= a_0(1-t)^3 + 3a_1t(1-t)^2 + 3a_2t^2(1-t) + a_3t^3 \\ y(t) &= b_0(1-t)^3 + 3b_1t(1-t)^2 + 3b_2t^2(1-t) + b_3t^3 \end{aligned}$$

We use the formula for the slope of the tangent line to find the slope of the tangent line at P_3. We obtain

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-3b_0(1-t)^2 + 3b_1((1-t)^2 - 2t(1-t)) + 3b_2(2t(1-t) - t^2) + 3b_3t^2}{-3a_0(1-t)^2 + 3a_1((1-t)^2 - 2t(1-t)) + 3a_2(2t(1-t) - t^2) + 3a_3t^2} \tag{1}$$

The slope of the tangent line at P_3 is obtained by setting $t = 1$ in (1). That is,

$$m_1 = \frac{0 + 0 - 3b_2 + 3b_3}{0 + 0 - 3a_2 + 3a_3} = \frac{b_3 - b_2}{a_3 - a_2} \tag{2}$$

We compute the slope of the segment $\overline{P_2P_3}$ for $P_2 = (a_2, b_2)$ and $P_3 = (a_3, b_3)$. We get

$$m_2 = \frac{b_3 - b_2}{a_3 - a_2}$$

Since the two slopes are equal, we conclude that the tangent line to the curve at the point P_3 is the segment $\overline{P_2P_3}$.

69. A bullet fired from a gun follows the trajectory

$$x = at, \qquad y = bt - 16t^2 \quad (a, b > 0)$$

Show that the bullet leaves the gun at an angle $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and lands at a distance $ab/16$ from the origin.

SOLUTION The height of the bullet equals the value of the y-coordinate. When the bullet leaves the gun, $y(t) = t(b - 16t) = 0$. The solutions to this equation are $t = 0$ and $t = \frac{b}{16}$, with $t = 0$ corresponding to the moment the bullet leaves the gun. We find the slope m of the tangent line at $t = 0$:

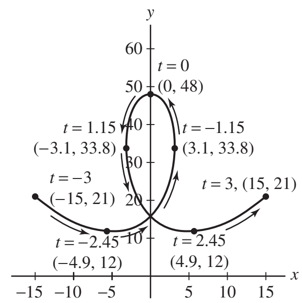
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{b - 32t}{a} \Rightarrow m = \left.\frac{b - 32t}{a}\right|_{t=0} = \frac{b}{a}$$

It follows that $\tan\theta = \frac{b}{a}$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. The bullet lands at $t = \frac{b}{16}$. We find the distance of the bullet from the origin at this time, by substituting $t = \frac{b}{16}$ in $x(t) = at$. This gives

$$x\left(\frac{b}{16}\right) = \frac{ab}{16}$$

70. CAS Plot $c(t) = (t^3 - 4t, t^4 - 12t^2 + 48)$ for $-3 \le t \le 3$. Find the points where the tangent line is horizontal or vertical.

SOLUTION The graph of $c(t) = (t^3 - 4t, t^4 - 12t^2 + 48)$, $-3 \le t \le 3$ is shown in the following figure:



We find the slope of the tangent line at t:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(x)} = \frac{\left(t^4 - 12t^2 + 48\right)'}{\left(t^3 - 4t\right)'} = \frac{4t^3 - 24t}{3t^2 - 4} \tag{1}$$

The tangent line is horizontal where $\frac{dy}{dx} = 0$. That is,

$$\frac{4t^3 - 24t}{3t^2 - 4} = 0 \Rightarrow 4t(t^2 - 6) = 0 \Rightarrow t = 0,\ t = -\sqrt{6},\ t = \sqrt{6}.$$

We find the corresponding points by substituting these values of t in $c(t)$. We obtain:

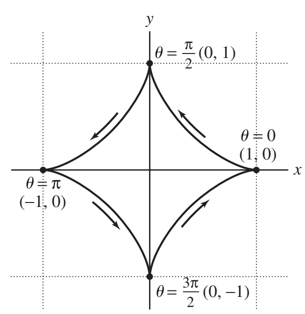
$$c(0) = (0, 48),\ c(-\sqrt{6}) \approx (-4.9, 12),\ c(\sqrt{6}) \approx (4.9, 12).$$

The tangent line is vertical where the slope in (1) is infinite, that is, where $3t^2 - 4 = 0$ or $t = \pm\frac{2}{\sqrt{3}} \approx \pm 1.15$. We find the points by setting $t = \pm\frac{2}{\sqrt{3}}$ in $c(t)$. We get

$$c\left(\frac{2}{\sqrt{3}}\right) \approx (-3.1, 33.8), \quad c\left(-\frac{2}{\sqrt{3}}\right) \approx (3.1, 33.8).$$

71. CAS Plot the astroid $x = \cos^3\theta$, $y = \sin^3\theta$ and find the equation of the tangent line at $\theta = \frac{\pi}{3}$.

SOLUTION The graph of the astroid $x = \cos^3\theta$, $y = \sin^3\theta$ is shown in the following figure:



The slope of the tangent line at $\theta = \frac{\pi}{3}$ is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \left.\frac{(\sin^3\theta)'}{(\cos^3\theta)'}\right|_{\theta=\pi/3} = \left.\frac{3\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)}\right|_{\theta=\pi/3} = -\left.\frac{\sin\theta}{\cos\theta}\right|_{\theta=\pi/3} = -\tan\theta\Big|_{\pi/3} = -\sqrt{3}$$

We find the point of tangency:

$$\left(x\left(\frac{\pi}{3}\right), y\left(\frac{\pi}{3}\right)\right) = \left(\cos^3\frac{\pi}{3}, \sin^3\frac{\pi}{3}\right) = \left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$$

The equation of the tangent line at $\theta = \frac{\pi}{3}$ is, thus,

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right) \Rightarrow y = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

72. Find the equation of the tangent line at $t = \frac{\pi}{4}$ to the cycloid generated by the unit circle with parametric equation (5).

SOLUTION We find the equation of the tangent line at $t = \frac{\pi}{4}$ to the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We first find the derivative $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t}$$

The slope of the tangent line at $t = \frac{\pi}{4}$ is therefore:

$$m = \left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sin\frac{\pi}{4}}{1-\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}-1}$$

We find the point of tangency:

$$\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4} - \sin\frac{\pi}{4}, 1-\cos\frac{\pi}{4}\right) = \left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$$

The equation of the tangent line is, thus,

$$y - \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{2}-1}\left(x - \left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)\right) \Rightarrow y = \frac{1}{\sqrt{2}-1}x + \left(2 - \frac{\frac{\pi}{4}}{\sqrt{2}-1}\right)$$

73. Find the points with horizontal tangent line on the cycloid with parametric equation (5).

SOLUTION The parametric equations of the cycloid are

$$x = t - \sin t, \quad y = 1 - \cos t$$

We find the slope of the tangent line at t:

$$\frac{dy}{dx} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t}$$

The tangent line is horizontal where it has slope zero. That is,

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t} = 0 \quad \Rightarrow \quad \begin{array}{c}\sin t = 0\\ \cos t \neq 1\end{array} \quad \Rightarrow \quad t = (2k-1)\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

We find the coordinates of the points with horizontal tangent line, by substituting $t = (2k-1)\pi$ in $x(t)$ and $y(t)$. This gives

$$x = (2k-1)\pi - \sin(2k-1)\pi = (2k-1)\pi$$
$$y = 1 - \cos((2k-1)\pi) = 1 - (-1) = 2$$

The required points are

$$((2k-1)\pi, 2), \quad k = 0, \pm 1, \pm 2, \ldots$$

74. Property of the Cycloid Prove that the tangent line at a point P on the cycloid always passes through the top point on the rolling circle as indicated in Figure 20. Assume the generating circle of the cycloid has radius 1.

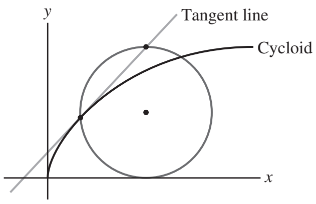


FIGURE 20

SOLUTION The definition of the cycloid is such that at time t, the top of the circle has coordinates $Q = (t, 2)$ (since at time $t = 2\pi$ the circle has rotated exactly once, and its circumference is 2π). Let L be the line through P and Q. To show that L is tangent to the cycloid at P it suffices to show that the slope of L equals the slope of the tangent at P. Recall that the cycloid is parametrized by $c(t) = (t - \sin t, 1 - \cos t)$. Then the slope of L is

$$\frac{2-(1-\cos t)}{t-(t-\sin t)} = \frac{1+\cos t}{\sin t}$$

and the slope of the tangent line is

$$\frac{y'(t)}{x'(t)} = \frac{(1-\cos t)'}{(t-\sin t)'} = \frac{\sin t}{1-\cos t} = \frac{\sin t(1+\cos t)}{1-\cos^2 t} = \frac{\sin t(1+\cos t)}{\sin^2 t} = \frac{1+\cos t}{\sin t}$$

and the two are equal.

75. A *curtate cycloid* (Figure 21) is the curve traced by a point at a distance h from the center of a circle of radius R rolling along the x-axis where $h < R$. Show that this curve has parametric equations $x = Rt - h\sin t$, $y = R - h\cos t$.

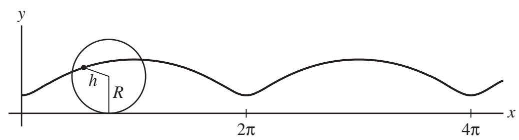
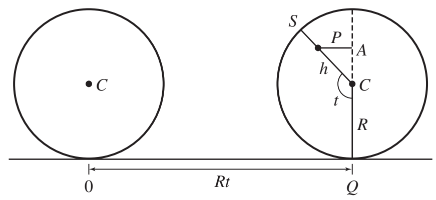


FIGURE 21 Curtate cycloid.

SOLUTION Let P be a point at a distance h from the center C of the circle. Assume that at $t = 0$, the line of CP is passing through the origin. When the circle rolls a distance Rt along the x-axis, the length of the arc $\widehat{SQ}$ (see figure) is also Rt and the angle $\angle SCQ$ has radian measure t. We compute the coordinates x and y of P.



$$x = Rt - \overline{PA} = Rt - h\sin(\pi - t) = Rt - h\sin t$$

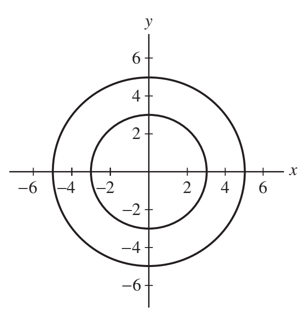
$$y = R + \overline{AC} = R + h\cos(\pi - t) = R - h\cos t$$

We obtain the following parametrization:

$$x = Rt - h\sin t, \;\; y = R - h\cos t.$$

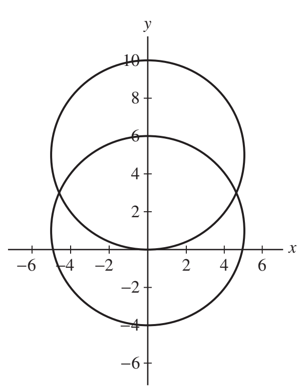
76. CAS Use a computer algebra system to explore what happens when $h > R$ in the parametric equations of Exercise 75. Describe the result.

SOLUTION Look first at the parametric equations $x = -h\sin t$, $y = -h\cos t$. These describe a circle of radius h. See for instance the graphs below obtained for $h = 3$ and $h = 5$.



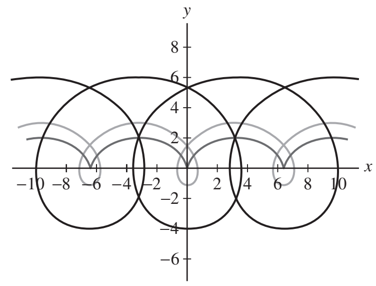
$c(t) = (-h^*\sin(t), -h^*\cos(t))$ $h = 3, 5$

Adding R to the y coordinate to obtain the parametric equations $x = -h\sin t$, $y = R - h\cos t$, yields a circle with its center moved up by R units:



$c(t) = (-h^*\sin(t), R-h^*\cos(t))$ $R = 1, 5$ $h = 5$

Now, we add Rt to the x coordinate to obtain the given parametric equation; the curve becomes a spring. The figure below shows the graphs obtained for $R = 1$ and various values of h. We see the inner loop formed for $h > R$.



77. Show that the line of slope t through $(-1, 0)$ intersects the unit circle in the point with coordinates

$$x = \frac{1 - t^2}{t^2 + 1}, \qquad y = \frac{2t}{t^2 + 1}$$ **10**

Conclude that these equations parametrize the unit circle with the point $(-1, 0)$ excluded (Figure 22). Show further that $t = y/(x + 1)$.

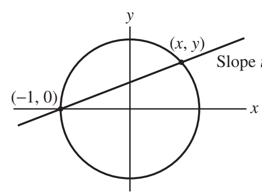


FIGURE 22 Unit circle.

SOLUTION The equation of the line of slope t through $(-1, 0)$ is $y = t(x + 1)$. The equation of the unit circle is $x^2 + y^2 = 1$. Hence, the line intersects the unit circle at the points (x, y) that satisfy the equations:

$$y = t(x + 1) \tag{1}$$

$$x^2 + y^2 = 1 \tag{2}$$

Substituting y from equation (1) into equation (2) and solving for x we obtain

$$\begin{aligned} x^2 + t^2(x + 1)^2 &= 1 \\ x^2 + t^2x^2 + 2t^2x + t^2 &= 1 \\ (1 + t^2)x^2 + 2t^2x + (t^2 - 1) &= 0 \end{aligned}$$

This gives

$$x_{1,2} = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2 + 1)(t^2 - 1)}}{2(1 + t^2)} = \frac{-2t^2 \pm 2}{2(1 + t^2)} = \frac{\pm 1 - t^2}{1 + t^2}$$

So $x_1 = -1$ and $x_2 = \dfrac{1 - t^2}{t^2 + 1}$. The solution $x = -1$ corresponds to the point $(-1, 0)$. We are interested in the second point of intersection that is varying as t varies. Hence the appropriate solution is

$$x = \frac{1 - t^2}{t^2 + 1}$$

We find the y-coordinate by substituting x in equation (1). This gives

$$y = t(x + 1) = t\left(\frac{1 - t^2}{t^2 + 1} + 1\right) = t \cdot \frac{1 - t^2 + t^2 + 1}{t^2 + 1} = \frac{2t}{t^2 + 1}$$

We conclude that the line and the unit circle intersect, besides at $(-1, 0)$, at the point with the following coordinates:

$$x = \frac{1 - t^2}{t^2 + 1}, \quad y = \frac{2t}{t^2 + 1} \tag{3}$$

Since these points determine all the points on the unit circle except for $(-1, 0)$ and no other points, the equations in (3) parametrize the unit circle with the point $(-1, 0)$ excluded.

We show that $t = \frac{y}{x+1}$. Using (3) we have

$$\frac{y}{x+1} = \frac{\frac{2t}{t^2+1}}{\frac{1-t^2}{t^2+1}+1} = \frac{\frac{2t}{t^2+1}}{\frac{1-t^2+t^2+1}{t^2+1}} = \frac{\frac{2t}{t^2+1}}{\frac{2}{t^2+1}} = \frac{2t}{2} = t.$$

78. The **folium of Descartes** is the curve with equation $x^3 + y^3 = 3axy$, where $a \neq 0$ is a constant (Figure 23).

(a) Show that the line $y = tx$ intersects the folium at the origin and at one other point P for all $t \neq -1, 0$. Express the coordinates of P in terms of t to obtain a parametrization of the folium. Indicate the direction of the parametrization on the graph.

(b) Describe the interval of t-values parametrizing the parts of the curve in quadrants I, II, and IV. Note that $t = -1$ is a point of discontinuity of the parametrization.

(c) Calculate dy/dx as a function of t and find the points with horizontal or vertical tangent.

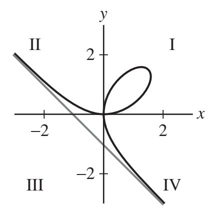


FIGURE 23 Folium $x^3 + y^3 = 3axy$.

SOLUTION

(a) We find the points where the line $y = tx$ $(t \neq -1, 0)$ and the folium intersect, by solving the following equations:

$$y = tx \tag{1}$$
$$x^3 + y^3 = 3axy \tag{2}$$

Substituting y from (1) in (2) and solving for x we get

$$x^3 + t^3x^3 = 3axtx$$
$$(1 + t^3)x^3 - 3atx^2 = 0$$
$$x^2(x(1 + t^3) - 3at) = 0 \Rightarrow x_1 = 0,\ x_2 = \frac{3at}{1+t^3}$$

Substituting in (1) we find the corresponding y-coordinates. That is,

$$y_1 = t \cdot 0 = 0,\ y_2 = t \cdot \frac{3at}{1+t^3} = \frac{3at^2}{1+t^3}$$

We conclude that the line $y = \text{tx}$, $t \neq 0, -1$ intersects the folium in a unique point P besides the origin. The coordinates of P are:

$$x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3},\quad t \neq 0, -1$$

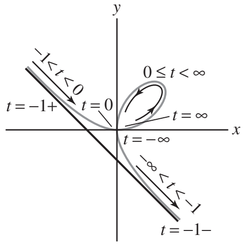
The coordinates of P determine a parametrization for the folium. We add the origin so $t = 0$ must be included in the interval of t. We get

$$c(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right),\quad t \neq -1$$

To indicate the direction on the curve (for $a > 0$), we first consider the following limits:

$$\lim_{t\to -1-} x(t) = \infty \qquad \lim_{t\to -1-} y(t) = -\infty$$
$$\lim_{t\to -\infty} x(t) = \lim_{t\to\infty} x(t) = 0 \qquad \lim_{t\to -\infty} y(t) = \lim_{t\to\infty} y(t) = 0$$
$$\lim_{t\to -1+} x(t) = -\infty \qquad \lim_{t\to -1+} y(t) = \infty$$
$$\lim_{t\to 0} x(t) = 0 \qquad \lim_{t\to 0} y(t) = 0$$

These limits determine the directions of the two parts of the folium in the second and fourth quadrant. The loop in the first quadrant, corresponds to the values $0 \le t < \infty$, and it is directed from $c(1) = (\frac{3a}{2}, \frac{3a}{2})$ to $c(2) = (\frac{2a}{3}, \frac{4a}{3})$ where $t = 1$ and $t = 2$ are two chosen values in the interval $0 \le t < \infty$. The following graph shows the directed folium:



(b) The limits computed in part (a) indicate that the parts of the curve in the second and fourth quadrants correspond to the values $-1 < t < 0$ and $-\infty < t < -1$ respectively. The loop in the first quadrant corresponds to the remaining interval $0 \le t < \infty$.

(c) We find the derivative $\frac{dy}{dx}$, using the Formula for the Slope of the Tangent Line. We get

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\frac{3at^2}{1+t^3}\right)'}{\left(\frac{3at}{1+t^3}\right)'} = \frac{\frac{6at(1+t^3)-3at^2\cdot 3t^2}{(1+t^3)^2}}{\frac{3a(1+t^3)-3at\cdot 3t^2}{(1+t^3)^2}} = \frac{6at - 3at^4}{3a - 6at^3} = \frac{t(2-t^3)}{1-2t^3}$$

Horizontal tangent occurs when $\frac{dy}{dx} = 0$. That is,

$$\frac{t(2-t^3)}{1-2t^3} = 0 \Rightarrow t(2-t^3) = 0,\ 1 - 2t^3 \ne 0 \Rightarrow t = 0,\ t = \sqrt[3]{2}.$$

The corresponding points are:

$$c(0) = (x(0), y(0)) = (0, 0)$$

$$c\left(\sqrt[3]{2}\right) = \left(x\left(\sqrt[3]{2}\right), y\left(\sqrt[3]{2}\right)\right) = \left(\frac{3a\sqrt[3]{2}}{1+2}, \frac{3a\sqrt[3]{4}}{1+2}\right) = \left(a\sqrt[3]{2}, a\sqrt[3]{4}\right)$$

Vertical tangent line occurs when $\frac{dy}{dx}$ is infinite. That is,

$$1 - 2t^3 = 0 \Rightarrow t = \frac{1}{\sqrt[3]{2}}$$

The corresponding point is

$$c\left(\frac{1}{\sqrt[3]{2}}\right) = \left(x\left(\frac{1}{\sqrt[3]{2}}\right), y\left(\frac{1}{\sqrt[3]{2}}\right)\right) = \left(\frac{\frac{3a}{\sqrt[3]{2}}}{1+\frac{1}{2}}, \frac{\frac{3a}{\sqrt[3]{4}}}{1+\frac{1}{2}}\right) = \left(\sqrt[3]{4}a, \sqrt[3]{2}a\right).$$

79. Use the results of Exercise 78 to show that the asymptote of the folium is the line $x + y = -a$. *Hint:* Show that $\lim_{t\to -1}(x + y) = -a$.

SOLUTION We must show that as $x \to \infty$ or $x \to -\infty$ the graph of the folium is getting arbitrarily close to the line $x + y = -a$, and the derivative $\frac{dy}{dx}$ is approaching the slope -1 of the line.

In Exercise 78 we showed that $x \to \infty$ when $t \to (-1^-)$ and $x \to -\infty$ when $t \to (-1^+)$. We first show that the graph is approaching the line $x + y = -a$ as $x \to \infty$ or $x \to -\infty$, by showing that $\lim_{t\to -1-} x + y = \lim_{t\to -1+} x + y = -a$.

For $x(t) = \dfrac{3at}{1+t^3}$, $y(t) = \dfrac{3at^2}{1+t^3}$, $a > 0$, calculated in Exercise 78, we obtain using L'Hôpital's Rule:

$$\lim_{t\to -1-}(x+y) = \lim_{t\to -1-}\frac{3at + 3at^2}{1+t^3} = \lim_{t\to -1-}\frac{3a + 6at}{3t^2} = \frac{3a - 6a}{3} = -a$$

$$\lim_{t\to -1+}(x+y) = \lim_{t\to -1+}\frac{3at + 3at^2}{1+t^3} = \lim_{t\to -1+}\frac{3a + 6at}{3t^2} = \frac{3a - 6a}{3} = -a$$

We now show that $\dfrac{dy}{dx}$ is approaching -1 as $t \to -1-$ and as $t \to -1+$. We use $\dfrac{dy}{dx} = \dfrac{6at - 3at^4}{3a - 6at^3}$ computed in Exercise 78 to obtain

$$\lim_{t\to -1-}\frac{dy}{dx} = \lim_{t\to -1-}\frac{6at - 3at^4}{3a - 6at^3} = \frac{-9a}{9a} = -1$$

$$\lim_{t\to -1+} \frac{dy}{dx} = \lim_{t\to -1+} \frac{6at - 3at^4}{3a - 6at^3} = \frac{-9a}{9a} = -1$$

We conclude that the line $x + y = -a$ is an asymptote of the folium as $x \to \infty$ and as $x \to -\infty$.

80. Find a parametrization of $x^{2n+1} + y^{2n+1} = ax^n y^n$, where a and n are constants.

SOLUTION Following the method in Exercise 78, we first find the coordinates of the point P where the curve and the line $y = tx$ intersect. We solve the following equations:

$$y = tx$$
$$x^{2n+1} + y^{2n+1} = ax^n y^n$$

Substituting $y = tx$ in the second equation and solving for x yields

$$x^{2n+1} + t^{2n+1}x^{2n+1} = ax^n \cdot t^n x^n$$
$$(1 + t^{2n+1})x^{2n+1} - at^n x^{2n} = 0$$
$$x^{2n}((1 + t^{2n+1})x - at^n) = 0 \Rightarrow x = 0,\ x = \frac{at^n}{1 + t^{2n+1}}$$

We assume that $t \neq -1$ (so $1 + t^{2n+1} \neq 0$) and obtain one solution besides the origin. The corresponding y coordinates are

$$y = tx = t \cdot \frac{at^n}{1 + t^{2n+1}} = \frac{at^{n+1}}{1 + t^{2n+1}}$$

Hence, the points $x = \dfrac{at^n}{1 + t^{2n+1}}$, $y = \dfrac{at^{n+1}}{1 + t^{2n+1}}$, $t \neq -1$, are exactly the points on the curve. We obtain the following parametrization:

$$x = \frac{at^n}{1 + t^{2n+1}},\ y = \frac{at^{n+1}}{1 + t^{2n+1}}, \quad t \neq -1.$$

81. Second Derivative for a Parametrized Curve Given a parametrized curve $c(t) = (x(t), y(t))$, show that

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^2}$$

Use this to prove the formula

$$\boxed{\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}} \qquad \boxed{11}$$

SOLUTION By the formula for the slope of the tangent line we have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Differentiating with respect to t, using the Quotient Rule, gives

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^2}$$

By the Chain Rule we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

Substituting into the above equation $\left(\text{and using } \dfrac{dt}{dx} = \dfrac{1}{dx/dt} = \dfrac{1}{x'(t)}\right)$ gives

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^2} \cdot \frac{1}{x'(t)} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}$$

82. The second derivative of $y = x^2$ is $dy^2/d^2x = 2$. Verify that Eq. (11) applied to $c(t) = (t, t^2)$ yields $dy^2/d^2x = 2$. In fact, any parametrization may be used. Check that $c(t) = (t^3, t^6)$ and $c(t) = (\tan t, \tan^2 t)$ also yield $dy^2/d^2x = 2$.

SOLUTION For the parametrization $c(t) = (t, t^2)$, we have

$$x'(t) = 1, \quad x''(t) = 0, \quad y'(t) = 2t, \quad y''(t) = 2$$

so that indeed

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} = \frac{1 \cdot 2 - 2t \cdot 0}{1^3} = 2$$

For $c(t) = (t^3, t^6)$, we have

$$x'(t) = 3t^2, \quad x''(t) = 6t, \quad y'(t) = 6t^5, \quad y''(t) = 30t^4$$

so that again

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} = \frac{3t^2 \cdot 30t^4 - 6t^5 \cdot 6t}{(3t^2)^3} = \frac{54t^6}{27t^6} = 2$$

Finally, for $c(t) = (\tan t, \tan^2 t)$,

$$x'(t) = \sec^2 t, \quad x''(t) = 2\tan t \sec^2 t, \quad y'(t) = 2\tan t \sec^2 t, \quad y''(t) = 6\sec^4 t - 4\sec^2 t$$

and

$$\begin{aligned}\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} &= \frac{\sec^2 t(6\sec^4 t - 4\sec^2 t) - 2\tan t \sec^2 t(2\tan t \sec^2 t)}{\sec^6 t} \\ &= \frac{6\sec^6 t - 4\sec^4 t - 4\sec^4 t \tan^2 t}{\sec^6 t} = \frac{6\sec^6 t - 4\sec^4 t(1 - (1 + \sec^2 t)))}{\sec^6 t} \\ &= \frac{2\sec^6 t}{\sec^6 t} = 2\end{aligned}$$

In Exercises 83–86, use Eq. (11) to find d^2y/dx^2.

83. $x = t^3 + t^2, \quad y = 7t^2 - 4, \quad t = 2$

SOLUTION We find the first and second derivatives of $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 3t^2 + 2t \Rightarrow x'(2) = 3 \cdot 2^2 + 2 \cdot 2 = 16 \\ x''(t) &= 6t + 2 \quad \Rightarrow x''(2) = 6 \cdot 2 + 2 = 14 \\ y'(t) &= 14t \qquad \Rightarrow y'(2) = 14 \cdot 2 = 28 \\ y''(t) &= 14 \qquad \Rightarrow y''(2) = 14 \end{aligned}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \left.\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}\right|_{t=2} = \frac{16 \cdot 14 - 28 \cdot 14}{16^3} = \frac{-21}{512}$$

84. $x = s^{-1} + s, \quad y = 4 - s^{-2}, \quad s = 1$

SOLUTION Since $x'(s) = -s^{-2} + 1 = 1 - \frac{1}{s^2}$, we have $x'(1) = 0$. Hence, Eq. (11) cannot be used to compute $\frac{d^2y}{dx^2}$ at $s = 1$.

85. $x = 8t + 9, \quad y = 1 - 4t, \quad t = -3$

SOLUTION We compute the first and second derivatives of $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 8 \quad \Rightarrow x'(-3) = 8 \\ x''(t) &= 0 \quad \Rightarrow x''(-3) = 0 \\ y'(t) &= -4 \Rightarrow y'(-3) = -4 \\ y''(t) &= 0 \quad \Rightarrow y''(-3) = 0 \end{aligned}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{t=-3} = \frac{x'(-3)y''(-3) - y'(-3)x''(-3)}{x'(-3)^3} = \frac{8 \cdot 0 - (-4) \cdot 0}{8^3} = 0$$

86. $x = \cos\theta, \quad y = \sin\theta, \quad \theta = \frac{\pi}{4}$

SOLUTION We find the first and second derivatives of $x(\theta)$ and $y(\theta)$:

$$x'(\theta) = -\sin\theta \Rightarrow x'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$x''(\theta) = -\cos\theta \Rightarrow x''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$y'(\theta) = \cos\theta \quad \Rightarrow y'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$y''(\theta) = -\sin\theta \Rightarrow y''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Using Eq. (11) we get

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{x'\left(\frac{\pi}{4}\right)y''\left(\frac{\pi}{4}\right) - y'\left(\frac{\pi}{4}\right)x''\left(\frac{\pi}{4}\right)}{\left(x'\left(\frac{\pi}{4}\right)\right)^3} = \frac{\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)^3} = -2\sqrt{2}$$

87. Use Eq. (11) to find the t-intervals on which $c(t) = (t^2, t^3 - 4t)$ is concave up.

SOLUTION The curve is concave up where $\frac{d^2y}{dx^2} > 0$. Thus,

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} > 0 \tag{1}$$

We compute the first and second derivatives:

$$x'(t) = 2t, \qquad x''(t) = 2$$

$$y'(t) = 3t^2 - 4, \quad y''(t) = 6t$$

Substituting in (1) and solving for t gives

$$\frac{12t^2 - (6t^2 - 8)}{8t^3} = \frac{6t^2 + 8}{8t^3}$$

Since $6t^2 + 8 > 0$ for all t, the quotient is positive if $8t^3 > 0$. We conclude that the curve is concave up for $t > 0$.

88. Use Eq. (11) to find the t-intervals on which $c(t) = (t^2, t^4 - 4t)$ is concave up.

SOLUTION The curve is concave up where $\frac{d^2y}{dx^2} > 0$. That is,

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} > 0 \tag{1}$$

We compute the first and second derivatives:

$$x'(t) = 2t, \qquad x''(t) = 2$$

$$y'(t) = 4t^3 - 4, \quad y''(t) = 12t^2$$

Substituting in (1) and solving for t gives

$$\frac{24t^3 - (8t^3 - 8)}{8t^3} = \frac{16t^3 + 8}{8t^3} = 1 + \frac{1}{2t^3}$$

This is clearly positive for $t > 0$. For $t < 0$, we want $1 + \frac{1}{2t^3} > 0$, which means $\frac{1}{2t^3} > -1$, so $2t^3 < -1$ (by taking the reciprocal of both sides), so $t < -\frac{1}{\sqrt[3]{2}}$. Thus, we see that our curve is concave up for $t < -\frac{1}{\sqrt[3]{2}}$ and for $t > 0$.

89. Area Under a Parametrized Curve Let $c(t) = (x(t), y(t))$, where $y(t) > 0$ and $x'(t) > 0$ (Figure 24). Show that the area A under $c(t)$ for $t_0 \le t \le t_1$ is

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt \qquad \boxed{12}$$

Hint: Because it is increasing, the function $x(t)$ has an inverse $t = g(x)$ and $c(t)$ is the graph of $y = y(g(x))$. Apply the change-of-variables formula to $A = \int_{x(t_0)}^{x(t_1)} y(g(x))\,dx$.

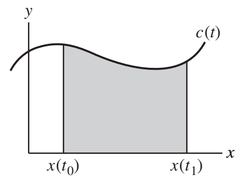


FIGURE 24

SOLUTION Let $x_0 = x(t_0)$ and $x_1 = x(t_1)$. We are given that $x'(t) > 0$, hence $x = x(t)$ is an increasing function of t, so it has an inverse function $t = g(x)$. The area A is given by $\int_{x_0}^{x_1} y(g(x))\,dx$. Recall that y is a function of t and $t = g(x)$, so the height y at any point x is given by $y = y(g(x))$. We find the new limits of integration. Since $x_0 = x(t_0)$ and $x_1 = x(t_1)$, the limits for t are t_0 and t_1, respectively. Also since $x'(t) = \frac{dx}{dt}$, we have $dx = x'(t)dt$. Performing this substitution gives

$$A = \int_{x_0}^{x_1} y(g(x))\,dx = \int_{t_0}^{t_1} y(g(x))x'(t)\,dt.$$

Since $g(x) = t$, we have $A = \displaystyle\int_{t_0}^{t_1} y(t)x'(t)\,dt$.

90. Calculate the area under $y = x^2$ over $[0, 1]$ using Eq. (12) with the parametrizations (t^3, t^6) and (t^2, t^4).

SOLUTION The area A under $y = x^2$ on $[0, 1]$ is given by the integral

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt$$

where $x(t_0) = 0$ and $x(t_1) = 1$. We first use the parametrization (t^3, t^6). We have $x(t) = t^3$, $y(t) = t^6$. Hence,

$$0 = x(t_0) = t_0^3 \Rightarrow t_0 = 0$$
$$1 = x(t_1) = t_1^3 \Rightarrow t_1 = 1$$

Also $x'(t) = 3t^2$. Substituting these values in Eq. (12) we obtain

$$A = \int_0^1 t^6 \cdot 3t^2\,dt = \int_0^1 3t^8\,dt = \frac{3}{9}t^9\Big|_0^1 = \frac{3}{9} = \frac{1}{3}$$

Using the parametrization $x(t) = t^2$, $y(t) = t^4$, we have $x'(t) = 2t$. We find t_0 and t_1:

$$0 = x(t_0) = t_0^2 \Rightarrow t_0 = 0$$
$$1 = x(t_1) = t_1^2 \Rightarrow t_1 = 1 \quad \text{or} \quad t_1 = -1.$$

Equation (12) is valid if $x'(t) > 0$, that is if $t > 0$. Hence we choose the positive value, $t_1 = 1$. We now use Eq. (12) to obtain

$$A = \int_0^1 t^4 \cdot 2t\,dt = \int_0^1 2t^5\,dt = \frac{2}{6}t^6\Big|_0^1 = \frac{2}{6} = \frac{1}{3}$$

Both answers agree, as expected.

91. What does Eq. (12) say if $c(t) = (t, f(t))$?

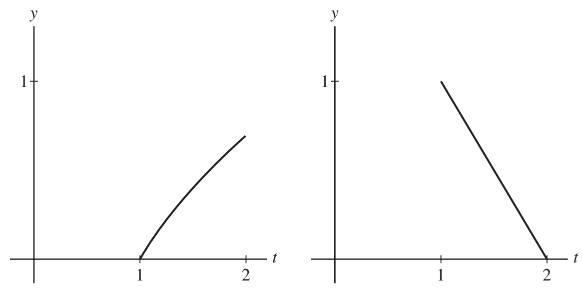
SOLUTION In the parametrization $x(t) = t$, $y(t) = f(t)$ we have $x'(t) = 1$, $t_0 = x(t_0)$, $t_1 = x(t_1)$. Hence Eq. (12) becomes

$$A = \int_{t_0}^{t_1} y(t)x'(t)\,dt = \int_{x(t_0)}^{x(t_1)} f(t)\,dt$$

We see that in this parametrization Eq. (12) is the familiar formula for the area under the graph of a positive function.

92. Sketch the graph of $c(t) = (\ln t, 2 - t)$ for $1 \le t \le 2$ and compute the area under the graph using Eq. (12).

SOLUTION We use the following graphs of $x(t) = \ln t$ and $y(t) = 2 - t$ for $1 \le t \le 2$:



$x(t) = \ln t$ $\quad$ $y(t) = 2 - t$

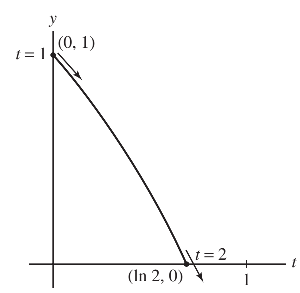
We see that for $1 < t < 2$, $x(t)$ is positive and increasing and $y(t)$ is positive and decreasing. Also $c(1) = (\ln 1, 2 - 1) = (0, 1)$ and $c(2) = (\ln 2, 2 - 2) = (\ln 2, 0)$. Additional information is obtained from the derivative

$$\frac{dy}{dx} = \frac{(2-t)'}{(\ln t)'} = -\frac{1}{1/t} = -t,$$

yielding

$$\left.\frac{dy}{dx}\right|_{t=1} - 1 \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{t=2} - 2.$$

We obtain the following graph:



We now use Eq. (12) to compute the area A under the graph. We have $x(t) = \ln t$, $x'(t) = \frac{1}{t}$, $y(t) = 2 - t$, $t_0 = 1$, $t_1 = 2$. Hence,

$$\begin{aligned} A &= \int_{t_0}^{t_1} y(t)x'(t)\,dt = \int_1^2 (2-t)\cdot\frac{1}{t}\,dt = \int_1^2 \left(\frac{2}{t} - 1\right) dt \\ &= 2\ln t - t\Big|_1^2 = (2\ln 2 - 2) - (2\ln 1 - 1) = 2\ln 2 - 1 \approx 0.386 \end{aligned}$$

93. Galileo tried unsuccessfully to find the area under a cycloid. Around 1630, Gilles de Roberval proved that the area under one arch of the cycloid $c(t) = (Rt - R\sin t, R - R\cos t)$ generated by a circle of radius R is equal to three times the area of the circle (Figure 25). Verify Roberval's result using Eq. (12).

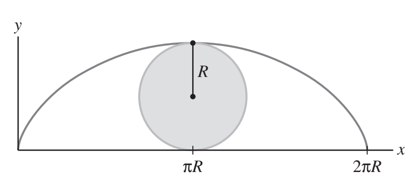


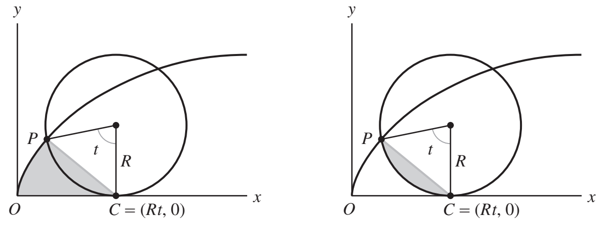
FIGURE 25 The area of one arch of the cycloid equals three times the area of the generating circle.

SOLUTION This reduces to

$$\int_0^{2\pi} (R - R\cos t)(Rt - R\sin t)'\,dt = \int_0^{2\pi} R^2(1-\cos t)^2\,dt = 3\pi R^2.$$

Further Insights and Challenges

94. Prove the following generalization of Exercise 93: For all $t > 0$, the area of the cycloidal sector OPC is equal to three times the area of the circular segment cut by the chord PC in Figure 26.



(A) Cycloidal sector OPC (B) Circular segment cut by the chord PC

FIGURE 26

SOLUTION Drop a perpendicular from point P to the x-axis and label the point of intersection T, and denote by D the center of the circle. Then the area of the cycloidal sector is equal to the area of OPT plus the area of PTC. The latter is a triangle with height $y(t) = R - R\cos t$ and base $Rt - (Rt - R\sin t) = R\sin t$, so its area is $\frac{1}{2}R^2\sin t(1-\cos t)$. The area of OPT, using Eq. (12), is

$$\int_0^t y(u)x'(u)\,du = \int_0^t (R - R\cos u)(Ru - R\sin u)'\,du = R^2\int_0^t (1-\cos u)^2\,du$$
$$= R^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{2}\sin t\cos t\right)$$

so that the total area of the cycloidal sector is

$$R^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{2}\sin t\cos t\right) + R^2\frac{1}{2}\sin t(1-\cos t) = 3\left(\frac{1}{2}R^2 t - \frac{1}{2}R^2\sin t\right) = 3\cdot\frac{1}{2}R^2(t - \sin t)$$

The area of the circular segment is the area of the circular sector DPC subtended by the angle t less the area of the triangle DPC. The triangle DPC has height $R\cos\frac{t}{2}$ and base $2R\sin\frac{t}{2}$ so that its area is $R^2\cos\frac{t}{2}\sin\frac{t}{2} = \frac{1}{2}R^2\sin t$, and the area of the circular sector is $\pi R^2\cdot\frac{t}{2\pi} = \frac{1}{2}R^2 t$. Thus the area of the circular segment is

$$\frac{1}{2}R^2(t - \sin t)$$

which is one third the area of the cycloidal sector.

95.  Derive the formula for the slope of the tangent line to a parametric curve $c(t) = (x(t), y(t))$ using a method different from that presented in the text. Assume that $x'(t_0)$ and $y'(t_0)$ exist and that $x'(t_0) \neq 0$. Show that

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)} = \frac{y'(t_0)}{x'(t_0)}$$

Then explain why this limit is equal to the slope dy/dx. Draw a diagram showing that the ratio in the limit is the slope of a secant line.

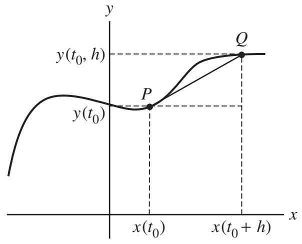
SOLUTION Since $y'(t_0)$ and $x'(t_0)$ exist, we have the following limits:

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{h} = y'(t_0), \quad \lim_{h\to 0}\frac{x(t_0+h) - x(t_0)}{h} = x'(t_0) \tag{1}$$

We use Basic Limit Laws, the limits in (1) and the given data $x'(t_0) \neq 0$, to write

$$\lim_{h\to 0}\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)} = \lim_{h\to 0}\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} = \frac{\lim_{h\to 0}\frac{y(t_0+h)-y(t_0)}{h}}{\lim_{h\to 0}\frac{x(t_0+h)-x(t_0)}{h}} = \frac{y'(t_0)}{x'(t_0)}$$

Notice that the quotient $\frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)}$ is the slope of the secant line determined by the points $P = (x(t_0), y(t_0))$ and $Q = (x(t_0+h), y(t_0+h))$. Hence, the limit of the quotient as $h \to 0$ is the slope of the tangent line at P, that is the derivative $\frac{dy}{dx}$.



96. Verify that the **tractrix** curve ($\ell > 0$)

$$c(t) = \left(t - \ell \tanh \frac{t}{\ell}, \ell \operatorname{sech} \frac{t}{\ell}\right)$$

has the following property: For all t, the segment from $c(t)$ to $(t, 0)$ is tangent to the curve and has length ℓ (Figure 27).

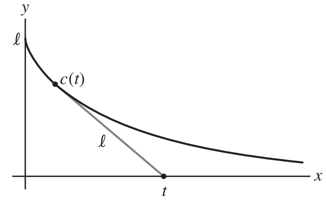
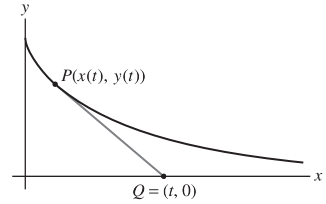


FIGURE 27 The tractrix $c(t) = \left(t - \ell \tanh \frac{t}{\ell}, \ell \operatorname{sech} \frac{t}{\ell}\right)$.

SOLUTION Let $P = c(t)$ and $Q = (t, 0)$.



The slope of the segment $\overline{PQ}$ is

$$m_1 = \frac{y(t) - 0}{x(t) - t} = \frac{\ell \operatorname{sech}\left(\frac{t}{\ell}\right)}{-\ell \tanh\left(\frac{t}{\ell}\right)} = -\frac{1}{\sinh\left(\frac{t}{\ell}\right)}$$

We compute the slope of the tangent line at P:

$$m_2 = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\left(\ell \operatorname{sech}\left(\frac{t}{\ell}\right)\right)'}{\left(t - \ell \tanh\left(\frac{t}{\ell}\right)\right)'} = \frac{\ell \cdot \frac{1}{\ell}\left(-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)\right)}{1 - \ell \cdot \frac{1}{\ell}\operatorname{sech}^2\left(\frac{t}{\ell}\right)}$$

$$= -\frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)}{1 - \operatorname{sech}^2\left(\frac{t}{\ell}\right)} = \frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)\tanh\left(\frac{t}{\ell}\right)}{\tanh^2\left(\frac{t}{\ell}\right)} = \frac{-\operatorname{sech}\left(\frac{t}{\ell}\right)}{\tanh\left(\frac{t}{\ell}\right)} = -\frac{1}{\sinh\left(\frac{t}{\ell}\right)}$$

Since $m_1 = m_2$, we conclude that the segment from $c(t)$ to $(t, 0)$ is tangent to the curve.

We now show that $|\overline{PQ}| = \ell$:

$$|\overline{PQ}| = \sqrt{(x(t) - t)^2 + (y(t) - 0)^2} = \sqrt{\left(-\ell \tanh \frac{t}{\ell}\right)^2 + \left(\ell \operatorname{sech}\left(\frac{t}{\ell}\right)\right)^2}$$

$$= \sqrt{\ell^2\left(\tanh^2\left(\frac{t}{\ell}\right) + \operatorname{sech}^2\left(\frac{t}{\ell}\right)\right)} = \ell\sqrt{\operatorname{sech}^2\left(\frac{t}{\ell}\right)\sinh^2\left(\frac{t}{\ell}\right) + \operatorname{sech}^2\left(\frac{t}{\ell}\right)}$$

$$= \ell \operatorname{sech}\left(\frac{t}{\ell}\right)\sqrt{\sinh^2\left(\frac{t}{\ell}\right) + 1} = \ell \operatorname{sech}\left(\frac{t}{\ell}\right)\cosh\left(\frac{t}{\ell}\right) = \ell \cdot 1 = \ell$$

97. In Exercise 54 of Section 10.1 (ET Exercise 54 of Section 9.1), we described the tractrix by the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{\ell^2 - y^2}}$$

Show that the curve $c(t)$ identified as the tractrix in Exercise 96 satisfies this differential equation. Note that the derivative on the left is taken with respect to x, not t.

SOLUTION Note that $dx/dt = 1 - \operatorname{sech}^2(t/\ell) = \tanh^2(t/\ell)$ and $dy/dt = -\operatorname{sech}(t/\ell)\tanh(t/\ell)$. Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\operatorname{sech}(t/\ell)}{\tanh(t/\ell)} = \frac{-y/\ell}{\sqrt{1 - y^2/\ell^2}}$$

Multiplying top and bottom by ℓ/ℓ gives

$$\frac{dy}{dx} = \frac{-y}{\sqrt{\ell^2 - y^2}}$$

In Exercises 98 and 99, refer to Figure 28.

98. In the parametrization $c(t) = (a\cos t, b\sin t)$ of an ellipse, t is *not* an angular parameter unless $a = b$ (in which case the ellipse is a circle). However, t can be interpreted in terms of area: Show that if $c(t) = (x, y)$, then $t = (2/ab)A$, where A is the area of the shaded region in Figure 28. *Hint:* Use Eq. (12).

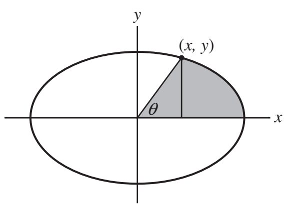
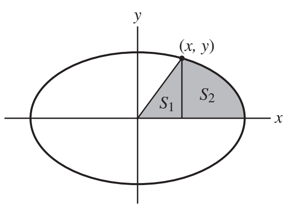


FIGURE 28 The parameter θ on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION We compute the area A of the shaded region as the sum of the area S_1 of the triangle and the area S_2 of the region under the curve. The area of the triangle is

$$S_1 = \frac{xy}{2} = \frac{(a\cos t)(b\sin t)}{2} = \frac{ab\sin 2t}{4} \qquad (1)$$



The area S_2 under the curve can be computed using Eq. (12). The lower limit of the integration is $t_0 = 0$ (corresponds to $(a, 0)$) and the upper limit is t (corresponds to $(x(t), y(t))$). Also $y(t) = b\sin t$ and $x'(t) = -a\sin t$. Since $x'(t) < 0$ on the interval $0 < t < \frac{\pi}{2}$ (which represents the ellipse on the first quadrant), we use the positive value $a\sin t$ to obtain a positive value for the area. This gives

$$\begin{aligned} S_2 &= \int_0^t b\sin u \cdot a\sin u\,du = ab\int_0^t \sin^2 u\,du \\ &= ab\int_0^t \left(\frac{1}{2} - \frac{1}{2}\cos 2u\right) du = ab\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]\Bigg|_0^t \\ &= ab\left[\frac{t}{2} - \frac{\sin 2t}{4} - 0\right] = \frac{abt}{2} - \frac{ab\sin 2t}{4} \end{aligned} \qquad (2)$$

Combining (1) and (2) we obtain

$$A = S_1 + S_2 = \frac{ab\sin 2t}{4} + \frac{abt}{2} - \frac{ab\sin 2t}{4} = \frac{abt}{2}$$

Hence, $t = \frac{2A}{ab}$.

99. Show that the parametrization of the ellipse by the angle θ is

$$x = \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$
$$y = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

SOLUTION We consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

For the angle θ we have $\tan\theta = \frac{y}{x}$, hence,

$$y = x\tan\theta \tag{1}$$

Substituting in the equation of the ellipse and solving for x we obtain

$$\frac{x^2}{a^2} + \frac{x^2\tan^2\theta}{b^2} = 1$$
$$b^2x^2 + a^2x^2\tan^2\theta = a^2b^2$$
$$(a^2\tan^2\theta + b^2)x^2 = a^2b^2$$
$$x^2 = \frac{a^2b^2}{a^2\tan^2\theta + b^2} = \frac{a^2b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

We now take the square root. Since the sign of the x-coordinate is the same as the sign of $\cos\theta$, we take the positive root, obtaining

$$x = \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{2}$$

Hence by (1), the y-coordinate is

$$y = x\tan\theta = \frac{ab\cos\theta\tan\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{3}$$

Equalities (2) and (3) give the following parametrization for the ellipse:

$$c_1(\theta) = \left(\frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}, \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right)$$

11.2 Arc Length and Speed

Preliminary Questions

1. What is the definition of arc length?

SOLUTION A curve can be approximated by a polygonal path obtained by connecting points

$$p_0 = c(t_0),\ p_1 = c(t_1),\ \ldots,\ p_N = c(t_N)$$

on the path with segments. One gets an approximation by summing the lengths of the segments. The definition of arc length is the limit of that approximation when increasing the number of points so that the lengths of the segments approach zero. In doing so, we obtain the following theorem for the arc length:

$$S = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

which is the length of the curve $c(t) = (x(t), y(t))$ for $a \le t \le b$.

2. What is the interpretation of $\sqrt{x'(t)^2 + y'(t)^2}$ for a particle following the trajectory $(x(t), y(t))$?

SOLUTION The expression $\sqrt{x'(t)^2 + y'(t)^2}$ denotes the speed at time t of a particle following the trajectory $(x(t), y(t))$.

3. A particle travels along a path from (0, 0) to (3, 4). What is the displacement? Can the distance traveled be determined from the information given?

SOLUTION The net displacement is the distance between the initial point (0, 0) and the endpoint (3, 4). That is

$$\sqrt{(3-0)^2+(4-0)^2}=\sqrt{25}=5.$$

The distance traveled can be determined only if the trajectory $c(t)=(x(t),y(t))$ of the particle is known.

4. A particle traverses the parabola $y=x^2$ with constant speed 3 cm/s. What is the distance traveled during the first minute? *Hint:* No computation is necessary.

SOLUTION Since the speed is constant, the distance traveled is the following product: $L=st=3\cdot 60=180$ cm.

Exercises

In Exercises 1–10, use Eq. (3) to find the length of the path over the given interval.

1. $(3t+1,9-4t)$, $\quad 0\le t\le 2$

SOLUTION Since $x=3t+1$ and $y=9-4t$ we have $x'=3$ and $y'=-4$. Hence, the length of the path is

$$S=\int_0^2\sqrt{3^2+(-4)^2}\,dt=5\int_0^2 dt=10.$$

2. $(1+2t,2+4t)$, $\quad 1\le t\le 4$

SOLUTION We have $x=1+2t$ and $y=2+4t$, hence $x'=2$ and $y'=4$. Using the formula for arc length we obtain

$$S=\int_1^4\sqrt{2^2+4^2}\,dt=\int_1^4\sqrt{20}\,dt=\sqrt{20}(4-1)=6\sqrt{5}$$

3. $(2t^2,3t^2-1)$, $\quad 0\le t\le 4$

SOLUTION Since $x=2t^2$ and $y=3t^2-1$, we have $x'=4t$ and $y'=6t$. By the formula for the arc length we get

$$S=\int_0^4\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_0^4\sqrt{16t^2+36t^2}\,dt=\sqrt{52}\int_0^4 t\,dt=\sqrt{52}\cdot\frac{t^2}{2}\Big|_0^4=16\sqrt{13}$$

4. $(3t,4t^{3/2})$, $\quad 0\le t\le 1$

SOLUTION We have $x=3t$ and $y=4t^{3/2}$, hence $x'=3$ and $y'=6t^{1/2}$. Using the formula for the arc length we obtain

$$S=\int_0^1\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_0^1\sqrt{3^2+\left(6t^{1/2}\right)^2}\,dt=\int_0^1\sqrt{9+36t}\,dt=3\int_0^1\sqrt{1+4t}\,dt$$

Setting $u=1+4t$ we get

$$S=\frac{3}{4}\int_1^5\sqrt{u}\,du=\frac{3}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^5=\frac{1}{2}(5^{3/2}-1)\approx 5.09$$

5. $(3t^2,4t^3)$, $\quad 1\le t\le 4$

SOLUTION We have $x=3t^2$ and $y=4t^3$. Hence $x'=6t$ and $y'=12t^2$. By the formula for the arc length we get

$$S=\int_1^4\sqrt{x'(t)^2+y'(t)^2}\,dt=\int_1^4\sqrt{36t^2+144t^4}\,dt=6\int_1^4\sqrt{1+4t^2}\,t\,dt.$$

Using the substitution $u=1+4t^2$, $du=8t\,dt$ we obtain

$$S=\frac{6}{8}\int_5^{65}\sqrt{u}\,du=\frac{3}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{65}=\frac{1}{2}(65^{3/2}-5^{3/2})\approx 256.43$$

6. $(t^3+1, t^2-3), \quad 0 \le t \le 1$

SOLUTION We have $x = t^3+1$, $y = t^2-3$, hence, $x' = 3t^2$ and $y' = 2t$. By the formula for the arc length we get

$$S = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^1 \sqrt{9t^4+4t^2}\,dt = \int_0^1 t\sqrt{9t^2+4}\,dt$$

We compute the integral using the substitution $u = 4+9t^2$. This gives

$$S = \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_4^{13} = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8) \approx 1.44.$$

7. $(\sin 3t, \cos 3t), \quad 0 \le t \le \pi$

SOLUTION We have $x = \sin 3t$, $y = \cos 3t$, hence $x' = 3\cos 3t$ and $y' = -3\sin 3t$. By the formula for the arc length we obtain:

$$S = \int_0^\pi \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^\pi \sqrt{9\cos^2 3t + 9\sin^2 3t}\,dt = \int_0^\pi \sqrt{9}\,dt = 3\pi$$

8. $(\sin\theta - \theta\cos\theta, \cos\theta + \theta\sin\theta), \quad 0 \le \theta \le 2$

SOLUTION We have $x = \sin\theta - \theta\cos\theta$ and $y = \cos\theta + \theta\sin\theta$. Hence, $x' = \cos\theta - (\cos\theta - \theta\sin\theta) = \theta\sin\theta$ and $y' = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$. Using the formula for the arc length we obtain:

$$S = \int_0^2 \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \int_0^2 \sqrt{(\theta\sin\theta)^2 + (\theta\cos\theta)^2}\,d\theta$$

$$= \int_0^2 \sqrt{\theta^2(\sin^2\theta + \cos^2\theta)}\,d\theta = \int_0^2 \theta\,d\theta = \frac{\theta^2}{2}\Big|_0^2 = 2$$

In Exercises 9 and 10, use the identity

$$\frac{1-\cos t}{2} = \sin^2\frac{t}{2}$$

9. $(2\cos t - \cos 2t, 2\sin t - \sin 2t), \quad 0 \le t \le \frac{\pi}{2}$

SOLUTION We have $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$. Thus, $x' = -2\sin t + 2\sin 2t$ and $y' = 2\cos t - 2\cos 2t$. We get

$$\begin{aligned}
x'(t)^2 + y'(t)^2 &= (-2\sin t + 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2 \\
&= 4\sin^2 t - 8\sin t\sin 2t + 4\sin^2 2t + 4\cos^2 t - 8\cos t\cos 2t + 4\cos^2 2t \\
&= 4(\sin^2 t + \cos^2 t) + 4(\sin^2 2t + \cos^2 2t) - 8(\sin t\sin 2t + \cos t\cos 2t) \\
&= 4 + 4 - 8\cos(2t - t) = 8 - 8\cos t = 8(1-\cos t)
\end{aligned}$$

We now use the formula for the arc length to obtain

$$S = \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2} = \int_0^{\pi/2} \sqrt{8(1-\cos t)}\,dt = \int_0^{\pi/2} \sqrt{16\sin^2\frac{t}{2}}\,dt = 4\int_0^{\pi/2} \sin\frac{t}{2}\,dt$$

$$= -8\cos\frac{t}{2}\Big|_0^{\pi/2} = -8\left(\cos\frac{\pi}{4} - \cos 0\right) = -8\left(\frac{\sqrt{2}}{2} - 1\right) \approx 2.34$$

10. $(5(\theta - \sin\theta), 5(1-\cos\theta)), \quad 0 \le \theta \le 2\pi$

SOLUTION Since $x = 5(\theta - \sin\theta)$ and $y = 5(1-\cos\theta)$, we have $x' = 5(1-\cos\theta)$ and $y' = 5\sin\theta$. Using the formula for the arc length we obtain:

$$\begin{aligned}
S &= \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{25(1-\cos\theta)^2 + 25\sin^2\theta}\,d\theta \\
&= 5\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta = 5\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\,d\theta \\
&= 5\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\,d\theta = 10\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = 20\int_0^{\pi} \sin u\,du \\
&= 20(-\cos u)\Big|_0^{\pi} = -20(-1-1) = 40.
\end{aligned}$$

11. Show that one arch of a cycloid generated by a circle of radius R has length $8R$.

SOLUTION Recall from earlier that the cycloid generated by a circle of radius R has parametric equations $x = Rt - R\sin t$, $y = R - R\cos t$. Hence, $x' = R - R\cos t$, $y' = R\sin t$. Using the identity $\sin^2 \frac{t}{2} = \frac{1-\cos t}{2}$, we get

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= R^2(1-\cos t)^2 + R^2\sin^2 t = R^2(1 - 2\cos t + \cos^2 t + \sin^2 t) \\ &= R^2(1 - 2\cos t + 1) = 2R^2(1-\cos t) = 4R^2 \sin^2 \frac{t}{2} \end{aligned}$$

One arch of the cycloid is traced as t varies from 0 to 2π. Hence, using the formula for the arc length we obtain:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{4R^2 \sin^2 \frac{t}{2}}\,dt = 2R\int_0^{2\pi} \sin\frac{t}{2}\,dt = 4R\int_0^{\pi} \sin u\,du \\ &= -4R\cos u\Big|_0^{\pi} = -4R(\cos\pi - \cos 0) = 8R \end{aligned}$$

12. Find the length of the spiral $c(t) = (t\cos t, t\sin t)$ for $0 \le t \le 2\pi$ to three decimal places (Figure 7). *Hint:* Use the formula

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)$$

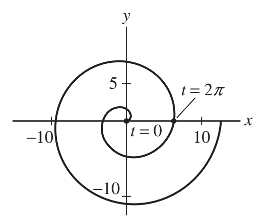


FIGURE 7 The spiral $c(t) = (t\cos t, t\sin t)$.

SOLUTION We use the formula for the arc length:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt \tag{1}$$

Differentiating $x = t\cos t$ and $y = t\sin t$ yields

$$\begin{aligned} x'(t) &= \frac{d}{dt}(t\cos t) = \cos t - t\sin t \\ y'(t) &= \frac{d}{dt}(t\sin t) = \sin t + t\cos t \end{aligned}$$

Thus,

$$\begin{aligned} \sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\ &= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} \\ &= \sqrt{(\cos^2 t + \sin^2 t)(1+t^2)} = \sqrt{1+t^2} \end{aligned}$$

We substitute into (1) and use the integral given in the hint to obtain the following arc length:

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{1+t^2}\,dt = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\Big|_0^{2\pi} \\ &= \frac{1}{2}\cdot 2\pi\sqrt{1+(2\pi)^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+(2\pi)^2}\right) - \left(0 + \frac{1}{2}\ln 1\right) \\ &= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right) \approx 21.256 \end{aligned}$$

13. Find the length of the tractrix (see Figure 6)

$$c(t) = (t - \tanh(t), \operatorname{sech}(t)), \qquad 0 \le t \le A$$

SOLUTION Since $x = t - \tanh(t)$ and $y = \operatorname{sech}(t)$ we have $x' = 1 - \operatorname{sech}^2(t)$ and $y' = -\operatorname{sech}(t)\tanh(t)$. Hence,

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= \left(1 - \operatorname{sech}^2(t)\right)^2 + \operatorname{sech}^2(t)\tanh^2(t) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^4(t) + \operatorname{sech}^2(t)\tanh^2(t) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^2(t)(\operatorname{sech}^2(t) + \tanh^2(t)) \\ &= 1 - 2\operatorname{sech}^2(t) + \operatorname{sech}^2(t) = 1 - \operatorname{sech}^2(t) = \tanh^2(t) \end{aligned}$$

Hence, using the formula for the arc length we get:

$$\begin{aligned} S &= \int_0^A \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^A \sqrt{\tanh^2(t)}\,dt = \int_0^A \tanh(t)\,dt = \ln(\cosh(t))\Big|_0^A \\ &= \ln(\cosh(A)) - \ln(\cosh(0)) = \ln(\cosh(A)) - \ln 1 = \ln(\cosh(A)) \end{aligned}$$

14. CAS Find a numerical approximation to the length of $c(t) = (\cos 5t, \sin 3t)$ for $0 \le t \le 2\pi$ (Figure 8).

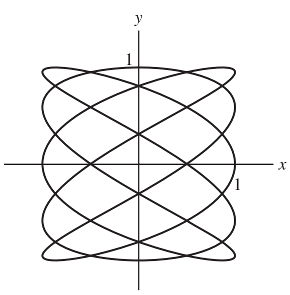


FIGURE 8

SOLUTION Since $x = \cos 5t$ and $y = \sin 3t$, we have

$$x'(t) = -5\sin 5t, \qquad y'(t) = 3\cos 3t$$

so that

$$x'(t)^2 + y'(t)^2 = 25\sin^2 5t + 9\cos^2 3t$$

Then the arc length is

$$\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{25\sin^2 5t + 9\cos^2 3t}\,dt \approx 24.60296$$

In Exercises 15–18, determine the speed s at time t (assume units of meters and seconds).

15. $(t^3, t^2), \quad t = 2$

SOLUTION We have $x(t) = t^3$, $y(t) = t^2$ hence $x'(t) = 3t^2$, $y'(t) = 2t$. The speed of the particle at time t is thus, $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^2} = t\sqrt{9t^2 + 4}$. At time $t = 2$ the speed is

$$\left.\frac{ds}{dt}\right|_{t=2} = 2\sqrt{9 \cdot 2^2 + 4} = 2\sqrt{40} = 4\sqrt{10} \approx 12.65 \text{ m/s}.$$

16. $(3\sin 5t, 8\cos 5t), \quad t = \frac{\pi}{4}$

SOLUTION We have $x = 3\sin 5t$, $y = 8\cos 5t$, hence $x' = 15\cos 5t$, $y' = -40\sin 5t$. Thus, the speed of the particle at time t is

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{225\cos^2 5t + 1600\sin^2 5t} \\ &= \sqrt{225(\cos^2 5t + \sin^2 5t) + 1375\sin^2 5t} = 5\sqrt{9 + 55\sin^2 5t} \end{aligned}$$

Thus,

$$\frac{ds}{dt} = 5\sqrt{9 + 55\sin^2 5t}.$$

The speed at time $t = \frac{\pi}{4}$ is thus

$$\left.\frac{ds}{dt}\right|_{t=\pi/4} = 5\sqrt{9 + 55\sin^2\left(5 \cdot \frac{\pi}{4}\right)} \cong 30.21 \text{ m/s}$$

17. $(5t + 1, 4t - 3), \quad t = 9$

SOLUTION Since $x = 5t + 1$, $y = 4t - 3$, we have $x' = 5$ and $y' = 4$. The speed of the particle at time t is

$$\frac{ds}{dt} = \sqrt{x'(t) + y'(t)} = \sqrt{5^2 + 4^2} = \sqrt{41} \approx 6.4 \text{ m/s}.$$

We conclude that the particle has constant speed of 6.4 m/s.

18. $(\ln(t^2 + 1), t^3), \quad t = 1$

SOLUTION We have $x = \ln(t^2 + 1)$, $y = t^3$, so $x' = \dfrac{2t}{t^2 + 1}$ and $y' = 3t^2$. The speed of the particle at time t is thus

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\frac{4t^2}{\left(t^2 + 1\right)^2} + 9t^4} = t\sqrt{\frac{4}{\left(t^2 + 1\right)^2} + 9t^2}.$$

The speed at time $t = 1$ is

$$\left.\frac{ds}{dt}\right|_{t=1} = \sqrt{\frac{4}{2^2} + 9} = \sqrt{10} \approx 3.16 \text{ m/s}.$$

19. Find the minimum speed of a particle with trajectory $c(t) = (t^3 - 4t, t^2 + 1)$ for $t \geq 0$. *Hint:* It is easier to find the minimum of the square of the speed.

SOLUTION We first find the speed of the particle. We have $x(t) = t^3 - 4t$, $y(t) = t^2 + 1$, hence $x'(t) = 3t^2 - 4$ and $y'(t) = 2t$. The speed is thus

$$\frac{ds}{dt} = \sqrt{\left(3t^2 - 4\right)^2 + (2t)^2} = \sqrt{9t^4 - 24t^2 + 16 + 4t^2} = \sqrt{9t^4 - 20t^2 + 16}.$$

The square root function is an increasing function, hence the minimum speed occurs at the value of t where the function $f(t) = 9t^4 - 20t^2 + 16$ has minimum value. Since $\lim_{t\to\infty} f(t) = \infty$, f has a minimum value on the interval $0 \leq t < \infty$, and it occurs at a critical point or at the endpoint $t = 0$. We find the critical point of f on $t \geq 0$:

$$f'(t) = 36t^3 - 40t = 4t(9t^2 - 10) = 0 \Rightarrow t = 0, t = \sqrt{\frac{10}{9}}.$$

We compute the values of f at these points:

$$f(0) = 9 \cdot 0^4 - 20 \cdot 0^2 + 16 = 16$$

$$f\left(\sqrt{\frac{10}{9}}\right) = 9\left(\sqrt{\frac{10}{9}}\right)^4 - 20\left(\sqrt{\frac{10}{9}}\right)^2 + 16 = \frac{44}{9} \approx 4.89$$

We conclude that the minimum value of f on $t \geq 0$ is 4.89. The minimum speed is therefore

$$\left(\frac{ds}{dt}\right)_{\min} \approx \sqrt{4.89} \approx 2.21.$$

20. Find the minimum speed of a particle with trajectory $c(t) = (t^3, t^{-2})$ for $t \geq 0.5$.

SOLUTION We first compute the speed of the particle. Since $x(t) = t^3$ and $y(t) = t^{-2}$, we have $x'(t) = 3t^2$ and $y'(t) = -2t^{-3}$. The speed is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^{-6}}.$$

The square root function is an increasing function, hence the minimum value of $\frac{ds}{dt}$ occurs at the point where the function $f(t) = 9t^4 + 4t^{-6}$ attains its minimum value. We find the critical points of f on the interval $t \geq 0.5$:

$$f'(t) = 36t^3 - 24t^{-7} = 0$$

$$3t^{10} - 2 = 0 \Rightarrow t = \sqrt[10]{\frac{2}{3}} \approx 0.96$$

Since $\lim_{t\to\infty} f(t) = \infty$, the minimum value on $0.5 \le t < \infty$ exists, and it occurs at the critical point $t = 0.96$ or at the endpoint $t = 0.5$. We compute the values of f at these points:

$$f(0.96) = 9 \cdot (0.96)^4 + 4 \cdot (0.96)^{-6} = 12.75$$
$$f(0.5) = 9(0.5)^4 + 4(0.5)^{-6} = 256.56$$

We conclude that the minimum value of f on the interval $t \ge 0.5$ is 12.75. The minimum speed for $t \ge 0.5$ is therefore

$$\left(\frac{ds}{dt}\right)_{\min} = \sqrt{12.75} \approx 3.57$$

21. Find the speed of the cycloid $c(t) = (4t - 4\sin t, 4 - 4\cos t)$ at points where the tangent line is horizontal.

SOLUTION We first find the points where the tangent line is horizontal. The slope of the tangent line is the following quotient:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\sin t}{4 - 4\cos t} = \frac{\sin t}{1 - \cos t}.$$

To find the points where the tangent line is horizontal we solve the following equation for $t \ge 0$:

$$\frac{dy}{dx} = 0, \quad \frac{\sin t}{1 - \cos t} = 0 \Rightarrow \sin t = 0 \quad \text{and} \quad \cos t \ne 1.$$

Now, $\sin t = 0$ and $t \ge 0$ at the points $t = \pi k$, $k = 0, 1, 2, \dots$. Since $\cos \pi k = (-1)^k$, the points where $\cos t \ne 1$ are $t = \pi k$ for k odd. The points where the tangent line is horizontal are, therefore:

$$t = \pi(2k - 1), \quad k = 1, 2, 3, \dots$$

The speed at time t is given by the following expression:

$$\begin{aligned}
\frac{ds}{dt} &= \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(4 - 4\cos t)^2 + (4\sin t)^2} \\
&= \sqrt{16 - 32\cos t + 16\cos^2 t + 16\sin^2 t} = \sqrt{16 - 32\cos t + 16} \\
&= \sqrt{32(1 - \cos t)} = \sqrt{32 \cdot 2\sin^2 \frac{t}{2}} = 8\left|\sin \frac{t}{2}\right|
\end{aligned}$$

That is, the speed of the cycloid at time t is

$$\frac{ds}{dt} = 8\left|\sin \frac{t}{2}\right|.$$

We now substitute

$$t = \pi(2k - 1), \quad k = 1, 2, 3, \dots$$

to obtain

$$\frac{ds}{dt} = 8\left|\sin \frac{\pi(2k - 1)}{2}\right| = 8|(-1)^{k+1}| = 8$$

22. Calculate the arc length integral $s(t)$ for the *logarithmic spiral* $c(t) = (e^t \cos t, e^t \sin t)$.

SOLUTION We have $x'(t) = e^t(\cos t - \sin t)$, $y'(t) = e^t(\cos t + \sin t)$ so that

$$x'(t)^2 + y'(t)^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t) = 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

so that the arc length integral is

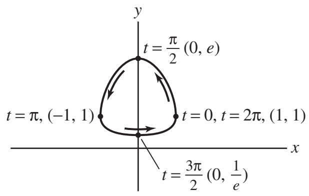
$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{2}\int_a^b e^t\, dt$$

If neither a nor b is $\pm\infty$, then this equals $\sqrt{2}(e^b - e^a)$. Note that the origin corresponds to $t = -\infty$.

CAS *In Exercises 23–26, plot the curve and use the Midpoint Rule with N = 10, 20, 30, and 50 to approximate its length.*

23. $c(t) = (\cos t, e^{\sin t})$ for $0 \le t \le 2\pi$

SOLUTION The curve of $c(t) = (\cos t, e^{\sin t})$ for $0 \le t \le 2\pi$ is shown in the figure below:



$c(t) = (\cos t, e^{\sin t}), 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t\, e^{\sin t})^2}\, dt.$$

That is, $S = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t\, e^{2\sin t}}\, dt$. We approximate the integral using the Mid-Point Rule with $N = 10, 20, 30, 50$. For $f(t) = \sqrt{\sin^2 t + \cos^2 t\, e^{2\sin t}}$ we obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 6.903734$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 6.915035$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

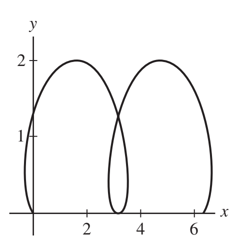
$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 6.914949$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25},\ c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 6.914951$$

24. $c(t) = (t - \sin 2t, 1 - \cos 2t)$ for $0 \le t \le 2\pi$

SOLUTION The curve is shown in the figure below:



$c(t) = (t - \sin 2t, 1 - \cos 2t), 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{(1 - 2\cos 2t)^2 + (2\sin 2t)^2}\, dt = \int_0^{2\pi} \sqrt{1 - 4\cos 2t + 4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} \sqrt{5 - 4\cos 2t}\, dt.$$

That is,

$$S = \int_0^{2\pi} \sqrt{5 - 4\cos 2t}\, dt.$$

Approximating the length using the Mid-Point Rule with $N = 10, 20, 30, 50$ for $f(t) = \sqrt{5 - 4\cos 2t}$ we obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 13.384047$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 13.365095$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

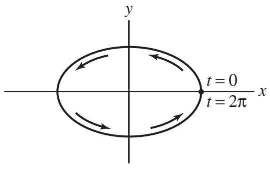
$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 13.364897$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 13.364893$$

25. The ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

SOLUTION We use the parametrization given in Example 4, section 12.1, that is, $c(t) = (5\cos t, 3\sin t)$, $0 \le t \le 2\pi$. The curve is shown in the figure below:



$c(t) = (5\cos t, 3\sin t)$, $0 \le t \le 2\pi$.

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (3\cos t)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{25\sin^2 t + 9\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t) + 16\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{9 + 16\sin^2 t}\, dt.$$

That is,

$$S = \int_0^{2\pi} \sqrt{9 + 16\sin^2 t}\, dt.$$

We approximate the integral using the Mid-Point Rule with $N = 10, 20, 30, 50$, for $f(t) = \sqrt{9 + 16\sin^2 t}$. We obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 25.528309$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 25.526999$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

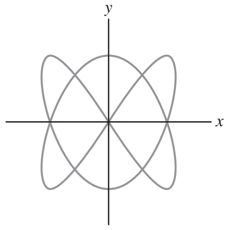
$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 25.526999$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 25.526999$$

26. $x = \sin 2t, \quad y = \sin 3t \quad$ for $0 \le t \le 2\pi$

SOLUTION The curve is shown in the figure below:



$c(t) = (\sin 2t, \sin 3t), 0 \le t \le 2\pi.$

The length of the curve is given by the following integral:

$$S = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (3\cos 3t)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 9\cos^2 3t}\, dt.$$

We approximate the length using the Mid-Point Rule with $N = 10, 20, 30, 50$ for $f(t) = \sqrt{4\cos^2 2t + 9\cos^2 3t}$. We obtain

$$(N = 10): \quad \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{5}$$

$$M_{10} = \frac{\pi}{5} \sum_{i=1}^{10} f(c_i) = 15.865169$$

$$(N = 20): \quad \Delta x = \frac{2\pi}{20} = \frac{\pi}{10}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{10}$$

$$M_{20} = \frac{\pi}{10} \sum_{i=1}^{20} f(c_i) = 15.324697$$

$$(N = 30): \quad \Delta x = \frac{2\pi}{30} = \frac{\pi}{15}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{15}$$

$$M_{30} = \frac{\pi}{15} \sum_{i=1}^{30} f(c_i) = 15.279322$$

$$(N = 50): \quad \Delta x = \frac{2\pi}{50} = \frac{\pi}{25}, c_i = \left(i - \frac{1}{2}\right) \cdot \frac{\pi}{25}$$

$$M_{50} = \frac{\pi}{25} \sum_{i=1}^{50} f(c_i) = 15.287976$$

27. If you unwind thread from a stationary circular spool, keeping the thread taut at all times, then the endpoint traces a curve $\mathcal{C}$ called the **involute** of the circle (Figure 9). Observe that $\overline{PQ}$ has length $R\theta$. Show that $\mathcal{C}$ is parametrized by

$$c(\theta) = \big(R(\cos\theta + \theta\sin\theta),\, R(\sin\theta - \theta\cos\theta)\big)$$

Then find the length of the involute for $0 \le \theta \le 2\pi$.

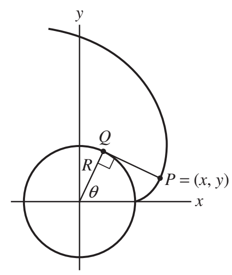


FIGURE 9 Involute of a circle.

SOLUTION Suppose that the arc $\widehat{QT}$ corresponding to the angle θ is unwound. Then the length of the segment $\overline{QP}$ equals the length of this arc. That is, $\overline{QP} = R\theta$. With the help of the figure we can see that

$$x = \overline{OA} + \overline{AB} = \overline{OA} + \overline{EP} = R\cos\theta + \overline{QP}\sin\theta = R\cos\theta + R\theta\sin\theta = R(\cos\theta + \theta\sin\theta).$$

Furthermore,

$$y = \overline{QA} - \overline{QE} = R\sin\theta - \overline{QP}\cos\theta = R\sin\theta - R\theta\cos\theta = R(\sin\theta - \theta\cos\theta)$$

The coordinates of P with respect to the parameter θ form the following parametrization of the curve:

$$c(\theta) = (R(\cos\theta + \theta\sin\theta), R(\sin\theta - \theta\cos\theta)), \qquad 0 \le \theta \le 2\pi.$$

We find the length of the involute for $0 \le \theta \le 2\pi$, using the formula for the arc length:

$$S = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta.$$

We compute the integrand:

$$x'(\theta) = \frac{d}{d\theta}(R(\cos\theta + \theta\sin\theta)) = R(-\sin\theta + \sin\theta + \theta\cos\theta) = R\theta\cos\theta$$
$$y'(\theta) = \frac{d}{d\theta}(R(\sin\theta - \theta\cos\theta)) = R(\cos\theta - (\cos\theta - \theta\sin\theta)) = R\theta\sin\theta$$
$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(R\theta\cos\theta)^2 + (R\theta\sin\theta)^2} = \sqrt{R^2\theta^2(\cos^2\theta + \sin^2\theta)} = \sqrt{R^2\theta^2} = R\theta$$

We now compute the arc length:

$$S = \int_0^{2\pi} R\theta\, d\theta = \frac{R\theta^2}{2}\bigg|_0^{2\pi} = \frac{R\cdot(2\pi)^2}{2} = 2\pi^2 R.$$

28. Let $a > b$ and set

$$k = \sqrt{1 - \frac{b^2}{a^2}}$$

Use a parametric representation to show that the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ has length $L = 4aG\left(\frac{\pi}{2}, k\right)$, where

$$G(\theta, k) = \int_0^{\theta} \sqrt{1 - k^2\sin^2 t}\, dt$$

is the *elliptic integral of the second kind*.

SOLUTION Since the ellipse is symmetric with respect to the x and y axis, its length L is four times the length of the part of the ellipse which is in the first quadrant. This part is represented by the following parametrization: $x(t) = a\sin t$, $y(t) = b\cos t$, $0 \le t \le \frac{\pi}{2}$. Using the formula for the arc length we get:

$$L = 4\int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt = 4\int_0^{\pi/2} \sqrt{(a\cos t)^2 + (-b\sin t)^2}\, dt$$
$$= 4\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt$$

We rewrite the integrand as follows:

$$\begin{aligned}
L &= 4\int_0^{\pi/2}\sqrt{a^2\cos^2 t + a^2\sin^2 t + (b^2-a^2)\sin^2 t}\,dt \\
&= 4\int_0^{\pi/2}\sqrt{a^2(\cos^2 t + \sin^2 t) + (b^2-a^2)\sin^2 t}\,dt \\
&= 4\int_0^{\pi/2}\sqrt{a^2 + (b^2-a^2)\sin^2 t}\,dt = 4a\int_0^{\pi/2}\sqrt{\frac{a^2}{a^2} + \frac{b^2-a^2}{a^2}\sin^2 t}\,dt \\
&= 4a\int_0^{\pi/2}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2 t}\,dt = 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt = 4aG\left(\frac{\pi}{2},k\right)
\end{aligned}$$

where $k = \sqrt{1-\frac{b^2}{a^2}}$.

In Exercises 29–32, use Eq. (4) to compute the surface area of the given surface.

29. The cone generated by revolving $c(t) = (t, mt)$ about the x-axis for $0 \le t \le A$

SOLUTION Substituting $y(t) = mt$, $y'(t) = m$, $x'(t) = 1$, $a = 0$, and $b = 0$ in the formula for the surface area, we get

$$S = 2\pi\int_0^A mt\sqrt{1+m^2}\,dt = 2\pi\sqrt{1+m^2}\,m\int_0^A t\,dt = 2\pi m\sqrt{1+m^2}\cdot\frac{t^2}{2}\bigg|_0^A = m\sqrt{1+m^2}\pi A^2$$

30. A sphere of radius R

SOLUTION The sphere of radius R is generated by revolving the half circle $c(t) = (R\cos t, R\sin t)$, $0 \le t \le \pi$ about the x-axis. We have $x(t) = R\cos t$, $x'(t) = -R\sin t$, $y(t) = R\sin t$, $y'(t) = R\cos t$. Using the formula for the surface area, we get

$$\begin{aligned}
S &= 2\pi\int_0^{\pi} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{\pi} R\sin t\sqrt{R^2\sin^2 t + R^2\cos^2 t}\,dt \\
&= 2\pi R^2\int_0^{\pi}\sin t\,dt = -2\pi R^2\cos t\bigg|_0^{\pi} = -2\pi R^2(-1-1) = 4\pi R^2
\end{aligned}$$

31. The surface generated by revolving one arch of the cycloid $c(t) = (t - \sin t, 1 - \cos t)$ about the x-axis

SOLUTION One arch of the cycloid is traced as t varies from 0 to 2π. Since $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$, we have $x'(t) = 1 - \cos t$ and $y'(t) = \sin t$. Hence, using the identity $1 - \cos t = 2\sin^2\frac{t}{2}$, we get

$$x'(t)^2 + y'(t)^2 = (1-\cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t = 4\sin^2\frac{t}{2}$$

By the formula for the surface area we obtain:

$$\begin{aligned}
S &= 2\pi\int_0^{2\pi} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{2\pi}(1-\cos t)\cdot 2\sin\frac{t}{2}\,dt \\
&= 2\pi\int_0^{2\pi} 2\sin^2\frac{t}{2}\cdot 2\sin\frac{t}{2}\,dt = 8\pi\int_0^{2\pi}\sin^3\frac{t}{2}\,dt = 16\pi\int_0^{\pi}\sin^3 u\,du
\end{aligned}$$

We use a reduction formula to compute this integral, obtaining

$$S = 16\pi\left[\frac{1}{3}\cos^3 u - \cos u\right]\bigg|_0^{\pi} = 16\pi\left[\frac{4}{3}\right] = \frac{64\pi}{3}$$

32. The surface generated by revolving the astroid $c(t) = (\cos^3 t, \sin^3 t)$ about the x-axis for $0 \le t \le \frac{\pi}{2}$

SOLUTION We have $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $x'(t) = -3\cos^2 t\sin t$, $y'(t) = 3\sin^2 t\cos t$. Hence,

$$x'(t)^2 + y'(t)^2 = 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t = 9\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) = 9\cos^2 t\sin^2 t$$

Using the formula for the surface area we get

$$S = 2\pi\int_0^{\pi/2} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\pi\int_0^{\pi/2}\sin^3 t\cdot 3\cos t\sin t\,dt = 6\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

We compute the integral using the substitution $u = \sin t$ $du = \cos t\,dt$. We obtain

$$S = 6\pi\int_0^1 u^4\,du = 6\pi\frac{u^5}{5}\bigg|_0^1 = \frac{6\pi}{5}.$$

Further Insights and Challenges

33. CAS Let $b(t)$ be the "Butterfly Curve":

$$x(t) = \sin t \left(e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5 \right)$$

$$y(t) = \cos t \left(e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5 \right)$$

(a) Use a computer algebra system to plot $b(t)$ and the speed $s'(t)$ for $0 \le t \le 12\pi$.
(b) Approximate the length $b(t)$ for $0 \le t \le 10\pi$.

SOLUTION

(a) Let $f(t) = e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5$, then

$$x(t) = \sin t f(t)$$
$$y(t) = \cos t f(t)$$

and so

$$(x'(t))^2 + (y'(t))^2 = [\sin t f'(t) + \cos t f(t)]^2 + [\cos t f'(t) - \sin t f(t)]^2$$

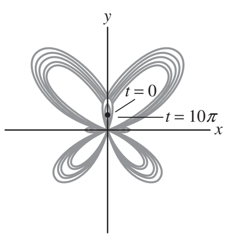
Using the identity $\sin^2 t + \cos^2 t = 1$, we get

$$(x'(t))^2 + (y'(t))^2 = (f'(t))^2 + (f(t))^2.$$

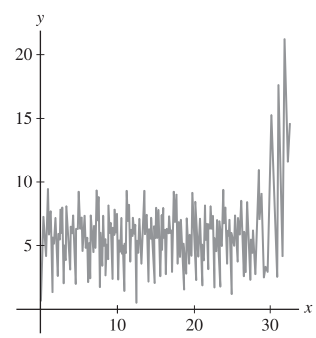
Thus, $s'(t)$ is the following:

$$\sqrt{\left[e^{\cos t} - 2\cos 4t - \sin\left(\frac{t}{12}\right)^5\right]^2 + \left[-\sin t e^{\cos t} + 8\sin 4t - \frac{5}{12}\left(\frac{t}{12}\right)^4 \cos\left(\frac{t}{12}\right)^5\right]^2}.$$

The following figures show the curves of $b(t)$ and the speed $s'(t)$ for $0 \le t \le 10\pi$:



The "Butterfly Curve" $b(t)$, $0 \le t \le 10\pi$



$s'(t)$, $0 \le t \le 10\pi$

Looking at the graph, we see it would be difficult to compute the length using numeric integration; due to the high frequency oscillations, very small steps would be needed.

(b) The length of $b(t)$ for $0 \le t \le 10\pi$ is given by the integral: $L = \int_0^{10\pi} s'(t)\,dt$ where $s'(t)$ is given in part (a). We approximate the length using the Midpoint Rule with $N = 30$. The numerical methods in Mathematica approximate the answer by 211.952. Using the Midpoint Rule with $N = 50$, we get 204.48; with $N = 500$, we get 211.6; and with $N = 5000$, we get 212.09.

34. CAS Let $a \ge b > 0$ and set $k = \dfrac{2\sqrt{ab}}{a-b}$. Show that the **trochoid**

$$x = at - b\sin t, \qquad y = a - b\cos t, \quad 0 \le t \le T$$

has length $2(a-b)G\left(\frac{T}{2}, k\right)$ with $G(\theta, k)$ as in Exercise 28.

SOLUTION We have $x'(t) = a - b\cos t$, $y'(t) = b\sin t$. Hence,

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= (a - b\cos t)^2 + (b\sin t)^2 = a^2 - 2ab\cos t + b^2\cos^2 t + b^2\sin^2 t \\ &= a^2 + b^2 - 2ab\cos t \end{aligned}$$

The length of the trochoid for $0 \le t \le T$ is

$$L = \int_0^T \sqrt{a^2 + b^2 - 2ab\cos t}\, dt$$

We rewrite the integrand as follows to bring it to the required form. We use the identity $1 - \cos t = 2\sin^2 \frac{t}{2}$ to obtain

$$\begin{aligned} L &= \int_0^T \sqrt{(a-b)^2 + 2ab - 2ab\cos t}\, dt = \int_0^T \sqrt{(a-b)^2 + 2ab(1-\cos t)}\, dt \\ &= \int_0^T \sqrt{(a-b)^2 + 4ab\sin^2 \frac{t}{2}}\, dt = \int_0^T \sqrt{(a-b)^2 \left(1 + \frac{4ab}{(a-b)^2}\sin^2 \frac{t}{2}\right)}\, dt \\ &= (a-b)\int_0^T \sqrt{1 + k^2 \sin^2 \frac{t}{2}}\, dt \end{aligned}$$

(where $k = \frac{2\sqrt{ab}}{a-b}$).

Substituting $u = \frac{t}{2}$, $du = \frac{1}{2}\,dt$, we get

$$L = 2(a-b)\int_0^{T/2} \sqrt{1 + k^2 \sin^2 u}\, du = 2(a-b)E(T/2, k)$$

35. A satellite orbiting at a distance R from the center of the earth follows the circular path $x = R\cos\omega t$, $y = R\sin\omega t$.

(a) Show that the period T (the time of one revolution) is $T = 2\pi/\omega$.

(b) According to Newton's laws of motion and gravity,

$$x''(t) = -Gm_e \frac{x}{R^3}, \qquad y''(t) = -Gm_e \frac{y}{R^3}$$

where G is the universal gravitational constant and m_e is the mass of the earth. Prove that $R^3/T^2 = Gm_e/4\pi^2$. Thus, R^3/T^2 has the same value for all orbits (a special case of Kepler's Third Law).

SOLUTION

(a) As shown in Example 4, the circular path has constant speed of $\frac{ds}{dt} = \omega R$. Since the length of one revolution is $2\pi R$, the period T is

$$T = \frac{2\pi R}{\omega R} = \frac{2\pi}{\omega}.$$

(b) Differentiating $x = R\cos\omega t$ twice with respect to t gives

$$\begin{aligned} x'(t) &= -R\omega\sin\omega t \\ x''(t) &= -R\omega^2\cos\omega t \end{aligned}$$

Substituting $x(t)$ and $x''(t)$ in the equation $x''(t) = -Gm_e \dfrac{x}{R^3}$ and simplifying, we obtain

$$\begin{aligned} -R\omega^2\cos\omega t &= -Gm_e \cdot \frac{R\cos\omega t}{R^3} \\ -R\omega^2 &= -\frac{Gm_e}{R^2} \Rightarrow R^3 = \frac{Gm_e}{\omega^2} \end{aligned}$$

By part (a), $T = \dfrac{2\pi}{\omega}$. Hence, $\omega = \dfrac{2\pi}{T}$. Substituting yields

$$R^3 = \frac{Gm_e}{\frac{4\pi^2}{T^2}} = \frac{T^2 Gm_e}{4\pi^2} \Rightarrow \frac{R^3}{T^2} = \frac{Gm_e}{4\pi^2}$$

36. The acceleration due to gravity on the surface of the earth is

$$g = \frac{Gm_e}{R_e^2} = 9.8 \text{ m/s}^2, \quad \text{where } R_e = 6378 \text{ km}$$

Use Exercise 35(b) to show that a satellite orbiting at the earth's surface would have period $T_e = 2\pi\sqrt{R_e/g} \approx 84.5$ min. Then estimate the distance R_m from the moon to the center of the earth. Assume that the period of the moon (sidereal month) is $T_m \approx 27.43$ days.

SOLUTION By part (b) of Exercise 35, it follows that

$$\frac{R_e^3}{T_e^2} = \frac{Gm_e}{4\pi^2} \Rightarrow T_e^2 = \frac{4\pi^2 R_e^3}{Gm_e} = \frac{4\pi^2 R_e}{\frac{Gm_e}{R_e^2}} = \frac{4\pi^2 R_e}{g}$$

Hence,

$$T_e = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6378 \cdot 10^3}{9.8}} \approx 5068.8 \text{ s} \approx 84.5 \text{ min}.$$

In part (b) of Exercise 35 we showed that $\frac{R^3}{T^2}$ is the same for all orbits. It follows that this quotient is the same for the satellite orbiting at the earth's surface and for the moon orbiting around the earth. Thus,

$$\frac{R_m^3}{T_m^2} = \frac{R_e^3}{T_e^2} \Rightarrow R_m = R_e\left(\frac{T_m}{T_e}\right)^{2/3}.$$

Setting $T_m = 27.43 \cdot 1440 = 39{,}499.2$ minutes, $T_e = 84.5$ minutes, and $R_e = 6378$ km we get

$$R_m = 6378\left(\frac{39{,}499.2}{84.5}\right)^{2/3} \approx 384{,}154 \text{ km}.$$

11.3 Polar Coordinates

Preliminary Questions

1. Points P and Q with the same radial coordinate (choose the correct answer):

(a) Lie on the same circle with the center at the origin.

(b) Lie on the same ray based at the origin.

SOLUTION Two points with the same radial coordinate are equidistant from the origin, therefore they lie on the same circle centered at the origin. The angular coordinate defines a ray based at the origin. Therefore, if the two points have the same angular coordinate, they lie on the same ray based at the origin.

2. Give two polar representations for the point $(x, y) = (0, 1)$, one with negative r and one with positive r.

SOLUTION The point $(0, 1)$ is on the y-axis, distant one unit from the origin, hence the polar representation with positive r is $(r, \theta) = \left(1, \frac{\pi}{2}\right)$. The point $(r, \theta) = \left(-1, \frac{\pi}{2}\right)$ is the reflection of $(r, \theta) = \left(1, \frac{\pi}{2}\right)$ through the origin, hence we must add π to return to the original point.

We obtain the following polar representation of $(0, 1)$ with negative r:

$$(r, \theta) = \left(-1, \frac{\pi}{2} + \pi\right) = \left(-1, \frac{3\pi}{2}\right).$$

3. Describe each of the following curves:

(a) $r = 2$ **(b)** $r^2 = 2$ **(c)** $r\cos\theta = 2$

SOLUTION

(a) Converting to rectangular coordinates we get

$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 2^2.$$

This is the equation of the circle of radius 2 centered at the origin.

(b) We convert to rectangular coordinates, obtaining $x^2 + y^2 = 2$. This is the equation of the circle of radius $\sqrt{2}$, centered at the origin.

(c) We convert to rectangular coordinates. Since $x = r\cos\theta$ we obtain the following equation: $x = 2$. This is the equation of the vertical line through the point $(2, 0)$.

4. If $f(-\theta) = f(\theta)$, then the curve $r = f(\theta)$ is symmetric with respect to the (choose the correct answer):

(a) x-axis **(b)** y-axis **(c)** origin

SOLUTION The equality $f(-\theta) = f(\theta)$ for all θ implies that whenever a point (r, θ) is on the curve, also the point $(r, -\theta)$ is on the curve. Since the point $(r, -\theta)$ is the reflection of (r, θ) with respect to the x-axis, we conclude that the curve is symmetric with respect to the x-axis.

Exercises

1. Find polar coordinates for each of the seven points plotted in Figure 16.

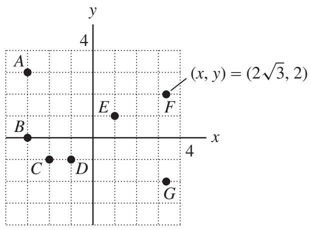
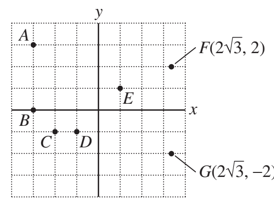


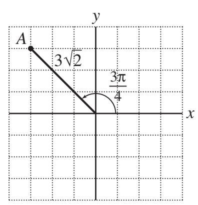
FIGURE 16

SOLUTION We mark the points as shown in the figure.

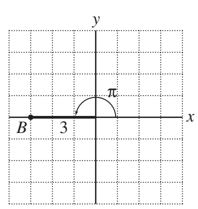


Using the data given in the figure for the x and y coordinates and the quadrants in which the point are located, we obtain:

(A), with rectangular coordinates $(-3, 4)$: $\begin{array}{l} r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} \\ \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \end{array} \Rightarrow (r, \theta) = \left(3\sqrt{2}, \frac{3\pi}{4}\right)$

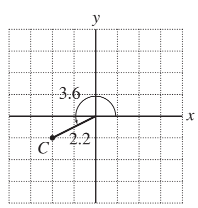


(B), with rectangular coordinates $(-3, 0)$: $\begin{array}{l} r = 3 \\ \theta = \pi \end{array} \Rightarrow (r, \theta) = (3, \pi)$

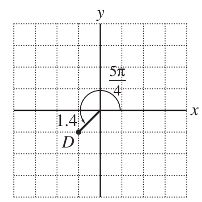


(C), with rectangular coordinates $(-2, -1)$:

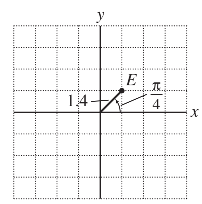
$$\begin{array}{l} r = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.2 \\ \theta = \tan^{-1}\left(\frac{-1}{-2}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \pi + 0.46 \approx 3.6 \end{array} \Rightarrow (r, \theta) \approx \left(\sqrt{5}, 3.6\right)$$



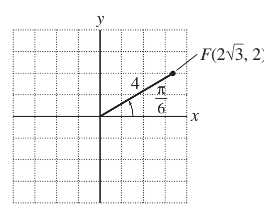
(D), with rectangular coordinates $(-1,-1)$: $\begin{array}{l} r=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.4 \\ \theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4} \end{array}$ $\Rightarrow (r,\theta)\approx\left(\sqrt{2},\frac{5\pi}{4}\right)$



(E), with rectangular coordinates $(1,1)$: $\begin{array}{l} r=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.4 \\ \theta=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4} \end{array}$ $\Rightarrow (r,\theta)\approx\left(\sqrt{2},\frac{\pi}{4}\right)$



(F), with rectangular coordinates $(2\sqrt{3},2)$: $\begin{array}{l} r=\sqrt{\left(2\sqrt{3}\right)^2+2^2}=\sqrt{16}=4 \\ \theta=\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6} \end{array}$ $\Rightarrow (r,\theta)=\left(4,\frac{\pi}{6}\right)$

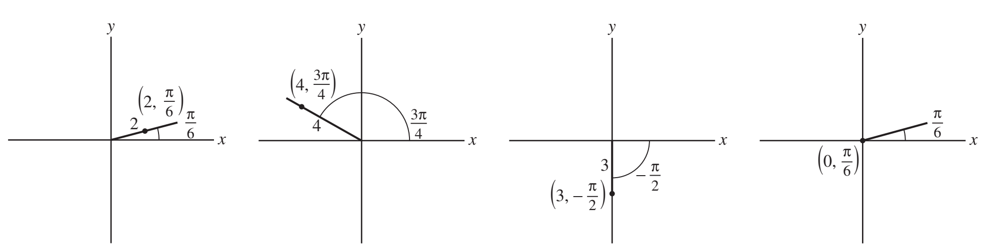


(G), with rectangular coordinates $(2\sqrt{3},-2)$: G is the reflection of F about the x axis, hence the two points have equal radial coordinates, and the angular coordinate of G is obtained from the angular coordinate of F: $\theta=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$. Hence, the polar coordinates of G are $\left(4,\frac{11\pi}{6}\right)$.

2. Plot the points with polar coordinates:

(a) $\left(2,\frac{\pi}{6}\right)$ **(b)** $\left(4,\frac{3\pi}{4}\right)$ **(c)** $\left(3,-\frac{\pi}{2}\right)$ **(d)** $\left(0,\frac{\pi}{6}\right)$

SOLUTION We first plot the ray $\theta=\theta_0$ for the given angle θ_0, and then mark the point on this line distanced $r=r_0$ from the origin. We obtain the following points:



(a) (b) (c) (d)

$R=0$ is the point $(0,0)$ in rect. coords.

3. Convert from rectangular to polar coordinates.

(a) $(1,0)$ **(b)** $(3,\sqrt{3})$ **(c)** $(-2,2)$ **(d)** $(-1,\sqrt{3})$

SOLUTION

(a) The point $(1,0)$ is on the positive x axis distanced one unit from the origin. Hence, $r=1$ and $\theta=0$. Thus, $(r,\theta)=(1,0)$.

(b) The point $\left(3,\sqrt{3}\right)$ is in the first quadrant so $\theta=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}$. Also, $r=\sqrt{3^2+\left(\sqrt{3}\right)^2}=\sqrt{12}$. Hence, $(r,\theta)=\left(\sqrt{12},\frac{\pi}{6}\right)$.

(c) The point $(-2, 2)$ is in the second quadrant. Hence,

$$\theta = \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Also, $r = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$. Hence, $(r, \theta) = \left(\sqrt{8}, \frac{3\pi}{4}\right)$.

(d) The point $\left(-1, \sqrt{3}\right)$ is in the second quadrant, hence,

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Also, $r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$. Hence, $(r, \theta) = \left(2, \frac{2\pi}{3}\right)$.

4. Convert from rectangular to polar coordinates using a calculator (make sure your choice of θ gives the correct quadrant).

(a) $(2, 3)$ **(b)** $(4, -7)$ **(c)** $(-3, -8)$ **(d)** $(-5, 2)$

SOLUTION

(a) The point $(2, 3)$ is in the first quadrant, with $x = 2$ and $y = 3$. Hence

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{3}{2}\right) \approx 0.98 \\ r &= \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.6 \end{aligned} \quad \Rightarrow \quad (r, \theta) \approx (3.6, 0.98)\,.$$

(b) The point $(4, -7)$ is in the fourth quadrant with $x = 4$ and $y = -7$. We have

$$\tan^{-1}\left(\frac{-7}{4}\right) \approx -1.05$$

$$r = \sqrt{(-7)^2 + 4^2} = \sqrt{65} \approx 8.1$$

Note that $\tan^{-1}$ an angle less that zero in the fourth quadrant; since we want an angle between 0 and 2π, we add 2π to get $\theta \approx 2\pi - 1.05 \approx 5.232$. Thus $(r, \theta) \approx (8.1, 5.2)$.

(c) The point $(-3, -8)$ is in the third quadrant, with $x = -3$ and $y = -8$. We have

$$\tan^{-1}\left(\frac{-8}{-3}\right) = \tan^{-1}\left(\frac{8}{3}\right) \approx 1.212$$

$$r = \sqrt{(-3)^2 + (-8)^2} = \sqrt{73} \approx 8.54$$

Note that $\tan^{-1}$ produced an angle in the first quadrant; we want the third quadrant angle with the same tangent, so we add π to get $\theta \approx \pi + 1.212 \approx 4.35$. Thus $(r, \theta) \approx (8.54, 4.35)$

(d) The point $(-5, 2)$ is in the second quadrant, with $x = -5$ and $y = 2$. We have

$$\tan^{-1}\left(\frac{2}{-5}\right) \approx -0.38$$

$$r = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.39$$

Note that the angle is in the fourth quadrant; to get the second quadrant angle with the same tangent and in the range $[0, 2\pi)$, we add π to get $\theta \approx \pi - 0.38 \approx 2.76$. Thus $(r, \theta) \approx (5.39, 2.76)$.

5. Convert from polar to rectangular coordinates:

(a) $\left(3, \frac{\pi}{6}\right)$ **(b)** $\left(6, \frac{3\pi}{4}\right)$ **(c)** $\left(0, \frac{\pi}{5}\right)$ **(d)** $\left(5, -\frac{\pi}{2}\right)$

SOLUTION

(a) Since $r = 3$ and $\theta = \frac{\pi}{6}$, we have:

$$\begin{aligned} x &= r\cos\theta = 3\cos\frac{\pi}{6} = 3 \cdot \frac{\sqrt{3}}{2} \approx 2.6 \\ y &= r\sin\theta = 3\sin\frac{\pi}{6} = 3 \cdot \frac{1}{2} = 1.5 \end{aligned} \quad \Rightarrow \quad (x, y) \approx (2.6, 1.5)\,.$$

(b) For $\left(6, \frac{3\pi}{4}\right)$ we have $r = 6$ and $\theta = \frac{3\pi}{4}$. Hence,

$$\begin{aligned} x &= r\cos\theta = 6\cos\frac{3\pi}{4} \approx -4.24 \\ y &= r\sin\theta = 6\sin\frac{3\pi}{4} \approx 4.24 \end{aligned} \quad \Rightarrow \quad (x, y) \approx (-4.24, 4.24)\,.$$

(c) For $\left(0, \frac{\pi}{5}\right)$, we have $r = 0$, so that the rectangular coordinates are $(x, y) = (0, 0)$.

(d) Since $r = 5$ and $\theta = -\frac{\pi}{2}$ we have

$$\begin{aligned} x &= r\cos\theta = 5\cos\left(-\frac{\pi}{2}\right) = 5\cdot 0 = 0 \\ y &= r\sin\theta = 5\sin\left(-\frac{\pi}{2}\right) = 5\cdot(-1) = -5 \end{aligned} \quad \Rightarrow \quad (x, y) = (0, -5)$$

6. Which of the following are possible polar coordinates for the point P with rectangular coordinates $(0, -2)$?

(a) $\left(2, \frac{\pi}{2}\right)$

(b) $\left(2, \frac{7\pi}{2}\right)$

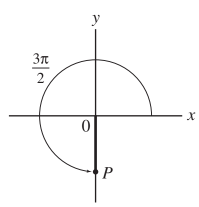
(c) $\left(-2, -\frac{3\pi}{2}\right)$

(d) $\left(-2, \frac{7\pi}{2}\right)$

(e) $\left(-2, -\frac{\pi}{2}\right)$

(f) $\left(2, -\frac{7\pi}{2}\right)$

SOLUTION The point P has distance 2 from the origin and the angle between $\overline{OP}$ and the positive x-axis in the positive direction is $\frac{3\pi}{2}$. Hence, $(r, \theta) = \left(2, \frac{3\pi}{2}\right)$ is one choice for the polar coordinates for P.



The polar coordinates $(2, \theta)$ are possible for P if $\theta - \frac{3\pi}{2}$ is a multiple of 2π. The polar coordinate $(-2, \theta)$ are possible for P if $\theta - \frac{3\pi}{2}$ is an odd multiple of π. These considerations lead to the following conclusions:

(a) $\left(2, \frac{\pi}{2}\right)$ $\frac{\pi}{2} - \frac{3\pi}{2} = -\pi \Rightarrow \left(2, \frac{\pi}{2}\right)$ does not represent P.

(b) $\left(2, \frac{7\pi}{2}\right)$ $\frac{7\pi}{2} - \frac{3\pi}{2} = 2\pi \Rightarrow \left(2, \frac{7\pi}{2}\right)$ represents P.

(c) $\left(-2, -\frac{3\pi}{2}\right)$ $-\frac{3\pi}{2} - \frac{3\pi}{2} = -3\pi \Rightarrow \left(-2, -\frac{3\pi}{2}\right)$ represents P.

(d) $\left(-2, \frac{7\pi}{2}\right)$ $\frac{7\pi}{2} - \frac{3\pi}{2} = 2\pi \Rightarrow \left(-2, \frac{7\pi}{2}\right)$ does not represent P.

(e) $\left(-2, -\frac{\pi}{2}\right)$ $-\frac{\pi}{2} - \frac{3\pi}{2} = -2\pi \Rightarrow \left(-2, -\frac{\pi}{2}\right)$ does not represent P.

(f) $\left(2, -\frac{7\pi}{2}\right)$ $-\frac{7\pi}{2} - \frac{3\pi}{2} = -5\pi \Rightarrow \left(2, -\frac{7\pi}{2}\right)$ does not represent P.

7. Describe each shaded sector in Figure 17 by inequalities in r and θ.

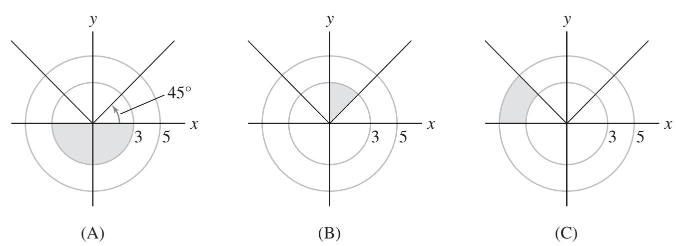


FIGURE 17

SOLUTION

(a) In the sector shown below r is varying between 0 and 3 and θ is varying between π and 2π. Hence the following inequalities describe the sector:

$$0 \le r \le 3$$
$$\pi \le \theta \le 2\pi$$

(b) In the sector shown below r is varying between 0 and 3 and θ is varying between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Hence, the inequalities for the sector are:

$$0 \le r \le 3$$
$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

(c) In the sector shown below r is varying between 3 and 5 and θ is varying between $\frac{3\pi}{4}$ and π. Hence, the inequalities are:

$$3 \le r \le 5$$
$$\frac{3\pi}{4} \le \theta \le \pi$$

8. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{2}$.

SOLUTION A line of slope $m = \frac{1}{2}$ makes an angle $\theta_0 = \tan^{-1}\frac{1}{2} \approx 0.46$ with the positive x-axis. The equation of the line is $\theta \approx 0.46$, while r is arbitrary.

9. What is the slope of the line $\theta = \frac{3\pi}{5}$?

SOLUTION This line makes an angle $\theta_0 = \frac{3\pi}{5}$ with the positive x-axis, hence the slope of the line is $m = \tan\frac{3\pi}{5} \approx -3.1$.

10. Which of $r = 2\sec\theta$ and $r = 2\csc\theta$ defines a horizontal line?

SOLUTION The equation $r = 2\csc\theta$ is the polar equation of a horizontal line, as it can be written as $r = 2/\sin\theta$, so $r\sin\theta = 2$, which becomes $y = 2$. On the other hand, the equation $r = 2\sec\theta$ is the polar equation of a vertical line, as it can be written as $r = 2/\cos\theta$, so $r\cos\theta = 2$, which becomes $x = 2$.

In Exercises 11–16, convert to an equation in rectangular coordinates.

11. $r = 7$

SOLUTION $r = 7$ describes the points having distance 7 from the origin, that is, the circle with radius 7 centered at the origin. The equation of the circle in rectangular coordinates is

$$x^2 + y^2 = 7^2 = 49.$$

12. $r = \sin\theta$

SOLUTION Multiplying by r and substituting $y = r\sin\theta$ and $r^2 = x^2 + y^2$ gives

$$r^2 = r\sin\theta$$
$$x^2 + y^2 = y$$

We move the y and then complete the square to obtain

$$x^2 + y^2 - y = 0$$
$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Thus, $r = \sin\theta$ is the equation of a circle of radius $\frac{1}{2}$ and center $\left(0, \frac{1}{2}\right)$.

13. $r = 2\sin\theta$

SOLUTION We multiply the equation by r and substitute $r^2 = x^2 + y^2$, $r\sin\theta = y$. This gives

$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$

Moving the $2y$ and completing the square yield: $x^2 + y^2 - 2y = 0$ and $x^2 + (y-1)^2 = 1$. Thus, $r = 2\sin\theta$ is the equation of a circle of radius 1 centered at $(0, 1)$.

14. $r = 2\csc\theta$

SOLUTION We multiply the equation by $\sin\theta$ and substitute $y = r\sin\theta$. We get

$$r\sin\theta = 2$$
$$y = 2$$

Thus, $r = 2\csc\theta$ is the equation of the line $y = 2$.

15. $r = \dfrac{1}{\cos\theta - \sin\theta}$

SOLUTION We multiply the equation by $\cos\theta - \sin\theta$ and substitute $y = r\sin\theta$, $x = r\cos\theta$. This gives

$$r(\cos\theta - \sin\theta) = 1$$
$$r\cos\theta - r\sin\theta = 1$$

$x - y = 1 \Rightarrow y = x - 1$. Thus,

$$r = \frac{1}{\cos\theta - \sin\theta}$$

is the equation of the line $y = x - 1$.

16. $r = \dfrac{1}{2 - \cos\theta}$

SOLUTION We multiply the equation by $2 - \cos\theta$. Then we substitute $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$, to obtain

$$r(2 - \cos\theta) = 1$$
$$2r - r\cos\theta = 1$$
$$2\sqrt{x^2 + y^2} - x = 1$$

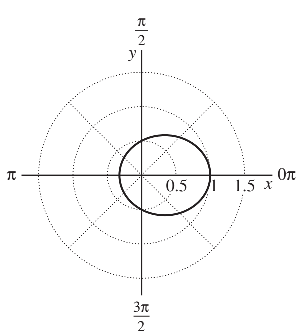
Moving the x, then squaring and simplifying, we obtain

$$2\sqrt{x^2 + y^2} = x + 1$$
$$4\left(x^2 + y^2\right) = x^2 + 2x + 1$$
$$3x^2 - 2x + 4y^2 = 1$$

We complete the square:

$$3\left(x^2 - \frac{2}{3}x\right) + 4y^2 = 1$$
$$3\left(x - \frac{1}{3}\right)^2 + 4y^2 = \frac{4}{3}$$
$$\frac{\left(x - \frac{1}{3}\right)^2}{\frac{4}{9}} + \frac{y^2}{\frac{1}{3}} = 1$$

This is the equation of the ellipse shown in the figure:



In Exercises 17–20, convert to an equation in polar coordinates.

17. $x^2 + y^2 = 5$

SOLUTION We make the substitution $x^2 + y^2 = r^2$ to obtain; $r^2 = 5$ or $r = \sqrt{5}$.

18. $x = 5$

SOLUTION Substituting $x = r\cos\theta$ gives the polar equation $r\cos\theta = 5$ or $r = 5\sec\theta$.

19. $y = x^2$

SOLUTION Substituting $y = r\sin\theta$ and $x = r\cos\theta$ yields

$$r\sin\theta = r^2\cos^2\theta.$$

Then, dividing by $r\cos^2\theta$ we obtain,

$$\frac{\sin\theta}{\cos^2\theta} = r \qquad \text{so} \qquad r = \tan\theta\sec\theta$$

20. $xy = 1$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ to obtain

$$(r\cos\theta)(r\sin\theta) = 1$$
$$r^2\cos\theta\sin\theta = 1$$

Using the identity $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$ yields

$$r^2\cdot\frac{\sin 2\theta}{2} = 1 \Rightarrow r^2 = 2\csc 2\theta.$$

21. Match each equation with its description.

(a) $r = 2$	**(i)** Vertical line
(b) $\theta = 2$	**(ii)** Horizontal line
(c) $r = 2\sec\theta$	**(iii)** Circle
(d) $r = 2\csc\theta$	**(iv)** Line through origin

SOLUTION

(a) $r = 2$ describes the points 2 units from the origin. Hence, it is the equation of a circle.

(b) $\theta = 2$ describes the points P so that $\overline{OP}$ makes an angle of $\theta_0 = 2$ with the positive x-axis. Hence, it is the equation of a line through the origin.

(c) This is $r\cos\theta = 2$, which is $x = 2$, a vertical line.

(d) Converting to rectangular coordinates, we get $r = 2\csc\theta$, so $r\sin\theta = 2$ and $y = 2$. This is the equation of a horizontal line.

22. Find the values of θ in the plot of $r = 4\cos\theta$ corresponding to points A, B, C, D in Figure 18. Then indicate the portion of the graph traced out as θ varies in the following intervals:

(a) $0 \le \theta \le \frac{\pi}{2}$ **(b)** $\frac{\pi}{2} \le \theta \le \pi$ **(c)** $\pi \le \theta \le \frac{3\pi}{2}$

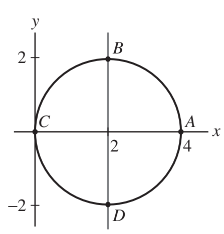


FIGURE 18 Plot of $r = 4\cos\theta$.

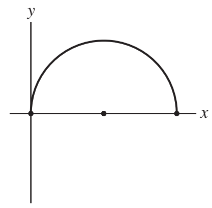
SOLUTION The point A is on the x-axis hence $\theta = 0$. The point B is in the first quadrant with $x = y = 2$ hence $\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$. The point C is at the origin. Thus,

$$r = 0 \Rightarrow 4\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2},\ \frac{3\pi}{2}.$$

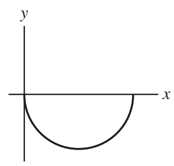
The point D is in the fourth quadrant with $x = 2$, $y = -2$, hence

$$\theta = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}.$$

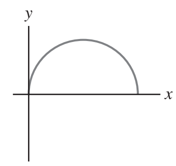
$0 \le \theta \le \frac{\pi}{2}$ represents the first quadrant, hence the points (r,θ) where $r = 4\cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$ are the points on the circle which are in the first quadrant, as shown below:



If we insist that $r \geq 0$, then since $\frac{\pi}{2} \leq \theta \leq \pi$ represents the second quadrant and $\pi \leq \theta \leq \frac{3\pi}{2}$ represents the third quadrant, and since the circle $r = 4\cos\theta$ has no points in the left xy -plane, then there are no points for (b) and (c). However, if we allow $r < 0$ then (b) represents the semi-circle



and (c) like (a) represent

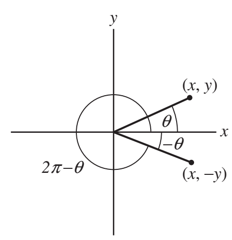


23. Suppose that $P = (x, y)$ has polar coordinates (r, θ). Find the polar coordinates for the points:

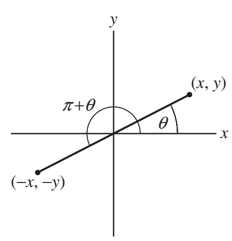
(a) $(x, -y)$ **(b)** $(-x, -y)$ **(c)** $(-x, y)$ **(d)** (y, x)

SOLUTION

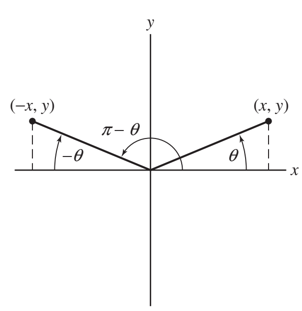
(a) $(x, -y)$ is the symmetric point of (x, y) with respect to the x-axis, hence the two points have the same radial coordinate, and the angular coordinate of $(x, -y)$ is $2\pi - \theta$. Hence, $(x, -y) = (r, 2\pi - \theta)$.



(b) $(-x, -y)$ is the symmetric point of (x, y) with respect to the origin. Hence, $(-x, -y) = (r, \theta + \pi)$.



(c) $(-x, y)$ is the symmetric point of (x, y) with respect to the y-axis. Hence the two points have the same radial coordinates and the angular coordinate of $(-x, y)$ is $\pi - \theta$. Hence, $(-x, y) = (r, \pi - \theta)$.

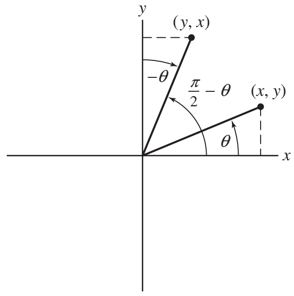


(d) Let (r_1, θ_1) denote the polar coordinates of (y, x). Hence,

$$r_1 = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2} = r$$

$$\tan\theta_1 = \frac{x}{y} = \frac{1}{y/x} = \frac{1}{\tan\theta} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Since the points (x, y) and (y, x) are in the same quadrant, the solution for θ_1 is $\theta_1 = \frac{\pi}{2} - \theta$. We obtain the following polar coordinates: $(y, x) = \left(r, \frac{\pi}{2} - \theta\right)$.



24. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$	**(i)** $r^2(1 - 2\sin^2\theta) = 4$
(b) $x^2 + (y-1)^2 = 1$	**(ii)** $r(\cos\theta + \sin\theta) = 4$
(c) $x^2 - y^2 = 4$	**(iii)** $r = 2\sin\theta$
(d) $x + y = 4$	**(iv)** $r = 2$

SOLUTION

(a) Since $x^2 + y^2 = r^2$, we have $r^2 = 4$ or $r = 2$.
(b) Using Example 7, the equation of the circle $x^2 + (y-1)^2 = 1$ has polar equation $r = 2\sin\theta$.
(c) Setting $x = r\cos\theta$, $y = r\sin\theta$ in $x^2 - y^2 = 4$ gives

$$x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\left(\cos^2\theta - \sin^2\theta\right) = 4.$$

We now use the identity $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$ to obtain the following equation:

$$r^2\left(1 - 2\sin^2\theta\right) = 4.$$

(d) Setting $x = r\cos\theta$ and $y = r\sin\theta$ in $x + y = 4$ we get:

$$x + y = 4$$
$$r\cos\theta + r\sin\theta = 4$$

so

$$r(\cos\theta + \sin\theta) = 4$$

25. What are the polar equations of the lines parallel to the line $r\cos\left(\theta - \frac{\pi}{3}\right) = 1$?

SOLUTION The line $r\cos\left(\theta - \frac{\pi}{3}\right) = 1$, or $r = \sec\left(\theta - \frac{\pi}{3}\right)$, is perpendicular to the ray $\theta = \frac{\pi}{3}$ and at distance $d = 1$ from the origin. Hence, the lines parallel to this line are also perpendicular to the ray $\theta = \frac{\pi}{3}$, so the polar equations of these lines are $r = d\sec\left(\theta - \frac{\pi}{3}\right)$ or $r\cos\left(\theta - \frac{\pi}{3}\right) = d$.

26. Show that the circle with center at $\left(\frac{1}{2}, \frac{1}{2}\right)$ in Figure 19 has polar equation $r = \sin\theta + \cos\theta$ and find the values of θ between 0 and π corresponding to points A, B, C, and D.

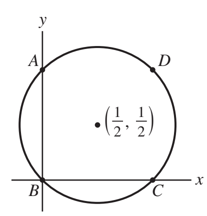


FIGURE 19 Plot of $r = \sin\theta + \cos\theta$.

SOLUTION We show that the rectangular equation of $r = \sin\theta + \cos\theta$ is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

We multiply the polar equation by r and substitute $r^2 = x^2 + y^2$, $r\sin\theta = y$, $r\cos\theta = x$. This gives

$$r = \sin\theta + \cos\theta$$
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$

Transferring sides and completing the square yields

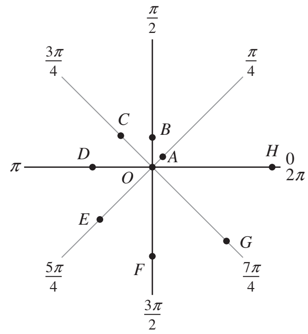
$$x^2 - x + y^2 - y = 0$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Clearly point C corresponds to $\theta = 0$ since $\cos 0 + \sin 0 = 1$. The circle is traced out counterclockwise as θ increases to π, so A corresponds to $\theta = \frac{\pi}{2}$ since again $\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0$. Next, D clearly corresponds to $\theta = \frac{\pi}{4}$, and indeed $\cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \sqrt{2}$, which is the diameter of the circle. Finally, point A corresponds to $\theta = \frac{3\pi}{4}$, since there $\cos\theta = -\sin\theta$.

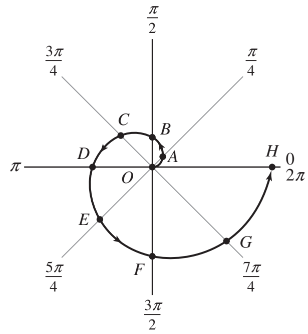
27. Sketch the curve $r = \frac{1}{2}\theta$ (the spiral of Archimedes) for θ between 0 and 2π by plotting the points for $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, 2\pi$.

SOLUTION We first plot the following points (r, θ) on the spiral:

$$O = (0,0),\ A = \left(\frac{\pi}{8}, \frac{\pi}{4}\right),\ B = \left(\frac{\pi}{4}, \frac{\pi}{2}\right),\ C = \left(\frac{3\pi}{8}, \frac{3\pi}{4}\right),\ D = \left(\frac{\pi}{2}, \pi\right),$$
$$E = \left(\frac{5\pi}{8}, \frac{5\pi}{4}\right),\ F = \left(\frac{3\pi}{4}, \frac{3\pi}{2}\right),\ G = \left(\frac{7\pi}{8}, \frac{7\pi}{4}\right),\ H = (\pi, 2\pi).$$



Since $r(0) = \frac{0}{2} = 0$, the graph begins at the origin and moves toward the points A, B, C, D, E, F, G and H as θ varies from $\theta = 0$ to the other values stated above. Connecting the points in this direction we obtain the following graph for $0 \le \theta \le 2\pi$:



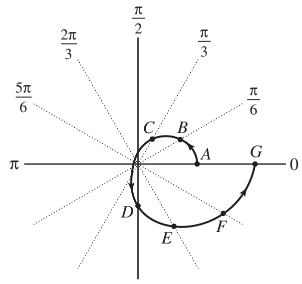
28. Sketch $r = 3\cos\theta - 1$ (see Example 8).

SOLUTION We first choose some values of θ between 0 and π and mark the corresponding points on the graph. Then we use symmetry (due to $\cos(2\pi - \theta) = \cos\theta$) to plot the other half of the graph by reflecting the first half through the x-axis. Since $r = 3\cos\theta - 1$ is periodic, the entire curve is obtained as θ varies from 0 to 2π. We start with the values $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$, and compute the corresponding values of r:

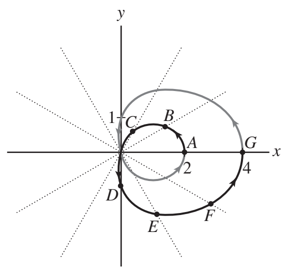
$$r = 3\cos 0 - 1 = 3 - 1 = 2 \Rightarrow A = (2, 0)$$
$$r = 3\cos\frac{\pi}{6} - 1 = \frac{3\sqrt{3}}{2} - 1 \approx 1.6 \Rightarrow B = \left(1.6, \frac{\pi}{6}\right)$$
$$r = 3\cos\frac{\pi}{3} - 1 = \frac{3}{2} - 1 = 0.5 \Rightarrow C = \left(0.5, \frac{\pi}{3}\right)$$
$$r = 3\cos\frac{\pi}{2} - 1 = 3 \cdot 0 - 1 = -1 \Rightarrow D = \left(-1, \frac{\pi}{2}\right)$$

$$r = 3\cos\frac{2\pi}{3} - 1 = -2.5 \Rightarrow E = \left(-2.5, \frac{2\pi}{3}\right)$$
$$r = 3\cos\frac{5\pi}{6} - 1 = -3.6 \Rightarrow F = \left(-3.6, \frac{5\pi}{6}\right)$$
$$r = 3\cos\pi - 1 = -4 \Rightarrow G = (-4, \pi)$$

The graph begins at the point $(r, \theta) = (2, 0)$ and moves toward the other points in this order, as θ varies from 0 to π. Since r is negative for $\frac{\pi}{2} \le \theta \le \pi$, the curve continues into the fourth quadrant, rather than into the second quadrant. We obtain the following graph:



Now we have half the curve and we use symmetry to plot the rest. Reflecting the first half through the x axis we obtain the whole curve:

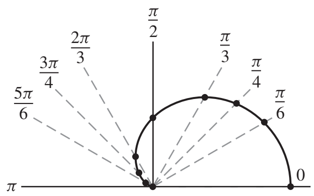


29. Sketch the cardioid curve $r = 1 + \cos\theta$.

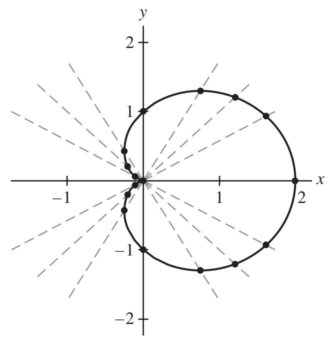
SOLUTION Since $\cos\theta$ is period with period 2π, the entire curve will be traced out as θ varies from 0 to 2π. Additionally, since $\cos(2\pi - \theta) = \cos(\theta)$, we can sketch the curve for θ between 0 and π and reflect the result through the x axis to obtain the whole curve. Use the values $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$, and π:

θ	r	point
0	$1 + \cos 0 = 2$	$(2, 0)$
$\frac{\pi}{6}$	$1 + \cos\frac{\pi}{6} = \frac{2+\sqrt{3}}{2}$	$\left(\frac{2+\sqrt{3}}{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$1 + \cos\frac{\pi}{4} = \frac{2+\sqrt{2}}{2}$	$\left(\frac{2+\sqrt{2}}{2}, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$1 + \cos\frac{\pi}{3} = \frac{3}{2}$	$\left(\frac{3}{2}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$1 + \cos\frac{\pi}{2} = 1$	$\left(1, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$1 + \cos\frac{2\pi}{3} = \frac{1}{2}$	$\left(\frac{1}{2}, \frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$	$1 + \cos\frac{3\pi}{4} = \frac{2-\sqrt{2}}{2}$	$\left(\frac{2-\sqrt{2}}{2}, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	$1 + \cos\frac{5\pi}{6} = \frac{2-\sqrt{3}}{2}$	$\left(\frac{2-\sqrt{3}}{2}, \frac{5\pi}{6}\right)$

$\theta = 0$ corresponds to the point $(2, 0)$, and the graph moves clockwise as θ increases from 0 to π. Thus the graph is



Reflecting through the x axis gives the other half of the curve:



30. Show that the cardioid of Exercise 29 has equation

$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

in rectangular coordinates.

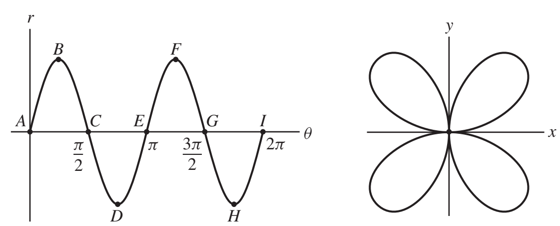
SOLUTION Multiply through by r and substitute for r, r^2, and $r\cos\theta$ to get

$$r = 1 + \cos\theta$$
$$r^2 = r + r\cos\theta$$
$$x^2 + y^2 = \sqrt{x^2 + y^2} + x$$
$$x^2 + y^2 - x = \sqrt{x^2 + y^2}$$
$$(x^2 + y^2 - x)^2 = x^2 + y^2$$

31. Figure 20 displays the graphs of $r = \sin 2\theta$ in rectangular coordinates and in polar coordinates, where it is a "rose with four petals." Identify:

(a) The points in (B) corresponding to points A–I in (A).

(b) The parts of the curve in (B) corresponding to the angle intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$, and $\left[\frac{3\pi}{2}, 2\pi\right]$.



(A) Graph of r as a function of θ, where $r = \sin 2\theta$

(B) Graph of $r = \sin 2\theta$ in polar coordinates

FIGURE 20

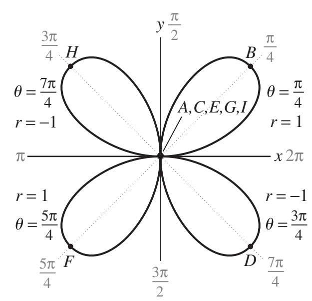
SOLUTION

(a) The graph (A) gives the following polar coordinates of the labeled points:

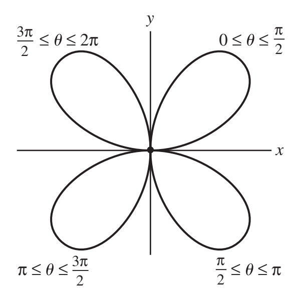
$$A: \quad \theta = 0, \quad r = 0$$
$$B: \quad \theta = \frac{\pi}{4}, \quad r = \sin\frac{2\pi}{4} = 1$$
$$C: \quad \theta = \frac{\pi}{2}, \quad r = 0$$
$$D: \quad \theta = \frac{3\pi}{4}, \quad r = \sin\frac{2\cdot 3\pi}{4} = -1$$
$$E: \quad \theta = \pi, \quad r = 0$$
$$F: \quad \theta = \frac{5\pi}{4}, \quad r = 1$$
$$G: \quad \theta = \frac{3\pi}{2}, \quad r = 0$$

$$H: \quad \theta = \frac{7\pi}{4}, \quad r = -1$$
$$I: \quad \theta = 2\pi, \quad r = 0.$$

Since the maximal value of $|r|$ is 1, the points with $r = 1$ or $r = -1$ are the furthest points from the origin. The corresponding quadrant is determined by the value of θ and the sign of r. If $r_0 < 0$, the point (r_0, θ_0) is on the ray $\theta = -\theta_0$. These considerations lead to the following identification of the points in the xy plane. Notice that A, C, G, E, and I are the same point.



(b) We use the graph (A) to find the sign of $r = \sin 2\theta$: $0 \le \theta \le \frac{\pi}{2} \Rightarrow r \ge 0 \Rightarrow (r, \theta)$ is in the first quadrant. $\frac{\pi}{2} \le \theta \le \pi \Rightarrow r \le 0 \Rightarrow (r, \theta)$ is in the fourth quadrant. $\pi \le \theta \le \frac{3\pi}{2} \Rightarrow r \ge 0 \Rightarrow (r, \theta)$ is in the third quadrant. $\frac{3\pi}{2} \le \theta \le 2\pi \Rightarrow r \le 0 \Rightarrow (r, \theta)$ is in the second quadrant. That is,



32. Sketch the curve $r = \sin 3\theta$. First fill in the table of r-values below and plot the corresponding points of the curve. Notice that the three petals of the curve correspond to the angle intervals $\left[0, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, and $\left[\frac{\pi}{3}, \pi\right]$. Then plot $r = \sin 3\theta$ in rectangular coordinates and label the points on this graph corresponding to (r, θ) in the table.

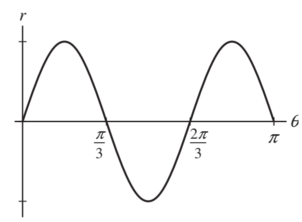
θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\cdots$	$\frac{11\pi}{12}$	π
r									

SOLUTION We compute the values of r corresponding to the given values of θ:

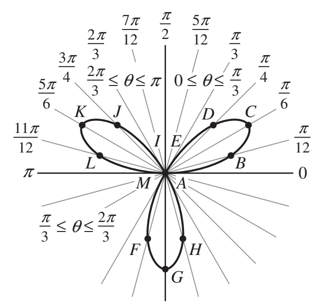
$$\theta = 0, \qquad r = \sin 0 = 0 \qquad (A)$$
$$\theta = \frac{\pi}{12}, \qquad r = \sin \frac{3\pi}{12} \approx 0.71 \qquad (B)$$
$$\theta = \frac{\pi}{6}, \qquad r = \sin \frac{3\pi}{6} = 1 \qquad (C)$$
$$\theta = \frac{\pi}{4}, \qquad r = \sin \frac{3\pi}{4} \approx 0.71 \qquad (D)$$
$$\theta = \frac{\pi}{3}, \qquad r = \sin \frac{3\pi}{3} = 0 \qquad (E)$$
$$\theta = \frac{5\pi}{12}, \qquad r = \sin \frac{15\pi}{12} \approx -0.71 \qquad (F)$$
$$\theta = \frac{\pi}{2}, \qquad r = \sin \frac{3\pi}{2} = -1 \qquad (G)$$
$$\theta = \frac{7\pi}{12}, \qquad r = \sin \frac{21\pi}{12} \approx -0.71 \qquad (H)$$
$$\theta = \frac{3\pi}{2}, \qquad r = \sin \frac{9\pi}{2} = 0 \qquad (I)$$

$$\theta = \frac{3\pi}{4}, \quad r = \sin\frac{9\pi}{4} \approx 0.71 \qquad (J)$$
$$\theta = \frac{5\pi}{6}, \quad r = \sin\frac{15\pi}{6} = 1 \qquad (K)$$
$$\theta = \frac{11\pi}{12}, \quad r = \sin\frac{33\pi}{12} \approx 0.71 \qquad (L)$$
$$\theta = \pi, \quad r = \sin 3\pi = 0 \qquad (M)$$

We plot the points on the xy -plane and join them to obtain the following curve:



Using the graph of $r = \sin 3\theta$ we find the sign of r and determine the parts of the graph corresponding to the angle intervals. We get

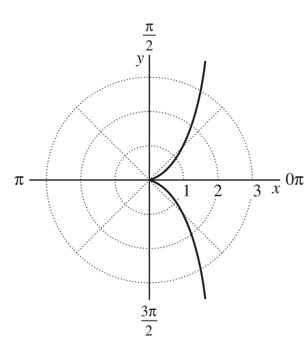


$$r = \sin 3\theta \quad \begin{cases} 0 \le \theta \le \frac{\pi}{3} \Rightarrow r \ge 0 \Rightarrow (r,\theta) \text{ in the first quadrant.} \\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \Rightarrow r \le 0 \Rightarrow (r,\theta) \text{ in the third and fourth quadrant.} \\ \frac{2\pi}{3} \le \theta \le \pi \Rightarrow r \ge 0 \Rightarrow (r,\theta) \text{ in the second quadrant.} \end{cases}$$

33. CAS Plot the **cissoid** $r = 2\sin\theta\tan\theta$ and show that its equation in rectangular coordinates is

$$y^2 = \frac{x^3}{2-x}$$

SOLUTION Using a CAS we obtain the following curve of the cissoid:



We substitute $\sin\theta = \frac{y}{r}$ and $\tan\theta = \frac{y}{x}$ in $r = 2\sin\theta\tan\theta$ to obtain

$$r = 2\frac{y}{r}\cdot\frac{y}{x}.$$

Multiplying by rx, setting $r^2 = x^2 + y^2$ and simplifying, yields

$$r^2 x = 2y^2$$
$$(x^2 + y^2)x = 2y^2$$

$$x^3 + y^2x = 2y^2$$
$$y^2(2-x) = x^3$$

so

$$y^2 = \frac{x^3}{2-x}$$

34. Prove that $r = 2a\cos\theta$ is the equation of the circle in Figure 21 using only the fact that a triangle inscribed in a circle with one side a diameter is a right triangle.

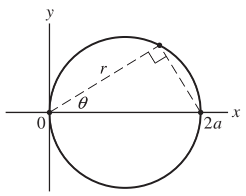


FIGURE 21

SOLUTION Since the triangle inscribed in the circle has a diameter as one of its sides, it is a right triangle, so we may use the definition of cosine for angles in right triangles to write

$$\cos\theta = \frac{r}{2a} \Rightarrow r = 2a\cos\theta.$$

35. Show that

$$r = a\cos\theta + b\sin\theta$$

is the equation of a circle passing through the origin. Express the radius and center (in rectangular coordinates) in terms of a and b.

SOLUTION We multiply the equation by r and then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$. This gives

$$r^2 = ar\cos\theta + br\sin\theta$$
$$x^2 + y^2 = ax + by$$

Transferring sides and completing the square yields

$$x^2 - ax + y^2 - by = 0$$
$$\left(x^2 - 2\cdot\frac{a}{2}x + \left(\frac{a}{2}\right)^2\right) + \left(y^2 - 2\cdot\frac{b}{2}y + \left(\frac{b}{2}\right)^2\right) = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$
$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2+b^2}{4}$$

This is the equation of the circle with radius $\frac{\sqrt{a^2+b^2}}{2}$ centered at the point $\left(\frac{a}{2}, \frac{b}{2}\right)$. By plugging in $x = 0$ and $y = 0$ it is clear that the circle passes through the origin.

36. Use the previous exercise to write the equation of the circle of radius 5 and center $(3, 4)$ in the form $r = a\cos\theta + b\sin\theta$.

SOLUTION In the previous exercise we showed that $r = a\cos\theta + b\sin\theta$ is the equation of the circle with radius $\frac{\sqrt{a^2+b^2}}{2}$ centered at $\left(\frac{a}{2}, \frac{b}{2}\right)$. Thus, we must have

$$\left(\frac{a}{2}, \frac{b}{2}\right) = (3, 4) \Rightarrow \frac{a}{2} = 3, \frac{b}{2} = 4 \Rightarrow a = 6, b = 8.$$

The radius of the circle is $\frac{\sqrt{a^2+b^2}}{2} = \frac{\sqrt{6^2+8^2}}{2} = 5$. Thus, the corresponding equation is $r = 6\cos\theta + 8\sin\theta$.

37. Use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to find a polar equation of the hyperbola $x^2 - y^2 = 1$.

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ in $x^2 - y^2 = 1$ to obtain

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$

Using the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we obtain the following equation of the hyperbola:

$$r^2\cos 2\theta = 1 \quad \text{or} \quad r^2 = \sec 2\theta.$$

38. Find an equation in rectangular coordinates for the curve $r^2 = \cos 2\theta$.

SOLUTION We first use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to rewrite the equation of the curve as follows: $r^2 = \cos^2\theta - \sin^2\theta$. Multiplying by r^2 and substituting $r^2 = x^2 + y^2$, $r\cos\theta = x$ and $r\sin\theta = y$, we get

$$r^4 = (r\cos\theta)^2 - (r\sin\theta)^2(x^2 + y^2)^2 = x^2 - y^2.$$

Thus, the curve has the equation $(x^2 + y^2)^2 = x^2 - y^2$ in rectangular coordinates.

39. Show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and use this identity to find an equation in rectangular coordinates for the curve $r = \cos 3\theta$.

SOLUTION We use the identities $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, and $\sin 2\alpha = 2\sin\alpha\cos\alpha$ to write

$$\begin{aligned}\cos 3\theta = \cos(2\theta + \theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\ &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta\end{aligned}$$

Using this identity we may rewrite the equation $r = \cos 3\theta$ as follows:

$$r = \cos^3\theta - 3\sin^2\theta\cos\theta \tag{1}$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Substituting into (1) gives:

$$r = \left(\frac{x}{r}\right)^3 - 3\left(\frac{y}{r}\right)^2\left(\frac{x}{r}\right)$$
$$r = \frac{x^3}{r^3} - \frac{3y^2x}{r^3}$$

We now multiply by r^3 and make the substitution $r^2 = x^2 + y^2$ to obtain the following equation for the curve:

$$r^4 = x^3 - 3y^2x$$
$$(x^2 + y^2)^2 = x^3 - 3y^2x$$

40. Use the addition formula for the cosine to show that the line $\mathcal{L}$ with polar equation $r\cos(\theta - \alpha) = d$ has the equation in rectangular coordinates $(\cos\alpha)x + (\sin\alpha)y = d$. Show that $\mathcal{L}$ has slope $m = -\cot\alpha$ and y-intercept $d/\sin\alpha$.

SOLUTION We use the identity $\cos(a - b) = \cos a\cos b + \sin a\sin b$ to rewrite the equation $r\cos(\theta - \alpha) = d$ as follows:

$$r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = d$$
$$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = d$$

We now substitute $r\cos\theta = x$ and $r\sin\theta = y$ to obtain: $x\cos\alpha + y\sin\alpha = d$. Dividing by $\cos\alpha$, transferring sides and simplifying yields

$$x + y\tan\alpha = \frac{d}{\cos\alpha}$$
$$y\tan\alpha = -x + \frac{d}{\cos\alpha}$$
$$y = -\frac{x}{\tan\alpha} + \frac{d}{\tan\alpha\cos\alpha}$$

so

$$y = (-\cot\alpha)\, x + \frac{d}{\sin\alpha}$$

This equation of the line implies that $\mathcal{L}$ has slope $m = -\cot\alpha$ and y -intercept $\frac{d}{\sin\alpha}$.

In Exercises 41–44, find an equation in polar coordinates of the line $\mathcal{L}$ with the given description.

41. The point on $\mathcal{L}$ closest to the origin has polar coordinates $\left(2, \frac{\pi}{9}\right)$.

SOLUTION In Example 5, it is shown that the polar equation of the line where (r, α) is the point on the line closest to the origin is $r = d\sec(\theta - \alpha)$. Setting $(d, \alpha) = \left(2, \frac{\pi}{9}\right)$ we obtain the following equation of the line:

$$r = 2\sec\left(\theta - \frac{\pi}{9}\right).$$

42. The point on $\mathcal{L}$ closest to the origin has rectangular coordinates $(-2, 2)$.

SOLUTION We first convert the rectangular coordinates $(-2, 2)$ to polar coordinates (d, α). This point is in the second quadrant so $\frac{\pi}{2} < \alpha < \pi$. Hence,

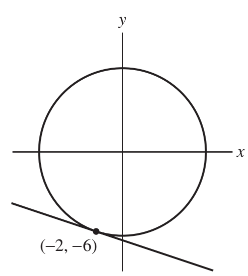
$$\left.\begin{aligned} d &= \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \\ \alpha &= \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \end{aligned}\right. \quad \Rightarrow \quad (d, \alpha) = \left(2\sqrt{2}, \frac{3\pi}{4}\right).$$

Substituting $d = 2\sqrt{2}$ and $\alpha = \frac{3\pi}{4}$ in the equation $r = d\sec(\theta - \alpha)$ gives us

$$r = 2\sqrt{2}\sec\left(\theta - \frac{3\pi}{4}\right).$$

43. $\mathcal{L}$ is tangent to the circle $r = 2\sqrt{10}$ at the point with rectangular coordinates $(-2, -6)$.

SOLUTION



Since $\mathcal{L}$ is tangent to the circle at the point $(-2, -6)$, this is the point on $\mathcal{L}$ closest to the center of the circle which is at the origin. Therefore, we may use the polar coordinates (d, α) of this point in the equation of the line:

$$r = d\sec(\theta - \alpha) \tag{1}$$

We thus must convert the coordinates $(-2, -6)$ to polar coordinates. This point is in the third quadrant so $\pi < \alpha < \frac{3\pi}{2}$. We get

$$d = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} = 2\sqrt{10}$$

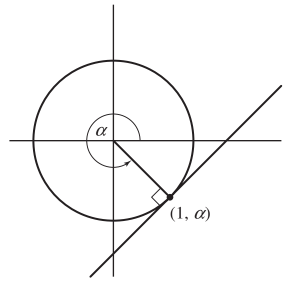
$$\alpha = \tan^{-1}\left(\frac{-6}{-2}\right) = \tan^{-1} 3 \approx \pi + 1.25 \approx 4.39$$

Substituting in (1) yields the following equation of the line:

$$r = 2\sqrt{10}\sec(\theta - 4.39).$$

44. $\mathcal{L}$ has slope 3 and is tangent to the unit circle in the fourth quadrant.

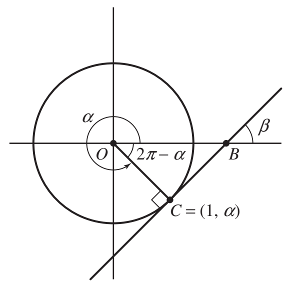
SOLUTION We denote the point of tangency by $P_0 = (1, \alpha)$ (in polar coordinates).



Since $\mathcal{L}$ is the tangent line to the circle at P_0, P_0 is the point on $\mathcal{L}$ closest to the center of the circle at the origin. Thus, the polar equation of $\mathcal{L}$ is

$$r = \sec(\theta - \alpha) \tag{1}$$

We now must find α. Let β be the given angle shown in the figure.



By the given information, $\tan\beta = 3$. Also, since the point of tangency is in the fourth quadrant, β must be an acute angle. Hence

$$\tan\beta = 3,\ 0 < \beta < \frac{\pi}{2} \Rightarrow \beta = 1.25 \text{ rad.}$$

Now, since $\frac{3\pi}{2} < \alpha < 2\pi$, we have for the triangle OBC

$$(2\pi - \alpha) + \frac{\pi}{2} + 1.25 = \pi \Rightarrow \alpha = \frac{3\pi}{2} + 1.25 = 5.96 \text{ rad.}$$

Substituting into (1) we obtain the following polar equation of the tangent line:

$$r = \sec(\theta - 5.96).$$

45. Show that every line that does not pass through the origin has a polar equation of the form

$$r = \frac{b}{\sin\theta - a\cos\theta}$$

where $b \neq 0$.

SOLUTION Write the equation of the line in rectangular coordinates as $y = ax + b$. Since the line does not pass through the origin, we have $b \neq 0$. Substitute for y and x to convert to polar coordinates, and simplify:

$$\begin{aligned} y &= ax + b \\ r\sin\theta &= ar\cos\theta + b \\ r(\sin\theta - a\cos\theta) &= b \\ r &= \frac{b}{\sin\theta - a\cos\theta} \end{aligned}$$

46. By the Law of Cosines, the distance d between two points (Figure 22) with polar coordinates (r, θ) and (r_0, θ_0) is

$$d^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$$

Use this distance formula to show that

$$r^2 - 10r\cos\left(\theta - \frac{\pi}{4}\right) = 56$$

is the equation of the circle of radius 9 whose center has polar coordinates $\left(5, \frac{\pi}{4}\right)$.

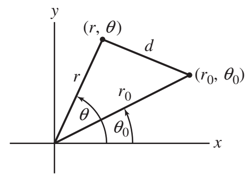


FIGURE 22

SOLUTION The distance d between a point (r, θ) on the circle and the center $(r_0, \theta_0) = \left(5, \frac{\pi}{4}\right)$ is the radius 9. Setting $d = 9$, $r_0 = 5$ and $\theta_0 = \frac{\pi}{4}$ in the distance formula we get

$$d^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$$
$$9^2 = r^2 + 5^2 - 2 \cdot r \cdot 5 \cos\left(\theta - \frac{\pi}{4}\right)$$

Transferring sides we get

$$r^2 - 10r \cos\left(\theta - \frac{\pi}{4}\right) = 56.$$

47. For $a > 0$, a **lemniscate curve** is the set of points P such that the product of the distances from P to $(a, 0)$ and $(-a, 0)$ is a^2. Show that the equation of the lemniscate is

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

Then find the equation in polar coordinates. To obtain the simplest form of the equation, use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Plot the lemniscate for $a = 2$ if you have a computer algebra system.

SOLUTION We compute the distances d_1 and d_2 of $P(x, y)$ from the points $(a, 0)$ and $(-a, 0)$ respectively. We obtain:

$$d_1 = \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + y^2}$$
$$d_2 = \sqrt{(x+a)^2 + (y-0)^2} = \sqrt{(x+a)^2 + y^2}$$

For the points $P(x, y)$ on the lemniscate we have $d_1 d_2 = a^2$. That is,

$$\begin{aligned}
a^2 &= \sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} = \sqrt{\left[(x-a)^2 + y^2\right]\left[(x+a)^2 + y^2\right]} \\
&= \sqrt{(x-a)^2(x+a)^2 + y^2(x-a)^2 + y^2(x+a)^2 + y^4} \\
&= \sqrt{(x^2 - a^2)^2 + y^2\left[(x-a)^2 + (x+a)^2\right] + y^4} \\
&= \sqrt{x^4 - 2a^2x^2 + a^4 + y^2\left(x^2 - 2xa + a^2 + x^2 + 2xa + a^2\right) + y^4} \\
&= \sqrt{x^4 - 2a^2x^2 + a^4 + 2y^2x^2 + 2y^2a^2 + y^4} \\
&= \sqrt{x^4 + 2x^2y^2 + y^4 + 2a^2(y^2 - x^2) + a^4} \\
&= \sqrt{(x^2 + y^2)^2 + 2a^2(y^2 - x^2) + a^4}.
\end{aligned}$$

Squaring both sides and simplifying yields

$$a^4 = (x^2 + y^2)^2 + 2a^2(y^2 - x^2) + a^4$$
$$0 = (x^2 + y^2)^2 + 2a^2(y^2 - x^2)$$

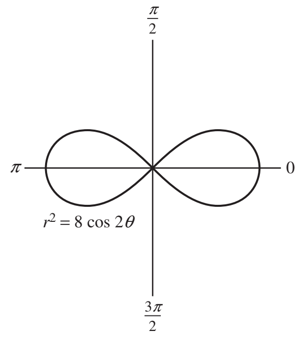
so

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

We now find the equation in polar coordinates. We substitute $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$ into the equation of the lemniscate. This gives

$$(r^2)^2 = 2a^2(r^2\cos^2\theta - r^2\sin^2\theta) = 2a^2r^2(\cos^2\theta - \sin^2\theta) = 2a^2r^2\cos 2\theta$$
$$r^4 = 2a^2r^2\cos 2\theta$$

$r = 0$ is a solution, hence the origin is on the curve. For $r \neq 0$ we divide the equation by r^2 to obtain $r^2 = 2a^2 \cos 2\theta$. This curve also includes the origin ($r = 0$ is obtained for $\theta = \frac{\pi}{4}$ for example), hence this is the polar equation of the lemniscate. Setting $a = 2$ we get $r^2 = 8 \cos 2\theta$.



48. Let c be a fixed constant. Explain the relationship between the graphs of:

(a) $y = f(x + c)$ and $y = f(x)$ (rectangular)

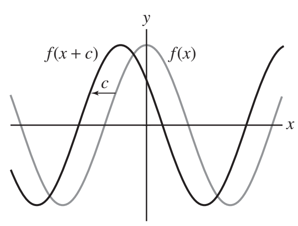
(b) $r = f(\theta + c)$ and $r = f(\theta)$ (polar)

(c) $y = f(x) + c$ and $y = f(x)$ (rectangular)

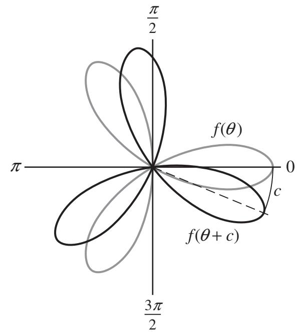
(d) $r = f(\theta) + c$ and $r = f(\theta)$ (polar)

SOLUTION

(a) For $c > 0$, $y = f(x + c)$ shifts the graph of $y = f(x)$ by c units to the left. If $c < 0$, the result is a shift to the right. It is a horizontal translation.

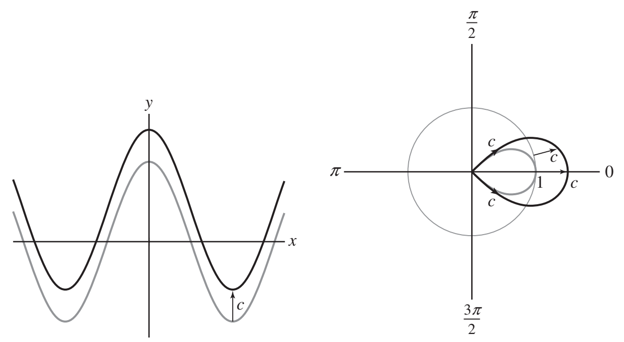


(b) As in part (a), the graph of $r = f(\theta + c)$ is a shift of the graph of $r = f(\theta)$ by c units in θ. Thus, the graph in polar coordinates is rotated by angle c as shown in the following figure:



(c) $y = f(x) + c$ shifts the graph vertically upward by c units if $c > 0$, and downward by $(-c)$ units if $c < 0$. It is a vertical translation.

(d) The graph of $r = f(\theta) + c$ is a shift of the graph of $r = f(\theta)$ by c units in r. In the corresponding graph, in polar coordinates, each point with $f(\theta) > 0$ moves on the ray connecting it to the origin c units away from the origin if $c > 0$ and $(-c)$ units toward the origin if $c < 0$, and vice-versa for $f(\theta) < 0$.



$c > 0$

49. The Derivative in Polar Coordinates Show that a polar curve $r = f(\theta)$ has parametric equations

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta$$

Then apply Theorem 2 of Section 11.1 to prove

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} \tag{2}$$

where $f'(\theta) = df/d\theta$.

SOLUTION Multiplying both sides of the given equation by $\cos\theta$ yields $r\cos\theta = f(\theta)\cos\theta$; multiplying both sides by $\sin\theta$ yields $r\sin\theta = f(\theta)\sin\theta$. The left-hand sides of these two equations are the x and y coordinates in rectangular coordinates, so for any θ we have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, showing that the parametric equations are as claimed. Now, by the formula for the derivative we have

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} \tag{1}$$

We differentiate the functions $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ using the Product Rule for differentiation. This gives

$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$$
$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

Substituting in (1) gives

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

50. Use Eq. (2) to find the slope of the tangent line to $r = \sin\theta$ at $\theta = \frac{\pi}{3}$.

SOLUTION We have $f(\theta) = \sin\theta$, $f'(\theta) = \cos\theta$ and, by Eq. (2), the slope of the tangent line is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\sin\theta\cos\theta + \cos\theta\sin\theta}{-\sin^2\theta + \cos^2\theta} = \frac{\sin 2\theta}{\cos 2\theta}$$

Evaluating at $\theta = \frac{\pi}{3}$ gives

$$\frac{dy}{dx} = \frac{\sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3}} = \frac{\sqrt{3}2}{-1/2} = -\sqrt{3}$$

Thus the slope of the tangent line to $r = \sin\theta$ at $\theta = \frac{\pi}{3}$ is $-\sqrt{3}$.

51. Use Eq. (2) to find the slope of the tangent line to $r = \theta$ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

SOLUTION In the given curve we have $r = f(\theta) = \theta$. Using Eq. (2) we obtain the following derivative, which is the slope of the tangent line at (r, θ).

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\theta\cos\theta + 1\cdot\sin\theta}{-\theta\sin\theta + 1\cdot\cos\theta} \tag{1}$$

The slope, m, of the tangent line at $\theta = \frac{\pi}{2}$ and $\theta = \pi$ is obtained by substituting these values in (1). We get ($\theta = \frac{\pi}{2}$):

$$m = \frac{\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}}{-\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = \frac{\frac{\pi}{2}\cdot 0 + 1}{-\frac{\pi}{2}\cdot 1 + 0} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}.$$

$(\theta = \pi)$:

$$m = \frac{\pi\cos\pi + \sin\pi}{-\pi\sin\pi + \cos\pi} = \frac{-\pi}{-1} = \pi.$$

52. Find the equation in rectangular coordinates of the tangent line to $r = 4\cos 3\theta$ at $\theta = \frac{\pi}{6}$.

SOLUTION We have $f(\theta) = 4\cos 3\theta$. By Eq. (2),

$$m = \frac{4\cos 3\theta\cos\theta - 12\sin 3\theta\sin\theta}{-4\cos 3\theta\sin\theta - 12\sin 3\theta\cos\theta}.$$

Setting $\theta = \frac{\pi}{6}$ yields

$$m = \frac{4\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) - 12\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)}{-4\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) - 12\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)} = \frac{-12\sin\frac{\pi}{6}}{-12\cos\frac{\pi}{6}} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

We identify the point of tangency. For $\theta = \frac{\pi}{6}$ we have $r = 4\cos\frac{3\pi}{6} = 4\cos\frac{\pi}{2} = 0$. The point of tangency is the origin. The tangent line is the line through the origin with slope $\frac{1}{\sqrt{3}}$. This is the line $y = \frac{x}{\sqrt{3}}$.

53. Find the polar coordinates of the points on the lemniscate $r^2 = \cos 2t$ in Figure 23 where the tangent line is horizontal.

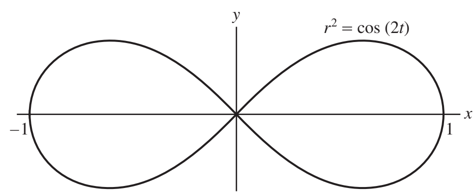


FIGURE 23

SOLUTION This curve is defined for $-\frac{\pi}{2} \le 2t \le \frac{\pi}{2}$ (where $\cos 2t \ge 0$), so for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. For each θ in that range, there are two values of r satisfying the equation ($\pm\sqrt{\cos 2t}$). By symmetry, we need only calculate the coordinates of the points corresponding to the positive square root (i.e. to the right of the y axis). Then the equation becomes $r = \sqrt{\cos 2t}$. Now, by Eq. (2), with $f(t) = \sqrt{\cos(2t)}$ and $f'(t) = -\sin(2t)(\cos(2t))^{-1/2}$, we have

$$\frac{dy}{dx} = \frac{f(t)\cos t + f'(t)\sin t}{-f(t)\sin t + f'(t)\cos t} = \frac{\cos t\sqrt{\cos(2t)} - \sin(2t)\sin t(\cos(2t))^{-1/2}}{-\sin t\sqrt{\cos(2t)} - \sin(2t)\cos t(\cos(2t))^{-1/2}}$$

The tangent line is horizontal when this derivative is zero, which occurs when the numerator of the fraction is zero and the denominator is not. Multiply top and bottom of the fraction by $\sqrt{\cos(2t)}$, and use the identities $\cos 2t = \cos^2 t - \sin^2 t$, $\sin 2t = 2\sin t\cos t$ to get

$$-\frac{\cos t\cos 2t - \sin t\sin 2t}{\sin t\cos 2t + \cos t\sin 2t} = -\frac{\cos t(\cos^2 t - 3\sin^2 t)}{\sin t\cos 2t + \cos t\sin 2t}$$

The numerator is zero when $\cos t = 0$, so when $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, or when $\tan t = \pm\frac{1}{\sqrt{3}}$, so when $t = \pm\frac{\pi}{6}$ or $t = \pm\frac{5\pi}{6}$. Of these possibilities, only $t = \pm\frac{\pi}{6}$ lie in the range $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. Note that the denominator is nonzero for $t = \pm\frac{\pi}{6}$, so these are the two values of t for which the tangent line is horizontal. The corresponding values of r are solutions to

$$r^2 = \cos\left(2\cdot\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$r^2 = \cos\left(2\cdot\frac{-\pi}{6}\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

Finally, the four points are $(r, t) =$

$$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(-\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, -\frac{pi}{6}\right), \qquad \left(-\frac{1}{\sqrt{2}}, -\frac{\pi}{6}\right)$$

If desired, we can change the second and fourth points by adding π to the angle and making r positive, to get

$$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, \frac{7\pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, -\frac{pi}{6}\right), \qquad \left(\frac{1}{\sqrt{2}}, \frac{5\pi}{6}\right)$$

54. Find the polar coordinates of the points on the cardioid $r = 1 + \cos\theta$ where the tangent line is horizontal (see Figure 24).

SOLUTION Use Eq. (2) with $f(\theta) = 1 + \cos\theta$ and $f'(\theta) = -\sin\theta$. Then

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - \cos\theta\sin\theta - \sin\theta\cos\theta} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$$

The tangent line is horizontal when the numerator is zero but the denominator is not. The numerator is zero when $\cos\theta + \cos 2\theta = 0$. But

$$\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = \left(\cos\theta - \frac{1}{2}\right)(\cos\theta + 1)$$

So for $0 \le \theta < 2\pi$, the numerator is zero when $\theta = \pi$ and when $\theta = \pm\frac{\pi}{3}$. For the latter two points, the denominator is nonzero, so the tangent is horizontal at the points

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right), \qquad \left(\frac{3}{2}, -\frac{\pi}{3}\right) = \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

When $\theta = \pi$, both numerator and denominator vanish. However, using L'Hôpital's Rule, we have

$$-\lim_{\theta\to\pi} \frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta} = -\lim_{\theta\to\pi} \frac{-\sin\theta - 2\sin 2\theta}{\cos\theta + 2\cos 2\theta} = 0$$

so that the tangent is defined at $\theta = \pi$, and it is horizontal. Thus the tangent is also horizontal at the point

$$(r, \theta) = (0, \pi)$$

55. Use Eq. (2) to show that for $r = \sin\theta + \cos\theta$,

$$\frac{dy}{dx} = \frac{\cos 2\theta + \sin 2\theta}{\cos 2\theta - \sin 2\theta}$$

Then calculate the slopes of the tangent lines at points A, B, C in Figure 19.

SOLUTION In Exercise 49 we proved that for a polar curve $r = f(\theta)$ the following formula holds:

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} \tag{1}$$

For the given circle we have $r = f(\theta) = \sin\theta + \cos\theta$, hence $f'(\theta) = \cos\theta - \sin\theta$. Substituting in (1) we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\sin\theta + \cos\theta)\cos\theta + (\cos\theta - \sin\theta)\sin\theta}{-(\sin\theta + \cos\theta)\sin\theta + (\cos\theta - \sin\theta)\cos\theta} = \frac{\sin\theta\cos\theta + \cos^2\theta + \cos\theta\sin\theta - \sin^2\theta}{-\sin^2\theta - \cos\theta\sin\theta + \cos^2\theta - \sin\theta\cos\theta} \\ &= \frac{\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta}\end{aligned}$$

We use the identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$ to obtain

$$\frac{dy}{dx} = \frac{\cos 2\theta + \sin 2\theta}{\cos 2\theta - \sin 2\theta} \tag{2}$$

The derivative $\frac{dy}{dx}$ is the slope of the tangent line at (r, θ). The slopes of the tangent lines at the points with polar coordinates $A = \left(1, \frac{\pi}{2}\right)$ $B = \left(0, \frac{3\pi}{4}\right)$ $C = (1, 0)$ are computed by substituting the values of θ in (2). This gives

$$\begin{aligned}\left.\frac{dy}{dx}\right|_A &= \frac{\cos\left(2\cdot\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{\pi}{2}\right)}{\cos\left(2\cdot\frac{\pi}{2}\right) - \sin\left(2\cdot\frac{\pi}{2}\right)} = \frac{\cos\pi + \sin\pi}{\cos\pi - \sin\pi} = \frac{-1+0}{-1-0} = 1 \\ \left.\frac{dy}{dx}\right|_B &= \frac{\cos\left(2\cdot\frac{3\pi}{4}\right) + \sin\left(2\cdot\frac{3\pi}{4}\right)}{\cos\left(2\cdot\frac{3\pi}{4}\right) - \sin\left(2\cdot\frac{3\pi}{4}\right)} = \frac{\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}}{\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}} = \frac{0-1}{0+1} = -1 \\ \left.\frac{dy}{dx}\right|_C &= \frac{\cos(2\cdot 0) + \sin(2\cdot 0)}{\cos(2\cdot 0) - \sin(2\cdot 0)} = \frac{\cos 0 + \sin 0}{\cos 0 - \sin 0} = \frac{1+0}{1-0} = 1\end{aligned}$$

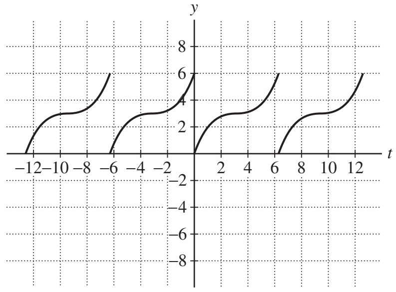
Further Insights and Challenges

56. Let $f(x)$ be a periodic function of period 2π—that is, $f(x) = f(x + 2\pi)$. Explain how this periodicity is reflected in the graph of:

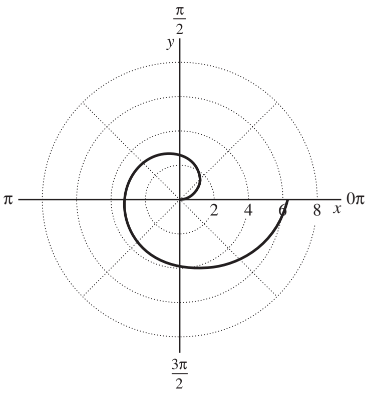
(a) $y = f(x)$ in rectangular coordinates
(b) $r = f(\theta)$ in polar coordinates

SOLUTION

(a) The graph of $y = f(x)$ on an interval of length 2π repeats itself on successive intervals of length 2π. For instance:

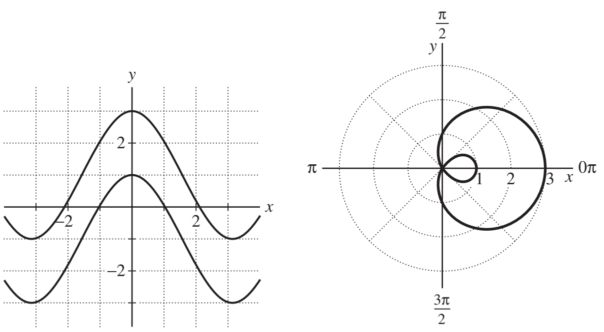


(b) Shown below is the graph of the function above, this time drawn in polar coordinates. The graphs of the various branches repeat themselves and are drawn one on the top of the other.



57. GU Use a graphing utility to convince yourself that the polar equations $r = f_1(\theta) = 2\cos\theta - 1$ and $r = f_2(\theta) = 2\cos\theta + 1$ have the same graph. Then explain why. *Hint:* Show that the points $(f_1(\theta + \pi), \theta + \pi)$ and $(f_2(\theta), \theta)$ coincide.

SOLUTION The graphs of $r = 2\cos\theta - 1$ and $r = 2\cos\theta + 1$ in the xy -plane coincide as shown in the graph obtained using a CAS.



Recall that (r, θ) and $(-r, \theta + \pi)$ represent the same point. Replacing θ by $\theta + \pi$ and r by $(-r)$ in $r = 2\cos\theta - 1$ we obtain

$$-r = 2\cos(\theta + \pi) - 1$$
$$-r = -2\cos\theta - 1$$
$$r = 2\cos\theta + 1$$

Thus, the two equations define the same graph. (One could also convert both equations to rectangular coordinates and note that they come out identical.)

58. CAS We investigate how the shape of the limaçon curve $r = b + \cos\theta$ depends on the constant b (see Figure 24).

(a) Argue as in Exercise 57 to show that the constants b and $-b$ yield the same curve.

(b) Plot the limaçon for $b = 0, 0.2, 0.5, 0.8, 1$ and describe how the curve changes.

(c) Plot the limaçon for $b = 1.2, 1.5, 1.8, 2, 2.4$ and describe how the curve changes.

(d) Use Eq. (2) to show that

$$\frac{dy}{dx} = -\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta$$

(e) Find the points where the tangent line is vertical. Note that there are three cases: $0 \le b < 2$, $b = 1$, and $b > 2$. Do the plots constructed in (b) and (c) reflect your results?

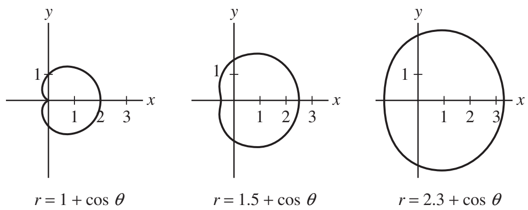


FIGURE 24

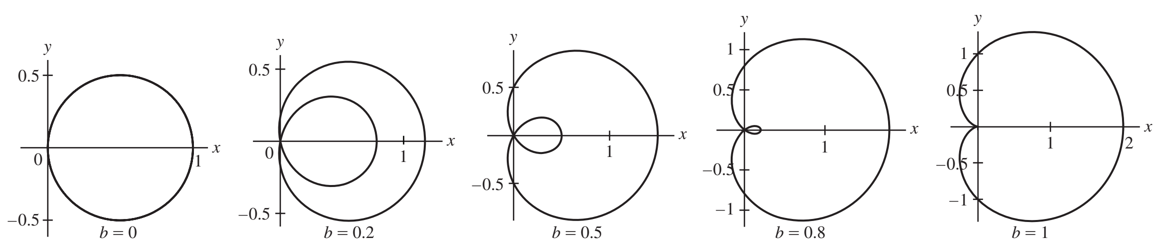
SOLUTION

(a) If (r, θ) is on the curve $r = -b + \cos\theta$, then so is $(-r, \theta + \pi)$ since they represent the same point. Thus

$$-r = -b + \cos(\theta + \pi)$$
$$-r = -b - \cos\theta$$
$$r = b + \cos\theta$$

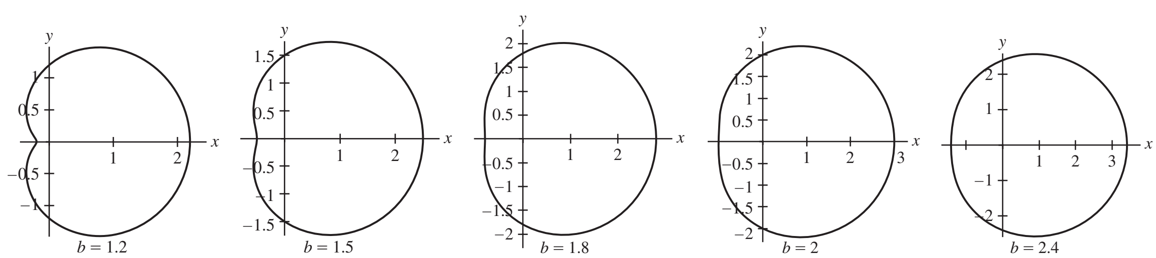
Thus the same set of points lie on the graph of both equations, so they define the same curve.

(b)



For $0 < b < 1$, there is a "loop" inside the curve. For $b = 0$, the curve is a circle, although actually for $0 \le \theta \le 2\pi$ the circle is traversed twice, so in fact the loop is as large as the circle and overlays it. When $b = 1$, the loop is pinched to a point.

(c)



For b between 1 and 2, the pinch at $b = 1$ smooths out into a concavity in the curve, which decreases in size. By $b = 2$ it appears to be gone; further increases in b push the left-hand section of the curve out, making it more convex.

(d) By Eq. (2), with $f(\theta) = b + \cos\theta$ and $f'(\theta) = -\sin\theta$, we have (using the double-angle identities for sin and cos)

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta} = \frac{(b + \cos\theta)\cos\theta - \sin^2\theta}{-(b + \cos\theta)\sin\theta - \sin\theta\cos\theta} = \frac{b\cos\theta + \cos 2\theta}{-b\sin\theta - 2\sin\theta\cos\theta}$$

$$= -\frac{b\cos\theta + \cos 2\theta}{\sin\theta(b + 2\cos\theta)} = -\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta$$

(e) From part **(d)**, the tangent line is vertical when either $\csc\theta$ is undefined or when $b + 2\cos\theta = 0$ (as long as the numerator $b\cos\theta + \cos 2\theta \neq 0$). Consider first the case when $\csc\theta$ is undefined, so that $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, the numerator of the fraction is $b + 1 \neq 0$ and the denominator is $b + 2 \neq 0$, so that the tangent is vertical here.

$$\text{For any } b, \text{ the limaçon has a vertical tangent at } (b + \cos 0, 0) = (b + 1, 0)$$

If $\theta = \pi$, the numerator of the fraction is $1 - b$ and the denominator is $b + 2 \neq 0$. As long as $b \neq 1$, the numerator does not vanish and we have found a point of vertical tangency. If $b = 1$, then by L'Hôpital's Rule,

$$-\lim_{\theta\to\pi}\left(\frac{b\cos\theta + \cos 2\theta}{b + 2\cos\theta}\right)\csc\theta = -\lim_{\theta\to\pi}\left(\frac{b\cos\theta + \cos 2\theta}{(b + 2\cos\theta)\sin\theta}\right) = \lim_{\theta\to\pi}\frac{\sin t + \sin 2t}{2\cos^2 t - 2\sin^2 t + \cos t} = 0$$

so that the tangent is not vertical here. Thus

$$\text{If } b \neq 1, \text{ the limaçon has a vertical tangent at } (b + \cos\pi, \pi) = (b - 1, \pi)$$

Next consider the possibility that $b + 2\cos\theta = 0$; this happens when $\cos\theta = -\frac{b}{2}$. First assume that $0 \le b < 2$. This equation holds for two values of θ: $\cos^{-1}\left(-\frac{b}{2}\right)$ and $-\cos^{-1}\left(-\frac{b}{2}\right)$. Neither of these angles is 0 or π, so that $\csc\theta$ is defined. Additionally, the numerator is

$$b\cos\theta + \cos 2\theta = b\cos\theta + 2\cos^2\theta - 1 = -\frac{b^2}{2} + 2\cdot\frac{b^2}{4} - 1 = -1$$

so that the numerator does not vanish. Thus

$$\text{For } 0 \le b < 2, \text{ the limaçon has a vertical tangent at } \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right) \text{ and } \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right)$$

Next assume that $b = 2$; then $\cos\theta = -1$ holds for $\theta = \pi$; we have considered that case above. Finally assume that $b > 2$; then $\cos\theta = -\frac{b}{2}$ has no solutions. Thus, in summary, vertical tangents of the limaçon occur as follows:

$$\begin{aligned}
0 \le b < 2, b \neq 1: &\quad \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right), \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right), (b - 1, \pi), (b + 1, 0)\\
b = 1: &\quad \left(\frac{b}{2}, \cos^{-1}\left(-\frac{b}{2}\right)\right), \left(\frac{b}{2}, -\cos^{-1}\left(-\frac{b}{2}\right)\right), (b + 1, 0)\\
b \ge 2: &\quad (b + 1, 0), (b - 1, \pi)
\end{aligned}$$

These do correspond to the figures in parts **(b)** and **(c)**.

11.4 Area and Arc Length in Polar Coordinates

Preliminary Questions

1. Polar coordinates are suited to finding the area (choose one):

(a) Under a curve between $x = a$ and $x = b$.

(b) Bounded by a curve and two rays through the origin.

SOLUTION Polar coordinates are best suited to finding the area bounded by a curve and two rays through the origin. The formula for the area in polar coordinates gives the area of this region.

2. Is the formula for area in polar coordinates valid if $f(\theta)$ takes negative values?

SOLUTION The formula for the area

$$\frac{1}{2}\int_\alpha^\beta f(\theta)^2\,d\theta$$

always gives the actual (positive) area, even if $f(\theta)$ takes on negative values.

3. The horizontal line $y = 1$ has polar equation $r = \csc\theta$. Which area is represented by the integral $\frac{1}{2}\int_{\pi/6}^{\pi/2} \csc^2\theta\, d\theta$ (Figure 12)?

(a) $\square ABCD$ **(b)** $\triangle ABC$ **(c)** $\triangle ACD$

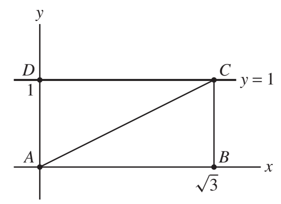


FIGURE 12

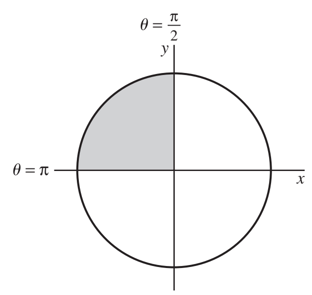
SOLUTION This integral represents an area taken from $\theta = \pi/6$ to $\theta = \pi/2$, which can only be the triangle $\triangle ACD$, as seen in part (c).

Exercises

1. Sketch the area bounded by the circle $r = 5$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = \pi$, and compute its area as an integral in polar coordinates.

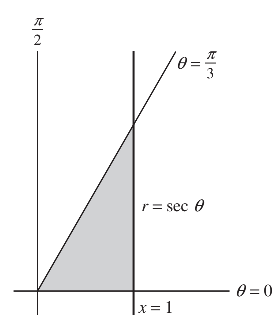
SOLUTION The region bounded by the circle $r = 5$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = \pi$ is the shaded region in the figure. The area of the region is given by the following integral:

$$\frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} 5^2\, d\theta = \frac{25}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{25\pi}{4}$$



2. Sketch the region bounded by the line $r = \sec\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{3}$. Compute its area in two ways: as an integral in polar coordinates and using geometry.

SOLUTION The region bounded by the line $r = \sec\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{3}$ is shown here:

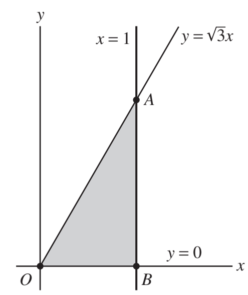


Using the area in polar coordinates, the area of the region is given by the following integral:

$$A = \frac{1}{2}\int_0^{\pi/3} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/3} \sec^2\theta\, d\theta = \frac{1}{2}\tan\theta\Big|_0^{\pi/3} = \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right) = \frac{\sqrt{3}}{2}$$

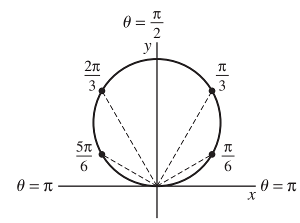
We now compute the area using the formula for the area of a triangle. The equations of the lines $\theta = \frac{\pi}{3}$, $\theta = 0$, and $r = \sec\theta$ in rectangular coordinates are $y = \sqrt{3}x$, $y = 0$ and $x = 1$ respectively (see Example 5 in Section 12.3 for the equation of the line $r = \sec\theta$). Denoting the vertices of the triangle by O, A, B (see figure) we have $O = (0, 0)$, $A = \left(1, \sqrt{3}\right)$ and $B = (1, 0)$. The area of the triangle is thus

$$A = \frac{OB \cdot AB}{2} = \frac{1 \cdot \sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$



3. Calculate the area of the circle $r = 4\sin\theta$ as an integral in polar coordinates (see Figure 4). Be careful to choose the correct limits of integration.

SOLUTION The equation $r = 4\sin\theta$ defines a circle of radius 2 tangent to the x-axis at the origin as shown in the figure:



The circle is traced as θ varies from 0 to π. We use the area in polar coordinates and the identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

to obtain the following area:

$$A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} (4\sin\theta)^2\, d\theta = 8\int_0^{\pi} \sin^2\theta\, d\theta = 4\int_0^{\pi} (1 - \cos 2\theta)\, d\theta = 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= 4\left(\left(\pi - \frac{\sin 2\pi}{2}\right) - 0\right) = 4\pi.$$

4. Find the area of the shaded triangle in Figure 13 as an integral in polar coordinates. Then find the rectangular coordinates of P and Q and compute the area via geometry.

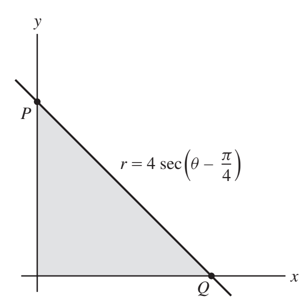


FIGURE 13

SOLUTION The boundary of the region is traced as θ varies from 0 to $\frac{\pi}{2}$, so the area is

$$\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} 16\sec^2\left(\theta - \frac{\pi}{4}\right) d\theta = 8\tan\left(\theta - \frac{\pi}{4}\right)\Big|_0^{\pi/2} = 8(1 + 1) = 16$$

5. Find the area of the shaded region in Figure 14. Note that θ varies from 0 to $\frac{\pi}{2}$.

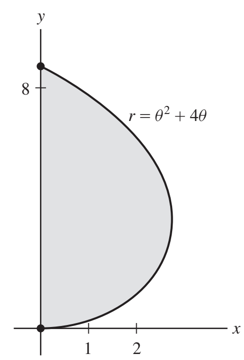


FIGURE 14

SOLUTION Since θ varies from 0 to $\frac{\pi}{2}$, the area is

$$\frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (\theta^2 + 4\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} \theta^4 + 8\theta^3 + 16\theta^2\,d\theta$$
$$= \frac{1}{2}\left(\frac{1}{5}\theta^5 + 2\theta^4 + \frac{16}{3}\theta^3\right)\Big|_0^{\pi/2} = \frac{\pi^5}{320} + \frac{\pi^4}{16} + \frac{\pi^2}{3}$$

6. Which interval of θ-values corresponds to the the shaded region in Figure 15? Find the area of the region.

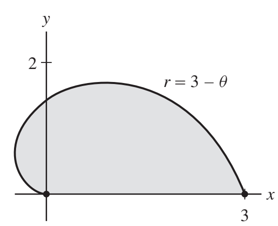
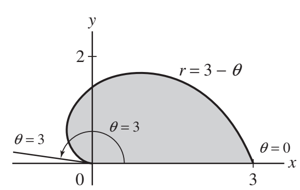


FIGURE 15

SOLUTION We first find the interval of θ. At the origin $r = 0$, so $\theta = 3$. At the endpoint on the x-axis, $\theta = 0$. Thus, θ varies from 0 to 3.



Using the area in polar coordinates we obtain

$$A = \frac{1}{2}\int_0^3 r^2\,d\theta = \frac{1}{2}\int_0^3 (3-\theta)^2\,d\theta = -\frac{(3-\theta)^3}{6}\Big|_0^3 = 4.5.$$

7. Find the total area enclosed by the cardioid in Figure 16.

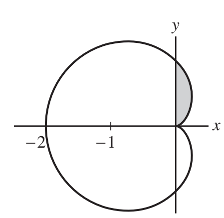
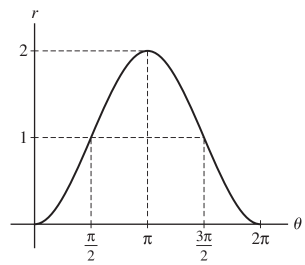


FIGURE 16 The cardioid $r = 1 - \cos\theta$.

SOLUTION We graph $r = 1 - \cos\theta$ in r and θ (cartesian, not polar, this time):



We see that as θ varies from 0 to π, the radius r increases from 0 to 2, so we get the upper half of the cardioid (the lower half is obtained as θ varies from π to 2π and consequently r decreases from 2 to 0). Since the cardioid is symmetric with respect to the x-axis we may compute the upper area and double the result. Using

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

we get

$$\begin{aligned} A &= 2\cdot\frac{1}{2}\int_0^{\pi} r^2\,d\theta = \int_0^{\pi}(1-\cos\theta)^2\,d\theta = \int_0^{\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \\ &= \int_0^{\pi}\left(1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right)d\theta = \int_0^{\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta \\ &= \frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\Big|_0^{\pi} = \frac{3\pi}{2} \end{aligned}$$

The total area enclosed by the cardioid is $A = \frac{3\pi}{2}$.

8. Find the area of the shaded region in Figure 16.

SOLUTION The shaded region is traced as θ varies from 0 to $\frac{\pi}{2}$. Using the formula for the area in polar coordinates we get:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left(1 - 2\cos\theta + \frac{\cos 2\theta + 1}{2}\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta \\ &= \frac{1}{2}\left(\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{1}{2}\left(\left(\frac{3}{2}\cdot\frac{\pi}{2} - 2\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi\right) - 0\right) \\ &= \frac{1}{2}\left(\frac{3\pi}{4} - 2\right) = \frac{3\pi}{8} - 1 \approx 0.18 \end{aligned}$$

9. Find the area of one leaf of the "four-petaled rose" $r = \sin 2\theta$ (Figure 17). Then prove that the total area of the rose is equal to one-half the area of the circumscribed circle.

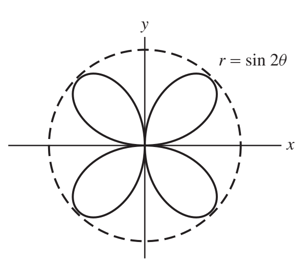
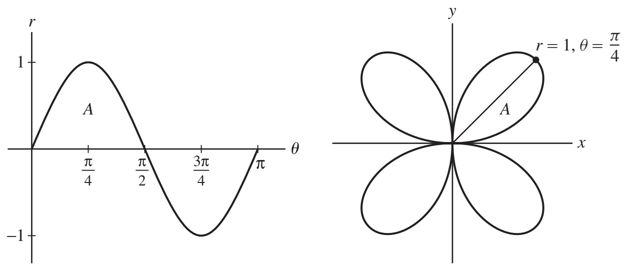


FIGURE 17 Four-petaled rose $r = \sin 2\theta$.

SOLUTION We consider the graph of $r = \sin 2\theta$ in cartesian and in polar coordinates:



We see that as θ varies from 0 to $\frac{\pi}{4}$ the radius r is increasing from 0 to 1, and when θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r is decreasing back to zero. Hence, the leaf in the first quadrant is traced as θ varies from 0 to $\frac{\pi}{2}$. The area of the leaf (the four leaves have equal areas) is thus

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta.$$

Using the identity

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

we get

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right) d\theta = \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right)\bigg|_0^{\pi/2} = \frac{1}{2}\left(\left(\frac{\pi}{4} - \frac{\sin 2\pi}{8}\right) - 0\right) = \frac{\pi}{8}$$

The area of one leaf is $A = \frac{\pi}{8} \approx 0.39$. It follows that the area of the entire rose is $\frac{\pi}{2}$. Since the "radius" of the rose (the point where $\theta = \frac{\pi}{4}$) is 1, and the circumscribed circle is tangent there, the circumscribed circle has radius 1 and thus area π. Hence the area of the rose is half that of the circumscribed circle.

10. Find the area enclosed by one loop of the lemniscate with equation $r^2 = \cos 2\theta$ (Figure 18). Choose your limits of integration carefully.

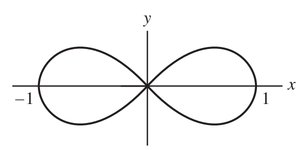
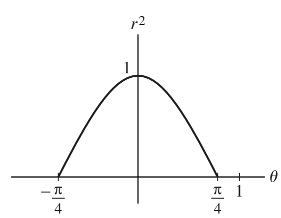
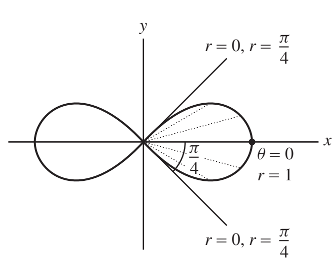


FIGURE 18 The lemniscate $r^2 = \cos 2\theta$.

SOLUTION We sketch the graph of $r^2 = \cos 2\theta$ in the $\left(r^2, \theta\right)$ plane; for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$:



We see that as θ varies from $-\frac{\pi}{4}$ to 0, r^2 increases from 0 to 1, hence r also increases from 0 to 1. Then, as θ varies from 0 to $\frac{\pi}{4}$, r^2, so r decreases from 1 to 0. This gives the right-hand loop of the lemniscate.

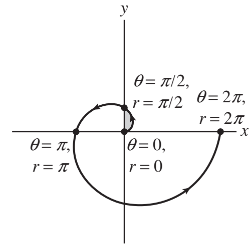


Therefore, the area enclosed by the right-hand loop is:

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{2}\frac{\sin 2\theta}{2}\bigg|_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}$$

11. Sketch the spiral $r=\theta$ for $0\le\theta\le 2\pi$ and find the area bounded by the curve and the first quadrant.

SOLUTION The spiral $r=\theta$ for $0\le\theta\le 2\pi$ is shown in the following figure in the xy-plane:



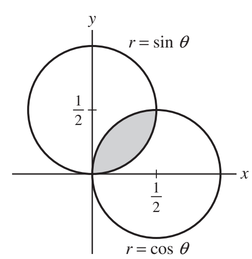
The spiral $r=\theta$

We must compute the area of the shaded region. This region is traced as θ varies from 0 to $\frac{\pi}{2}$. Using the formula for the area in polar coordinates we get

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\theta^2\,d\theta = \frac{1}{2}\frac{\theta^3}{3}\bigg|_0^{\pi/2} = \frac{1}{6}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{48}$$

12. Find the area of the intersection of the circles $r=\sin\theta$ and $r=\cos\theta$.

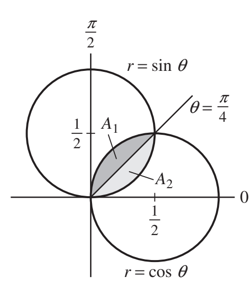
SOLUTION The region of intersection between the two circles is shown in the following figure:



We first find the value of θ at the point of intersection (besides the origin) of the two circles, by solving the following equation for $0\le a\le\frac{\pi}{2}$:

$$\sin\theta = \cos\theta$$
$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

We now compute the area as the sum of the two areas A_1 and A_2, shown in the figure:



Using the formula for the area in polar coordinates we get

$$A_1 = \frac{1}{2}\int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/2}^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{4}\int_{\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{4}\left(\theta+\frac{\sin 2\theta}{2}\right)\bigg|_{\pi/2}^{\pi/2} = \frac{1}{4}\left(\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(\frac{\pi}{4}+\frac{\sin\frac{\pi}{2}}{2}\right)\right) = \frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi}{16}-\frac{1}{8}$$

$$A_2 = \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{4}\int_0^{\pi/4}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{1}{4}\left(\left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - 0\right) = \frac{\pi}{16} - \frac{1}{8}$$

Notice that $A_2 = A_1$ as shown in the figure due to symmetry. The total area enclosed by the two circles is the sum

$$A = A_1 + A_2 = \left(\frac{\pi}{16} - \frac{1}{8}\right) + \left(\frac{\pi}{16} - \frac{1}{8}\right) = \frac{\pi}{8} - \frac{1}{4} \approx 0.14.$$

13. Find the area of region A in Figure 19.

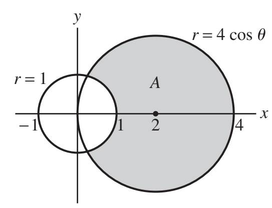
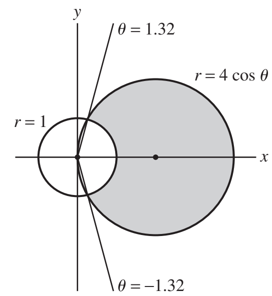


FIGURE 19

SOLUTION We first find the values of θ at the points of intersection of the two circles, by solving the following equation for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:

$$4\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{4} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{4}\right)$$



We now compute the area using the formula for the area between two curves:

$$A = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left((4\cos\theta)^2 - 1^2\right)d\theta = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left(16\cos^2\theta - 1\right)d\theta$$

Using the identity $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ we get

$$A = \frac{1}{2}\int_{-\theta_1}^{\theta_1}\left(\frac{16(\cos 2\theta + 1)}{2} - 1\right)d\theta = \frac{1}{2}\int_{-\theta_1}^{\theta_1}(8\cos 2\theta + 7)\,d\theta = \frac{1}{2}(4\sin 2\theta + 7\theta)\Big|_{-\theta_1}^{\theta_1}$$

$$= 4\sin 2\theta_1 + 7\theta_1 = 8\sin\theta_1\cos\theta_1 + 7\theta_1 = 8\sqrt{1-\cos^2\theta_1}\cos\theta_1 + 7\theta_1$$

Using the fact that $\cos\theta_1 = \frac{1}{4}$ we get

$$A = \frac{\sqrt{15}}{2} + 7\cos^{-1}\left(\frac{1}{4}\right) \approx 11.163$$

14. Find the area of the shaded region in Figure 20, enclosed by the circle $r = \frac{1}{2}$ and a petal of the curve $r = \cos 3\theta$. *Hint:* Compute the area of both the petal and the region inside the petal and outside the circle.

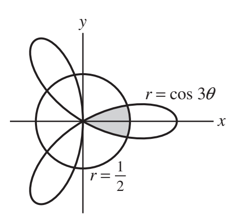
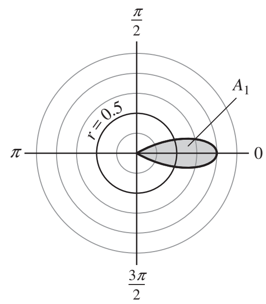
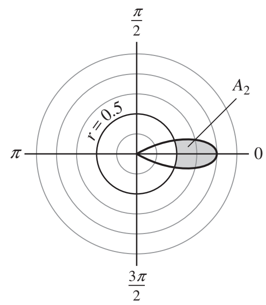


FIGURE 20

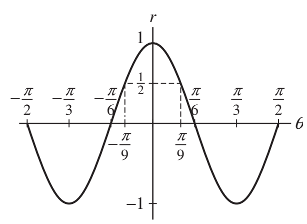
SOLUTION We compute the area A of the given region as the difference between the area A_1 of the leaf, shown here:



The area, A_2, of the region inside the leaf and outside the circle, shown here:



Computing A_1: To determine the limits of integration we use the following graph of $r = \cos 3\theta$:



$r = \cos 3\theta$

As θ varies from $-\frac{\pi}{6}$ to 0, r increases from 0 to 1. Then, as θ varies from 0 to $\frac{\pi}{6}$, r decreases from 1 back to zero. Hence the leaf is traced as θ varies from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$. We use the formula for the area in polar coordinates to obtain

$$A_1 = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta = \frac{1}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) \, d\theta$$
$$= \frac{1}{4}\left(\theta + \frac{\sin 6\theta}{6}\right)\Big|_{-\pi/6}^{\pi/6} = \frac{1}{4}\left(\left(\frac{\pi}{6} + \frac{\sin \pi}{6}\right) - \left(-\frac{\pi}{6} + \frac{\sin(-\pi)}{6}\right)\right) = \frac{1}{4}\cdot\frac{2\pi}{6} = \frac{\pi}{12}$$

Computing A_2: The two curves intersect at the points where $\cos 3\theta = \frac{1}{2}$, that is, $\theta = \pm\frac{\pi}{9}$ (see the graph of $r = \cos 3\theta$ in the $r\theta$-plane). Using the formula for the area between two curves we get

$$A_2 = \frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\cos^2 3\theta - \left(\frac{1}{2}\right)^2\right) d\theta = \frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta - \frac{1}{4}\right) d\theta$$
$$= \frac{1}{8}\int_{-\pi/9}^{\pi/9} (1 + 2\cos 6\theta) \, d\theta = \frac{1}{8}\left(\theta + \frac{\sin 6\theta}{3}\right)\Big|_{-\pi/9}^{\pi/9}$$
$$= \frac{1}{8}\left(\left(\frac{\pi}{9} + \frac{\sin\frac{6\pi}{9}}{3}\right) - \left(-\frac{\pi}{9} + \frac{\sin\left(-\frac{6\pi}{9}\right)}{3}\right)\right) = \frac{1}{4}\left(\frac{\pi}{9} + \frac{\sqrt{3}}{6}\right) = \frac{\pi}{36} + \frac{\sqrt{3}}{24}$$

The required area is the difference between A_1 and A_2, that is,

$$A = A_1 - A_2 = \frac{\pi}{12} - \left(\frac{\pi}{36} + \frac{\sqrt{3}}{24}\right) = \frac{\pi}{18} - \frac{\sqrt{3}}{24} \approx 0.102.$$

15. Find the area of the inner loop of the limaçon with polar equation $r = 2\cos\theta - 1$ (Figure 21).

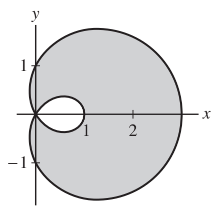
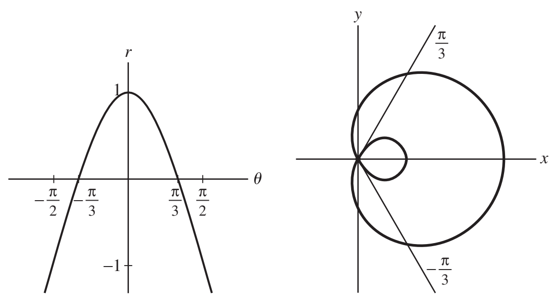


FIGURE 21 The limaçon $r = 2\cos\theta - 1$.

SOLUTION We consider the graph of $r = 2\cos\theta - 1$ in cartesian and in polar, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:



$r = 2\cos\theta - 1$

As θ varies from $-\frac{\pi}{3}$ to 0, r increases from 0 to 1. As θ varies from 0 to $\frac{\pi}{3}$, r decreases from 1 back to 0. Hence, the inner loop of the limaçon is traced as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. The area of the shaded region is thus

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} r^2\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)^2\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2\theta - 4\cos\theta + 1\right)\,d\theta$$

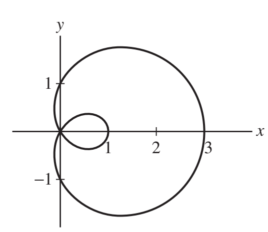
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\left(\cos 2\theta + 1\right) - 4\cos\theta + 1)\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos 2\theta - 4\cos\theta + 3)\,d\theta$$

$$= \frac{1}{2}\left(\sin 2\theta - 4\sin\theta + 3\theta\right)\Big|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(\left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi\right) - \left(\sin\left(-\frac{2\pi}{3}\right) - 4\sin\left(-\frac{\pi}{3}\right) - \pi\right)\right)$$

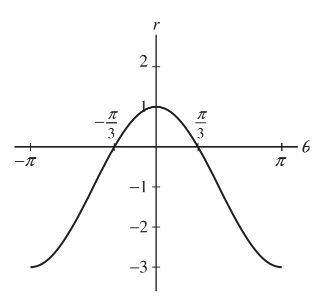
$$= \frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2} + \pi = \pi - \frac{3\sqrt{3}}{2} \approx 0.54$$

16. Find the area of the shaded region in Figure 21 between the inner and outer loop of the limaçon $r = 2\cos\theta - 1$.

SOLUTION The region is shown in the figure below.

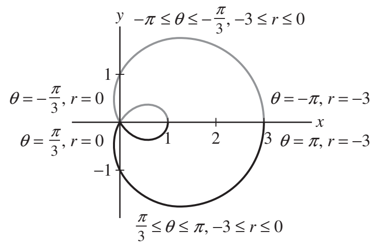


We use the following graph.



Graph of $r = 2\cos\theta - 1$

As θ varies from $\frac{\pi}{3}$ to π, r is negative and $|r|$ increases from 0 to 3. This gives the outer loop of the limaçon which is in the lower half plane. Similarly, the outer loop which is in the upper half plane is traced for $-\pi \le \theta \le -\frac{\pi}{3}$.



Using symmetry with respect to the x-axis, we obtain the following for the area of the outer loop:

$$A = 2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi} r^2\,d\theta = \int_{\pi/3}^{\pi} (2\cos\theta - 1)^2\,d\theta = \int_{\pi/3}^{\pi}\left(4\cos^2\theta - 4\cos\theta + 1\right)\,d\theta$$

$$= \int_{\pi/3}^{\pi} (2\,(1+\cos 2\theta) - 4\cos\theta + 1)\;d\theta = \int_{\pi/3}^{\pi} (2\cos 2\theta - 4\cos\theta + 3)\;d\theta = \sin 2\theta - 4\sin\theta + 3\theta\Big|_{\pi/3}^{\pi}$$

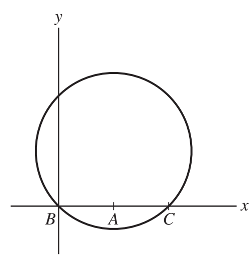
$$= (\sin 2\pi - 4\sin\pi + 3\pi) - \left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi\right) = 3\pi - \left(\frac{\sqrt{3}}{2} - 2\sqrt{3} + \pi\right) = 2\pi + \frac{3\sqrt{3}}{2}$$

Finally, to find the area of the region between the inner and outer loop of the limaçon, we subtract the area of the inner loop, obtained in the previous exercise, from the area of the outer loop:

$$\left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$$

17. Find the area of the part of the circle $r = \sin\theta + \cos\theta$ in the fourth quadrant (see Exercise 26 in Section 11.3).

SOLUTION The value of θ corresponding to the point B is the solution of $r = \sin\theta + \cos\theta = 0$ for $-\pi \le \theta \le \pi$.



That is,

$$\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = -\frac{\pi}{4}$$

At the point C, we have $\theta = 0$. The part of the circle in the fourth quadrant is traced if θ varies between $-\frac{\pi}{4}$ and 0. This leads to the following area:

$$A = \frac{1}{2}\int_{-\pi/4}^{0} r^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{0} (\sin\theta + \cos\theta)^2\,d\theta = \frac{1}{2}\int_{-\pi/4}^{0}\left(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\right)\,d\theta$$

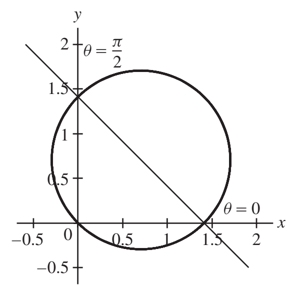
Using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$ we get:

$$A = \frac{1}{2}\int_{-\pi/4}^{0} (1 + \sin 2\theta)\;d\theta = \frac{1}{2}\left(\theta - \frac{\cos 2\theta}{2}\right)\Big|_{-\pi/4}^{0}$$

$$= \frac{1}{2}\left(\left(0 - \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{\cos\left(\frac{-\pi}{2}\right)}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4} \approx 0.14.$$

18. Find the area of the region inside the circle $r = 2\sin\left(\theta + \frac{\pi}{4}\right)$ and above the line $r = \sec\left(\theta - \frac{\pi}{4}\right)$.

SOLUTION The line $r = \sec\left(\theta - \frac{\pi}{4}\right)$ intersects the circle $r = 2\sin\left(\theta + \frac{\pi}{4}\right)$ when $\theta = 0$ and $\theta = 2\pi$.



Thus the area of the region inside the circle and above the line is

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2}\left(\left(2\sin\left(\theta+\frac{\pi}{4}\right)\right)^2 - \left(\sec\left(\theta-\frac{\pi}{4}\right)\right)^2\right)d\theta &= \frac{1}{2}\int_0^{\pi/2} 4\sin^2\left(\theta+\frac{\pi}{4}\right) - \sec^2\left(\theta-\frac{\pi}{4}\right)d\theta \\ &= \frac{1}{2}\left(2t - 2\sin\left(t+\frac{\pi}{4}\right)\cos\left(t+\frac{\pi}{4}\right) - \tan\left(t-\frac{\pi}{4}\right)\right)\Big|_0^{\pi/2} \\ &= \frac{1}{2}\left(\pi - 2\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right) - \left(-2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)\right)\right) \\ &= \frac{1}{2}(\pi + 1 - 1 + 1 - 1) = \frac{\pi}{2}\end{aligned}$$

19. Find the area between the two curves in Figure 22(A).

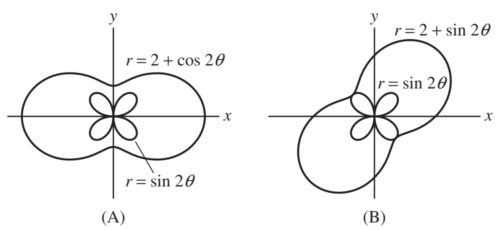
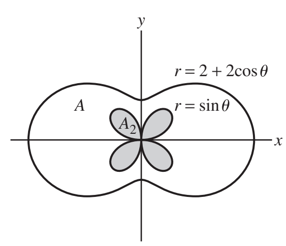


FIGURE 22

SOLUTION We compute the area A between the two curves as the difference between the area A_1 of the region enclosed in the outer curve $r = 2 + \cos 2\theta$ and the area A_2 of the region enclosed in the inner curve $r = \sin 2\theta$. That is,

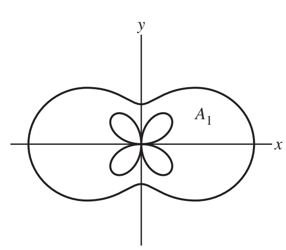
$$A = A_1 - A_2.$$



In Exercise 9 we showed that $A_2 = \frac{\pi}{2}$, hence,

$$A = A_1 - \frac{\pi}{2} \tag{1}$$

We compute the area A_1.



Using symmetry, the area is four times the area enclosed in the first quadrant. That is,

$$A_1 = 4 \cdot \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = 2\int_0^{\pi/2} (2+\cos 2\theta)^2\,d\theta = 2\int_0^{\pi/2}\left(4+4\cos 2\theta+\cos^2 2\theta\right)d\theta$$

Using the identity $\cos^2 2\theta = \frac{1}{2}\cos 4\theta + \frac{1}{2}$ we get

$$A_1 = 2\int_0^{\pi/2}\left(4+4\cos 2\theta+\frac{1}{2}\cos 4\theta+\frac{1}{2}\right)d\theta = 2\int_0^{\pi/2}\left(\frac{9}{2}+\frac{1}{2}\cos 4\theta+4\cos 2\theta\right)d\theta$$

$$= 2\left(\frac{9\theta}{2}+\frac{\sin 4\theta}{8}+2\sin 2\theta\right)\Big|_0^{\pi/2} = 2\left(\left(\frac{9\pi}{4}+\frac{\sin 2\pi}{8}+2\sin\pi\right)-0\right) = \frac{9\pi}{2} \qquad (2)$$

Combining (1) and (2) we obtain

$$A = \frac{9\pi}{2} - \frac{\pi}{2} = 4\pi.$$

20. Find the area between the two curves in Figure 22(B).

SOLUTION Since

$$2+\cos 2\left(\theta-\frac{\pi}{4}\right) = 2+\cos\left(2\theta-\frac{\pi}{2}\right) = 2+\cos\left(\frac{\pi}{2}-2\theta\right) = 2+\sin 2\theta$$

it follows that the curve $r = 2+\sin 2\theta$ is obtained by rotating the curve $r = 2+\cos\theta$ by $\frac{\pi}{4}$ about the origin. Therefore the area between the curves $r = 2+\sin 2\theta$ and $r = \sin 2\theta$ is the same as the area between the curves $r = 2+\cos\theta$ and $r = \sin 2\theta$ computed in Exercise 19. That is, $A = 4\pi$. (Notice that if the inner curve remains inside the rotated curve, the area between the curves is not changed).

21. Find the area inside both curves in Figure 23.

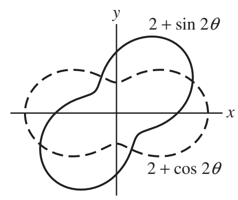
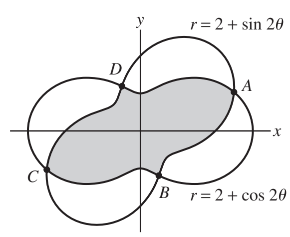


FIGURE 23

SOLUTION The area we need to find is the area of the shaded region in the figure.



We first find the values of θ at the points of intersection A, B, C, and D of the two curves, by solving the following equation for $-\pi \le \theta \le \pi$:

$$2+\cos 2\theta = 2+\sin 2\theta$$
$$\cos 2\theta = \sin 2\theta$$
$$\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4}+\pi k \Rightarrow \theta = \frac{\pi}{8}+\frac{\pi k}{2}$$

The solutions for $-\pi \le \theta \le \pi$ are

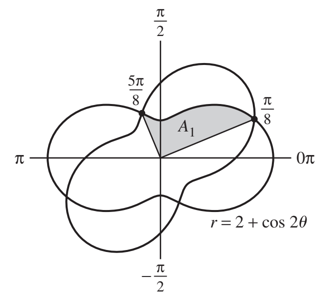
$$A:\quad \theta = \frac{\pi}{8}.$$

$$B:\quad \theta = -\frac{3\pi}{8}.$$

$$C:\quad \theta = -\frac{7\pi}{8}.$$

$$D:\quad \theta = \frac{5\pi}{8}.$$

Using symmetry, we compute the shaded area in the figure below and multiply it by 4:



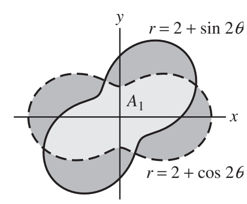
$$A = 4 \cdot A_1 = 4 \cdot \frac{1}{2} \cdot \int_{\pi/8}^{5\pi/8} (2+\cos 2\theta)^2\, d\theta = 2\int_{\pi/8}^{5\pi/8} \left(4 + 4\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= 2\int_{\pi/8}^{5\pi/8} \left(4 + 4\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta = \int_{\pi/8}^{5\pi/8} (9 + 8\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= 9\theta + 4\sin 2\theta + \frac{\sin 4\theta}{4}\bigg|_{\pi/8}^{5\pi/8} = 9\left(\frac{5\pi}{8} - \frac{\pi}{8}\right) + 4\left(\sin\frac{5\pi}{4} - \sin\frac{\pi}{4}\right) + \frac{1}{4}\left(\sin\frac{5\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{9\pi}{2} - 4\sqrt{2}$$

22. Find the area of the region that lies inside one but not both of the curves in Figure 23.

SOLUTION The area we need to find is the area of the shaded region in the following figure:



We denote by A_1 the area inside both curves. In Exercise 20 we showed that the curve $r = 2 + \sin 2\theta$ is obtained by rotating the curve $r = \cos 2\theta$ by $\frac{\pi}{4}$ around the origin. Hence, the areas enclosed in these curves are equal. We denote it by A_2. It follows that the area A that we need to find is

$$A = 2A_2 - 2A_1 = 2(A_2 - A_1) \tag{1}$$

In Exercise 20 we found that $A_2 = \frac{9\pi}{2}$, and in Exercise 21 we showed that $A_1 = \frac{9\pi}{2} - 4\sqrt{2}$. Substituting in (1) we obtain

$$A = 2\left(\frac{9\pi}{2} - \left(\frac{9\pi}{2} - 4\sqrt{2}\right)\right) = 8\sqrt{2} \approx 11.3.$$

23. Calculate the total length of the circle $r = 4\sin\theta$ as an integral in polar coordinates.

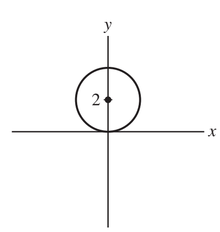
SOLUTION We use the formula for the arc length:

$$S = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \tag{1}$$

In this case, $f(\theta) = 4\sin\theta$ and $f'(\theta) = 4\cos\theta$, hence

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(4\sin\theta)^2 + (4\cos\theta)^2} = \sqrt{16} = 4$$

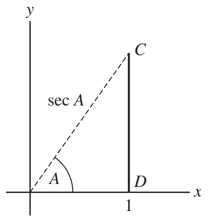
The circle is traced as θ is varied from 0 to π. Substituting $\alpha = 0$, $\beta = \pi$ in (1) yields $S = \int_0^{\pi} 4\, d\theta = 4\pi$.



The circle $r = 4\sin\theta$

24. Sketch the segment $r = \sec\theta$ for $0 \le \theta \le A$. Then compute its length in two ways: as an integral in polar coordinates and using trigonometry.

SOLUTION The line $r = \sec\theta$ has the rectangular equation $x = 1$. The segment AB for $0 \le \theta \le A$ is shown in the figure.



Using trigonometry, the length of the segment $\overline{AB}$ is

$$L = \overline{AB} = \overline{0B}\tan A = 1 \cdot \tan A = \tan A$$

Alternatively, we use the integral in polar coordinates with $f(\theta) = \sec(\theta)$ and $f'(\theta) = \tan\theta\sec\theta$. This gives

$$L = \int_0^A \sqrt{(\sec\theta)^2 + (\tan\theta\sec\theta)^2}\,d\theta = \int_0^A \sqrt{1+\tan^2\theta}\,\sec\theta\,d\theta = \int_0^A \sec^2\theta\,d\theta = \tan\theta\Big|_0^A = \tan A.$$

The two answers agree, as expected.

In Exercises 25–30, compute the length of the polar curve.

25. The length of $r = \theta^2$ for $0 \le \theta \le \pi$

SOLUTION We use the formula for the arc length. In this case $f(\theta) = \theta^2$, $f'(\theta) = 2\theta$, so we obtain

$$S = \int_0^\pi \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\,d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^\pi \theta\sqrt{\theta^2+4}\,d\theta$$

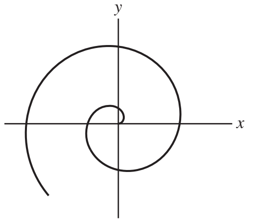
We compute the integral using the substitution $u = \theta^2 + 4$, $du = 2\theta\,d\theta$. This gives

$$S = \frac{1}{2}\int_4^{\pi^2+4} \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^{\pi^2+4} = \frac{1}{3}\left(\left(\pi^2+4\right)^{3/2} - 4^{3/2}\right) = \frac{1}{3}\left(\left(\pi^2+4\right)^{3/2} - 8\right) \approx 14.55$$

26. The spiral $r = \theta$ for $0 \le \theta \le A$

SOLUTION We use the formula for the arc length. In this case $f(\theta) = \theta$, $f'(\theta) = 1$. Using integration formulas we get:

$$\begin{aligned} S &= \int_0^A \sqrt{\theta^2 + 1^2}\,d\theta = \int_0^A \sqrt{\theta^2+1}\,d\theta = \frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left|\theta + \sqrt{\theta^2+1}\right|\Big|_0^A \\ &= \frac{A}{2}\sqrt{A^2+1} + \frac{1}{2}\ln\left|A + \sqrt{A^2+1}\right| \end{aligned}$$



The spiral $r = \theta$

27. The equiangular spiral $r = e^\theta$ for $0 \le \theta \le 2\pi$

SOLUTION Since $f(\theta) = e^\theta$, by the formula for the arc length we have:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{f'(\theta)^2 + f(\theta)}\,d\theta + \int_0^{2\pi} \sqrt{\left(e^\theta\right)^2 + \left(e^\theta\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\,d\theta \\ &= \sqrt{2}\int_0^{2\pi} e^\theta\,d\theta = \sqrt{2}e^\theta\Big|_0^{2\pi} = \sqrt{2}\left(e^{2\pi} - e^0\right) = \sqrt{2}\left(e^{2\pi} - 1\right) \approx 755.9 \end{aligned}$$

28. The inner loop of $r = 2\cos\theta - 1$ in Figure 21

SOLUTION In Exercise 15 it is shown that the inner loop of the limaçon $r = 2\cos\theta - 1$ is traced as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$. Also,

$$f(\theta) = 2\cos\theta - 1 \quad \text{and} \quad f'(\theta) = -2\sin\theta.$$

Using the integral for the arc length we obtain

$$\begin{aligned} L &= \int_{-\pi/3}^{\pi/3} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_{-\pi/3}^{\pi/3} \sqrt{(2\cos\theta - 1)^2 + (-2\sin\theta)^2}\,d\theta \\ &= \int_{-\pi/3}^{\pi/3} \sqrt{4\cos^2\theta - 4\cos\theta + 1 + 4\sin^2\theta}\,d\theta = \int_{-\pi/3}^{\pi/3} \sqrt{5 - 4\cos\theta}\,d\theta \end{aligned}$$

29. The cardioid $r = 1 - \cos\theta$ in Figure 16

SOLUTION In the equation of the cardioid, $f(\theta) = 1 - \cos\theta$. Using the formula for arc length in polar coordinates we have:

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta \tag{1}$$

We compute the integrand:

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2\,(1 - \cos\theta)}$$

We identify the interval of θ. Since $-1 \le \cos\theta \le 1$, every $0 \le \theta \le 2\pi$ corresponds to a nonnegative value of r. Hence, θ varies from 0 to 2π. By (1) we obtain

$$L = \int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\,d\theta$$

Now, $1 - \cos\theta = 2\sin^2(\theta/2)$, and on the interval $0 \le \theta \le \pi$, $\sin(\theta/2)$ is nonnegative, so that $\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\sin(\theta/2)$ there. The graph is symmetric, so it suffices to compute the integral for $0 \le \theta \le \pi$, and we have

$$L = 2\int_0^{\pi} \sqrt{2(1 - \cos\theta)}\,d\theta = 2\int_0^{\pi} 2\sin(\theta/2)\,d\theta = 8\sin\frac{\theta}{2}\bigg|_0^{\pi} = 8$$

30. $r = \cos^2\theta$

SOLUTION Since $\cos\theta = \cos(-\theta)$ and $\cos^2(\pi - \theta) = \cos^2\theta$ the curve is symmetric with respect to the x and y-axis. Therefore, we may compute the length as four times the length of the part of the curve in the first quadrant. We use the formula for the arc length in polar coordinates. In this case, $f(\theta) = \cos^2\theta$, $f'(\theta) = 2\cos\theta\,(-\sin\theta)$, so we obtain

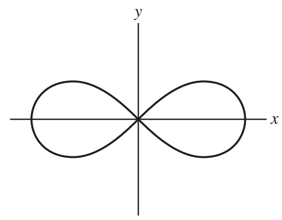
$$\begin{aligned} \sqrt{f(\theta)^2 + f'(\theta)^2} &= \sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta} = \cos\theta\sqrt{\cos^2\theta + 4\sin^2\theta} \\ &= \cos\theta\sqrt{\cos^2\theta + \sin^2\theta + 3\sin^2\theta} = \cos\theta\sqrt{1 + 3\sin^2\theta} \end{aligned}$$

Thus,

$$L = \int_0^{\pi/2} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_0^{\pi/2} \cos\theta\sqrt{1 + 3\sin^2\theta}\,d\theta.$$

We compute the integral using the substitution $u = \sqrt{3}\sin\theta$ we get

$$\begin{aligned} L &= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \sqrt{1 + u^2}\,du = \frac{1}{\sqrt{3}}\left(\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2}\ln|u + \sqrt{1 + u^2}|\right)\bigg|_0^{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\sqrt{1 + 3} + \frac{1}{2}\ln\left(\sqrt{3} + \sqrt{1 + 3}\right) - 0\right) = 1 + \frac{1}{2\sqrt{3}}\ln\left(2 + \sqrt{3}\right) \end{aligned}$$



Graph of $r = \cos^2\theta$

Thus the total length equals $4L = 4 + \frac{2}{\sqrt{3}} \ln\left(2 + \sqrt{3}\right) \approx 5.52$.

In Exercises 31 and 32, express the length of the curve as an integral but do not evaluate it.

31. $r = (2 - \cos\theta)^{-1}, \quad 0 \le \theta \le 2\pi$

SOLUTION We have $f(\theta) = (2-\cos\theta)^{-1}$, $f'(\theta) = -(2-\cos\theta)^{-2}\sin\theta$, hence,

$$\sqrt{f^2(\theta) + f'(\theta)^2} = \sqrt{(2-\cos\theta)^{-2} + (2-\cos\theta)^{-4}\sin^2\theta} = \sqrt{(2-\cos\theta)^{-4}\left((2-\cos\theta)^2 + \sin^2\theta\right)}$$

$$= (2-\cos\theta)^{-2}\sqrt{4 - 4\cos\theta + \cos^2\theta + \sin^2\theta} = (2-\cos\theta)^{-2}\sqrt{5 - 4\cos\theta}$$

Using the integral for the arc length we get

$$L = \int_0^{2\pi} \sqrt{5 - 4\cos\theta}(2-\cos\theta)^{-2}\, d\theta.$$

32. $r = \sin^3 t, \quad 0 \le \theta \le 2\pi$

SOLUTION We have $f(t) = \sin^3 t$, $f'(t) = 3\sin^2 t\cos t$, so that

$$\sqrt{f(t)^2 + f'(t)^2} = \sqrt{\sin^6 t + 9\sin^4 t\cos^2 t} = \sin^2 t\sqrt{\sin^2 t + 9\cos^2 t}$$

$$= \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 8\cos^2 t} = \sin^2 t\sqrt{1 + 8\cos^2 t}$$

Using the formula for arc length integral we get

$$L = \int_0^{2\pi} \sin^2 t\sqrt{1 + 8\cos^2 t}\, dt$$

In Exercises 33–36, use a computer algebra system to calculate the total length to two decimal places.

33. CAS The three-petal rose $r = \cos 3\theta$ in Figure 20

SOLUTION We have $f(\theta) = \cos 3\theta$, $f'(\theta) = -3\sin 3\theta$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\cos^2 3\theta + 9\sin^2 3\theta} = \sqrt{\cos^2 3\theta + \sin^2 3\theta + 8\sin^2 3\theta} = \sqrt{1 + 8\sin^2 3\theta}$$

Note that the curve is traversed completely for $0 \le \theta \le \pi$. Using the arc length formula and evaluating with Maple gives

$$L = \int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{1 + 8\sin^2 3\theta}\, d\theta \approx 6.682446608$$

34. CAS The curve $r = 2 + \sin 2\theta$ in Figure 23

SOLUTION We have $f(\theta) = 2 + \sin 2\theta$, $f'(\theta) = 2\cos 2\theta$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(2 + \sin 2\theta)^2 + 4\cos^2 2\theta} = \sqrt{4 + 4\sin 2\theta + \sin^2 2\theta + 4\cos^2 2\theta}$$

$$= \sqrt{4 + 4\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta + 3\cos^2 2\theta}$$

$$= \sqrt{5 + 4\sin 2\theta + 3\cos^2 2\theta}$$

The curve is traversed completely for $0 \le \theta \le 2\pi$. Using the arc length formula and evaluating with Maple gives

$$L = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \approx 15.40375907$$

35. CAS The curve $r = \theta \sin\theta$ in Figure 24 for $0 \le \theta \le 4\pi$

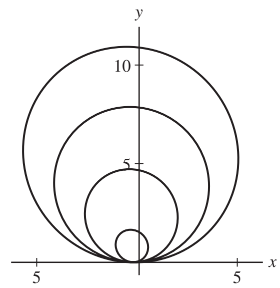


FIGURE 24 $r = \theta \sin\theta$ for $0 \le \theta \le 4\pi$.

SOLUTION We have $f(\theta) = \theta\sin\theta$, $f'(\theta) = \sin\theta + \theta\cos\theta$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\theta^2\sin^2\theta + (\sin\theta + \theta\cos\theta)^2} = \sqrt{\theta^2\sin^2\theta + \sin^2\theta + 2\theta\sin\theta\cos\theta + \theta^2\cos^2\theta}$$
$$= \sqrt{\theta^2 + \sin^2\theta + \theta\sin 2\theta}$$

using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$. Thus by the arc length formula and evaluating with Maple, we have

$$L = \int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_0^{4\pi} \sqrt{\theta^2 + \sin^2\theta + \theta\sin 2\theta}\,d\theta \approx 79.56423976$$

36. CAS $r = \sqrt{\theta}$, $0 \le \theta \le 4\pi$

SOLUTION We have $f(\theta) = \sqrt{\theta}$, $f'(\theta) = \frac{1}{2}\theta^{-1/2}$, so that

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{\theta + \frac{1}{4\theta}}$$

so that by the arc length formula, evaluating with Maple, we have

$$L = \int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta = \int_0^{4\pi} \sqrt{\theta + \frac{1}{4\theta}}\,d\theta \approx 30.50125041$$

Further Insights and Challenges

37. Suppose that the polar coordinates of a moving particle at time t are $(r(t), \theta(t))$. Prove that the particle's speed is equal to $\sqrt{(dr/dt)^2 + r^2(d\theta/dt)^2}$.

SOLUTION The speed of the particle in rectangular coordinates is:

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \tag{1}$$

We need to express the speed in polar coordinates. The x and y coordinates of the moving particles as functions of t are

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t)$$

We differentiate $x(t)$ and $y(t)$, using the Product Rule for differentiation. We obtain (omitting the independent variable t)

$$x' = r'\cos\theta - r(\sin\theta)\,\theta'$$
$$y' = r'\sin\theta - r(\cos\theta)\,\theta'$$

Hence,

$$x'^2 + y'^2 = \left(r'\cos\theta - r\theta'\sin\theta\right)^2 + \left(r'\sin\theta + r\theta'\cos\theta\right)^2$$
$$= r'^2\cos^2\theta - 2r'r\theta'\cos\theta\sin\theta + r^2\theta'^2\sin^2\theta + r'^2\sin^2\theta + 2r'r\theta'\sin^2\theta\cos\theta + r^2\theta'^2\cos^2\theta$$
$$= r'^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\theta'^2\left(\sin^2\theta + \cos^2\theta\right) = r'^2 + r^2\theta'^2 \tag{2}$$

Substituting (2) into (1) we get

$$\frac{ds}{dt} = \sqrt{r'^2 + r^2\theta'^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}$$

38. Compute the speed at time $t = 1$ of a particle whose polar coordinates at time t are $r = t, \theta = t$ (use Exercise 37). What would the speed be if the particle's rectangular coordinates were $x = t$, $y = t$? Why is the speed increasing in one case and constant in the other?

SOLUTION By Exercise 37 the speed of the particle is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2} \tag{1}$$

In this case $r = t$ and $\theta = t$ so $\frac{dr}{dt} = 1$ and $\frac{d\theta}{dt} = 1$. Substituting into (1) gives the following function of the speed:

$$\frac{ds}{dt} = \sqrt{1 + r(t)^2}$$

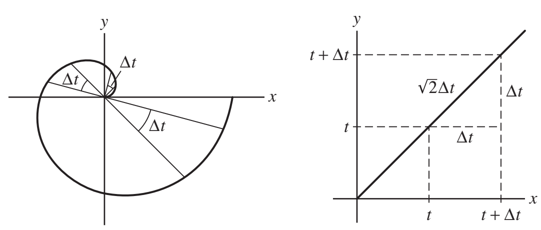
The speed expressed in rectangular coordinates is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

If $x = t$ and $y = t$, then $x'(t) = 1$ and $y'(t) = 1$. So the speed of the particle at time t is

$$\frac{ds}{dt} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

On the curve $x = t$, $y = t$ the particle travels the same distance $\Delta t\sqrt{2}$ for all time intervals Δt, hence, it has a constant speed. However, on the spiral $r = t, \theta = t$ the particle travels greater distances for time intervals $(t, t + \Delta t)$ as t increases, hence the speed is an increasing function of t.



$x = t, y = t$ $\qquad$ $r = t, \theta = t$

11.5 Conic Sections

Preliminary Questions

1. Which of the following equations defines an ellipse? Which does not define a conic section?

(a) $4x^2 - 9y^2 = 12$ **(b)** $-4x + 9y^2 = 0$

(c) $4y^2 + 9x^2 = 12$ **(d)** $4x^3 + 9y^3 = 12$

SOLUTION

(a) This is the equation of the hyperbola $\left(\frac{x}{\sqrt{3}}\right)^2 - \left(\frac{y}{\frac{2}{\sqrt{3}}}\right)^2 = 1$, which is a conic section.

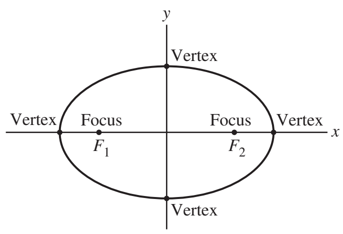
(b) The equation $-4x + 9y^2 = 0$ can be rewritten as $x = \frac{9}{4}y^2$, which defines a parabola. This is a conic section.

(c) The equation $4y^2 + 9x^2 = 12$ can be rewritten in the form $\left(\frac{y}{\sqrt{3}}\right)^2 + \left(\frac{x}{\frac{2}{\sqrt{3}}}\right)^2 = 1$, hence it is the equation of an ellipse, which is a conic section.

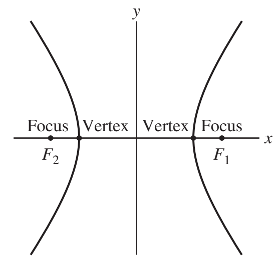
(d) This is not the equation of a conic section, since it is not an equation of degree two in x and y.

2. For which conic sections do the vertices lie between the foci?

SOLUTION If the vertices lie between the foci, the conic section is a hyperbola.



ellipse: foci between vertices

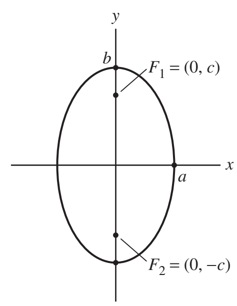


hyperbola: vertices between foci

3. What are the foci of

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{if } a < b?$$

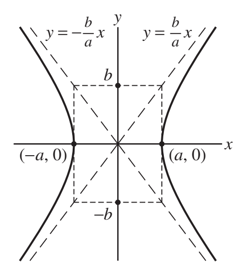
SOLUTION If $a < b$ the foci of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ are at the points $(0, c)$ and $(0, -c)$ on the y-axis, where $c = \sqrt{b^2 - a^2}$.



$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1; a < b$

4. What is the geometric interpretation of b/a in the equation of a hyperbola in standard position?

SOLUTION The vertices, i.e., the points where the focal axis intersects the hyperbola, are at the points $(a, 0)$ and $(-a, 0)$. The values $\pm\frac{b}{a}$ are the slopes of the two asymptotes of the hyperbola.



Hyperbola in standard position

Exercises

In Exercises 1–6, find the vertices and foci of the conic section.

1. $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

SOLUTION This is an ellipse in standard position with $a = 9$ and $b = 4$. Hence, $c = \sqrt{9^2 - 4^2} = \sqrt{65} \approx 8.06$. The foci are at $F_1 = (-8.06, 0)$ and $F_2 = (8.06, 0)$, and the vertices are $(9, 0)$, $(-9, 0)$, $(0, 4)$, $(0, -4)$.

2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

SOLUTION Writing the equation in the from $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ we get an ellipse with $a = 3$ and $b = 2$. Hence $c = \sqrt{3^2 - 2^2} = \sqrt{5} \approx 2.24$. The foci are at $F_1 = (-2.24, 0)$ and $F_2 = (2.24, 0)$ and the vertices are $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$.

3. $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{9}\right)^2 = 1$

SOLUTION This is a hyperbola in standard position with $a = 4$ and $b = 9$. Hence, $c = \sqrt{a^2 + b^2} = \sqrt{97} \approx 9.85$. The foci are at $(\pm\sqrt{97}, 0)$ and the vertices are $(\pm 2, 0)$.

4. $\frac{x^2}{4} - \frac{y^2}{9} = 36$

SOLUTION Putting this equation in standard form gives

$$\left(\frac{x}{12}\right)^2 - \left(\frac{y}{18}\right)^2 = 1$$

so this is a hyperbola in standard position with $a = 12$ and $b = 18$. Thus

$$c = \sqrt{a^2 + b^2} = 6\sqrt{13} \approx 21.633$$

The foci are at $(\pm 6\sqrt{13}, 0)$ and the vertices are at $(\pm 12, 0)$.

5. $\left(\frac{x-3}{7}\right)^2 - \left(\frac{y+1}{4}\right)^2 = 1$

SOLUTION We first consider the hyperbola $\left(\frac{x}{7}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$. For this hyperbola, $a = 7$, $b = 4$ and $c = \sqrt{7^2 + 4^2} \approx 8.06$. Hence, the foci are at $(8.06, 0)$ and $(-8.06, 0)$ and the vertices are at $(7, 0)$ and $(-7, 0)$. Since the given hyperbola is obtained by translating the center of the hyperbola $\left(\frac{x}{7}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$ to the point $(3, -1)$, the foci are at $F_1 = (8.06 + 3, 0 - 1) = (11.06, -1)$ and $F_2 = (-8.06 + 3, 0 - 1) = (-5.06, -1)$ and the vertices are $A = (7 + 3, 0 - 1) = (10, -1)$ and $A' = (-7 + 3, 0 - 1) = (-4, -1)$.

6. $\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{7}\right)^2 = 1$

SOLUTION We first consider the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$. Hence, $a = 4$ and $b = 7$ so $a < b$ and the focal axis is vertical. $c = \sqrt{7^2 - 4^2} \approx 5.74$ hence the foci are at $(0, 5.74)$ and $(0, -5.74)$. The vertices are $(4, 0)$, $(-4, 0)$, $(0, 7)$, $(0, -7)$. When we translate the ellipse so that its center is $(3, -1)$, the points above are translated so that the new vertices are $(4 + 3, 0 - 1) = (7, -1)$, $(-4 + 3, 0 - 1) = (-1, -1)$, $(0 + 3, 7 - 1) = (3, 6)$ and $(0 + 3, -7 - 1) = (3, -8)$. The new foci are at $(3, 4.74)$ and $(3, -6.74)$.

In Exercises 7–10, find the equation of the ellipse obtained by translating (as indicated) the ellipse

$$\left(\frac{x-8}{6}\right)^2 + \left(\frac{y+4}{3}\right)^2 = 1$$

7. Translated with center at the origin

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). Thus, for our ellipse to be located at the origin, it must have equation

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$$

8. Translated with center at $(-2, -12)$

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

describes an ellipse with center (h, k). Thus, for our ellipse to have center $(-2, -12)$, it must have equation

$$\frac{(x+2)^2}{6^2} + \frac{(y+12)^2}{3^2} = 1$$

9. Translated to the right six units

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

describes an ellipse with center (h, k). The original ellipse has center at $(8, -4)$, so we want an ellipse with center $(14, -4)$. Thus its equation is

$$\frac{(x-14)^2}{6^2}+\frac{(y+4)^2}{3^2}=1$$

10. Translated down four units

SOLUTION Recall that the equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

describes an ellipse with center (h, k). The original ellipse has center at $(8, -4)$, so we want an ellipse with center $(8, -8)$. Thus its equation is

$$\frac{(x-8)^2}{6^2}+\frac{(y+8)^2}{3^2}=1$$

In Exercises 11–14, find the equation of the given ellipse.

11. Vertices $(\pm 5, 0)$ and $(0, \pm 7)$

SOLUTION Since both sets of vertices are symmetric around the origin, the center of the ellipse is at $(0, 0)$. We have $a = 5$ and $b = 7$, so the equation of the ellipse is

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{7}\right)^2=1$$

12. Foci $(\pm 6, 0)$ and focal vertices $(\pm 10, 0)$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ with $a = 10$. The foci are $(\pm c, 0)$ with $c = 6$, so we use the relation $c=\sqrt{a^2-b^2}$ to find b:

$$b^2=a^2-c^2=10^2-6^2=64 \Rightarrow b=8$$

Therefore the equation of the ellipse is

$$\left(\frac{x}{10}\right)^2+\left(\frac{y}{8}\right)^2=1.$$

13. Foci $(0, \pm 10)$ and eccentricity $e = \frac{3}{5}$

SOLUTION Since the foci are on the y axis, this ellipse has a vertical major axis with center $(0, 0)$, so its equation is

$$\left(\frac{x}{b}\right)^2+\left(\frac{y}{a}\right)^2=1$$

We have $a=\frac{c}{e}=\frac{10}{3/5}=\frac{50}{3}$ and

$$b=\sqrt{a^2-c^2}=\sqrt{\frac{2500}{9}-100}=\frac{1}{3}\sqrt{2500-900}=\frac{40}{3}$$

Thus the equation of the ellipse is

$$\left(\frac{x}{40/3}\right)^2+\left(\frac{y}{50/3}\right)^2=1$$

14. Vertices $(4, 0)$, $(28, 0)$ and eccentricity $e = \frac{2}{3}$

SOLUTION This ellipse has a horizontal major axis with center midway between the vertices, at $(16, 0)$. Thus if the center were at $(0, 0)$, the ellipse would have vertices $(\pm 12, 0)$, so that $a = 12$ and $c = ae = 12 \cdot \frac{2}{3} = 8$. Then

$$b=\sqrt{a^2-c^2}=\sqrt{12^2-8^2}=\sqrt{80}=4\sqrt{5}$$

Finally, translating the center to (16, 0), the equation of the ellipse is

$$\left(\frac{(x-16)}{12}\right)^2 + \left(\frac{y}{4\sqrt{5}}\right)^2 = 1$$

In Exercises 15–20, find the equation of the given hyperbola.

15. Vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. The vertices are $(\pm a, 0)$ with $a = 3$ and the foci $(\pm c, 0)$ with $c = 5$. We use the relation $c = \sqrt{a^2 + b^2}$ to find b:

$$b = \sqrt{c^2 - a^2} = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$$

Therefore, the equation of the hyperbola is

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = 1.$$

16. Vertices $(\pm 3, 0)$ and asymptotes $y = \pm\frac{1}{2}x$

SOLUTION The equation is $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$. The vertices are $(\pm a, 0)$ with $a = 3$ and the asymptotes are $y = \pm\frac{b}{a}x$ with $\frac{b}{a} = \frac{1}{2}$. Hence, $b = \frac{a}{2} = \frac{3}{2}$ so the equation of the hyperbola is

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{3/2}\right)^2 = 1$$

17. Foci $(\pm 4, 0)$ and eccentricity $e = 2$

SOLUTION We have $c = 4$ and $e = 2$; from $c = ae$ we get $a = 2$, and then

$$b = \sqrt{c^2 - a^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}$$

The hyperbola has center at (0, 0) and horizontal axis, so its equation is

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{2\sqrt{3}}\right)^2 = 1$$

18. Vertices $(0, \pm 6)$ and eccentricity $e = 3$

SOLUTION The hyperbola has a vertical focal axis and center at (0, 0), so has equation

$$\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$$

$b = 6$ and $e = 3$ implies, since $be = c$, that $c = 18$, and

$$a = \sqrt{c^2 - b^2} = \sqrt{18^2 - 6^2} = \sqrt{288} = 12\sqrt{2}$$

Thus the equation of the hyperbola is

$$\left(\frac{y}{6}\right)^2 - \left(\frac{x}{12\sqrt{2}}\right)^2 = 1$$

19. Vertices $(-3, 0)$, $(7, 0)$ and eccentricity $e = 3$

SOLUTION The center is at $\frac{-3+7}{2} = 2$ with a horizontal focal axis, so the equation is

$$\left(\frac{x-2}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

Then $a = 7 - 2 = 5$, and $c = ae = 5 \cdot 3 = 15$. Finally,

$$b = \sqrt{c^2 - a^2} = \sqrt{15^2 - 5^2} = 10\sqrt{2}$$

so that the equation of the hyperbola is

$$\left(\frac{x-2}{5}\right)^2 - \left(\frac{y}{10\sqrt{2}}\right)^2 = 1$$

20. Vertices $(0, -6)$, $(0, 4)$ and foci $(0, -9)$, $(0, 7)$

SOLUTION The center of the hyperbola is at $\frac{-6+4}{2} = -1$ along the y axis; we write the equation as

$$\left(\frac{y+1}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$$

$b = 5$ since it is the distance from the given vertex to the center, and $c = 8$ since it is the distance from the foci to the center. Then

$$a = \sqrt{c^2 - b^2} = \sqrt{64 - 25} = \sqrt{39}$$

so that the equation of the hyperbola is

$$\left(\frac{y+1}{5}\right)^2 - \left(\frac{x}{\sqrt{39}}\right)^2 = 1$$

In Exercises 21–28, find the equation of the parabola with the given properties.

21. Vertex $(0, 0)$, focus $\left(\frac{1}{12}, 0\right)$

SOLUTION Since the focus is on the x-axis rather than the y-axis, and the vertex is $(0, 0)$, the equation is $x = \frac{1}{4c}y^2$. The focus is $(0, c)$ with $c = \frac{1}{12}$, so the equation is

$$x = \frac{1}{4 \cdot \frac{1}{12}}y^2 = 3y^2$$

22. Vertex $(0, 0)$, focus $(0, 2)$

SOLUTION The vertex is at $(0, 0)$, so the equation is $y = \frac{1}{4c}x^2 = \frac{1}{8}x^2$.

23. Vertex $(0, 0)$, directrix $y = -5$

SOLUTION The equation is $y = \frac{1}{4c}x^2$. The directrix is $y = -c$ with $c = 5$, hence $y = \frac{1}{20}x^2$.

24. Vertex $(3, 4)$, directrix $y = -2$

SOLUTION If the graph were translated to the origin, the vertex would be $(0, 0)$ and the directrix would be translated down 4 units so would be $y = -6$. Then $c = 6$ so the equation is $y = \frac{1}{4c}x^2 = \frac{1}{24}x^2$. Translating back to $(3, 4)$ gives

$$y = \frac{1}{24}(x-3)^2 + 4$$

25. Focus $(0, 4)$, directrix $y = -4$

SOLUTION The focus is $(0, c)$ with $c = 4$ and the directrix is $y = -c$ with $c = 4$, hence the equation of the parabola is

$$y = \frac{1}{4c}x^2 = \frac{x^2}{16}.$$

26. Focus $(0, -4)$, directrix $y = 4$

SOLUTION The focus is at $(0, c)$ with $c = -4$ and the directrix is $y = -c$ with $c = -4$, hence the equation $y = \frac{x^2}{4c}$ of the parabola becomes $y = -\frac{x^2}{16}$. Since $c < 0$, the parabola is open downward.

27. Focus $(2, 0)$, directrix $x = -2$

SOLUTION The focus is on the x-axis rather than on the y-axis and the directrix is a vertical line rather than horizontal as in the parabola in standard position. Therefore, the equation of the parabola is obtained by interchanging x and y in $y = \frac{1}{4c}x^2$. Also, by the given information $c = 2$. Hence, $x = \frac{1}{4c}y^2 = \frac{1}{4 \cdot 2}y^2$ or $x = \frac{y^2}{8}$.

28. Focus $(-2, 0)$, vertex $(2, 0)$

SOLUTION The vertex is always midway between the focus and the directrix, so the directrix must be the vertical line $x = 6$, and $c = -2 - 2 = -4$. Since the directrix is a vertical line, the parabola is obtained by interchanging x and y in the equation for a parabola in standard position. Finally, $c = -2 - 2 = -4$ is the distance from the vertex to the focus, so the equation is

$$x - 2 = \frac{1}{4c}y^2 = -\frac{1}{16}y^2, \quad \text{so} \quad x = 2 - \frac{1}{16}y^2$$

In Exercises 29–38, find the vertices, foci, center (if an ellipse or a hyperbola), and asymptotes (if a hyperbola).

29. $x^2 + 4y^2 = 16$

SOLUTION We first divide the equation by 16 to convert it to the equation in standard form:

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

For this ellipse, $a = 4$ and $b = 2$ hence $c = \sqrt{4^2 - 2^2} = \sqrt{12} \approx 3.5$. Since $a > b$ we have:

- The vertices are at $(\pm 4, 0)$, $(0, \pm 2)$.
- The foci are $F_1 = (-3.5, 0)$ and $F_2 = (3.5, 0)$.
- The focal axis is the x-axis and the conjugate axis is the y-axis.
- The ellipse is centered at the origin.

30. $4x^2 + y^2 = 16$

SOLUTION We divide the equation by 16 to rewrite it in the standard form:

$$\frac{4x^2}{16} + \frac{y^2}{16} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

This is the equation of an ellipse with $a = 2$, $b = 4$. Since $a < b$ the focal axis is the y-axis. Also, $c = \sqrt{4^2 - 2^2} = \sqrt{12} \approx 3.5$. We get:

- The vertices are at $(\pm 2, 0)$, $(0, \pm 4)$.
- The foci are $(0, \pm 3.5)$.
- The focal axis is the y-axis and the conjugate axis is the x-axis.
- The center is at the origin.

31. $\left(\frac{x-3}{4}\right)^2 - \left(\frac{y+5}{7}\right)^2 = 1$

SOLUTION For this hyperbola $a = 4$ and $b = 7$ so $c = \sqrt{4^2 + 7^2} = \sqrt{65} \approx 8.06$. For the standard hyperbola $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{7}\right)^2 = 1$, we have

- The vertices are $A = (4, 0)$ and $A' = (-4, 0)$.
- The foci are $F = (\sqrt{65}, 0)$ and $F' = (-\sqrt{65}, 0)$.
- The focal axis is the x-axis $y = 0$, and the conjugate axis is the y-axis $x = 0$.
- The center is at the midpoint of $\overline{FF'}$; that is, at the origin.
- The asymptotes $y = \pm\frac{b}{a}x$ are $y = \pm\frac{7}{4}x$.

The given hyperbola is a translation of the standard hyperbola, 3 units to the right and 5 units downward. Hence the following holds:

- The vertices are at $A = (7, -5)$ and $A' = (-1, -5)$.
- The foci are at $F = (3 + \sqrt{65}, -5)$ and $F' = (3 - \sqrt{65}, -5)$.
- The focal axis is $y = -5$ and the conjugate axis is $x = 3$.
- The center is at $(3, -5)$.
- The asymptotes are $y + 5 = \pm\frac{7}{4}(x - 3)$.

32. $3x^2 - 27y^2 = 12$

SOLUTION We first rewrite the equation in the standard form:

$$\frac{3x^2}{12} - \frac{27y^2}{12} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{\frac{4}{9}} = 1 \Rightarrow \left(\frac{x}{2}\right)^2 - \left(\frac{y}{\frac{2}{3}}\right)^2 = 1$$

This is the equation of an hyperbola in standard position. We have $a = 2$, $b = \frac{2}{3}$ and $c = \sqrt{2^2 + \left(\frac{2}{3}\right)^2} \approx 2.1$. Hence:

- The vertices are $(\pm 2, 0)$.
- The foci are $(\pm 2.1, 0)$.

- The focal axis is the x-axis and the conjugate axis is the y-axis.
- The center is at the origin.
- The asymptotes are $y = \pm\frac{b}{a}x$, that is, $y = \pm\frac{1}{3}x$.

33. $4x^2 - 3y^2 + 8x + 30y = 215$

SOLUTION Since there is no cross term, we complete the square of the terms involving x and y separately:

$$4x^2 - 3y^2 + 8x + 30y = 4\left(x^2 + 2x\right) - 3\left(y^2 - 10y\right) = 4(x+1)^2 - 4 - 3(y-5)^2 + 75 = 215$$

Hence,

$$4(x+1)^2 - 3(y-5)^2 = 144$$
$$\frac{4(x+1)^2}{144} - \frac{3(y-5)^2}{144} = 1$$
$$\left(\frac{x+1}{6}\right)^2 - \left(\frac{y-5}{\sqrt{48}}\right)^2 = 1$$

This is the equation of the hyperbola obtained by translating the hyperbola $\left(\frac{x}{6}\right)^2 - \left(\frac{y}{\sqrt{48}}\right)^2 = 1$ one unit to the left and five units upwards. Since $a = 6$, $b = \sqrt{48}$, we have $c = \sqrt{36+48} = \sqrt{84} \sim 9.2$. We obtain the following table:

	Standard position	Translated hyperbola
vertices	$(6, 0), (-6, 0)$	$(5, 5), (-7, 5)$
foci	$(\pm 9.2, 0)$	$(8.2, 5), (-10.2, 5)$
focal axis	The x-axis	$y = 5$
conjugate axis	The y-axis	$x = -1$
center	The origin	$(-1, 5)$
asymptotes	$y = \pm 1.15x$	$y = -1.15x + 3.85$ $y = 1.15x + 6.15$

34. $y = 4x^2$

SOLUTION This is the parabola in standard position $y = \frac{1}{4c}x^2$ with $c = \frac{1}{16}$. The vertex of the parabola is at the origin, the focus is $F = \left(0, \frac{1}{16}\right)$ and the axis is the y-axis.

35. $y = 4(x-4)^2$

SOLUTION By Exercise 34, the parabola $y = 4x^2$ has the vertex at the origin, the focus at $\left(0, \frac{1}{16}\right)$ and its axis is the y-axis. Our parabola is a translation of the standard parabola four units to the right. Hence its vertex is at $(4, 0)$, the focus is at $\left(4, \frac{1}{16}\right)$ and its axis is the vertical line $x = 4$.

36. $8y^2 + 6x^2 - 36x - 64y + 134 = 0$

SOLUTION We first identify the conic section. Since there is no cross term, we complete the square of the terms involving x and y terms separately:

$$\begin{aligned}8y^2 + 6x^2 - 36x - 64y + 134 &= 6\left(x^2 - 6x\right) + 8\left(y^2 - 8y\right) + 134\\ &= 6(x-3)^2 - 54 + 8(y-4)^2 - 128 + 134\\ &= 6(x-3)^2 + 8(y-4)^2 - 48\end{aligned}$$

We obtain the following equation:

$$6(x-3)^2 + 8(y-4)^2 - 48 = 0$$
$$3(x-3)^2 + 4(y-4)^2 = 24$$
$$\left(\frac{x-3}{\sqrt{8}}\right)^2 + \left(\frac{y-4}{\sqrt{6}}\right)^2 = 1$$

We identify the conic as a translation of the ellipse $\left(\frac{x}{\sqrt{8}}\right)^2 + \left(\frac{y}{\sqrt{6}}\right)^2 = 1$, so that the center is at $c = (3, 4)$. Since $a = \sqrt{8}$, $b = \sqrt{6}$ and $a > b$ the foci of the standard ellipse are $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$ for $\sqrt{2} = c = \sqrt{a^2 - b^2}$. Hence the foci of the translated ellipse are $\left(3-\sqrt{2},4\right)$ and $\left(3+\sqrt{2},4\right)$. The vertices $\left(\pm\sqrt{8},0\right)$ and $\left(0,\pm\sqrt{6}\right)$ of the standard ellipse are translated to the points $\left(3\pm\sqrt{8},4\right)$ and $\left(3,4\pm\sqrt{6}\right)$. The focal axis is the line $y=4$, and the conjugate axis is the line $x=3$.

37. $4x^2+25y^2-8x-10y=20$

SOLUTION Since there are no cross terms this conic section is obtained by translating a conic section in standard position. To identify the conic section we complete the square of the terms involving x and y separately:

$$\begin{aligned}4x^2+25y^2-8x-10y &= 4\left(x^2-2x\right)+25\left(y^2-\frac{2}{5}y\right)\\ &= 4(x-1)^2-4+25\left(y-\frac{1}{5}\right)^2-1\\ &= 4(x-1)^2+25\left(y-\frac{1}{5}\right)^2-5=20\end{aligned}$$

Hence,

$$\begin{aligned}4(x-1)^2+25\left(y-\frac{1}{5}\right)^2&=25\\ \frac{4}{25}(x-1)^2+\left(y-\frac{1}{5}\right)^2&=1\\ \left(\frac{x-1}{\frac{5}{2}}\right)^2+\left(y-\frac{1}{5}\right)^2&=1\end{aligned}$$

This is the equation of the ellipse obtained by translating the ellipse in standard position $\left(\frac{x}{\frac{5}{2}}\right)^2+y^2=1$ one unit to the right and $\frac{1}{5}$ unit upward. Since $a=\frac{5}{2}$, $b=1$ we have $c=\sqrt{\left(\frac{5}{2}\right)^2-1}\approx 2.3$, so we obtain the following table:

	Standard position	Translated ellipse
Vertices	$\left(\pm\frac{5}{2},0\right),(0,\pm 1)$	$\left(1\pm\frac{5}{2},\frac{1}{5}\right),\left(1,\frac{1}{5}\pm 1\right)$
Foci	$(-2.3,0),(2.3,0)$	$\left(-1.3,\frac{1}{5}\right),\left(3.3,\frac{1}{5}\right)$
Focal axis	The x-axis	$y=\frac{1}{5}$
Conjugate axis	The y-axis	$x=1$
Center	The origin	$\left(1,\frac{1}{5}\right)$

38. $16x^2+25y^2-64x-200y+64=0$

SOLUTION There is no cross term in this equation, so the conic section is obtained by translating a conic section in standard position. Complete the square in each variable:

$$\begin{aligned}-64 &= 16x^2+25y^2-64x-200y=16x^2-64x+64+25y^2-200y+400-64-400\\ &= 16(x^2-4x+4)+25(y^2-8y+16)-464=16(x-2)^2+25(y-4)^2-464\end{aligned}$$

Collecting constants gives

$$16(x-2)^2+25(y-4)^2=400$$

and dividing through by 400 gives an ellipse whose equation in standard form is so that the curve is an ellipse whose equation in standard form is

$$\left(\frac{x-2}{5}\right)^2+\left(\frac{y-4}{4}\right)^2=1$$

Thus the center of the ellipse is $(2,4)$. The focal axis is $y=4$, because $a=5$ and $b=4$ imply that the focal axis is horizontal. Thus the conjugate axis is $x=2$. $c=\sqrt{a^2-b^2}=\sqrt{25-16}=3$. Thus

- The vertices are $(2\pm 5,4)$ and $(2,4\pm 4)$, so are $(-3,4)$, $(7,4)$, $(2,0)$, and $(2,8)$.
- The foci are $(2\pm 3,4)$ so are $(5,4)$ and $(-1,4)$.

In Exercises 39–42, use the Discriminant Test to determine the type of the conic section (in each case, the equation is nondegenerate). Plot the curve if you have a computer algebra system.

39. $4x^2 + 5xy + 7y^2 = 24$

SOLUTION Here, $D = 25 - 4 \cdot 4 \cdot 7 = -87$, so the conic section is an ellipse.

40. $x^2 - 2xy + y^2 + 24x - 8 = 0$

SOLUTION Here, $D = 4 - 4 \cdot 1 \cdot 1 = 0$, giving us a parabola.

41. $2x^2 - 8xy + 3y^2 - 4 = 0$

SOLUTION Here, $D = 64 - 4 \cdot 2 \cdot 3 = 40$, giving us a hyperbola.

42. $2x^2 - 3xy + 5y^2 - 4 = 0$

SOLUTION Here, $D = 9 - 4 \cdot 2 \cdot (5) = -31$, giving us an ellipse or a circle. Since the coefficients of x^2 and y^2 are different, the curve is an ellipse.

43. Show that the "conic" $x^2 + 3y^2 - 6x + 12y + 23 = 0$ has no points.

SOLUTION Complete the square in each variable separately:

$$-23 = x^2 - 6x + 3y^2 + 12y = (x^2 - 6x + 9) + (3y^2 + 12y + 12) - 9 - 12 = (x-3)^2 + 3(y+2)^2 - 21$$

Collecting constants and reversing sides gives

$$(x-3)^2 + 3(y+2)^2 = -2$$

which has no solutions since the left-hand side is a sum of two squares so is always nonnegative.

44. For which values of a does the conic $3x^2 + 2y^2 - 16y + 12x = a$ have at least one point?

SOLUTION Complete the square in each variable:

$$a = 3x^2 + 2y^2 - 16y + 12x = 3x^2 + 12x + 12 + 2y^2 - 16y + 32 - 12 - 32 = 3(x+2)^2 + 2(x-4)^2 - 44$$

so that, collecting constants,

$$3(x+2)^2 + 2(x-4)^2 = a + 44$$

The left-hand side is a sum of two squares, so is always nonnegative, so in order for the conic (ellipse) to have at least one point, we must have $a + 44 \ge 0$, or $a \ge -44$.

45. Show that $\frac{b}{a} = \sqrt{1 - e^2}$ for a standard ellipse of eccentricity e.

SOLUTION By the definition of eccentricity:

$$e = \frac{c}{a} \tag{1}$$

For the ellipse in standard position, $c = \sqrt{a^2 - b^2}$. Substituting into (1) and simplifying yields

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

We square the two sides and solve for $\frac{b}{a}$:

$$e^2 = 1 - \left(\frac{b}{a}\right)^2 \Rightarrow \left(\frac{b}{a}\right)^2 = 1 - e^2 \Rightarrow \frac{b}{a} = \sqrt{1 - e^2}$$

46. Show that the eccentricity of a hyperbola in standard position is $e = \sqrt{1 + m^2}$, where $\pm m$ are the slopes of the asymptotes.

SOLUTION By the definition of eccentricity, we have:

$$e = \frac{c}{a} \tag{1}$$

For the hyperbola in standard position, $c = \sqrt{a^2 + b^2}$, by substituting in (1) we get

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2} \tag{2}$$

The slopes of the asymptotes are $\pm\frac{b}{a}$. Setting $m = \frac{b}{a}$ we get

$$e = \sqrt{1 + m^2}$$

47. Explain why the dots in Figure 23 lie on a parabola. Where are the focus and directrix located?

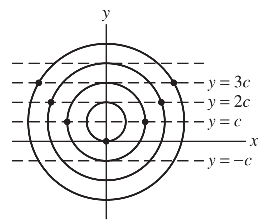
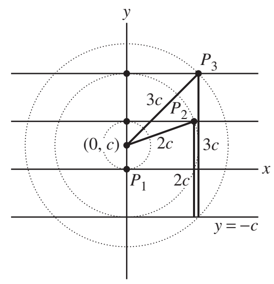


FIGURE 23

SOLUTION All the circles are centered at $(0, c)$ and the kth circle has radius kc. Hence the indicated point P_k on the kth circle has a distance kc from the point $F = (0, c)$. The point P_k also has distance kc from the line $y = -c$. That is, the indicated point on each circle is equidistant from the point $F = (0, c)$ and the line $y = -c$, hence it lies on the parabola with focus at $F = (0, c)$ and directrix $y = -c$.



48. Find the equation of the ellipse consisting of points P such that $PF_1 + PF_2 = 12$, where $F_1 = (4, 0)$ and $F_2 = (-2, 0)$.

SOLUTION This is a translation one unit to the right of an ellipse in standard position with foci $F_1 = (3, 0)$ and $F_2 = (-3, 0)$; points P on this ellipse therefore also satisfy the equation $PF_1 + PF_2 = 12$. But $PF_1 + PF_2 = 2a$ so that $a = 6$; since $(3, 0)$ is a focus, $c = 3$, so that $b = \sqrt{a^2 - c^2} = \sqrt{36 - 9} = 3\sqrt{3}$. The equation of the ellipse in standard position is therefore

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

so that the equation of the desired ellipse is

$$\frac{(x-1)^2}{36} + \frac{y^2}{27} = 1$$

49. A **latus rectum** of a conic section is a chord through a focus parallel to the directrix. Find the area bounded by the parabola $y = x^2/(4c)$ and its latus rectum (refer to Figure 8).

SOLUTION The directrix is $y = -c$, and the focus is $(0, c)$. The chord through the focus parallel to $y = -c$ is clearly $y = c$; this line intersects the parabola when $c = x^2/(4c)$ or $4c^2 = x^2$, so when $x = \pm 2c$. The desired area is then

$$\int_{-2c}^{2c} c - \frac{1}{4c}x^2\,dx = \left(c\,x - \frac{1}{12c}x^3\right)\Big|_{-2c}^{2c}$$
$$= 2c^2 - \frac{8c^3}{12c} - \left(-2c^2 - \frac{(-2c)^3}{12c}\right) = 4c^2 - \frac{4}{3}c^2 = \frac{8}{3}c^2$$

50. Show that the tangent line at a point $P = (x_0, y_0)$ on the hyperbola $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ has equation

$$Ax - By = 1$$

where $A = \dfrac{x_0}{a^2}$ and $B = \dfrac{y_0}{b^2}$.

SOLUTION The equation of the tangent line is

$$y - y_0 = m\,(x - x_0)\,;\;\; m = \frac{dy}{dx}\Big|_{x=x_0,\,y=y_0} \tag{1}$$

To find the slope m we first implicitly differentiate the equation of the hyperbola with respect to x, which gives

$$2\left(\frac{x}{a}\right)\cdot\frac{1}{a}-2\left(\frac{y}{b}\right)\cdot\frac{1}{b}y'=0$$
$$\frac{x}{a^2}=\frac{y}{b^2}y'\Rightarrow y'=\frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

We substitute $x=x_0$, $y=y_0$ to obtain the following slope of the tangent line:

$$m=\frac{b^2}{a^2}\frac{x_0}{y_0}=\frac{x_0}{a^2}\cdot\frac{b^2}{y_0}=A\cdot\frac{1}{B}=\frac{A}{B} \tag{2}$$

Substituting (2) in (1) gives

$$y-y_0=\frac{A}{B}(x-x_0)$$
$$By-By_0=Ax-Ax_0\Rightarrow Ax-By=Ax_0-By_0 \tag{3}$$

Now,

$$Ax_0-By_0=\frac{x_0}{a^2}x_0-\frac{y_0}{b^2}y_0=\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}$$

and the point (x_0, y_0) lies on the hyperbola so

$$\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1,$$

therefore $Ax_0-By_0=1$. Substituting in (3) we obtain $Ax-By=1$.

In Exercises 51–54, find the polar equation of the conic with the given eccentricity and directrix, and focus at the origin.

51. $e=\frac{1}{2}$, $x=3$

SOLUTION Substituting $e=\frac{1}{2}$ and $d=3$ in the polar equation of a conic section we obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{1}{2}\cdot 3}{1+\frac{1}{2}\cos\theta}=\frac{3}{2+\cos\theta}\Rightarrow r=\frac{3}{2+\cos\theta}$$

52. $e=\frac{1}{2}$, $x=-3$

SOLUTION We use the polar equation of a conic section with $e=\frac{1}{2}$ and $d=-3$ to obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{1}{2}\cdot(-3)}{1+\frac{1}{2}\cos\theta}=\frac{-3}{2+\cos\theta}\Rightarrow r=\frac{-3}{2+\cos\theta}$$

53. $e=1$, $x=4$

SOLUTION We substitute $e=1$ and $d=4$ in the polar equation of a conic section to obtain

$$r=\frac{ed}{1+e\cos\theta}=\frac{1\cdot 4}{1+1\cdot\cos\theta}=\frac{4}{1+\cos\theta}\Rightarrow r=\frac{4}{1+\cos\theta}$$

54. $e=\frac{3}{2}$, $x=-4$

SOLUTION Substituting $e=\frac{3}{2}$ and $d=-4$ in the polar equation of the conic section gives

$$r=\frac{ed}{1+e\cos\theta}=\frac{\frac{3}{2}\cdot(-4)}{1+\frac{3}{2}\cos\theta}=\frac{-12}{2+3\cos\theta}\Rightarrow r=\frac{-12}{2+3\cos\theta}$$

In Exercises 55–58, identify the type of conic, the eccentricity, and the equation of the directrix.

55. $r=\dfrac{8}{1+4\cos\theta}$

SOLUTION Matching with the polar equation $r=\frac{ed}{1+e\cos\theta}$ we get $ed=8$ and $e=4$ yielding $d=2$. Since $e>1$, the conic section is a hyperbola, having eccentricity $e=4$ and directrix $x=2$ (referring to the focus-directrix definition (11)).

56. $r = \dfrac{8}{4+\cos\theta}$

SOLUTION To identify the values of e and d we first rewrite the equation in the form $r = \frac{ed}{1+e\cos\theta}$:

$$r = \frac{8}{4+\cos\theta} = \frac{2}{1+\frac{1}{4}\cos\theta}$$

Thus, $ed = 2$ and $e = \frac{1}{4}$, yielding $d = 8$. Since $e < 1$, the conic is an ellipse, having eccentricity $e = \frac{1}{4}$ and directrix $x = 8$.

57. $r = \dfrac{8}{4+3\cos\theta}$

SOLUTION We first rewrite the equation in the form $r = \frac{ed}{1+e\cos\theta}$, obtaining

$$r = \frac{2}{1+\frac{3}{4}\cos\theta}$$

Hence, $ed = 2$ and $e = \frac{3}{4}$ yielding $d = \frac{8}{3}$. Since $e < 1$, the conic section is an ellipse, having eccentricity $e = \frac{3}{4}$ and directrix $x = \frac{8}{3}$.

58. $r = \dfrac{12}{4+3\cos\theta}$

SOLUTION We rewrite the equation in the form of the polar equation $r = \frac{ed}{1+e\cos\theta}$:

$$r = \frac{12}{4+3\cos\theta} = \frac{3}{1+\frac{3}{4}\cos\theta}$$

Hence, $ed = 3$ and $e = \frac{3}{4}$ which implies $d = 4$. Since $e < 1$, the conic section is an ellipse having eccentricity $e = \frac{3}{4}$ and directrix $x = 4$.

59. Find a polar equation for the hyperbola with focus at the origin, directrix $x = -2$, and eccentricity $e = 1.2$.

SOLUTION We substitute $d = -2$ and $e = 1.2$ in the polar equation $r = \frac{ed}{1+e\cos\theta}$ and use Exercise 40 to obtain

$$r = \frac{1.2\cdot(-2)}{1+1.2\cos\theta} = \frac{-2.4}{1+1.2\cos\theta} = \frac{-12}{5+6\cos\theta} = \frac{12}{5-6\cos\theta}$$

60. Let $\mathcal{C}$ be the ellipse $r = de/(1+e\cos\theta)$, where $e < 1$. Show that the x-coordinates of the points in Figure 24 are as follows:

Point	A	C	F_2	A'
x-coordinate	$\dfrac{de}{e+1}$	$-\dfrac{de^2}{1-e^2}$	$-\dfrac{2de^2}{1-e^2}$	$-\dfrac{de}{1-e}$

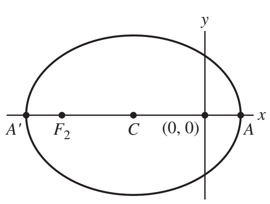


FIGURE 24

SOLUTION To find the x coordinate of A we substitute $\theta = 0$ in the polar equation $r = \frac{de}{1+e\cos\theta}$. This gives

$$x_A = r\cos 0 = r = \frac{de}{1+e\cos 0} = \frac{de}{1+e} \tag{1}$$

The point A' corresponds to $\theta = \pi$, so

$$x_{A'} = r\cos\pi = -r = -\frac{de}{1+e\cos\pi} = -\frac{de}{1-e} \tag{2}$$

The center C is the midpoint of $\overline{A'A}$. From (1) and (2) we obtain

$$x_C = \frac{x_A + x_{A'}}{2} = \frac{1}{2}\left(\frac{de}{1+e} - \frac{de}{1-e}\right) = \frac{de(1-e) - de(1+e)}{2(1+e)(1-e)} = \frac{-de^2}{1-e^2} \tag{3}$$

Finally, one focus is at the origin; the center C is the midpoint of $\overline{F_1F_2}$. Thus

$$x_C = \frac{x_{F_1} + x_{F_2}}{2} = \frac{0 + x_{F_2}}{2} = \frac{x_{F_2}}{2} \Rightarrow x_{F_2} = 2x_C$$

Using (3), we obtain

$$x_{F_2} = \frac{-2de^2}{1-e^2}$$

61. Find an equation in rectangular coordinates of the conic

$$r = \frac{16}{5 + 3\cos\theta}$$

Hint: Use the results of Exercise 60.

SOLUTION Put this equation in the form of the referenced exercise:

$$\frac{16}{5+3\cos\theta} = \frac{\frac{16}{5}}{1 + \frac{3}{5}\cos\theta} = \frac{\frac{16}{3}\cdot\frac{3}{5}}{1 + \frac{3}{5}\cos\theta}$$

so that $e = \frac{3}{5}$ and $d = \frac{16}{3}$. Then the center of the ellipse has x-coordinate

$$-\frac{de^2}{1-e^2} = -\frac{\frac{16}{3}\cdot\frac{9}{25}}{1 - \frac{9}{25}} = -\frac{16}{3}\cdot\frac{9}{25}\cdot\frac{25}{16} = -3$$

and y-coordinate 0, and A' has x-coordinate

$$-\frac{de}{1-e} = -\frac{\frac{16}{3}\cdot\frac{3}{5}}{1 - \frac{3}{5}} = -\frac{16}{3}\cdot\frac{3}{5}\cdot\frac{5}{2} = -8$$

and y-coordinate 0, so $a = -3 - (-8) = 5$, and the equation is

$$\left(\frac{x+3}{5}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

To find b, set $\theta = \frac{\pi}{2}$; then $r = \frac{16}{5}$. But the point corresponding to $\theta = \frac{\pi}{2}$ lies on the y-axis, so has coordinates $\left(0, \frac{16}{5}\right)$. This point is on the ellipse, so that

$$\left(\frac{0+3}{5}\right)^2 + \left(\frac{\frac{16}{5}}{b}\right)^2 = 1 \quad\Rightarrow\quad \frac{256}{25\cdot b^2} = \frac{16}{25} \quad\Rightarrow\quad \frac{256}{b^2} = 16 \quad\Rightarrow\quad b = 4$$

and the equation is

$$\left(\frac{x+3}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

62. Let $e > 1$. Show that the vertices of the hyperbola $r = \dfrac{de}{1 + e\cos\theta}$ have x-coordinates $\dfrac{ed}{e+1}$ and $\dfrac{ed}{e-1}$.

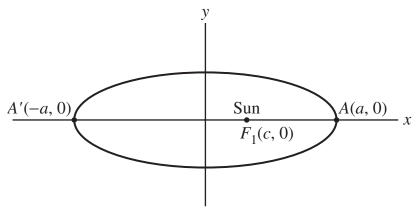
SOLUTION Since the focus is at the origin and the hyperbola is to the right (see figure), the two vertices have positive x coordinates. The corresponding values of θ at the vertices are $\theta = 0$ and $\theta = \pi$. Hence, since $e > 1$ we obtain

$$x_A = |r(0)| = \left|\frac{de}{1 + e\cos 0}\right| = \frac{de}{1+e}$$

$$x_{A'} = |r(\pi)| = \left|\frac{de}{1 + e\cos\pi}\right| = \left|\frac{de}{1-e}\right| = \frac{de}{e-1}$$

63. Kepler's First Law states that planetary orbits are ellipses with the sun at one focus. The orbit of Pluto has eccentricity $e \approx 0.25$. Its **perihelion** (closest distance to the sun) is approximately 2.7 billion miles. Find the **aphelion** (farthest distance from the sun).

SOLUTION We define an xy-coordinate system so that the orbit is an ellipse in standard position, as shown in the figure.



The aphelion is the length of $\overline{A'F_1}$, that is $a + c$. By the given data, we have

$$0.25 = e = \frac{c}{a} \Rightarrow c = 0.25a$$
$$a - c = 2.7 \Rightarrow c = a - 2.7$$

Equating the two expressions for c we get

$$0.25a = a - 2.7$$
$$0.75a = 2.7 \Rightarrow a = \frac{2.7}{0.75} = 3.6, \;\; c = 3.6 - 2.7 = 0.9$$

The aphelion is thus

$$\overline{A'F_0} = a + c = 3.6 + 0.9 = 4.5 \text{ billion miles.}$$

64. Kepler's Third Law states that the ratio $T/a^{3/2}$ is equal to a constant C for all planetary orbits around the sun, where T is the period (time for a complete orbit) and a is the semimajor axis.

(a) Compute C in units of days and kilometers, given that the semimajor axis of the earth's orbit is 150×10^6 km.

(b) Compute the period of Saturn's orbit, given that its semimajor axis is approximately 1.43×10^9 km.

(c) Saturn's orbit has eccentricity $e = 0.056$. Find the perihelion and aphelion of Saturn (see Exercise 63).

SOLUTION

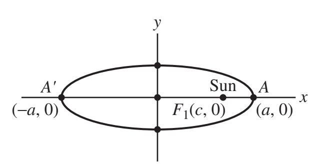
(a) By Kepler's Law, $\frac{T}{a^{3/2}} = C$. For the earth's orbit $a = 150 \times 10^6$ km and $T = 365$ days. Hence,

$$C = \frac{T}{a^{3/2}} = \frac{365}{(150 \times 10^6)^{3/2}} = \frac{365}{1837.12 \times 10^9} = 1.987 \cdot 10^{-10} \text{ days/km}$$

(b) By Kepler's Third Law and using the constant C computed in part (a) we get

$$\frac{T}{a^{3/2}} = C$$
$$\frac{T}{(1.43 \times 10^9)^{3/2}} = 1.987 \times 10^{-10}$$
$$T = (1.987 \times 10^{-10})(1.43 \times 10^9)^{3/2} = 10{,}745 \text{ days.}$$

(c) We define the xy-coordinate system so that the orbit is in standard position (see figure). (The sun is at one focus.)



The perihelion is $a - c$ and the aphelion is $a + c$. By the given information $a = 1.43 \times 10^9$ km and $e = 0.056$. Hence

$$e = \frac{c}{a} \Rightarrow 0.056 = \frac{c}{1.43 \times 10^9} \Rightarrow c = 0.08 \times 10^9 \text{ km}$$

We obtain the following solutions:

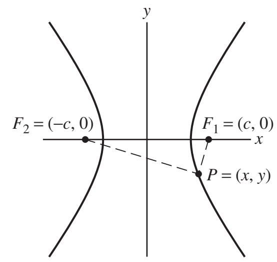
$$\text{perihelion} = a - c = 1.43 \times 10^9 - 0.08 \times 10^9 = 1.35 \times 10^9 \text{ km}$$
$$\text{aphelion} = a + c = 1.43 \times 10^9 + 0.08 \times 10^9 = 1.51 \times 10^9 \text{ km}$$

Further Insights and Challenges

65. Verify Theorem 2.

SOLUTION Let $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ and let $P(x, y)$ be an arbitrary point on the hyperbola. Then for some constant a,

$$\overline{PF_1} - \overline{PF_2} = \pm 2a$$



Using the distance formula we write this as

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a.$$

Moving the second term to the right and squaring both sides gives

$$\sqrt{(x-c)^2 + y^2} = \sqrt{(x+c)^2 + y^2} \pm 2a$$
$$(x-c)^2 + y^2 = (x+c)^2 + y^2 \pm 4a\sqrt{(x+c)^2 + y^2} + 4a^2$$
$$(x-c)^2 - (x+c)^2 - 4a^2 = \pm 4a\sqrt{(x+c)^2 + y^2}$$
$$xc + a^2 = \pm a\sqrt{(x+c)^2 + y^2}$$

We square and simplify to obtain

$$x^2c^2 + 2xca^2 + a^4 = a^2\left((x+c)^2 + y^2\right)$$
$$= a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$
$$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right)$$
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

For $b = \sqrt{c^2 - a^2}$ (or $c = \sqrt{a^2 + b^2}$) we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

66. Verify Theorem 5 in the case $0 < e < 1$. *Hint:* Repeat the proof of Theorem 5, but set $c = d/(e^{-2} - 1)$.

SOLUTION We follow closely the proof of Theorem 5 in the book, which covered the case $e > 1$. This time, for $0 < e < 1$, we prove that $PF = ePD$ defines an ellipse. We choose our coordinate axes so that the focus F lies on the x-axis with coordinates $F = (c, 0)$ and so that the directrix is vertical, lying to the right of F at a distance d from F. As suggested by the hint, we set $c = \frac{d}{e^{-2}-1}$, but since we are working towards an ellipse, we will also need to let $b = \sqrt{a^2 - c^2}$ as opposed to the $\sqrt{c^2 - a^2}$ from the original proof of Theorem 5. Here's the complete list of definitions:

$$c = \frac{d}{e^{-2} - 1}, \quad a = \frac{c}{e}, \quad b = \sqrt{a^2 - c^2}$$

The directrix is the line

$$x = c + d = c + c(e^{-2} - 1) = ce^{-2} = \frac{a}{e}$$

Now, the equation

$$PF = e \cdot PD$$

for the points $P = (x, y)$, $F = (c, 0)$, and $D = (a/e, y)$ becomes

$$\sqrt{(x-c)^2 + y^2} = e \cdot \sqrt{(x-(a/e))^2}$$

Returning to the proof of Theorem 5, we see that this is the same equation that appears in the middle of the proof of the Theorem. As seen there, this equation can be transformed into

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

and this is equivalent to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Since $a^2(1-e^2) = a^2 - a^2e^2 = a^2 - c^2 = b^2$, then we obtain the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

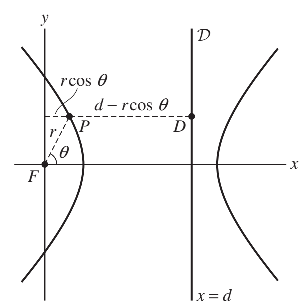
67. Verify that if $e > 1$, then Eq. (11) defines a hyperbola of eccentricity e, with its focus at the origin and directrix at $x = d$.

SOLUTION The points $P = (r, \theta)$ on the hyperbola satisfy $\overline{PF} = e\overline{PD}$, $e > 1$. Referring to the figure we see that

$$\overline{PF} = r,\ \overline{PD} = d - r\cos\theta \tag{1}$$

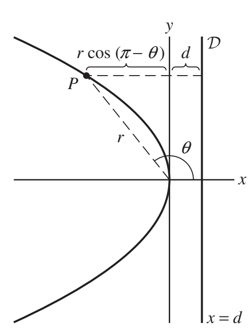
Hence

$$r = e(d - r\cos\theta)$$
$$r = ed - er\cos\theta$$
$$r(1 + e\cos\theta) = ed \Rightarrow r = \frac{ed}{1 + e\cos\theta}$$



Remark: Equality (1) holds also for $\theta > \frac{\pi}{2}$. For example, in the following figure, we have

$$\overline{PD} = d + r\cos(\pi - \theta) = d - r\cos\theta$$



Reflective Property of the Ellipse *In Exercises 68–70, we prove that the focal radii at a point on an ellipse make equal angles with the tangent line $\mathcal{L}$. Let $P = (x_0, y_0)$ be a point on the ellipse in Figure 25 with foci $F_1 = (-c, 0)$ and $F_2 = (c, 0)$, and eccentricity $e = c/a$.*

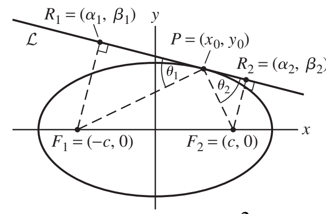


FIGURE 25 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

68. Show that the equation of the tangent line at P is $Ax + By = 1$, where $A = \dfrac{x_0}{a^2}$ and $B = \dfrac{y_0}{b^2}$.

SOLUTION The equation of the tangent line is

$$y - y_0 = m(x - x_0); \quad m = \left.\frac{dy}{dx}\right|_{x=x_0, y=y_0} \tag{1}$$

To find the slope m we implicitly differentiate the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x. We get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2} \Rightarrow y' = -\frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

We substitute $x = x_0$, $y = y_0$ to obtain the following slope of the tangent line:

$$m = -\frac{b^2}{a^2}\frac{x_0}{y_0} = -\frac{x_0}{a^2} \cdot \frac{b^2}{y_0} = -\frac{A}{B}$$

Substituting in (1) and simplifying gives

$$y - y_0 = -\frac{A}{B}(x - x_0)$$
$$By - By_0 = -Ax + Ax_0$$
$$Ax + By = Ax_0 + By_0$$

Now,

$$Ax_0 + By_0 = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2},$$

so we get $Ax + By = 1$.

69. Points R_1 and R_2 in Figure 25 are defined so that $\overline{F_1R_1}$ and $\overline{F_2R_2}$ are perpendicular to the tangent line.

(a) Show, with A and B as in Exercise 68, that

$$\frac{\alpha_1 + c}{\beta_1} = \frac{\alpha_2 - c}{\beta_2} = \frac{A}{B}$$

(b) Use (a) and the distance formula to show that

$$\frac{F_1R_1}{F_2R_2} = \frac{\beta_1}{\beta_2}$$

(c) Use (a) and the equation of the tangent line in Exercise 68 to show that

$$\beta_1 = \frac{B(1 + Ac)}{A^2 + B^2}, \qquad \beta_2 = \frac{B(1 - Ac)}{A^2 + B^2}$$

SOLUTION

(a) Since $R_1 = (\alpha_1, \beta_1)$ and $R_2 = (\alpha_2, \beta_2)$ lie on the tangent line at P, that is on the line $Ax + By = 1$, we have

$$A\alpha_1 + B\beta_1 = 1 \quad \text{and} \quad A\alpha_2 + B\beta_2 = 1$$

The slope of the line R_1F_1 is $\frac{\beta_1}{\alpha_1 + c}$ and it is perpendicular to the tangent line having slope $-\frac{A}{B}$. Similarly, the slope of the line R_2F_2 is $\frac{\beta_2}{\alpha_2 - c}$ and it is also perpendicular to the tangent line. Hence,

$$\frac{\alpha_1 + c}{\beta_1} = \frac{A}{B} \quad \text{and} \quad \frac{\alpha_2 - c}{\beta_2} = \frac{A}{B}.$$

(b) Using the distance formula, we have

$$\overline{R_1F_1}^2 = (\alpha_1 + c)^2 + \beta_1^2$$

Thus,

$$\overline{R_1F_1}^2 = \beta_1^2 \left(\left(\frac{\alpha_1 + c}{\beta_1} \right)^2 + 1 \right) \tag{1}$$

By part (a), $\frac{\alpha_1+c}{\beta_1} = \frac{A}{B}$. Substituting in (1) gives

$$\overline{R_1F_1}^2 = \beta_1^2 \left(\frac{A^2}{B^2} + 1 \right) \tag{2}$$

Likewise,

$$\overline{R_2F_2}^2 = (\alpha_2 - c)^2 + {\beta_2}^2 = {\beta_2}^2 \left(\left(\frac{\alpha_2 - c}{\beta_2} \right)^2 + 1 \right) \tag{3}$$

but since $\frac{\alpha_2-c}{\beta_2} = \frac{A}{B}$, substituting in (3) gives

$$\overline{R_2F_2}^2 = \beta_2^2 \left(\frac{A^2}{B^2} + 1 \right). \tag{4}$$

Dividing, we find that

$$\frac{\overline{R_1F_1}^2}{\overline{R_2F_2}^2} = \frac{\beta_1^2}{\beta_2^2} \quad \text{so} \quad \frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\beta_1}{\beta_2},$$

as desired.

(c) In part (a) we showed that

$$\begin{cases} A\alpha_1 + B\beta_1 = 1 \\ \dfrac{\beta_1}{\alpha_1 + c} = \dfrac{B}{A} \end{cases}$$

Eliminating α_1 and solving for β_1 gives

$$\beta_1 = \frac{B(1 + Ac)}{A^2 + B^2}. \tag{5}$$

Similarly, we have

$$\begin{cases} A\alpha_2 + B\beta_2 = 1 \\ \dfrac{\beta_2}{\alpha_2 - c} = \dfrac{B}{A} \end{cases}$$

Eliminating α_2 and solving for β_2 yields

$$\beta_2 = \frac{B\,(1 - Ac)}{A^2 + B^2} \tag{6}$$

70. (a) Prove that $PF_1 = a + x_0e$ and $PF_2 = a - x_0e$. *Hint:* Show that $PF_1{}^2 - PF_2{}^2 = 4x_0c$. Then use the defining property $PF_1 + PF_2 = 2a$ and the relation $e = c/a$.

(b) Verify that $\dfrac{F_1R_1}{PF_1} = \dfrac{F_2R_2}{PF_2}$.

(c) Show that $\sin\theta_1 = \sin\theta_2$. Conclude that $\theta_1 = \theta_2$.

SOLUTION

(a) Using the distance formula we have

$$\overline{PF_1}^2 = (x_0 + c)^2 + y^2; \quad \overline{PF_2}^2 = (x_0 - c)^2 + y^2$$

Hence,

$$\overline{PF_1}^2 - \overline{PF_2}^2 = (x_0 + c)^2 + y^2 - (x_0 - c)^2 - y^2$$

$$= (x_0 + c)^2 - (x_0 - c)^2$$
$$= x_0{}^2 + 2x_0c + c^2 - x_0{}^2 + 2x_0c - c^2 = 4x_0c$$

That is, $\overline{PF_1}^2 - \overline{PF_2}^2 = 4x_0c$. Now use the identity $u^2 - v^2 = (u - v)(u + v)$ to write this as

$$\left(\overline{PF_1} - \overline{PF_2}\right)\left(\overline{PF_1} + \overline{PF_2}\right) = 4x_0c \tag{1}$$

Since P lies on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ we have

$$\overline{PF_1} + \overline{PF_2} = 2a \tag{2}$$

Substituting in (1) gives

$$\left(\overline{PF_1} - \overline{PF_2}\right) \cdot 2a = 4x_0c$$

We divide by a and use the eccentricity $e = \frac{c}{a}$ to obtain

$$\overline{PF_1} - \overline{PF_2} = 2x_0e$$

Solve this equation together with equation (2) to see that

$$\overline{PF_1} = a + x_0e, \qquad \overline{PF_2} = a - x_0e$$

(b) Substituting the expression for β_1 from Eq. (5) in Exercise 69 into Eq. (2) in Exercise 69 yields

$$\overline{R_1F_1}^2 = \frac{B^2(1 + Ac)^2}{(A^2 + B^2)^2}\left(\frac{A^2}{B^2} + 1\right) = \frac{B^2(1 + Ac)^2(A^2 + B^2)}{(A^2 + B^2)^2B^2} = \frac{(1 + Ac)^2}{A^2 + B^2}$$

and similarly, substituting the expression for β_2 from Eq. (6) in Exercise 69 into Eq. (4) in Exercise 69 yields

$$\overline{R_2F_2}^2 = \frac{B^2(1 - Ac)^2}{(A^2 + B^2)^2}\left(\frac{A^2}{B^2} + 1\right) = \frac{B^2(1 - Ac)^2(A^2 + B^2)}{(A^2 + B^2)^2B^2} = \frac{(1 - Ac)^2}{A^2 + B^2}$$

Taking square roots and dividing these two formulas gives

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\frac{1+Ac}{\sqrt{A^2+B^2}}}{\frac{1-Ac}{\sqrt{A^2+B^2}}} = \frac{1 + Ac}{1 - Ac}$$

Substitute $c = ea$ and $A = \frac{x_0}{a^2}$ (from Exercise 68) to get

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{1 + \frac{x_0ea}{a^2}}{1 - \frac{x_0ea}{a^2}} = \frac{1 + \frac{x_0e}{a}}{1 - \frac{x_0e}{a}} = \frac{a + x_0e}{a - x_0e}$$

But part (a) showed that $\overline{PF_1} = a + x_0e$ and $\overline{PF_2} = a - x_0e$, so that

$$\frac{\overline{R_1F_1}}{\overline{R_2F_2}} = \frac{\overline{PF_1}}{\overline{PF_2}} \qquad \Rightarrow \qquad \frac{\overline{R_1F_1}}{\overline{PF_1}} = \frac{\overline{R_2F_2}}{\overline{PF_2}}$$

(c) Since $\frac{\overline{R_1F_1}}{\overline{PF_1}} = \sin\theta_1$ and $\frac{\overline{R_2F_2}}{\overline{PF_2}} = \sin\theta_2$, we get $\sin\theta_1 = \sin\theta_2$, which implies that $\theta_1 = \theta_2$ since the two angles are acute.

71. Here is another proof of the Reflective Property.

(a) Figure 25 suggests that $\mathcal{L}$ is the unique line that intersects the ellipse only in the point P. Assuming this, prove that $QF_1 + QF_2 > PF_1 + PF_2$ for all points Q on the tangent line other than P.

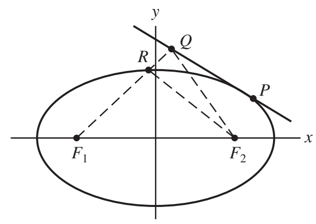
(b) Use the Principle of Least Distance (Example 6 in Section 4.7) to prove that $\theta_1 = \theta_2$.

SOLUTION

(a) Consider a point $Q \neq P$ on the line $\mathcal{L}$ (see figure). Since $\mathcal{L}$ intersects the ellipse in only one point, the remainder of the line lies outside the ellipse, so that QR does not have zero length, and F_2QR is a triangle. Thus

$$QF_1 + QF_2 = QR + RF_1 + QF_2 = RF_1 + (QR + QF_2) > RF_1 + RF_2$$

since the sum of lengths of two sides of a triangle exceeds the length of the third side. But since point R lies on the ellipse, $RF_2 + RF_2 = PF_1 + PF_2$, and we are done.



(b) Consider a beam of light traveling from F_1 to F_2 by reflection off of the line $\mathcal{L}$. By the principle of least distance, the light takes the shortest path, which by part (a) is the path through P. By Example 6 in Section 4.7, this shortest path has the property that the angle of incidence (θ_1) is equal to the angle of reflection (θ_2).

72. Show that the length QR in Figure 26 is independent of the point P.

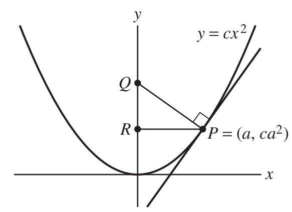


FIGURE 26

SOLUTION We find the slope m of the tangent line at $P = \left(a, ca^2\right)$:

$$m = (cx^2)'\Big|_{x=a} = 2cx\Big|_{x=a} = 2ca$$

The slope of the perpendicular line PQ is, thus, $-\frac{1}{2ca}$, and the equation of this line is

$$y - ca^2 = -\frac{1}{2ca}(x-a) \Rightarrow y = -\frac{x}{2ac} + ca^2 + \frac{1}{2c}$$

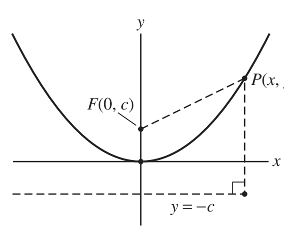
The y-intercept of the line PQ is $y = ca^2 + \frac{1}{2c}$. We now find the length $\overline{QR}$, by computing the distance between the points $Q(0, ca^2 + \frac{1}{2c})$ and $P(0, ca^2)$. This gives

$$\overline{QR} = ca^2 + \frac{1}{2c} - ca^2 = \frac{1}{2c}$$

Indeed, the length $\overline{QR}$ is independent of a, i.e. it is independent of the point P.

73. Show that $y = x^2/4c$ is the equation of a parabola with directrix $y = -c$, focus $(0, c)$, and the vertex at the origin, as stated in Theorem 3.

SOLUTION The points $P = (x, y)$ on the parabola are equidistant from $F = (0, c)$ and the line $y = -c$.



That is, by the distance formula, we have

$$\overline{PF} = \overline{PD}$$
$$\sqrt{x^2 + (y-c)^2} = |y + c|$$

Squaring and simplifying yields

$$x^2 + (y-c)^2 = (y+c)^2$$
$$x^2 + y^2 - 2yc + c^2 = y^2 + 2yc + c^2$$
$$x^2 - 2yc = 2yc$$

$$x^2 = 4yc \Rightarrow y = \frac{x^2}{4c}$$

Thus, we showed that the points that are equidistant from the focus $F = (0, c)$ and the directrix $y = -c$ satisfy the equation $y = \frac{x^2}{4c}$.

74. Consider two ellipses in standard position:

$$E_1: \quad \left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

$$E_2: \quad \left(\frac{x}{a_2}\right)^2 + \left(\frac{y}{b_2}\right)^2 = 1$$

We say that E_1 is similar to E_2 under scaling if there exists a factor $r > 0$ such that for all (x, y) on E_1, the point (rx, ry) lies on E_2. Show that E_1 and E_2 are similar under scaling if and only if they have the same eccentricity. Show that any two circles are similar under scaling.

SOLUTION If E_1 and E_2 are similar under scaling, then since $(a_1, 0)$ and $(0, b_1)$ are points on the first ellipse, the scaled points $(ra_1, 0)$ and $(0, rb_1)$ must be on the second ellipse. This implies that $(ra_1/a_2)^2 + (0/b_2)^2 = 1$ and that $(0/a_1)^2 + (rb_1/b_2)^2 = 1$, which means that $ra_1 \pm a_2$ and $rb_1 = \pm b_2$. But, since r, a_1, and a_2 are all positive, then this implies that $a_2 = ra_1$ and $b_2 = rb_1$, and so

$$c_2 = \sqrt{a_2^2 - b_2^2} = r\sqrt{a_1^2 - b_1^2} = rc_1.$$

Thus,

$$e_2 = \frac{c_2}{a_2} = \frac{r_1 c_1}{r_1 a_1} = \frac{c_1}{a_1} = e_1$$

and so the two ellipses have the same eccentricity. On the other hand, if the two ellipses have the same eccentricity, then

$$\sqrt{1 - \frac{b_2^2}{a_2^2}} = \frac{c_2}{a_2} = e_2 = e_1 = \frac{c_1}{a_1} = \sqrt{1 - \frac{b_1^2}{a_1^2}}$$

which implies

$$\sqrt{1 - \frac{b_2^2}{a_2^2}} = \sqrt{1 - \frac{b_1^2}{a_1^2}}$$

and this implies that $b_2/a_2 = \pm b_1/a_1$ and so $b_2/a_2 = b_1/a_1$ (recall that all constants are positive). Define $r = b_2/b_1$. Then, $b_2 = rb_1$, but since $b_2/a_2 = b_1/a_1$ we get that $a_2 = ra_1$ as well. Thus, for the point (x, y) on the first ellipse, we have that

$$\left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

If we put the scaled point (rx, ry) into the second ellipse, we get

$$\left(\frac{rx}{a_2}\right)^2 + \left(\frac{ry}{b_2}\right)^2 = \left(\frac{rx}{ra_1}\right)^2 + \left(\frac{ry}{rb_1}\right)^2 = \left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{b_1}\right)^2 = 1$$

which implies that E_2 is a scaled version of E_1. Since all circles have eccentricity 0, then they are all similar under scaling.

75. Derive Eqs. (13) and (14) in the text as follows. Write the coordinates of P with respect to the rotated axes in Figure 21 in polar form $x' = r\cos\alpha$, $y' = r\sin\alpha$. Explain why P has polar coordinates $(r, \alpha + \theta)$ with respect to the standard x and y-axes and derive Eqs. (13) and (14) using the addition formulas for cosine and sine.

SOLUTION If the polar coordinates of P with respect to the rotated axes are (r, α), then the line from the origin to P has length r and makes an angle of α with the rotated x-axis (the x'-axis). Since the x'-axis makes an angle of θ with the x-axis, it follows that the line from the origin to P makes an angle of $\alpha + \theta$ with the x-axis, so that the polar coordinates of P with respect to the standard axes are $(r, \alpha + \theta)$. Write (x', y') for the rectangular coordinates of P with respect to the rotated axes and (x, y) for the rectangular coordinates of P with respect to the standard axes. Then

$$x = r\cos(\alpha + \theta) = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta = x'\cos\theta - y'\sin\theta$$

$$y = r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta = x'\sin\theta + y'\cos\theta$$

76. If we rewrite the general equation of degree 2 (Eq. 12) in terms of variables x' and y' that are related to x and y by Eqs. (13) and (14), we obtain a new equation of degree 2 in x' and y' of the same form but with different coefficients:

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$$

(a) Show that $b' = b\cos 2\theta + (c-a)\sin 2\theta$.
(b) Show that if $b \neq 0$, then we obtain $b' = 0$ for

$$\theta = \frac{1}{2}\cot^{-1}\frac{a-c}{b}$$

This proves that it is always possible to eliminate the cross term bxy by rotating the axes through a suitable angle.

SOLUTION

(a) If we plug in $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, we will get a very ugly mess. Fortunately, we only care about the $x'y'$ term, so we really only need to look at the $ax^2 + bxy + cy^2$ part of the formula. In fact, we only need to pull out those terms which have an $x'y'$ in them. Thus

$$\begin{aligned} ax^2 &\quad \text{becomes} \quad a(x'\cos\theta - y'\sin\theta)^2 = -2ax'y'\cos\theta\sin\theta + \dots \\ bxy &\quad \text{becomes} \quad b(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) = bx'y'(\cos^2\theta - \sin^2\theta) + \dots \\ cy^2 &\quad \text{becomes} \quad c(x'\sin\theta + y'\cos\theta)^2 = 2cx'y'\cos\theta\sin\theta + \dots \end{aligned}$$

so that

$$ax^2 + bxy + cy^2 = ((c-a)2\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta))x'y' + \cdots = ((c-a)\sin 2\theta + b\cos 2\theta)x'y' + \dots$$

and thus b', the coefficient of $x'y'$, is $b\cos 2\theta + (c-a)\sin 2\theta$, as desired.
(b) Setting $b' = 0$, we get $0 = b\cos 2\theta + (c-a)\sin 2\theta$, so $b\cos 2\theta = (a-c)\sin 2\theta$, so $\cot 2\theta = \frac{a-c}{b}$, giving us $2\theta = \cot^{-1}\frac{a-c}{b}$, and thus $\theta = \frac{1}{2}\cot^{-1}\frac{a-c}{b}$.

CHAPTER REVIEW EXERCISES

1. Which of the following curves pass through the point $(1, 4)$?

(a) $c(t) = (t^2, t+3)$
(b) $c(t) = (t^2, t-3)$
(c) $c(t) = (t^2, 3-t)$
(d) $c(t) = (t-3, t^2)$

SOLUTION To check whether it passes through the point $(1, 4)$, we solve the equations $c(t) = (1, 4)$ for the given curves.

(a) Comparing the second coordinate of the curve and the point yields:

$$\begin{aligned} t + 3 &= 4 \\ t &= 1 \end{aligned}$$

We substitute $t = 1$ in the first coordinate, to obtain

$$t^2 = 1^2 = 1$$

Hence the curve passes through $(1, 4)$.

(b) Comparing the second coordinate of the curve and the point yields:

$$\begin{aligned} t - 3 &= 4 \\ t &= 7 \end{aligned}$$

We substitute $t = 7$ in the first coordinate to obtain

$$t^2 = 7^2 = 49 \neq 1$$

Hence the curve does not pass through $(1, 4)$.

(c) Comparing the second coordinate of the curve and the point yields

$$\begin{aligned} 3 - t &= 4 \\ t &= -1 \end{aligned}$$

We substitute $t = -1$ in the first coordinate, to obtain

$$t^2 = (-1)^2 = 1$$

Hence the curve passes through $(1, 4)$.

(d) Comparing the first coordinate of the curve and the point yields

$$\begin{aligned} t - 3 &= 1 \\ t &= 4 \end{aligned}$$

We substitute $t = 4$ in the second coordinate, to obtain:

$$t^2 = 4^2 = 16 \neq 4$$

Hence the curve does not pass through $(1, 4)$.

2. Find parametric equations for the line through $P = (2, 5)$ perpendicular to the line $y = 4x - 3$.

SOLUTION The line perpendicular to $y = 4x - 3$ at $P = (2, 5)$ is the line of slope $-\frac{1}{4}$ passing through P. This line has the equation

$$y - 5 = -\frac{1}{4}(x - 2)$$

A bit of calculation shows that the parametric equations of the line are

$$c(t) = \left(2 + t, 5 - \frac{1}{4}t\right)$$

or

$$\begin{aligned} x &= 2 + t \\ y &= 5 - \frac{1}{4}t \end{aligned}$$

3. Find parametric equations for the circle of radius 2 with center $(1, 1)$. Use the equations to find the points of intersection of the circle with the x- and y-axes.

SOLUTION Using the standard technique for parametric equations of curves, we obtain

$$c(t) = (1 + 2\cos t, 1 + 2\sin t)$$

We compare the x coordinate of $c(t)$ to 0:

$$\begin{aligned} 1 + 2\cos t &= 0 \\ \cos t &= -\frac{1}{2} \\ t &= \pm\frac{2\pi}{3} \end{aligned}$$

Substituting in the y coordinate yields

$$1 + 2\sin\left(\pm\frac{2\pi}{3}\right) = 1 \pm 2\frac{\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

Hence, the intersection points with the y-axis are $(0, 1 \pm \sqrt{3})$. We compare the y coordinate of $c(t)$ to 0:

$$\begin{aligned} 1 + 2\sin t &= 0 \\ \sin t &= -\frac{1}{2} \\ t &= -\frac{\pi}{6} \quad \text{or} \quad \frac{7}{6}\pi \end{aligned}$$

Substituting in the x coordinates yields

$$\begin{aligned} 1 + 2\cos\left(-\frac{\pi}{6}\right) &= 1 + 2\frac{\sqrt{3}}{2} = 1 + \sqrt{3} \\ 1 + 2\cos\left(\frac{7}{6}\pi\right) &= 1 - 2\cos\left(\frac{\pi}{6}\right) = 1 - 2\frac{\sqrt{3}}{2} = 1 - \sqrt{3} \end{aligned}$$

Hence, the intersection points with the x-axis are $(1 \pm \sqrt{3}, 0)$.

4. Find a parametrization $c(t)$ of the line $y = 5 - 2x$ such that $c(0) = (2, 1)$.

SOLUTION The line is passing through $P = (0, 5)$ with slope -2, hence (by one of the examples in section 12.1) it has the parametrization

$$c(t) = (t, 5 - 2t)$$

This parametrization does not satisfy $c(0) = (2, 1)$. We replace the parameter t by a parameter s, so that $t = s + \beta$, to obtain another parametrization for the line:

$$c^*(s) = (s + \beta, 5 - 2(s + \beta)) = (s + \beta, 5 - 2\beta - 2s) \tag{1}$$

We require that $c^*(0) = (2, 1)$. That is,

$$c^*(0) = (\beta, 5 - 2\beta) = (2, 1)$$

or

$$\begin{aligned} \beta &= 2 \\ 5 - 2\beta &= 1 \end{aligned} \quad \Rightarrow \quad \beta = 2$$

Substituting in (1) gives the parametrization

$$c^*(s) = (s + 2, 1 - 2s)$$

5. Find a parametrization $c(\theta)$ of the unit circle such that $c(0) = (-1, 0)$.

SOLUTION The unit circle has the parametrization

$$c(t) = (\cos t, \sin t)$$

This parametrization does not satisfy $c(0) = (-1, 0)$. We replace the parameter t by a parameter θ so that $t = \theta + \alpha$, to obtain another parametrization for the circle:

$$c^*(\theta) = (\cos(\theta + \alpha), \sin(\theta + \alpha)) \tag{1}$$

We need that $c^*(0) = (1, 0)$, that is,

$$c^*(0) = (\cos\alpha, \sin\alpha) = (-1, 0)$$

Hence

$$\begin{aligned} \cos\alpha &= -1 \\ \sin\alpha &= 0 \end{aligned} \quad \Rightarrow \quad \alpha = \pi$$

Substituting in (1) we obtain the following parametrization:

$$c^*(\theta) = (\cos(\theta + \pi), \sin(\theta + \pi))$$

6. Find a path $c(t)$ that traces the parabolic arc $y = x^2$ from $(0, 0)$ to $(3, 9)$ for $0 \le t \le 1$.

SOLUTION The second coordinates of the points on the parabolic arc are the square of the first coordinates. Therefore the points on the arc have the form:

$$c(t) = (\alpha t, \alpha^2 t^2) \tag{1}$$

We need that $c(1) = (3, 9)$. That is,

$$c(1) = (\alpha, \alpha^2) = (3, 9) \Rightarrow \alpha = 3$$

Substituting in (1) gives the following parametrization:

$$c(t) = (3t, 9t^2)$$

7. Find a path $c(t)$ that traces the line $y = 2x + 1$ from $(1, 3)$ to $(3, 7)$ for $0 \le t \le 1$.

SOLUTION Solution 1: By one of the examples in section 12.1, the line through $P = (1, 3)$ with slope 2 has the parametrization

$$c(t) = (1 + t, 3 + 2t)$$

But this parametrization does not satisfy $c(1) = (3, 7)$. We replace the parameter t by a parameter s so that $t = \alpha s + \beta$. We get

$$c^*(s) = (1 + \alpha s + \beta, 3 + 2(\alpha s + \beta)) = (\alpha s + \beta + 1, 2\alpha s + 2\beta + 3)$$

We need that $c^*(0) = (1, 3)$ and $c^*(1) = (3, 7)$. Hence,

$$c^*(0) = (1 + \beta, 3 + 2\beta) = (1, 3)$$
$$c^*(1) = (\alpha + \beta + 1, 2\alpha + 2\beta + 3) = (3, 7)$$

We obtain the equations

$$\begin{aligned} 1 + \beta &= 1 \\ 3 + 2\beta &= 3 \\ \alpha + \beta + 1 &= 3 \\ 2\alpha + 2\beta + 3 &= 7 \end{aligned} \quad \Rightarrow \quad \beta = 0, \alpha = 2$$

Substituting in (1) gives

$$c^*(s) = (2s + 1, 4s + 3)$$

Solution 2: The segment from $(1, 3)$ to $(3, 7)$ has the following vector parametrization:

$$(1 - t)\langle 1, 3\rangle + t\langle 3, 7\rangle = \langle 1 - t + 3t, 3(1 - t) + 7t\rangle = \langle 1 + 2t, 3 + 4t\rangle$$

The parametrization is thus

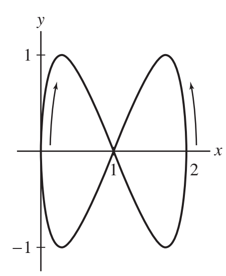
$$c(t) = (1 + 2t, 3 + 4t)$$

8. Sketch the graph $c(t) = (1 + \cos t, \sin 2t)$ for $0 \le t \le 2\pi$ and draw arrows specifying the direction of motion.

SOLUTION From $x = 1 + \cos t$ we have $x - 1 = \cos t$. We substitute this in the y coordinate to obtain

$$y = \sin 2t = 2 \sin t \cos t = \pm 2\sqrt{\sin^2 t} \cos t = \pm 2\sqrt{1 - \cos^2 t} \cos t = \pm 2\sqrt{1 - (x - 1)^2}(x - 1)$$

We can see that the graph is symmetric with respect to the x-axis, hence we plot the function $y = 2\sqrt{1 - (x - 1)^2}(x - 1)$ and reflect it with respect to the x-axis. When $t = 0$ we have $c(0) = (2, 0)$. when t increases near 0, $\cos t$ is decreasing and $\sin 2t$ is increasing, hence the general direction at the point $(2, 0)$ is upwards and left. As t approaches $\pi/2$, the x-coordinate decreases to 1 and the y-coordinate to 0. Likewise, as t moves from $\pi/2$ to π, the x-coordinate moves to 0 while the y-coordinate falls to -1 and then rises to 0. The resulting graph is seen here in the corresponding figure.



Plot of Exercise 8

In Exercises 9–12, express the parametric curve in the form $y = f(x)$.

9. $c(t) = (4t - 3, 10 - t)$

SOLUTION We use the given equation to express t in terms of x.

$$\begin{aligned} x &= 4t - 3 \\ 4t &= x + 3 \\ t &= \frac{x + 3}{4} \end{aligned}$$

Substituting in the equation of y yields

$$y = 10 - t = 10 - \frac{x + 3}{4} = -\frac{x}{4} + \frac{37}{4}$$

That is,

$$y = -\frac{x}{4} + \frac{37}{4}$$

10. $c(t) = (t^3 + 1, t^2 - 4)$

SOLUTION The parametric equations are $x = t^3 + 1$ and $y = t^2 - 4$. We express t in terms of x:

$$\begin{aligned} x &= t^3 + 1 \\ t^3 &= x - 1 \\ t &= (x-1)^{1/3} \end{aligned}$$

Substituting in the equation of y yields

$$y = t^2 - 4 = (x-1)^{2/3} - 4$$

That is,

$$y = (x-1)^{2/3} - 4$$

11. $c(t) = \left(3 - \frac{2}{t}, t^3 + \frac{1}{t}\right)$

SOLUTION We use the given equation to express t in terms of x:

$$\begin{aligned} x &= 3 - \frac{2}{t} \\ \frac{2}{t} &= 3 - x \\ t &= \frac{2}{3-x} \end{aligned}$$

Substituting in the equation of y yields

$$y = \left(\frac{2}{3-x}\right)^3 + \frac{1}{2/(3-x)} = \frac{8}{(3-x)^3} + \frac{3-x}{2}$$

12. $x = \tan t, \quad y = \sec t$

SOLUTION We use the trigonometric identity

$$1 + \tan^2 t = \sec^2 t$$

Substituting the parametric equations $x = \tan t$ and $y = \sec t$ we obtain

$$1 + x^2 = y^2 \quad \text{or} \quad y = \pm\sqrt{x^2+1}$$

In Exercises 13–16, calculate dy/dx at the point indicated.

13. $c(t) = (t^3 + t, t^2 - 1), \quad t = 3$

SOLUTION The parametric equations are $x = t^3 + t$ and $y = t^2 - 1$. We use the theorem on the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2+1}$$

We now substitute $t = 3$ to obtain

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{2 \cdot 3}{3 \cdot 3^2 + 1} = \frac{3}{14}$$

14. $c(\theta) = (\tan^2\theta, \cos\theta), \quad \theta = \frac{\pi}{4}$

SOLUTION The parametric equations are $x = \tan^2\theta$, $y = \cos\theta$. We use the theorem on the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{2\tan\theta\sec^2\theta} = -\frac{\cos^3\theta}{2}$$

We now substitute $\theta = \frac{\pi}{4}$ to obtain

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\frac{\cos^3\frac{\pi}{4}}{2} = -\frac{1}{4\sqrt{2}}$$

15. $c(t) = (e^t - 1, \sin t), \quad t = 20$

SOLUTION We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(\sin t)'}{(e^t - 1)'} = \frac{\cos t}{e^t}$$

We now substitute $t = 20$:

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\cos 20}{e^{20}}$$

16. $c(t) = (\ln t, 3t^2 - t), \quad P = (0, 2)$

SOLUTION The parametric equations are $x = \ln t$, $y = 3t^2 - t$. We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t - 1}{\frac{1}{t}} = 6t^2 - t \tag{1}$$

We now must identify the value of t corresponding to the point $P = (0, 2)$ on the curve. We solve the following equations:

$$\begin{aligned} \ln t &= 0 \\ 3t^2 - t &= 2 \end{aligned} \quad \Rightarrow \quad t = 1$$

Substituting $t = 1$ in (1) we obtain

$$\left.\frac{dy}{dx}\right|_{P} = 6 \cdot 1^2 - 1 = 5$$

17. CAS Find the point on the cycloid $c(t) = (t - \sin t, 1 - \cos t)$ where the tangent line has slope $\frac{1}{2}$.

SOLUTION Since $x = t - \sin t$ and $y = 1 - \cos t$, the theorem on the slope of the tangent line gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

The points where the tangent line has slope $\frac{1}{2}$ are those where $\frac{dy}{dx} = \frac{1}{2}$. We solve for t:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \\ \frac{\sin t}{1 - \cos t} &= \frac{1}{2} \\ 2\sin t &= 1 - \cos t \end{aligned} \tag{1}$$

We let $u = \sin t$. Then $\cos t = \pm\sqrt{1 - \sin^2 t} = \pm\sqrt{1 - u^2}$. Hence

$$2u = 1 \pm \sqrt{1 - u^2}$$

We transfer sides and square to obtain

$$\pm\sqrt{1 - u^2} = 2u - 1$$

$$\begin{aligned} 1-u^2 &= 4u^2-4u+1 \\ 5u^2-4u &= u(5u-4)=0 \\ u=0,\ u &= \frac{4}{5} \end{aligned}$$

We find t by the relation $u=\sin t$:

$$\begin{aligned} u=0: &\quad \sin t = 0 \Rightarrow t=0, t=\pi \\ u=\frac{4}{5}: &\quad \sin t = \frac{4}{5} \Rightarrow t\approx 0.93, t\approx 2.21 \end{aligned}$$

These correspond to the points $(0,1)$, $(\pi,2)$, $(0.13,0.40)$, and $(1.41,1.60)$, respectively, for $0<t<2\pi$.

18. Find the points on $(t+\sin t, t-2\sin t)$ where the tangent is vertical or horizontal.

SOLUTION We use the theorem for the slope of the tangent line to find $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1-2\cos t}{1+\cos t}$$

We find the values of t for which the denominator is zero. We ignore the numerator, since when $1+\cos t=0$, $1-2\cos t=3\neq 0$.

$$\begin{aligned} 1+\cos t &= 0 \\ \cos t &= -1 \\ t &= \pi+2\pi k \quad \text{where } k\in\mathbb{Z} \end{aligned}$$

We now find the values of t for which the numerator is 0:

$$\begin{aligned} 1-2\cos t &= 0 \\ 1 &= 2\cos t \\ \frac{1}{2} &= \cos t \\ t &= \pm\frac{\pi}{3}+2\pi k \quad \text{where } k\in\mathbb{Z} \end{aligned}$$

Note that the denominator is not zero at these points. Thus, we have vertical tangents at $t=\pi+2\pi k$ and horizontal tangents at $t=\pm\pi/3+2\pi k$.

19. Find the equation of the Bézier curve with control points

$$P_0=(-1,-1),\quad P_1=(-1,1),\quad P_2=(1,1),\quad P_3(1,-1)$$

SOLUTION We substitute the given points in the appropriate formulas in the text to find the parametric equations of the Bézier curve. We obtain

$$\begin{aligned} x(t) &= -(1-t)^3-3t(1-t)^2+t^2(1-t)+t^3 \\ &= -(1-3t+3t^2-t^3)-(3t-6t^2+3t^3)+(t^2-t^3)+t^3 \\ &= (-2t^3+4t^2-1) \\ y(t) &= -(1-t)^3+3t(1-t)^2+t^2(1-t)-t^3 \\ &= -(1-3t+3t^2-t^3)+(3t-6t^2+3t^3)+(t^2-t^3)-t^3 \\ &= (2t^3-8t^2+6t-1) \end{aligned}$$

20. Find the speed at $t=\frac{\pi}{4}$ of a particle whose position at time t seconds is $c(t)=(\sin 4t, \cos 3t)$.

SOLUTION We use the parametric definition to find the speed. We obtain

$$\frac{ds}{dt}=\sqrt{((\sin 4t)')^2+((\cos 3t)')^2}=\sqrt{(4\cos 4t)^2+(-3\sin 3t)^2}=\sqrt{16\cos^2 4t+9\sin^2 3t}$$

At time $t=\frac{\pi}{4}$ the speed is

$$\left.\frac{ds}{dt}\right|_{t=\pi/4}=\sqrt{16\cos^2\pi+9\sin^2\frac{3\pi}{4}}=\sqrt{16+9\cdot\frac{1}{2}}=\sqrt{20.5}\approx 4.53$$

21. Find the speed (as a function of t) of a particle whose position at time t seconds is $c(t) = (\sin t + t, \cos t + t)$. What is the particle's maximal speed?

SOLUTION We use the parametric definition to find the speed. We obtain

$$\frac{ds}{dt} = \sqrt{((\sin t + t)')^2 + ((\cos t + t)')^2} = \sqrt{(\cos t + 1)^2 + (1 - \sin t)^2}$$
$$= \sqrt{\cos^2 t + 2\cos t + 1 + 1 - 2\sin t + \sin^2 t} = \sqrt{3 + 2(\cos t - \sin t)}$$

We now differentiate the speed function to find its maximum:

$$\frac{d^2 s}{dt^2} = \left(\sqrt{3 + 2(\cos t - \sin t)}\right)' = \frac{-\sin t - \cos t}{\sqrt{3 + 2(\cos t - \sin t)}}$$

We equate the derivative to zero, to obtain the maximum point:

$$\begin{aligned}
\frac{d^2 s}{dt^2} &= 0 \\
\frac{-\sin t - \cos t}{\sqrt{3 + 2(\cos t - \sin t)}} &= 0 \\
-\sin t - \cos t &= 0 \\
-\sin t &= \cos t \\
\sin(-t) &= \cos(-t) \\
-t &= \frac{\pi}{4} + \pi k \\
t &= -\frac{\pi}{4} + \pi k
\end{aligned}$$

Substituting t in the function of speed we obtain the value of the maximal speed:

$$\sqrt{3 + 2\left(\cos -\frac{\pi}{4} - \sin -\frac{\pi}{4}\right)} = \sqrt{3 + 2\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right)} = \sqrt{3 + 2\sqrt{2}}$$

22. Find the length of $(3e^t - 3, 4e^t + 7)$ for $0 \le t \le 1$.

SOLUTION We use the formula for arc length, to obtain

$$s = \int_0^1 \sqrt{((3e^t - 3)')^2 + ((4e^t + 7)')^2}\, dt = \int_0^1 \sqrt{(3e^t)^2 + (4e^t)^2}\, dt$$
$$= \int_0^1 \sqrt{9e^{2t} + 16e^{2t}}\, dt = \int_0^1 \sqrt{25e^{2t}}\, dt = \int_0^1 5e^t\, dt = 5e^t\Big|_0^1 = 5(e - 1)$$

In Exercises 23 and 24, let $c(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

23. Show that $c(t)$ for $0 \le t < \infty$ has finite length and calculate its value.

SOLUTION We use the formula for arc length, to obtain:

$$\begin{aligned}
s &= \int_0^\infty \sqrt{((e^{-t}\cos t)')^2 + ((e^{-t}\sin t)')^2} dt \\
&= \int_0^\infty \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2} dt \\
&= \int_0^\infty \sqrt{e^{-2t}(\cos t + \sin t)^2 + e^{-2t}(\cos t - \sin t)^2} dt \\
&= \int_0^\infty e^{-t}\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t}\, dt \\
&= \int_0^\infty e^{-t}\sqrt{2} dt = \sqrt{2}(-e^{-t})\Big|_0^\infty = -\sqrt{2}\left(\lim_{t\to\infty} e^{-t} - e^0\right) \\
&= -\sqrt{2}(0 - 1) = \sqrt{2}
\end{aligned}$$

24. Find the first positive value of t_0 such that the tangent line to $c(t_0)$ is vertical, and calculate the speed at $t = t_0$.

SOLUTION The curve has a vertical tangent where $\lim_{t\to t_0}\left|\frac{dy}{dx}\right| = \infty$. We first find $\frac{dy}{dx}$ using the theorem for the slope of a tangent line:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(e^{-t}\sin t)'}{(e^{-t}\cos t)'} = \frac{-e^{-t}\sin t + e^{-t}\cos t}{-e^{-t}\cos t - e^{-t}\sin t}$$
$$= -\frac{\cos t - \sin t}{\cos t + \sin t} = \frac{\sin t - \cos t}{\sin t + \cos t}$$

We now search for t_0 such that $\lim_{t\to t_0}\left|\frac{dy}{dx}\right| = \infty$. In our case, this happens when the denominator is 0, but the numerator is not, thus:

$$\sin t_0 + \cos t_0 = 0$$
$$\cos t_0 = -\sin t_0$$
$$\cos -t_0 = \sin -t_0$$
$$-t_0 = \frac{\pi}{4} - \pi$$
$$t_0 = \frac{3}{4}\pi$$

We now use the formula for the speed, to find the speed at t_0.

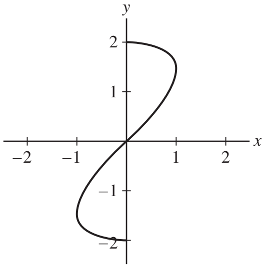
$$\frac{ds}{dt} = \sqrt{((e^{-t}\sin t)')^2 + ((e^{-t}\cos t)')^2}$$
$$= \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2}$$
$$= \sqrt{e^{-2t}(\cos t + \sin t)^2 + e^{-2t}(\cos t - \sin t)^2}$$
$$= e^{-t}\sqrt{\cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t} = e^{-t}\sqrt{2}$$

Next we substitute $t = \frac{3}{4}\pi$, to obtain

$$e^{-t_0}\sqrt{2} = e^{-3\pi/4}\sqrt{2}$$

25. CAS Plot $c(t) = (\sin 2t, 2\cos t)$ for $0 \le t \le \pi$. Express the length of the curve as a definite integral, and approximate it using a computer algebra system.

SOLUTION We use a CAS to plot the curve. The resulting graph is shown here.



Plot of the curve $(\sin 2t, 2\cos t)$

To calculate the arc length we use the formula for the arc length to obtain

$$s = \int_0^{\pi} \sqrt{(2\cos 2t)^2 + (-2\sin t)^2}\,dt = 2\int_0^{\pi} \sqrt{\cos^2 2t + \sin^2 t}\,dt$$

We use a CAS to obtain $s = 6.0972$.

26. Convert the points $(x, y) = (1, -3), (3, -1)$ from rectangular to polar coordinates.

SOLUTION We convert the given points from cartesian coordinates to polar coordinates. For the first point we have

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$
$$\theta = \arctan \frac{y}{x} = \arctan -3 = 5.034$$

For the second point we have

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$
$$\theta = \arctan \frac{y}{x} = \arctan \frac{-1}{3} = -0.321,\ 5.961$$

27. Convert the points $(r, \theta) = \left(1, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{4}\right)$ from polar to rectangular coordinates.

SOLUTION We convert the points from polar coordinates to cartesian coordinates. For the first point we have

$$x = r\cos\theta = 1 \cdot \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$y = r\sin\theta = 1 \cdot \sin\frac{\pi}{6} = \frac{1}{2}$$

For the second point we have

$$x = r\cos\theta = 3\cos\frac{5\pi}{4} = -\frac{3\sqrt{2}}{2}$$
$$y = r\sin\theta = 3\sin\frac{5\pi}{4} = -\frac{3\sqrt{2}}{2}$$

28. Write $(x + y)^2 = xy + 6$ as an equation in polar coordinates.

SOLUTION We use the formula for converting from cartesian coordinates to polar coordinates to substitute r and θ for x and y:

$$\begin{aligned}
(x+y)^2 &= xy + 6 \\
x^2 + 2xy + y^2 &= xy + 6 \\
x^2 + y^2 &= -xy + 6 \\
r^2 &= -(r\cos\theta)(r\sin\theta) + 6 \\
r^2 &= -r^2\cos\theta\sin\theta + 6 \\
r^2(1 + \sin\theta\cos\theta) &= 6 \\
r^2 &= \frac{6}{1 + \sin\theta\cos\theta} \\
r^2 &= \frac{6}{1 + \frac{\sin 2\theta}{2}} \\
r^2 &= \frac{12}{2 + \sin 2\theta}
\end{aligned}$$

29. Write $r = \dfrac{2\cos\theta}{\cos\theta - \sin\theta}$ as an equation in rectangular coordinates.

SOLUTION We use the formula for converting from polar coordinates to cartesian coordinates to substitute x and y for r and θ:

$$\begin{aligned}
r &= \frac{2\cos\theta}{\cos\theta - \sin\theta} \\
\sqrt{x^2 + y^2} &= \frac{2r\cos\theta}{r\cos\theta - r\sin\theta} \\
\sqrt{x^2 + y^2} &= \frac{2x}{x - y}
\end{aligned}$$

30. Show that $r = \dfrac{4}{7\cos\theta - \sin\theta}$ is the polar equation of a line.

SOLUTION We use the formula for converting from polar coordinates to cartesian coordinates to substitute x and y for r and θ:

$$\begin{aligned} r &= \frac{4}{7\cos\theta - \sin\theta} \\ 1 &= \frac{4}{7r\cos\theta - r\sin\theta} \\ 1 &= \frac{4}{7x - y} \\ 7x - y &= 4 \\ y &= 7x - 4 \end{aligned}$$

We obtained a linear function. Since the original equation in polar coordinates represents the same curve, it represents a straight line as well.

31. GU Convert the equation

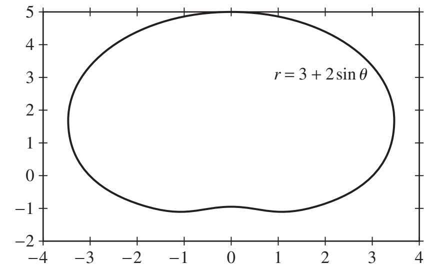
$$9(x^2 + y^2) = (x^2 + y^2 - 2y)^2$$

to polar coordinates, and plot it with a graphing utility.

SOLUTION We use the formula for converting from cartesian coordinates to polar coordinates to substitute r and θ for x and y:

$$\begin{aligned} 9(x^2 + y^2) &= (x^2 + y^2 - 2y)^2 \\ 9r^2 &= (r^2 - 2r\sin\theta)^2 \\ 3r &= r^2 - 2r\sin\theta \\ 3 &= r - 2\sin\theta \\ r &= 3 + 2\sin\theta \end{aligned}$$

The plot of $r = 3 + 2\sin\theta$ is shown here:



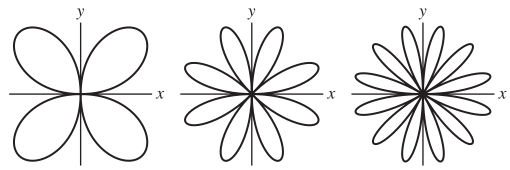
Plot of $r = 3 + 2\sin\theta$

32. Calculate the area of the circle $r = 3\sin\theta$ bounded by the rays $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

SOLUTION We use the formula for area in polar coordinates to obtain

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/3}^{2\pi/3} (3\sin\theta)^2\,d\theta = \frac{9}{2}\int_{\pi/3}^{2\pi/3} \sin^2\theta\,d\theta = \frac{9}{4}\int_{\pi/3}^{2\pi/3} (1 - \cos 2\theta)\,d\theta = \frac{9}{4}\left(\theta - \frac{\sin 2\theta}{2}\Big|_{\pi/3}^{2\pi/3}\right) \\ &= \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\left(\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right) = \frac{9}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \end{aligned}$$

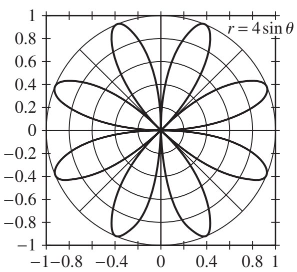
33. Calculate the area of one petal of $r = \sin 4\theta$ (see Figure 1).



$n = 2$ (4 petals) $n = 4$ (8 petals) $n = 6$ (12 petals)

FIGURE 1 Plot of $r = \sin(n\theta)$.

SOLUTION We use a CAS to generate the plot, as shown here.



Plot of $r = \sin 4\theta$

We can see that one leaf lies between the rays $\theta = 0$ and $\theta = \dfrac{\theta}{4}$. We now use the formula for area in polar coordinates to obtain

$$A = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{4}\int_0^{\pi/4} (1 - \cos 8\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{\sin 8\theta}{8}\Big|_0^{\pi/4}\right)$$

$$= \frac{\pi}{16} - \frac{1}{32}(\sin 2\pi - \sin 0) = \frac{\pi}{16}$$

34. The equation $r = \sin(n\theta)$, where $n \geq 2$ is even, is a "rose" of $2n$ petals (Figure 1). Compute the total area of the flower, and show that it does not depend on n.

SOLUTION We calculate the total area of the flower, that is, the area between the rays $\theta = 0$ and $\theta = 2\pi$, using the formula for area in polar coordinates:

$$A = \frac{1}{2}\int_0^{2\pi} \sin^2 2n\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} (1 - \cos 4n\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{\sin 4n\theta}{4n}\Big|_0^{2\pi}\right)$$

$$= \frac{\pi}{2} - \frac{1}{16n}(\sin 8n\pi - \sin 0) = \frac{\pi}{2}$$

Since the area is $\dfrac{\pi}{2}$ for every $n \in \mathbb{Z}$, the area is independent of n.

35. Calculate the total area enclosed by the curve $r^2 = \cos\theta e^{\sin\theta}$ (Figure 2).

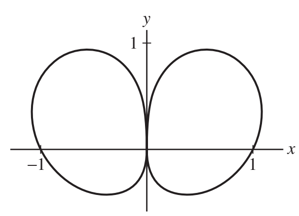


FIGURE 2 Graph of $r^2 = \cos\theta e^{\sin\theta}$.

SOLUTION Note that this is defined only for θ between $-\pi/2$ and $\pi/2$. We use the formula for area in polar coordinates to obtain:

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta e^{\sin\theta}\, d\theta$$

We evaluate the integral by making the substitution $x = \sin\theta\, dx = \cos\theta\, d\theta$:

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta e^{\sin\theta} d\theta = \frac{1}{2}e^x\Big|_{-1}^{1} = \frac{1}{2}\left(e - e^{-1}\right)$$

36. Find the shaded area in Figure 3.

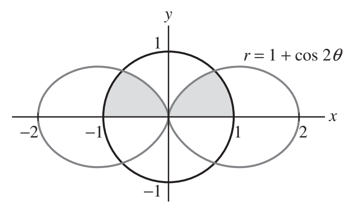


FIGURE 3

SOLUTION We first find the points of intersection between the unit circle and the function.

$$\begin{aligned}1 &= 1 + \cos 2\theta\\ \cos 2\theta &= 0\\ 2\theta &= \frac{\pi}{2} + \pi n\\ \theta &= \frac{\pi}{4} + \frac{\pi}{2}n\end{aligned}$$

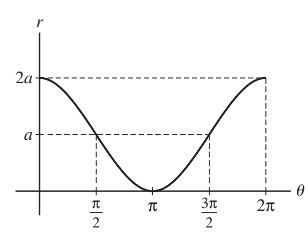
We now find the area of the shaded figure in the first quadrant. This has two parts. The first, from 0 to $\pi/4$, is just an octant of the unit circle, and thus has area $\pi/8$. The second, from $\pi/4$ to $\pi/2$, is found as follows:

$$\begin{aligned}A &= \frac{1}{2}\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)^2\, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\, d\theta\\ &= \frac{1}{2}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Big|_{\pi/4}^{\pi/2} = \frac{1}{2}\left(\frac{3\pi}{8} - 1\right)\end{aligned}$$

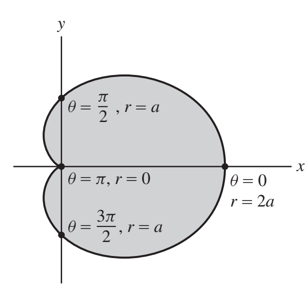
The total area in the first quadrant is thus $\frac{5\pi}{16} - \frac{1}{2}$; multiply by 2 to get the total area of $\frac{5\pi}{8} - 1$.

37. Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$, where $a > 0$.

SOLUTION The graph of $r = a\,(1 + \cos\theta)$ in the $r\theta$-plane for $0 \le \theta \le 2\pi$ and the cardioid in the xy-plane are shown in the following figures:



$r = a\,(1 + \cos\theta)$



The cardioid $r = a\,(1 + \cos\theta)$, $a > 0$

As θ varies from 0 to π the radius r decreases from $2a$ to 0, and this gives the upper part of the cardioid.

The lower part is traced as θ varies from π to 2π and consequently r increases from 0 back to $2a$. We compute the area enclosed by the upper part of the cardioid and the x-axis, using the following integral (we use the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$):

$$\begin{aligned}\frac{1}{2}\int_0^{\pi} r^2\, d\theta &= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{\pi}\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta\\ &= \frac{a^2}{2}\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{a^2}{2}\int_0^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta\\ &= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \frac{a^2}{2}\left[\frac{3\pi}{2} + 2\sin\pi + \frac{1}{4}\sin 2\pi - 0\right] = \frac{3\pi a^2}{4}\end{aligned}$$

Using symmetry, the total area A enclosed by the cardioid is

$$A = 2 \cdot \frac{3\pi a^2}{4} = \frac{3\pi a^2}{2}$$

38. Calculate the length of the curve with polar equation $r = \theta$ in Figure 4.

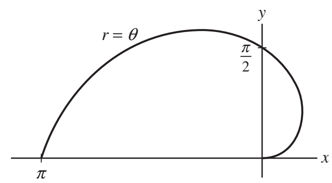


FIGURE 4

SOLUTION The interval of θ values is $0 \le \theta \le \pi$. We use the formula for the arc length in polar coordinates, with $r = f(\theta) = \theta$. We get

$$S = \int_0^{\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{\theta^2 + \left(\theta'\right)^2}\, d\theta = \int_0^{\pi} \sqrt{\theta^2 + 1}\, d\theta$$
$$= \frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left|\theta + \sqrt{\theta^2+1}\right|\Bigg|_{\theta=0}^{\pi} = \frac{\pi}{2}\sqrt{\pi^2+1} + \frac{1}{2}\ln\left(\pi + \sqrt{\pi^2+1}\right)$$

39. CAS Figure 5 shows the graph of $r = e^{0.5\theta}\sin\theta$ for $0 \le \theta \le 2\pi$. Use a computer algebra system to approximate the difference in length between the outer and inner loops.

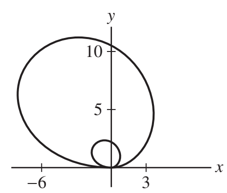


FIGURE 5

SOLUTION We note that the inner loop is the curve for $\theta \in [0, \pi]$, and the outer loop is the curve for $\theta \in [\pi, 2\pi]$. We express the length of these loops using the formula for the arc length. The length of the inner loop is

$$s_1 = \int_0^{\pi} \sqrt{(e^{0.5\theta}\sin\theta)^2 + ((e^{0.5\theta}\sin\theta)')^2} d\theta = \int_0^{\pi} \sqrt{e^{\theta}\sin^2\theta + \left(\frac{e^{0.5\theta}\sin\theta}{2} + e^{0.5\theta}\cos\theta\right)^2}\, d\theta$$

and the length of the outer loop is

$$s_2 = \int_{\pi}^{2\pi} \sqrt{e^{\theta}\sin^2\theta + \left(\frac{e^{0.5\theta}\sin\theta}{2} + e^{0.5\theta}\cos\theta\right)^2}\, d\theta$$

We now use the CAS to calculate the arc length of each of the loops. We obtain that the length of the inner loop is 7.5087 and the length of the outer loop is 36.121, hence the outer one is 4.81 times longer than the inner one.

40. Show that $r = f_1(\theta)$ and $r = f_2(\theta)$ define the same curves in polar coordinates if $f_1(\theta) = -f_2(\theta + \pi)$. Use this to show that the following define the same conic section:

$$r = \frac{de}{1 - e\cos\theta}, \qquad r = \frac{-de}{1 + e\cos\theta}$$

SOLUTION Suppose (r, θ) lies on the curve $r = f_2(\theta)$. Since (r, θ) and $(-r, \theta + \pi)$ define the same point in polar coordinates, we have $-r = f_2(\theta + \pi) = -f_1(\theta)$, so that $r = f_1(\theta)$, Thus (r, θ) lies on f_1 as well. Conversely, suppose (r, θ) lies on $r = f_1(\theta)$. Since (r, θ) and $(-r, \theta - \pi)$ define the same point in polar coordinates, we have $-r = f_1(\theta - \pi) = -f_2(\theta - \pi + \pi) = -f_2(\theta)$ so that $r = f_2(\theta)$ and (r, θ) lies on f_2 as well. Thus the two equations define exactly the same set of points.

Now set

$$f_1(\theta) = \frac{de}{1 - e\cos\theta} \qquad f_2(\theta) = -\frac{de}{1 + e\cos\theta}$$

and consider the polar equations $r = f_1(\theta)$ and $r = f_2(\theta)$. We have

$$-f_2(\theta + \pi) = -\frac{-de}{1 + e\cos(\theta + \pi)} = \frac{de}{1 - e\cos\theta} = f_1(\theta)$$

so that by the above, the two equations define the same conic section.

In Exercises 41–44, identify the conic section. Find the vertices and foci.

41. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

SOLUTION This is an ellipse in standard position. Its foci are $(\pm\sqrt{3^2 - 2^2}, 0) = (\pm\sqrt{5}, 0)$ and its vertices are $(\pm 3, 0)$, $(0, \pm 2)$.

42. $x^2 - 2y^2 = 4$

SOLUTION We divide the equation by 4 to obtain

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$

This is a hyperbola in standard position, its foci are $\left(\pm\sqrt{2^2 + \sqrt{2}^2}, 0\right) = (\pm\sqrt{6}, 0)$, and its vertices are $(\pm 2, 0)$.

43. $\left(2x + \frac{1}{2}y\right)^2 = 4 - (x - y)^2$

SOLUTION We simplify the equation:

$$\begin{aligned}
\left(2x + \frac{1}{2}y\right)^2 &= 4 - (x - y)^2 \\
4x^2 + 2xy + \frac{1}{4}y^2 &= 4 - x^2 + 2xy - y^2 \\
5x^2 + \frac{5}{4}y^2 &= 4 \\
\frac{5x^2}{4} + \frac{5y^2}{16} &= 1 \\
\left(\frac{x}{\frac{2}{\sqrt{5}}}\right)^2 + \left(\frac{y}{\frac{4}{\sqrt{5}}}\right)^2 &= 1
\end{aligned}$$

This is an ellipse in standard position, with foci $\left(0, \pm\sqrt{\left(\frac{4}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2}\right) = \left(0, \pm\sqrt{\frac{12}{5}}\right)$ and vertices $\left(\pm\frac{2}{\sqrt{5}}, 0\right)$, $\left(0, \pm\frac{4}{\sqrt{5}}\right)$.

44. $(y - 3)^2 = 2x^2 - 1$

SOLUTION We simplify the equation:

$$\begin{aligned}
(y - 3)^2 &= 2x^2 - 1 \\
2x^2 - (y - 3)^2 &= 1 \\
\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 - (y - 3)^2 &= 1
\end{aligned}$$

This is a hyperbola shifted 3 units on the y-axis. Therefore, its foci are $\left(\pm\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 1}, 3\right) = \left(\pm\sqrt{\frac{3}{2}}, 3\right)$ and its vertices are $\left(\pm\frac{1}{\sqrt{2}}, 3\right)$.

In Exercises 45–50, find the equation of the conic section indicated.

45. Ellipse with vertices $(\pm 8, 0)$ and foci $(\pm\sqrt{3}, 0)$

SOLUTION Since the foci of the desired ellipse are on the x-axis, we conclude that $a > b$. We are given that the points $(\pm 8, 0)$ are vertices of the ellipse, and since they are on the x-axis, $a = 8$. We are given that the foci are $(\pm\sqrt{3}, 0)$ and we have shown that $a > b$, hence we have that $\sqrt{a^2 - b^2} = \sqrt{3}$. Solving for b yields

$$\sqrt{a^2 - b^2} = \sqrt{3}$$
$$a^2 - b^2 = 3$$
$$8^2 - b^2 = 3$$
$$b^2 = 61$$
$$b = \sqrt{61}$$

Next we use a and b to construct the equation of the ellipse:

$$\left(\frac{x}{8}\right)^2 + \left(\frac{y}{\sqrt{61}}\right)^2 = 1.$$

46. Ellipse with foci $(\pm 8, 0)$, eccentricity $\frac{1}{8}$

SOLUTION If the foci are on the x-axis, then $a > b$, and $c = \sqrt{a^2 - b^2}$. We are given that $e = \frac{1}{8}$, and $c = 8$. Substituting and solving for a and b yields

$$e = \frac{c}{a}$$
$$c = \sqrt{a^2 - b^2}$$
$$\frac{1}{8} = \frac{8}{a}$$
$$64 = a$$
$$8 = \sqrt{64^2 - b^2}$$
$$64 = 64^2 - b^2$$
$$b^2 = 64 \cdot 63$$
$$b = 8\sqrt{63}$$

We use a and b to construct the equation of the ellipse:

$$\left(\frac{x}{64}\right)^2 + \left(\frac{y}{8\sqrt{63}}\right)^2 = 1.$$

47. Hyperbola with vertices $(\pm 8, 0)$, asymptotes $y = \pm \frac{3}{4}x$

SOLUTION Since the asymptotes of the hyperbola are $y = \pm \frac{3}{4}x$, and the equation of the asymptotes for a general hyperbola in standard position is $y = \pm \frac{b}{a}x$, we conclude that $\frac{b}{a} = \frac{3}{4}$. We are given that the vertices are $(\pm 8, 0)$, thus $a = 8$. We substitute and solve for b:

$$\frac{b}{a} = \frac{3}{4}$$
$$\frac{b}{8} = \frac{3}{4}$$
$$b = 6$$

Next we use a and b to construct the equation of the hyperbola:

$$\left(\frac{x}{8}\right)^2 - \left(\frac{y}{6}\right)^2 = 1.$$

48. Hyperbola with foci $(2, 0)$ and $(10, 0)$, eccentricity $e = 4$

SOLUTION Since the foci lie on the x axis, the x is the focal axis. The center of the hyperbola is midway between the foci, so lies at $(6, 0)$, and $c = 4$. Then $c = ae$ gives $a = 1$; then $b = \sqrt{c^2 - a^2} = \sqrt{15}$, so that the equation of the hyperbola is

$$(x - 6)^2 - \left(\frac{y}{\sqrt{15}}\right)^2 = 1$$

49. Parabola with focus $(8, 0)$, directrix $x = -8$

SOLUTION This is similar to the usual equation of a parabola, but we must use y as x, and x as y, to obtain

$$x = \frac{1}{32}y^2.$$

50. Parabola with vertex $(4, -1)$, directrix $x = 15$

SOLUTION The directrix is a vertical line and the vertex is $(4, -1)$, so the equation is of the form

$$x - 4 = \frac{1}{4c}(y + 1)^2$$

The directrix is to the right of the vertex; the distance from the directrix to the vertex is -11, so $c = -11$ and the equation is

$$x = 4 - \frac{1}{44}(y + 1)^2$$

51. Find the asymptotes of the hyperbola $3x^2 + 6x - y^2 - 10y = 1$.

SOLUTION We complete the squares and simplify:

$$\begin{aligned} 3x^2 + 6x - y^2 - 10y &= 1 \\ 3(x^2 + 2x) - (y^2 + 10y) &= 1 \\ 3(x^2 + 2x + 1 - 1) - (y^2 + 10y + 25 - 25) &= 1 \\ 3(x + 1)^2 - 3 - (y + 5)^2 + 25 &= 1 \\ 3(x + 1)^2 - (y + 5)^2 &= -21 \\ \left(\frac{y + 5}{\sqrt{21}}\right)^2 - \left(\frac{x + 1}{\sqrt{7}}\right)^2 &= 1 \end{aligned}$$

We obtained a hyperbola with focal axis that is parallel to the y-axis, and is shifted -5 units on the y-axis, and -1 units in the x-axis. Therefore, the asymptotes are

$$x + 1 = \pm\frac{\sqrt{7}}{\sqrt{21}}(y + 5) \quad \text{or} \quad y + 5 = \pm\sqrt{3}(x + 1).$$

52. Show that the "conic section" with equation $x^2 - 4x + y^2 + 5 = 0$ has no points.

SOLUTION We complete the squares in the given equation:

$$\begin{aligned} x^2 - 4x + 4y^2 + 5 &= 0 \\ x^2 - 4x + 4 - 4 + 4y^2 + 5 &= 0 \\ (x - 2)^2 + 4y^2 &= -1 \end{aligned}$$

Since $(x - 2)^2 \geq 0$ and $y^2 \geq 0$, there is no point satisfying the equation, hence it cannot represent a conic section.

53. Show that the relation $\frac{dy}{dx} = (e^2 - 1)\frac{x}{y}$ holds on a standard ellipse or hyperbola of eccentricity e.

SOLUTION We differentiate the equations of the standard ellipse and the hyperbola with respect to x:

Ellipse:	Hyperbola:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$	$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$
$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$	$\frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$

The eccentricity of the ellipse is $e = \frac{\sqrt{a^2 - b^2}}{a}$, hence $e^2a^2 = a^2 - b^2$ or $e^2 = 1 - \frac{b^2}{a^2}$ yielding $\frac{b^2}{a^2} = 1 - e^2$.

The eccentricity of the hyperbola is $e = \frac{\sqrt{a^2+b^2}}{a}$, hence $e^2a^2 = a^2 + b^2$ or $e^2 = 1 + \frac{b^2}{a^2}$, giving $\frac{b^2}{a^2} = e^2 - 1$. Combining with the expressions for $\frac{dy}{dx}$ we get:

Ellipse: $$\frac{dy}{dx} = -(1-e^2)\frac{x}{y} = (e^2-1)\frac{x}{y}$$

Hyperbola: $$\frac{dy}{dx} = (e^2-1)\frac{x}{y}$$

We, thus, proved that the relation $\frac{dy}{dx} = (e^2 - 1)\frac{x}{y}$ holds on a standard ellipse or hyperbola of eccentricity e.

54. The orbit of Jupiter is an ellipse with the sun at a focus. Find the eccentricity of the orbit if the perihelion (closest distance to the sun) equals 740×10^6 km and the aphelion (farthest distance from the sun) equals 816×10^6 km.

SOLUTION For the sake of simplicity, we treat all numbers in units of 10^6 km. By Kepler's First Law we conclude that the sun is at one of the foci of the ellipse. Therefore, the closest and farthest points to the sun are vertices. Moreover, they are the vertices on the x-axis, hence we conclude that the distance between the two vertices is

$$2a = 740 + 816 = 1556$$

Since the distance between each focus and the vertex that is closest to it is the same distance, and since $a = 778$, we conclude that the distance between the foci is

$$c = a - 740 = 38$$

We substitute this in the formula for the eccentricity to obtain:

$$e = \frac{c}{a} = 0.0488.$$

55. Refer to Figure 25 in Section 11.5. Prove that the product of the perpendicular distances F_1R_1 and F_2R_2 from the foci to a tangent line of an ellipse is equal to the square b^2 of the semiminor axes.

SOLUTION We first consider the ellipse in standard position:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of the tangent line at $P = (x_0, y_0)$ is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

or

$$b^2x_0x + a^2y_0y - a^2b^2 = 0$$

The distances of the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ from the tangent line are

$$\overline{F_1R_1} = \frac{|b^2x_0c - a^2b^2|}{\sqrt{b^4x_0^2 + a^4y_0^2}}; \quad \overline{F_2R_2} = \frac{|b^2x_0c + a^2b^2|}{\sqrt{b^4x_0^2 + a^4y_0^2}}$$

We compute the product of the distances:

$$\overline{F_1R_1} \cdot \overline{F_2R_2} = \left|\frac{\left(b^2x_0c - a^2b^2\right)\left(b^2x_0c + a^2b^2\right)}{b^4x_0^2 + a^4y_0^2}\right| = \left|\frac{b^4x_0^2c^2 - a^4b^4}{b^4x_0^2 + a^4y_0^2}\right| \tag{1}$$

The point $P = (x_0, y_0)$ lies on the ellipse, hence:

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \Rightarrow a^4y_0^2 = a^4b^2 - a^2b^2x_0^2$$

We substitute in (1) to obtain (notice that $b^2 - a^2 = -c^2$)

$$\begin{aligned}\overline{F_1R_1} \cdot \overline{F_2R_2} &= \frac{|b^4x_0^2c^2 - a^4b^4|}{|b^4x_0^2 + a^4b^2 - a^2b^2x_0^2|} = \frac{|b^4x_0^2c^2 - a^4b^4|}{|b^2(b^2 - a^2)x_0^2 + a^4b^2|} \\ &= \frac{|b^4x_0^2c^2 - a^4b^4|}{|-b^2x_0^2c^2 + a^4b^2|} = \frac{|b^2(x_0^2c^2 - a^4)|}{|-(x_0^2c^2 - a^4)|} = |-b^2| = b^2\end{aligned}$$

The product $\overline{F_1R_1} \cdot \overline{F_2R_2}$ remains unchanged if we translate the standard ellipse.

12 VECTOR GEOMETRY

12.1 Vectors in the Plane (LT Section 13.1)

Preliminary Questions

1. Answer true or false. Every nonzero vector is:

(a) Equivalent to a vector based at the origin.

(b) Equivalent to a unit vector based at the origin.

(c) Parallel to a vector based at the origin.

(d) Parallel to a unit vector based at the origin.

SOLUTION

(a) This statement is true. Translating the vector so that it is based on the origin, we get an equivalent vector based at the origin.

(b) Equivalent vectors have equal lengths, hence vectors that are not unit vectors, are not equivalent to a unit vector.

(c) This statement is true. A vector based at the origin such that the line through this vector is parallel to the line through the given vector, is parallel to the given vector.

(d) Since parallel vectors do not necessarily have equal lengths, the statement is true by the same reasoning as in (c).

2. What is the length of $-3\mathbf{a}$ if $\|\mathbf{a}\| = 5$?

SOLUTION Using properties of the length we get

$$\|-3\mathbf{a}\| = |-3|\|\mathbf{a}\| = 3\|\mathbf{a}\| = 3 \cdot 5 = 15$$

3. Suppose that $\mathbf{v}$ has components $\langle 3, 1\rangle$. How, if at all, do the components change if you translate $\mathbf{v}$ horizontally two units to the left?

SOLUTION Translating $\mathbf{v} = \langle 3, 1\rangle$ yields an equivalent vector, hence the components are not changed.

4. What are the components of the zero vector based at $P = (3, 5)$?

SOLUTION The components of the zero vector are always $\langle 0, 0\rangle$, no matter where it is based.

5. True or false?

(a) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ are parallel.

(b) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ point in the same direction.

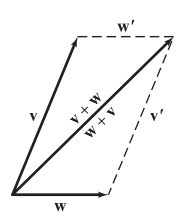
SOLUTION

(a) The lines through $\mathbf{v}$ and $-2\mathbf{v}$ are parallel, therefore these vectors are parallel.

(b) The vector $-2\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, where the scalar is negative. Therefore $-2\mathbf{v}$ points in the opposite direction as $\mathbf{v}$.

6. Explain the commutativity of vector addition in terms of the Parallelogram Law.

SOLUTION To determine the vector $\mathbf{v} + \mathbf{w}$, we translate $\mathbf{w}$ to the equivalent vector $\mathbf{w}'$ whose tail coincides with the head of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is the vector pointing from the tail of $\mathbf{v}$ to the head of $\mathbf{w}'$.



To determine the vector $\mathbf{w} + \mathbf{v}$, we translate $\mathbf{v}$ to the equivalent vector $\mathbf{v}'$ whose tail coincides with the head of $\mathbf{w}$. Then $\mathbf{w} + \mathbf{v}$ is the vector pointing from the tail of $\mathbf{w}$ to the head of $\mathbf{v}'$. In either case, the resulting vector is the vector with the tail at the basepoint of $\mathbf{v}$ and $\mathbf{w}$, and head at the opposite vertex of the parallelogram. Therefore $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

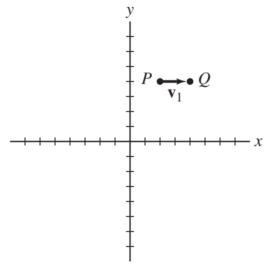
Exercises

1. Sketch the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ with tail P and head Q, and compute their lengths. Are any two of these vectors equivalent?

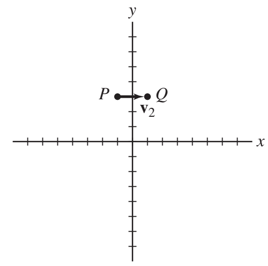
	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
P	$(2, 4)$	$(-1, 3)$	$(-1, 3)$	$(4, 1)$
Q	$(4, 4)$	$(1, 3)$	$(2, 4)$	$(6, 3)$

SOLUTION Using the definitions we obtain the following answers:

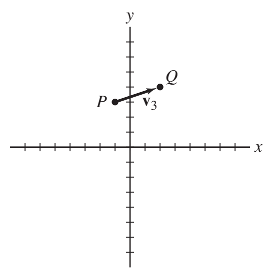
$$\mathbf{v}_1 = \overrightarrow{PQ} = \langle 4-2, 4-4\rangle = \langle 2, 0\rangle$$
$$\|\mathbf{v}_1\| = \sqrt{2^2+0^2} = 2$$



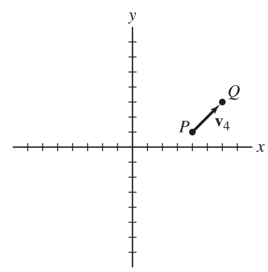
$$\mathbf{v}_2 = \langle 1-(-1), 3-3\rangle = \langle 2, 0\rangle$$
$$\|\mathbf{v}_2\| = \sqrt{2^2+0^2} = 2$$



$$\mathbf{v}_3 = \langle 2-(-1), 4-3\rangle = \langle 3, 1\rangle$$
$$\|\mathbf{v}_3\| = \sqrt{3^2+1^2} = \sqrt{10}$$



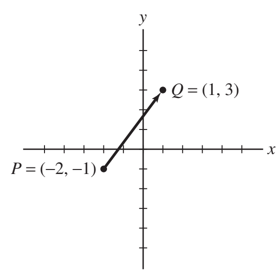
$$\mathbf{v}_4 = \langle 6-4, 3-1\rangle = \langle 2, 2\rangle$$
$$\|\mathbf{v}_4\| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$



$\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel and have the same length, hence they are equivalent.

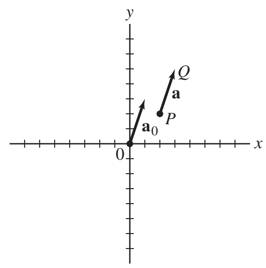
2. Sketch the vector $\mathbf{b} = \langle 3, 4\rangle$ based at $P = (-2, -1)$.

SOLUTION The vector $\mathbf{b} = \langle 3, 4\rangle$ based at P has terminal point Q, located 3 units to the right and 4 units up from P. Therefore $Q = (-2+3, -1+4) = (1, 3)$. The vector equivalent to $\mathbf{b}$ is PQ shown in the figure.



3. What is the terminal point of the vector $\mathbf{a} = \langle 1, 3\rangle$ based at $P = (2, 2)$? Sketch $\mathbf{a}$ and the vector $\mathbf{a}_0$ based at the origin and equivalent to $\mathbf{a}$.

SOLUTION The terminal point Q of the vector $\mathbf{a}$ is located 1 unit to the right and 3 units up from $P = (2, 2)$. Therefore, $Q = (2+1, 2+3) = (3, 5)$. The vector $\mathbf{a}_0$ equivalent to $\mathbf{a}$ based at the origin is shown in the figure, along with the vector $\mathbf{a}$.

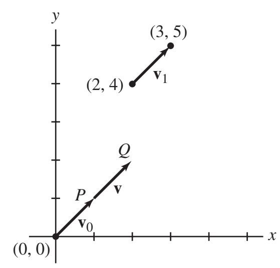


4. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (1, 1)$ and $Q = (2, 2)$. What is the head of the vector $\mathbf{v}'$ equivalent to $\mathbf{v}$ based at $(2, 4)$? What is the head of the vector $\mathbf{v}_0$ equivalent to $\mathbf{v}$ based at the origin? Sketch $\mathbf{v}$, $\mathbf{v}_0$, and $\mathbf{v}'$.

SOLUTION We first find the components of $\mathbf{v}$:

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1, 1\rangle$$

Since $\mathbf{v}'$ is equivalent to $\mathbf{v}$, the two vectors have the same components, hence the head of $\mathbf{v}'$ is located one unit to the right and one unit up from $(2, 4)$. This is the point $(3, 5)$. The head of $\mathbf{v}_0$ is located one unit to the right and one unit up from the origin; that is, the head is at the point $(1, 1)$.



In Exercises 5–8, find the components of $\overrightarrow{PQ}$.

5. $P = (3, 2), \quad Q = (2, 7)$

SOLUTION Using the definition of the components of a vector we have $\overrightarrow{PQ} = \langle 2 - 3, 7 - 2\rangle = \langle -1, 5\rangle$.

6. $P = (1, -4), \quad Q = (3, 5)$

SOLUTION The components of $\overrightarrow{PQ}$ are $\overrightarrow{PQ} = \langle 3 - 1, 5 - (-4)\rangle = \langle 2, 9\rangle$.

7. $P = (3, 5), \quad Q = (1, -4)$

SOLUTION By the definition of the components of a vector, we obtain $\overrightarrow{PQ} = \langle 1 - 3, -4 - 5\rangle = \langle -2, -9\rangle$.

8. $P = (0, 2), \quad Q = (5, 0)$

SOLUTION The components of the vector $\overrightarrow{PQ}$ are $\overrightarrow{PQ} = \langle 5 - 0, 0 - 2\rangle = \langle 5, -2\rangle$.

In Exercises 9–14, calculate.

9. $\langle 2, 1\rangle + \langle 3, 4\rangle$

SOLUTION Using vector algebra we have $\langle 2, 1\rangle + \langle 3, 4\rangle = \langle 2 + 3, 1 + 4\rangle = \langle 5, 5\rangle$.

10. $\langle -4, 6\rangle - \langle 3, -2\rangle$

SOLUTION $\langle -4, 6\rangle - \langle 3, -2\rangle = \langle -4 - 3, 6 - (-2)\rangle = \langle -7, 8\rangle$

11. $5\,\langle 6, 2\rangle$

SOLUTION $5\langle 6, 2\rangle = \langle 5 \cdot 6, 5 \cdot 2\rangle = \langle 30, 10\rangle$

12. $4(\langle 1, 1\rangle + \langle 3, 2\rangle)$

SOLUTION Using vector algebra we obtain

$$4\,(\langle 1, 1\rangle + \langle 3, 2\rangle) = 4\langle 1 + 3, 1 + 2\rangle = 4\langle 4, 3\rangle = \langle 4 \cdot 4, 4 \cdot 3\rangle = \langle 16, 12\rangle$$

13. $\left\langle -\frac{1}{2}, \frac{5}{3}\right\rangle + \left\langle 3, \frac{10}{3}\right\rangle$

SOLUTION The vector sum is $\left\langle -\dfrac{1}{2}, \dfrac{5}{3}\right\rangle + \left\langle 3, \dfrac{10}{3}\right\rangle = \left\langle -\dfrac{1}{2} + 3, \dfrac{5}{3} + \dfrac{10}{3}\right\rangle = \left\langle \dfrac{5}{2}, 5\right\rangle$.

14. $\langle \ln 2, e \rangle + \langle \ln 3, \pi \rangle$

SOLUTION The vector sum is $\langle \ln 2, e \rangle + \langle \ln 3, \pi \rangle = \langle \ln 2 + \ln 3, e + \pi \rangle = \langle \ln 6, e + \pi \rangle$.

15. Which of the vectors (A)–(C) in Figure 21 is equivalent to $\mathbf{v} - \mathbf{w}$?

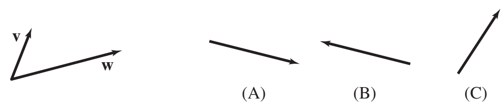
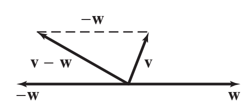


FIGURE 21

SOLUTION The vector $-\mathbf{w}$ has the same length as $\mathbf{w}$ but points in the opposite direction. The sum $\mathbf{v} + (-\mathbf{w})$, which is the difference $\mathbf{v} - \mathbf{w}$, is obtained by the parallelogram law. This vector is the vector shown in (b).



16. Sketch $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ for the vectors in Figure 22.

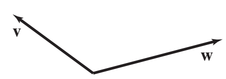
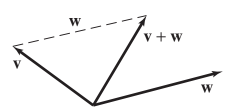
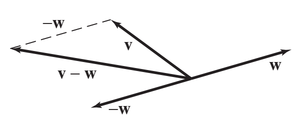


FIGURE 22

SOLUTION The vector $\mathbf{v} + \mathbf{w}$ is obtained by the parallelogram law:



Since $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$, we first sketch the vector $-\mathbf{w}$, which has the same length as $\mathbf{w}$ but points to the opposite direction. Then we add $-\mathbf{w}$ to $\mathbf{v}$ using the parallelogram law. This gives:



17. Sketch $2\mathbf{v}$, $-\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and $2\mathbf{v} - \mathbf{w}$ for the vectors in Figure 23.

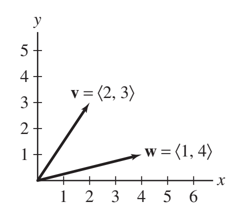
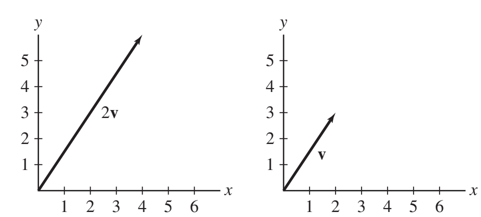
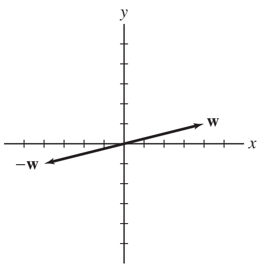


FIGURE 23

SOLUTION The scalar multiple $2\mathbf{v}$ points in the same direction as $\mathbf{v}$ and its length is twice the length of $\mathbf{v}$. It is the vector $2\mathbf{v} = \langle 4, 6 \rangle$.



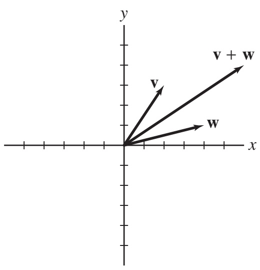
$-\mathbf{w}$ has the same length as $\mathbf{w}$ but points to the opposite direction. It is the vector $-\mathbf{w} = \langle -4, -1 \rangle$.



The vector sum $\mathbf{v} + \mathbf{w}$ is the vector:

$$\mathbf{v} + \mathbf{w} = \langle 2, 3\rangle + \langle 4, 1\rangle = \langle 6, 4\rangle.$$

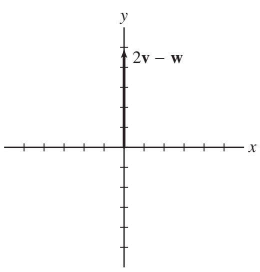
This vector is shown in the following figure:



The vector $2\mathbf{v} - \mathbf{w}$ is

$$2\mathbf{v} - \mathbf{w} = 2\langle 2, 3\rangle - \langle 4, 1\rangle = \langle 4, 6\rangle - \langle 4, 1\rangle = \langle 0, 5\rangle$$

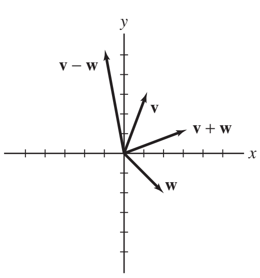
It is shown next:



18. Sketch $\mathbf{v} = \langle 1, 3\rangle$, $\mathbf{w} = \langle 2, -2\rangle$, $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$.

SOLUTION We compute the sum $\mathbf{v} + \mathbf{w}$ and the difference $\mathbf{v} - \mathbf{w}$ and then sketch the vectors. This gives:

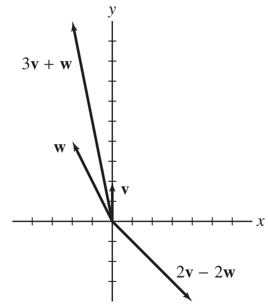
$$\mathbf{v} + \mathbf{w} = \langle 1, 3\rangle + \langle 2, -2\rangle = \langle 1 + 2, 3 - 2\rangle = \langle 3, 1\rangle$$
$$\mathbf{v} - \mathbf{w} = \langle 1, 3\rangle - \langle 2, -2\rangle = \langle 1 - 2, 3 + 2\rangle = \langle -1, 5\rangle$$



19. Sketch $\mathbf{v} = \langle 0, 2\rangle$, $\mathbf{w} = \langle -2, 4\rangle$, $3\mathbf{v} + \mathbf{w}$, $2\mathbf{v} - 2\mathbf{w}$.

SOLUTION We compute the vectors and then sketch them:

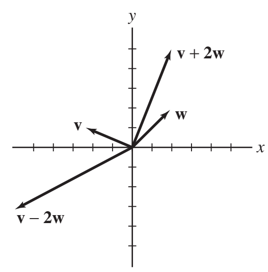
$$3\mathbf{v} + \mathbf{w} = 3\langle 0, 2\rangle + \langle -2, 4\rangle = \langle 0, 6\rangle + \langle -2, 4\rangle = \langle -2, 10\rangle$$
$$2\mathbf{v} - 2\mathbf{w} = 2\langle 0, 2\rangle - 2\langle -2, 4\rangle = \langle 0, 4\rangle - \langle -4, 8\rangle = \langle 4, -4\rangle$$



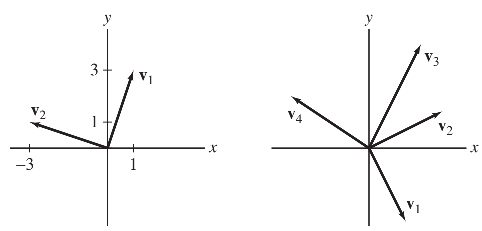
20. Sketch $\mathbf{v} = \langle -2, 1\rangle$, $\mathbf{w} = \langle 2, 2\rangle$, $\mathbf{v} + 2\mathbf{w}$, $\mathbf{v} - 2\mathbf{w}$.

SOLUTION We compute the linear combinations $\mathbf{v} + 2\mathbf{w}$ and $\mathbf{v} - 2\mathbf{w}$ and then sketch the vectors:

$$\mathbf{v} + 2\mathbf{w} = \langle -2, 1\rangle + 2\langle 2, 2\rangle = \langle -2, 1\rangle + \langle 4, 4\rangle = \langle 2, 5\rangle$$
$$\mathbf{v} - 2\mathbf{w} = \langle -2, 1\rangle - 2\langle 2, 2\rangle = \langle -2, 1\rangle - \langle 4, 4\rangle = \langle -6, -3\rangle$$



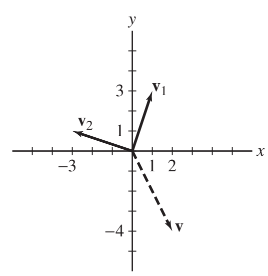
21. Sketch the vector $\mathbf{v}$ such that $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ for $\mathbf{v}_1$ and $\mathbf{v}_2$ in Figure 24(A).



(A) (B)

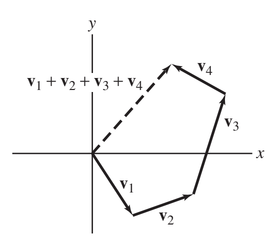
FIGURE 24

SOLUTION Since $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2 = 0$, we have that $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2$, and since $\mathbf{v}_1 = \langle 1, 3\rangle$ and $\mathbf{v}_2 = \langle -3, 1\rangle$, then $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2 = \langle 2, -4\rangle$, as seen in this picture.



22. Sketch the vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$ in Figure 24(B).

SOLUTION If we place the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ tip to tail, as shown in the following figure, it is easy to sketch in the sum $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$, as shown.



23. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (-2, 5)$, $Q = (1, -2)$. Which of the following vectors with the given tails and heads are equivalent to $\mathbf{v}$?

(a) $(-3, 3)$, $(0, 4)$ **(b)** $(0, 0)$, $(3, -7)$
(c) $(-1, 2)$, $(2, -5)$ **(d)** $(4, -5)$, $(1, 4)$

SOLUTION Two vectors are equivalent if they have the same components. We thus compute the vectors and check whether this condition is satisfied.

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), -2 - 5 \rangle = \langle 3, -7 \rangle$$

(a) $\langle 0 - (-3), 4 - 3 \rangle = \langle 3, 1 \rangle$ **(b)** $\langle 3 - 0, -7 - 0 \rangle = \langle 3, -7 \rangle$
(c) $\langle 2 - (-1), -5 - 2 \rangle = \langle 3, -7 \rangle$ **(d)** $\langle 1 - 4, 4 - (-5) \rangle = \langle -3, 9 \rangle$

We see that the vectors in (b) and (c) are equivalent to $\mathbf{v}$.

24. Which of the following vectors are parallel to $\mathbf{v} = \langle 6, 9 \rangle$ and which point in the same direction?

(a) $\langle 12, 18 \rangle$ **(b)** $\langle 3, 2 \rangle$ **(c)** $\langle 2, 3 \rangle$
(d) $\langle -6, -9 \rangle$ **(e)** $\langle -24, -27 \rangle$ **(f)** $\langle -24, -36 \rangle$

SOLUTION Two vectors are parallel if they are scalar multiples of each other. The vectors point in the same direction if the multiplying scalar is positive. We use this to obtain the following conclusions:

(a) $\langle 12, 18 \rangle = 2\langle 6, 9 \rangle = 2\mathbf{v} \Rightarrow$ both vectors point in the same direction.
(b) $\langle 3, 2 \rangle$ is not a scalar multiple of $\mathbf{v}$, hence the vectors are not parallel.
(c) $\langle 2, 3 \rangle = \frac{1}{3}\langle 6, 9 \rangle = \frac{1}{3}\mathbf{v} \Rightarrow$ both vectors point in the same direction.
(d) $\langle -6, -9 \rangle = -\langle 6, 9 \rangle = -\mathbf{v} \Rightarrow$ parallel to $\mathbf{v}$ and points in the opposite direction.
(e) $\langle -24, -27 \rangle$ is not a scalar multiple of $\mathbf{v}$, hence the vectors are not parallel.
(f) $\langle -24, -36 \rangle = -4\langle 6, 9 \rangle = -4\mathbf{v} \Rightarrow$ parallel to $\mathbf{v}$ and points in the opposite direction.

In Exercises 25–28, sketch the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, and determine whether they are equivalent.

25. $A = (1, 1)$, $B = (3, 7)$, $P = (4, -1)$, $Q = (6, 5)$

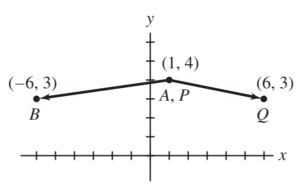
SOLUTION We compute the vectors and check whether they have the same components:

$$\begin{aligned} \overrightarrow{AB} &= \langle 3 - 1, 7 - 1 \rangle = \langle 2, 6 \rangle \\ \overrightarrow{PQ} &= \langle 6 - 4, 5 - (-1) \rangle = \langle 2, 6 \rangle \end{aligned} \quad \Rightarrow \quad \text{The vectors are equivalent.}$$

26. $A = (1, 4)$, $B = (-6, 3)$, $P = (1, 4)$, $Q = (6, 3)$

SOLUTION We compute $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ and see if they have the same components:

$$\begin{aligned} \overrightarrow{AB} &= \langle -6 - 1, 3 - 4 \rangle = \langle -7, -1 \rangle \\ \overrightarrow{PQ} &= \langle 6 - 1, 3 - 4 \rangle = \langle 5, -1 \rangle \end{aligned} \quad \Rightarrow \quad \text{The vectors are not equivalent.}$$



27. $A = (-3, 2)$, $B = (0, 0)$, $P = (0, 0)$, $Q = (3, -2)$

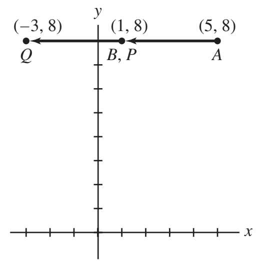
SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\begin{aligned} \overrightarrow{AB} &= \langle 0 - (-3), 0 - 2 \rangle = \langle 3, -2 \rangle \\ \overrightarrow{PQ} &= \langle 3 - 0, -2 - 0 \rangle = \langle 3, -2 \rangle \end{aligned} \quad \Rightarrow \quad \text{The vectors are equivalent.}$$

28. $A = (5, 8)$, $B = (1, 8)$, $P = (1, 8)$, $Q = (-3, 8)$

SOLUTION Computing $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ gives:

$$\begin{aligned} \overrightarrow{AB} &= \langle 1 - 5, 8 - 8 \rangle = \langle -4, 0 \rangle \\ \overrightarrow{PQ} &= \langle -3 - 1, 8 - 8 \rangle = \langle -4, 0 \rangle \end{aligned} \quad \Rightarrow \quad \text{The vectors are equivalent.}$$



In Exercises 29–32, are $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ parallel? And if so, do they point in the same direction?

29. $A = (1, 1), \quad B = (3, 4), \quad P = (1, 1), \quad Q = (7, 10)$

SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\overrightarrow{AB} = \langle 3 - 1, 4 - 1 \rangle = \langle 2, 3 \rangle$$
$$\overrightarrow{PQ} = \langle 7 - 1, 10 - 1 \rangle = \langle 6, 9 \rangle$$

Since $\overrightarrow{AB} = \frac{1}{3}\langle 6, 9 \rangle$, the vectors are parallel and point in the same direction.

30. $A = (-3, 2), \quad B = (0, 0), \quad P = (0, 0), \quad Q = (3, 2)$

SOLUTION We compute the two vectors:

$$\overrightarrow{AB} = \langle 0 - (-3), 0 - 2 \rangle = \langle 3, -2 \rangle$$
$$\overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

The vectors are not scalar multiples of each other, hence they are not parallel.

31. $A = (2, 2), \quad B = (-6, 3), \quad P = (9, 5), \quad Q = (17, 4)$

SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\overrightarrow{AB} = \langle -6 - 2, 3 - 2 \rangle = \langle -8, 1 \rangle$$
$$\overrightarrow{PQ} = \langle 17 - 9, 4 - 5 \rangle = \langle 8, -1 \rangle$$

Since $\overrightarrow{AB} = -\overrightarrow{PQ}$, the vectors are parallel and point in opposite directions.

32. $A = (5, 8), \quad B = (2, 2), \quad P = (2, 2), \quad Q = (-3, 8)$

SOLUTION Computing $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ gives:

$$\overrightarrow{AB} = \langle 2 - 5, 2 - 8 \rangle = \langle -3 - 6 \rangle$$
$$\overrightarrow{PQ} = \langle -3 - 2, 8 - 2 \rangle = \langle -5, 6 \rangle$$

The vectors are not scalar multiples of each other, hence they are not parallel.

In Exercises 33–36, let $R = (-2, 7)$. Calculate the following.

33. The length of $\overrightarrow{OR}$

SOLUTION Since $\overrightarrow{OR} = \langle -2, 7 \rangle$, the length of the vector is $\|\overrightarrow{OR}\| = \sqrt{(-2)^2 + 7^2} = \sqrt{53}$.

34. The components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2)$

SOLUTION We compute the components of the vector to obtain:

$$\mathbf{u} = \overrightarrow{PR} = \langle -2 - 1, 7 - 2 \rangle = \langle -3, 5 \rangle$$

35. The point P such that $\overrightarrow{PR}$ has components $\langle -2, 7 \rangle$

SOLUTION Denoting $P = (x_0, y_0)$ we have:

$$\overrightarrow{PR} = \langle -2 - x_0, 7 - y_0 \rangle = \langle -2, 7 \rangle$$

Equating corresponding components yields:

$$\begin{aligned} -2 - x_0 &= -2 \\ 7 - y_0 &= 7 \end{aligned} \quad \Rightarrow \quad x_0 = 0,\ y_0 = 0 \quad \Rightarrow \quad P = (0, 0)$$

36. The point Q such that $\overrightarrow{RQ}$ has components $\langle 8, -3\rangle$

SOLUTION We denote $Q = (x_0, y_0)$ and have:

$$\overrightarrow{RQ} = \langle x_0 - (-2), y_0 - 7\rangle = \langle x_0 + 2, y_0 - 7\rangle = \langle 8, -3\rangle$$

Equating the corresponding components of the two vectors yields:

$$\begin{array}{l} x_0 + 2 = 8 \\ y_0 - 7 = -3 \end{array} \quad \Rightarrow \quad x_0 = 6, \; y_0 = 4 \quad \Rightarrow \quad Q = (6, 4)$$

In Exercises 37–42, find the given vector.

37. Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4\rangle$

SOLUTION The unit vector $\mathbf{e_v}$ is the following vector:

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We find the length of $\mathbf{v} = \langle 3, 4\rangle$:

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus

$$\mathbf{e_v} = \frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle.$$

38. Unit vector $\mathbf{e_w}$ where $\mathbf{w} = \langle 24, 7\rangle$

SOLUTION The unit vector $\mathbf{e_w}$ is the following vector:

$$\mathbf{e_w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w}$$

We find the length of $\mathbf{w} = \langle 24, 7\rangle$:

$$\|\mathbf{w}\| = \sqrt{24^2 + 7^2} = \sqrt{625} = 25$$

Thus

$$\mathbf{e_w} = \frac{1}{25}\langle 24, 7\rangle = \left\langle \frac{24}{25}, \frac{7}{25}\right\rangle.$$

39. Vector of length 4 in the direction of $\mathbf{u} = \langle -1, -1\rangle$

SOLUTION Since $\|u\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, the unit vector in the direction of $\mathbf{u}$ is $\mathbf{e_u} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$. We multiply $\mathbf{e_u}$ by 4 to obtain the desired vector:

$$4\mathbf{e_u} = 4\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle = \left\langle -2\sqrt{2}, -2\sqrt{2}\right\rangle$$

40. Unit vector in the direction opposite to $\mathbf{v} = \langle -2, 4\rangle$

SOLUTION We first compute the unit vector $\mathbf{e_v}$ in the direction of $\mathbf{v}$ and then multiply by -1 to obtain a unit vector in the opposite direction. This gives:

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(-2)^2 + 4^2}}\langle -2, 4\rangle = \frac{1}{\sqrt{20}}\langle -2, 4\rangle = \left\langle -\frac{2}{2\sqrt{5}}, \frac{4}{2\sqrt{5}}\right\rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$$

The desired vector is thus

$$-\mathbf{e_v} = -\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle.$$

41. Unit vector $\mathbf{e}$ making an angle of $\frac{4\pi}{7}$ with the x-axis

SOLUTION The unit vector $\mathbf{e}$ is the following vector:

$$\mathbf{e} = \left\langle \cos\frac{4\pi}{7}, \sin\frac{4\pi}{7}\right\rangle = \langle -0.22, 0.97\rangle.$$

42. Vector **v** of length 2 making an angle of 30° with the x-axis

SOLUTION The desired vector is

$$\mathbf{v} = 2\langle \cos 30^\circ, \sin 30^\circ \rangle = 2\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \left\langle \sqrt{3}, 1 \right\rangle.$$

43. Find all scalars λ such that $\lambda\langle 2, 3\rangle$ has length 1.

SOLUTION We have:

$$\|\lambda\langle 2, 3\rangle\| = |\lambda|\|\langle 2, 3\rangle\| = |\lambda|\sqrt{2^2+3^2} = |\lambda|\sqrt{13}$$

The scalar λ must satisfy

$$\begin{aligned} |\lambda|\sqrt{13} &= 1 \\ |\lambda| &= \frac{1}{\sqrt{13}} \end{aligned} \quad\Rightarrow\quad \lambda_1 = \frac{1}{\sqrt{13}},\ \lambda_2 = -\frac{1}{\sqrt{13}}$$

44. Find a vector **v** satisfying $3\mathbf{v} + \langle 5, 20\rangle = \langle 11, 17\rangle$.

SOLUTION Write $\mathbf{v} = \langle x, y\rangle$ to get the equation $3\langle x, y\rangle + \langle 5, 20\rangle = \langle 11, 17\rangle$, which gives us $3x + 5 = 11$ (and thus $x = 2$) and also $3y + 20 = 17$ (and so $y = -1$). Thus, $\mathbf{v} = \langle 2, -1\rangle$.

45. What are the coordinates of the point P in the parallelogram in Figure 25(A)?

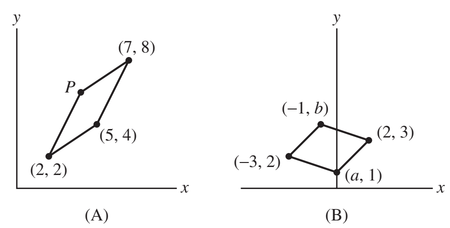
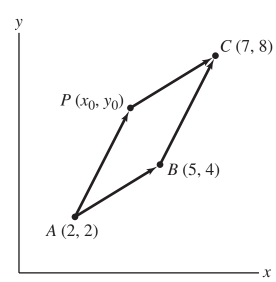


FIGURE 25

SOLUTION We denote by A, B, C the points in the figure.



Let $P = (x_0, y_0)$. We compute the following vectors:

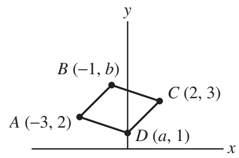
$$\overrightarrow{PC} = \langle 7 - x_0, 8 - y_0\rangle$$
$$\overrightarrow{AB} = \langle 5 - 2, 4 - 2\rangle = \langle 3, 2\rangle$$

The vectors $\overrightarrow{PC}$ and $\overrightarrow{AB}$ are equivalent, hence they have the same components. That is:

$$\begin{aligned} 7 - x_0 &= 3 \\ 8 - y_0 &= 2 \end{aligned} \quad\Rightarrow\quad x_0 = 4,\ y_0 = 6 \quad\Rightarrow\quad P = (4, 6)$$

46. What are the coordinates a and b in the parallelogram in Figure 25(B)?

SOLUTION We denote the points in the figure by A, B, C and D.



We compute the following vectors:

$$\overrightarrow{AB} = \langle -1 - (-3), b - 2 \rangle = \langle 2, b - 2 \rangle$$
$$\overrightarrow{DC} = \langle 2 - a, 3 - 1 \rangle = \langle 2 - a, 2 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{DC}$, the two vectors have the same components. That is,

$$\begin{aligned} 2 &= 2 - a \\ b - 2 &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 0 \\ b &= 4 \end{aligned}$$

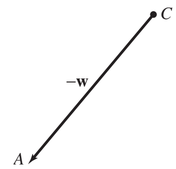
47. Let $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$, where A, B, C are three distinct points in the plane. Match (a)–(d) with (i)–(iv). (*Hint:* Draw a picture.)

(a) $-\mathbf{w}$ **(b)** $-\mathbf{v}$ **(c)** $\mathbf{w} - \mathbf{v}$ **(d)** $\mathbf{v} - \mathbf{w}$

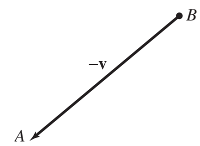
(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{CA}$ **(iii)** $\overrightarrow{BC}$ **(iv)** $\overrightarrow{BA}$

SOLUTION

(a) $-\mathbf{w}$ has the same length as $\mathbf{w}$ and points in the opposite direction. Hence: $-\mathbf{w} = \overrightarrow{CA}$.



(b) $-\mathbf{v}$ has the same length as $\mathbf{v}$ and points in the opposite direction. Hence: $-\mathbf{v} = \overrightarrow{BA}$.

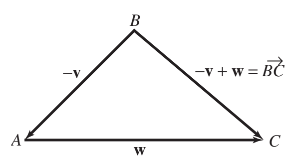


(c) By the parallelogram law we have:

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\mathbf{v} + \mathbf{w} = \mathbf{w} - \mathbf{v}$$

That is,

$$\mathbf{w} - \mathbf{v} = \overrightarrow{BC}$$

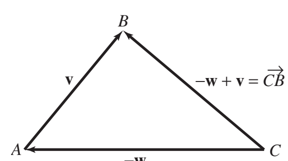


(d) By the parallelogram law we have:

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\mathbf{w} + \mathbf{v} = \mathbf{v} - \mathbf{w}$$

That is,

$$\mathbf{v} - \mathbf{w} = \overrightarrow{CB}.$$



48. Find the components and length of the following vectors:

(a) $4\mathbf{i} + 3\mathbf{j}$ **(b)** $2\mathbf{i} - 3\mathbf{j}$ **(c)** $\mathbf{i} + \mathbf{j}$ **(d)** $\mathbf{i} - 3\mathbf{j}$

SOLUTION

(a) Since $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, using vector algebra we have:

$$4\mathbf{i} + 3\mathbf{j} = 4\langle 1, 0 \rangle + 3\langle 0, 1 \rangle = \langle 4, 0 \rangle + \langle 0, 3 \rangle = \langle 4 + 0, 0 + 3 \rangle = \langle 4, 3 \rangle$$

The length of the vector is:

$$\|4\mathbf{i}+3\mathbf{j}\| = \sqrt{4^2+3^2} = 5$$

(b) We use vector algebra and the definition of the standard basis vector to compute the components of the vector $2\mathbf{i}-3\mathbf{j}$:

$$2\mathbf{i}-3\mathbf{j} = 2\langle 1,0\rangle - 3\langle 0,1\rangle = \langle 2,0\rangle - \langle 0,3\rangle = \langle 2-0, 0-3\rangle = \langle 2,-3\rangle$$

The length of this vector is:

$$\|2\mathbf{i}-3\mathbf{j}\| = \sqrt{2^2+(-3)^2} = \sqrt{13}$$

(c) We find the components of the vector $\mathbf{i}+\mathbf{j}$:

$$\mathbf{i}+\mathbf{j} = \langle 1,0\rangle + \langle 0,1\rangle = \langle 1+0, 0+1\rangle = \langle 1,1\rangle$$

The length of this vector is:

$$\|\mathbf{i}+\mathbf{j}\| = \sqrt{1^2+1^2} = \sqrt{2}$$

(d) We find the components of the vector $\mathbf{i}-3\mathbf{j}$, using vector algebra:

$$\mathbf{i}-3\mathbf{j} = \langle 1,0\rangle - 3\langle 0,1\rangle = \langle 1,0\rangle - \langle 0,3\rangle = \langle 1-0, 0-3\rangle = \langle 1,-3\rangle$$

The length of this vector is

$$\|\mathbf{i}-3\mathbf{j}\| = \sqrt{1^2+(-3)^2} = \sqrt{10}$$

In Exercises 49–52, calculate the linear combination.

49. $3\mathbf{j}+(9\mathbf{i}+4\mathbf{j})$

SOLUTION We have:

$$3\mathbf{j}+(9\mathbf{i}+4\mathbf{j}) = 3\langle 0,1\rangle + 9\langle 1,0\rangle + 4\langle 0,1\rangle = \langle 9,7\rangle$$

50. $-\frac{3}{2}\mathbf{i}+5\left(\frac{1}{2}\mathbf{j}-\frac{1}{2}\mathbf{i}\right)$

SOLUTION We have:

$$-\frac{3}{2}\mathbf{i}+5\left(\frac{1}{2}\mathbf{j}-\frac{1}{2}\mathbf{i}\right) = -\frac{3}{2}\langle 1,0\rangle + 5\left(\frac{1}{2}\langle 0,1\rangle - \frac{1}{2}\langle 1,0\rangle\right) = \left\langle -4, \frac{5}{2}\right\rangle$$

51. $(3\mathbf{i}+\mathbf{j})-6\mathbf{j}+2(\mathbf{j}-4\mathbf{i})$

SOLUTION We have:

$$(3\mathbf{i}+\mathbf{j})-6\mathbf{j}+2(\mathbf{j}-4\mathbf{i}) = (\langle 3,0\rangle + \langle 0,1\rangle) - \langle 0,6\rangle + 2(\langle 0,1\rangle - \langle 4,0\rangle) = \langle -5,-3\rangle$$

52. $3(3\mathbf{i}-4\mathbf{j})+5(\mathbf{i}+4\mathbf{j})$

SOLUTION We have:

$$3(3\mathbf{i}-4\mathbf{j})+5(\mathbf{i}+4\mathbf{j}) = 3(\langle 3,0\rangle - \langle 0,4\rangle) + 5(\langle 1,0\rangle + \langle 0,4\rangle) = \langle 14,8\rangle$$

53. For each of the position vectors $\mathbf{u}$ with endpoints A, B, and C in Figure 26, indicate with a diagram the multiples $r\mathbf{v}$ and $s\mathbf{w}$ such that $\mathbf{u} = r\mathbf{v}+s\mathbf{w}$. A sample is shown for $\mathbf{u} = \overrightarrow{OQ}$.

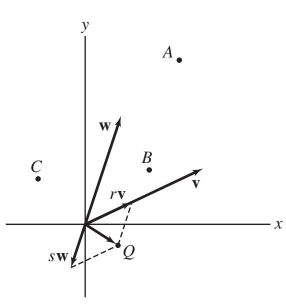
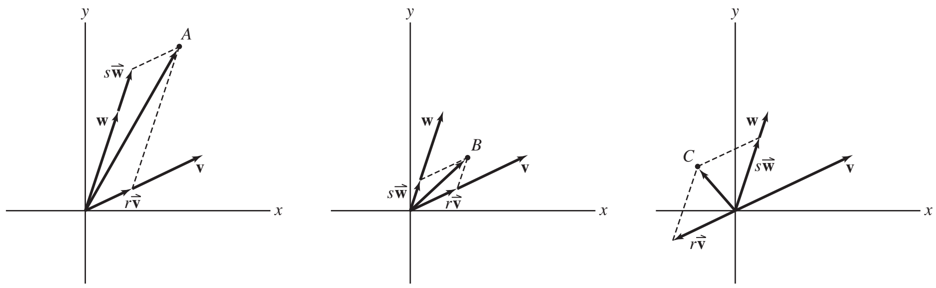


FIGURE 26

SOLUTION See the following three figures:



54. Sketch the parallelogram spanned by $\mathbf{v} = \langle 1, 4 \rangle$ and $\mathbf{w} = \langle 5, 2 \rangle$. Add the vector $\mathbf{u} = \langle 2, 3 \rangle$ to the sketch and express $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

SOLUTION We have

$$\mathbf{u} = \langle 2, 3 \rangle = r\mathbf{v} + s\mathbf{w} = r\langle 1, 4 \rangle + s\langle 5, 2 \rangle$$

which becomes the two equations

$$2 = r + 5s$$
$$3 = 4r + 2s$$

Solving the first equation for r gives

$$r = 2 - 5s$$

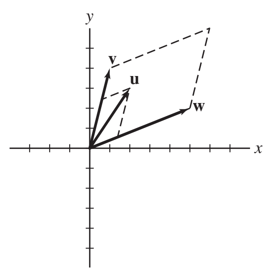
and substituting that into the first equation gives

$$3 = 4(2 - 5s) + 2s = 8 - 18s$$

So $18s = 5$, so $s = 5/18$, and thus $r = 11/18$. In other words,

$$\mathbf{u} = \langle 2, 3 \rangle = \frac{11}{18}\langle 1, 4 \rangle + \frac{5}{18}\langle 5, 2 \rangle$$

as seen in this picture:



In Exercises 55 and 56, express $\mathbf{u}$ *as a linear combination* $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$. *Then sketch* $\mathbf{u}, \mathbf{v}, \mathbf{w}$, *and the parallelogram formed by* $r\mathbf{v}$ *and* $s\mathbf{w}$.

55. $\mathbf{u} = \langle 3, -1 \rangle$; $\mathbf{v} = \langle 2, 1 \rangle, \mathbf{w} = \langle 1, 3 \rangle$

SOLUTION We have

$$\mathbf{u} = \langle 3, -1 \rangle = r\mathbf{v} + s\mathbf{w} = r\langle 2, 1 \rangle + s\langle 1, 3 \rangle$$

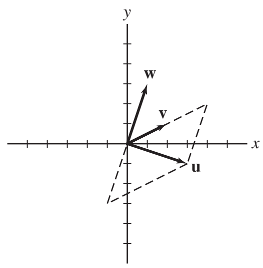
which becomes the two equations

$$3 = 2r + s$$
$$-1 = r + 3s$$

Solving the second equation for r gives $r = -1 - 3s$, and substituting that into the first equation gives $3 = 2(-1 - 3s) + s = -2 - 6s + s$, so $5 = -5s$, so $s = -1$, and thus $r = 2$. In other words,

$$\mathbf{u} = \langle 3, -1 \rangle = 2\langle 2, 1 \rangle - 1\langle 1, 3 \rangle$$

as seen in this sketch:



56. $\mathbf{u} = \langle 6, -2 \rangle$; $\mathbf{v} = \langle 1, 1 \rangle$, $\mathbf{w} = \langle 1, -1 \rangle$

SOLUTION We have

$$\mathbf{u} = \langle 6, -2 \rangle = r\mathbf{v} + s\mathbf{w} = r\langle 1, 1 \rangle + s\langle 1, -1 \rangle$$

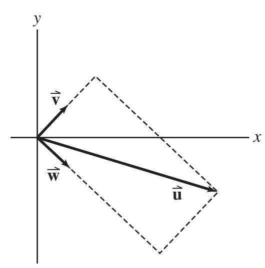
which becomes the two equations

$$6 = r + s$$
$$-2 = r - s$$

Adding gives $4 = 2r$, so $r = 2$ and thus $s = 4$. In other words,

$$\mathbf{u} = \langle 6, -2 \rangle = 2\langle 1, 1 \rangle + 4\langle 1, -1 \rangle$$

as seen in this sketch:



57. Calculate the magnitude of the force on cables 1 and 2 in Figure 27.

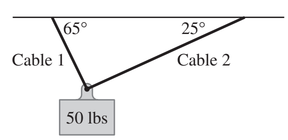
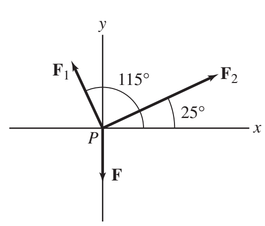


FIGURE 27

SOLUTION The three forces acting on the point P are:

- The force $\mathbf{F}$ of magnitude 50 lb that acts vertically downward.
- The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ that act through cables 1 and 2 respectively.



Since the point P is not in motion we have

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F} = 0 \tag{1}$$

We compute the forces. Letting $\|\mathbf{F}_1\| = f_1$ and $\|\mathbf{F}_2\| = f_2$ we have:

$$\mathbf{F}_1 = f_1\langle \cos 115^\circ, \sin 115^\circ \rangle = f_1\langle -0.423, 0.906 \rangle$$
$$\mathbf{F}_2 = f_2\langle \cos 25^\circ, \sin 25^\circ \rangle = f_2\langle 0.906, 0.423 \rangle$$
$$\mathbf{F} = \langle 0, -50 \rangle$$

Substituting the forces in (1) gives

$$f_1\langle -0.423, 0.906\rangle + f_2\langle 0.906, 0.423\rangle + \langle 0, -50\rangle = \langle 0, 0\rangle$$
$$\langle -0.423 f_1 + 0.906 f_2, 0.906 f_1 + 0.423 f_2 - 50\rangle = \langle 0, 0\rangle$$

We equate corresponding components and get

$$-0.423 f_1 + 0.906 f_2 = 0$$
$$0.906 f_1 + 0.423 f_2 - 50 = 0$$

By the first equation, $f_2 = 0.467 f_1$. Substituting in the second equation and solving for f_1 yields

$$0.906 f_1 + 0.423 \cdot 0.467 f_1 - 50 = 0$$
$$1.104 f_1 = 50 \quad \Rightarrow \quad f_1 = 45.29, \; f_2 = 0.467 f_1 = 21.15$$

We conclude that the magnitude of the force on cable 1 is $f_1 = 45.29$ lb and the magnitude of the force on cable 2 is $f_2 = 21.15$ lb.

58. Determine the magnitude of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Figure 28, assuming that there is no net force on the object.

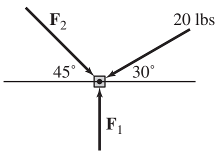
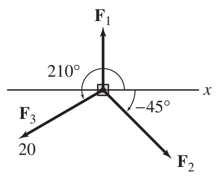


FIGURE 28

SOLUTION We denote $\|\mathbf{F}_1\| = f_1$ and $\|\mathbf{F}_2\| = f_2$. It is convenient (but not necessary) to redraw the vectors as being centered at the object, giving us the following figure.



Since there is no net force on the object, we have

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0 \tag{1}$$

We find the forces:

$$\mathbf{F}_1 = f_1\langle 0, 1\rangle = \langle 0, f_1\rangle$$
$$\mathbf{F}_2 = f_2\langle \cos(-45^\circ), \sin(-45^\circ)\rangle = f_2\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle = \langle 0.707 f_2, -0.707 f_2\rangle$$
$$\mathbf{F}_3 = 20\langle \cos 210^\circ, \sin 210^\circ\rangle = \langle -17.32, -10\rangle$$

We substitute the forces in (1):

$$\langle 0, f_1\rangle + \langle 0.707 f_2, -0.707 f_2\rangle + \langle -17.32, -10\rangle = \langle 0, 0\rangle$$
$$\langle 0.707 f_2 - 17.32, f_1 - 0.707 f_2 - 10\rangle = \langle 0, 0\rangle$$

Equating corresponding components we obtain

$$0.707 f_2 - 17.32 = 0$$
$$f_1 - 0.707 f_2 - 10 = 0$$

The first equation gives $f_2 = 24.5$. Substituting in the second equation and solving for f_1 gives

$$f_1 - 0.707 \cdot 24.5 - 10 = 0 \Rightarrow f_1 = 27.32$$

The magnitude of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are $f_1 = 27.32$ lb and $f_2 = 24.5$ lb respectively.

59. A plane flying due east at 200 km/h encounters a 40-km/h wind blowing in the north-east direction. The resultant velocity of the plane is the vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the velocity vector of the plane and $\mathbf{v}_2$ is the velocity vector of the wind (Figure 29). The angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{\pi}{4}$. Determine the resultant *speed* of the plane (the length of the vector $\mathbf{v}$).

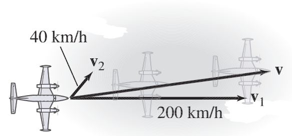
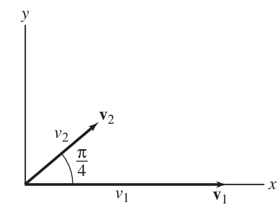


FIGURE 29

SOLUTION The resultant speed of the plane is the length of the sum vector $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We place the xy-coordinate system as shown in the figure, and compute the components of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. This gives

$$\mathbf{v}_1 = \langle v_1, 0 \rangle$$

$$\mathbf{v}_2 = \left\langle v_2 \cos\frac{\pi}{4}, v_2 \sin\frac{\pi}{4} \right\rangle = \left\langle v_2 \cdot \frac{\sqrt{2}}{2}, v_2 \cdot \frac{\sqrt{2}}{2} \right\rangle$$



We now compute the sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$:

$$\mathbf{v} = \langle v_1, 0 \rangle + \left\langle \frac{\sqrt{2}v_2}{2}, \frac{\sqrt{2}v_2}{2} \right\rangle = \left\langle \frac{\sqrt{2}}{2}v_2 + v_1, \frac{\sqrt{2}}{2}v_2 \right\rangle$$

The resultant speed is the length of $\mathbf{v}$, that is,

$$v = \|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}v_2}{2}\right)^2 + \left(v_1 + \frac{\sqrt{2}v_2}{2}\right)^2} = \sqrt{\frac{v_2^2}{2} + v_1^2 + 2 \cdot \frac{\sqrt{2}}{2}v_2 v_1 + \frac{v_2^2}{2}} = \sqrt{v_1^2 + v_2^2 + \sqrt{2}v_1 v_2}$$

Finally, we substitute the given information $v_1 = 200$ and $v_2 = 40$ in the equation above, to obtain

$$v = \sqrt{200^2 + 40^2 + \sqrt{2} \cdot 200 \cdot 40} \approx 230 \text{ km/hr}$$

Further Insights and Challenges

In Exercises 60–62, refer to Figure 30, which shows a robotic arm consisting of two segments of lengths L_1 and L_2.

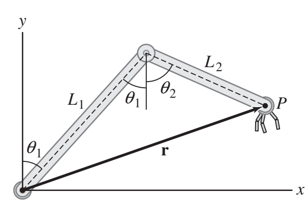
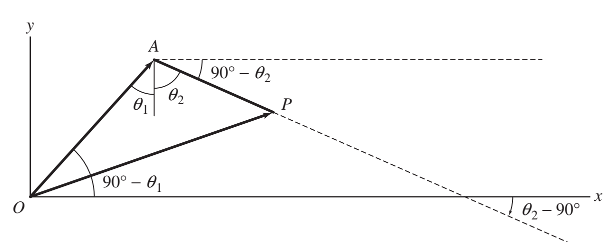


FIGURE 30

60. Find the components of the vector $\mathbf{r} = \overrightarrow{OP}$ in terms of θ_1 and θ_2.

SOLUTION We denote by A the point in the figure.



By the parallelogram law we have

$$\mathbf{r} = \overrightarrow{OA} + \overrightarrow{AP} \tag{1}$$

We find the vectors $\overrightarrow{OA}$ and $\overrightarrow{AP}$:

- The vector $\overrightarrow{OA}$ has length L_1 and it makes an angle of $90^\circ - \theta_1$ with the x-axis.
- The vector $\overrightarrow{AP}$ has length L_2 and it makes an angle of $-(90^\circ - \theta_2) = \theta_2 - 90^\circ$ with the x-axis.

Hence,

$$\overrightarrow{OA} = L_1 \left\langle \cos(90^\circ - \theta_1), \sin(90^\circ - \theta_1) \right\rangle = L_1 \langle \sin\theta_1, \cos\theta_1 \rangle = \langle L_1 \sin\theta_1, L_1 \cos\theta_1 \rangle$$

$$\overrightarrow{AP} = L_2 \left\langle \cos(\theta_2 - 90^\circ), \sin(\theta_2 - 90^\circ) \right\rangle = L_2 \langle \sin\theta_2, -\cos\theta_2 \rangle = \langle L_2 \sin\theta_2, -L_2 \cos\theta_2 \rangle$$

Substituting into (1) we obtain

$$\mathbf{r} = \langle L_1 \sin\theta_1, L_1 \cos\theta_1 \rangle + \langle L_2 \sin\theta_2 - L_2 \cos\theta_2 \rangle$$
$$\mathbf{r} = \langle L_1 \sin\theta_1 + L_2 \sin\theta_2, L_1 \cos\theta_1 - L_2 \cos\theta_2 \rangle$$

Thus, the x component of $\mathbf{r}$ is $L_1 \sin\theta_1 + L_2 \sin\theta_2$ and the y component is $L_1 \cos\theta_1 - L_2 \cos\theta_2$.

61. Let $L_1 = 5$ and $L_2 = 3$. Find $\mathbf{r}$ for $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$.

SOLUTION In Exercise 60 we showed that

$$\mathbf{r} = \langle L_1 \sin\theta_1 + L_2 \sin\theta_2, L_1 \cos\theta_1 - L_2 \cos\theta_2 \rangle$$

Substituting the given information we obtain

$$\mathbf{r} = \left\langle 5 \sin\frac{\pi}{3} + 3 \sin\frac{\pi}{4}, 5 \cos\frac{\pi}{3} - 3 \cos\frac{\pi}{4} \right\rangle = \left\langle \frac{5\sqrt{3}}{2} + \frac{3\sqrt{2}}{2}, \frac{5}{2} - \frac{3\sqrt{2}}{2} \right\rangle \approx \langle 6.45, 0.38 \rangle$$

62. Let $L_1 = 5$ and $L_2 = 3$. Show that the set of points reachable by the robotic arm with $\theta_1 = \theta_2$ is an ellipse.

SOLUTION Substituting $L_1 = 5$, $L_2 = 3$, and $\theta_1 = \theta_2 = \theta$ in the formula for $\mathbf{r}$ obtained in Exercise 60 we get

$$\begin{aligned} \mathbf{r} &= \langle L_1 \sin\theta_1 + L_2 \sin\theta_2, L_1 \cos\theta_1 - L_2 \cos\theta_2 \rangle \\ &= \langle 5\sin\theta + 3\sin\theta, 5\cos\theta - 3\cos\theta \rangle = \langle 8\sin\theta, 2\cos\theta \rangle \end{aligned}$$

Thus, the x and y components of $\mathbf{r}$ are

$$x = 8\sin\theta, \; y = 2\cos\theta$$

so $\frac{x}{8} = \sin\theta$, $\frac{y}{2} = \cos\theta$. Using the identity $\sin^2\theta + \cos^2\theta = 1$ we get

$$\left(\frac{x}{8}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

which is the formula of an ellipse.

63. Use vectors to prove that the diagonals $\overline{AC}$ and $\overline{BD}$ of a parallelogram bisect each other (Figure 31). *Hint:* Observe that the midpoint of $\overline{BD}$ is the terminal point of $\mathbf{w} + \frac{1}{2}(\mathbf{v} - \mathbf{w})$.

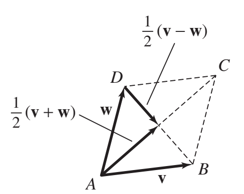
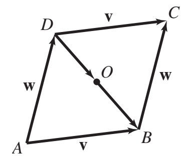


FIGURE 31

SOLUTION We denote by O the midpoint of $\overline{BD}$. Hence,

$$\overrightarrow{DO} = \frac{1}{2}\overrightarrow{DB}$$



Using the Parallelogram Law we have

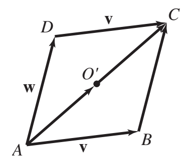
$$\overrightarrow{AO} = \overrightarrow{AD} + \overrightarrow{DO} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DB}$$

Since $\overrightarrow{AD} = \mathbf{w}$ and $\overrightarrow{DB} = \mathbf{v} - \mathbf{w}$ we get

$$\overrightarrow{AO} = \mathbf{w} + \frac{1}{2}(\mathbf{v} - \mathbf{w}) = \frac{\mathbf{w} + \mathbf{v}}{2} \tag{1}$$

On the other hand, $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \mathbf{w} + \mathbf{v}$, hence the midpoint O' of the diagonal $\overline{AC}$ is the terminal point of $\frac{\mathbf{w}+\mathbf{v}}{2}$. That is,

$$\overrightarrow{AO'} = \frac{\mathbf{w} + \mathbf{v}}{2} \tag{2}$$



We combine (1) and (2) to conclude that O and O' are the same point. That is, the diagonal $\overline{AC}$ and $\overline{BD}$ bisect each other.

64. Use vectors to prove that the segments joining the midpoints of opposite sides of a quadrilateral bisect each other (Figure 32). *Hint:* Show that the midpoints of these segments are the terminal points of

$$\frac{1}{4}(2\mathbf{u} + \mathbf{v} + \mathbf{z}) \quad \text{and} \quad \frac{1}{4}(2\mathbf{v} + \mathbf{w} + \mathbf{u})$$

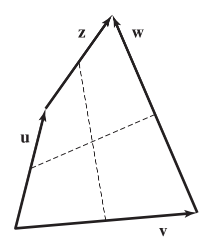
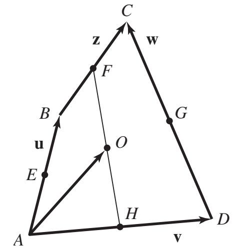


FIGURE 32

SOLUTION We denote by A, B, C, D the corresponding points in the figure and by E, F, G, H the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$, respectively. Also, O is the midpoint of $\overline{FH}$ and O' is the midpoint of $\overline{EG}$.



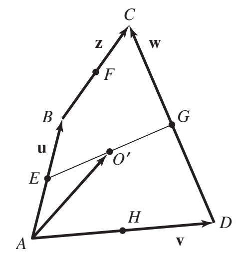
We must show that O and O' are the same point. Using the Parallelogram Law we have

$$\overrightarrow{AO} = \overrightarrow{AH} + \overrightarrow{HO} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\overrightarrow{HF}$$

$$\overrightarrow{HF} = \overrightarrow{HA} + \overrightarrow{AB} + \overrightarrow{BF} = -\frac{1}{2}\mathbf{v} + \mathbf{u} + \frac{1}{2}\mathbf{z}$$

Hence,

$$\overrightarrow{AO} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\left(-\frac{1}{2}\mathbf{v} + \mathbf{u} + \frac{1}{2}\mathbf{z}\right) = \frac{1}{4}\mathbf{v} + \frac{1}{2}\mathbf{u} + \frac{1}{4}\mathbf{z} = \frac{1}{4}(2\mathbf{u} + \mathbf{v} + \mathbf{z}) \tag{1}$$



Similarly,

$$\overrightarrow{AO'} = \overrightarrow{AD} + \overrightarrow{DG} + \overrightarrow{GO'} = \mathbf{v} + \frac{1}{2}\mathbf{w} + \frac{1}{2}\overrightarrow{GE}$$

$$\overrightarrow{GE} = \overrightarrow{GD} + \overrightarrow{DA} + \overrightarrow{AE} = -\frac{1}{2}\mathbf{w} - \mathbf{v} + \frac{1}{2}\mathbf{u}$$

Hence,

$$\overrightarrow{AO'} = \mathbf{v} + \frac{1}{2}\mathbf{w} + \frac{1}{2}\left(-\frac{1}{2}\mathbf{w} - \mathbf{v} + \frac{1}{2}\mathbf{u}\right) = \frac{1}{2}\mathbf{v} + \frac{1}{4}\mathbf{w} + \frac{1}{4}\mathbf{u} = \frac{1}{4}(2\mathbf{v} + \mathbf{w} + \mathbf{u}) \tag{2}$$

To show that $\overrightarrow{AO} = \overrightarrow{AO'}$ we must express $\mathbf{z}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. We have

$$\mathbf{v} + \mathbf{w} - \mathbf{z} - \mathbf{u} = 0 \Rightarrow \mathbf{z} = \mathbf{v} + \mathbf{w} - \mathbf{u}$$

Substituting into (1) we get

$$\overrightarrow{AO} = \frac{1}{4}(2\mathbf{u} + \mathbf{v} + (\mathbf{v} + \mathbf{w} - \mathbf{u})) = \frac{1}{4}(2\mathbf{v} + \mathbf{w} + \mathbf{u}) \tag{3}$$

By (2) and (3) we conclude that $\overrightarrow{AO} = \overrightarrow{AO'}$. It means that the points O and O' are the same point, in other words, the segment $\overline{FH}$ and $\overline{EG}$ bisect each other.

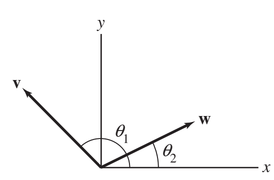
65. Prove that two vectors $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$ are perpendicular if and only if

$$ac + bd = 0$$

SOLUTION Suppose that the vectors $\mathbf{v}$ and $\mathbf{w}$ make angles θ_1 and θ_2, which are not $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, respectively, with the positive x-axis. Then their components satisfy

$$\begin{aligned} a &= \|\mathbf{v}\|\cos\theta_1 \\ b &= \|\mathbf{v}\|\sin\theta_1 \end{aligned} \quad \Rightarrow \quad \frac{b}{a} = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

$$\begin{aligned} c &= \|\mathbf{w}\|\cos\theta_2 \\ d &= \|\mathbf{w}\|\sin\theta_2 \end{aligned} \quad \Rightarrow \quad \frac{d}{c} = \frac{\sin\theta_2}{\cos\theta_2} = \tan\theta_2$$



That is, the vectors $\mathbf{v}$ and $\mathbf{w}$ are on the lines with slopes $\frac{b}{a}$ and $\frac{d}{c}$, respectively. The lines are perpendicular if and only if their slopes satisfy

$$\frac{b}{a} \cdot \frac{d}{c} = -1 \quad \Rightarrow \quad bd = -ac \quad \Rightarrow \quad ac + bd = 0$$

We now consider the case where one of the vectors, say $\mathbf{v}$, is perpendicular to the x-axis. In this case $a = 0$, and the vectors are perpendicular if and only if $\mathbf{w}$ is parallel to the x-axis, that is, $d = 0$. So $ac + bd = 0 \cdot c + b \cdot 0 = 0$.

12.2 Vectors in Three Dimensions (LT Section 13.2)

Preliminary Questions

1. What is the terminal point of the vector $\mathbf{v} = \langle 3, 2, 1\rangle$ based at the point $P = (1, 1, 1)$?

SOLUTION We denote the terminal point by $Q = (a, b, c)$. Then by the definition of components of a vector, we have

$$\langle 3, 2, 1\rangle = \langle a - 1, b - 1, c - 1\rangle$$

Equivalent vectors have equal components respectively, thus,

$$\begin{aligned} 3 &= a - 1 & & a = 4 \\ 2 &= b - 1 & \Rightarrow \quad & b = 3 \\ 1 &= c - 1 & & c = 2 \end{aligned}$$

The terminal point of $\mathbf{v}$ is thus $Q = (4, 3, 2)$.

2. What are the components of the vector $\mathbf{v} = \langle 3, 2, 1\rangle$ based at the point $P = (1, 1, 1)$?

SOLUTION The component of $\mathbf{v} = \langle 3, 2, 1\rangle$ are $\langle 3, 2, 1\rangle$ regardless of the base point. The component of $\mathbf{v}$ and the base point $P = (1, 1, 1)$ determine the head $Q = (a, b, c)$ of the vector, as found in the previous exercise.

3. If $\mathbf{v} = -3\mathbf{w}$, then (choose the correct answer):

(a) $\mathbf{v}$ and $\mathbf{w}$ are parallel.

(b) $\mathbf{v}$ and $\mathbf{w}$ point in the same direction.

SOLUTION The vectors $\mathbf{v}$ and $\mathbf{w}$ lie on parallel lines, hence these vectors are parallel. Since $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$ by a negative scalar, $\mathbf{v}$ and $\mathbf{w}$ point in opposite directions. Thus, (a) is correct and (b) is not.

4. Which of the following is a direction vector for the line through $P = (3, 2, 1)$ and $Q = (1, 1, 1)$?

(a) $\langle 3, 2, 1\rangle$ **(b)** $\langle 1, 1, 1\rangle$ **(c)** $\langle 2, 1, 0\rangle$

SOLUTION Any vector that is parallel to the vector $\overrightarrow{PQ}$ is a direction vector for the line through P and Q. We compute the vector $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = \langle 1 - 3, 1 - 2, 1 - 1\rangle = \langle -2, -1, 0\rangle.$$

The vectors $\langle 3, 2, 1\rangle$ and $\langle 1, 1, 1\rangle$ are not constant multiples of $\overrightarrow{PQ}$, hence they are not parallel to $\overrightarrow{PQ}$. However $\langle 2, 1, 0\rangle = -1\langle -2, -1, 0\rangle = -\overrightarrow{PQ}$, hence the vector $\langle 2, 1, 0\rangle$ is parallel to $\overrightarrow{PQ}$. Therefore, the vector $\langle 2, 1, 0\rangle$ is a direction vector for the line through P and Q.

5. How many different direction vectors does a line have?

SOLUTION All the vectors that are parallel to a line are also direction vectors for that line. Therefore, there are infinitely many direction vectors for a line.

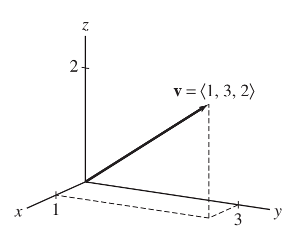
6. True or false? If $\mathbf{v}$ is a direction vector for a line $\mathcal{L}$, then $-\mathbf{v}$ is also a direction vector for $\mathcal{L}$.

SOLUTION True. Every vector that is parallel to $\mathbf{v}$ is a direction vector for the line L. Since $-\mathbf{v}$ is parallel to $\mathbf{v}$, it is also a direction vector for L.

Exercises

1. Sketch the vector $\mathbf{v} = \langle 1, 3, 2\rangle$ and compute its length.

SOLUTION The vector $\mathbf{v} = \langle 1, 3, 2\rangle$ is shown in the following figure:



The length of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

2. Let $\mathbf{v} = \overrightarrow{P_0Q_0}$, where $P_0 = (1, -2, 5)$ and $Q_0 = (0, 1, -4)$. Which of the following vectors (with tail P and head Q) are equivalent to $\mathbf{v}$?

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
P	$(1, 2, 4)$	$(1, 5, 4)$	$(0, 0, 0)$	$(2, 4, 5)$
Q	$(0, 5, -5)$	$(0, -8, 13)$	$(-1, 3, -9)$	$(1, 7, 4)$

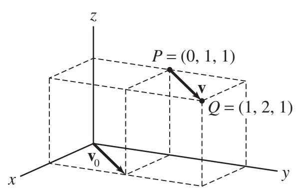
SOLUTION We compute the vectors $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$\begin{aligned}\mathbf{v} = \overrightarrow{P_0Q_0} &= \langle 0-1, 1-(-2), -4-5\rangle = \langle -1, 3, -9\rangle \\ \mathbf{v}_1 &= \langle 0-1, 5-2, -5-4\rangle = \langle -1, 3, -9\rangle \\ \mathbf{v}_2 &= \langle 0-1, -8-5, 13-4\rangle = \langle -1, -13, 9\rangle \\ \mathbf{v}_3 &= \langle -1-0, 3-0, -9-0\rangle = \langle -1, 3, -9\rangle \\ \mathbf{v}_4 &= \langle 1-2, 7-4, 4-5\rangle = \langle -1, 3, -1\rangle\end{aligned}$$

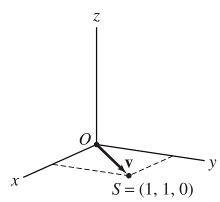
The vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ are equivalent to $\mathbf{v}$.

3. Sketch the vector $\mathbf{v} = \langle 1, 1, 0\rangle$ based at $P = (0, 1, 1)$. Describe this vector in the form $\overrightarrow{PQ}$ for some point Q, and sketch the vector $\mathbf{v}_0$ based at the origin equivalent to $\mathbf{v}$.

SOLUTION The vector $\mathbf{v} = \langle 1, 1, 0\rangle$ based at $P = (0, 1, 1)$ is shown in the figure:



The head Q of the vector $\mathbf{v} = \overrightarrow{PQ}$ is at the point $Q = (0+1, 1+1, 1+0) = (1, 2, 1)$.



The vector $\mathbf{v}_0$ based at the origin and equivalent to $\mathbf{v}$ is

$$v_0 = \langle 1, 1, 0\rangle = \overrightarrow{OS}, \text{ where } S = (1, 1, 0).$$

4. Determine whether the coordinate systems (A)–(C) in Figure 17 satisfy the right-hand rule.

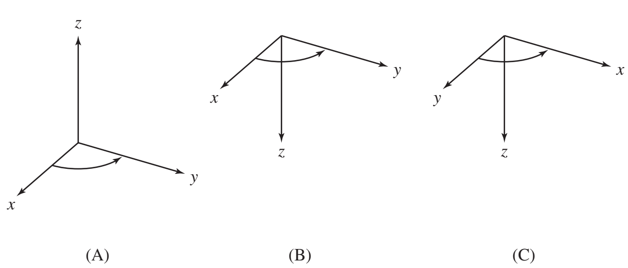


FIGURE 17

SOLUTION The coordinate systems (A) and (C) satisfy the right-hand rule, since when the right hand is positioned so that the fingers curl from the positive x-axis toward the positive y-axis, the thumb points in the positive z-direction. Similarly, system (B) does not satisfy the right-hand rule.

In Exercises 5–8, find the components of the vector $\overrightarrow{PQ}$.

5. $P = (1, 0, 1)$, $\quad Q = (2, 1, 0)$

SOLUTION By the definition of the vector components we have

$$\overrightarrow{PQ} = \langle 2-1, 1-0, 0-1\rangle = \langle 1, 1, -1\rangle$$

6. $P = (-3, -4, 2)$, $Q = (1, -4, 3)$

SOLUTION The components of the vector $\overrightarrow{PQ}$ are

$$\overrightarrow{PQ} = \langle 1 - (-3), -4 - (-4), 3 - 2 \rangle = \langle 4, 0, 1 \rangle$$

7. $P = (4, 6, 0)$, $Q = \left(-\frac{1}{2}, \frac{9}{2}, 1\right)$

SOLUTION Using the definition of vector components we have

$$\overrightarrow{PQ} = \left\langle -\frac{1}{2} - 4, \frac{9}{2} - 6, 1 - 0 \right\rangle = \left\langle -\frac{9}{2}, -\frac{3}{2}, 1 \right\rangle$$

8. $P = \left(-\frac{1}{2}, \frac{9}{2}, 1\right)$, $Q = (4, 6, 0)$

SOLUTION The components of the vector with the head at $Q = (4, 6, 0)$ and tail at $P = \left(-\frac{1}{2}, \frac{9}{2}, 1\right)$ are

$$\overrightarrow{PQ} = \left\langle 4 - \left(-\frac{1}{2}\right), 6 - \frac{9}{2}, 0 - 1 \right\rangle = \left\langle \frac{9}{2}, \frac{3}{2}, -1 \right\rangle.$$

In Exercises 9–12, let $R = (1, 4, 3)$.

9. Calculate the length of $\overrightarrow{OR}$.

SOLUTION The length of $\overrightarrow{OR}$ is the distance from $R = (1, 4, 3)$ to the origin. That is,

$$\|\overrightarrow{OR}\| = \sqrt{(1-0)^2 + (4-0)^2 + (3-0)^2} = \sqrt{26} \approx 5.1.$$

10. Find the point Q such that $\mathbf{v} = \overrightarrow{RQ}$ has components $\langle 4, 1, 1 \rangle$, and sketch $\mathbf{v}$.

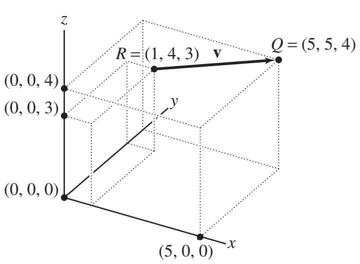
SOLUTION Denoting $Q = (x_0, y_0, z_0)$ we have

$$\overrightarrow{RQ} = \langle x_0 - 1, y_0 - 4, z_0 - 3 \rangle = \langle 4, 1, 1 \rangle$$

Equating corresponding components, we get

$$\begin{aligned} x_0 - 1 &= 4 \\ y_0 - 4 &= 1 \quad \Rightarrow \quad x_0 = 5,\ y_0 = 5,\ z_0 = 4 \\ z_0 - 3 &= 1 \end{aligned}$$

The point Q is, thus, $Q = (5, 5, 4)$.



11. Find the point P such that $\mathbf{w} = \overrightarrow{PR}$ has components $\langle 3, -2, 3 \rangle$, and sketch $\mathbf{w}$.

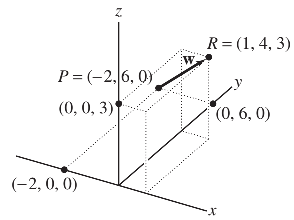
SOLUTION Denoting $P = (x_0, y_0, z_0)$ we get

$$\overrightarrow{PR} = \langle 1 - x_0, 4 - y_0, 3 - z_0 \rangle = \langle 3, -2, 3 \rangle$$

Equating corresponding components gives

$$\begin{aligned} 1 - x_0 &= 3 \\ 4 - y_0 &= -2 \quad \Rightarrow \quad x_0 = -2,\ y_0 = 6,\ z_0 = 0 \\ 3 - z_0 &= 3 \end{aligned}$$

The point P is, thus, $P = (-2, 6, 0)$.



12. Find the components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2, 2)$.

SOLUTION The components of $\mathbf{u} = \overrightarrow{PR}$ where $P = (1, 2, 2)$ and $R = (1, 4, 3)$ are

$$\mathbf{u} = \overrightarrow{PR} = \langle 1 - 1, 4 - 2, 3 - 2 \rangle = \langle 0, 2, 1 \rangle$$

13. Let $\mathbf{v} = \langle 4, 8, 12 \rangle$. Which of the following vectors is parallel to $\mathbf{v}$? Which point in the same direction?

(a) $\langle 2, 4, 6 \rangle$ **(b)** $\langle -1, -2, 3 \rangle$

(c) $\langle -7, -14, -21 \rangle$ **(d)** $\langle 6, 10, 14 \rangle$

SOLUTION A vector is parallel to $\mathbf{v}$ if it is a scalar multiple of $\mathbf{v}$. It points in the same direction if the multiplying scalar is positive. Using these properties we obtain the following answer:

(a) $\langle 2, 4, 6 \rangle = \frac{1}{2}\mathbf{v} \Rightarrow$ The vectors are parallel and point in the same direction.

(b) $\langle -1, -2, 3 \rangle$ is not a scalar multiple of $\mathbf{v}$, hence these vectors are not parallel.

(c) $\langle -7, -14, -21 \rangle = -\frac{7}{4}\mathbf{v} \Rightarrow$ The vectors are parallel but point in opposite directions.

(d) $\langle 6, 10, 14 \rangle$ is not a constant multiple of $\mathbf{v}$, hence these vectors are not parallel.

In Exercises 14–17, determine whether $\overrightarrow{AB}$ is equivalent to $\overrightarrow{PQ}$.

14. $A = (1, 1, 1)$ $B = (3, 3, 3)$
$P = (1, 4, 5)$ $Q = (3, 6, 7)$

SOLUTION Two vectors are equivalent if one is a translation of the other, that is, if they have the same components. We compute $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\left.\begin{aligned}\overrightarrow{AB} &= \langle 3 - 1, 3 - 1, 3 - 1 \rangle = \langle 2, 2, 2 \rangle \\ \overrightarrow{PQ} &= \langle 3 - 1, 6 - 4, 7 - 5 \rangle = \langle 2, 2, 2 \rangle\end{aligned}\right. \quad \Rightarrow \quad \text{The vectors are equivalent.}$$

15. $A = (1, 4, 1)$ $B = (-2, 2, 0)$
$P = (2, 5, 7)$ $Q = (-3, 2, 1)$

SOLUTION We compute the two vectors:

$$\overrightarrow{AB} = \langle -2 - 1, 2 - 4, 0 - 1 \rangle = \langle -3, -2, -1 \rangle$$

$$\overrightarrow{PQ} = \langle -3 - 2, 2 - 5, 1 - 7 \rangle = \langle -5, -3, -6 \rangle$$

The components of $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are not equal, hence they are not a translate of each other, that is, the vectors are not equivalent.

16. $A = (0, 0, 0)$ $B = (-4, 2, 3)$
$P = (4, -2, -3)$ $Q = (0, 0, 0)$

SOLUTION We compute $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\left.\begin{aligned}\overrightarrow{AB} &= \langle -4 - 0, 2 - 0, 3 - 0 \rangle = \langle -4, 2, 3 \rangle \\ \overrightarrow{PQ} &= \langle 0 - 4, 0 - (-2), 0 - (-3) \rangle = \langle -4, 2, 3 \rangle\end{aligned}\right. \quad \Rightarrow \quad \text{The vectors are equivalent.}$$

17. $A = (1, 1, 0)$ $B = (3, 3, 5)$
$P = (2, -9, 7)$ $Q = (4, -7, 13)$

SOLUTION The vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are the following vectors:

$$\overrightarrow{AB} = \langle 3 - 1, 3 - 1, 5 - 0 \rangle = \langle 2, 2, 5 \rangle$$

$$\overrightarrow{PQ} = \langle 4 - 2, -7 - (-9), 13 - 7 \rangle = \langle 2, 2, 6 \rangle$$

The z-coordinates of the vectors are not equal, hence the vectors are not equivalent.

In Exercises 18–23, calculate the linear combinations.

18. $5\langle 2, 2, -3\rangle + 3\langle 1, 7, 2\rangle$

SOLUTION Using vector algebra we get

$$5\langle 2, 2, -3\rangle + 3\langle 1, 7, 2\rangle = \langle 10, 10, -15\rangle + \langle 3, 21, 6\rangle = \langle 13, 31, -9\rangle.$$

19. $-2\langle 8, 11, 3\rangle + 4\langle 2, 1, 1\rangle$

SOLUTION Using the operations of vector addition and scalar multiplication we have

$$-2\langle 8, 11, 3\rangle + 4\langle 2, 1, 1\rangle = \langle -16, -22, -6\rangle + \langle 8, 4, 4\rangle = \langle -8, -18, -2\rangle.$$

20. $6(4\mathbf{j} + 2\mathbf{k}) - 3(2\mathbf{i} + 7\mathbf{k})$

SOLUTION Using vector algebra, we have

$$6(4\mathbf{j} + 2\mathbf{k}) - 3(2\mathbf{i} + 7\mathbf{k}) = (24\mathbf{j} + 12\mathbf{k}) - (6\mathbf{i} + 21\mathbf{k}) = -6\mathbf{i} + 24\mathbf{j} - 9\mathbf{k}.$$

21. $\frac{1}{2}\langle 4, -2, 8\rangle - \frac{1}{3}\langle 12, 3, 3\rangle$

SOLUTION Using the operations on vectors we have

$$\frac{1}{2}\langle 4, -2, 8\rangle - \frac{1}{3}\langle 12, 3, 3\rangle = \langle 2, -1, 4\rangle - \langle 4, 1, 1\rangle = \langle -2, -2, 3\rangle.$$

22. $5(\mathbf{i} + 2\mathbf{j}) - 3(2\mathbf{j} + \mathbf{k}) + 7(2\mathbf{k} - \mathbf{i})$

SOLUTION Using the operations on vectors we have

$$5(\mathbf{i} + 2\mathbf{j}) - 3(2\mathbf{j} + \mathbf{k}) + 7(2\mathbf{k} - \mathbf{i}) = 5\mathbf{i} + 10\mathbf{j} - 6\mathbf{j} - 3\mathbf{k} + 14\mathbf{k} - 7\mathbf{i} = -2\mathbf{i} + 4\mathbf{j} + 11\mathbf{k}.$$

23. $4\langle 6, -1, 1\rangle - 2\langle 1, 0, -1\rangle + 3\langle -2, 1, 1\rangle$

SOLUTION Using the operations of vector addition and scalar multiplication we have

$$\begin{aligned} 4\langle 6, -1, 1\rangle - 2\langle 1, 0, -1\rangle + 3\langle -2, 1, 1\rangle &= \langle 24, -4, 4\rangle + \langle -2, 0, 2\rangle + \langle -6, 3, 3\rangle \\ &= \langle 16, -1, 9\rangle. \end{aligned}$$

In Exercises 24–27, find the given vector.

24. $\mathbf{e_v}$, where $\mathbf{v} = \langle 1, 1, 2\rangle$

SOLUTION $\mathbf{e_v}$ is the vector

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We find the length of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

Hence,

$$\mathbf{e_v} = \frac{1}{\sqrt{6}}\langle 1, 1, 2\rangle = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

25. $\mathbf{e_w}$, where $\mathbf{w} = \langle 4, -2, -1\rangle$

SOLUTION We first find the length of $\mathbf{w}$:

$$\|\mathbf{w}\| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

Hence,

$$\mathbf{e_w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \left\langle \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \right\rangle$$

26. Unit vector in the direction of $\mathbf{u} = \langle 1, 0, 7 \rangle$

SOLUTION A unit vector in the direction of $\mathbf{u} = \langle 1, 0, 7 \rangle$ is the vector

$$\mathbf{e_u} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$$

We compute the length of $\mathbf{u}$:

$$\|\mathbf{u}\| = \sqrt{1^2 + 0^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$$

Hence,

$$\mathbf{e_u} = \frac{1}{5\sqrt{2}}\langle 1, 0, 7 \rangle = \left\langle \frac{1}{5\sqrt{2}}, 0, \frac{7}{5\sqrt{2}} \right\rangle$$

27. Unit vector in the direction opposite to $\mathbf{v} = \langle -4, 4, 2 \rangle$

SOLUTION A unit vector in the direction opposite to $\mathbf{v} = \langle -4, 4, 2 \rangle$ is the following vector:

$$-\mathbf{e_v} = -\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We compute the length of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 4^2 + 2^2} = 6$$

The desired vector is, thus,

$$-\mathbf{e_v} = -\frac{1}{6}\langle -4, 4, 2 \rangle = \left\langle \frac{-4}{-6}, \frac{4}{-6}, \frac{2}{-6} \right\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$$

28. Sketch the following vectors, and find their components and lengths.

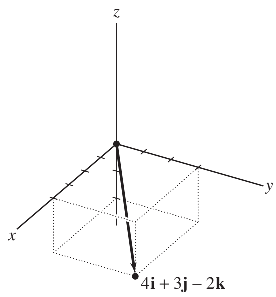
(a) $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ **(b)** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(c) $4\mathbf{j} + 3\mathbf{k}$ **(d)** $12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

SOLUTION By the definition of the standard basis vectors in R^3 and the definition of vector length, we obtain the following answers:

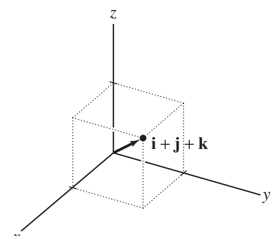
(a) $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \langle 4, 3, -2 \rangle$

$\|4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$



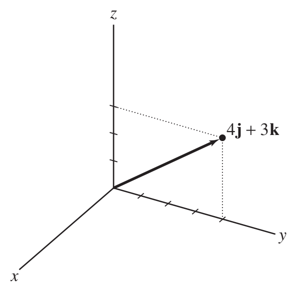
(b) $\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle$

$\|\mathbf{i} + \mathbf{j} + \mathbf{k}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$



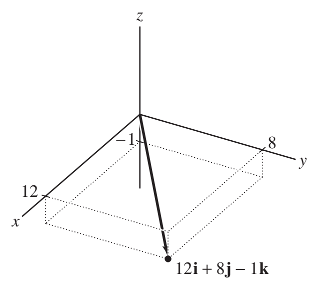
(c) $4\mathbf{j} + 3\mathbf{k} = \langle 0, 4, 3 \rangle$

$\|4\mathbf{j} + 3\mathbf{k}\| = \sqrt{0^2 + 4^2 + 3^2} = 5$



(d) $12\mathbf{i} + 8\mathbf{j} - \mathbf{k} = \langle 12, 8, -1 \rangle$

$\|12\mathbf{i} + 8\mathbf{j} - \mathbf{k}\| = \sqrt{12^2 + 8^2 + (-1)^2} = \sqrt{209}$



In Exercises 29–36, find a vector parametrization for the line with the given description.

29. Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3 \rangle$

SOLUTION The vector parametrization for the line is

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v}$$

Inserting the given data we get

$$\mathbf{r}(t) = \langle 1, 2, -8 \rangle + t\langle 2, 1, 3 \rangle = \langle 1 + 2t, 2 + t, -8 + 3t \rangle$$

30. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = \langle 1, 0, 1 \rangle$

SOLUTION A vector parametrization of the line is

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 4, 0, 8 \rangle + t\langle 1, 0, 1 \rangle = \langle 4 + t, 0, 8 + t \rangle$$

31. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k}$

SOLUTION Since $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k} = \langle 7, 0, 4 \rangle$ we obtain the following parametrization:

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 4, 0, 8 \rangle + t\langle 7, 0, 4 \rangle = \langle 4 + 7t, 0, 8 + 4t \rangle$$

32. Passes through O, direction vector $\mathbf{v} = \langle 3, -1, -4 \rangle$

SOLUTION Since $\mathbf{v} = \langle 3, -1, -4 \rangle$ and $P = O$, we obtain the following parametrization:

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 0, 0, 0 \rangle + t\langle 3, -1, -4 \rangle = \langle 3t, -t, -4t \rangle$$

33. Passes through $(1, 1, 1)$ and $(3, -5, 2)$

SOLUTION We use the equation of the line through two points P and Q:

$$\mathbf{r}(t) = (1 - t)\overrightarrow{OP} + t\overrightarrow{OQ}$$

Since $\overrightarrow{OP} = \langle 1, 1, 1 \rangle$ and $\overrightarrow{OQ} = \langle 3, -5, 2 \rangle$ we obtain

$$\mathbf{r}(t) = (1 - t)\langle 1, 1, 1 \rangle + t\langle 3, -5, 2 \rangle = \langle 1 - t, 1 - t, 1 - t \rangle + \langle 3t, -5t, 2t \rangle = \langle 1 + 2t, 1 - 6t, 1 + t \rangle$$

34. Passes through $(-2, 0, -2)$ and $(4, 3, 7)$

SOLUTION Using the equation of the line through two points P and Q, with $\overrightarrow{OP} = \langle -2, 0, -2 \rangle$ and $\overrightarrow{OQ} = \langle 4, 3, 7 \rangle$ we obtain

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1 - t)\langle -2, 0, -2 \rangle + t\langle 4, 3, 7 \rangle \\ &= \langle -2(1 - t), 0, -2(1 - t) \rangle + \langle 4t, 3t, 7t \rangle = \langle -2 + 6t, 3t, -2 + 9t \rangle\end{aligned}$$

35. Passes through O and $(4, 1, 1)$

SOLUTION By the equation of the line through two points we get

$$\mathbf{r}(t) = (1-t)\langle 0, 0, 0\rangle + t\langle 4, 1, 1\rangle = \langle 0, 0, 0\rangle + \langle 4t, t, t\rangle = \langle 4t, t, t\rangle$$

36. Passes through $(1, 1, 1)$ parallel to the line through $(2, 0, -1)$ and $(4, 1, 3)$

SOLUTION The direction vector is $\mathbf{v} = \langle 4-2, 1-0, 3-(-1)\rangle = \langle 2, 1, 4\rangle$. Hence, using the equation of a line we obtain

$$\mathbf{r}(t) = \langle 1, 1, 1\rangle + t\langle 2, 1, 4\rangle = \langle 1+2t, 1+t, 1+4t\rangle$$

In Exercises 37–40, find parametric equations for the lines with the given description.

37. Perpendicular to the xy-plane, passes through the origin

SOLUTION A direction vector for the line is a vector parallel to the z-axis, for instance, we may choose $\mathbf{v} = \langle 0, 0, 1\rangle$. The line passes through the origin $(0, 0, 0)$, hence we obtain the following parametrization:

$$\mathbf{r}(t) = \langle 0, 0, 0\rangle + t\langle 0, 0, 1\rangle = \langle 0, 0, t\rangle$$

or $x = 0$, $y = 0$, $z = t$.

38. Perpendicular to the yz-plane, passes through $(0, 0, 2)$

SOLUTION The direction vector is parallel to the x-axis. We may choose $\mathbf{v} = \langle 1, 0, 0\rangle$. Also $\overrightarrow{OP_0} = \langle 0, 0, 2\rangle$ so we obtain

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v} = \langle 0, 0, 2\rangle + t\langle 1, 0, 0\rangle = \langle t, 0, 2\rangle$$

or $x = t$, $y = 0$, $z = 2$.

39. Parallel to the line through $(1, 1, 0)$ and $(0, -1, -2)$, passes through $(0, 0, 4)$

SOLUTION The direction vector is $\mathbf{v} = \langle 0-1, -1-1, -2-0\rangle = \langle -1, -2, -2\rangle$. Hence, using the equation of a line we obtain

$$\mathbf{r}(t) = \langle 0, 0, 4\rangle + t\langle -1, -2, -2\rangle = \langle -t, -2t, 4-2t\rangle$$

40. Passes through $(1, -1, 0)$ and $(0, -1, 2)$

SOLUTION Using the equation of the line through two points P and Q, with $\overrightarrow{OP} = \langle 1, -1, 0\rangle$ and $\overrightarrow{OQ} = \langle 0, -1, 2\rangle$ we obtain

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\overrightarrow{OP} + t\overrightarrow{OQ} = (1-t)\langle 1, -1, 0\rangle + t\langle 0, -1, 2\rangle \\ &= \langle (1-t), -(1-t), 0\rangle + \langle 0, -t, 2t\rangle = \langle 1-t, -1, 2t\rangle\end{aligned}$$

41. Which of the following is a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the xz-plane (Figure 18)?

(a) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 1, 0, 1\rangle$
(b) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 0, 0, 1\rangle$
(c) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 0, 1, 0\rangle$
(d) $\mathbf{r}(t) = \langle 4, 9, 8\rangle + t\langle 1, 1, 0\rangle$

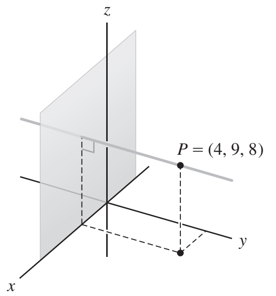


FIGURE 18

SOLUTION Since the direction vector must be perpendicular to the xz-plane, then the direction vector for the line must be parallel to $\mathbf{j}$, which is only satisfied by solution (c).

42. Find a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the yz-plane.

SOLUTION The direction vector is parallel to the x-axis. We may choose $\mathbf{v} = \langle 1, 0, 0\rangle$. Also $\overrightarrow{OP} = \langle 4, 9, 8\rangle$ so we obtain

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 4, 9, 8\rangle + t\langle 1, 0, 0\rangle = \langle 4+t, 9, 8\rangle$$

or $x = 4+t$, $y = 9$, $z = 8$.

In Exercises 43–46, let $P = (2, 1, -1)$ *and* $Q = (4, 7, 7)$*. Find the coordinates of each of the following.*

43. The midpoint of $\overline{PQ}$

SOLUTION We first parametrize the line through $P = (2, 1, -1)$ and $Q = (4, 7, 7)$:

$$\mathbf{r}(t) = (1-t)\langle 2, 1, -1\rangle + t\langle 4, 7, 7\rangle = \langle 2+2t, 1+6t, -1+8t\rangle$$

The midpoint of $\overline{PQ}$ occurs at $t = \frac{1}{2}$, that is,

$$\text{midpoint} = \mathbf{r}\left(\frac{1}{2}\right) = \left\langle 2 + 2\cdot\frac{1}{2}, 1 + 6\cdot\frac{1}{2}, -1 + 8\cdot\frac{1}{2}\right\rangle = \langle 3, 4, 3\rangle$$

The midpoint of $\overline{PQ}$ is the terminal point of the vector $\mathbf{r}(t)$, that is, $(3, 4, 3)$. (One could also use the midpoint formula to arrive at the same solution.)

44. The point on $\overline{PQ}$ lying two-thirds of the way from P to Q

SOLUTION We first parametrize the line through $P = (2, 1, -1)$ and $Q = (4, 7, 7)$:

$$\mathbf{r}(t) = (1-t)\langle 2, 1, -1\rangle + t\langle 4, 7, 7\rangle = \langle 2+2t, 1+6t, -1+8t\rangle$$

The point on $\overline{PQ}$ lying two-thirds of the way from P to Q is at $t = \frac{2}{3}$. That is,

$$\mathbf{r}\left(\frac{2}{3}\right) = \left\langle 2 + 2\cdot\frac{2}{3}, 1 + 6\cdot\frac{2}{3}, -1 + 8\cdot\frac{2}{3}\right\rangle = \left\langle \frac{10}{3}, 5, \frac{13}{3}\right\rangle$$

The desired point is the head of the vector $\mathbf{r}(\frac{2}{3})$ which is $\left\langle \frac{10}{3}, 5, \frac{13}{3}\right\rangle$.

45. The point R such that Q is the midpoint of $\overline{PR}$

SOLUTION We denote $R = (x_0, y_0, z_0)$. By the formula for the midpoint of a segment we have

$$\langle 4, 7, 7\rangle = \left\langle \frac{2+x_0}{2}, \frac{1+y_0}{2}, \frac{-1+z_0}{2}\right\rangle$$

Equating corresponding components we get

$$\begin{aligned} 4 &= \frac{2+x_0}{2} \\ 7 &= \frac{1+y_0}{2} \\ 7 &= \frac{-1+z_0}{2} \end{aligned} \quad \Rightarrow \quad x_0 = 6,\ y_0 = 13,\ z_0 = 15 \quad \Rightarrow \quad R = (6, 13, 15)$$

46. The two points on the line through $\overline{PQ}$ whose distance from P is twice its distance from Q

SOLUTION In Exercise 44 we showed that the parametric equation of the line through P and Q is

$$\mathbf{r}(t) = \langle 2+2t, 1+6t, -1+8t\rangle \tag{1}$$

Hence an arbitrary point R on the line is $R = (2+2t, 1+6t, -1+8t)$. We must find the points R on the line satisfying

$$\|\overrightarrow{PR}\| = 2\|\overrightarrow{QR}\| \tag{2}$$

We compute the lengths of the vectors:

$$\begin{aligned} \|\overrightarrow{PR}\| &= \|\langle 2t, 6t, 8t\rangle\| = |t|\sqrt{2^2+6^2+8^2} = \sqrt{104}|t| \\ \|\overrightarrow{QR}\| &= \|\langle -2+2t, -6+6t, -8+8t\rangle\| = \|(-1+t)\langle 2, 6, 8\rangle\| \\ &= |-1+t|\sqrt{2^2+6^2+8^2} = |-1+t|\sqrt{104} \end{aligned}$$

Substituting in (2) and solving for t gives $\sqrt{104}|t| = 2\sqrt{104}|-1+t|$ and so $|t| = 2|t-1|$. Thus, either $t = 2(t-1)$ (and so $t = 2$) or $t = -2(t-1)$ (and so $t = \frac{2}{3}$). We now compute $\mathbf{r}(t)$ for these values of t. Using (1) we get

$$\begin{aligned} \mathbf{r}(2) &= \langle 2+2\cdot 2, 1+6\cdot 2, -1+8\cdot 2\rangle = \langle 6, 13, 15\rangle \\ \mathbf{r}\left(\frac{2}{3}\right) &= \left\langle 2+2\cdot\frac{2}{3}, 1+6\cdot\frac{2}{3}, -1+8\cdot\frac{2}{3}\right\rangle = \left\langle \frac{10}{3}, 5, \frac{13}{3}\right\rangle \end{aligned}$$

The desired points are the terminal points on these vectors. That is, $(6, 13, 15)$ and $\left(\frac{10}{3}, 5, \frac{13}{3}\right)$.

47. Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ define the same line, where

$$\mathbf{r}_1(t) = \langle 3, -1, 4\rangle + t\langle 8, 12, -6\rangle$$
$$\mathbf{r}_2(t) = \langle 11, 11, -2\rangle + t\langle 4, 6, -3\rangle$$

Hint: Show that $\mathbf{r}_2$ passes through $(3, -1, 4)$ and that the direction vectors for $\mathbf{r}_1$ and $\mathbf{r}_2$ are parallel.

SOLUTION We observe first that the direction vectors of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are multiples of each other:

$$\langle 8, 12, -6\rangle = 2\langle 4, 6, -3\rangle$$

Therefore $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are parallel. To show they coincide, it suffices to prove that they share a point in common, so we verify that $\mathbf{r}_1(0) = \langle 3, -1, 4\rangle$ lies on $\mathbf{r}_2(t)$ by solving for t:

$$\begin{aligned}\langle 3, -1, 4\rangle &= \langle 11, 11, -2\rangle + t\langle 4, 6, -3\rangle\\ \langle 3, -1, 4\rangle - \langle 11, 11, -2\rangle &= t\langle 4, 6, -3\rangle\\ \langle -8, -12, 6\rangle &= t\langle 4, 6, -3\rangle\end{aligned}$$

This equation is satisfied for $t = -2$, so $\mathbf{r}_1$ and $\mathbf{r}_2$ coincide.

48. Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ define the same line, where

$$\mathbf{r}_1(t) = t\langle 2, 1, 3\rangle, \qquad \mathbf{r}_2(t) = \langle -6, -3, -9\rangle + t\langle 8, 4, 12\rangle$$

SOLUTION Note that $\mathbf{r}_1(-3) = \mathbf{r}_2(0)$ and $\mathbf{r}_1(0) = \mathbf{r}_2(3/4)$. Since both lines go through these two distinct points, then they must be the same line. (One could also solve this problem by showing that they share a point and have parallel direction vectors.)

49. Find two different vector parametrizations of the line through $P = (5, 5, 2)$ with direction vector $\mathbf{v} = \langle 0, -2, 1\rangle$.

SOLUTION Two different parameterizations are

$$\mathbf{r}_1(t) = \langle 5, 5, 2\rangle + t\langle 0, -2, 1\rangle$$
$$\mathbf{r}_2(t) = \langle 5, 5, 2\rangle + t\langle 0, -20, 10\rangle$$

50. Find the point of intersection of the lines $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle -3, 1, 0\rangle$ and $\mathbf{s}(t) = \langle 0, 1, 1\rangle + t\langle 2, 0, 1\rangle$.

SOLUTION The two lines intersect if there exist parameter values t_1 and t_2 such that

$$\begin{aligned}\langle 1, 0, 0\rangle + t_1\langle -3, 1, 0\rangle &= \langle 0, 1, 1\rangle + t_2\langle 2, 0, 1\rangle\\ \langle 1, 0, 0\rangle + \langle -3t_1, t_1, 0\rangle &= \langle 0, 1, 1\rangle + \langle 2t_2, 0, t_2\rangle\\ \langle 1 - 3t_1, t_1, 0\rangle &= \langle 2t_2, 1, 1 + t_2\rangle\end{aligned}$$

We obtain the following equations for the components of the two vectors:

$$1 - 3t_1 = 2t_2 \tag{1}$$
$$t_1 = 1 \tag{2}$$
$$0 = 1 + t_2 \tag{3}$$

The third equation yields $t_2 = -1$ and the second yields $t_1 = 1$. We now must check whether these values satisfy the first equation:

$$\begin{aligned}1 - 3t_1 &= 1 - 3 = -2\\ 2t_2 &= 2 \cdot (-1) = -2\end{aligned} \quad \Rightarrow \quad 1 - 3t_1 = 2t_2 \quad \text{for} \quad t_1 = 1,\ t_2 = -1$$

Therefore, $t_1 = 1$ and $t_2 = -1$ are solutions of the equations in (1)–(3). To find the point of intersection of the given lines we first set one of the solutions t_1 or t_2 in the equation of corresponding line. Setting $t_1 = 1$ in $\langle 1, 0, 0\rangle + t\langle -3, 1, 0\rangle$ yields

$$\langle 1, 0, 0\rangle + 1 \cdot \langle -3, 1, 0\rangle = \langle -2, 1, 0\rangle$$

The head $(-2, 1, 0)$ is the required point.

51. Show that the lines $\mathbf{r}_1(t) = \langle -1, 2, 2\rangle + t\,\langle 4, -2, 1\rangle$ and $\mathbf{r}_2(t) = \langle 0, 1, 1\rangle + t\,\langle 2, 0, 1\rangle$ do not intersect.

SOLUTION The two lines intersect if there exist parameter values t_1 and t_2 such that

$$\begin{aligned}\langle -1, 2, 2\rangle + t_1\langle 4, -2, 1\rangle &= \langle 0, 1, 1\rangle + t_2\langle 2, 0, 1\rangle\\ \langle -1 + 4t_1, 2 - 2t_1, 2 + t_1\rangle &= \langle 2t_2, 1, 1 + t_2\rangle\end{aligned}$$

Equating corresponding components yields

$$\begin{aligned}-1 + 4t_1 &= 2t_2\\ 2 - 2t_1 &= 1\\ 2 + t_1 &= 1 + t_2\end{aligned}$$

The second equation implies $t_1 = \frac{1}{2}$. Substituting into the first and third equations we get

$$\begin{aligned}-1 + 4\cdot\frac{1}{2} = 2t_2 \quad &\Rightarrow \quad t_2 = \frac{1}{2}\\ 2 + \frac{1}{2} = 1 + t_2 \quad &\Rightarrow \quad t_2 = \frac{3}{2}\end{aligned}$$

We conclude that the equations do not have solutions, which means that the two lines do not intersect.

52. Determine whether the lines $\mathbf{r}_1(t) = \langle 2, 1, 1\rangle + t\,\langle -4, 0, 1\rangle$ and $\mathbf{r}_2(s) = \langle -4, 1, 5\rangle + s\,\langle 2, 1, -2\rangle$ intersect, and if so, find the point of intersection.

SOLUTION The lines intersect if there exist parameter values t and s such that

$$\begin{aligned}\langle 2, 1, 1\rangle + t\langle -4, 0, 1\rangle &= \langle -4, 1, 5\rangle + s\langle 2, 1, -2\rangle\\ \langle 2 - 4t, 1, 1 + t\rangle &= \langle -4 + 2s, 1 + s, 5 - 2s\rangle\end{aligned}$$

We equate corresponding components to obtain

$$\begin{aligned}2 - 4t &= -4 + 2s\\ 1 &= 1 + s\\ 1 + t &= 5 - 2s\end{aligned}$$

The second equation implies $s = 0$. Substituting in the first and third equations we get

$$\begin{aligned}2 - 4t = -4 + 2\cdot 0 \quad &\Rightarrow \quad t = \frac{3}{2}\\ 1 + t = 5 - 2\cdot 0 \quad &\Rightarrow \quad t = 4\end{aligned}$$

We conclude that the equations do not have solutions, that is, the lines do not intersect.

53. Determine whether the lines $\mathbf{r}_1(t) = \langle 0, 1, 1\rangle + t\,\langle 1, 1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 3\rangle + s\,\langle 1, 4, 4\rangle$ intersect, and if so, find the point of intersection.

SOLUTION The lines intersect if there exist parameter values t and s such that

$$\begin{aligned}\langle 0, 1, 1\rangle + t\langle 1, 1, 2\rangle &= \langle 2, 0, 3\rangle + s\langle 1, 4, 4\rangle\\ \langle t, 1 + t, 1 + 2t\rangle &= \langle 2 + s, 4s, 3 + 4s\rangle \qquad (1)\end{aligned}$$

Equating corresponding components we get

$$\begin{aligned}t &= 2 + s\\ 1 + t &= 4s\\ 1 + 2t &= 3 + 4s\end{aligned}$$

Substituting t from the first equation into the second equation we get

$$\begin{aligned}1 + 2 + s &= 4s\\ 3s &= 3\end{aligned} \quad \Rightarrow \quad s = 1,\ t = 2 + s = 3$$

We now check whether $s = 1$, $t = 3$ satisfy the third equation:

$$\begin{aligned}1 + 2\cdot 3 &= 3 + 4\cdot 1\\ 7 &= 7\end{aligned}$$

We conclude that $s = 1$, $t = 3$ is the solution of (1), hence the two lines intersect. To find the point of intersection we substitute $s = 1$ in the right-hand side of (1) to obtain

$$\langle 2 + 1, 4 \cdot 1, 3 + 4 \cdot 1 \rangle = \langle 3, 4, 7 \rangle$$

The point of intersection is the terminal point of this vector, that is, (3, 4, 7).

54. Find the intersection of the lines $\mathbf{r}_1(t) = \langle -1, 1 \rangle + t\,\langle 2, 4 \rangle$ and $\mathbf{r}_2(s) = \langle 2, 1 \rangle + s\,\langle -1, 6 \rangle$ in $\mathbf{R}^2$.

SOLUTION We must find the parameter values t and s such that

$$\begin{aligned} \langle -1, 1 \rangle + t\langle 2, 4 \rangle &= \langle 2, 1 \rangle + s\langle -1, 6 \rangle \\ \langle -1 + 2t, 1 + 4t \rangle &= \langle 2 - s, 1 + 6s \rangle \end{aligned} \tag{1}$$

Equating corresponding components we get

$$\begin{aligned} -1 + 2t &= 2 - s \\ 1 + 4t &= 1 + 6s \end{aligned}$$

The first equation implies $s = 3 - 2t$. We substitute in the second equation:

$$\begin{aligned} 1 + 4t &= 1 + 6(3 - 2t) \\ 1 + 4t &= 19 - 12t \\ 16t &= 18 \end{aligned} \quad \Rightarrow \quad t = \frac{9}{8},\ s = 3 - 2 \cdot \frac{9}{8} = \frac{3}{4}$$

To find the point of intersection we substitute $s = \frac{3}{4}$ in the right-hand side of (1). This gives

$$\left\langle 2 - \frac{3}{4}, 1 + 6 \cdot \frac{3}{4} \right\rangle = \left\langle \frac{5}{4}, \frac{11}{2} \right\rangle.$$

The point of intersection is the terminal point of this vector which is $\left(\frac{5}{4}, \frac{11}{2}\right)$.

55. Find the components of the vector $\mathbf{v}$ whose tail and head are the midpoints of segments $\overline{AC}$ and $\overline{BC}$ in Figure 19.

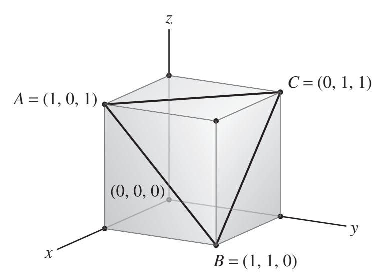
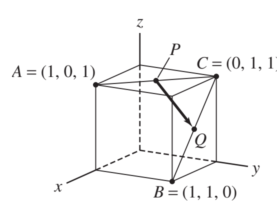


FIGURE 19

SOLUTION We denote by P and Q the midpoints of the segments $\overline{AC}$ and $\overline{BC}$ respectively. Thus,

$$\mathbf{v} = \overrightarrow{PQ} \tag{1}$$



We use the formula for the midpoint of a segment to find the coordinates of the points P and Q. This gives

$$P = \left(\frac{1+0}{2}, \frac{0+1}{2}, \frac{1+1}{2} \right) = \left(\frac{1}{2}, \frac{1}{2}, 1 \right)$$

$$Q = \left(\frac{1+0}{2}, \frac{1+1}{2}, \frac{0+1}{2} \right) = \left(\frac{1}{2}, 1, \frac{1}{2} \right)$$

Substituting in (1) yields the following vector:

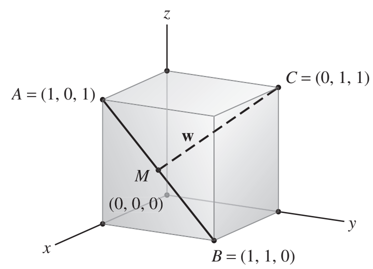
$$\mathbf{v} = \overrightarrow{PQ} = \left\langle \frac{1}{2} - \frac{1}{2}, 1 - \frac{1}{2}, \frac{1}{2} - 1 \right\rangle = \left\langle 0, \frac{1}{2}, -\frac{1}{2} \right\rangle.$$

56. Find the components of the vector **w** whose tail is C and head is the midpoint of $\overline{AB}$ in Figure 19.

SOLUTION We denote the midpoint of $\overline{AB}$ by $M = (a, b, c)$. To find the coordinates of M we first parametrize the line through $A = (1, 0, 1)$ and $B = (1, 1, 0)$:

$$\mathbf{r}(t) = (1-t)\langle 1, 0, 1\rangle + t\langle 1, 1, 0\rangle = \langle 1-t, 0, 1-t\rangle + \langle t, t, 0\rangle = \langle 1, t, 1-t\rangle$$

The midpoint of $\overline{AB}$ occurs at $t = \frac{1}{2}$, hence the vector $\overline{OM}$ is $\langle 1, \frac{1}{2}, \frac{1}{2}\rangle$.



The point M is the terminal point of $\overline{OM}$, that is, $M = \left(1, \frac{1}{2}, \frac{1}{2}\right)$. We now find the vector $\mathbf{w} = \overline{CM}$:

$$\mathbf{w} = \left\langle 1-0, \frac{1}{2}-1, \frac{1}{2}-1\right\rangle = \left\langle 1, -\frac{1}{2}, -\frac{1}{2}\right\rangle.$$

Further Insights and Challenges

In Exercises 57–63, we consider the equations of a line in ***symmetric form****, when $a \neq 0$, $b \neq 0$, $c \neq 0$.*

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \qquad \boxed{12}$$

57. Let $\mathcal{L}$ be the line through $P_0 = (x_0, y_0, c_0)$ with direction vector $\mathbf{v} = \langle a, b, c\rangle$. Show that $\mathcal{L}$ is defined by the symmetric Eq. (12). *Hint:* Use the vector parametrization to show that every point on $\mathcal{L}$ satisfies Eq. (12).

SOLUTION $\mathcal{L}$ is given by vector parametrization

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle$$

which gives us the equations

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct.$$

Solving for t gives

$$t = \frac{x-x_0}{a}$$
$$t = \frac{y-y_0}{b}$$
$$t = \frac{z-z_0}{c}$$

Setting each equation equal to the other gives Eq. (12).

58. Find the symmetric equations of the line through $P_0 = (-2, 3, 3)$ with direction vector $\mathbf{v} = \langle 2, 4, 3\rangle$.

SOLUTION Using $(x_0, y_0, z_0) = (-2, 3, 3)$ and $\langle a, b, c\rangle = \langle 2, 4, 3\rangle$ in Eq. (12) gives

$$\frac{x+2}{2} = \frac{y-3}{4} = \frac{z-3}{3}$$

59. Find the symmetric equations of the line through $P = (1, 1, 2)$ and $Q = (-2, 4, 0)$.

SOLUTION This line has direction vector $\overrightarrow{PQ} = \langle -3, 3, -2\rangle$. Using $(x_0, y_0, z_0) = P = (1, 1, 2)$ and $\langle a, b, c\rangle = \overrightarrow{PQ} = \langle -3, 3, -2\rangle$ in Eq. (12) gives

$$\frac{x-1}{-3} = \frac{y-1}{3} = \frac{z-2}{-2}$$

60. Find the symmetric equations of the line

$$x = 3 + 2t, \quad y = 4 - 9t, \quad z = 12t$$

SOLUTION If we solve each equation fot t, we get:

$$t = \frac{x-3}{2}, \quad t = \frac{4-y}{9}, \quad t = \frac{z}{12}$$

When we set these equations equal to each other, we get:

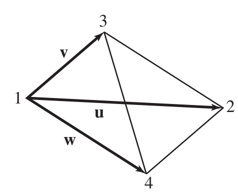
$$\frac{x-3}{2} = \frac{4-y}{9} = \frac{z}{12}$$

61. Find a vector parametrization for the line

$$\frac{x-5}{9} = \frac{y+3}{7} = z - 10$$

SOLUTION Using $(x_0, y_0, z_0) = (5, -3, 10)$ and $\langle a, b, c\rangle = \langle 9, 7, 1\rangle$ gives

$$\mathbf{r}(t) = \langle 5, -3, 10\rangle + t\,\langle 9, 7, 1\rangle$$



62. Find a vector parametrization for the line $\frac{x}{2} = \frac{y}{7} = \frac{z}{8}$.

SOLUTION If we let t equal these three terms, as follows:

$$t = \frac{x}{2} = \frac{y}{7} = \frac{z}{8}$$

then we can break it up into three equations:

$$t = \frac{x}{2}, \quad t = \frac{y}{7}, \quad t = \frac{z}{8}$$

and solving for x, y, and z gives us:

$$x = 2t, \quad y = 7t, \quad z = 8t$$

and writing this in vector form gives us

$$\mathbf{r}(t) = t\,\langle 2, 7, 8\rangle$$

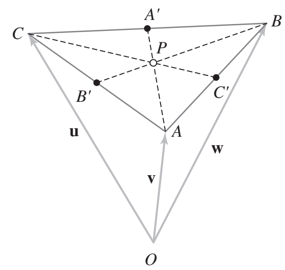
63. Show that the line in the plane through (x_0, y_0) of slope m has symmetric equations

$$x - x_0 = \frac{y - y_0}{m}$$

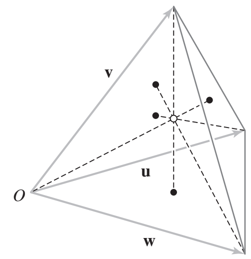
SOLUTION The line through (x_0, y_0) of slope m has equation $y - y_0 = m(x - x_0)$, which becomes $x - x_0 = \frac{1}{m}(y - y_0)$, which becomes

$$\frac{x - x_0}{1} = \frac{y - y_0}{m}$$

64. A median of a triangle is a segment joining a vertex to the midpoint of the opposite side. Referring to Figure 20(A), prove that three medians of triangle ABC intersect at the terminal point P of the vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. The point P is the *centroid* of the triangle. *Hint:* Show, by parametrizing the segment $\overline{AA'}$, that P lies two-thirds of the way from A to A'. It will follow similarly that P lies on the other two medians.



(A)



(B)

FIGURE 20

SOLUTION We denote the vertices of the triangle by A, B, C as shown in Figure 20(A). Hence,

$$\overrightarrow{OA'} = \overrightarrow{OB} + \overrightarrow{BA'} = \mathbf{w} + \frac{1}{2}(\mathbf{u} - \mathbf{w}) = \frac{1}{2}(\mathbf{u} + \mathbf{w})$$

$$\overrightarrow{OA} = \mathbf{v}$$

Thus, the line through the points A and A' has the parametrization

$$t\mathbf{v} + (1 - t)\frac{\mathbf{u} + \mathbf{w}}{2} \tag{1}$$

Similarly, the line through B and B' has the parametrization

$$t\mathbf{w} + (1 - t)\frac{\mathbf{v} + \mathbf{u}}{2} \tag{2}$$

And the line through C and C' has the parametrization

$$t\mathbf{u} + (1 - t)\frac{\mathbf{v} + \mathbf{w}}{2} \tag{3}$$

Now, setting $t = \frac{1}{3}$ in (1), (2) and (3) yields $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. We conclude that the terminal point of this vector lies on each one of the lines, hence it is their point of intersection.

65. A median of a tetrahedron is a segment joining a vertex to the centroid of the opposite face. The tetrahedron in Figure 20(B) has vertices at the origin and at the terminal points of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Show that the medians intersect at the terminal point of $\frac{1}{4}(\mathbf{u} + \mathbf{v} + \mathbf{w})$.

SOLUTION We first find vectors from the origin to the centroids of the four faces (labelled 1,2,3,4 after their opposite vertices, also labelled 1,2,3,4). Now, by the previous problem (Exercise 64), a vector from the origin (vertex 1) to the centroid of the opposite face (face 1) is $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. As for face 2, a vector from vertex 2 to the centroid of face 2 is $\frac{1}{3}(-\mathbf{u} + (\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{u}))$, but since vertex 2 is at the head of vector $\mathbf{u}$, then a vector from the origin to the centroid of face 2 is $\mathbf{u} + \frac{1}{3}(-\mathbf{u} + (\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{u})) = \frac{1}{3}(\mathbf{v} + \mathbf{w})$. Similarly, a vector from the origin to the centroid of face 3 is $\mathbf{v} + \frac{1}{3}(-\mathbf{v} + (\mathbf{u} - \mathbf{v}) + (\mathbf{w} - \mathbf{v})) = \frac{1}{3}(\mathbf{u} + \mathbf{w})$, and from the origin to the centroid of face 4 is $\frac{1}{3}(\mathbf{u} + \mathbf{v})$.

We now find the paramentric equations of four lines $\ell_1, \dots, \ell_4$, each from vertex i to the centroid of the (opposite) face i.

$$\ell_1(t) = t\mathbf{0} + (1 - t)\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

$$\ell_2(t) = t\mathbf{u} + (1 - t)\frac{1}{3}(\mathbf{v} + \mathbf{w})$$

$$\ell_3(t) = t\mathbf{v} + (1 - t)\frac{1}{3}(\mathbf{u} + \mathbf{w})$$

$$\ell_4(t) = t\mathbf{w} + (1 - t)\frac{1}{3}(\mathbf{u} + \mathbf{v})$$

By substituting $t = 1/4$ into each line, we find that they all intersect in the same point:

$$\ell_1(1/4) = 1/4\mathbf{0} + (1 - 1/4)\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 1/4(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

$$\ell_2(1/4) = 1/4\mathbf{u} + (1 - 1/4)\frac{1}{3}(\mathbf{v} + \mathbf{w}) = 1/4(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

$$\ell_3(1/4) = 1/4\mathbf{v} + (1 - 1/4)\frac{1}{3}(\mathbf{u} + \mathbf{w}) = 1/4(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

$$\ell_4(1/4) = 1/4\mathbf{w} + (1 - 1/4)\frac{1}{3}(\mathbf{u} + \mathbf{v}) = 1/4(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

We conclude that all four lines intersect at the terminal point of the vector $1/4(\mathbf{u} + \mathbf{v} + \mathbf{w})$, as desired.

12.3 Dot Product and the Angle between Two Vectors (LT Section 13.3)

Preliminary Questions

1. Is the dot product of two vectors a scalar or a vector?

SOLUTION The dot product of two vectors is the sum of products of scalars, hence it is a scalar.

2. What can you say about the angle between **a** and **b** if $\mathbf{a} \cdot \mathbf{b} < 0$?

SOLUTION Since the cosine of the angle between **a** and **b** satisfies $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$, also $\cos\theta < 0$. By definition $0 \le \theta \le \pi$, but since $\cos\theta < 0$ then θ is in $(\pi/2, \pi]$. In other words, the angle between **a** and **b** is obtuse.

3. Which property of dot products allows us to conclude that if **v** is orthogonal to both **u** and **w**, then **v** is orthogonal to $\mathbf{u} + \mathbf{w}$?

SOLUTION One property is that two vectors are orthogonal if and only if the dot product of the two vectors is zero. The second property is the Distributive Law. Since **v** is orthogonal to **u** and **w**, we have $\mathbf{v} \cdot \mathbf{u} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$. Therefore,

$$\mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0$$

We conclude that **v** is orthogonal to $\mathbf{u} + \mathbf{w}$.

4. Which is the projection of **v** along **v**: (a) **v** or (b) $\mathbf{e_v}$?

SOLUTION The projection of **v** along itself is **v**, since

$$\mathbf{v}_{||} = \left(\frac{\mathbf{v} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \mathbf{v}$$

Also, the projection of **v** along $\mathbf{e_v}$ is the same answer, **v**, because

$$\mathbf{v}_{||} = \left(\frac{\mathbf{v} \cdot \mathbf{e_v}}{\mathbf{e_v} \cdot \mathbf{e_v}}\right)\mathbf{e_v} = \|\mathbf{v}\|\mathbf{e_v} = \mathbf{v}$$

5. Let $\mathbf{u}_{||}$ be the projection of **u** along **v**. Which of the following is the projection **u** along the vector $2\mathbf{v}$ and which is the projection of $2\mathbf{u}$ along **v**?

(a) $\frac{1}{2}\mathbf{u}_{||}$ **(b)** $\mathbf{u}_{||}$ **(c)** $2\mathbf{u}_{||}$

SOLUTION Since $\mathbf{u}_{||}$ is the projection of **u** along **v**, we have,

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$$

The projection of **u** along the vector $2\mathbf{v}$ is

$$\left(\frac{\mathbf{u} \cdot 2\mathbf{v}}{2\mathbf{v} \cdot 2\mathbf{v}}\right)2\mathbf{v} = \left(\frac{2\mathbf{u} \cdot \mathbf{v}}{4\mathbf{v} \cdot \mathbf{v}}\right)2\mathbf{v} = \left(\frac{4\mathbf{u} \cdot \mathbf{v}}{4\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \mathbf{u}_{||}$$

That is, $u_{||}$ is the projection of **u** along $2\mathbf{v}$, so our answer is (b) for the first part. Notice that the projection of **u** along **v** is the projection of **u** along the unit vector $\mathbf{e_v}$, hence it depends on the direction of **v** rather than on the length of **v**. Therefore, the projection of **u** along **v** and along $2\mathbf{v}$ is the same vector.

On the other hand, the projection of $2\mathbf{u}$ along **v** is as follows:

$$\left(\frac{2\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = 2\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = 2\mathbf{u}_{||}$$

giving us answer (c) for the second part.

6. Which of the following is equal to $\cos\theta$, where θ is the angle between **u** and **v**?

(a) $\mathbf{u} \cdot \mathbf{v}$ **(b)** $\mathbf{u} \cdot \mathbf{e_v}$ **(c)** $\mathbf{e_u} \cdot \mathbf{e_v}$

SOLUTION By the Theorems on the Dot Product and the Angle Between Vectors, we have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \mathbf{e_u} \cdot \mathbf{e_v}$$

The correct answer is (c).

Exercises

In Exercises 1–12, compute the dot product.

1. $\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle$

SOLUTION Using the definition of the dot product we obtain

$$\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle = 1 \cdot 4 + 2 \cdot 3 + 1 \cdot 5 = 15$$

2. $\langle 3, -2, 2\rangle \cdot \langle 1, 0, 1\rangle$

SOLUTION By the definition of the dot product we have

$$\langle 3, -2, 2\rangle \cdot \langle 1, 0, 1\rangle = 3 \cdot 1 + (-2) \cdot 0 + 2 \cdot 1 = 5$$

3. $\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle$

SOLUTION The dot product is

$$\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle = 0 \cdot 7 + 1 \cdot 41 + 0 \cdot (-3) = 41$$

4. $\langle 1, 1, 1\rangle \cdot \langle 6, 4, 2\rangle$

SOLUTION We multiply corresponding components and add to obtain

$$\langle 1, 1, 1\rangle \cdot \langle 6, 4, 2\rangle = 1 \cdot 6 + 1 \cdot 4 + 1 \cdot 2 = 12$$

5. $\langle 3, 1\rangle \cdot \langle 4, -7\rangle$

SOLUTION The dot product of the two vectors is the following scalar:

$$\langle 3, 1\rangle \cdot \langle 4, -7\rangle = 3 \cdot 4 + 1 \cdot (-7) = 5$$

6. $\left\langle \frac{1}{6}, \frac{1}{2}\right\rangle \cdot \left\langle 3, \frac{1}{2}\right\rangle$

SOLUTION The dot product is

$$\left\langle \frac{1}{6}, \frac{1}{2}\right\rangle \cdot \left\langle 3, \frac{1}{2}\right\rangle = \frac{1}{6} \cdot 3 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

7. $\mathbf{k} \cdot \mathbf{j}$

SOLUTION By the orthogonality of $\mathbf{j}$ and $\mathbf{k}$, we have $\mathbf{k} \cdot \mathbf{j} = 0$

8. $\mathbf{k} \cdot \mathbf{k}$

SOLUTION Since $\mathbf{k}$ has length 1, we have $\mathbf{k} \cdot \mathbf{k} = 1$

9. $(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{j} + \mathbf{k})$

SOLUTION By the distributive law and the orthogonality of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we have

$$(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{j} + \mathbf{k}) = \mathbf{i} \cdot \mathbf{j} + \mathbf{i} \cdot \mathbf{k} + \mathbf{j} \cdot \mathbf{j} + \mathbf{j} \cdot \mathbf{k} = 0 + 0 + 1 + 0 = 1$$

10. $(3\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} - 4\mathbf{k})$

SOLUTION By the distributive law and the orthogonality of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we have

$$(3\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} - 4\mathbf{k}) = 3\mathbf{j} \cdot \mathbf{i} - 12\mathbf{j} \cdot \mathbf{k} + 2\mathbf{k} \cdot \mathbf{i} - 8\mathbf{k} \cdot \mathbf{k} = 0 - 0 + 0 - 8\|\mathbf{k}\|^2 = -8 \cdot 1^2 = -8$$

11. $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$

SOLUTION We use properties of the dot product to obtain

$$\begin{aligned}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) &= 3\mathbf{i} \cdot \mathbf{i} + 2\mathbf{i} \cdot \mathbf{j} - 5\mathbf{i} \cdot \mathbf{k} + 3\mathbf{j} \cdot \mathbf{i} + 2\mathbf{j} \cdot \mathbf{j} - 5\mathbf{j} \cdot \mathbf{k} + 3\mathbf{k} \cdot \mathbf{i} + 2\mathbf{k} \cdot \mathbf{j} - 5\mathbf{k} \cdot \mathbf{k} \\ &= 3\|\mathbf{i}\|^2 + 2\|\mathbf{j}\|^2 - 5\|\mathbf{k}\|^2 = 3 \cdot 1 + 2 \cdot 1 - 5 \cdot 1 = 0\end{aligned}$$

12. $(-\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 7\mathbf{k})$

SOLUTION Using the distributive law we have

$$(-\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = -\mathbf{k} \cdot \mathbf{i} + 2\mathbf{k} \cdot \mathbf{j} - 7\mathbf{k} \cdot \mathbf{k} = 0 + 0 - 7\|\mathbf{k}\|^2 = -7 \cdot 1^2 = -7$$

In Exercises 13–18, determine whether the two vectors are orthogonal and, if not, whether the angle between them is acute or obtuse.

13. $\langle 1, 1, 1\rangle, \quad \langle 1, -2, -2\rangle$

SOLUTION We compute the dot product of the two vectors:

$$\langle 1, 1, 1\rangle \cdot \langle 1, -2, -2\rangle = 1 \cdot 1 + 1 \cdot (-2) + 1 \cdot (-2) = -3$$

Since the dot product is negative, the angle between the vectors is obtuse.

14. $\langle 0, 2, 4\rangle, \quad \langle -5, 0, 0\rangle$

SOLUTION Computing the dot product gives

$$\langle 0, 2, 4\rangle \cdot \langle -5, 0, 0\rangle = 0 \cdot (-5) + 2 \cdot 0 + 4 \cdot 0 = 0$$

The dot product is zero, hence the vectors are orthogonal.

15. $\langle 1, 2, 1\rangle, \quad \langle 7, -3, -1\rangle$

SOLUTION We compute the dot product:

$$\langle 1, 2, 1\rangle \cdot \langle 7, -3, -1\rangle = 1 \cdot 7 + 2 \cdot (-3) + 1 \cdot (-1) = 0$$

The dot product is zero, hence the vectors are orthogonal.

16. $\langle 0, 2, 4\rangle, \quad \langle 3, 1, 0\rangle$

SOLUTION We find the dot product of the two vectors:

$$\langle 0, 2, 4\rangle \cdot \langle 3, 1, 0\rangle = 0 \cdot 3 + 2 \cdot 1 + 4 \cdot 0 = 2$$

The dot product is positive, hence the angle between the vectors is acute.

17. $\left\langle \frac{12}{5}, -\frac{4}{5}\right\rangle, \quad \left\langle \frac{1}{2}, -\frac{7}{4}\right\rangle$

SOLUTION We find the dot product of the two vectors:

$$\left\langle \frac{12}{5}, -\frac{4}{5}\right\rangle \cdot \left\langle \frac{1}{2}, -\frac{7}{4}\right\rangle = \frac{12}{5} \cdot \frac{1}{2} + \left(-\frac{4}{5}\right) \cdot \left(-\frac{7}{4}\right) = \frac{12}{10} + \frac{28}{20} = \frac{13}{5}$$

The dot product is positive, hence the angle between the vectors is acute.

18. $\langle 12, 6\rangle, \quad \langle 2, -4\rangle$

SOLUTION Since $\langle 12, 6\rangle \cdot \langle 2, -4\rangle = 12 \cdot 2 + 6 \cdot (-4) = 0$, the vectors are orthogonal.

In Exercises 19–22, find the cosine of the angle between the vectors.

19. $\langle 0, 3, 1\rangle, \quad \langle 4, 0, 0\rangle$

SOLUTION Since $\langle 0, 3, 1\rangle \cdot \langle 4, 0, 0\rangle = 0 \cdot 4 + 3 \cdot 0 + 1 \cdot 0 = 0$, the vectors are orthogonal, that is, the angle between them is $\theta = 90^\circ$ and $\cos\theta = 0$.

20. $\langle 1, 1, 1\rangle, \quad \langle 2, -1, 2\rangle$

SOLUTION We write $\mathbf{v} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle 2, -1, 2\rangle$ and use the formula for the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

We compute the values in this formula:

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$\|\mathbf{w}\| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$
$$\mathbf{v} \cdot \mathbf{w} = \langle 1, 1, 1 \rangle \cdot \langle 2, -1, 2 \rangle = 1 \cdot 2 + 1 \cdot (-1) + 1 \cdot 2 = 3$$

Hence,

$$\cos\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

21. $\mathbf{i} + \mathbf{j}, \quad \mathbf{j} + 2\mathbf{k}$

SOLUTION We use the formula for the cosine of the angle between two vectors. Let $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = \mathbf{j} + 2\mathbf{k}$. We compute the following values:

$$\|\mathbf{v}\| = \|\mathbf{i} + \mathbf{j}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\|\mathbf{w}\| = \|\mathbf{j} + 2\mathbf{k}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
$$\mathbf{v} \cdot \mathbf{w} = (\mathbf{i} + \mathbf{j}) \cdot (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} \cdot \mathbf{j} + 2\mathbf{i} \cdot \mathbf{k} + \mathbf{j} \cdot \mathbf{j} + 2\mathbf{j} \cdot \mathbf{k} = \|\mathbf{j}\|^2 = 1$$

Hence,

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}.$$

22. $3\mathbf{i} + \mathbf{k}, \quad \mathbf{i} + \mathbf{j} + \mathbf{k}$

SOLUTION We write $\mathbf{v} = 3\mathbf{i} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{3^2 + 1^2} = \sqrt{10}$$
$$\|\mathbf{w}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$\mathbf{v} \cdot \mathbf{w} = (3\mathbf{i} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = (3\mathbf{i} + 0\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 4$$

Hence,

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{4}{\sqrt{10}\sqrt{3}} = \frac{4}{\sqrt{30}}$$

In Exercises 23–28, find the angle between the vectors. Use a calculator if necessary.

23. $\langle 2, \sqrt{2} \rangle, \quad \langle 1 + \sqrt{2}, 1 - \sqrt{2} \rangle$

SOLUTION We write $\mathbf{v} = \left\langle 2, \sqrt{2} \right\rangle$ and $\mathbf{w} = \left\langle 2, \sqrt{2} \right\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{4 + 2} = \sqrt{6}$$
$$\|\mathbf{w}\| = \sqrt{(1 + \sqrt{2})^2 + (1 - \sqrt{2})^2} = \sqrt{6}$$
$$\mathbf{v} \cdot \mathbf{w} = 2 + 2\sqrt{2} + \sqrt{2} - 2 = 3\sqrt{2}$$

Hence,

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{3\sqrt{2}}{\sqrt{6}\sqrt{6}} = \frac{\sqrt{2}}{2}$$

and so,

$$\theta = \cos^{-1}\frac{\sqrt{2}}{2} = \pi/4$$

24. $\langle 5, \sqrt{3}\rangle, \quad \langle \sqrt{3}, 2\rangle$

SOLUTION We denote $\mathbf{v} = \left\langle 5, \sqrt{3}\right\rangle$ and $\mathbf{w} = \left\langle \sqrt{3}, 2\right\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{25+3} = \sqrt{28}$$
$$\|\mathbf{w}\| = \sqrt{3+4} = \sqrt{7}$$
$$\mathbf{v}\cdot\mathbf{w} = 5\cdot\sqrt{3} + \sqrt{3}\cdot 2 = 7\sqrt{3}$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{7\sqrt{3}}{\sqrt{28}\sqrt{7}} = \frac{\sqrt{3}}{2}$$

and so,

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = \pi/6$$

25. $\langle 1, 1, 1\rangle, \quad \langle 1, 0, 1\rangle$

SOLUTION We denote $\mathbf{v} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle 1, 0, 1\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$
$$\|\mathbf{w}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$
$$\mathbf{v}\cdot\mathbf{w} = 1+0+1 = 2$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{\sqrt{6}}{3}$$

and so,

$$\theta = \cos^{-1}\frac{\sqrt{6}}{3} \approx 0.615$$

26. $\langle 3, 1, 1\rangle, \quad \langle 2, -4, 2\rangle$

SOLUTION We denote $\mathbf{v} = \langle 3, 1, 1\rangle$ and $\mathbf{w} = \langle 2, -4, 2\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{3^2+1^2+1^2} = \sqrt{11}$$
$$\|\mathbf{w}\| = \sqrt{2^2+(-4)^2+2^2} = \sqrt{24}$$
$$\mathbf{v}\cdot\mathbf{w} = 6-4+2 = 4$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{4}{\sqrt{11}\sqrt{24}} = \frac{2}{\sqrt{66}}$$

and so,

$$\theta = \cos^{-1}\frac{2}{\sqrt{66}} \approx 1.322$$

27. $\langle 0, 1, 1\rangle, \quad \langle 1, -1, 0\rangle$

SOLUTION We denote $\mathbf{v} = \langle 0, 1, 1\rangle$ and $\mathbf{w} = \langle 1, -1, 0\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\|\mathbf{v}\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$$
$$\|\mathbf{w}\| = \sqrt{1^2+(-1)^2+0^2} = \sqrt{2}$$
$$\mathbf{v}\cdot\mathbf{w} = 0+(-1)+0 = -1$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2}$$

and so,

$$\theta = \cos^{-1} -\frac{1}{2} = \frac{2\pi}{3}$$

28. $\langle 1, 1, -1\rangle, \quad \langle 1, -2, -1\rangle$

SOLUTION Since $\langle 1, 1, -1\rangle \cdot \langle 1, -2, -1\rangle = 1 - 2 + 1 = 0$, then the two vectors are orthogonal. Thus, the angle between them is $\pi/2$.

29. Find all values of b for which the vectors are orthogonal.

(a) $\langle b, 3, 2\rangle, \quad \langle 1, b, 1\rangle$

(b) $\langle 4, -2, 7\rangle, \quad \left\langle b^2, b, 0\right\rangle$

SOLUTION

(a) The vectors are orthogonal if and only if the scalar product is zero. That is,

$$\begin{aligned} \langle b, 3, 2\rangle \cdot \langle 1, b, 1\rangle &= 0 \\ b\cdot 1 + 3\cdot b + 2\cdot 1 &= 0 \\ 4b + 2 &= 0 \quad \Rightarrow \quad b = -\frac{1}{2} \end{aligned}$$

(b) We set the scalar product of the two vectors equal to zero and solve for b. This gives

$$\begin{aligned} \langle 4, -2, 7\rangle \cdot \langle b^2, b, 0\rangle &= 0 \\ 4b^2 - 2b + 7\cdot 0 &= 0 \\ 2b(2b-1) &= 0 \quad \Rightarrow \quad b = 0 \text{ or } b = \frac{1}{2} \end{aligned}$$

30. Find a vector that is orthogonal to $\langle -1, 2, 2\rangle$.

SOLUTION We must find a vector $\mathbf{v} = \langle a, b, c\rangle$ such that

$$\begin{aligned} \langle a, b, c\rangle \cdot \langle -1, 2, 2\rangle &= 0 \\ -a + 2b + 2c &= 0 \end{aligned}$$

We choose $c = 1$, $b = 1$, hence $a = 4$. The vector $\mathbf{v} = \langle 4, 1, 1\rangle$ is orthogonal to $\langle -1, 2, 2\rangle$.

31. Find two vectors that are not multiples of each other and are both orthogonal to $\langle 2, 0, -3\rangle$.

SOLUTION We denote by $\langle a, b, c\rangle$, a vector orthogonal to $\langle 2, 0, -3\rangle$. Hence,

$$\begin{aligned} \langle a, b, c\rangle \cdot \langle 2, 0, -3\rangle &= 0 \\ 2a + 0 - 3c &= 0 \\ 2a - 3c &= 0 \quad \Rightarrow \quad a = \frac{3}{2}c \end{aligned}$$

Thus, the vectors orthogonal to $\langle 2, 0, -3\rangle$ are of the form

$$\left\langle \frac{3}{2}c, b, c\right\rangle.$$

We may find two such vectors by setting $c = 0$, $b = 1$ and $c = 2$, $b = 2$. We obtain

$$\mathbf{v}_1 = \langle 0, 1, 0\rangle, \quad \mathbf{v}_2 = \langle 3, 2, 2\rangle.$$

32. Find a vector that is orthogonal to $\mathbf{v} = \langle 1, 2, 1\rangle$ but not to $\mathbf{w} = \langle 1, 0, -1\rangle$.

SOLUTION We want a vector $\langle a, b, c\rangle$ such that $\langle a, b, c\rangle \cdot \langle 1, 2, 1\rangle = 0$ but $\langle a, b, c\rangle \cdot \langle 1, 0, -1\rangle \neq 0$. While we could set up some equations, it's easy to note that if we let $\langle a, b, c\rangle = \mathbf{w} = \langle 1, 0, -1\rangle$, then both conditions are satisfied.

33. Find $\mathbf{v}\cdot\mathbf{e}$ where $\|\mathbf{v}\| = 3$, $\mathbf{e}$ is a unit vector, and the angle between $\mathbf{e}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

SOLUTION Since $\mathbf{v}\cdot\mathbf{e} = \|\mathbf{v}\|\|\mathbf{e}\|\cos 2\pi/3$, and $\|\mathbf{v}\| = 3$ and $\|\mathbf{e}\| = 1$, we have $\mathbf{v}\cdot\mathbf{e} = 3\cdot 1\cdot(-1/2) = -3/2$.

34. Assume that $\mathbf{v}$ lies in the yz-plane. Which of the following dot products is equal to zero for all choices of $\mathbf{v}$?

(a) $\mathbf{v} \cdot \langle 0, 2, 1\rangle$

(b) $\mathbf{v} \cdot \mathbf{k}$

(c) $\mathbf{v} \cdot \langle -3, 0, 0\rangle$

(d) $\mathbf{v} \cdot \mathbf{j}$

SOLUTION Since $\mathbf{v}$ lies in the yz-plane, then it must be of the form $\mathbf{v} = \langle 0, v_2, v_3\rangle$. The only dot product which is always equal to zero is (c), since $\mathbf{v} \cdot \langle -3, 0, 0\rangle = 0$.

In Exercises 35–38, simplify the expression.

35. $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$

SOLUTION By properties of the dot product we obtain

$$(\mathbf{v} - \mathbf{w}) \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|^2$$

36. $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w}$

SOLUTION Using properties of the dot product we obtain

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w} &= \mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 + \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2 - 2\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\end{aligned}$$

37. $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}$

SOLUTION We use properties of the dot product to write

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w} &= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 + \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} - \|\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2\end{aligned}$$

38. $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{w}) \cdot \mathbf{w}$

SOLUTION By properties of the dot product we get

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{w}) \cdot \mathbf{w} &= (\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - \mathbf{w} \cdot (\mathbf{v} - \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\end{aligned}$$

In Exercises 39–42, use the properties of the dot product to evaluate the expression, assuming that $\mathbf{u} \cdot \mathbf{v} = 2$, $\|\mathbf{u}\| = 1$, and $\|\mathbf{v}\| = 3$.

39. $\mathbf{u} \cdot (4\mathbf{v})$

SOLUTION Using properties of the dot product we get

$$\mathbf{u} \cdot (4\mathbf{v}) = 4(\mathbf{u} \cdot \mathbf{v}) = 4 \cdot 2 = 8.$$

40. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$

SOLUTION Using the distributive law and the dot product relation with length we obtain

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 = 2 + 3^2 = 11.$$

41. $2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v})$

SOLUTION By properties of the dot product we obtain

$$\begin{aligned}2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v}) &= (2\mathbf{u}) \cdot (3\mathbf{u}) - (2\mathbf{u}) \cdot \mathbf{v} = 6(\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{u} \cdot \mathbf{v}) \\ &= 6\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) = 6 \cdot 1^2 - 2 \cdot 2 = 2\end{aligned}$$

42. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

SOLUTION We use the distributive law, commutativity and the relation with length to write

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 1^2 - 3^2 = -8\end{aligned}$$

43. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\| \, \|\mathbf{w}\|$.

SOLUTION Using the formula for dot product, and the given equation $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\| \, \|\mathbf{w}\|$, we get:

$$\|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta = -\|\mathbf{v}\| \, \|\mathbf{w}\|,$$

which implies $\cos\theta = -1$, and so the angle between the two vectors is $\theta = \pi$.

44. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\|\mathbf{v}\|\ \|\mathbf{w}\|$.

SOLUTION Using the formula for dot product, and the given equation $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\|\mathbf{v}\|\ \|\mathbf{w}\|$, we get:

$$\|\mathbf{v}\|\ \|\mathbf{w}\| \cos\theta = \frac{1}{2}\|\mathbf{v}\|\ \|\mathbf{w}\|,$$

which implies $\cos\theta = \frac{1}{2}$, and so the angle between the two vectors is $\theta = \pi/3$.

45. Assume that $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 5$ and that the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\theta = \frac{\pi}{3}$.

(a) Use the relation $\|\mathbf{v}+\mathbf{w}\|^2 = (\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w})$ to show that $\|\mathbf{v}+\mathbf{w}\|^2 = 3^2 + 5^2 + 2\mathbf{v}\cdot\mathbf{w}$.

(b) Find $\|\mathbf{v}+\mathbf{w}\|$.

SOLUTION For part (a), we use the distributive property to get:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2\\ &= 3^2 + 5^2 + 2\mathbf{v}\cdot\mathbf{w}\end{aligned}$$

For part (b), we use the definition of dot product on the previous equation to get:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= 3^2 + 5^2 + 2\mathbf{v}\cdot\mathbf{w}\\ &= 34 + 2\cdot 3\cdot 5\cdot \cos\pi/3\\ &= 34 + 15 = 49\end{aligned}$$

Thus, $\|\mathbf{v}+\mathbf{w}\| = \sqrt{49} = 7$.

46. Assume that $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $120°$. Determine:

(a) $\mathbf{v}\cdot\mathbf{w}$ **(b)** $\|2\mathbf{v}+\mathbf{w}\|$ **(c)** $\|2\mathbf{v}-3\mathbf{w}\|$

SOLUTION

(a) We use the relation between the dot product and the angle between two vectors to write

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = 2\cdot 3\cos 120° = 6\cdot\left(-\frac{1}{2}\right) = -3$$

(b) By the relation of the dot product with length and by properties of the dot product we have

$$\begin{aligned}\|2\mathbf{v}+\mathbf{w}\|^2 &= (2\mathbf{v}+\mathbf{w})\cdot(2\mathbf{v}+\mathbf{w}) = 4\mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{w} + 2\mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}\\ &= 4\|\mathbf{v}\|^2 + 4\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2\end{aligned}$$

We now substitute $\mathbf{v}\cdot\mathbf{w} = -3$ from part (a) and the given information, obtaining

$$\|2\mathbf{v}+\mathbf{w}\|^2 = 4\cdot 2^2 + 4(-3) + 3^2 = 13 \quad\Rightarrow\quad \|2\mathbf{v}+\mathbf{w}\| = \sqrt{13} \approx 3.61$$

(c) We express the length in terms of a dot product and use properties of the dot product. This gives

$$\begin{aligned}\|2\mathbf{v}-3\mathbf{w}\|^2 &= (2\mathbf{v}-3\mathbf{w})\cdot(2\mathbf{v}-3\mathbf{w}) = 4\mathbf{v}\cdot\mathbf{v} - 6\mathbf{v}\cdot\mathbf{w} - 6\mathbf{w}\cdot\mathbf{v} + 9\mathbf{w}\cdot\mathbf{w}\\ &= 4\|\mathbf{v}\|^2 - 12\mathbf{v}\cdot\mathbf{w} + 9\|\mathbf{w}\|^2\end{aligned}$$

Substituting $\mathbf{v}\cdot\mathbf{w} = -3$ from part (a) and the given values yields

$$\|2\mathbf{v}-3\mathbf{w}\|^2 = 4\cdot 2^2 - 12(-3) + 9\cdot 3^2 = 133 \quad\Rightarrow\quad \|2\mathbf{v}-3\mathbf{w}\| = \sqrt{133} \approx 11.53$$

47. Show that if $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \frac{3}{2}$, then $\|\mathbf{e}-\mathbf{f}\| = \frac{\sqrt{7}}{2}$. *Hint:* Show that $\mathbf{e}\cdot\mathbf{f} = \frac{1}{8}$.

SOLUTION We use the relation of the dot product with length and properties of the dot product to write

$$\begin{aligned}9/4 = \|\mathbf{e}+\mathbf{f}\|^2 &= (\mathbf{e}+\mathbf{f})\cdot(\mathbf{e}+\mathbf{f}) = \mathbf{e}\cdot\mathbf{e} + \mathbf{e}\cdot\mathbf{f} + \mathbf{f}\cdot\mathbf{e} + \mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2 + 2\mathbf{e}\cdot\mathbf{f} + \|\mathbf{f}\|^2 = 1^2 + 2\mathbf{e}\cdot\mathbf{f} + 1^2 = 2 + 2\mathbf{e}\cdot\mathbf{f}\end{aligned}$$

We now find $\mathbf{e}\cdot\mathbf{f}$:

$$9/4 = 2 + 2\mathbf{e}\cdot\mathbf{f} \quad\Rightarrow\quad \mathbf{e}\cdot\mathbf{f} = 1/8$$

Hence, using the same method as above, we have:

$$\begin{aligned}\|\mathbf{e}-\mathbf{f}\|^2 &= (\mathbf{e}-\mathbf{f})\cdot(\mathbf{e}-\mathbf{f}) = \mathbf{e}\cdot\mathbf{e}-\mathbf{e}\cdot\mathbf{f}-\mathbf{f}\cdot\mathbf{e}+\mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2-2\mathbf{e}\cdot\mathbf{f}+\|\mathbf{f}\|^2 = 1^2-2\mathbf{e}\cdot\mathbf{f}+1^2 = 2-2\mathbf{e}\cdot\mathbf{f} = 2-2/8 = 7/4.\end{aligned}$$

Taking square roots, we get:

$$\|\mathbf{e}-\mathbf{f}\| = \frac{\sqrt{7}}{2}$$

48. Find $\|2\mathbf{e}-3\mathbf{f}\|$ assuming that $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \sqrt{3/2}$.

SOLUTION We use the relation of the dot product with length and properties of the dot product to write

$$\begin{aligned}3/2 = \|\mathbf{e}+\mathbf{f}\|^2 &= (\mathbf{e}+\mathbf{f})\cdot(\mathbf{e}+\mathbf{f}) = \mathbf{e}\cdot\mathbf{e}+\mathbf{e}\cdot\mathbf{f}+\mathbf{f}\cdot\mathbf{e}+\mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2+2\mathbf{e}\cdot\mathbf{f}+\|\mathbf{f}\|^2 = 1^2+2\mathbf{e}\cdot\mathbf{f}+1^2 = 2+2\mathbf{e}\cdot\mathbf{f}\end{aligned}$$

We now find $\mathbf{e}\cdot\mathbf{f}$:

$$3/2 = 2+2\mathbf{e}\cdot\mathbf{f} \quad\Rightarrow\quad \mathbf{e}\cdot\mathbf{f} = -1/4$$

Hence, using the same method as above, we have:

$$\begin{aligned}\|2\mathbf{e}-3\mathbf{f}\|^2 &= (2\mathbf{e}-3\mathbf{f})\cdot(2\mathbf{e}-3\mathbf{f})\\ &= \|2\mathbf{e}\|^2-2\cdot 2\mathbf{e}\cdot 3\mathbf{f}+\|3\mathbf{f}\|^2 = 2^2-12\mathbf{e}\cdot\mathbf{f}+3^2 = 13+3 = 16.\end{aligned}$$

Taking square roots, we get:

$$\|2\mathbf{e}-3\mathbf{f}\| = 4$$

49. Find the angle θ in the triangle in Figure 12.

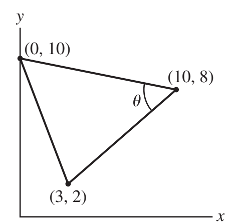
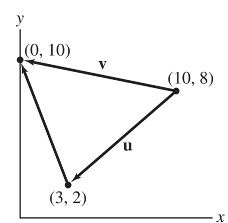


FIGURE 12

SOLUTION We denote by $\mathbf{u}$ and $\mathbf{v}$ the vectors in the figure.



Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|\|\mathbf{u}\|} \tag{1}$$

We find the vectors $\mathbf{v}$ and $\mathbf{u}$, and then compute their length and the dot product $\mathbf{v}\cdot\mathbf{u}$. This gives

$$\begin{aligned}\mathbf{v} &= \langle 0-10, 10-8\rangle = \langle -10, 2\rangle\\ \mathbf{u} &= \langle 3-10, 2-8\rangle = \langle -7, -6\rangle\\ \|\mathbf{v}\| &= \sqrt{(-10)^2+2^2} = \sqrt{104}\\ \|\mathbf{u}\| &= \sqrt{(-7)^2+(-6)^2} = \sqrt{85}\\ \mathbf{v}\cdot\mathbf{u} &= \langle -10, 2\rangle\cdot\langle -7, -6\rangle = (-10)\cdot(-7)+2\cdot(-6) = 58\end{aligned}$$

Substituting these values in (1) yields

$$\cos\theta = \frac{58}{\sqrt{104}\sqrt{85}} \approx 0.617$$

Hence the angle of the triangle is 51.91°.

50. Find all three angles in the triangle in Figure 13.

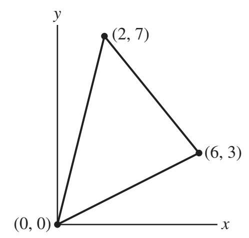
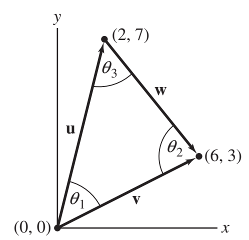


FIGURE 13

SOLUTION We denote by **u**, **v** and **w** the vectors and by θ_1, θ_2, and θ_3 the angles shown in the figure. We compute the vectors:

$$\mathbf{u} = \langle 2, 7\rangle$$
$$\mathbf{v} = \langle 6, 3\rangle$$
$$\mathbf{w} = \langle 6-2, 3-7\rangle = \langle 4, -4\rangle$$



Since the angles are acute the cosines are positive, so we have

$$\cos\theta_1 = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|},$$
$$\cos\theta_2 = \frac{|\mathbf{v}\cdot\mathbf{w}|}{\|\mathbf{v}\|\|\mathbf{w}\|},$$
$$\cos\theta_3 = 180 - (\theta_1 + \theta_2) \qquad (1)$$

We compute the lengths and the dot products in (1):

$$\mathbf{u}\cdot\mathbf{v} = \langle 2, 7\rangle\cdot\langle 6, 3\rangle = 2\cdot 6 + 7\cdot 3 = 33$$
$$\mathbf{v}\cdot\mathbf{w} = \langle 6, 3\rangle\cdot\langle 4, -4\rangle = 6\cdot 4 + 3\cdot(-4) = 12$$
$$\|\mathbf{u}\| = \sqrt{2^2+7^2} = \sqrt{53}$$
$$\|\mathbf{v}\| = \sqrt{6^2+3^2} = \sqrt{45}$$
$$\|\mathbf{w}\| = \sqrt{4^2+(-4)^2} = \sqrt{32}$$

Substituting in (1) and solving for acute angles yields

$$\cos\theta_1 = \frac{33}{\sqrt{53}\sqrt{45}} \approx 0.676 \quad\Rightarrow\quad \theta_1 \approx 47.47^\circ$$
$$\cos\theta_2 = \frac{12}{\sqrt{45}\sqrt{32}} \approx 0.316 \quad\Rightarrow\quad \theta_2 \approx 71.58^\circ$$

The sum of the angles in a triangle is 180°, hence

$$\theta_3 = 180^\circ - (47.47 + 71.58) \approx 60.95^\circ.$$

In Exercises 51–58, find the projection of **u** *along* **v**.

51. $\mathbf{u} = \langle 2, 5 \rangle, \quad \mathbf{v} = \langle 1, 1 \rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, 5 \rangle \cdot \langle 1, 1 \rangle = 7$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 1^2 = 2$$

The projection of **u** along **v** is the following vector:

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \frac{7}{2}\mathbf{v} = \left\langle \frac{7}{2}, \frac{7}{2} \right\rangle$$

52. $\mathbf{u} = \langle 2, -3 \rangle, \quad \mathbf{v} = \langle 1, 2 \rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, -3 \rangle \cdot \langle 1, 2 \rangle = -4$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 2^2 = 5$$

The projection of **u** along **v** is the following vector:

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \frac{-4}{5}\mathbf{v} = \left\langle \frac{-4}{5}, \frac{-8}{5} \right\rangle$$

53. $\mathbf{u} = \langle -1, 2, 0 \rangle, \quad \mathbf{v} = \langle 2, 0, 1 \rangle$

SOLUTION The projection of **u** along **v** is the following vector:

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$$

We compute the values in this expression:

$$\mathbf{u} \cdot \mathbf{v} = \langle -1, 2, 0 \rangle \cdot \langle 2, 0, 1 \rangle = -1 \cdot 2 + 2 \cdot 0 + 0 \cdot 1 = -2$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 2^2 + 0^2 + 1^2 = 5$$

Hence,

$$\mathbf{u}_{||} = -\frac{2}{5}\langle 2, 0, 1 \rangle = \left\langle -\frac{4}{5}, 0, -\frac{2}{5} \right\rangle.$$

54. $\mathbf{u} = \langle 1, 1, 1 \rangle, \quad \mathbf{v} = \langle 1, 1, 0 \rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 0 \rangle = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 2$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 1^2 + 0^2 = 2$$

The projection of **u** along **v** is the following vector:

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \frac{2}{2}\mathbf{v} = \mathbf{v} = \langle 1, 1, 0 \rangle$$

55. $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}, \quad \mathbf{v} = \mathbf{k}$

SOLUTION The projection of **u** along **v** is the following vector:

$$\mathbf{u}_{||} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$$

We compute the dot products:

$$\mathbf{u} \cdot \mathbf{v} = (5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}) \cdot \mathbf{k} = -4\mathbf{k} \cdot \mathbf{k} = -4$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = \|\mathbf{k}\|^2 = 1$$

Hence,

$$\mathbf{u}_{||} = \frac{-4}{1}\mathbf{k} = -4\mathbf{k}$$

56. $\mathbf{u} = \mathbf{i} + 29\mathbf{k}, \quad \mathbf{v} = \mathbf{j}$

SOLUTION Since $\mathbf{u} \cdot \mathbf{v} = (\mathbf{i} + 29\mathbf{k}) \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{j} + 29\mathbf{k} \cdot \mathbf{j} = 0$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$, the projection of $\mathbf{u}$ along $\mathbf{v}$ is the zero vector

$$\mathbf{u}_{||} = \mathbf{0}$$

57. $\mathbf{u} = \langle a, b, c \rangle, \quad \mathbf{v} = \mathbf{i}$

SOLUTION The component of $\mathbf{u}$ along $\mathbf{v}$ is a, since

$$\mathbf{u} \cdot \mathbf{e_v} = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{i} = a$$

Therefore, the projection of $\mathbf{u}$ along $\mathbf{v}$ is the vector

$$\mathbf{u}_{||} = (\mathbf{u} \cdot \mathbf{e_v})\mathbf{e_v} = a\mathbf{i}$$

58. $\mathbf{u} = \langle a, a, b \rangle, \quad \mathbf{v} = \mathbf{i} - \mathbf{j}$

SOLUTION We compute the following dot product:

$$\mathbf{u} \cdot \mathbf{v} = (a\mathbf{i} + a\mathbf{j} + b\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = (a\mathbf{i} + a\mathbf{j} + b\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 0\mathbf{k}) = a \cdot 1 + a \cdot (-1) + b \cdot 0 = 0$$

The dot product is zero, hence the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, and the projection of $\mathbf{u}$ along $\mathbf{v}$ is the zero vector:

$$\mathbf{u}_{||} = \mathbf{0}$$

In Exercises 59 and 60, compute the component of $\mathbf{u}$ *along* $\mathbf{v}$.

59. $\mathbf{u} = \langle 3, 2, 1 \rangle, \quad \mathbf{v} = \langle 1, 0, 1 \rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 3, 2, 1 \rangle \cdot \langle 1, 0, 1 \rangle = 4$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 1^2 = 2$$

The component of $\mathbf{u}$ along $\mathbf{v}$ is the length of the projection of $\mathbf{u}$ along $\mathbf{v}$

$$\left\| \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \right\| = \frac{4}{2} \|\mathbf{v}\| = 2\|\mathbf{v}\| = 2\sqrt{2}$$

60. $\mathbf{u} = \langle 3, 0, 9 \rangle, \quad \mathbf{v} = \langle 1, 2, 2 \rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 3, 0, 9 \rangle \cdot \langle 1, 2, 2 \rangle = 21$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 2^2 + 2^2 = 9$$

The component of $\mathbf{u}$ along $\mathbf{v}$ is the length of the projection of $\mathbf{u}$ along $\mathbf{v}$

$$\left\| \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \right\| = \frac{21}{9} \|\mathbf{v}\| = \frac{7}{3} \cdot 3 = 7$$

61. Find the length of $\overline{OP}$ in Figure 14.

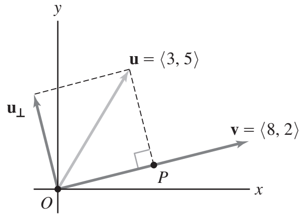


FIGURE 14

SOLUTION This is just the component of $\mathbf{u} = \langle 3, 5\rangle$ along $\mathbf{v} = \langle 8, 2\rangle$. We first compute the following dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle 3, 5\rangle \cdot \langle 8, 2\rangle = 34$$
$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 8^2 + 2^2 = 68$$

The component of $\mathbf{u}$ along $\mathbf{v}$ is the length of the projection of $\mathbf{u}$ along $\mathbf{v}$

$$\left\| \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \right\| = \frac{34}{68}\|\mathbf{v}\| = \frac{34}{68}\sqrt{68}$$

62. Find $\|\mathbf{u}_\perp\|$ in Figure 14.

SOLUTION From the previous problem (see solution above) we know that the component of $\mathbf{u}$ along $\mathbf{v}$ is $1/2$, and thus the projection is $\mathbf{u}_\| = \langle 4, 1\rangle$. Using the standard formula for $\mathbf{u}_\perp$, we obtain

$$\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\| = \langle 3, 5\rangle - \langle 4, 1\rangle = \langle -1, 4\rangle$$

In Exercises 63–68, find the decomposition $\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_\perp$ *with respect to* $\mathbf{b}$.

63. $\mathbf{a} = \langle 1, 0\rangle, \quad \mathbf{b} = \langle 1, 1\rangle$

SOLUTION

Step 1. We compute $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 0\rangle \cdot \langle 1, 1\rangle = 1 \cdot 1 + 0 \cdot 1 = 1$$
$$\mathbf{b} \cdot \mathbf{b} = \|\mathbf{b}\|^2 = 1^2 + 1^2 = 2$$

Step 2. We find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b} = \frac{1}{2}\langle 1, 1\rangle = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle$$

Step 3. We find the orthogonal part as the difference:

$$\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_{||} = \langle 1, 0\rangle - \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Hence,

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_\perp = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle + \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

64. $\mathbf{a} = \langle 2, -3\rangle, \quad \mathbf{b} = \langle 5, 0\rangle$

SOLUTION We first compute $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{b} = \langle 2, -3\rangle \cdot \langle 5, 0\rangle = 10$$
$$\mathbf{b} \cdot \mathbf{b} = \|\mathbf{b}\|^2 = 5^2 + 0^2 = 25$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b} = \frac{10}{25}\langle 5, 0\rangle = \langle 2, 0\rangle$$

We now find the vector $\mathbf{a}_\perp$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle 2, -3\rangle - \langle 2, 0\rangle = \langle 0, -3\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_\perp = \langle 2, 0\rangle + \langle 0, -3\rangle$$

65. $\mathbf{a} = \langle 4, -1, 0\rangle, \quad \mathbf{b} = \langle 0, 1, 1\rangle$

SOLUTION We first compute $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{b} = \langle 4, -1, 0\rangle \cdot \langle 0, 1, 1\rangle = 4 \cdot 0 + (-1) \cdot 1 + 0 \cdot 1 = -1$$
$$\mathbf{b} \cdot \mathbf{b} = \|\mathbf{b}\|^2 = 0^2 + 1^2 + 1^2 = 2$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{-1}{2}\langle 0, 1, 1\rangle = \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle$$

We now find the vector $\mathbf{a}_{\perp}$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle 4, -1, 0\rangle - \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle = \left\langle 4, -\frac{1}{2}, \frac{1}{2}\right\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} = \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle 4, -\frac{1}{2}, \frac{1}{2}\right\rangle$$

66. $\mathbf{a} = \langle 4, -1, 5\rangle, \quad \mathbf{b} = \langle 2, 1, 1\rangle$

SOLUTION We first compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = \langle 4, -1, 5\rangle \cdot \langle 2, 1, 1\rangle = 12$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 2^2 + 1^2 + 1^2 = 6$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = 2\langle 2, 1, 1\rangle = \langle 4, 2, 2\rangle$$

We now find the vector $\mathbf{a}_{\perp}$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle 4, -1, 5\rangle - \langle 4, 2, 2\rangle = \langle 0, -3, 3\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} = \langle 4, 2, 2\rangle + \langle 0, -3, 3\rangle$$

67. $\mathbf{a} = \langle x, y\rangle, \quad \mathbf{b} = \langle 1, -1\rangle$

SOLUTION We first compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = \langle x, y\rangle \cdot \langle 1, -1\rangle = x - y$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 1^2 + (-1)^2 = 2$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{x-y}{2}\langle 1, -1\rangle = \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle$$

We now find the vector $\mathbf{a}_{\perp}$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle x, y\rangle - \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle = \left\langle \frac{x+y}{2}, \frac{x+y}{2}\right\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} = \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle + \left\langle \frac{x+y}{2}, \frac{x+y}{2}\right\rangle$$

68. $\mathbf{a} = \langle x, y, z\rangle, \quad \mathbf{b} = \langle 1, 1, 1\rangle$

SOLUTION We first compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = \langle x, y, z\rangle \cdot \langle 1, 1, 1\rangle = x + y + z$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 1^2 + 1^2 + 1^2 = 3$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{x+y+z}{3}\langle 1, 1, 1\rangle = \left\langle \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right\rangle$$

We now find the vector $\mathbf{a}_\perp$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle x, y, z\rangle - \left\langle \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3} \right\rangle = \left\langle \frac{2x-y-z}{3}, \frac{-x+2y-z}{3}, \frac{-x-y+2z}{3} \right\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_\perp = \left\langle \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3} \right\rangle + \left\langle \frac{2x-y-z}{3}, \frac{-x+2y-z}{3}, \frac{-x-y+2z}{3} \right\rangle$$

69. Let $\mathbf{e}_\theta = \langle \cos\theta, \sin\theta\rangle$. Show that $\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \cos(\theta - \psi)$ for any two angles θ and ψ.

SOLUTION First, $\mathbf{e}_\theta$ is a unit vector since by a trigonometric identity we have

$$\|\mathbf{e}_\theta\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

The cosine of the angle α between $\mathbf{e}_\theta$ and the vector $\mathbf{i}$ in the direction of the positive x-axis is

$$\cos\alpha = \frac{\mathbf{e}_\theta \cdot \mathbf{i}}{\|\mathbf{e}_\theta\| \cdot \|\mathbf{i}\|} = \mathbf{e}_\theta \cdot \mathbf{i} = ((\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}) \cdot \mathbf{i} = \cos\theta$$

The solution of $\cos\alpha = \cos\theta$ for angles between 0 and π is $\alpha = \theta$. That is, the vector $\mathbf{e}_\theta$ makes an angle θ with the x-axis. We now use the trigonometric identity

$$\cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi)$$

to obtain the following equality:

$$\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \langle \cos\theta, \sin\theta\rangle \cdot \langle \cos\psi, \sin\psi\rangle = \cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi)$$

70. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in the plane.

(a) Use Theorem 2 to explain why the dot product $\mathbf{v} \cdot \mathbf{w}$ does not change if both $\mathbf{v}$ and $\mathbf{w}$ are rotated by the same angle θ.

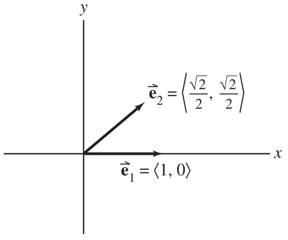
(b) Sketch the vectors $\mathbf{e}_1 = \langle 1, 0\rangle$ and $\mathbf{e}_2 = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$, and determine the vectors $\mathbf{e}_1', \mathbf{e}_2'$ obtained by rotating $\mathbf{e}_1, \mathbf{e}_2$ through an angle $\frac{\pi}{4}$. Verify that $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1' \cdot \mathbf{e}_2'$.

SOLUTION

(a) By Theorem 2,

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\alpha$$

where α is the angle between $\mathbf{v}$ and $\mathbf{w}$. Since rotation by an angle θ does not change the angle between the vectors, nor the norms of the vectors, the dot product $\mathbf{v} \cdot \mathbf{w}$ remains unchanged.



(b) Notice from the picture that if we rotate $\mathbf{e}_1$ by $\pi/4$, we get $\mathbf{e}_2$, and when we rotate $\mathbf{e}_2$ by the same amount we get a unit vector along the y axis. Thus, $\mathbf{e}_1' = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\mathbf{e}_2' = \langle 0, 1\rangle$. Note that $\mathbf{e}_1 \cdot \mathbf{e}_2 = 1 \cdot \frac{\sqrt{2}}{2} + 0 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$ and $\mathbf{e}_1' \cdot \mathbf{e}_2' = 0 \cdot \frac{\sqrt{2}}{2} + 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$. Thus, $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1' \cdot \mathbf{e}_2'$.

In Exercises 71–74, refer to Figure 15.

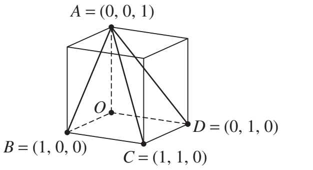
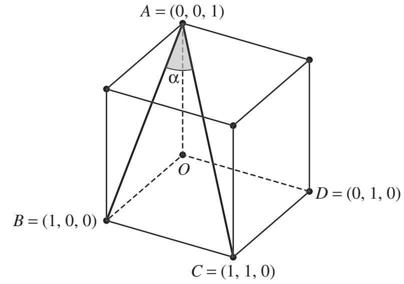


FIGURE 15 Unit cube in $\mathbf{R}^3$.

71. Find the angle between $\overline{AB}$ and $\overline{AC}$.

SOLUTION The cosine of the angle α between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is

$$\cos\alpha = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} \tag{1}$$



We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and then calculate their dot product and lengths. We get

$$\begin{aligned}
\overrightarrow{AB} &= \langle 1-0, 0-0, 0-1\rangle = \langle 1, 0, -1\rangle\\
\overrightarrow{AC} &= \langle 1-0, 1-0, 0-1\rangle = \langle 1, 1, -1\rangle\\
\overrightarrow{AB}\cdot\overrightarrow{AC} &= \langle 1, 0, -1\rangle\cdot\langle 1, 1, -1\rangle = 1\cdot 1 + 0\cdot 1 + (-1)\cdot(-1) = 2\\
\|\overrightarrow{AB}\| &= \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}\\
\|\overrightarrow{AC}\| &= \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}
\end{aligned}$$

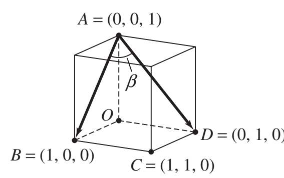
Substituting in (1) and solving for $0 \le \alpha \le 90^\circ$ gives

$$\cos\alpha = \frac{2}{\sqrt{2}\cdot\sqrt{3}} \approx 0.816 \quad\Rightarrow\quad \alpha \approx 35.31^\circ.$$

72. Find the angle between $\overline{AB}$ and $\overline{AD}$.

SOLUTION The cosine of the angle β between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is

$$\cos\beta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{\|\overrightarrow{AB}\|\|\overrightarrow{AD}\|} \tag{1}$$



We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and then calculate their dot product and lengths. This gives

$$\begin{aligned}
\overrightarrow{AB} &= \langle 1-0, 0-0, 0-1\rangle = \langle 1, 0, -1\rangle\\
\overrightarrow{AD} &= \langle 0-0, 1-0, 0-1\rangle = \langle 0, 1, -1\rangle\\
\overrightarrow{AB}\cdot\overrightarrow{AD} &= \langle 1, 0, -1\rangle\cdot\langle 0, 1, -1\rangle = 1\cdot 0 + 0\cdot 1 + (-1)\cdot(-1) = 1\\
\|\overrightarrow{AB}\| &= \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}\\
\|\overrightarrow{AD}\| &= \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}
\end{aligned}$$

Substituting in (1) and solving for $0 \le \beta \le 90^\circ$ gives

$$\cos\beta = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \quad\Rightarrow\quad \beta = 60^\circ.$$

It's interesting to note that we could have done this problem in a much simpler way. The triangle *ABD* is equilateral since each side is the diagonal of a unit square. Hence, all interior angles of the triangle are 60 degrees!

73. Calculate the projection of $\overrightarrow{AC}$ along $\overrightarrow{AD}$.

SOLUTION $\overline{DC}$ is perpendicular to the face OAD of the cube. Hence, it is orthogonal to the segment $\overline{AD}$ on this face. Therefore, the projection of the vector $\overrightarrow{AC}$ along $\overrightarrow{AD}$ is the vector $\overrightarrow{AD}$ itself.

74. Calculate the projection of $\overrightarrow{AD}$ along $\overrightarrow{AB}$.

SOLUTION The projection of $\overrightarrow{AD}$ along $\overrightarrow{AB}$ is the following vector:

$$\overrightarrow{AD}_{\|} = \left(\frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB} \tag{1}$$

We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and then calculate the dot product appearing in (1). We obtain

$$\begin{aligned}
\overrightarrow{AB} &= \langle 1-0, 0-0, 0-1\rangle = \langle 1, 0, -1\rangle \\
\overrightarrow{AD} &= \langle 0-0, 1-0, 0-1\rangle = \langle 0, 1, -1\rangle \\
\overrightarrow{AB}\cdot\overrightarrow{AD} &= \langle 1, 0, -1\rangle\cdot\langle 0, 1, -1\rangle = 1\cdot 0 + 0\cdot 1 + (-1)\cdot(-1) = 1 \\
\overrightarrow{AB}\cdot\overrightarrow{AB} &= \|\overrightarrow{AB}\|^2 = 1^2 + 0^2 + (-1)^2 = 2
\end{aligned}$$

Substituting in (1) gives

$$\overrightarrow{AD}_{\|} = \frac{1}{2}\langle 1, 0, -1\rangle = \left\langle \frac{1}{2}, 0, -\frac{1}{2}\right\rangle.$$

75. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and set $\mathbf{u} = \mathbf{e_v} + \mathbf{e_w}$. Use the dot product to show that the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{u}$ and $\mathbf{w}$. Explain this result geometrically with a diagram.

SOLUTION We denote by α the angle between $\mathbf{u}$ and $\mathbf{v}$ and by β the angle between $\mathbf{u}$ and $\mathbf{w}$. Since $\mathbf{e_v}$ and $\mathbf{e_w}$ are vectors in the directions of $\mathbf{v}$ and $\mathbf{w}$ respectively, α is the angle between $\mathbf{u}$ and $\mathbf{e_v}$ and β is the angle between $\mathbf{u}$ and $\mathbf{e_w}$. The cosines of these angles are thus

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{e_v}}{\|\mathbf{u}\|\|\mathbf{e_v}\|} = \frac{\mathbf{u}\cdot\mathbf{e_v}}{\|\mathbf{u}\|}; \quad \cos\beta = \frac{\mathbf{u}\cdot\mathbf{e_w}}{\|\mathbf{u}\|\|\mathbf{e_w}\|} = \frac{\mathbf{u}\cdot\mathbf{e_w}}{\|\mathbf{u}\|}$$

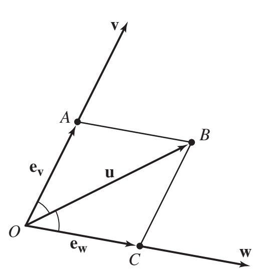
To show that $\cos\alpha = \cos\beta$ (which implies that $\alpha = \beta$) we must show that

$$\mathbf{u}\cdot\mathbf{e_v} = \mathbf{u}\cdot\mathbf{e_w}.$$

We compute the two dot products:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{e_v} &= (\mathbf{e_v}+\mathbf{e_w})\cdot\mathbf{e_v} = \mathbf{e_v}\cdot\mathbf{e_v} + \mathbf{e_w}\cdot\mathbf{e_v} = 1 + \mathbf{e_w}\cdot\mathbf{e_v} \\
\mathbf{u}\cdot\mathbf{e_w} &= (\mathbf{e_v}+\mathbf{e_w})\cdot\mathbf{e_w} = \mathbf{e_v}\cdot\mathbf{e_w} + \mathbf{e_w}\cdot\mathbf{e_w} = \mathbf{e_v}\cdot\mathbf{e_w} + 1
\end{aligned}$$

We see that $\mathbf{u}\cdot\mathbf{e_v} = \mathbf{u}\cdot\mathbf{e_w}$. We conclude that $\cos\alpha = \cos\beta$, hence $\alpha = \beta$. Geometrically, $\mathbf{u}$ is a diagonal in the rhombus $OABC$ (see figure), hence it bisects the angle $\sphericalangle AOC$ of the rhombus.



76. Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{a}$ be nonzero vectors such that $\mathbf{v}\cdot\mathbf{a}=\mathbf{w}\cdot\mathbf{a}$. Is it true that $\mathbf{v}=\mathbf{w}$? Either prove this or give a counterexample.

SOLUTION The equality $\mathbf{v}\cdot\mathbf{a}=\mathbf{w}\cdot\mathbf{a}$ is equivalent to the following equality:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a}&=\mathbf{w}\cdot\mathbf{a}\\ \mathbf{v}\cdot\mathbf{a}-\mathbf{w}\cdot\mathbf{a}&=0\\ (\mathbf{v}-\mathbf{w})\cdot\mathbf{a}&=0\end{aligned}$$

That is, $\mathbf{v}-\mathbf{w}$ is orthogonal to $\mathbf{a}$ rather than $\mathbf{v}=\mathbf{w}$. Consider the following counterexample:

$$\mathbf{a}=\langle 1,0,1\rangle;\quad \mathbf{v}=\langle 3,1,1\rangle;\quad \mathbf{w}=\langle 4,1,0\rangle$$

Obviously, $\mathbf{v}\neq\mathbf{w}$, but $\mathbf{v}\cdot\mathbf{a}=\mathbf{w}\cdot\mathbf{a}$ since

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a}&=\langle 3,1,1\rangle\cdot\langle 1,0,1\rangle=3\cdot 1+1\cdot 0+1\cdot 1=4\\ \mathbf{w}\cdot\mathbf{a}&=\langle 4,1,0\rangle\cdot\langle 1,0,1\rangle=4\cdot 1+1\cdot 0+0\cdot 1=4\end{aligned}$$

77. Calculate the force (in newtons) required to push a 40-kg wagon up a 10° incline (Figure 16).

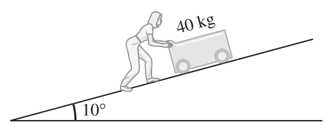
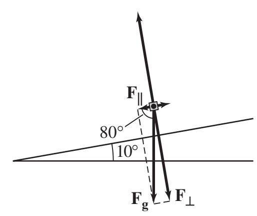


FIGURE 16

SOLUTION Gravity exerts a force $\mathbf{F}_g$ of magnitude $40g$ newtons where $g=9.8$. The magnitude of the force required to push the wagon equals the component of the force $\mathbf{F}_g$ along the ramp. Resolving $\mathbf{F}_g$ into a sum $\mathbf{F}_g=\mathbf{F}_{||}+\mathbf{F}_{\perp}$, where $\mathbf{F}_{||}$ is the force along the ramp and $\mathbf{F}_{\perp}$ is the force orthogonal to the ramp, we need to find the magnitude of $\mathbf{F}_{||}$. The angle between $\mathbf{F}_g$ and the ramp is $90^\circ-10^\circ=80^\circ$. Hence,

$$\mathbf{F}_{||}=\|\mathbf{F}_g\|\cos 80^\circ=40\cdot 9.8\cdot\cos 80^\circ\approx 68.07\text{ N}.$$



Therefore the minimum force required to push the wagon is 68.07 N. (Actually, this is the force required to keep the wagon from sliding down the hill; any slight amount greater than this force will serve to push it up the hill.)

78. A force $\mathbf{F}$ is applied to each of two ropes (of negligible weight) attached to opposite ends of a 40-kg wagon and making an angle of 35° with the horizontal (Figure 17). What is the maximum magnitude of $\mathbf{F}$ (in newtons) that can be applied without lifting the wagon off the ground?

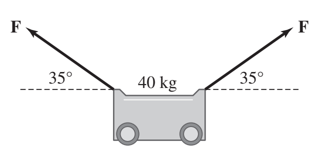
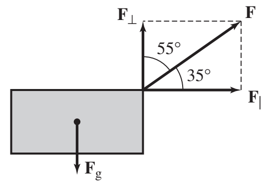


FIGURE 17

SOLUTION With two ropes at either end, both at the same angle with the horizontal and both with the same force, pulling on the 40-kg wagon, each rope will need to lift 20 kg. Let's look at the situation on the right-hand side of the wagon. We resolve the force $\mathbf{F}$ on the right-hand rope into a sum $\mathbf{F}=\mathbf{F}_{||}+\mathbf{F}_{\perp}$ where $\mathbf{F}_{||}$ is the horizontal force and $\mathbf{F}_{\perp}$ is the force orthogonal to the ground. The wagon will not be lifted off the ground if the magnitude of $\mathbf{F}_{\perp}$, that is the component of $\mathbf{F}$ along the direction orthogonal to the ground, is equal to (but not more than) the magnitude of the force due to gravity from 20 kg (remember, each rope needs to only lift half of the wagon, and remember also that the acceleration due to gravity is 9.8 meters per second squared). That is,

$$20(9.8)=\|\mathbf{F}_{\perp}\| \qquad (1)$$

The angle between $\mathbf{F}$ and a vector orthogonal to the ground is $90^\circ-35^\circ=55^\circ$, hence, $20(9.8)=196=\|\mathbf{F}\|\cos 55^\circ$



This gives us

$$196 = \|\mathbf{F}\| \cos 55^\circ \quad \Rightarrow \quad \|\mathbf{F}\| = \frac{196}{\cos 55^\circ} \approx 341 \text{ Newtons}$$

The maximum force that can be applied is of magnitude 341 newtons on each rope.

79. A light beam travels along the ray determined by a unit vector $\mathbf{L}$, strikes a flat surface at point P, and is reflected along the ray determined by a unit vector $\mathbf{R}$, where $\theta_1 = \theta_2$ (Figure 18). Show that if $\mathbf{N}$ is the unit vector orthogonal to the surface, then

$$\mathbf{R} = 2(\mathbf{L} \cdot \mathbf{N})\mathbf{N} - \mathbf{L}$$

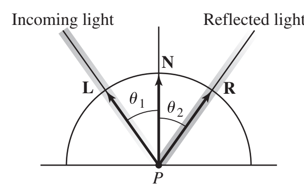
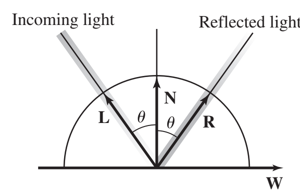


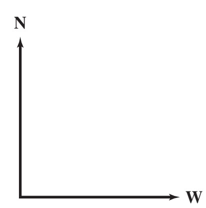
FIGURE 18

SOLUTION We denote by $\mathbf{W}$ a unit vector orthogonal to $\mathbf{N}$ in the direction shown in the figure, and let $\theta_1 = \theta_2 = \theta$.



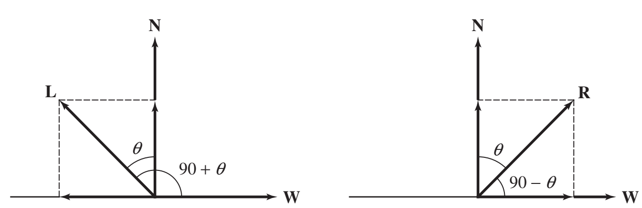
We resolve the unit vectors $\mathbf{R}$ and $\mathbf{L}$ into a sum of forces along $\mathbf{N}$ and $\mathbf{W}$. This gives

$$\begin{aligned} \mathbf{R} &= \cos(90 - \theta)\mathbf{W} + \cos\theta\mathbf{N} = \sin\theta\mathbf{W} + \cos\theta\mathbf{N} \\ \mathbf{L} &= -\cos(90 - \theta)\mathbf{W} + \cos\theta\mathbf{N} = -\sin\theta\mathbf{W} + \cos\theta\mathbf{N} \end{aligned} \tag{1}$$



Now, since

$$\mathbf{L} \cdot \mathbf{N} = \|\mathbf{L}\|\|\mathbf{N}\| \cos\theta = 1 \cdot 1 \cos\theta = \cos\theta$$



we have by (1):

$$\begin{aligned} 2(\mathbf{L} \cdot \mathbf{N})\mathbf{N} - \mathbf{L} &= (2\cos\theta)\mathbf{N} - \mathbf{L} = (2\cos\theta)\mathbf{N} - ((-\sin\theta)\mathbf{W} + (\cos\theta)\mathbf{N}) \\ &= (2\cos\theta)\mathbf{N} + (\sin\theta)\mathbf{W} - (\cos\theta)\mathbf{N} = (\sin\theta)\mathbf{W} + (\cos\theta)\mathbf{N} = \mathbf{R} \end{aligned}$$

80. Let P and Q be antipodal (opposite) points on a sphere of radius r centered at the origin and let R be a third point on the sphere (Figure 19). Prove that $\overline{PR}$ and $\overline{QR}$ are orthogonal.

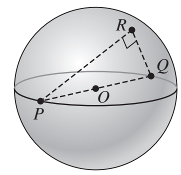
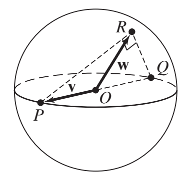


FIGURE 19

SOLUTION We denote the vectors $\overrightarrow{OP}$ and $\overrightarrow{OR}$ by

$$\mathbf{v} = \overrightarrow{OP}, \quad \mathbf{w} = \overrightarrow{OR}$$



Thus,

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\mathbf{v} + \mathbf{w}$$
$$\overrightarrow{RQ} = \overrightarrow{RO} + \overrightarrow{OQ} = -\mathbf{w} - \mathbf{v}$$

We now show that $\overrightarrow{PR} \cdot \overrightarrow{RQ} = 0$:

$$\begin{aligned}\overrightarrow{PR} \cdot \overrightarrow{RQ} &= (-\mathbf{v} + \mathbf{w}) \cdot (-\mathbf{w} - \mathbf{v}) = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2\end{aligned}$$

The lengths $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ equal the radius r of the sphere, hence,

$$\overrightarrow{PR} \cdot \overrightarrow{RQ} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = r^2 - r^2 = 0$$

The dot product of $\overrightarrow{PR}$ and $\overrightarrow{RQ}$ is zero, therefore the two vectors are orthogonal.

81. Prove that $\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = 4\mathbf{v} \cdot \mathbf{w}$.

SOLUTION We compute the following values:

$$\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$$
$$\|\mathbf{v} - \mathbf{w}\|^2 = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$$

Hence,

$$\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = (\|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2) - (\|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2) = 4\mathbf{v} \cdot \mathbf{w}$$

82. Use Exercise 81 to show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\|\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} + \mathbf{w}\|$.

SOLUTION In Exercise 81 we showed that

$$\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = 4\mathbf{v} \cdot \mathbf{w}$$

The vectors $\mathbf{v} \cdot \mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. That is, if and only if

$$\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = 0$$

or

$$\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|.$$

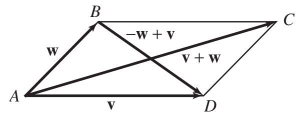
83. Show that the two diagonals of a parallelogram are perpendicular if and only if its sides have equal length. *Hint:* Use Exercise 82 to show that $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are orthogonal if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$.

SOLUTION We denote the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ by

$$\mathbf{w} = \overrightarrow{AB}, \quad \mathbf{v} = \overrightarrow{AD}.$$

Then,

$$\overrightarrow{AC} = \mathbf{w} + \mathbf{v}, \quad \overrightarrow{BD} = -\mathbf{w} + \mathbf{v}.$$



The diagonals are perpendicular if and only if the vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are orthogonal. By Exercise 82 these vectors are orthogonal if and only if the norms of the sum $(\mathbf{v} + \mathbf{w}) + (\mathbf{v} - \mathbf{w}) = 2\mathbf{v}$ and the difference $(\mathbf{v} + \mathbf{w}) - (\mathbf{v} - \mathbf{w}) = 2\mathbf{w}$ are equal, that is,

$$\|2\mathbf{v}\| = \|2\mathbf{w}\|$$
$$2\|\mathbf{v}\| = 2\|\mathbf{w}\| \quad \Rightarrow \quad \|\mathbf{v}\| = \|\mathbf{w}\|$$

84. Verify the Distributive Law:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

SOLUTION We denote the components of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ by

$$\mathbf{u} = \langle a_1, a_2, a_3 \rangle; \quad \mathbf{v} = \langle b_1, b_2, b_3 \rangle; \quad \mathbf{w} = \langle c_1, c_2, c_3 \rangle$$

We compute the left-hand side:

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle a_1, a_2, a_3 \rangle \left(\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle \right) \\ &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= \langle a_1(b_1 + c_1), a_2(b_2 + c_2), a_3(b_3 + c_3) \rangle\end{aligned}$$

Using the distributive law for scalars and the definitions of vector sum and the dot product we get

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle a_1b_1 + a_1c_1, a_2b_2 + a_2c_2, a_3b_3 + a_3c_3 \rangle \\ &= \langle a_1b_1, a_2b_2, a_3b_3 \rangle + \langle a_1c_1, a_2c_2, a_3c_3 \rangle \\ &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle + \langle a_1, a_2, a_3 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}\end{aligned}$$

85. Verify that $(\lambda\mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w})$ for any scalar λ.

SOLUTION We denote the components of the vectors $\mathbf{v}$ and $\mathbf{w}$ by

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle \quad \mathbf{w} = \langle b_1, b_2, b_3 \rangle$$

Thus,

$$\begin{aligned}(\lambda\mathbf{v}) \cdot \mathbf{w} &= (\lambda\langle a_1, a_2, a_3 \rangle) \cdot \langle b_1, b_2, b_3 \rangle = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle \\ &= \lambda a_1b_1 + \lambda a_2b_2 + \lambda a_3b_3\end{aligned}$$

Recalling that λ, a_i, and b_i are scalars and using the definitions of scalar multiples of vectors and the dot product, we get

$$(\lambda\mathbf{v}) \cdot \mathbf{w} = \lambda(a_1b_1 + a_2b_2 + a_3b_3) = \lambda\left(\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle\right) = \lambda(\mathbf{v} \cdot \mathbf{w})$$

Further Insights and Challenges

86. Prove the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$, by referring to Figure 20. *Hint:* Consider the right triangle $\triangle PQR$.

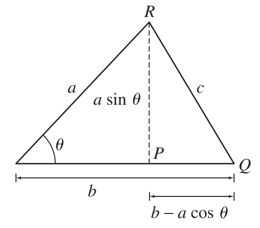
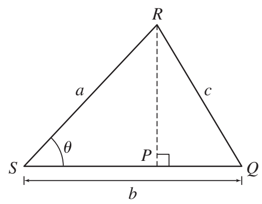


FIGURE 20

SOLUTION We denote the vertices of the triangle by S, Q, and R. Since $\overrightarrow{RQ} = \overrightarrow{RS} + \overrightarrow{SQ}$, we have

$$\begin{aligned} c^2 &= \|\overrightarrow{RQ}\|^2 = \overrightarrow{RQ}\cdot\overrightarrow{RQ} = \left(\overrightarrow{RS}+\overrightarrow{SQ}\right)\cdot\left(\overrightarrow{RS}+\overrightarrow{SQ}\right) \\ &= \overrightarrow{RS}\cdot\overrightarrow{RS} + \overrightarrow{RS}\cdot\overrightarrow{SQ} + \overrightarrow{SQ}\cdot\overrightarrow{RS} + \overrightarrow{SQ}\cdot\overrightarrow{SQ} \\ &= \|\overrightarrow{RS}\|^2 + 2\overrightarrow{RS}\cdot\overrightarrow{SQ} + \|\overrightarrow{SQ}\|^2 \\ c^2 &= a^2 + 2\overrightarrow{RS}\cdot\overrightarrow{SQ} + b^2 \end{aligned} \tag{1}$$



We find the dot product $\overrightarrow{RS}\cdot\overrightarrow{SQ}$. The angle between the vectors $\overrightarrow{RS}$ and $\overrightarrow{SQ}$ is θ, hence,

$$\overrightarrow{SR}\cdot\overrightarrow{SQ} = \|\overrightarrow{SR}\|\cdot\|\overrightarrow{SQ}\|\cos\theta = ab\cos\theta.$$

Therefore,

$$\overrightarrow{RS}\cdot\overrightarrow{SQ} = -\overrightarrow{SR}\cdot\overrightarrow{SQ} = -ab\cos\theta \tag{2}$$

Substituting (2) in (1) yields

$$c^2 = a^2 - 2ab\cos\theta + b^2 = a^2 + b^2 - 2ab\cos\theta.$$

(Note that we did not need to use the point P.)

87. In this exercise, we prove the Cauchy–Schwarz inequality: If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\| \tag{6}$$

(a) Let $f(x) = \|x\mathbf{v} + \mathbf{w}\|^2$ for x a scalar. Show that $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v}\cdot\mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) Conclude that $b^2 - 4ac \le 0$. *Hint:* Observe that $f(x) \ge 0$ for all x.

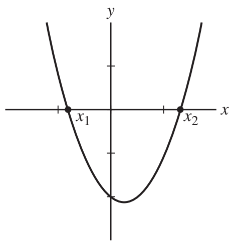
SOLUTION

(a) We express the norm as a dot product and compute it:

$$\begin{aligned} f(x) &= \|x\mathbf{v}+\mathbf{w}\|^2 = (x\mathbf{v}+\mathbf{w})\cdot(x\mathbf{v}+\mathbf{w}) \\ &= x^2\mathbf{v}\cdot\mathbf{v} + x\mathbf{v}\cdot\mathbf{w} + x\mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2x^2 + 2(\mathbf{v}\cdot\mathbf{w})x + \|\mathbf{w}\|^2 \end{aligned}$$

Hence, $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v}\cdot\mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) If f has distinct real roots x_1 and x_2, then $f(x)$ is negative for x between x_1 and x_2, but this is impossible since f is the square of a length.



$f(x) = ax^2 + bx + c,\ a > 0$

Using properties of quadratic functions, it follows that f has a nonpositive discriminant. That is, $b^2 - 4ac \le 0$. Substituting the values for a, b, and c, we get

$$4(\mathbf{v}\cdot\mathbf{w})^2 - 4\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \le 0$$
$$(\mathbf{v}\cdot\mathbf{w})^2 \le \|\mathbf{v}\|^2\|\mathbf{w}\|^2$$

Taking the square root of both sides we obtain

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$$

88. Use (6) to prove the Triangle Inequality

$$\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$$

Hint: First use the Triangle Inequality for numbers to prove

$$|(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})| \le |(\mathbf{v}+\mathbf{w})\cdot\mathbf{v}| + |(\mathbf{v}+\mathbf{w})\cdot\mathbf{w}|$$

SOLUTION Using the relation between the length and dot product we have

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2 \qquad (1)\end{aligned}$$

Obviously, $\mathbf{v}\cdot\mathbf{w} \le |\mathbf{v}\cdot\mathbf{w}|$. Also, by the Cauchy–Schwarz inequality $|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$. Therefore, $\mathbf{v}\cdot\mathbf{w} \le \|\mathbf{v}\|\|\mathbf{w}\|$, and combining this with (1) we get

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2 \le \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2$$

That is,

$$\|\mathbf{v}+\mathbf{w}\|^2 \le (\|\mathbf{v}\| + \|\mathbf{w}\|)^2$$

Taking the square roots of both sides and recalling that the length is nonnegative, we get

$$\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$$

89. This exercise gives another proof of the relation between the dot product and the angle θ between two vectors $\mathbf{v} = \langle a_1, b_1\rangle$ and $\mathbf{w} = \langle a_2, b_2\rangle$ in the plane. Observe that $\mathbf{v} = \|\mathbf{v}\|\langle\cos\theta_1, \sin\theta_1\rangle$ and $\mathbf{w} = \|\mathbf{w}\|\langle\cos\theta_2, \sin\theta_2\rangle$, with θ_1 and θ_2 as in Figure 21. Then use the addition formula for the cosine to show that

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$$

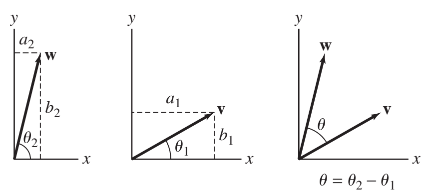


FIGURE 21

SOLUTION Using the trigonometric function for angles in right triangles, we have

$$a_2 = \|\mathbf{v}\|\sin\theta_1, \qquad a_1 = \|\mathbf{v}\|\cos\theta_1$$
$$b_2 = \|\mathbf{w}\|\sin\theta_2, \qquad b_1 = \|\mathbf{w}\|\cos\theta_2$$

Hence, using the given identity we obtain

$$\mathbf{v}\cdot\mathbf{w} = \langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2 = \|\mathbf{v}\|\cos\theta_1\|\mathbf{w}\|\cos\theta_2 + \|\mathbf{v}\|\sin\theta_1\|\mathbf{w}\|\sin\theta_2$$
$$= \|\mathbf{v}\|\|\mathbf{w}\|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta_1 - \theta_2)$$

That is,

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$$

90. Let $\mathbf{v} = \langle x, y\rangle$ and

$$\mathbf{v}_\theta = \langle x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle$$

Prove that the angle between $\mathbf{v}$ and $\mathbf{v}_\theta$ is θ.

SOLUTION The dot product of the vectors $\mathbf{v}$ and $\mathbf{v}_\theta$ is

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v}_\theta &= \langle x, y\rangle \cdot \langle x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle\\ &= x(x\cos\theta + y\sin\theta) + y(-x\sin\theta + y\cos\theta)\\ &= x^2\cos\theta + xy\sin\theta - xy\sin\theta + y^2\cos\theta\\ &= (x^2 + y^2)\cos\theta\end{aligned}$$

That is,

$$\mathbf{v}\cdot\mathbf{v}_\theta = (x^2 + y^2)\cos\theta \tag{1}$$

On the other hand, if α denotes the angle between $\mathbf{v}$ and $\mathbf{v}_\theta$, we have

$$\mathbf{v}\cdot\mathbf{v}_\theta = \|\mathbf{v}\|\|\mathbf{v}_\theta\|\cos\alpha \tag{2}$$

We compute the lengths. Using the identities $\cos^2\theta + \sin^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, we obtain

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{\langle x, y\rangle} = \sqrt{x^2 + y^2}\\ \|\mathbf{v}_\theta\| &= \sqrt{(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2}\\ &= \sqrt{x^2\cos^2\theta + xy\sin 2\theta + y^2\sin^2\theta + x^2\sin^2\theta - xy\sin 2\theta + y^2\cos^2\theta}\\ &= \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)} = \sqrt{x^2\cdot 1 + y^2\cdot 1} = \sqrt{x^2 + y^2}\end{aligned}$$

Substituting the lengths in (2) yields

$$\mathbf{v}\cdot\mathbf{v}_\theta = \sqrt{x^2 + y^2}\cdot\sqrt{x^2 + y^2}\cos\alpha = (x^2 + y^2)\cos\alpha \tag{3}$$

We now equate (1) and (3) to obtain

$$\begin{aligned}(x^2 + y^2)\cos\theta &= (x^2 + y^2)\cos\alpha\\ \cos\theta &= \cos\alpha\end{aligned}$$

The solution for angles between 0° and 180° is $\alpha = 0$. That is, the angle between $\mathbf{v}$ and $\mathbf{v}_\theta$ is θ.

91. Let $\mathbf{v}$ be a nonzero vector. The angles α, β, γ between $\mathbf{v}$ and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are called the direction angles of $\mathbf{v}$ (Figure 22). The cosines of these angles are called the **direction cosines** of $\mathbf{v}$. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

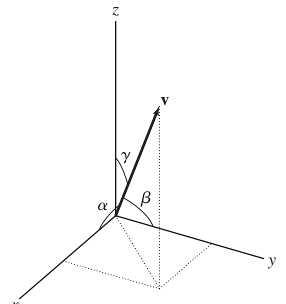


FIGURE 22 Direction angles of $\mathbf{v}$.

SOLUTION We use the relation between the dot product and the angle between two vectors to write

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|}$$
$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|} \tag{1}$$
$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|}$$

We compute the values involved in (1). Letting $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ we get

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$
$$\mathbf{v}\cdot\mathbf{j} = \langle v_1, v_2, v_3\rangle \cdot \langle 0, 1, 0\rangle = v_2$$
$$\mathbf{v}\cdot\mathbf{k} = \langle v_1, v_2, v_3\rangle \cdot \langle 0, 0, 1\rangle = v_3$$
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

We now substitute (2) into (1) to obtain

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

Finally, we compute the sum of squares of the direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_3}{\|\mathbf{v}\|}\right)^2 = \frac{1}{\|\mathbf{v}\|^2}(v_1^2 + v_2^2 + v_3^2) = \frac{1}{\|\mathbf{v}\|^2}\cdot\|\mathbf{v}\|^2 = 1$$

92. Find the direction cosines of $\mathbf{v} = \langle 3, 6, -2\rangle$.

SOLUTION Let α, β, γ denote the angles between $\mathbf{v}$ and the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. We need to compute $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Using the formula for the angle between two vectors and the lengths $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$, $\|\mathbf{v}\| = \sqrt{3^2 + 6^2 + (-2)^2} = 7$ we get

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})\cdot\mathbf{i}}{7} = \frac{3}{7}$$
$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})\cdot\mathbf{j}}{7} = \frac{6}{7}$$
$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})\cdot\mathbf{k}}{7} = -\frac{2}{7}$$

93. The set of all points $X = (x, y, z)$ equidistant from two points P, Q in $\mathbf{R}^3$ is a plane (Figure 23). Show that X lies on this plane if

$$\overrightarrow{PQ}\cdot\overrightarrow{OX} = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right) \tag{7}$$

Hint: If R is the midpoint of $\overline{PQ}$, then X is equidistant from P and Q if and only if $\overrightarrow{XR}$ is orthogonal to $\overrightarrow{PQ}$.

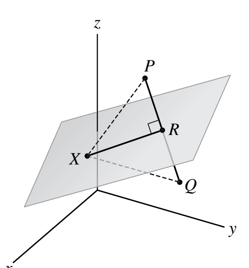


FIGURE 23

SOLUTION Let R be the midpoint of the segment $\overline{PQ}$. The points $X = (x, y, z)$ that are equidistant from P and Q are the points for which the vector $\overrightarrow{XR}$ is orthogonal to $\overrightarrow{PQ}$. That is,

$$\overrightarrow{XR}\cdot\overrightarrow{PQ} = 0 \tag{1}$$

Since $\overrightarrow{XR} = \overrightarrow{XO} + \overrightarrow{OR}$ we have by (1):

$$O = \left(\overrightarrow{XO} + \overrightarrow{OR}\right) \cdot \overrightarrow{PQ} = \overrightarrow{XO} \cdot \overrightarrow{PQ} + \overrightarrow{OR} \cdot \overrightarrow{PQ} = -\overrightarrow{OX} \cdot \overrightarrow{PQ} + \overrightarrow{OR} \cdot \overrightarrow{PQ}$$

Transferring sides we get

$$\overrightarrow{OX} \cdot \overrightarrow{PQ} = \overrightarrow{OR} \cdot \overrightarrow{PQ} \tag{2}$$

We now write $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ on the right-hand-side of (2), and $\overrightarrow{OR} = \dfrac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$. We get

$$\begin{aligned}\overrightarrow{OX} \cdot \overrightarrow{PQ} &= \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) \cdot \left(\overrightarrow{PO} + \overrightarrow{OQ}\right) = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) \cdot \left(\overrightarrow{OQ} - \overrightarrow{OP}\right) \\ &= \frac{1}{2}\left(\overrightarrow{OP} \cdot \overrightarrow{OQ} - \overrightarrow{OP} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OQ} - \overrightarrow{OQ} \cdot \overrightarrow{OP}\right) = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right)\end{aligned}$$

Thus, we showed that the vector equation of the plane is

$$\overrightarrow{OX} \cdot \overrightarrow{PQ} = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right).$$

94. Sketch the plane consisting of all points $X = (x, y, z)$ equidistant from the points $P = (0, 1, 0)$ and $Q = (0, 0, 1)$. Use Eq. (7) to show that X lies on this plane if and only if $y = z$.

SOLUTION As seen in the solution to Problem 93, the point $X = (x, y, z)$ lies on the plane iff Eq. (7) holds. Using this equation with $X = (x, y, z)$, $P = (0, 1, 0)$, and $Q = (0, 0, 1)$ gives

$$\langle x, y, z\rangle \cdot \langle 0, -1, 1\rangle = \frac{1}{2}(1^2 - 1^2) = 0$$

This gives us $0x - 1y + 1z = 0$, which gives us $y = z$, as desired.

95. Use Eq. (7) to find the equation of the plane consisting of all points $X = (x, y, z)$ equidistant from $P = (2, 1, 1)$ and $Q = (1, 0, 2)$.

SOLUTION Using Eq. (7) with $X = (x, y, z)$, $P = (2, 1, 1)$, and $Q = (1, 0, 2)$ gives

$$\langle x, y, z\rangle \cdot \langle -1, -1, 1\rangle = \frac{1}{2}\left((\sqrt{5})^2 - (\sqrt{6})^2\right) = -\frac{1}{2}$$

This gives us $-1x - 1y + 1z = -\frac{1}{2}$, which leads to $2x + 2y - 2z = 1$.

12.4 The Cross Product (LT Section 13.4)

Preliminary Questions

1. What is the (1, 3) minor of the matrix $\begin{vmatrix} 3 & 4 & 2 \\ -5 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix}$?

SOLUTION The (1, 3) minor is obtained by crossing out the first row and third column of the matrix. That is,

$$\begin{vmatrix} 3 & 4 & 2 \\ -5 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix} \quad \Rightarrow \quad \begin{vmatrix} -5 & -1 \\ 4 & 0 \end{vmatrix}$$

2. The angle between two unit vectors $\mathbf{e}$ and $\mathbf{f}$ is $\frac{\pi}{6}$. What is the length of $\mathbf{e} \times \mathbf{f}$?

SOLUTION We use the Formula for the Length of the Cross Product:

$$\|\mathbf{e} \times \mathbf{f}\| = \|\mathbf{e}\|\|\mathbf{f}\| \sin\theta$$

Since $\mathbf{e}$ and $\mathbf{f}$ are unit vectors, $\|\mathbf{e}\| = \|\mathbf{f}\| = 1$. Also $\theta = \frac{\pi}{6}$, therefore,

$$\|\mathbf{e} \times \mathbf{f}\| = 1 \cdot 1 \cdot \sin\frac{\pi}{6} = \frac{1}{2}$$

The length of $\mathbf{e} \times \mathbf{f}$ is $\frac{1}{2}$.

3. What is $\mathbf{u} \times \mathbf{w}$, assuming that $\mathbf{w} \times \mathbf{u} = \langle 2, 2, 1 \rangle$?

SOLUTION By anti-commutativity of the cross product, we have

$$\mathbf{u} \times \mathbf{w} = -\mathbf{w} \times \mathbf{u} = -\langle 2, 2, 1 \rangle = \langle -2, -2, -1 \rangle$$

4. Find the cross product without using the formula:

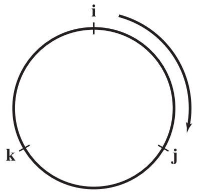
(a) $\langle 4, 8, 2 \rangle \times \langle 4, 8, 2 \rangle$ **(b)** $\langle 4, 8, 2 \rangle \times \langle 2, 4, 1 \rangle$

SOLUTION By properties of the cross product, the cross product of parallel vectors is the zero vector. In particular, the cross product of a vector with itself is the zero vector. Since $\langle 4, 8, 2 \rangle = 2\langle 2, 4, 1 \rangle$, the vectors $\langle 4, 8, 2 \rangle$ and $\langle 2, 4, 1 \rangle$ are parallel. We conclude that

$$\langle 4, 8, 2 \rangle \times \langle 4, 8, 2 \rangle = \mathbf{0} \quad \text{and} \quad \langle 4, 8, 2 \rangle \times \langle 2, 4, 1 \rangle = \mathbf{0}.$$

5. What are $\mathbf{i} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$?

SOLUTION The cross product $\mathbf{i} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$ are determined by the right-hand rule. We can also use the following figure to determine these cross-products:



We get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \text{ and } \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

6. When is the cross product $\mathbf{v} \times \mathbf{w}$ equal to zero?

SOLUTION The cross product $\mathbf{v} \times \mathbf{w}$ is equal to zero if one of the vectors $\mathbf{v}$ or $\mathbf{w}$ (or both) is the zero vector, or if $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors.

Exercises

In Exercises 1–4, calculate the 2×2 determinant.

1. $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$

SOLUTION Using the definition of 2×2 determinant we get

$$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 4 = -5$$

2. $\begin{vmatrix} \frac{2}{3} & \frac{1}{6} \\ -5 & 2 \end{vmatrix}$

SOLUTION Using the definition we get

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{6} \\ -5 & 2 \end{vmatrix} = \frac{2}{3} \cdot 2 - \frac{1}{6} \cdot (-5) = \frac{4}{3} + \frac{5}{6} = \frac{13}{6}$$

3. $\begin{vmatrix} -6 & 9 \\ 1 & 1 \end{vmatrix}$

SOLUTION We evaluate the determinant to obtain

$$\begin{vmatrix} -6 & 9 \\ 1 & 1 \end{vmatrix} = -6 \cdot 1 - 9 \cdot 1 = -15$$

4. $\begin{vmatrix} 9 & 25 \\ 5 & 14 \end{vmatrix}$

SOLUTION The value of the 2×2 determinant is

$$\begin{vmatrix} 9 & 25 \\ 5 & 14 \end{vmatrix} = 9 \cdot 14 - 5 \cdot 25 = 1$$

In Exercises 5–8, calculate the 3×3 *determinant.*

5. $\begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

SOLUTION Using the definition of 3×3 determinant we obtain

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix} &= 1\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} \\ &= 1 \cdot (-3 \cdot 1 - 0 \cdot 0) - 2 \cdot (4 \cdot 1 - 0 \cdot 1) + 1 \cdot (4 \cdot 0 - (-3) \cdot 1) \\ &= -3 - 8 + 3 = -8 \end{aligned}$$

6. $\begin{vmatrix} 1 & 0 & 1 \\ -2 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix}$

SOLUTION We evaluate the 3×3 determinant to obtain

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 1 \\ -2 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix} &= 1\begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - 0\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} \\ &= 1 \cdot (0 \cdot (-1) - 3 \cdot 3) - 0 + 1 \cdot (-2 \cdot 3 - 0 \cdot 1) \\ &= -9 - 6 = -15 \end{aligned}$$

7. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -4 & 2 \end{vmatrix}$

SOLUTION We have

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -4 & 2 \end{vmatrix} &= 1\begin{vmatrix} 4 & 6 \\ -4 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ -3 & -4 \end{vmatrix} \\ &= 1(4 \cdot 2 - 6 \cdot (-4)) - 2(2 \cdot 2 - 6 \cdot (-3)) + 3(2 \cdot (-4) - 4 \cdot (-3)) \\ &= 32 - 44 + 12 = 0 \end{aligned}$$

8. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$

SOLUTION We have

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} &= 1\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1(0 \cdot 0 - 1 \cdot (-1)) \\ &= 1 \end{aligned}$$

In Exercises 9–12, calculate $\mathbf{v} \times \mathbf{w}$.

9. $\mathbf{v} = \langle 1, 2, 1 \rangle, \quad \mathbf{w} = \langle 3, 1, 1 \rangle$

SOLUTION Using the definition of the cross product we get

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k}$$
$$= (2-1)\mathbf{i} - (1-3)\mathbf{j} + (1-6)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$$

10. $\mathbf{v} = \langle 2, 0, 0 \rangle, \quad \mathbf{w} = \langle -1, 0, 1 \rangle$

SOLUTION By the definition of the cross product we have

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \mathbf{k}$$
$$= (0-0)\,\mathbf{i} - (2-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = -2\mathbf{j}$$

11. $\mathbf{v} = \left\langle \frac{2}{3}, 1, \frac{1}{2} \right\rangle, \quad \mathbf{w} = \langle 4, -6, 3 \rangle$

SOLUTION We have

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & 1 & \frac{1}{2} \\ 4 & -6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} \\ -6 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{2}{3} & \frac{1}{2} \\ 4 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{2}{3} & 1 \\ 4 & -6 \end{vmatrix} \mathbf{k}$$
$$= (3+3)\,\mathbf{i} - (2-2)\,\mathbf{j} + (-4-4)\,\mathbf{k} = 6\mathbf{i} - 8\mathbf{k}$$

12. $\mathbf{v} = \langle 1, 1, 0 \rangle, \quad \mathbf{w} = \langle 0, 1, 1 \rangle$

SOLUTION The cross product $\mathbf{v} \times \mathbf{w}$ is the following vector:

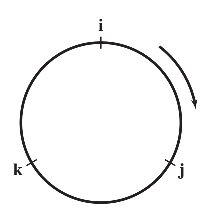
$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$
$$= (1-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + (1-0)\,\mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

In Exercises 13–16, use the relations in Eq. (5) to calculate the cross product.

13. $(\mathbf{i} + \mathbf{j}) \times \mathbf{k}$

SOLUTION We use basic properties of the cross product to obtain

$$(\mathbf{i} + \mathbf{j}) \times \mathbf{k} = \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{k} = -\mathbf{j} + \mathbf{i}$$



$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

14. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{j} + \mathbf{k})$

SOLUTION Using properties of the cross product we get

$$(\mathbf{j} - \mathbf{k}) \times (\mathbf{j} + \mathbf{k}) = (\mathbf{j} - \mathbf{k}) \times \mathbf{j} + (\mathbf{j} - \mathbf{k}) \times \mathbf{k} = \mathbf{j} \times \mathbf{j} - \mathbf{k} \times \mathbf{j} + \mathbf{j} \times \mathbf{k} - \mathbf{k} \times \mathbf{k}$$
$$= \mathbf{0} + \mathbf{i} + \mathbf{i} - \mathbf{0} = 2\mathbf{i}$$

15. $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times (\mathbf{j} - \mathbf{k})$

SOLUTION Using the distributive law we obtain

$$\begin{aligned}(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times (\mathbf{j} - \mathbf{k}) &= (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times \mathbf{j} - (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times (\mathbf{k}) \\ &= \mathbf{i} \times \mathbf{j} + 2\mathbf{k} \times \mathbf{j} - \mathbf{i} \times \mathbf{k} - (-3\mathbf{j}) \times \mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

16. $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 7\mathbf{k})$

SOLUTION We use the distributive law to obtain

$$\begin{aligned}(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 7\mathbf{k}) &= (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{i} + (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{j} + (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times (-7\mathbf{k}) \\ &= 2\mathbf{i} \times \mathbf{i} - 3\mathbf{j} \times \mathbf{i} + 4\mathbf{k} \times \mathbf{i} + 2\mathbf{i} \times \mathbf{j} - 3\mathbf{j} \times \mathbf{j} + 4\mathbf{k} \times \mathbf{j} - 14\mathbf{i} \times \mathbf{k} \\ &\quad + 21\mathbf{j} \times \mathbf{k} - 28\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + 3\mathbf{i} \times \mathbf{j} - 4\mathbf{i} \times \mathbf{k} + 2\mathbf{i} \times \mathbf{j} - \mathbf{0} - 4\mathbf{j} \times \mathbf{k} - 14\mathbf{i} \times \mathbf{k} + 21\mathbf{j} \times \mathbf{k} - \mathbf{0} \\ &= 5\mathbf{i} \times \mathbf{j} - 18\mathbf{i} \times \mathbf{k} + 17\mathbf{j} \times \mathbf{k} = 5\mathbf{k} + 18\mathbf{j} + 17\mathbf{i} \\ &= 17\mathbf{i} + 18\mathbf{j} + 5\mathbf{k}\end{aligned}$$

In Exercises 17–22, calculate the cross product assuming that

$$\mathbf{u} \times \mathbf{v} = \langle 1, 1, 0 \rangle, \quad \mathbf{u} \times \mathbf{w} = \langle 0, 3, 1 \rangle, \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, 1 \rangle$$

17. $\mathbf{v} \times \mathbf{u}$

SOLUTION Using the properties of the cross product we obtain

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = \langle -1, -1, 0 \rangle$$

18. $\mathbf{v} \times (\mathbf{u} + \mathbf{v})$

SOLUTION Using the properties of the cross product we obtain

$$\mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{v} = -\mathbf{u} \times \mathbf{v} + \mathbf{0} = \langle -1, -1, 0 \rangle$$

19. $\mathbf{w} \times (\mathbf{u} + \mathbf{v})$

SOLUTION Using the properties of the cross product we obtain

$$\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v} = -\mathbf{u} \times \mathbf{w} - \mathbf{v} \times \mathbf{w} = \langle -2, -2, -2 \rangle .$$

20. $(3\mathbf{u} + 4\mathbf{w}) \times \mathbf{w}$

SOLUTION Using the properties of the cross product we obtain

$$(3\mathbf{u} + 4\mathbf{w}) \times \mathbf{w} = 3\mathbf{u} \times \mathbf{w} + 4\mathbf{w} \times \mathbf{w} = \langle 0, 9, 3 \rangle$$

21. $(\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 2\mathbf{v})$

SOLUTION Using the properties of the cross product we obtain

$$\begin{aligned}(\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 2\mathbf{v}) &= (\mathbf{u} - 2\mathbf{v}) \times \mathbf{u} + (\mathbf{u} - 2\mathbf{v}) \times 2\mathbf{v} = \mathbf{u} \times \mathbf{u} - 2\mathbf{v} \times \mathbf{u} + \mathbf{u} \times 2\mathbf{v} - 4\mathbf{v} \times \mathbf{v} \\ &= 0 + 2\mathbf{u} \times \mathbf{v} + 2\mathbf{u} \times \mathbf{v} - 0 = 0 + 4\mathbf{u} \times \mathbf{v} = \langle 4, 4, 0 \rangle\end{aligned}$$

22. $(\mathbf{v}+\mathbf{w})\times(3\mathbf{u}+2\mathbf{v})$

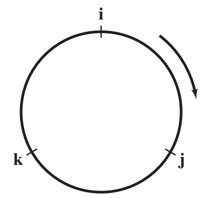
SOLUTION Using the properties of the cross product we obtain

$$\begin{aligned}(\mathbf{v}+\mathbf{w})\times(3\mathbf{u}+2\mathbf{v}) &= \mathbf{v}\times 3\mathbf{u}+\mathbf{w}\times 3\mathbf{u}+\mathbf{v}\times 2\mathbf{v}+\mathbf{w}\times 2\mathbf{v}\\ &= -3\mathbf{u}\times\mathbf{v}-3\mathbf{u}\times\mathbf{w}+0-2\mathbf{v}\times\mathbf{w}=\langle -7,-10,-5\rangle\end{aligned}$$

23. Let $\mathbf{v}=\langle a,b,c\rangle$. Calculate $\mathbf{v}\times\mathbf{i}$, $\mathbf{v}\times\mathbf{j}$, and $\mathbf{v}\times\mathbf{k}$.

SOLUTION We write $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and use the distributive law:

$$\mathbf{v}\times\mathbf{i}=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{i}=a\mathbf{i}\times\mathbf{i}+b\mathbf{j}\times\mathbf{i}+c\mathbf{k}\times\mathbf{i}=a\cdot\mathbf{0}-b\mathbf{k}+c\mathbf{j}=-b\mathbf{k}+c\mathbf{j}=\langle 0,c,-b\rangle$$
$$\mathbf{v}\times\mathbf{j}=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{j}=a\mathbf{i}\times\mathbf{j}+b\mathbf{j}\times\mathbf{j}+c\mathbf{k}\times\mathbf{j}=a\mathbf{k}+b\mathbf{0}-c\mathbf{i}=a\mathbf{k}-c\mathbf{i}=\langle -c,0,a\rangle$$
$$\mathbf{v}\times\mathbf{k}=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{k}=a\mathbf{i}\times\mathbf{k}+b\mathbf{j}\times\mathbf{k}+c\mathbf{k}\times\mathbf{k}=-a\mathbf{j}+b\mathbf{i}+c\mathbf{0}=-a\mathbf{j}+b\mathbf{i}=\langle b,-a,0\rangle$$



24. Find $\mathbf{v}\times\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors of length 3 in the xz-plane, oriented as in Figure 15, and $\theta=\frac{\pi}{6}$.

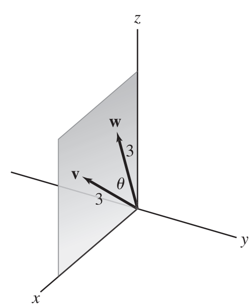


FIGURE 15

SOLUTION Recall that $\|\mathbf{v}\times\mathbf{w}\|=\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$, and since the vectors $\mathbf{v}$, $\mathbf{w}$ have length 3 and since the angle θ is $\pi/6$, we get that the cross product $\mathbf{v}\times\mathbf{w}$ has length $3\cdot 3\cdot\frac{1}{2}$ which is $\frac{9}{2}$. By the Right-Hand Rule, $\mathbf{v}\times\mathbf{w}$ must point along the negative y-axis. Thus, $\mathbf{v}\times\mathbf{w}=\langle 0,-\frac{9}{2},0\rangle$.

In Exercises 25 and 26, refer to Figure 16.

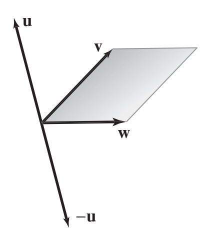


FIGURE 16

25. Which of $\mathbf{u}$ and $-\mathbf{u}$ is equal to $\mathbf{v}\times\mathbf{w}$?

SOLUTION The direction of $\mathbf{v}\times\mathbf{w}$ is determined by the right-hand rule, that is, our thumb points in the direction of $\mathbf{v}\times\mathbf{w}$ when the fingers of our right hand curl from $\mathbf{v}$ to $\mathbf{w}$. Therefore $\mathbf{v}\times\mathbf{w}$ equals $-\mathbf{u}$ rather than $\mathbf{u}$.

26. Which of the following form a right-handed system?

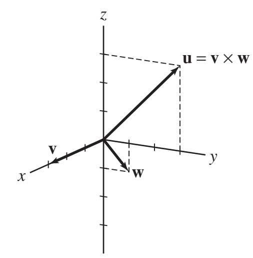
(a) $\{\mathbf{v},\mathbf{w},\mathbf{u}\}$ **(b)** $\{\mathbf{w},\mathbf{v},\mathbf{u}\}$ **(c)** $\{\mathbf{v},\mathbf{u},\mathbf{w}\}$

(d) $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ **(e)** $\{\mathbf{w},\mathbf{v},-\mathbf{u}\}$ **(f)** $\{\mathbf{v},-\mathbf{u},\mathbf{w}\}$

SOLUTION Applying the right-hand rule (and assuming that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all mutually perpendicular), we see that only (b) and (c) form a right-handed system.

27. Let $\mathbf{v} = \langle 3, 0, 0\rangle$ and $\mathbf{w} = \langle 0, 1, -1\rangle$. Determine $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ using the geometric properties of the cross product rather than the formula.

SOLUTION The cross product $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$.



Since $\mathbf{v}$ lies along the x-axis, $\mathbf{u}$ lies in the yz-plane, therefore $\mathbf{u} = \langle 0, b, c\rangle$. $\mathbf{u}$ is also orthogonal to $\mathbf{w}$, so $\mathbf{u} \cdot \mathbf{w} = 0$. This gives $\mathbf{u} \cdot \mathbf{w} = \langle 0, b, c\rangle \cdot \langle 0, 1, -1\rangle = b - c = 0 \Rightarrow b = c$. Thus, $\mathbf{u} = \langle 0, b, b\rangle$. By the right-hand rule, $\mathbf{u}$ points to the positive z-direction so $b > 0$. We compute the length of $\mathbf{u}$. Since $\mathbf{v} \cdot \mathbf{w} = \langle 3, 0, 0\rangle \cdot \langle 0, 1, -1\rangle = 0$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Hence,

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\frac{\pi}{2} = \|\mathbf{v}\|\|\mathbf{w}\| = 3 \cdot \sqrt{2}.$$

Also since $b > 0$, we have

$$\|\mathbf{u}\| = \|\langle 0, b, b\rangle\| = \sqrt{2b^2} = b\sqrt{2}$$

Equating the lengths gives

$$b\sqrt{2} = 3\sqrt{2} \quad \Rightarrow \quad b = 3.$$

We conclude that $\mathbf{u} = \mathbf{v} \times \mathbf{w} = \langle 0, 3, 3\rangle$.

28. What are the possible angles θ between two unit vectors $\mathbf{e}$ and $\mathbf{f}$ if $\|\mathbf{e} \times \mathbf{f}\| = \frac{1}{2}$?

SOLUTION Using the length of the cross product we have

$$\frac{1}{2} = \|\mathbf{e} \times \mathbf{f}\| = \|\mathbf{e}\|\|\mathbf{f}\| \sin\theta = 1 \cdot 1 \sin\theta.$$

That is, $\sin\theta = \frac{1}{2}$. The solution for $0 \le \theta \le \pi$ are $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. We conclude that the possible angles between $\mathbf{e}$ and $\mathbf{f}$ are $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

29. Show that if $\mathbf{v}$ and $\mathbf{w}$ lie in the yz-plane, then $\mathbf{v} \times \mathbf{w}$ is a multiple of $\mathbf{i}$.

SOLUTION $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{v}$ and $\mathbf{w}$ lie in the yz-plane, $\mathbf{v} \times \mathbf{w}$ must lie along the x axis which is perpendicular to yz-plane. That is, $\mathbf{v} \times \mathbf{w}$ is a scalar multiple of the unit vector $\mathbf{i}$.

30. Find the two unit vectors orthogonal to both $\mathbf{a} = \langle 3, 1, 1\rangle$ and $\mathbf{b} = \langle -1, 2, 1\rangle$.

SOLUTION The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, therefore the desired vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are the unit vectors in the direction of $\mathbf{a} \times \mathbf{b}$ and $-\mathbf{a} \times \mathbf{b}$ respectively. That is,

$$\mathbf{u}_1 = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}, \quad \mathbf{u}_2 = -\frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \tag{1}$$

We compute the cross product vector $\mathbf{a} \times \mathbf{b}$ and its length:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{k}$$

$$= (1-2)\mathbf{i} - (3+1)\mathbf{j} + (6+1)\mathbf{k} = -\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} = \langle -1, -4, 7\rangle$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(-1)^2 + (-4)^2 + 7^2} = \sqrt{66}$$

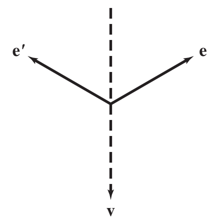
Substituting into (1) we get

$$\mathbf{u}_1 = \left\langle -\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}} \right\rangle, \quad \mathbf{u}_2 = \left\langle \frac{1}{\sqrt{66}}, \frac{4}{\sqrt{66}}, -\frac{7}{\sqrt{66}} \right\rangle$$

31. Let $\mathbf{e}$ and $\mathbf{e}'$ be unit vectors in $\mathbf{R}^3$ such that $\mathbf{e} \perp \mathbf{e}'$. Use the geometric properties of the cross product to compute $\mathbf{e} \times (\mathbf{e}' \times \mathbf{e})$.

SOLUTION Let $\mathbf{u} = \mathbf{e} \times \left(\mathbf{e}' \times \mathbf{e}\right)$ and $\mathbf{v} = \mathbf{e}' \times \mathbf{e}$. The vector $\mathbf{v}$ is orthogonal to $\mathbf{e}'$ and $\mathbf{e}$, hence $\mathbf{v}$ is orthogonal to the plane π defined by $\mathbf{e}'$ and $\mathbf{e}$. Now $\mathbf{u}$ is orthogonal to $\mathbf{v}$, hence $\mathbf{u}$ lies in the plane π orthogonal to $\mathbf{v}$. $\mathbf{u}$ is orthogonal to $\mathbf{e}$, which is in this plane, hence $\mathbf{u}$ is a multiple of $\mathbf{e}'$:

$$\mathbf{u} = \lambda \mathbf{e}' \tag{1}$$



The right-hand rule implies that $\mathbf{u}$ is in the direction of $\mathbf{e}'$, hence $\lambda > 0$. To find λ, we compute the length of $\mathbf{u}$:

$$\|\mathbf{v}\| = \|\mathbf{e}' \times \mathbf{e}\| = \|\mathbf{e}'\|\|\mathbf{e}\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1$$

$$\|\mathbf{u}\| = \|\mathbf{e} \times \mathbf{v}\| = \|\mathbf{e}\|\|\mathbf{v}\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1 \tag{2}$$

Combining (1), (2), and $\lambda > 0$ we conclude that

$$\mathbf{u} = \mathbf{e} \times \left(\mathbf{e}' \times \mathbf{e}\right) = \mathbf{e}'.$$

32. Calculate the force $\mathbf{F}$ on an electron (charge $q = -1.6 \times 10^{-19}$ C) moving with velocity 10^5 m/s in the direction $\mathbf{i}$ in a uniform magnetic field $\mathbf{B}$, where $\mathbf{B} = 0.0004\mathbf{i} + 0.0001\mathbf{j}$ teslas (see Example 5).

SOLUTION The force $\mathbf{F}$ on an electron moving at velocity $\mathbf{v}$ in a uniform magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \quad \text{where} \quad q = -1.6 \cdot 10^{-19} \text{ coulombs}.$$

In our example, $\mathbf{v} = 10^5\mathbf{i}$ and $\mathbf{B} = 0.0004\mathbf{i} + 0.0001\mathbf{j}$, hence,

$$\begin{aligned} \mathbf{F} &= q10^5\mathbf{i} \times (0.0004\mathbf{i} + 0.0001\mathbf{j}) = 10q\mathbf{i} \times (4\mathbf{i} + \mathbf{j}) = 10q\,(4\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j}) \\ &= 10q\,(\mathbf{0} + \mathbf{k}) = 10q\mathbf{k} = (-1.6 \cdot 10^{-18})\mathbf{k} \end{aligned}$$

33. An electron moving with velocity $\mathbf{v}$ in the plane experiences a force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, where q is the charge on the electron and $\mathbf{B}$ is a uniform magnetic field pointing directly out of the page. Which of the two vectors $\mathbf{F}_1$ or $\mathbf{F}_2$ in Figure 17 represents the force on the electron? Remember that q is negative.

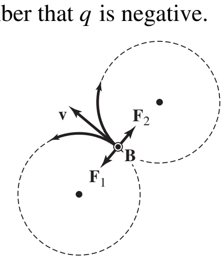
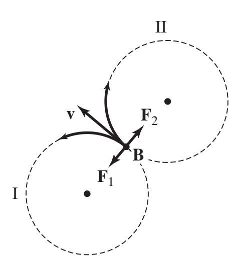


FIGURE 17 The magnetic field vector $\mathbf{B}$ points directly out of the page.

SOLUTION Since the magnetic field $\mathbf{B}$ points directly out of the page (toward us), the right-hand rule implies that the cross product $\mathbf{v} \times \mathbf{B}$ is in the direction of $\mathbf{F}_2$ (see figure).



Since $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$ and $q < 0$, the force $\mathbf{F}$ on the electron is represented by the opposite vector $\mathbf{F}_1$.

34. Calculate the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, where $\mathbf{u}=\langle 1,1,0\rangle$, $\mathbf{v}=\langle 3,-2,2\rangle$, and $\mathbf{w}=\langle 4,-1,2\rangle$.

SOLUTION The scalar triple product is the following 3×3 determinant:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}1&1&0\\3&-2&2\\4&-1&2\end{vmatrix}=1\begin{vmatrix}-2&2\\-1&2\end{vmatrix}-1\begin{vmatrix}3&2\\4&2\end{vmatrix}+0\begin{vmatrix}3&-2\\4&-1\end{vmatrix}$$
$$=1\cdot(-4+2)-1\cdot(6-8)+0=-2+2=0$$

35. Verify identity (10) for vectors $\mathbf{v}=\langle 3,-2,2\rangle$ and $\mathbf{w}=\langle 4,-1,2\rangle$.

SOLUTION We compute the cross product $\mathbf{v}\times\mathbf{w}$:

$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&2\\4&-1&2\end{vmatrix}=\begin{vmatrix}-2&2\\-1&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&2\\4&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&-2\\4&-1\end{vmatrix}\mathbf{k}$$
$$=(-4+2)\mathbf{i}-(6-8)\mathbf{j}+(-3+8)\mathbf{k}=-2\mathbf{i}+2\mathbf{j}+5\mathbf{k}=\langle -2,2,5\rangle$$

We now find the dot product $\mathbf{v}\cdot\mathbf{w}$:

$$\mathbf{v}\cdot\mathbf{w}=\langle 3,-2,2\rangle\cdot\langle 4,-1,2\rangle=3\cdot 4+(-2)\cdot(-1)+2\cdot 2=18$$

Finally we compute the squares of the lengths of $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}\times\mathbf{w}$:

$$\|\mathbf{v}\|^2=3^2+(-2)^2+2^2=17$$
$$\|\mathbf{w}\|^2=4^2+(-1)^2+2^2=21$$
$$\|\mathbf{v}\times\mathbf{w}\|^2=(-2)^2+2^2+5^2=33$$

We now verify the equality:

$$\|\mathbf{v}\|^2\|\mathbf{w}\|^2-(\mathbf{v}\cdot\mathbf{w})^2=17\cdot 21-18^2=33=\|\mathbf{v}\times\mathbf{w}\|^2$$

36. Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in Figure 18.

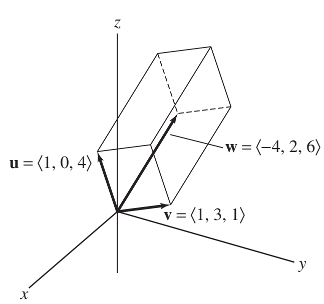


FIGURE 18

SOLUTION Using $\mathbf{u}=\langle 1,0,4\rangle$, $\mathbf{v}=\langle 1,3,1\rangle$ and $\mathbf{w}=\langle -4,2,6\rangle$, the scalar triple product is the following 3×3 determinant:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}1&0&4\\1&3&1\\-4&2&6\end{vmatrix}=1\cdot(18-2)-0+4(2+12)=16+56=72$$

37. Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ in Figure 18.

SOLUTION The area of the parallelogram equals the length of the cross product of the two vectors $\mathbf{v}=\langle 1,3,1\rangle$ and $\mathbf{w}=\langle -4,2,6\rangle$. We calculate the cross product as follows:

$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&1\\-4&2&6\end{vmatrix}=(18-2)\mathbf{i}-(6+4)\mathbf{j}+(2+12)\mathbf{k}=16\mathbf{i}-10\mathbf{j}+14\mathbf{k}$$

The length of this vector $16\mathbf{i}-10\mathbf{j}+14\mathbf{k}$ is $\sqrt{16^2+10^2+14^2}=2\sqrt{138}$. Thus, the area of the parallelogram is $2\sqrt{138}$.

38. Calculate the volume of the parallelepiped spanned by

$$\mathbf{u} = \langle 2, 2, 1 \rangle, \qquad \mathbf{v} = \langle 1, 0, 3 \rangle, \qquad \mathbf{w} = \langle 0, -4, 0 \rangle$$

SOLUTION Using $\mathbf{u} = \langle 2, 2, 1 \rangle$, $\mathbf{v} = \langle 1, 0, 3 \rangle$, and $\mathbf{w} = \langle 0, -4, 0 \rangle$, the volume is given by the following scalar triple product:

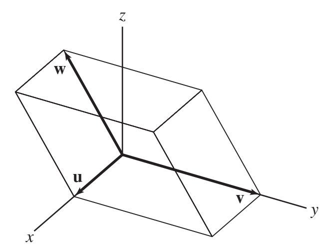
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & -4 & 0 \end{vmatrix} = 2(0+12) - 2(0-0) + 1(-4-0) = 24 - 4 = 20.$$

39. Sketch and compute the volume of the parallelepiped spanned by

$$\mathbf{u} = \langle 1, 0, 0 \rangle, \qquad \mathbf{v} = \langle 0, 2, 0 \rangle, \qquad \mathbf{w} = \langle 1, 1, 2 \rangle$$

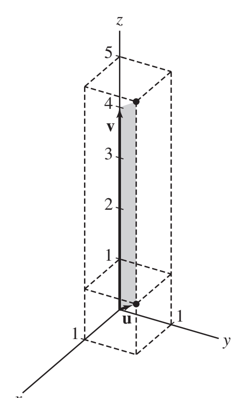
SOLUTION Using $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 2, 0 \rangle$, and $\mathbf{w} = \langle 1, 1, 2 \rangle$, the volume is given by the following scalar triple product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4.$$



40. Sketch the parallelogram spanned by $\mathbf{u} = \langle 1, 1, 1 \rangle$ and $\mathbf{v} = \langle 0, 0, 4 \rangle$, and compute its area.

SOLUTION The parallelogram spanned by $\mathbf{u} = \langle 1, 1, 1 \rangle$ and $\mathbf{v} = \langle 0, 0, 4 \rangle$ is shown in the figure.



We find its area A using the formula for the area of a parallelogram:

$$A = \|\mathbf{u} \times \mathbf{v}\|$$

We first find the cross product vector $\mathbf{u} \times \mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times 4\mathbf{k} = 4\mathbf{i} \times \mathbf{k} + 4\mathbf{j} \times \mathbf{k} + 4\mathbf{k} \times \mathbf{k} = -4\mathbf{j} + 4\mathbf{i} + \mathbf{0} = 4\mathbf{i} - 4\mathbf{j} = 4\langle 1, -1, 0 \rangle$$

Hence,

$$A = \|4\langle 1, -1, 0 \rangle\| = 4\|\langle 1, -1, 0 \rangle\| = 4\sqrt{1^2 + (-1)^2 + 0^2} = 4\sqrt{2}$$

41. Calculate the area of the parallelogram spanned by $\mathbf{u} = \langle 1, 0, 3 \rangle$ and $\mathbf{v} = \langle 2, 1, 1 \rangle$.

SOLUTION The area of the parallelogram is the length of the vector $\mathbf{u} \times \mathbf{v}$. We first compute this vector:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \mathbf{k} = -3\mathbf{i} - (1-6)\mathbf{j} + \mathbf{k} = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

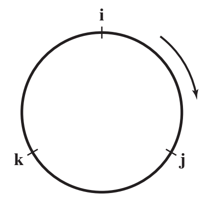
The area A is the length

$$A = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 5^2 + 1^2} = \sqrt{35} \approx 5.92.$$

42. Find the area of the parallelogram determined by the vectors $\langle a, 0, 0\rangle$ and $\langle 0, b, c\rangle$.

SOLUTION The area A is the length of the cross product of the two vectors. We first compute the cross product:

$$\langle a, 0, 0\rangle \times \langle 0, b, c\rangle = a\mathbf{i} \times (b\mathbf{j} + c\mathbf{k}) = ab\mathbf{i} \times \mathbf{j} + ac\mathbf{i} \times \mathbf{k} = ab\mathbf{k} - ac\mathbf{j} = \langle 0, -ac, ab\rangle$$

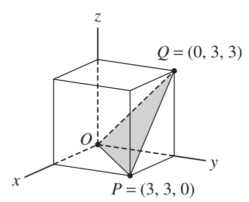


The area of the parallelogram is therefore

$$A = \|\langle 0, -ac, ab\rangle\| = \sqrt{0^2 + (-ac)^2 + (ab)^2} = \sqrt{a^2c^2 + a^2b^2} = |a|\sqrt{b^2 + c^2}$$

43. Sketch the triangle with vertices at the origin O, $P = (3, 3, 0)$, and $Q = (0, 3, 3)$, and compute its area using cross products.

SOLUTION The triangle OPQ is shown in the following figure.



The area S of the triangle is half of the area of the parallelogram determined by the vectors $\overrightarrow{OP} = \langle 3, 3, 0\rangle$ and $\overrightarrow{OQ} = \langle 0, 3, 3\rangle$. Thus,

$$S = \frac{1}{2}\|\overrightarrow{OP} \times \overrightarrow{OQ}\| \tag{1}$$

We compute the cross product:

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 0 & 3 \end{vmatrix}\mathbf{k}$$

$$= 9\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} = 9\langle 1, -1, 1\rangle$$

Substituting into (1) gives

$$S = \frac{1}{2}\|9\langle 1, -1, 1\rangle\| = \frac{9}{2}\|\langle 1, -1, 1\rangle\| = \frac{9}{2}\sqrt{1^2 + (-1)^2 + 1^2} = \frac{9\sqrt{3}}{2} \approx 7.8$$

The area of the triangle is $S = \frac{9\sqrt{3}}{2} \approx 7.8$.

44. Use the cross product to find the area of the triangle with vertices $P = (1, 1, 5)$, $Q = (3, 4, 3)$, and $R = (1, 5, 7)$ (Figure 19).

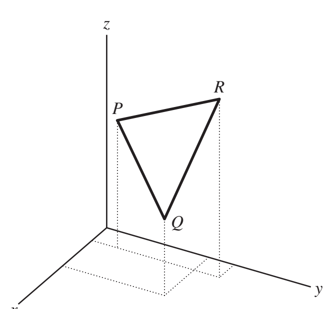


FIGURE 19

SOLUTION The area S of the triangle is half of the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. We use the formula for the area of a parallelogram via cross product to write

$$S = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$$

We compute the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \langle 3-1, 4-1, 3-5\rangle = \langle 2, 3, -2\rangle$$
$$\overrightarrow{PR} = \langle 1-1, 5-1, 7-5\rangle = \langle 0, 4, 2\rangle$$

We now find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ by computing the following determinant:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ 0 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix}\mathbf{k} = 14\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$$

Thus, we get

$$S = \frac{1}{2}\|14\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}\| = \frac{1}{2}\sqrt{14^2 + (-4)^2 + 8^2} = \frac{1}{2}\cdot 2\sqrt{69} = \sqrt{69} \approx 8.3$$

In Exercises 45–47, verify the identity using the formula for the cross product.

45. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

SOLUTION Let $\mathbf{v} = \langle a, b, c\rangle$ and $\mathbf{w} = \langle d, e, f\rangle$. By the definition of the cross product we have

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix}\mathbf{i} - \begin{vmatrix} a & c \\ d & f \end{vmatrix}\mathbf{j} + \begin{vmatrix} a & b \\ d & e \end{vmatrix}\mathbf{k} = (bf - ec)\mathbf{i} - (af - dc)\mathbf{j} + (ae - db)\mathbf{k}$$

We also have

$$-\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -d & -e & -f \\ a & b & c \end{vmatrix} = (-ec + bf)\mathbf{i} - (-dc + af)\mathbf{j} + (-db + ea)\mathbf{k}$$

Thus, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, as desired.

46. $(\lambda\mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w})$ (λ a scalar)

SOLUTION Let $\mathbf{v} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{w} = \langle b_1, b_2, b_3\rangle$. We compute $(\lambda\mathbf{v}) \times \mathbf{w}$:

$$\begin{aligned}(\lambda\mathbf{v}) \times \mathbf{w} = \langle \lambda a_1, \lambda a_2, \lambda a_3\rangle \times \langle b_1, b_2, b_3\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} \lambda a_2 & \lambda a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \lambda a_1 & \lambda a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \lambda a_1 & \lambda a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\ &= (\lambda a_2 b_3 - \lambda a_3 b_2)\,\mathbf{i} - (\lambda a_1 b_3 - \lambda a_3 b_1)\,\mathbf{j} + (\lambda a_1 b_2 - \lambda a_2 b_1)\,\mathbf{k} \\ &= \lambda\,(a_2 b_3 - a_3 b_2)\,\mathbf{i} - \lambda\,(a_1 b_3 - a_3 b_1)\,\mathbf{j} + \lambda\,(a_1 b_2 - a_2 b_1)\,\mathbf{k} \\ &= \lambda\,((a_2 b_3 - a_3 b_2)\,\mathbf{i} - (a_1 b_3 - a_3 b_1)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k})\end{aligned}$$

We compute $\lambda\,(\mathbf{v} \times \mathbf{w})$:

$$\lambda\,(\mathbf{v} \times \mathbf{w}) = \lambda\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \lambda\,((a_2 b_3 - a_3 b_2)\,\mathbf{i} - (a_1 b_3 - a_3 b_1)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k})$$

The two vectors are equal.

47. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

SOLUTION We let $\mathbf{u} = \langle a_1, a_2, a_3\rangle$, $\mathbf{v} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{w} = \langle c_1, c_2, c_3\rangle$. Computing the left-hand side gives

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle \times \langle c_1, c_2, c_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2+b_2 & a_3+b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1+b_1 & a_3+b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1+b_1 & a_2+b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k}$$

$$= (c_3(a_2+b_2) - c_2(a_3+b_3))\,\mathbf{i} - (c_3\,(a_1+b_1) - c_1\,(a_3+b_3))\,\mathbf{j} + (c_2(a_1+b_1) - c_1(a_2+b_2))\,\mathbf{k}$$

We now compute the right-hand-side of the equality:

$$\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$$

$$+ (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$$

$$= (a_2c_3 - a_3c_2 + b_2c_3 - b_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1 + b_1c_3 - b_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1 + b_1c_2 - b_2c_1)\mathbf{k}$$

$$= (c_3(a_2+b_2) - c_2(a_3+b_3))\,\mathbf{i} - (c_3(a_1+b_1) - c_1(a_3+b_3))\,\mathbf{j} + (c_2(a_1+b_1) - c_1(a_2+b_2))\,\mathbf{k}$$

The results are the same. Hence,

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}.$$

48. Use the geometric description in Theorem 1 to prove Theorem 2 (iii): $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ or $\mathbf{v} = \mathbf{0}$.

SOLUTION $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ if and only if $\|\mathbf{v}\times\mathbf{w}\| = 0$, that is, using Theorem 2 (b), if and only if

$$\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta = 0$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. This equality holds only if at least one of the vectors $\mathbf{v}$ or $\mathbf{w}$ is the zero vector or $\sin\theta = 0$. The solutions of $\sin\theta = 0$ for angles between 0 and 180° are $\theta = 0$ and $\theta = 180°$, that is, $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors. To summarize, we conclude that $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or $\mathbf{w} = \lambda\mathbf{v}$. This can be written as

$$\mathbf{v} = \mathbf{0} \quad \text{or} \quad \mathbf{w} = \lambda\mathbf{v}.$$

49. Verify the relations (5).

SOLUTION We must verify the following relations:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}, \quad \mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

We compute the cross products using the definition of the cross product. This gives

$$\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k}$$

$$\mathbf{j}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{k} = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k} = \mathbf{j}$$

$$\mathbf{i}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k} = \mathbf{0}$$

$$\mathbf{j}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{0}$$

$$\mathbf{k}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k} = \mathbf{0}$$

50. Show that

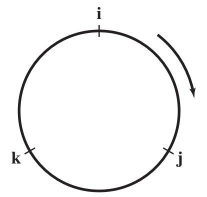
$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$$

Conclude that the Associative Law does not hold for cross products.

SOLUTION Using the cross products of the unit vectors **i**, **j**, and **k**, we obtain

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$



Since $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$ the associative law does not hold for cross products.

51. The components of the cross product have a geometric interpretation. Show that the absolute value of the **k**-component of $\mathbf{v} \times \mathbf{w}$ is equal to the area of the parallelogram spanned by the projections $\mathbf{v}_0$ and $\mathbf{w}_0$ onto the xy-plane (Figure 20).

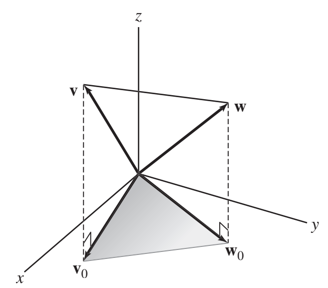


FIGURE 20

SOLUTION Let $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{w} = \langle b_1, b_2, b_3 \rangle$, hence, $\mathbf{v}_0 = \langle a_1, a_2, 0 \rangle$ and $\mathbf{w}_0 = \langle b_1, b_2, 0 \rangle$. The area S of the parallelogram spanned by $\mathbf{v}_0$ and $\mathbf{w}_0$ is the following value:

$$S = \|\mathbf{v}_0 \times \mathbf{w}_0\| \tag{1}$$

We compute the cross product:

$$\mathbf{v}_0 \times \mathbf{w}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & 0 \\ b_2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} = \langle 0, 0, a_1 b_2 - a_2 b_1 \rangle$$

Using (1) we have

$$S = \sqrt{0^2 + 0^2 + (a_1 b_2 - a_2 b_1)^2} = |a_1 b_2 - a_2 b_1| \tag{2}$$

We now compute $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The **k**-component of $\mathbf{v} \times \mathbf{w}$ is, thus,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{3}$$

By (2) and (3) we obtain the desired result.

52. Formulate and prove analogs of the result in Exercise 51 for the **i**- and **j**-components of $\mathbf{v} \times \mathbf{w}$.

SOLUTION The analogs for the **i** and **j** components of $\mathbf{v} \times \mathbf{w}$ are the following statements:

(a) The area of the parallelogram spanned by the projections $\mathbf{v}'$ and $\mathbf{w}'$ of vectors $\mathbf{v}$, $\mathbf{w}$ onto the xz-plane is equal to the absolute value of the **j**-component of $\mathbf{v} \times \mathbf{w}$.

(b) The area of the parallelogram spanned by the projections $\mathbf{v}'$ and $\mathbf{w}'$ of vectors $\mathbf{v}$, $\mathbf{w}$ onto the yz-plane is equal to the absolute value of the **i**-component of $\mathbf{v} \times \mathbf{w}$.

(c) If $\mathbf{v} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{w} = \langle b_1, b_2, b_3 \rangle$ then $\mathbf{v}' = \langle a_1, 0, a_3 \rangle$ and $\mathbf{w}' = \langle b_1, 0, b_3 \rangle$. The area S of the parallelogram spanned by $\mathbf{v}'$ and $\mathbf{w}'$ is

$$S = \|\mathbf{v}' \times \mathbf{w}'\| \tag{1}$$

We compute the cross product:

$$\mathbf{v}' \times \mathbf{w}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & 0 & a_3 \\ b_1 & 0 & b_3 \end{vmatrix} = \begin{vmatrix} 0 & a_3 \\ 0 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix} \mathbf{k}$$

$$= 0\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + 0\mathbf{k} = -(a_1 b_3 - a_3 b_1)\mathbf{j}$$

Combining with (1) we get

$$S = \| -(a_1 b_3 - a_3 b_1)\mathbf{j}\| = |a_1 b_3 - a_3 b_1| \tag{2}$$

We now find the cross product $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The $\mathbf{j}$-component of the cross product is

$$-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -(a_1 b_3 - a_3 b_1) \tag{3}$$

By (2) and (3) we obtain the desired result.

Proof of (b). In this case, $\mathbf{v}' = \langle 0, a_2, a_3 \rangle$ and $\mathbf{w}' = \langle 0, b_2, b_3 \rangle$. The area S of the parallelogram spanned by $\mathbf{v}'$ and $\mathbf{w}'$ is

$$S = \|\mathbf{v}' \times \mathbf{w}'\| \tag{4}$$

We compute the cross product:

$$\mathbf{v}' \times \mathbf{w}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & a_2 & a_3 \\ 0 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & a_3 \\ 0 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & a_2 \\ 0 & b_2 \end{vmatrix} \mathbf{k} = (a_2 b_3 - a_3 b_2)\mathbf{i}$$

Hence, by (4) we get

$$S = \|(a_2 b_3 - a_3 b_2)\,\mathbf{i}\| = |a_2 b_3 - a_3 b_2|\|\mathbf{i}\| = |a_2 b_3 - a_3 b_2| \tag{5}$$

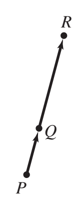
We now identify the $\mathbf{i}$ component of $\mathbf{v} \times \mathbf{w}$ as seen in the proof of part (a):

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2 \tag{6}$$

By (5) and (6) we obtain the desired result.

53. Show that three points P, Q, R are collinear (lie on a line) if and only if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.

SOLUTION The points P, Q, and R lie on one line if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. By basic properties of the cross product this is equivalent to $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.



54. Use the result of Exercise 53 to determine whether the points P, Q, and R are collinear, and if not, find a vector normal to the plane containing them.

(a) $P = (2, 1, 0)$, $Q = (1, 5, 2)$, $R = (-1, 13, 6)$

(b) $P = (2, 1, 0)$, $Q = (-3, 21, 10)$, $R = (5, -2, 9)$

(c) $P = (1, 1, 0)$, $Q = (1, -2, -1)$, $R = (3, 2, -4)$

SOLUTION

(a) Let $P = (2, 1, 0)$, $Q = (1, 5, 2)$, $R = (-1, 13, 6)$. By the result of Exercise 53, the points are collinear if and only if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$. We compute these vectors:

$$\overrightarrow{PQ} = \langle 1-2, 5-1, 2-0\rangle = \langle -1, 4, 2\rangle = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$\overrightarrow{PR} = \langle -1-2, 13-1, 6-0\rangle = \langle -3, 12, 6\rangle = -3\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$$

The cross product of these vectors is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 2 \\ -3 & 12 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 12 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 2 \\ -3 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 4 \\ -3 & 12 \end{vmatrix}\mathbf{k}$$
$$= (24-24)\mathbf{i} - (-6+6)\mathbf{j} + (-12+12)\mathbf{k} = \mathbf{0}$$

The cross product is the zero vector, hence the points P, Q, R are collinear.

(b) Let $P = (2, 1, 0)$, $Q = (-3, 21, 10)$, $R = (5, -2, 9)$. To check if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$ we first compute these vectors:

$$\overrightarrow{PQ} = \langle -3-2, 21-1, 10-0\rangle = \langle -5, 20, 10\rangle = -5\mathbf{i} + 20\mathbf{j} + 10\mathbf{k}$$
$$\overrightarrow{PR} = \langle 5-2, -2-1, 9-0\rangle = \langle 3, -3, 9\rangle = 3\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$$

We find the cross product:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 20 & 10 \\ 3 & -3 & 9 \end{vmatrix} = \begin{vmatrix} 20 & 10 \\ -3 & 9 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 10 \\ 3 & 9 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & 20 \\ 3 & -3 \end{vmatrix}\mathbf{k}$$
$$= (180+30)\mathbf{i} - (-45-30)\mathbf{j} + (15-60)\mathbf{k} = 210\mathbf{i} + 75\mathbf{j} - 45\mathbf{k} \neq \mathbf{0}$$

The cross product is not the zero vector, hence the points P, Q and R are not collinear.

(c) Let $P = (1, 1, 0)$, $Q = (1, -2, -1)$, $R = (3, 2, -4)$. These points are not collinear if $\overrightarrow{PQ} \times \overrightarrow{PR} \neq \mathbf{0}$. We find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \langle 1-1, -2-1, -1-0\rangle = \langle 0, -3, -1\rangle = -3\mathbf{j} - \mathbf{k}$$
$$\overrightarrow{PR} = \langle 3-1, 2-1, -4-0\rangle = \langle 2, 1, -4\rangle = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

Thus,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & -1 \\ 2 & 1 & -4 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 1 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ 2 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix}\mathbf{k}$$
$$= (12+1)\mathbf{i} - (0+2)\mathbf{j} + (0+6)\mathbf{k} = 13\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Since $\overrightarrow{PQ} \times \overrightarrow{PR} \neq \mathbf{0}$, the points P, Q, R are not collinear, rather, there is exactly one plane containing them. A vector normal to this plane is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product $\overrightarrow{PQ} \times \overrightarrow{PR} = 13\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ is such a vector.

55. Solve the equation $\langle 1, 1, 1\rangle \times \mathbf{X} = \langle 1, -1, 0\rangle$, where $\mathbf{X} = \langle x, y, z\rangle$. *Note:* There are infinitely many solutions.

SOLUTION Let $\mathbf{X} = \langle a, b, c\rangle$. We compute the cross product:

$$\langle 1, 1, 1\rangle \times \langle a, b, c\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}\mathbf{k}$$
$$= (c-b)\mathbf{i} - (c-a)\mathbf{j} + (b-a)\mathbf{k} = \langle c-b, a-c, b-a\rangle$$

The equation for $\mathbf{X}$ is, thus,

$$\langle c-b, a-c, b-a\rangle = \langle 1, -1, 0\rangle$$

Equating corresponding components we get

$$\begin{aligned} c-b &= 1 \\ a-c &= -1 \\ b-a &= 0 \end{aligned}$$

The third equation implies $a = b$. Substituting in the first and second equations gives

$$\begin{aligned} c-a &= 1 \\ a-c &= -1 \end{aligned} \quad \Rightarrow \quad c = a+1$$

The solution is thus, $b = a$, $c = a + 1$. The corresponding solutions $\mathbf{X}$ are

$$\mathbf{X} = \langle a, b, c \rangle = \langle a, a, a + 1 \rangle$$

One possible solution is obtained for $a = 0$, that is, $\mathbf{X} = \langle 0, 0, 1 \rangle$.

56. Explain geometrically why $\langle 1, 1, 1 \rangle \times \mathbf{X} = \langle 1, 0, 0 \rangle$ has no solution, where $\mathbf{X} = \langle x, y, z \rangle$.

SOLUTION The cross product vector $\langle 1, 0, 0 \rangle$ must be orthogonal to the vector $\langle 1, 1, 1 \rangle$. This condition is not satisfied since

$$\langle 1, 1, 1 \rangle \cdot \langle 1, 0, 0 \rangle = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = 1 \neq 0$$

Therefore, there is no vector $\mathbf{X}$ that satisfies the equation.

57. Let $\mathbf{X} = \langle x, y, z \rangle$. Show that $\mathbf{i} \times \mathbf{X} = \mathbf{v}$ has a solution if and only if $\mathbf{v}$ is contained in the yz-plane (the $\mathbf{i}$-component is zero).

SOLUTION The cross product vector $\mathbf{i} \times \mathbf{X} = \mathbf{v}$ must be orthogonal to the vector $\mathbf{i} = \langle 1, 0, 0 \rangle$. This condition is true if and only if $\langle 1, 0, 0 \rangle \cdot \mathbf{v} = 0$, which is true if and only if the $\mathbf{i}$-component of $\mathbf{v}$ is zero (that is, $\mathbf{v}$ is in the yz-plane).

58. Suppose that vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually orthogonal—that is, $\mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, and $\mathbf{v} \perp \mathbf{w}$. Prove that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$.

SOLUTION The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, hence it is parallel to $\mathbf{w}$. The cross product of parallel vectors is the zero vector, hence $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$. Similarly, the cross product $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, hence it is parallel to $\mathbf{u}$. Since the cross product of parallel vectors is the zero vector, we conclude that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$.

*In Exercises 59–62: The **torque** about the origin O due to a force $\mathbf{F}$ acting on an object with position vector $\mathbf{r}$ is the vector quantity $\tau = \mathbf{r} \times \mathbf{F}$. If several forces $\mathbf{F}_j$ act at positions $\mathbf{r}_j$, then the net torque (units: N-m or lb-ft) is the sum*

$$\tau = \sum \mathbf{r}_j \times \mathbf{F}_j$$

Torque measures how much the force causes the object to rotate. By Newton's Laws, τ is equal to the rate of change of angular momentum.

59. Calculate the torque τ about O acting at the point P on the mechanical arm in Figure 21(A), assuming that a 25-N force acts as indicated. Ignore the weight of the arm itself.

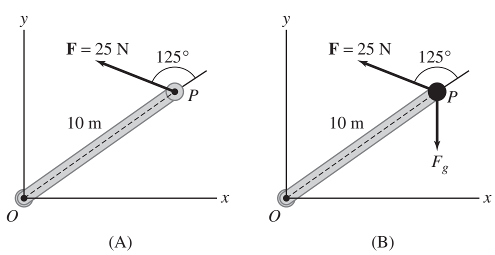
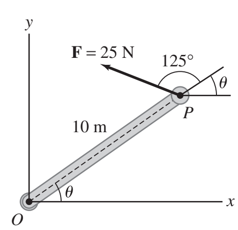


FIGURE 21

SOLUTION We denote by O and P the points shown in the figure and compute the position vector $\mathbf{r} = \overrightarrow{OP}$ and the force vector $\mathbf{F}$.



Denoting by θ the angle between the arm and the x-axis we have

$$\mathbf{r} = \overrightarrow{OP} = 10\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

The angle between the force vector $\mathbf{F}$ and the x-axis is $(\theta + 125^\circ)$, hence,

$$\mathbf{F} = 25\left(\cos\left(\theta + 125^\circ\right)\mathbf{i} + \sin\left(\theta + 125^\circ\right)\mathbf{j}\right)$$

The torque τ about O acting at the point P is the cross product $\tau = \mathbf{r} \times \mathbf{F}$. We compute it using the cross products of the unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\begin{aligned}
\tau = \mathbf{r} \times \mathbf{F} &= 10\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times 25\left(\cos\left(\theta + 125^\circ\right)\mathbf{i} + \sin\left(\theta + 125^\circ\right)\mathbf{j}\right) \\
&= 250\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times \left(\cos\left(\theta + 125^\circ\right)\mathbf{i} + \sin\left(\theta + 125^\circ\right)\mathbf{j}\right) \\
&= 250\left(\cos\theta\sin\left(\theta + 125^\circ\right)\mathbf{k} + \sin\theta\cos\left(\theta + 125^\circ\right)(-\mathbf{k})\right) \\
&= 250\left(\sin\left(\theta + 125^\circ\right)\cos\theta - \sin\theta\cos\left(\theta + 125^\circ\right)\right)\mathbf{k}
\end{aligned}$$

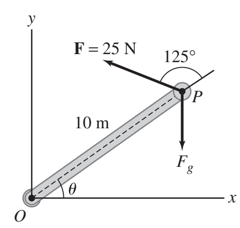
We now use the identity $\sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha - \beta)$ to obtain

$$\tau = 250\sin\left(\theta + 125^\circ - \theta\right)\mathbf{k} = 250\sin 125^\circ\mathbf{k} \approx 204.79\mathbf{k}$$

60. Calculate the net torque about O at P, assuming that a 30-kg mass is attached at P [Figure 21(B)]. The force $\mathbf{F}_g$ due to gravity on a mass m has magnitude $9.8m$ m/s^2 in the downward direction.

SOLUTION We denote by τ_1 and τ_2 the torques due to the forces $\mathbf{F}$ and $\mathbf{F}_g$ respectively. Let θ denote the angle between the arm and the x-axis, and $\mathbf{r} = \overrightarrow{OP}$ the position vector. The net torque about O at P is

$$\tau = \mathbf{r} \times \mathbf{F} + \mathbf{r} \times \mathbf{F}_g = \tau_1 + \tau_2 \tag{1}$$



In Exercise 59 we found that

$$\tau_1 = 204.79\mathbf{k} \tag{2}$$

We compute the torque τ_2:

$$\begin{aligned}
\tau_2 = \mathbf{r} \times \mathbf{F}_g &= 10\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times 9.8 \cdot 30\,(-\mathbf{j}) \\
&= 2940\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times (-\mathbf{j}) \\
&= 2940\cos\theta\,(-\mathbf{k}) = -2940\cos\theta\mathbf{k}
\end{aligned} \tag{3}$$

Combining (1), (2), and (3) we obtain

$$\tau = 204.79\mathbf{k} - 2940\cos\theta\mathbf{k} = (204.79 - 2940\cos\theta)\mathbf{k}$$

61. Let τ be the net torque about O acting on the robotic arm of Figure 22. Assume that the two segments of the arms have mass m_1 and m_2 (in kg) and that a weight of m_3 kg is located at the endpoint P. In calculating the torque, we may assume that the entire mass of each arm segment lies at the midpoint of the arm (its center of mass). Show that the position vectors of the masses m_1, m_2, and m_3 are

$$\begin{aligned}
\mathbf{r}_1 &= \frac{1}{2}L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) \\
\mathbf{r}_2 &= L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + \frac{1}{2}L_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j}) \\
\mathbf{r}_3 &= L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + L_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j})
\end{aligned}$$

Then show that

$$\tau = -g\left(L_1\left(\frac{1}{2}m_1 + m_2 + m_3\right)\sin\theta_1 + L_2\left(\frac{1}{2}m_2 + m_3\right)\sin\theta_2\right)\mathbf{k}$$

where $g = 9.8m/s^2$. To simplify the computation, note that all three gravitational forces act in the $-\mathbf{j}$ direction, so the $\mathbf{j}$-components of the position vectors $\mathbf{r}_i$ do not contribute to the torque.

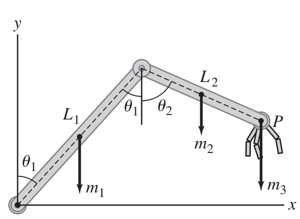
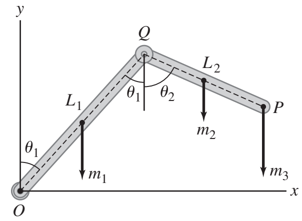


FIGURE 22

SOLUTION We denote by O, P, and Q the points shown in the figure.



The coordinates of O and Q are

$$O = (0, 0), \quad Q = (L_1 \sin\theta_1, L_1 \cos\theta_1)$$

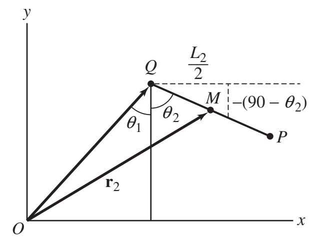
The midpoint of the segment OQ is, thus,

$$\left(\frac{0 + L_1 \sin\theta_1}{2}, \frac{0 + L_1 \cos\theta_1}{2}\right) = \left(\frac{L_1 \sin\theta_1}{2}, \frac{L_1 \cos\theta_1}{2}\right)$$

Since the mass m_1 is assumed to lie at the midpoint of the arm, the position vector of m_1 is

$$\mathbf{r}_1 = \frac{L_1}{2}(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) \tag{1}$$

We now find the position vector $\mathbf{r}_2$ of m_2. We have (see figure)



$$\mathbf{r}_2 = \overrightarrow{OQ} + \overrightarrow{QM} \tag{2}$$

$$\overrightarrow{OQ} = L_1 \sin\theta_1\mathbf{i} + L_1 \cos\theta_1\mathbf{j} = L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) \tag{3}$$

The vector $\overrightarrow{QM}$ makes an angle of $-\left(90^\circ - \theta_2\right)$ with the x axis and has length $\frac{L_2}{2}$, hence,

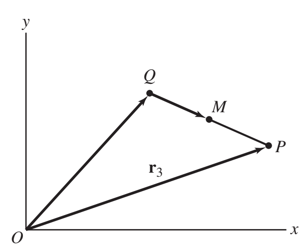
$$\overrightarrow{QM} = \frac{L_2}{2}\left(\cos\left(-\left(90^\circ - \theta_2\right)\right)\mathbf{i} + \sin\left(-\left(90^\circ - \theta_2\right)\right)\mathbf{j}\right) = \frac{L_2}{2}(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j}) \tag{4}$$

Combining (2), (3) and (4) we get

$$\mathbf{r}_2 = L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + \frac{L_2}{2}(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j}) \tag{5}$$

Finally, we find the position vector $\mathbf{r}_3$:

$$\mathbf{r}_3 = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OQ} + 2\overrightarrow{QM}$$



Substituting (3) and (4) we get

$$\mathbf{r}_3 = L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + L_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j}) \tag{6}$$

The net torque is the following vector:

$$\tau = \mathbf{r}_1 \times (-m_1 g\mathbf{j}) + \mathbf{r}_2 \times (-m_2 g\mathbf{j}) + \mathbf{r}_3 \times (-m_3 g\mathbf{j})$$

In computing the cross products, the $\mathbf{j}$ components of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ do not contribute to the torque since $\mathbf{j} \times \mathbf{j} = \mathbf{0}$. We thus consider only the $\mathbf{i}$ components of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in (1), (5) and (6). This gives

$$\begin{aligned}
\tau &= \frac{L_1}{2}\sin\theta_1 \mathbf{i} \times (-m_1 g\mathbf{j}) + \left(L_1 \sin\theta_1 + \frac{L_2}{2}\sin\theta_2\right)\mathbf{i} \times (-m_2 g\mathbf{j}) + (L_1 \sin\theta_1 + L_2 \sin\theta_2)\,\mathbf{i} \times (-m_3 g\mathbf{j}) \\
&= -\frac{L_1 m_1 g \sin\theta_1}{2}\mathbf{k} - \left(L_1 m_2 g \sin\theta_1 + \frac{L_2 m_2 g}{2}\sin\theta_2\right)\mathbf{k} - (L_1 m_3 g \sin\theta_1 + L_2 m_3 g \sin\theta_2)\,\mathbf{k} \\
&= -g\left(L_1\left(\frac{1}{2}m_1 + m_2 + m_3\right)\sin\theta_1 + L_2\left(\frac{1}{2}m_2 + m_3\right)\sin\theta_2\right)\mathbf{k}
\end{aligned}$$

62. Continuing with Exercise 61, suppose that $L_1 = 3$ m, $L_2 = 2$ m, $m_1 = 15$ kg, $m_2 = 20$ kg, and $m_3 = 18$ kg. If the angles θ_1, θ_2 are equal (say, to θ), what is the maximum allowable value of θ if we assume that the robotic arm can sustain a maximum torque of 1200 N-m?

SOLUTION Setting the given values $L_1 = 3$, $L_2 = 2$, $m_1 = 15$, $m_2 = 20$, $m_3 = 18$, and $\theta_1 = \theta_2 = \theta$ in the formula for τ obtained in Exercise 61 we get

$$\tau = -g\left(3\left(\frac{15}{2} + 20 + 18\right)\sin\theta + 2\left(\frac{20}{2} + 18\right)\sin\theta\right)\mathbf{k} = -1886.5\sin\theta\mathbf{k}$$

Thus,

$$\|\tau\| = 1886.5\sin\theta$$

Since the maximum torque sustained by the robotic arm is 1200 ft-lbs, we have

$$\begin{aligned}
1886.5\sin\theta &\le 1200 \\
\sin\theta &\le \frac{1200}{1886.5} \approx 0.636
\end{aligned}$$

The solution for acute angles is

$$\theta \le 39.5^\circ$$

The maximum allowable value of θ is $\theta = 39.5^\circ$.

Further Insights and Challenges

63. Show that 3×3 determinants can be computed using the **diagonal rule**: Repeat the first two columns of the matrix and form the products of the numbers along the six diagonals indicated. Then add the products for the diagonals that slant from left to right and subtract the products for the diagonals that slant from right to left.

$$\det(A) = \left|\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ - & - & - & + & + & + \end{array}\right.$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

SOLUTION Using the definition of 3×3 determinants given in Eq. (2) we get

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Using the definition of 2×2 determinants given in Eq. (1) we get

$$\begin{aligned}
\det(A) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}
\end{aligned}$$

64. Use the diagonal rule to calculate $\begin{vmatrix} 2 & 4 & 3 \\ 0 & 1 & -7 \\ -1 & 5 & 3 \end{vmatrix}$.

SOLUTION We form the following matrix:

$$\begin{array}{ccc|cc} 2 & 4 & 3 & 2 & 4 \\ 0 & 1 & -7 & 0 & 1 \\ -1 & 5 & 3 & -1 & 5 \end{array}$$

We now form the diagonals which slant from left to right and the diagonals which slant from right to left and assign corresponding sign to each diagonal:

$$\begin{array}{ccccc} 2 & 4 & 3 & 2 & 4 \\ 0 & 1 & -7 & 0 & 1 \\ -1 & 5 & 3 & -1 & 5 \end{array}$$
$$- \quad - \quad - \qquad + \quad + \quad +$$

We add the products for the diagonals with a positive sign and subtract the products for the diagonals with a negative sign. This gives

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & 1 & -7 \\ -1 & 5 & 3 \end{vmatrix} = 2 \cdot 1 \cdot 3 + 4 \cdot (-7) \cdot (-1) + 3 \cdot 0 \cdot 5 - 3 \cdot 1 \cdot (-1) - 2 \cdot (-7) \cdot 5 - 4 \cdot 0 \cdot 3 = 107$$

65. Prove that $\mathbf{v} \times \mathbf{w} = \mathbf{v} \times \mathbf{u}$ if and only if $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$ for some scalar λ. Assume that $\mathbf{v} \neq \mathbf{0}$.

SOLUTION Transferring sides and using the distributive law and the property of parallel vectors, we obtain the following equivalent equalities:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \mathbf{v} \times \mathbf{u} \\ \mathbf{0} &= \mathbf{v} \times \mathbf{u} - \mathbf{v} \times \mathbf{w} \\ \mathbf{0} &= \mathbf{v} \times (\mathbf{u} - \mathbf{w}) \end{aligned}$$

This holds if and only if there exists a scalar λ such that

$$\begin{aligned} \mathbf{u} - \mathbf{w} &= \lambda\mathbf{v} \\ \mathbf{u} &= \mathbf{w} + \lambda\mathbf{v} \end{aligned}$$

66. Use Eq. (10) to prove the Cauchy–Schwarz inequality:

$$|\mathbf{v} \cdot \mathbf{w}| \le \|\mathbf{v}\| \; \|\mathbf{w}\|$$

Show that equality holds if and only if $\mathbf{w}$ is a multiple of $\mathbf{v}$ or at least one of $\mathbf{v}$ and $\mathbf{w}$ is zero.

SOLUTION Transferring sides in Eq. (10) we get

$$(\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \|\mathbf{v} \times \mathbf{w}\|^2 \tag{1}$$

Since $\|\mathbf{v} \times \mathbf{w}\|^2 \ge 0$, we have

$$(\mathbf{v} \cdot \mathbf{w})^2 \le \|\mathbf{v}\|^2\|\mathbf{w}\|^2$$

Taking the square root of both sides gives

$$|\mathbf{v} \cdot \mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$$

Equality $|\mathbf{v} \cdot \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w}\|$ holds if and only if $(\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2$, that is by (1), if and only if $\|\mathbf{v} \times \mathbf{w}\| = 0$, or $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. This is equivalent to $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ, or $\mathbf{v} = \mathbf{0}$ (Theorem 3 (c)).

67. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$, then either (i) $\mathbf{u}$ and $\mathbf{v}$ are parallel, or (ii) $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

SOLUTION By the theorem on basic properties of the cross product, part (c), it follows that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ if and only if

- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ or
- $\mathbf{w} = \lambda\,(\mathbf{u} \times \mathbf{v})$

We consider the two possibilities.

1. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ is equivalent to $\mathbf{u}$ and $\mathbf{v}$ being parallel vectors or one of them being the zero vector.
2. The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, hence $\mathbf{w} = \lambda\,(\mathbf{u} \times \mathbf{v})$ implies that $\mathbf{w}$ is also orthogonal to $\mathbf{u}$ and $\mathbf{v}$ (for $\lambda \neq 0$) or $\mathbf{w} = \mathbf{0}$ (for $\lambda = 0$).

Conclusions: $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ implies that either $\mathbf{u}$ and $\mathbf{v}$ are parallel, or $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, or one of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is the zero vector.

68. Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are nonzero and

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$$

Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are either mutually parallel or mutually perpendicular. *Hint:* Use Exercise 67.

SOLUTION First notice that since $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$, also,

$$(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = -\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$$

Using Exercise 67 for nonzero vectors, we obtain

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0} \Rightarrow \mathbf{u} \parallel \mathbf{v} \text{ or } (\mathbf{w} \perp \mathbf{u} \text{ and } \mathbf{w} \perp \mathbf{v}) \tag{1}$$

$$(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = \mathbf{0} \Rightarrow \mathbf{w} \parallel \mathbf{v} \text{ or } (\mathbf{u} \perp \mathbf{v} \text{ and } \mathbf{u} \perp \mathbf{w}) \tag{2}$$

We consider each of the two possibilities in (1):

Case 1: $\mathbf{u} \parallel \mathbf{v}$ In this case, $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. Hence, by (2), $\mathbf{w} \parallel \mathbf{v}$ must hold. Thus, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Case 2: $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$ In this case, $\mathbf{w}$ and $\mathbf{v}$ are not parallel. Hence, by (2), $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$ must hold. Thus, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular.

Conclusion: If $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$ for nonzero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ then these vectors are parallel or mutually perpendicular.

69. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be nonzero vectors, and set

$$\mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \qquad \mathbf{w} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(a) Prove that

(i) $\mathbf{v}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$.

(ii) $\mathbf{v}$ is orthogonal to $\mathbf{a}$.

(b) Prove that $\mathbf{w}$ also satisfies (i) and (ii). Conclude that $\mathbf{v}$ and $\mathbf{w}$ are parallel.

(c) Show algebraically that $\mathbf{v} = \mathbf{w}$ (Figure 23).

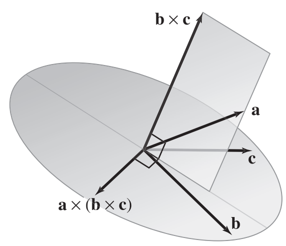


FIGURE 23

SOLUTION

(a) Since $\mathbf{v}$ is the cross product of $\mathbf{a}$ and another vector $(\mathbf{b} \times \mathbf{c})$, then $\mathbf{v}$ is orthogonal to $\mathbf{a}$. Furthermore, $\mathbf{v}$ is orthogonal to $(\mathbf{b} \times \mathbf{c})$, so it is orthogonal to the normal vector to the plane containing $\mathbf{b}$ and $\mathbf{c}$, so $\mathbf{v}$ must be in that plane.

(b) $\mathbf{w} \cdot \mathbf{a} = ((\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}) \cdot \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a}) = 0$ (since $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}$). Thus, $\mathbf{w}$ is orthogonal to $\mathbf{a}$. Also, $\mathbf{w}$ is a multiple of $\mathbf{b}$ and $\mathbf{c}$, so $\mathbf{w}$ must be in the plane containing $\mathbf{b}$ and $\mathbf{c}$.

Now, if $\mathbf{a}$ is perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, then $\mathbf{a}$ is parallel to $\mathbf{b} \times \mathbf{c}$ and so $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$, which means $\mathbf{v} = 0$, but also $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$ which means $\mathbf{w} = 0$. Thus, $\mathbf{v}$ and $\mathbf{w}$ are parallel (in fact, equal).

Now, if $\mathbf{a}$ is not perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, then the set of vectors on that plane that are also perpendicular to $\mathbf{a}$ form a line, and thus all such vectors are parallel. We conclude that $\mathbf{v}$ and $\mathbf{w}$, being on that plane and perpendicular to $\mathbf{a}$, are parallel.

(c) On the one hand,

$$\begin{aligned}
\mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix} \\
&= \big\langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\
&\quad a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\big\rangle
\end{aligned}$$

but on the other hand,

$$\begin{aligned}
\mathbf{w} &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
&= (a_1c_1 + a_2c_2 + a_3c_3)\langle b_1, b_2, b_3 \rangle - (a_1b_1 + a_2b_2 + a_3b_3)\langle c_1, c_2, c_3 \rangle
\end{aligned}$$

$$= \langle a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1, a_1c_1b_2 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2,$$
$$a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3\rangle$$
$$= \langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1),$$
$$a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\rangle$$

which is the same as $\mathbf{v}$.

70. Use Exercise 69 to prove the identity

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

SOLUTION We have

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\ &= -[(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] - [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \\ &= -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}\end{aligned}$$

as desired.

71. Show that if $\mathbf{a}, \mathbf{b}$ are nonzero vectors such that $\mathbf{a} \perp \mathbf{b}$, then there exists a vector $\mathbf{X}$ such that

$$\mathbf{a} \times \mathbf{X} = \mathbf{b} \qquad \boxed{13}$$

Hint: Show that if $\mathbf{X}$ is orthogonal to $\mathbf{b}$ and is not a multiple of $\mathbf{a}$, then $\mathbf{a} \times \mathbf{X}$ is a multiple of $\mathbf{b}$.

SOLUTION We define the following vectors:

$$\mathbf{X} = \frac{\mathbf{b} \times \mathbf{a}}{\|\mathbf{a}\|^2}, \quad \mathbf{c} = \mathbf{X} \times \mathbf{a} \qquad (1)$$

We show that $\mathbf{c} = \mathbf{b}$. Since $\mathbf{X}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{X}$ is orthogonal to the plane of $\mathbf{a}$ and $\mathbf{b}$. But $\mathbf{c}$ is orthogonal to $\mathbf{X}$, hence $\mathbf{c}$ is contained in the plane of $\mathbf{a}$ and $\mathbf{b}$, that is, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in the same plane. Now the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in one plane, and the vectors $\mathbf{c}$ and $\mathbf{b}$ are orthogonal to $\mathbf{a}$.

It follows that $\mathbf{c}$ and $\mathbf{b}$ are parallel. (2)

We now show that $\|\mathbf{c}\| = \|\mathbf{b}\|$. We use the cross-product identity to obtain

$$\|\mathbf{c}\|^2 = \|\mathbf{X} \times \mathbf{a}\|^2 = \|\mathbf{X}\|^2\|\mathbf{a}\|^2 - (\mathbf{X} \cdot \mathbf{a})^2$$

$\mathbf{X}$ is orthogonal to $\mathbf{a}$, hence $\mathbf{X} \cdot \mathbf{a} = 0$, and we obtain

$$\|\mathbf{c}\|^2 = \|\mathbf{X}\|^2\|\mathbf{a}\|^2 = \left\|\frac{\mathbf{b} \times \mathbf{a}}{\|\mathbf{a}\|^2}\right\|^2 \|\mathbf{a}\|^2 = \frac{1}{\|\mathbf{a}\|^4}\|\mathbf{b} \times \mathbf{a}\|^2\|\mathbf{a}\|^2 = \frac{1}{\|\mathbf{a}\|^2}\|\mathbf{b} \times \mathbf{a}\|^2$$

By the given data, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, so,

$$\|\mathbf{c}\|^2 = \frac{1}{\|\mathbf{a}\|^2}\left(\|\mathbf{b}\|^2\|\mathbf{a}\|^2\right) = \|\mathbf{b}\|^2 \Rightarrow \|\mathbf{c}\| = \|\mathbf{b}\| \qquad (3)$$

By (2) and (3) it follows that $\mathbf{c} = \mathbf{b}$ or $\mathbf{c} = -\mathbf{b}$. We thus proved that the vector $\mathbf{X} = \frac{\mathbf{b} \times \mathbf{a}}{\|\mathbf{a}\|^2}$ satisfies $\mathbf{X} \times \mathbf{a} = \mathbf{b}$ or $\mathbf{X} \times \mathbf{a} = -\mathbf{b}$. If $\mathbf{X} \times \mathbf{a} = -\mathbf{b}$, then $(-\mathbf{X}) \times \mathbf{a} = \mathbf{b}$. Hence, there exists a vector $\mathbf{X}$ such that $\mathbf{X} \times \mathbf{a} = \mathbf{b}$.

72. Show that if $\mathbf{a}, \mathbf{b}$ are nonzero vectors such that $\mathbf{a} \perp \mathbf{b}$, then the set of all solutions of Eq. (13) is a line with $\mathbf{a}$ as direction vector. *Hint:* Let $\mathbf{X}_0$ be any solution (which exists by Exercise 71), and show that every other solution is of the form $\mathbf{X}_0 + \lambda\mathbf{a}$ for some scalar λ.

SOLUTION By Exercise 71 there exists a solution $\mathbf{X}_0$ of the equation $\mathbf{X} \times \mathbf{a} = \mathbf{b}$. Let $\mathbf{X}$ be any solution of this equation. Thus,

$$\mathbf{X}_0 \times \mathbf{a} = \mathbf{b}$$
$$\mathbf{X} \times \mathbf{a} = \mathbf{b}$$

We subtract the first equation from the second and use the distributive law to obtain

$$\mathbf{X} \times \mathbf{a} - \mathbf{X}_0 \times \mathbf{a} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$
$$(\mathbf{X} - \mathbf{X}_0) \times \mathbf{a} = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} \times (\mathbf{X} - \mathbf{X}_0) = \mathbf{0}$$

We now use the basic properties of the cross product to conclude that there exists a scalar λ such that

$$\mathbf{X} - \mathbf{x_0} = \lambda\mathbf{a} \quad \text{or} \quad \mathbf{X} = \mathbf{x_0} + \lambda\mathbf{a}$$

This is a parametric equation of the line through $\mathbf{x_0}$ with $\mathbf{a}$ as a direction vector.

73. Assume that $\mathbf{v}$ and $\mathbf{w}$ lie in the first quadrant in $\mathbf{R}^2$ as in Figure 24. Use geometry to prove that the area of the parallelogram is equal to $\det\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix}$.

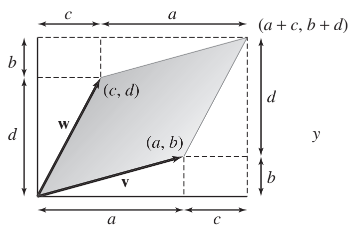
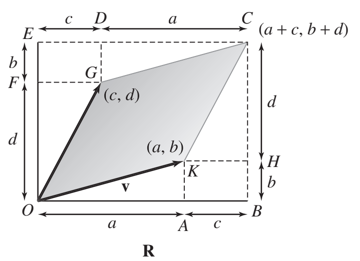


FIGURE 24

SOLUTION We denote the components of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} = \langle c, d\rangle$$
$$\mathbf{v} = \langle a, b\rangle$$

We also denote by $O, A, B, C, D, E, F, G, H, K$ the points shown in the figure.



Since $OGCK$ is a parallelogram, it follows by geometrical properties that the triangles OFG and KHC and also the triangles DGC and AKO are congruent. It also follows that the rectangles $EFDG$ and $ABHK$ have equal areas. We use the following notation:

A: The area of the parallelogram
S: The area of the rectangle $OBCE$
S_1: The area of the rectangle $EFDG$
S_2: The area of the triangle OFG
S_3: The area of the triangle DGC

Hence,

$$A = S - 2(S_1 + S_2 + S_3) \tag{1}$$

Using the formulas for the areas of rectangles and triangles we have (see figure)

$$S = OB \cdot OE = (a + c)(d + b)$$
$$S_1 = bc, \quad S_2 = \frac{cd}{2}, \quad S_3 = \frac{ab}{2}$$

Substituting into (1) we get

$$\begin{aligned} A &= (a + c)(d + b) - 2\left(bc + \frac{cd}{2} + \frac{ab}{2}\right) \\ &= ad + ab + cd + cb - 2bc - cd - ab \qquad (2) \\ &= ad - bc \end{aligned}$$

On the other hand,

$$\det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{3}$$

By (2) and (3) we obtain the desired result.

74. Consider the tetrahedron spanned by vectors **a**, **b**, and **c** as in Figure 25(A). Let A, B, C be the faces containing the origin O, and let D be the fourth face opposite O. For each face F, let $\mathbf{v}_F$ be the vector normal to the face, pointing outside the tetrahedron, of magnitude equal to twice the area of F. Prove the relations

$$\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$$

$$\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C + \mathbf{v}_D = 0$$

Hint: Show that $\mathbf{v}_D = (\mathbf{c} - \mathbf{b}) \times (\mathbf{b} - \mathbf{a})$.

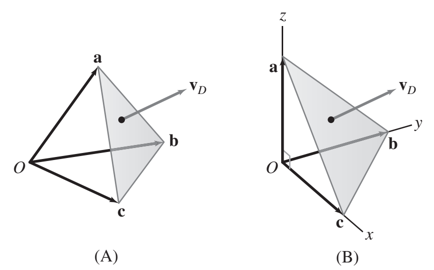
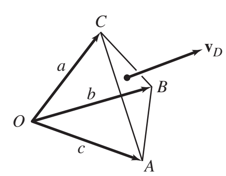


FIGURE 25 The vector $\mathbf{v}_D$ is perpendicular to the face.

SOLUTION We first show that $\mathbf{v}_D = (\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})$.



Since $\mathbf{v}_D$ is normal to the face D, it is orthogonal to the vectors $\mathbf{c} - \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$, hence it is parallel to the cross product of these two vectors. In other words, there exists a scalar $\lambda > 0$ such that (using the right-hand rule)

$$\mathbf{v}_D = \lambda\,(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})$$

The area of the face D is half of the area of the parallelogram spanned by $\mathbf{c} - \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$. The area of the parallelogram is $\|(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})\|$. Hence,

$$\|\mathbf{v}_D\| = \|\,(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})\,\|$$

Combining the above equations we have

$$\mathbf{v}_D = (\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})$$

Now, since $\mathbf{v}_A$ is normal to the face A, it is orthogonal to the vectors **a** and **c**, therefore it is parallel to $\mathbf{c} \times \mathbf{a}$. The area of the face A is $\frac{1}{2}\|\mathbf{c} \times \mathbf{a}\|$, which is also half the length of $\mathbf{v}_A$. Hence, using the right-hand rule we get

$$\mathbf{v}_A = \mathbf{c} \times \mathbf{a}$$

Similarly, we have

$$\mathbf{v}_B = \mathbf{a} \times \mathbf{b}, \quad \mathbf{v}_C = \mathbf{b} \times \mathbf{c}$$

Combining gives us

$$\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C = (\mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$$

We evaluate $\mathbf{v}_D$ using the distributive law:

$$\mathbf{v}_D = (\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = (\mathbf{c} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a}$$

Hence,

$$\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C + \mathbf{v}_D = (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a}) = \mathbf{0}$$

75. In the notation of Exercise 74, suppose that **a**, **b**, **c** are mutually perpendicular as in Figure 25(B). Let S_F be the area of face F. Prove the following three-dimensional version of the Pythagorean Theorem:

$$S_A^2 + S_B^2 + S_C^2 = S_D^2$$

SOLUTION Since $\|\mathbf{v}_D\| = S_D$ then using Exercise 74 we obtain

$$\begin{aligned} S_D^2 &= \|\mathbf{v}_D\|^2 = \mathbf{v}_D \cdot \mathbf{v}_D = (\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C) \cdot (\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C) \\ &= \mathbf{v}_A \cdot \mathbf{v}_A + \mathbf{v}_A \cdot \mathbf{v}_B + \mathbf{v}_A \cdot \mathbf{v}_C + \mathbf{v}_B \cdot \mathbf{v}_A + \mathbf{v}_B \cdot \mathbf{v}_B + \mathbf{v}_B \cdot \mathbf{v}_C + \mathbf{v}_C \cdot \mathbf{v}_A + \mathbf{v}_C \cdot \mathbf{v}_B + \mathbf{v}_C \cdot \mathbf{v}_C \\ &= \|\mathbf{v}_A\|^2 + \|\mathbf{v}_B\|^2 + \|\mathbf{v}_C\|^2 + 2\,(\mathbf{v}_A \cdot \mathbf{v}_B + \mathbf{v}_A \cdot \mathbf{v}_C + \mathbf{v}_B \cdot \mathbf{v}_C) \qquad (1) \end{aligned}$$

Now, the normals $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ to the coordinate planes are mutually orthogonal, hence,

$$\mathbf{v}_A \cdot \mathbf{v}_B = \mathbf{v}_A \cdot \mathbf{v}_C = \mathbf{v}_B \cdot \mathbf{v}_C = 0 \qquad (2)$$

Combining (1) and (2) and using the relations $\|\mathbf{v}_F\| = S_F$ we obtain

$$S_D^2 = S_A^2 + S_B^2 + S_C^2$$

12.5 Planes in Three-Space (LT Section 13.5)

Preliminary Questions

1. What is the equation of the plane parallel to $3x + 4y - z = 5$ passing through the origin?

SOLUTION The two planes are parallel, therefore the vector $\mathbf{n} = \langle 3, 4, -1 \rangle$ that is normal to the given plane is also normal to the plane we need to find. This plane is passing through the origin, hence we may substitute $\langle x_0, y_0, z_0 \rangle = \langle 0, 0, 0 \rangle$ in the vector form of the equation of the plane. This gives

$$\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$$

$$\langle 3, 4, -1 \rangle \cdot \langle x, y, z \rangle = \langle 3, 4, -1 \rangle \cdot \langle 0, 0, 0 \rangle = 0$$

or in scalar form

$$3x + 4y - z = 0$$

2. The vector **k** is normal to which of the following planes?

(a) $x = 1$ **(b)** $y = 1$ **(c)** $z = 1$

SOLUTION The planes $x = 1$, $y = 1$, and $z = 1$ are orthogonal to the x, y, and z-axes respectively. Since the plane $z = 1$ is orthogonal to the z-axis, the vector **k** is normal to this plane.

3. Which of the following planes is not parallel to the plane $x + y + z = 1$?

(a) $2x + 2y + 2z = 1$ **(b)** $x + y + z = 3$
(c) $x - y + z = 0$

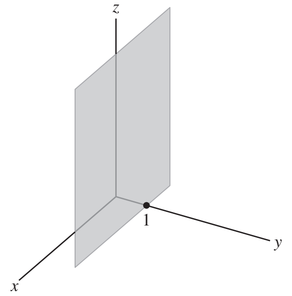
SOLUTION The two planes are parallel if vectors that are normal to the planes are parallel. The vector $\mathbf{n} = \langle 1, 1, 1 \rangle$ is normal to the plane $x + y + z = 1$. We identify the following normals:

- $\mathbf{v} = \langle 2, 2, 2 \rangle$ is normal to plane (a)
- $\mathbf{u} = \langle 1, 1, 1 \rangle$ is normal to plane (b)
- $\mathbf{w} = \langle 1, -1, 1 \rangle$ is normal to plane (c)

The vectors **v** and **u** are parallel to **n**, whereas **w** is not. (These vectors are not constant multiples of each other). Therefore, only plane (c) is not parallel to the plane $x + y + z = 1$.

4. To which coordinate plane is the plane $y = 1$ parallel?

SOLUTION The plane $y = 1$ is parallel to the xz-plane.



5. Which of the following planes contains the z-axis?

(a) $z = 1$ **(b)** $x + y = 1$ **(c)** $x + y = 0$

SOLUTION The points on the z-axis are the points with zero x and y coordinates. A plane contains the z-axis if and only if the points $(0, 0, c)$ satisfy the equation of the plane for all values of c.

(a) Plane (a) does not contain the z-axis, rather it is orthogonal to this axis. Only the point $(0, 0, 1)$ is on the plane.

(b) $x = 0$ and $y = 0$ do not satisfy the equation of the plane, since $0 + 0 \neq 1$. Therefore the plane does not contain the z-axis.

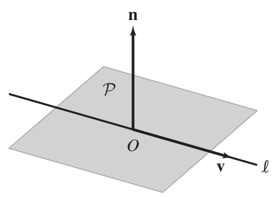
(c) The plane $x + y = 0$ contains the z-axis since $x = 0$ and $y = 0$ satisfy the equation of the plane.

6. Suppose that a plane $\mathcal{P}$ with normal vector $\mathbf{n}$ and a line $\mathcal{L}$ with direction vector $\mathbf{v}$ both pass through the origin and that $\mathbf{n} \cdot \mathbf{v} = 0$. Which of the following statements is correct?

(a) $\mathcal{L}$ is contained in $\mathcal{P}$.

(b) $\mathcal{L}$ is orthogonal to $\mathcal{P}$.

SOLUTION The direction vector of the line $\mathcal{L}$ is orthogonal to the vector $\mathbf{n}$ that is normal to the plane. Therefore, $\mathcal{L}$ is either parallel or contained in the plane. Since the origin is common to $\mathcal{L}$ and $\mathcal{P}$, the line is contained in the plane. That is, statement (a) is correct.



Exercises

In Exercises 1–8, write the equation of the plane with normal vector **n** *passing through the given point in each of the three forms (one vector form and two scalar forms).*

1. $\mathbf{n} = \langle 1, 3, 2\rangle$, $(4, -1, 1)$

SOLUTION The vector equation is

$$\langle 1, 3, 2\rangle \cdot \langle x, y, z\rangle = \langle 1, 3, 2\rangle \cdot \langle 4, -1, 1\rangle = 4 - 3 + 2 = 3$$

To obtain the scalar forms we compute the dot product on the left-hand side of the previous equation:

$$x + 3y + 2z = 3$$

or in the other scalar form:

$$(x - 4) + 3(y + 1) + 2(z - 1) + 4 - 3 + 2 = 3$$
$$(x - 4) + 3(y + 1) + 2(z - 1) = 0$$

2. $\mathbf{n} = \langle -1, 2, 1\rangle$, $(3, 1, 9)$

SOLUTION The vector equation is

$$\langle -1, 2, 1\rangle \cdot \langle x, y, z\rangle = \langle -1, 2, 1\rangle \cdot \langle 3, 1, 9\rangle = -3 + 2 + 9 = 8$$

To obtain the scalar form we compute the dot product on the left-hand side above:

$$-x + 2y + z = 8$$

or in the other scalar form:

$$-(x - 3) + 2(y - 1) + (z - 9) = 8 + 3 - 2 - 9 = 0$$
$$-(x - 3) + 2(y - 1) + (z - 9) = 0$$

3. $\mathbf{n} = \langle -1, 2, 1\rangle$, $(4, 1, 5)$

SOLUTION The vector form is

$$\langle -1, 2, 1\rangle \cdot \langle x, y, z\rangle = \langle -1, 2, 1\rangle \cdot \langle 4, 1, 5\rangle = -4 + 2 + 5 = 3$$

To obtain the scalar form we compute the dot product above:

$$-x + 2y + z = 3$$

or in the other scalar form:

$$-(x-4)+2(y-1)+(z-5)=3+4-2-5=0$$
$$-(x-4)+2(y-1)+(z-5)=0$$

4. $\mathbf{n}=\langle 2,-4,1\rangle$, $\left(\frac{1}{3},\frac{2}{3},1\right)$

SOLUTION The vector form is

$$\langle 2,-4,1\rangle\cdot\langle x,y,z\rangle=\langle 2,-4,1\rangle\cdot\left\langle\frac{1}{3},\frac{2}{3},1\right\rangle=\frac{2}{3}-\frac{8}{3}+1=-1$$

We find the scalar form by computing the dot product above:

$$2x-4y+z=-1$$

or in the form:

$$2\left(x-\frac{1}{3}\right)-4\left(y-\frac{2}{3}\right)+(z-1)=-1-\frac{2}{3}+\frac{8}{3}-1=0$$
$$2\left(x-\frac{1}{3}\right)-4\left(y-\frac{2}{3}\right)+(z-1)=0$$

5. $\mathbf{n}=\mathbf{i}$, $(3,1,-9)$

SOLUTION We find the vector form of the equation of the plane. We write the vector $\mathbf{n}=\mathbf{i}$ as $\mathbf{n}=\langle 1,0,0\rangle$ and obtain

$$\langle 1,0,0\rangle\cdot\langle x,y,z\rangle=\langle 1,0,0\rangle\cdot\langle 3,1,-9\rangle=3+0+0=3$$

Computing the dot product above gives the scalar form:

$$x+0+0=3$$
$$x=3$$

Or in the other scalar form:

$$(x-3)+0\cdot(y-1)+0\cdot(z+9)=3-3=0$$

6. $\mathbf{n}=\mathbf{j}$, $\left(-5,\frac{1}{2},\frac{1}{2}\right)$

SOLUTION Writing $\mathbf{n}=\mathbf{j}$ in the form $\mathbf{n}=\langle 0,1,0\rangle$ we obtain the following vector form of the equation of the plane:

$$\langle 0,1,0\rangle\cdot\langle x,y,z\rangle=\langle 0,1,0\rangle\cdot\left\langle -5,\frac{1}{2},\frac{1}{2}\right\rangle=0+\frac{1}{2}+0=\frac{1}{2}$$

We compute the dot product to obtain the scalar form:

$$0x+1y+0z=\frac{1}{2}$$
$$y=\frac{1}{2}$$

or in the other scalar form:

$$0(x+5)+\left(y-\frac{1}{2}\right)+0\left(z-\frac{1}{2}\right)=0$$

7. $\mathbf{n}=\mathbf{k}$, $(6,7,2)$

SOLUTION We write the normal $\mathbf{n}=\mathbf{k}$ in the form $\mathbf{n}=\langle 0,0,1\rangle$ and obtain the following vector form of the equation of the plane:

$$\langle 0,0,1\rangle\cdot\langle x,y,z\rangle=\langle 0,0,1\rangle\cdot\langle 6,7,2\rangle=0+0+2=2$$

We compute the dot product to obtain the scalar form:

$$0x+0y+1z=2$$
$$z=2$$

or in the other scalar form:

$$0(x-6)+0(y-7)+1(z-2)=0$$

8. $\mathbf{n} = \mathbf{i} - \mathbf{k}$, $(4, 2, -8)$

SOLUTION We write the normal $\mathbf{n} = \mathbf{i} - \mathbf{k}$ in the form $\mathbf{n} = \langle 1, 0, -1\rangle$ and obtain the following vector form of the equation of the plane:

$$\langle 1, 0, -1\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, -1\rangle \cdot \langle 4, 2, -8\rangle = 4 + 0 + 8 = 12$$

To find the scalar form we compute the dot product:

$$x - z = 12$$

or the other scalar form:

$$(x - 4) + 0(y - 2) - (z + 8) = 12 - 4 - 8 = 0$$

9. Write down the equation of any plane through the origin.

SOLUTION We can use any equation $ax + by + cz = d$ which contains the point $(x, y, z) = (0, 0, 0)$. One solution (and there are many) is $x + y + z = 0$.

10. Write down the equations of any two distinct planes with normal vector $\mathbf{n} = \langle 3, 2, 1\rangle$ that do not pass through the origin.

SOLUTION The equation of a plane with normal vector $\mathbf{n} = \langle 3, 2, 1\rangle$ is $\langle 3, 2, 1\rangle \cdot \langle x, y, z\rangle = d$, or in other words, $3x + 2y + z = d$. Since we do not want the planes to pass through the origin, we want values of d such that $(x, y, z) = (0, 0, 0)$ is not on the plane. This will hold for any $d \neq 0$, so two possible solutions (and there are many) are $3x + 2y + z = 1$ and $3x + 2y + z = 2$.

11. Which of the following statements are true of a plane that is parallel to the yz-plane?

(a) $\mathbf{n} = \langle 0, 0, 1\rangle$ is a normal vector.

(b) $\mathbf{n} = \langle 1, 0, 0\rangle$ is a normal vector.

(c) The equation has the form $ay + bz = d$

(d) The equation has the form $x = d$

SOLUTION

(a) For $\mathbf{n} = \langle 0, 0, 1\rangle$ a normal vector, the plane would be parallel to the xy-plane, not the yz-plane. This statement is false.

(b) For $\mathbf{n} = \langle 1, 0, 0\rangle$ a normal vector, the plane would be parallel to the yz-plane. This statement is true.

(c) For the equation $ay + bz = d$, this plane intersects the yz-plane at $y = 0$, $z = d/b$ or $y = d/a$, $z = 0$ depending on whether a or b is non-zero, but it is not equal to the yz-plane (which has equation $x = d$) Thus, it is not parallel to the yz-plane This statement is false.

(d) For the equation of the form $x = d$, this has $\langle 1, 0, 0\rangle$ as a normal vector and is parallel to the yz-plane. This statement is true.

12. Find a normal vector $\mathbf{n}$ and an equation for the planes in Figures 7(A)–(C).

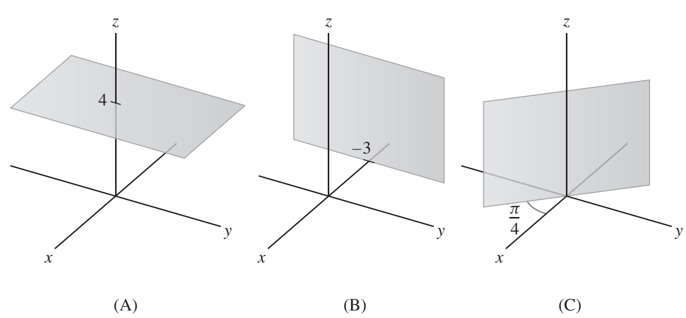


FIGURE 7

SOLUTION

(a) This plane has normal vector pointing straight up, so $\mathbf{n} = \langle 0, 0, 1\rangle$, and it contains the point $(0, 0, 4)$, so it has equation $0(x - 0) + 0(y - 0) + 1(z - 4) = 0$, which becomes $z = 4$.

(b) This plane has normal vector pointing along the x-axis, so $\mathbf{n} = \langle 1, 0, 0\rangle$, and it contains the point $(-3, 0, 0)$, so it has equation $1(x - -3) + 0(y - 0) + 0(z - 0) = 0$, which becomes $x = -3$.

(c) This plane has normal vector pointing along the line $y = x$, so $\mathbf{n} = \langle 1, 1, 0\rangle$, and it contains the point $(0, 0, 0)$, so it has equation $1(x - 0) + 1(y - 0) + 0(z - 0) = 0$, which becomes $x + y = 0$.

In Exercises 13–16, find a vector normal to the plane with the given equation.

13. $9x - 4y - 11z = 2$

SOLUTION Using the scalar form of the equation of the plane, a vector normal to the plane is the coefficients vector:

$$\mathbf{n} = \langle 9, -4, -11 \rangle$$

14. $x - z = 0$

SOLUTION We write the equation in vector form $\langle 1, 0, -1 \rangle \cdot \langle x, y, z \rangle = 0$ and identify $\mathbf{n} = \langle 1, 0, -1 \rangle$ as a vector normal to the plane.

15. $3(x - 4) - 8(y - 1) + 11z = 0$

SOLUTION Using the scalar form of the equation of the plane, $3x - 8y + 11z = 4$ a vector normal to the plane is the coefficients vector:

$$\mathbf{n} = \langle 3, -8, 11 \rangle$$

16. $x = 1$

SOLUTION The equation in vector form is

$$\langle 1, 0, 0 \rangle \cdot \langle x, y, z \rangle = 1$$

Therefore, the vector $\mathbf{n} = \langle 1, 0, 0 \rangle$ is normal to the plane.

In Exercises 17–20, find an equation of the plane passing through the three points given.

17. $P = (2, -1, 4), \quad Q = (1, 1, 1), \quad R = (3, 1, -2)$

SOLUTION We go through the steps below:

Step 1. Find the normal vector $\mathbf{n}$. The vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ lie on the plane, hence the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane. We compute the cross product:

$$\mathbf{a} = \overrightarrow{PQ} = \langle 1 - 2, 1 - (-1), 1 - 4 \rangle = \langle -1, 2, -3 \rangle$$
$$\mathbf{b} = \overrightarrow{PR} = \langle 3 - 2, 1 - (-1), -2 - 4 \rangle = \langle 1, 2, -6 \rangle$$
$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ 1 & 2 & -6 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 2 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -3 \\ 1 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k}$$
$$= -6\mathbf{i} - 9\mathbf{j} - 4\mathbf{k} = \langle -6, -9, -4 \rangle$$

Step 2. Choose a point on the plane. We choose any one of the three points on the plane, for instance $Q = (1, 1, 1)$. Using the vector form of the equation of the plane we get

$$\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$$
$$\langle -6, -9, -4 \rangle \cdot \langle x, y, z \rangle = \langle -6, -9, -4 \rangle \cdot \langle 1, 1, 1 \rangle$$

Computing the dot products we obtain the following equation:

$$-6x - 9y - 4z = -6 - 9 - 4 = -19$$
$$6x + 9y + 4z = 19$$

18. $P = (5, 1, 1), \quad Q = (1, 1, 2), \quad R = (2, 1, 1)$

SOLUTION Note that these three points all have a y-value of 1. So, consider the equation $y = 1$; it is the equation of a plane, and all three points P, Q, R satisfy the equation $y = 1$, so all three points P, Q, R are on this plane, so we are done! (We could also solve this problem using the traditional method of finding a normal vector, etc., and we will get the same answer of $y = 1$.)

19. $P = (1, 0, 0), \quad Q = (0, 1, 1), \quad R = (2, 0, 1)$

SOLUTION We use the vector form of the equation of the plane:

$$\mathbf{n} \cdot \langle x, y, z \rangle = d \tag{1}$$

To find the normal vector to the plane, $\mathbf{n}$, we first compute the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ that lie in the plane, and then find the cross product of these vectors. This gives

$$\overrightarrow{PQ} = \langle 0, 1, 1\rangle - \langle 1, 0, 0\rangle = \langle -1, 1, 1\rangle$$

$$\overrightarrow{PR} = \langle 2, 0, 1\rangle - \langle 1, 0, 0\rangle = \langle 1, 0, 1\rangle$$

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} + 2\mathbf{j} - \mathbf{k} = \langle 1, 2, -1\rangle \tag{2}$$

We now choose any one of the three points in the plane, say $P = (1, 0, 0)$, and compute d:

$$d = \mathbf{n} \cdot \overrightarrow{OP} = \langle 1, 2, -1\rangle \cdot \langle 1, 0, 0\rangle = 1 \cdot 1 + 2 \cdot 0 + (-1) \cdot 0 = 1 \tag{3}$$

Finally we substitute (2) and (3) into (1) to obtain the following equation of the plane:

$$\langle 1, 2, -1\rangle \cdot \langle x, y, z\rangle = 1$$

$$x + 2y - z = 1$$

20. $P = (2, 0, 0), \quad Q = (0, 4, 0), \quad R = (0, 0, 2)$

SOLUTION We go through the following steps:

Step 1. Find a normal vector $\mathbf{n}$. The normal vector $\mathbf{n}$ must be orthogonal to the vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ in the plane, hence we may choose $\mathbf{n}$ as the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. We compute $\mathbf{n}$:

$$\mathbf{a} = \overrightarrow{PQ} = \langle 0 - 2, 4 - 0, 0 - 0\rangle = \langle -2, 4, 0\rangle$$

$$\mathbf{b} = \overrightarrow{PR} = \langle 0 - 2, 0 - 0, 2 - 0\rangle = \langle -2, 0, 2\rangle$$

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 0 \\ -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 4 \\ -2 & 0 \end{vmatrix}\mathbf{k}$$

$$= 8\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} = \langle 8, 4, 8\rangle$$

Step 2. Choose a point on the plane. We choose a point on the plane, for instance $(x_0, y_0, z_0) = (2, 0, 0)$. Using the vector form of the equation we get

$$\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle$$

$$\langle 8, 4, 8\rangle \cdot \langle x, y, z\rangle = \langle 8, 4, 8\rangle \cdot \langle 2, 0, 0\rangle$$

We compute the dot products:

$$8x + 4y + 8z = 16 + 0 + 0 = 16$$

$$2x + y + 2z = 4$$

In Exercises 21–28, find the equation of the plane with the given description.

21. Passes through O and is parallel to $4x - 9y + z = 3$

SOLUTION The vector $\mathbf{n} = \langle 4, -9, 1\rangle$ is normal to the plane $4x - 9y + z = 3$, and so is also normal to the parallel plane. Setting $\mathbf{n} = \langle 4, -9, 1\rangle$ and $(x_0, y_0, z_0) = (0, 0, 0)$ in the vector equation of the plane yields

$$\langle 4, -9, 1\rangle \cdot \langle x, y, z\rangle = \langle 4, -9, 1\rangle \cdot \langle 0, 0, 0\rangle = 0$$

$$4x - 9y + z = 0$$

22. Passes through $(4, 1, 9)$ and is parallel to $x + y + z = 3$

SOLUTION We write the equation of the plane $x + y + z = 3$ in vector form:

$$\langle 1, 1, 1\rangle \cdot \langle x, y, z\rangle = 3$$

We identify $\mathbf{n} = \langle 1, 1, 1\rangle$ as a vector normal to the plane. This vector is also normal to the parallel plane. We substitute $\mathbf{n} = \langle 1, 1, 1\rangle$ and $(x_0, y_0, z_0) = (4, 1, 9)$ in the vector equation of the plane to obtain

$$\langle 1, 1, 1\rangle \cdot \langle x, y, z\rangle = \langle 1, 1, 1\rangle \cdot \langle 4, 1, 9\rangle$$

$$x + y + z = 4 + 1 + 9 = 14$$

$$x + y + z = 14$$

23. Passes through $(4, 1, 9)$ and is parallel to $x = 3$

SOLUTION The vector form of the plane $x = 3$ is

$$\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = 3$$

Hence, $\mathbf{n} = \langle 1, 0, 0\rangle$ is normal to this plane. This vector is also normal to the parallel plane. Setting $(x_0, y_0, z_0) = (4, 1, 9)$ and $\mathbf{n} = \langle 1, 0, 0\rangle$ in the vector equation of the plane yields

$$\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, 0\rangle \cdot \langle 4, 1, 9\rangle = 4 + 0 + 0 = 4$$

or

$$x + 0 + 0 = 4 \quad \Rightarrow \quad x = 4$$

24. Passes through $P = (3, 5, -9)$ and is parallel to the xz-plane

SOLUTION The xz-plane is the plane $y = 0$, which has normal vector $\mathbf{n} = \langle 0, 1, 0\rangle$. Using this in the scalar equation of the plane gives us:

$$0(x - 3) + 1(y - 5) + 0(z - -9) = 0 \quad \Rightarrow \quad y = 5$$

25. Passes through $(-2, -3, 5)$ and has normal vector $\mathbf{i} + \mathbf{k}$

SOLUTION We substitute $\mathbf{n} = \langle 1, 0, 1\rangle$ and $(x_0, y_0, z_0) = (-2, -3, 5)$ in the vector equation of the plane to obtain

$$\langle 1, 0, 1\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, 1\rangle \cdot \langle -2, -3, 5\rangle$$

or

$$\begin{aligned} x + 0 + z &= -2 + 0 + 5 = 3 \\ x + z &= 3 \end{aligned}$$

26. Contains the lines $\mathbf{r}_1(t) = \langle t, 2t, 3t\rangle$ and $\mathbf{r}_2(t) = \langle 3t, t, 8t\rangle$

SOLUTION Since the plane contains the lines $\ell_1(t) = \langle t, 2t, 3t\rangle$ and $\ell_2(t) = \langle 3t, t, 8t\rangle$, the direction vectors $\mathbf{v}_1 = \langle 1, 2, 3\rangle$ and $\mathbf{v}_2 = \langle 3, 1, 8\rangle$ of the lines lie in the plane. Therefore the cross product $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is normal to the plane. We compute the cross product:

$$\begin{aligned} \mathbf{n} = \langle 1, 2, 3\rangle \times \langle 3, 1, 8\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} \\ &= 13\mathbf{i} + \mathbf{j} - 5\mathbf{k} = \langle 13, 1, -5\rangle \end{aligned}$$

We now must choose a point on the plane. Since the line $\ell_1(t) = \langle t, 2t, 3t\rangle$ is contained in the plane, all of its points are on the plane. We choose the point corresponding to $t = 1$, that is,

$$\langle x_0, y_0, z_0\rangle = \langle 1, 2 \cdot 1, 3 \cdot 1\rangle = \langle 1, 2, 3\rangle$$

We now use the vector equation of the plane to determine the equation of the desired plane:

$$\begin{aligned} \mathbf{n} \cdot \langle x, y, z\rangle &= \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle \\ \langle 13, 1, -5\rangle \cdot \langle x, y, z\rangle &= \langle 13, 1, -5\rangle \cdot \langle 1, 2, 3\rangle \\ 13x + y - 5z &= 13 + 2 - 15 = 0 \\ 13x + y - 5z &= 0 \end{aligned}$$

27. Contains the lines $\mathbf{r}_1(t) = \langle 2, 1, 0\rangle + \langle t, 2t, 3t\rangle$ and $\mathbf{r}_2(t) = \langle 2, 1, 0\rangle + \langle 3t, t, 8t\rangle$

SOLUTION Since the plane contains the lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, the direction vectors $\mathbf{v}_1 = \langle 1, 2, 3\rangle$ and $\mathbf{v}_2 = \langle 3, 1, 8\rangle$ of the lines lie in the plane. Therefore the cross product $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is normal to the plane. We compute the cross product:

$$\begin{aligned} \mathbf{n} = \langle 1, 2, 3\rangle \times \langle 3, 1, 8\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} \\ &= 13\mathbf{i} + \mathbf{j} - 5\mathbf{k} = \langle 13, 1, -5\rangle \end{aligned}$$

We now must choose a point on the plane. Since the line $\mathbf{r}_1(t) = \langle 2+t, 1+2t, 3t\rangle$ is contained in the plane, all of its points are on the plane. We choose the point corresponding to $t = 0$, that is,

$$\langle x_0, y_0, z_0\rangle = \langle 2, 1, 0\rangle$$

We now use the vector equation of the plane to determine the equation of the desired plane:

$$\mathbf{n}\cdot\langle x, y, z\rangle = \mathbf{n}\cdot\langle x_0, y_0, z_0\rangle$$
$$\langle 13, 1, -5\rangle\cdot\langle x, y, z\rangle = \langle 13, 1, -5\rangle\cdot\langle 2, 1, 0\rangle$$
$$13x + y - 5z = 26 + 1 + 0 = 27$$
$$13x + y - 5z = 27$$

28. Contains $P = (-1, 0, 1)$ and $\mathbf{r}(t) = \langle t+1, 2t, 3t-1\rangle$

SOLUTION Since the line $\ell(t) = \langle t+1, 2t, 3t-1\rangle$ is contained in the plane, its direction vector $\mathbf{v} = \langle 1, 2, 3\rangle$ lies in the plane. To find a second vector on the plane we first choose a point on the line, say, the point corresponding to $t = 0$. That is,

$$Q = (0+1, 2\cdot 0, 0-1) = (1, 0, -1)$$

Now, since the points $P = (-1, 0, 1)$ and $Q = (1, 0, -1)$ are on the plane, the vector $\mathbf{u} = \overrightarrow{PQ}$ lies on the plane. We find it here:

$$\mathbf{u} = \overrightarrow{PQ} = \langle 1-(-1), 0-0, -1-1\rangle = \langle 2, 0, -2\rangle$$

We now compute a vector normal to the plane, by finding the cross product $\mathbf{n} = \mathbf{v}\times\mathbf{u}$:

$$\mathbf{n} = \mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$
$$= -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k} = \langle -4, 8-4\rangle$$

Finally we substitute $\mathbf{n} = \langle -4, 8, -4\rangle$ and the given point $P = (x_0, y_0, z_0) = (-1, 0, 1)$ in the vector equation of the plane to obtain

$$\mathbf{n}\cdot\langle x, y, z\rangle = \mathbf{n}\cdot\langle x_0, y_0, z_0\rangle$$
$$\langle -4, 8, -4\rangle\cdot\langle x, y, z\rangle = \langle -4, 8, -4\rangle\cdot\langle -1, 0, 1\rangle$$

or

$$-4x + 8y - 4z = 4 + 0 - 4 = 0$$
$$x - 2y + z = 0$$

29. Are the planes $\frac{1}{2}x + 2x - y = 5$ and $3x + 12x - 6y = 1$ parallel?

SOLUTION The planes $2\frac{1}{2}x - y = 5$ and $15x - 6y = 1$, are parallel if and only if the vectors $\mathbf{n}_1 = \left\langle 2\frac{1}{2}, -1, 0\right\rangle$ and $\mathbf{n}_2 = \langle 15, -6, 0\rangle$ normal to the planes are parallel. Since $\mathbf{n}_2 = 6\mathbf{n}_1$ the planes are parallel.

30. Let a, b, c be constants. Which two of the following equations define the plane passing through $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$?

(a) $ax + by + cz = 1$

(b) $bcx + acy + abz = abc$

(c) $bx + cy + az = 1$

(d) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

SOLUTION

(a) Substituting $x = a$, $y = 0$, $z = 0$ in the left-hand side of the equation of the plane $ax + by + cz = 1$ gives

$$a\cdot a + b\cdot 0 + c\cdot 0 = a^2$$

If $a \neq \pm 1$, the point $(a, 0, 0)$ does not lie on the plane.

(b) Substituting $x = a$, $y = z = 0$ in the left-hand side of the equation $bcx + acy + abz = abc$ gives

$$bca + ac\cdot 0 + ab\cdot 0 = bca$$

Thus, the point $(a, 0, 0)$ is on the plane. Similarly we check the other two points $(0, b, 0)$ and $(0, 0, c)$:

$$bc\cdot 0 + acb + ab\cdot 0 = acb$$
$$bc\cdot 0 + ac\cdot 0 + abc = abc$$

We conclude that the three points are on the plane.

(c) We substitute $x = a$, $y = 0$, $z = 0$ in the left-hand side of the equation of the plane $bx + cy + az = 1$:

$$ba + c \cdot 0 + a \cdot 0 = ab$$

If $ab \neq 1$, the point $(a, 0, 0)$ is not on the plane.

(d) We substitute coordinates of the points in the left-hand side of the equation of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$:

$$(a, 0, 0): \quad \frac{a}{a} + \frac{0}{b} + \frac{0}{c} = 1$$

$$(0, b, 0): \quad \frac{0}{a} + \frac{b}{b} + \frac{0}{c} = 1$$

$$(0, 0, c): \quad \frac{0}{a} + \frac{0}{b} + \frac{c}{c} = 1$$

We conclude that the three points are on the plane.

31. Find an equation of the plane $\mathcal{P}$ in Figure 8.

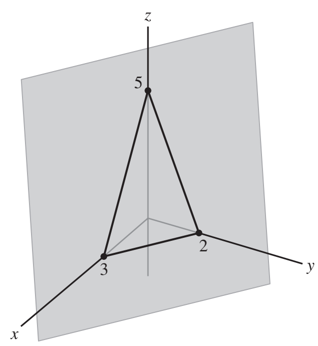
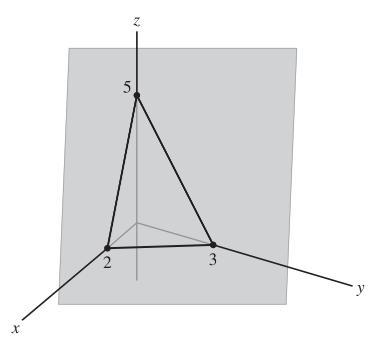


FIGURE 8

SOLUTION We must find the equation of the plane passing though the points $P = (3, 0, 0)$, $Q = (0, 2, 0)$, and $R = (0, 0, 5)$.



We use the following steps:

Step 1. Find a normal vector $\mathbf{n}$. The vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ lie in the plane, hence the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane. We compute the cross product:

$$\mathbf{a} = \overrightarrow{PQ} = \langle 0 - 3, 2 - 0, 0 - 0 \rangle = \langle -3, 2, 0 \rangle$$

$$\mathbf{b} = \overrightarrow{PR} = \langle 0 - 3, 0 - 0, 5 - 0 \rangle = \langle -3, 0, 5 \rangle$$

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 0 \\ -3 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 0 \\ -3 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ -3 & 0 \end{vmatrix} \mathbf{k}$$

$$= 10\mathbf{i} + 15\mathbf{j} + 6\mathbf{k} = \langle 10, 15, 6 \rangle$$

Step 2. Choose a point on the plane. We choose one of the points on the plane, say $P = (3, 0, 0)$. Substituting $\mathbf{n} = \langle 10, 15, 6 \rangle$ and $(x_0, y_0, z_0) = (3, 0, 0)$ in the vector form of the equation of the plane gives

$$\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$$

$$\langle 10, 15, 6 \rangle \cdot \langle x, y, z \rangle = \langle 10, 15, 6 \rangle \cdot \langle 3, 0, 0 \rangle$$

Computing the dot products we get the following scalar form of the equation of the plane:

$$10x + 15y + 6z = 10 \cdot 3 + 0 + 0 = 30$$
$$10x + 15y + 6z = 30$$

32. Verify that the plane $x - y + 5z = 10$ and the line $\mathbf{r}(t) = \langle 1, 0, 1\rangle + t\,\langle -2, 1, 1\rangle$ intersect at $P = (-3, 2, 3)$.

SOLUTION We verify that the point $P = (-3, 2, 3)$ is contained in both the plane and the line. Substituting the coordinates of P in the equation of the plane gives

$$-3 - 2 + 5 \cdot 3 = 10 \quad \Rightarrow \quad P \text{ is on the plane.}$$

The parametric equations of the line are

$$x = 1 - 2t, \quad y = t, \quad z = 1 + t$$

We verify that there exists a value of t such that

$$x = 1 - 2t = -3$$
$$y = t = 2$$
$$z = 1 + t = 3$$

$t = 2$ satisfies the three equations, hence the point $P = (-3, 2, 3)$ is on the line. Since P is on the plane and also on the line, the plane and the line intersect at this point. (Notice that we haven't proved that P is the only intersection point.)

In Exercises 33–36, find the intersection of the line and the plane.

33. $x + y + z = 14, \quad \mathbf{r}(t) = \langle 1, 1, 0\rangle + t\,\langle 0, 2, 4\rangle$

SOLUTION The line has parametric equations

$$x = 1, \quad y = 1 + 2t, \quad z = 4t$$

To find a value of t for which (x, y, z) lies on the plane, we substitute the parametric equations in the equation of the plane and solve for t:

$$x + y + z = 14$$
$$1 + (1 + 2t) + 4t = 14$$
$$6t = 12 \quad \Rightarrow \quad t = 2$$

The point P of intersection has coordinates

$$x = 1, \quad y = 1 + 2 \cdot 2 = 5, \quad z = 4 \cdot 2 = 8$$

That is, $P = (1, 5, 8)$.

34. $2x + y = 3, \quad \mathbf{r}(t) = \langle 2, -1, -1\rangle + t\,\langle 1, 2, -4\rangle$

SOLUTION The parametric equations of the line are

$$x = 2 + t, \quad y = -1 + 2t, \quad z = -1 - 4t \tag{1}$$

We substitute the parametric equations in the equation of the plane and solve for t, to find the value of t for which (x, y, z) lies on the plane. We obtain

$$2x + y = 3$$
$$2(2 + t) + (-1 + 2t) = 3$$
$$4 + 2t - 1 + 2t = 3$$
$$4t = 0 \quad \Rightarrow \quad t = 0$$

We find the coordinates of the point P of intersection by substituting $t = 0$ in the parametric equations (1). We obtain

$$x = 2 + 0 = 2, \quad y = -1 + 2 \cdot 0 = -1, \quad z = -1 - 4 \cdot 0 = -1$$

That is,

$$P = (2, -1, -1).$$

35. $z = 12, \quad \mathbf{r}(t) = t\,\langle -6, 9, 36\rangle$

SOLUTION The parametric equations of the line are

$$x = -6t, \quad y = 9t, \quad z = 36t \tag{1}$$

We substitute the parametric equations in the equation of the plane and solve for t:

$$z = 12$$
$$36t = 12 \quad \Rightarrow \quad t = \frac{1}{3}$$

The value of the parameter at the point of intersection is $t = \frac{1}{3}$. Substituting into (1) gives the coordinates of the point P of intersection:

$$x = -6 \cdot \frac{1}{3} = -2, \quad y = 9 \cdot \frac{1}{3} = 3, \quad z = 36 \cdot \frac{1}{3} = 12$$

That is,

$$P = (-2, 3, 12)\,.$$

36. $x - z = 6, \quad \mathbf{r}(t) = \langle 1, 0, -1\rangle + t\,\langle 4, 9, 2\rangle$

SOLUTION The parametric equations of the line are

$$x = 1 + 4t, \quad y = 9t, \quad z = -1 + 2t \tag{1}$$

We substitute the parametric equations in the equation of the plane and solve for t:

$$\begin{aligned} x - z &= 6 \\ 1 + 4t - (-1 + 2t) &= 6 \\ 1 + 4t + 1 - 2t &= 6 \\ 2t &= 4 \quad \Rightarrow \quad t = 2 \end{aligned}$$

The value of the parameter at the point P of intersection is $t = 2$. We find the coordinates of P by substituting $t = 2$ in (1). This gives

$$x = 1 + 4 \cdot 2 = 9, \quad y = 9 \cdot 2 = 18, \quad z = -1 + 2 \cdot 2 = 3$$

That is,

$$P = (9, 18, 3)\,.$$

In Exercises 37–42, find the trace of the plane in the given coordinate plane.

37. $3x - 9y + 4z = 5, \quad yz$

SOLUTION The yz-plane has the equation $x = 0$, hence the intersection of the plane with the yz-plane must satisfy both $x = 0$ and the equation of the plane $3x - 9y + 4z = 5$. That is, this is the set of all points $(0, y, z)$ in the yz-plane such that $-9y + 4z = 5$.

38. $3x - 9y + 4z = 5, \quad xz$

SOLUTION The trace of the plane in the xz coordinate plane is obtained by substituting $y = 0$ in the equation of the plane $3x - 9y + 4z = 5$. This gives the line $3x + 4z = 5$ in the xz-plane.

39. $3x + 4z = -2, \quad xy$

SOLUTION The trace of the plane $3x + 4z = -2$ in the xy coordinate plane is the set of all points that satisfy the equation of the plane and the equation $z = 0$ of the xy coordinate plane. Thus, we substitute $z = 0$ in $3x + 4z = -2$ to obtain the line $3x = -2$ or $x = -\frac{2}{3}$ in the xy-plane.

40. $3x + 4z = -2, \quad xz$

SOLUTION The xz-plane has equation $y = 0$, hence the intersection of the plane $3x + 4z = -2$ with the xz-plane is the set of all points $(x, 0, z)$ such that $3x + 4z = -2$. This is a line in the xz-plane.

41. $-x + y = 4, \quad xz$

SOLUTION The trace of the plane $-x + y = 4$ on the xz-plane is the set of all points that satisfy both the equation of the given plane and the equation $y = 0$ of the xz-plane. That is, the set of all points $(x, 0, z)$ such that $-x + 0 = 4$, or $x = -4$. This is a vertical line in the xz-plane.

42. $-x + y = 4, \quad yz$

SOLUTION The trace of the plane $-x + y = 4$ on the yz-plane is the set of all points that satisfy both the equation of the plane and the equation $x = 0$ of the yz-plane. That is, the set of all points $(0, y, z)$ such that $-0 + y = 4$, or $y = 4$. This is a vertical line in the yz-plane.

43. Does the plane $x = 5$ have a trace in the yz-plane? Explain.

SOLUTION The yz-plane has the equation $x = 0$, hence the x-coordinates of the points in this plane are zero, whereas the x-coordinates of the points in the plane $x = 5$ are 5. Thus, the two planes have no common points.

44. Give equations for two distinct planes whose trace in the xy-plane has equation $4x + 3y = 8$.

SOLUTION The xy-plane has the equation $z = 0$, hence the trace of a plane $ax + by + cz = 0$ in the xy-plane is obtained by substituting $z = 0$ in the equation of the plane. Therefore, the following two planes have trace $4x + 3y = 8$ in the xy-plane:

$$4x + 3y + z = 8; \quad 4x + 3y - 5z = 8$$

45. Give equations for two distinct planes whose trace in the yz-plane has equation $y = 4z$.

SOLUTION The yz-plane has the equation $x = 0$, hence the trace of a plane $ax + by + cz = 0$ in the yz-plane is obtained by substituting $x = 0$ in the equation of the plane. Therefore, the following two planes have trace $y = 4z$ (that is, $y - 4z = 0$) in the yz-plane:

$$x + y - 4z = 0; \quad 2x + y - 4z = 0$$

46. Find parametric equations for the line through $P_0 = (3, -1, 1)$ perpendicular to the plane $3x + 5y - 7z = 29$.

SOLUTION We need to find a direction vector for the line. Since the line is perpendicular to the plane $3x + 5y - 7z = 29$, it is parallel to the vector $\mathbf{n} = \langle 3, 5, -7\rangle$ normal to the plane. Hence, $\mathbf{n}$ is a direction vector for the line. The vector parametrization of the line is, thus,

$$\mathbf{r}(t) = \langle 3, -1, 1\rangle + t\langle 3, 5, -7\rangle$$

This yields the parametric equations

$$x = 3 + 3t, \quad y = -1 + 5t, \quad z = 1 - 7t$$

47. Find all planes in $\mathbf{R}^3$ whose intersection with the xz-plane is the line with equation $3x + 2z = 5$.

SOLUTION The intersection of the plane $ax + by + cz = d$ with the xz-plane is obtained by substituting $y = 0$ in the equation of the plane. This gives the following line in the xz-plane:

$$ax + cz = d$$

This is the equation of the line $3x + 2z = 5$ if and only if for some $\lambda \neq 0$,

$$a = 3\lambda, \quad c = 2\lambda, \quad d = 5\lambda$$

Notice that b can have any value. The planes are thus

$$(3\lambda)x + by + (2\lambda)z = 5\lambda, \quad \lambda \neq 0.$$

48. Find all planes in $\mathbf{R}^3$ whose intersection with the xy-plane is the line $\mathbf{r}(t) = t\,\langle 2, 1, 0\rangle$.

SOLUTION The intersection of the plane $ax + by + cz = d$ with the xy-plane is obtained by substituting $z = 0$ in the equation of the plane. This gives the line $ax + by = d$, in the xy-plane. We find the equation of the line $\mathbf{l}\,(t) = t\langle 2, 1, 0\rangle$. On this line we have

$$\begin{array}{l} x = 2t \\ y = t \end{array} \quad \Rightarrow \quad y = \frac{1}{2}x \Rightarrow x - 2y = 0$$

We thus must have $d = 0$ and $\frac{b}{a} = -2$, $a \neq 0$. That is, $d = 0$, $b = -2a$, $a \neq 0$. Notice that c can have any value. Hence, the planes are

$$ax - 2ay + cz = 0, a \neq 0$$

In Exercises 49–54, compute the angle between the two planes, defined as the angle θ (between 0 and π) between their normal vectors (Figure 9).

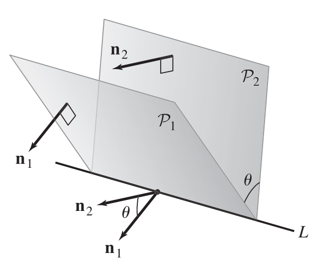


FIGURE 9 By definition, the angle between two planes is the angle between their normal vectors.

49. Planes with normals $\mathbf{n}_1 = \langle 1, 0, 1\rangle$, $\mathbf{n}_2 = \langle -1, 1, 1\rangle$

SOLUTION Using the formula for the angle between two vectors we get

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 1, 0, 1\rangle \cdot \langle -1, 1, 1\rangle}{\|\langle 1, 0, 1\rangle\|\|\langle -1, 1, 1\rangle\|} = \frac{-1+0+1}{\sqrt{1^2+0+1^2}\sqrt{(-1)^2+1^2+1^2}} = 0$$

The solution for $0 \le \theta < \pi$ is $\theta = \frac{\pi}{2}$.

50. Planes with normals $\mathbf{n}_1 = \langle 1, 2, 1\rangle$, $\mathbf{n}_2 = \langle 4, 1, 3\rangle$

SOLUTION By the formula for the angle between two vectors we get

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 1, 2, 1\rangle \cdot \langle 4, 1, 3\rangle}{\|\langle 1, 2, 1\rangle\|\|\langle 4, 1, 3\rangle\|} = \frac{4+2+3}{\sqrt{1^2+2^2+1^2}\sqrt{4^2+1^2+3^2}} = \frac{9}{\sqrt{6}\sqrt{26}} \approx 0.72$$

The solution for $0 \le \theta < \pi$ is $\theta = 0.766$ rad or $\theta = 43.9^\circ$.

51. $2x + 3y + 7z = 2$ and $4x - 2y + 2z = 4$

SOLUTION The planes $2x + 3y + 7z = 2$ and $4x - 2y + 2z = 4$ have the normals $\mathbf{n}_1 = \langle 2, 3, 7\rangle$ and $\mathbf{n}_2 = \langle 4, -2, 2\rangle$ respectively. The cosine of the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 2, 3, 7\rangle \cdot \langle 4, -2, 2\rangle}{\|\langle 2, 3, 7\rangle\|\|\langle 4, -2, 2\rangle\|} = \frac{8-6+14}{\sqrt{2^2+3^2+7^2}\sqrt{4^2+(-2)^2+2^2}} = \frac{16}{\sqrt{62}\sqrt{24}} \approx 0.415$$

The solution for $0 \le \theta < \pi$ is $\theta = 1.143$ rad or $\theta = 65.49^\circ$.

52. $x - 3y + z = 3$ and $2x - 3z = 4$

SOLUTION The planes $x - 3y + z = 3$ and $2x - 3z = 4$ have the normals $\mathbf{n}_1 = \langle 1, -3, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 0, -3\rangle$ respectively. We use the formula for the angle between two vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 1, -3, 1\rangle \cdot \langle 2, 0, -3\rangle}{\|\langle 1, -3, 1\rangle\|\|\langle 2, 0, -3\rangle\|} = \frac{2+0-3}{\sqrt{1^2+(-3)^2+1^2}\sqrt{2^2+0+(-3)^2}} = \frac{-1}{\sqrt{11}\sqrt{13}} \approx -0.084$$

The solution for $0 \le \theta < \pi$ is $\theta = 1.655$ rad or $\theta = 94.80^\circ$.

53. $3(x - 1) - 5y + 2(z - 12) = 0$ and the plane with normal $\mathbf{n} = \langle 1, 0, 1\rangle$

SOLUTION The plane $3(x - 1) - 5y + 2(z - 12) = 0$ has the normal $\mathbf{n}_1 = \langle 3, -5, 2\rangle$, and our second plane has given normal $\mathbf{n}_2 = \langle 1, 0, 1\rangle$. We use the formula for the angle between two vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 3, -5, 2\rangle \cdot \langle 1, 0, 1\rangle}{\|\langle 3, -5, 2\rangle\|\|\langle 1, 0, 1\rangle\|} = \frac{3+0+2}{\sqrt{3^2+(-5)^2+2^2}\sqrt{1^2+0+1^2}} = \frac{5}{\sqrt{38}\sqrt{2}} \approx 0.5735$$

The solution for $0 \le \theta < \pi$ is $\theta = 0.96$ rad or $\theta = 55^\circ$.

54. The plane through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and the yz-plane

SOLUTION We first must find normal vectors to the planes. The yz-plane has the equation $x = 0$, or in vector form $\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = 0$, hence $\mathbf{n}_1 = \langle 1, 0, 0\rangle$ is normal to the plane. We denote the points $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$ hence the vector $\mathbf{n}_2 = \overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane. We find it here:

$$\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$$

$$\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$$

$$\mathbf{n}_2 = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1\rangle$$

Using the formula for the angle between two vectors we have

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 1, 0, 0\rangle \cdot \langle 1, 1, 1\rangle}{\|\langle 1, 0, 0\rangle\|\|\langle 1, 1, 1\rangle\|} = \frac{1}{\sqrt{3}}$$

The solution for $0 \le \theta < \pi$ is $\theta = 0.955$ rad or $\theta = 54.74^\circ$.

55. Find an equation of a plane making an angle of $\frac{\pi}{2}$ with the plane $3x + y - 4z = 2$.

SOLUTION The angle θ between two planes (chosen so that $0 \le \theta < \pi$) is defined as the angle between their normal vectors. The following vector is normal to the plane $3x + y - 4z = 2$:

$$\mathbf{n}_1 = \langle 3, 1, -4 \rangle$$

Let $\mathbf{n} \cdot \langle x, y, z \rangle = d$ denote the equation of a plane making an angle of $\frac{\pi}{2}$ with the given plane, where $\mathbf{n} = \langle a, b, c \rangle$. Since the two planes are perpendicular, the dot product of their normal vectors is zero. That is,

$$\mathbf{n} \cdot \mathbf{n}_1 = \langle a, b, c \rangle \cdot \langle 3, 1, -4 \rangle = 3a + b - 4c = 0 \quad \Rightarrow \quad b = -3a + 4c$$

Thus, the required planes (there is more than one plane) have the following normal vector:

$$\mathbf{n} = \langle a, -3a + 4c, c \rangle$$

We obtain the following equation:

$$\begin{aligned} \mathbf{n} \cdot \langle x, y, c \rangle &= d \\ \langle a, -3a + 4c, c \rangle \cdot \langle x, y, z \rangle &= d \\ ax + (4c - 3a)y + cz &= d \end{aligned}$$

Every choice of the values of a, c and d yields a plane with the desired property. For example, we set $a = c = d = 1$ to obtain

$$x + y + z = 1$$

56. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Assume that the planes are not parallel, and let $\mathcal{L}$ be their intersection (a line). Show that $\mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector for $\mathcal{L}$.

SOLUTION A vector which is normal to a plane, is orthogonal to all the vectors in the plane. Since the line $\mathcal{L}$ is on both the planes $\mathcal{P}_1$ and $\mathcal{P}_2$, the normal $\mathbf{n}_1$ to the plane $\mathcal{P}_1$ is orthogonal to the direction vector of $\mathcal{L}$ and the normal $\mathbf{n}_2$ to the plane $\mathcal{P}_2$ is also orthogonal to this vector. That is, the line $\mathcal{L}$ is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. The cross product $\mathbf{n}_1 \times \mathbf{n}_2$ is also orthogonal to $\mathbf{n}_1$ and $\mathbf{n}_2$, hence it is parallel to $\mathcal{L}$. In other words $\mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector of $\mathcal{L}$.

57. Find a plane that is perpendicular to the two planes $x + y = 3$ and $x + 2y - z = 4$.

SOLUTION The vector forms of the equations of the planes are $\langle 1, 1, 0 \rangle \cdot \langle x, y, z \rangle = 3$ and $\langle 1, 2, -1 \rangle \cdot \langle x, y, z \rangle = 4$, hence the vectors $\mathbf{n}_1 = \langle 1, 1, 0 \rangle$ and $\mathbf{n}_2 = \langle 1, 2, -1 \rangle$ are normal to the planes. We denote the equation of the planes which are perpendicular to the two planes by

$$ax + by + cz = d \tag{1}$$

Then, the normal $\mathbf{n} = \langle a, b, c \rangle$ to the planes is orthogonal to the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ of the given planes. Therefore, $\mathbf{n} \cdot \mathbf{n}_1 = 0$ and $\mathbf{n} \cdot \mathbf{n}_2 = 0$ which gives us

$$\langle a, b, c \rangle \cdot \langle 1, 1, 0 \rangle = 0, \quad \langle a, b, c \rangle \cdot \langle 1, 2, -1 \rangle = 0$$

We obtain the following equations:

$$\begin{cases} a + b = 0 \\ a + 2b - c = 0 \end{cases}$$

The first equation implies that $b = -a$. Substituting in the second equation we get $a - 2a - c = 0$, or $c = -a$. Substituting $b = -a$ and $c = -a$ in (1) gives (for $a \ne 0$):

$$ax - ay - az = d \quad \Rightarrow \quad x - y - z = \frac{d}{a}$$

$\frac{d}{a}$ is an arbitrary constant which we denote by f. The planes which are perpendicular to the given planes are, therefore,

$$x - y - z = f$$

58. Let $\mathcal{L}$ be the intersection of the planes $x + y + z = 1$ and $x + 2y + 3z = 1$. Use Exercise 56 to find a direction vector for $\mathcal{L}$. Then find a point P on $\mathcal{L}$ by *inspection*, and write down the parametric equations for $\mathcal{L}$.

SOLUTION We identify $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, 2, 3 \rangle$ as normals to the planes $x + y + z = 1$ and $x + 2y + 3z = 1$ respectively. By Exercise 56, the cross product $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector of $\mathcal{L}$. We find it here:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = \langle 1, -2, 1 \rangle$$

We observe that $x = 1$, $y = 0$, $z = 0$ satisfy the equations of the two planes, hence the point $P = (1, 0, 0)$ is on the line $\mathcal{L}$. The vector parametrization of $\mathcal{L}$ is, thus,

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 1, -2, 1 \rangle$$

This yields the parametric equations

$$x = 1 + t, \quad y = -2t, \quad z = t.$$

59. Let $\mathcal{L}$ denote the intersection of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. Find parametric equations for the line $\mathcal{L}$. *Hint:* To find a point on $\mathcal{L}$, substitute an arbitrary value for z (say, $z = 2$) and then solve the resulting pair of equations for x and y.

SOLUTION We use Exercise 56 to find a direction vector for the line of intersection $\mathcal{L}$ of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. We identify the normals $\mathbf{n}_1 = \langle 1, -1, -1 \rangle$ and $\mathbf{n}_2 = \langle 2, 3, 1 \rangle$ to the two planes respectively. Hence, a direction vector for $\mathcal{L}$ is the cross product $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. We find it here:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} = \langle 2, -3, 5 \rangle$$

We now need to find a point on $\mathcal{L}$. We choose $z = 2$, substitute in the equations of the planes and solve the resulting equations for x and y. This gives

$$\begin{aligned} x - y - 2 &= 1 \\ 2x + 3y + 2 &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y &= 3 \\ 2x + 3y &= 0 \end{aligned}$$

The 1st equation implies that $y = x - 3$. Substituting in the 2nd equation and solving for x gives

$$\begin{aligned} 2x + 3(x - 3) &= 0 \\ 5x = 9 \quad &\Rightarrow \quad x = \frac{9}{5}, \quad y = \frac{9}{5} - 3 = -\frac{6}{5} \end{aligned}$$

We conclude that the point $\left(\frac{9}{5}, -\frac{6}{5}, 2\right)$ is on $\mathcal{L}$. We now use the vector parametrization of a line to obtain the following parametrization for $\mathcal{L}$:

$$\mathbf{r}(t) = \left\langle \frac{9}{5}, -\frac{6}{5}, 2 \right\rangle + t\langle 2, -3, 5 \rangle$$

This yields the parametric equations

$$x = \frac{9}{5} + 2t, \quad y = -\frac{6}{5} - 3t, \quad z = 2 + 5t$$

60. Find parametric equations for the intersection of the planes $2x + y - 3z = 0$ and $x + y = 1$.

SOLUTION We use Exercise 56 to determine a direction vector $\mathbf{v}$ for the line of intersection $\mathcal{L}$ of the two planes. The planes $2x + y - 3z = 0$ and $x + y = 1$ have normals $\mathbf{n}_1 = \langle 2, 1, -3 \rangle$ and $\mathbf{n}_2 = \langle 1, 1, 0 \rangle$ respectively, hence the cross product $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the line $\mathcal{L}$. We find it here:

$$\begin{aligned} \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} - 3\mathbf{j} + \mathbf{k} = \langle 3, -3, 1 \rangle \end{aligned}$$

We now must find a point P_0 on $\mathcal{L}$. We substitute an arbitrary value for z, say $z = 0$, in the equations of the planes $2x + y - 3z = 0$ and $x + y = 1$ and solve the resulting equations for x and y. This gives

$$\begin{aligned} 2x + y &= 0 \\ x + y &= 1 \end{aligned} \quad \Rightarrow \quad x = -1, \ y = 2$$

Thus, the point $P_0 = (-1, 2, 0)$ is on the line $\mathcal{L}$. We now use the vector parametrization of the line to obtain the following parametrization of $\mathcal{L}$:

$$\mathbf{r}(t) = \langle -1, 2, 0 \rangle + t\langle 3, -3, 1 \rangle$$

The parametric equation are thus,

$$x = -1 + 3t, \quad y = 2 - 3t, \quad z = t$$

61. Two vectors $\mathbf{v}$ and $\mathbf{w}$, each of length 12, lie in the plane $x + 2y - 2z = 0$. The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. This information determines $\mathbf{v} \times \mathbf{w}$ up to a sign ± 1. What are the two possible values of $\mathbf{v} \times \mathbf{w}$?

SOLUTION The length of $\mathbf{v} \times \mathbf{w}$ is $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, but since both vectors have length 12 and since the angle between them is $\pi/6$, then the length of $\mathbf{v} \times \mathbf{w}$ is $12 \cdot 12 \cdot 1/2 = 72$. The direction of $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane containing them, which is the plane $x + 2y - 2z = 0$, which has normal vector $\mathbf{n} = \langle 1, 2, -2\rangle$. Since $\mathbf{v} \times \mathbf{w}$ must have length 72 and must be parallel to $\langle 1, 2, -2\rangle$, then it must be ± 72 times the unit vector $\langle 1, 2, -2\rangle / \sqrt{1^2 + 2^2 + (-2)^2} = \langle 1/3, 2/3, -2/3\rangle$. Thus,

$$\mathbf{v} \times \mathbf{w} = \pm 72 \cdot \langle 1/3, 2/3, -2/3\rangle = \pm 24 \cdot \langle 1, 2, -2\rangle$$

62. The plane

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{3} = 1$$

intersects the x-, y-, and z-axes in points P, Q, and R. Find the area of the triangle $\triangle PQR$.

SOLUTION The points of intersection are found by setting two of the three variables in $\frac{x}{2} + \frac{y}{4} + \frac{z}{3} = 1$ equal to zero at a time. We get the following:

$$P = (2, 0, 0), \quad Q = (0, 4, 0), \quad R = (0, 0, 3)$$

and so we can find the two vectors that span the triangle:

$$\overrightarrow{PQ} = 4\mathbf{j} - 2\mathbf{i}, \quad \overrightarrow{PR} = 3\mathbf{k} - 2\mathbf{i}$$

We have

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= (4\mathbf{j} - 2\mathbf{i}) \times (3\mathbf{k} - 2\mathbf{i}) \\ &= 12\mathbf{j} \times \mathbf{k} - 8\mathbf{j} \times \mathbf{i} - 6\mathbf{i} \times \mathbf{k} + 4\mathbf{i} \times \mathbf{i} \\ &= 12\mathbf{i} + 8\mathbf{k} + 6\mathbf{j}\end{aligned}$$

The area of the triangle is

$$\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\sqrt{144 + 36 + 64} = \frac{1}{2}\sqrt{244} = \sqrt{61}$$

63. In this exercise, we show that the orthogonal distance D from the plane $\mathcal{P}$ with equation $ax + by + cz = d$ to the origin O is equal to (Figure 10)

$$D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

Let $\mathbf{n} = \langle a, b, c\rangle$, and let P be the point where the line through $\mathbf{n}$ intersects $\mathcal{P}$. By definition, the orthogonal distance from $\mathcal{P}$ to O is the distance from P to O.

(a) Show that P is the terminal point of $\mathbf{v} = \left(\frac{d}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n}$.

(b) Show that the distance from P to O is D.

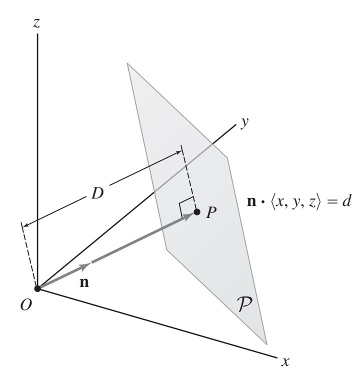


FIGURE 10

SOLUTION Let $\mathbf{v}$ be the vector $\mathbf{v} = \left(\frac{d}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n}$. Then $\mathbf{v}$ is parallel to $\mathbf{n}$ and the two vectors are on the same ray.

(a) First we must show that the terminal point of $\mathbf{v}$ lies on the plane $ax + by + cz = d$. Since the terminal point of $\mathbf{v}$ is the point

$$\left(\frac{d}{\mathbf{n} \cdot \mathbf{n}}\right)(a, b, c) = \left(\frac{da}{a^2 + b^2 + c^2}, \frac{db}{a^2 + b^2 + c^2}, \frac{dc}{a^2 + b^2 + c^2}\right)$$

then we need only show that this point satisfies $ax + by + cz = d$. Plugging in, we find:

$$ax + by + cz = a \cdot \frac{da}{a^2+b^2+c^2} + b \cdot \frac{db}{a^2+b^2+c^2} + c \cdot \frac{dc}{a^2+b^2+c^2} = \frac{a^2d+b^2d+c^2d}{a^2+b^2+c^2} = d$$

(b) We now show that the distance from P to O is D. This distance is just the length of the vector $\mathbf{v}$, which is:

$$\|\mathbf{v}\| = \left(\frac{|d|}{\mathbf{n}\cdot\mathbf{n}}\right)\|\mathbf{n}\| = \frac{|d|}{\|\mathbf{n}\|} = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

as desired.

64. Use Exercise 63 to compute the orthogonal distance from the plane $x + 2y + 3z = 5$ to the origin.

SOLUTION Using the formula

$$D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

and since our line is $x + 2y + 3z = 5$, we have $a = 1$, $b = 2$, $c = 3$, $d = 5$, and so we calculate

$$D = \frac{|5|}{\sqrt{1^2+2^2+3^2}} = \frac{5}{\sqrt{14}} \approx 1.3363$$

Further Insights and Challenges

In Exercises 65 and 66, let $\mathcal{P}$ be a plane with equation

$$ax + by + cz = d$$

and normal vector $\mathbf{n} = \langle a, b, c\rangle$. For any point Q, there is a unique point P on $\mathcal{P}$ that is closest to Q, and is such that $\overline{PQ}$ is orthogonal to $\mathcal{P}$ (Figure 11).

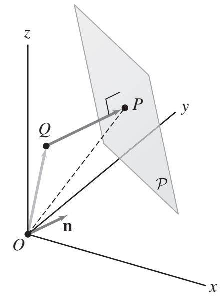


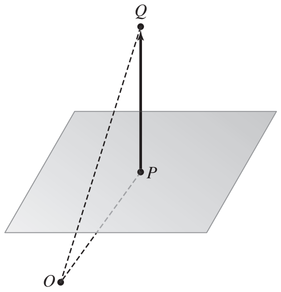
FIGURE 11

65. Show that the point P on $\mathcal{P}$ closest to Q is determined by the equation

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \overrightarrow{OQ}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} \qquad \boxed{7}$$

SOLUTION Since $\overrightarrow{PQ}$ is orthogonal to the plane $\mathcal{P}$, it is parallel to the vector $\mathbf{n} = \langle a, b, c\rangle$ which is normal to the plane. Hence,

$$\overrightarrow{PQ} = \lambda\mathbf{n} \qquad (1)$$



Since $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$, we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. Thus, by (1) we get

$$\overrightarrow{OQ} - \overrightarrow{OP} = \lambda \mathbf{n} \quad \Rightarrow \quad \overrightarrow{OP} = \overrightarrow{OQ} - \lambda \mathbf{n} \tag{2}$$

The point P is on the plane, hence $\overrightarrow{OP}$ satisfies the vector form of the equation of the plane, that is,

$$\mathbf{n} \cdot \overrightarrow{OP} = d \tag{3}$$

Substituting (2) into (3) and solving for λ yields

$$\mathbf{n} \cdot \left(\overrightarrow{OQ} - \lambda \mathbf{n}\right) = d$$

$$\mathbf{n} \cdot \overrightarrow{OQ} - \lambda \mathbf{n} \cdot \mathbf{n} = d$$

$$\lambda \mathbf{n} \cdot \mathbf{n} = \mathbf{n} \cdot \overrightarrow{OQ} - d \quad \Rightarrow \quad \lambda = \frac{\mathbf{n} \cdot \overrightarrow{OQ} - d}{\mathbf{n} \cdot \mathbf{n}} \tag{4}$$

Finally, we combine (2) and (4) to obtain

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \mathbf{n} \cdot \overrightarrow{OQ}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$$

66. By definition, the distance from a point $Q = (x_1, y_1, z_1)$ to the plane $\mathcal{P}$ is $\|QP\|$ where P is the point on $\mathcal{P}$ that is closest to Q. Prove:

$$\text{Distance from } Q \text{ to } \mathcal{P} = \frac{|ax_1 + by_1 + cz_1 - d|}{\|\mathbf{n}\|} \tag{8}$$

SOLUTION The distance l from Q to P is the length of the vector $\overrightarrow{PQ}$, that is,

$$\ell = \|\overrightarrow{PQ}\| = \|\overrightarrow{OQ} - \overrightarrow{OP}\| \tag{1}$$

By Eq. (7) in Exercise 65,

$$\overrightarrow{OP} - \overrightarrow{OQ} = \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n}$$

Combining with (1) and noticing that $\frac{\mathbf{n}}{\|\mathbf{n}\|}$ is a unit vector, we have

$$\ell = \left\| \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n} \right\| = \frac{|d - \overrightarrow{OQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \cdot \left\| \frac{\mathbf{n}}{\|\mathbf{n}\|} \right\| = \frac{|d - \overrightarrow{OQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \tag{2}$$

We compute the numerator in (2):

$$|d - \overrightarrow{OQ} \cdot \mathbf{n}| = |d - \langle x_1, y_1, z_1 \rangle \cdot \langle a, b, c \rangle| = |d - (ax_1 + by_1 + cz_1)| = |ax_1 + by_1 + cz_1 - d|$$

Substituting into (2) we obtain the following distance:

$$\ell = \frac{|ax_1 + by_1 + cz_1 - d|}{\|\mathbf{n}\|}$$

67. Use Eq. (7) to find the point P nearest to $Q = (2, 1, 2)$ on the plane $x + y + z = 1$.

SOLUTION We identify $\mathbf{n} = \langle 1, 1, 1 \rangle$ as a vector normal to the plane. By Eq. (7) the nearest point P to Q is determined by

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$$

We substitute $\mathbf{n} = \langle 1, 1, 1 \rangle$, $\overrightarrow{OQ} = \langle 2, 1, 2 \rangle$ and $d = 1$ in this equation to obtain

$$\begin{aligned} \overrightarrow{OP} &= \langle 2, 1, 2 \rangle + \frac{1 - \langle 2, 1, 2 \rangle \cdot \langle 1, 1, 1 \rangle}{\langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle} \langle 1, 1, 1 \rangle = \langle 2, 1, 2 \rangle + \frac{1 - (2 + 1 + 2)}{1 + 1 + 1} \langle 1, 1, 1 \rangle \\ &= \langle 2, 1, 2 \rangle - \frac{4}{3} \langle 1, 1, 1 \rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \end{aligned}$$

The terminal point $P = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ of $\overrightarrow{OP}$ is the nearest point to $Q = (2, 1, 2)$ on the plane.

68. Find the point P nearest to $Q = (-1, 3, -1)$ on the plane

$$x - 4z = 2$$

SOLUTION By Exercise 65, the nearest point P to Q on the plane $ax + by + cz = d$ is determined by the equation

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n}$$

We identify $\mathbf{n} = \langle 1, 0, -4 \rangle$ as a vector normal to the plane $x - 4z = 2$. We also substitute $d = 2$ and $\overrightarrow{OQ} = \langle -1, 3, -1 \rangle$ to obtain

$$\begin{aligned}\overrightarrow{OP} &= \langle -1, 3, -1 \rangle + \frac{2 - \langle -1, 3, -1 \rangle \cdot \langle 1, 0, -4 \rangle}{\langle 1, 0, -4 \rangle \cdot \langle 1, 0, -4 \rangle}\langle 1, 0, -4 \rangle \\ &= \langle -1, 3, -1 \rangle + \frac{2 - (-1 + 0 + 4)}{1 + 0 + 16}\langle 1, 0, -4 \rangle \\ &= \langle -1, 3, -1 \rangle - \frac{1}{17}\langle 1, 0, -4 \rangle = \left\langle -\frac{18}{17}, 3, -\frac{13}{17} \right\rangle\end{aligned}$$

The desired point P is the terminal point of $\overrightarrow{OP}$, that is, $P = \left(-\frac{18}{17}, 3, -\frac{13}{17}\right)$.

69. Use Eq. (8) to find the distance from $Q = (1, 1, 1)$ to the plane $2x + y + 5z = 2$.

SOLUTION By Eq. (8), the distance from $Q = \langle x_1, y_1, z_1 \rangle$ to the plane $ax + by + cz = d$ is

$$\ell = \frac{|ax_1 + by_1 + cz_1 - d|}{\|\mathbf{n}\|} \tag{1}$$

We identify the vector $\mathbf{n} = \langle 2, 1, 5 \rangle$ as a normal to the plane $2x + y + 5z = 2$. Also $a = 2$, $b = 1$, $c = 5$, $d = 2$, and $(x_1, y_1, z_1) = (1, 1, 1)$. Substituting in (1) above we get

$$\ell = \frac{|2 \cdot 1 + 1 \cdot 1 + 5 \cdot 1 - 2|}{\|\langle 2, 1, 5 \rangle\|} = \frac{6}{\sqrt{2^2 + 1^2 + 5^2}} = \frac{6}{\sqrt{30}} \approx 1.095$$

70. Find the distance from $Q = (1, 2, 2)$ to the plane $\mathbf{n} \cdot \langle x, y, z \rangle = 3$, where $\mathbf{n} = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle$.

SOLUTION We write the equation of the plane in scalar form:

$$\begin{aligned}\left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle \cdot \langle x, y, z \rangle &= 3 \\ \frac{3}{5}x + \frac{4}{5}y + 0 &= 3 \\ \frac{3}{5}x + \frac{4}{5}y &= 3\end{aligned}$$

We use Eq. (8) in for the distance ℓ from $Q = (x_1, y_1, z_1)$ to the plane $ax + by + cz = d$:

$$\ell = \frac{|ax_1 + by_1 + cz_1 - d|}{\|\mathbf{n}\|} \tag{1}$$

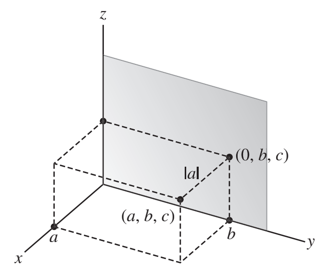
In our example, $a = \frac{3}{5}$, $b = \frac{4}{5}$, $c = 0$, $d = 3$. Also $(x_1, y_1, z_1) = (1, 2, 2)$ and $\mathbf{n} = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle$. Substituting these values in the formula (1) for the distance ℓ, we get

$$\ell = \frac{\left|\frac{3}{5} \cdot 1 + \frac{4}{5} \cdot 2 + 0 - 3\right|}{\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 0^2}} = \frac{\frac{4}{5}}{1} = \frac{4}{5}$$

We found that the distance from $Q = (1, 2, 2)$ to the given plane is $\ell = \frac{4}{5}$.

71. What is the distance from $Q = (a, b, c)$ to the plane $x = 0$? Visualize your answer geometrically and explain without computation. Then verify that Eq. (8) yields the same answer.

SOLUTION The plane $x = 0$ is the yz-coordinate plane. The nearest point to Q on the plane is the projection of Q on the plane, which is the point $Q' = (0, b, c)$.



Hence, the distance from Q to the plane is the length of the vector $\overrightarrow{Q'Q} = \langle a, 0, 0\rangle$ which is $|a|$. We now verify that Eq. (8) gives the same answer. The plane $x = 0$ has the vector parametrization $\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = 0$, hence $\mathbf{n} = \langle 1, 0, 0\rangle$. The coefficients of the plane $x = 0$ are $A = 1$, $B = C = D = 0$. Also $(x_1, y_1, z_1) = (a, b, c)$. Substituting this value in Eq. (8) we get

$$\frac{|Ax_1 + By_1 + Cz_1 - D|}{\|\mathbf{n}\|} = \frac{|1 \cdot a + 0 + 0 - 0|}{\|\langle 1, 0, 0\rangle\|} = \frac{|a|}{\sqrt{1^2 + 0^2 + 0^2}} = |a|$$

The two answers agree, as expected.

72. The equation of a plane $\mathbf{n} \cdot \langle x, y, z\rangle = d$ is said to be in **normal form** if $\mathbf{n}$ is a unit vector. Show that in this case, $|d|$ is the distance from the plane to the origin. Write the equation of the plane $4x - 2y + 4z = 24$ in normal form.

SOLUTION By Exercise 65 the point Q nearest to the origin on the plane $\mathbf{n} \cdot \langle x, y, z\rangle = d$ is the terminal point of the vector

$$\mathbf{v} = \left(\frac{d}{\|\mathbf{n}\|^2}\right)\mathbf{n}$$

If the equation of the plane is in normal form, $\mathbf{n}$ is a unit vector, hence $\|\mathbf{n}\| = 1$. Therefore Q is the terminal point of the vector

$$\mathbf{v} = d\mathbf{n}$$

The distance from the plane to the origin is the length of $\mathbf{v}$, that is,

$$\|\mathbf{v}\| = \|d\mathbf{n}\| = |d|\|\mathbf{n}\| = |d| \cdot 1 = |d|.$$

The plane $4x - 2x + 4z = 24$ can be written

$$\frac{4x - 2x + 4z}{\sqrt{4^2 + (-2)^2 + 4^2}} = \frac{24}{\sqrt{4^2 + (-2)^2 + 4^2}}$$

or

$$\frac{1}{6}\langle 4, -2, 4\rangle \cdot \langle x, y, z\rangle = 4$$

or

$$\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle \cdot \langle x, y, z\rangle = 4, \text{ in normal form.}$$

12.6 A Survey of Quadric Surfaces (LT Section 13.6)

Preliminary Questions

1. True or false? All traces of an ellipsoid are ellipses.

SOLUTION This statement is true, mostly. All traces of an ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ are ellipses, except for the traces obtained by intersecting the ellipsoid with the planes $x = \pm a$, $y = \pm b$ and $z = \pm c$. These traces reduce to the single points $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$ respectively.

2. True or false? All traces of a hyperboloid are hyperbolas.

SOLUTION The statement is false. For a hyperbola in the standard orientation, the horizontal traces are ellipses (or perhaps empty for a hyperbola of two sheets), and the vertical traces are hyperbolas.

3. Which quadric surfaces have both hyperbolas and parabolas as traces?

SOLUTION The hyperbolic paraboloid $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$ has vertical trace curves which are parabolas. If we set $x = x_0$ or $y = y_0$ we get

$$z = \left(\frac{x_0}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \quad \Rightarrow \quad z = -\left(\frac{y}{b}\right)^2 + C$$

$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y_0}{b}\right)^2 \quad \Rightarrow \quad z = \left(\frac{x}{a}\right)^2 + C$$

The hyperbolic paraboloid has vertical traces which are hyperbolas, since for $z = z_0$, $(z_0 > 0)$, we get

$$z_0 = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

4. Is there any quadric surface whose traces are all parabolas?

SOLUTION There is no quadric surface whose traces are all parabolas.

5. A surface is called **bounded** if there exists $M > 0$ such that every point on the surface lies at a distance of at most M from the origin. Which of the quadric surfaces are bounded?

SOLUTION The only quadric surface that is bounded is the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

All other quadric surfaces are not bounded, since at least one of the coordinates can increase or decrease without bound.

6. What is the definition of a parabolic cylinder?

SOLUTION A parabolic cylinder consists of all vertical lines passing through a parabola $\mathcal{C}$ in the xy-plane.

Exercises

In Exercises 1–6, state whether the given equation defines an ellipsoid or hyperboloid, and if a hyperboloid, whether it is of one or two sheets.

1. $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$

SOLUTION This equation is the equation of an ellipsoid.

2. $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 - \left(\frac{z}{7}\right)^2 = 1$

SOLUTION The given equation defines a hyperboloid of one sheet.

3. $x^2 + 3y^2 + 9z^2 = 1$

SOLUTION We rewrite the equation as follows:

$$x^2 + \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{z}{\frac{1}{3}}\right)^2 = 1$$

This equation defines an ellipsoid.

4. $-\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$

SOLUTION This is the equation of a hyperboloid of two sheets.

5. $x^2 - 3y^2 + 9z^2 = 1$

SOLUTION We rewrite the equation in the form

$$x^2 - \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{z}{\frac{1}{3}}\right)^2 = 1$$

This is the equation of a hyperboloid of one sheet.

6. $x^2 - 3y^2 - 9z^2 = 1$

SOLUTION Rewriting the equation in the form

$$x^2 - \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 - \left(\frac{z}{\frac{1}{3}}\right)^2 = 1$$

we identify it as the equation of a hyperboloid of two sheets.

In Exercises 7–12, state whether the given equation defines an elliptic paraboloid, a hyperbolic paraboloid, or an elliptic cone.

7. $z = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2$

SOLUTION This equation defines an elliptic paraboloid.

8. $z^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2$

SOLUTION This is the equation of an elliptic cone.

9. $z = \left(\frac{x}{9}\right)^2 - \left(\frac{y}{12}\right)^2$

SOLUTION This equation defines a hyperbolic paraboloid.

10. $4z = 9x^2 + 5y^2$

SOLUTION The equation can be rewritten as

$$z = \left(\frac{x}{\frac{2}{3}}\right)^2 + \left(\frac{y}{\frac{2}{\sqrt{5}}}\right)^2$$

hence it defines an elliptic paraboloid.

11. $3x^2 - 7y^2 = z$

SOLUTION Rewriting the equation as

$$z = \left(\frac{x}{\frac{1}{\sqrt{3}}}\right)^2 - \left(\frac{y}{\frac{1}{\sqrt{7}}}\right)^2$$

we identify it as the equation of a hyperbolic paraboloid.

12. $3x^2 + 7y^2 = 14z^2$

SOLUTION We rewrite the equations as follows:

$$\begin{aligned} 3x^2 + 7y^2 &= 14z^2 \\ 3x^2 &= -7y^2 + 14z^2 \\ \left(\frac{x}{\frac{1}{\sqrt{3}}}\right)^2 &= -\left(\frac{y}{\frac{1}{\sqrt{7}}}\right)^2 + \left(\frac{z}{\frac{1}{\sqrt{14}}}\right)^2 \end{aligned}$$

We identify it as the equation of an elliptic cone.

In Exercises 13–20, state the type of the quadric surface and describe the trace obtained by intersecting with the given plane.

13. $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1, \quad y = 0$

SOLUTION The equation $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1$ defines an ellipsoid. The xz-trace is obtained by substituting $y = 0$ in the equation of the ellipsoid. This gives the equation $x^2 + z^2 = 1$ which defines a circle in the xz-plane.

14. $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1, \quad y = 5$

SOLUTION This equation defines an ellipsoid. Substituting $y = 5$ gives

$$\begin{aligned} x^2 + \left(\frac{5}{4}\right)^2 + z^2 &= 1 \\ x^2 + z^2 &= -\left(\frac{3}{4}\right)^2 \end{aligned}$$

Since $x^2 + y^2 \geq 0$ for all x and z, the trace is an empty set.

15. $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1, \quad z = \frac{1}{4}$

SOLUTION The quadric surface is an ellipsoid, since its equation has the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ for $a = 1$, $b = 4$, $c = 1$. To find the trace obtained by intersecting the ellipsoid with the plane $z = \frac{1}{4}$, we set $z = \frac{1}{4}$ in the equation of the ellipsoid. This gives

$$lx^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = 1$$

$$x^2 + \frac{y^2}{16} = \frac{15}{16}$$

To get the standard form we divide by $\frac{15}{16}$ to obtain

$$\frac{x^2}{\frac{15}{16}} + \frac{y^2}{\frac{16\cdot 15}{16}} = 1 \quad \Rightarrow \quad \left(\frac{x}{\frac{\sqrt{15}}{4}}\right)^2 + \left(\frac{y}{\sqrt{15}}\right)^2 = 1 \tag{1}$$

We conclude that the trace is an ellipse on the xy-plane, whose equation is given in (1).

16. $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 - 5z^2 = 1, \quad x = 0$

SOLUTION The equation can be rewritten as

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{5}}}\right)^2 = 1$$

hence it defines a hyperboloid of one sheet. The yz-trace is obtained by substituting $x = 0$ in the equation of the hyperboloid. This gives

$$\left(\frac{0}{2}\right)^2 + \left(\frac{y}{5}\right)^2 - 5z^2 = 1 \quad \Rightarrow \quad \left(\frac{y}{5}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{5}}}\right)^2 = 1.$$

This is the equation of a hyperbola in the yz-plane.

17. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 - 5z^2 = 1, \quad y = 1$

SOLUTION Rewriting the equation in the form

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{5}}}\right)^2 = 1$$

we identify it as the equation of a hyperboloid of one sheet. Substituting $y = 1$ we get

$$\frac{x^2}{9} + \frac{1}{25} - 5z^2 = 1$$

$$\frac{x^2}{9} - 5z^2 = \frac{24}{25}$$

$$\frac{25}{24\cdot 9}x^2 - \frac{25\cdot 5}{24}z^2 = 1$$

$$\left(\frac{x}{\frac{6\sqrt{6}}{5}}\right)^2 - \left(\frac{z}{\frac{2}{5}\sqrt{\frac{6}{5}}}\right)^2 = 1$$

Thus, the trace on the plane $y = 1$ is a hyperbola.

18. $4x^2 + \left(\frac{y}{3}\right)^2 - 2z^2 = -1, \quad z = 1$

SOLUTION We rewrite the equation as follows:

$$\left(\frac{x}{\frac{1}{2}}\right)^2 + \left(\frac{y}{3}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2 = 1$$

This equation defines a hyperboloid of one sheet. To find the trace of the hyperboloid on the plane $z = 1$, we substitute $z = 1$ in the equation. This gives

$$4x^2 + \left(\frac{y}{3}\right)^2 - 2 \cdot 1^2 = 1$$

$$4x^2 + \frac{y^2}{9} = 3$$

$$\frac{4}{3}x^2 + \frac{y^2}{27} = 1 \quad \Rightarrow \quad \left(\frac{x}{\frac{\sqrt{3}}{2}}\right)^2 + \left(\frac{y}{3 \cdot \sqrt{3}}\right)^2 = 1$$

The trace is an ellipse on the plane $z = 1$.

19. $y = 3x^2, \quad z = 27$

SOLUTION This equation defines a parabolic cylinder, consisting of all vertical lines passing through the parabola $y = 3x^2$ in the xy-plane. Hence, the trace of the cylinder on the plane $z = 27$ is the parabola $y = 3x^2$ on this plane, that is, the following set:

$$\left\{(x, y, z) : y = 3x^2,\ z = 27\right\}.$$

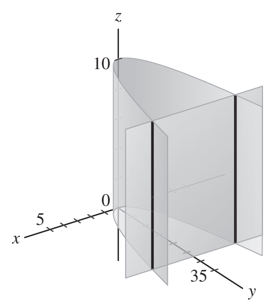
20. $y = 3x^2, \quad y = 27$

SOLUTION The equation $y = 3x^2$ defines a parabolic cylinder consisting of all vertical lines passing through the parabola $y = 3x^2$ in the xy-plane. To find the trace on the plane $y = 27$, we substitute $y = 27$ in the equation of the cylinder:

$$27 = 3x^2$$

$$9 = x^2 \quad \Rightarrow \quad x = 3,\ x = -3$$

Therefore, the trace is the two vertical lines through the points $(-3, 27)$ and $(3, 27)$ in the xy-plane.



21. Match each of the ellipsoids in Figure 12 with the correct equation:

(a) $x^2 + 4y^2 + 4z^2 = 16$

(b) $4x^2 + y^2 + 4z^2 = 16$

(c) $4x^2 + 4y^2 + z^2 = 16$

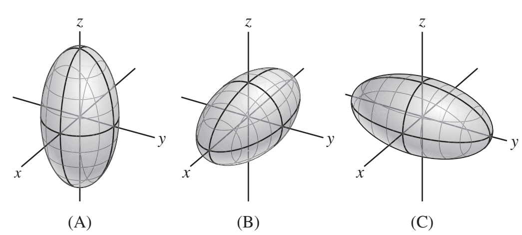


FIGURE 12

SOLUTION

(a) We rewrite the equation in the form

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

The ellipsoid intersects the x, y, and z axes at the points $(\pm 4, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 2)$, hence (B) is the corresponding figure.

(b) We rewrite the equation in the form

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

The x, y, and z intercepts are $(\pm 2, 0, 0)$, $(0, \pm 4, 0)$, and $(0, 0, \pm 2)$ respectively, hence (C) is the correct figure.

(c) We write the equation in the form

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$$

The x, y, and z intercepts are $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 4)$ respectively, hence the corresponding figure is (A).

22. Describe the surface that is obtained when, in the equation $\pm 8x^2 \pm 3y^2 \pm z^2 = 1$, we choose (a) all plus signs, (b) one minus sign, and (c) two minus signs.

SOLUTION

(a) Choosing all plus signs in the given equation yields $8x^2 + 3y^2 + z^2 = 1$, or

$$\left(\frac{x}{\sqrt{1/8}}\right)^2 + \left(\frac{y}{\sqrt{1/3}}\right)^2 + z^2 = 1$$

which is the equation of an ellipsoid.

(b) Choosing one minus sign gives the equation of a hyperboloid of one sheet.

(c) Choosing two minus signs gives the equation of a hyperboloid of two sheets.

23. What is the equation of the surface obtained when the elliptic paraboloid $z = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2$ is rotated about the x-axis by 90°? Refer to Figure 13.

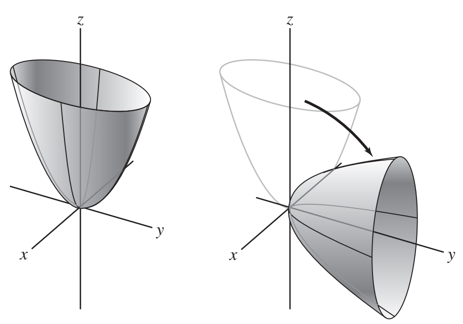


FIGURE 13

SOLUTION The axis of symmetry of the resulting surface is the y-axis rather than the z-axis. Interchanging y and z in the given equation gives the following equation of the rotated paraboloid:

$$y = \left(\frac{x}{2}\right)^2 + \left(\frac{z}{4}\right)^2$$

24. Describe the intersection of the horizontal plane $z = h$ and the hyperboloid $-x^2 - 4y^2 + 4z^2 = 1$. For which values of h is the intersection empty?

SOLUTION To find the intersection of the horizontal plane $z = h$ and the hyperboloid, we substitute $z = h$ in this equation. This gives

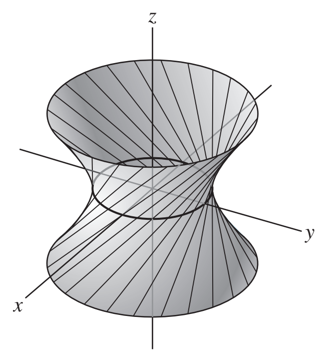
$$-x^2 - 4y^2 + 4h^2 = 1$$
$$4h^2 - 1 = x^2 + 4y^2$$

So, for $4h^2 - 1 > 0$, this is an ellipse, and for $4h^2 - 1 = 0$ this is a point, but for $4h^2 - 1 < 0$ there is no intersection. Solving that last equation for h, we get $4h^2 < 1$, so $h^2 < 1/4$, so $|h| < 1/2$ (also written as $-1/2 < h < 1/2$). Thus, for $|h| < 1/2$ there is no intersection, and for $|h| = 1/2$ the intersection is a point, and for $|h| > 1/2$ the intersection is an ellipse.

In Exercises 25–30, sketch the given surface.

25. $x^2 + y^2 - z^2 = 1$

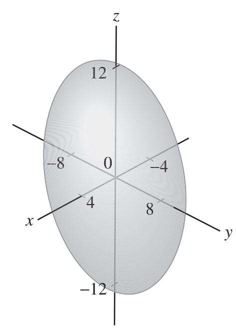
SOLUTION This equation defines a hyperboloid of one sheet. The trace on the plane $z = z_0$ is the circle $x^2 + y^2 = 1 + z_0^2$. The trace on the plane $y = y_0$ is the hyperbola $x^2 - z^2 = 1 - y_0^2$ and the trace on the plane $x = x_0$ is the hyperbola $y^2 - z^2 = 1 - x_0^2$. We obtain the following surface:



Graph of $x^2 + y^2 - z^2 = 1$

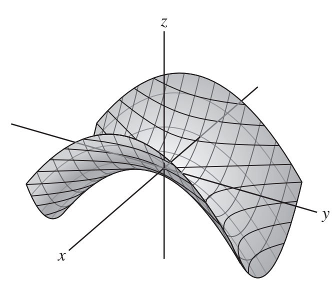
26. $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2 + \left(\frac{z}{12}\right)^2 = 1$

SOLUTION This equation defines and ellipsoid with x, y and z intercepts at the points $(\pm 4, 0, 0)$, $(0, \pm 8, 0)$ and $(0, 0, \pm 12)$. All the traces of the ellipsoid are ellipses. The surface is shown next:



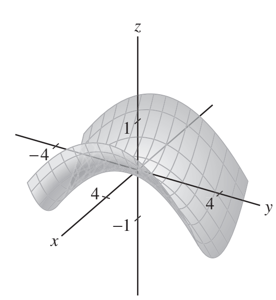
27. $z = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2$

SOLUTION This equation defines an elliptic paraboloid, as shown in the following figure:



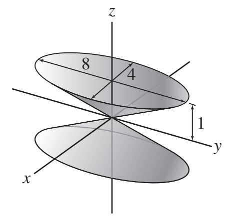
28. $z = \left(\frac{x}{4}\right)^2 - \left(\frac{y}{8}\right)^2$

SOLUTION The hyperbolic paraboloid defined by this equation is shown in the following figure:



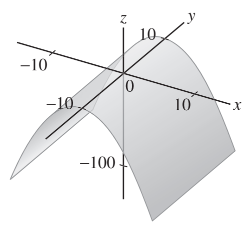
29. $z^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2$

SOLUTION This equation defines the following elliptic cone:



30. $z = -x^2$

SOLUTION This is the equation of a parabolic cylinder with base $\mathcal{C}$, where $\mathcal{C}$ is the parabola $z = -x^2$ in the xz-plane. The graph of the cylinder is shown next:



Graph of the parabolic cylinder $z = -x^2$

31. Find the equation of the ellipsoid passing through the points marked in Figure 14(A).

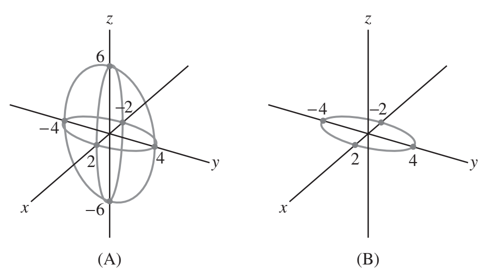


FIGURE 14

SOLUTION The equation of an ellipsoid is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{1}$$

The x, y and z intercepts are $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$ respectively. The x, y and z intercepts of the desired ellipsoid are $(\pm 2, 0, 0)$, $(0, \pm 4, 0)$ and $(0, 0, \pm 6)$ respectively, hence $a = 2$, $b = 4$ and $c = 6$. Substituting into (1) we get

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{6}\right)^2 = 1.$$

32. Find the equation of the elliptic cylinder passing through the points marked in Figure 14(B).

SOLUTION The equation of the elliptic cylinder in the xyz-coordinate system is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1}$$

The x and y intercepts are $(\pm a, 0)$ and $(0, \pm b)$ respectively. The x and y intercepts of the desired cylinder are $(\pm 2, 0)$ and $(0, \pm 4)$ respectively, hence $a = 2$ and $b = 4$. Substituting into (1) we obtain the following equation:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

33. Find the equation of the hyperboloid shown in Figure 15(A).

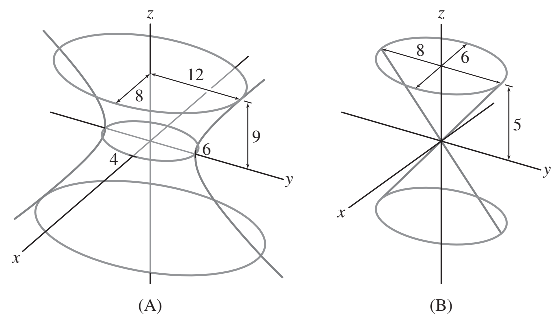


FIGURE 15

SOLUTION The hyperboloid in the figure is of one sheet and the intersections with the planes $z = z_0$ are ellipses. Hence, the equation of the hyperboloid has the form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1 \tag{1}$$

Substituting $z = 0$ we get

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

By the given information this ellipse has x and y intercepts at the points $(\pm 4, 0)$ and $(0, \pm 6)$ hence $a = 4$, $b = 6$. Substituting in (1) we get

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 - \left(\frac{z}{c}\right)^2 = 1 \tag{2}$$

Substituting $z = 9$ we get

$$\frac{x^2}{16} + \frac{y^2}{36} - \frac{9^2}{c^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1 + \frac{81}{c^2} = \frac{c^2 + 81}{c^2}$$

$$\frac{c^2x^2}{16(81 + c^2)} + \frac{c^2y^2}{36(81 + c^2)} = 1$$

$$\left(\frac{x}{\frac{4}{c}\sqrt{81 + c^2}}\right)^2 + \left(\frac{y}{\frac{6}{c}\sqrt{81 + c^2}}\right)^2 = 1$$

By the given information the following must hold:

$$\begin{matrix} \frac{4}{c}\sqrt{81 + c^2} = 8 \\ \frac{6}{c}\sqrt{81 + c^2} = 12 \end{matrix} \quad \Rightarrow \quad \frac{\sqrt{81 + c^2}}{c} = 2 \quad \Rightarrow \quad 81 + c^2 = 4c^2 \quad \Rightarrow \quad 3c^2 = 81$$

Thus, $c = 3\sqrt{3}$, and by substituting in (2) we obtain the following equation:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 - \left(\frac{z}{3\sqrt{3}}\right)^2 = 1$$

34. Find the equation of the quadric surface shown in Figure 15(B).

SOLUTION The quadratic surface in the figure is an elliptic cone. The horizontal trace curves are ellipses, hence the equation of the cone has the form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 \tag{1}$$

The trace on the plane $z = 5$ is the ellipse $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{8}\right)^2 = 1$. Substituting $z = 5$ in (1) gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{5}{c}\right)^2$$

$$\left(\frac{x}{5a/c}\right)^2 + \left(\frac{y}{5b/c}\right)^2 = 1$$

Thus, the following equalities must hold:

$$\frac{5a}{c} = 6 \quad \Rightarrow \quad a = \frac{6c}{5}$$
$$\frac{5b}{c} = 8 \quad \Rightarrow \quad b = \frac{8c}{5}$$

Substituting in (1) gives

$$\left(\frac{x}{6c/5}\right)^2 + \left(\frac{y}{8c/5}\right)^2 = \left(\frac{z}{c}\right)^2$$
$$\frac{1}{c^2}\left(\frac{x}{6/5}\right)^2 + \frac{1}{c^2}\left(\frac{y}{8/5}\right)^2 = \frac{1}{c^2}z^2$$
$$\left(\frac{x}{6/5}\right)^2 + \left(\frac{y}{8/5}\right)^2 = z^2$$
$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{8}\right)^2 = \left(\frac{z}{5}\right)^2$$

35. Determine the vertical traces of elliptic and parabolic cylinders in standard form.

SOLUTION The vertical traces of elliptic or parabolic cylinders are one or two vertical lines, or an empty set.

36. What is the equation of a hyperboloid of one or two sheets in standard form if every horizontal trace is a circle?

SOLUTION The equation of a hyperboloid (of one sheet) is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$$

or

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \left(\frac{z}{c}\right)^2$$

The horizontal traces are obtained by setting $z = z_0$, that is,

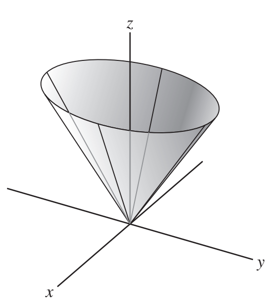
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \left(\frac{z_0}{c}\right)^2 = \text{constant}$$

This equation defines a circle when $a = b$. Thus, the corresponding hyperboloid is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$$

37. Let $\mathcal{C}$ be an ellipse in a horizonal plane lying above the xy-plane. Which type of quadric surface is made up of all lines passing through the origin and a point on $\mathcal{C}$?

SOLUTION The quadric surface is the upper part of an elliptic cone.



38. The eccentricity of a conic section is defined in Section 11.5. Show that the horizontal traces of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

are ellipses of the same eccentricity (apart from the traces at height $h = \pm c$, which reduce to a single point). Find the eccentricity.

SOLUTION The intersection of the ellipsoid with the horizonal plane $z = h$ for $|h| < c$ is obtained by substituting $z = h$ in the equation of the ellipsoid . This gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{h}{c}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 - \left(\frac{h}{c}\right)^2 = \frac{c^2 - h^2}{c^2}$$

$$\left(\frac{x}{\frac{a}{c}\sqrt{c^2 - h^2}}\right)^2 + \left(\frac{y}{\frac{b}{c}\sqrt{c^2 - h^2}}\right)^2 = 1$$

This is the equation of an ellipse in the plane $z = h$. Assume $a < b$. The eccentricity of the ellipse is

$$e = \frac{\sqrt{\left[\frac{a}{c}\sqrt{c^2 - h^2}\right]^2 - \left[\frac{b}{c}\sqrt{c^2 - h^2}\right]^2}}{\frac{a}{c}\sqrt{c^2 - h^2}} = \frac{\sqrt{\frac{a^2}{c^2}(c^2 - h^2) - \frac{b^2}{c^2}(c^2 - h^2)}}{\frac{a}{c}\sqrt{c^2 - h^2}}$$

$$= \frac{\sqrt{\frac{(c^2-h^2)}{c^2}(a^2 - b^2)}}{\frac{a}{c}\sqrt{c^2 - h^2}} = \frac{\frac{\sqrt{c^2-h^2}}{c}\sqrt{a^2 - b^2}}{\frac{\sqrt{c^2-h^2}}{c}a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

Since the eccentricity is independent of h, all the horizontal traces are ellipses with the same eccentricity $e = \sqrt{1 - \left(\frac{a}{b}\right)^2}$. If $a > b$, we obtain (in a similar manner) $e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$, again independent of h.

Further Insights and Challenges

39. Let $\mathcal{S}$ be the hyperboloid $x^2 + y^2 = z^2 + 1$ and let $P = (\alpha, \beta, 0)$ be a point on $\mathcal{S}$ in the (x, y)-plane. Show that there are precisely two lines through P entirely contained in $\mathcal{S}$ (Figure 16). *Hint:* Consider the line $\mathbf{r}(t) = \langle \alpha + at, \beta + bt, t\rangle$ through P. Show that $\mathbf{r}(t)$ is contained in $\mathcal{S}$ if (a, b) is one of the two points on the unit circle obtained by rotating (α, β) through $\pm\frac{\pi}{2}$. This proves that a hyperboloid of one sheet is a **doubly ruled surface**, which means that it can be swept out by moving a line in space in two different ways.

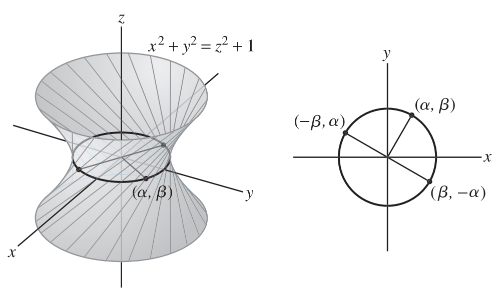


FIGURE 16

SOLUTION The parametric equations of the lines through $P = (\alpha, \beta, 0)$ have the form

$$x = \alpha + ks, \quad y = \beta + \ell s, \quad z = ms$$

Setting the parameter $t = ms$ and replacing $\frac{k}{m}$ and $\frac{\ell}{m}$ by a and b, respectively, we obtain the following (normalized) form

$$x = \alpha + at, \quad y = \beta + bt, \quad z = t$$

The line is entirely contained in S if and only if for all values of the parameter t, the following equality holds:

$$(\alpha + at)^2 + (\beta + bt)^2 = t^2 + 1$$

That is, for all t,

$$\alpha^2 + 2\alpha at + a^2t^2 + \beta^2 + 2\beta bt + b^2t^2 = t^2 + 1$$

$$(a^2 + b^2 - 1)t^2 + 2(\alpha a + \beta b)t + (\alpha^2 + \beta^2 - 1) = 0$$

This equality holds for all t if and only if all the coefficients are zero. That is, if and only if

$$\begin{cases} a^2 + b^2 - 1 = 0 \\ \alpha a + \beta b = 0 \\ \alpha^2 + \beta^2 - 1 = 0 \end{cases}$$

The first and the third equations imply that (a, b) and (α, β) are points on the unit circle $x^2 + y^2 = 1$. The second equation implies that the vector $\mathbf{u} = \langle a, b \rangle$ is orthogonal to the vector $\mathbf{v} = \langle \alpha, \beta \rangle$ (since $\mathbf{u} \cdot \mathbf{v} = a\alpha + b\beta = 0$).

Conclusions: There are precisely two lines through P entirely contained in S. For the direction vectors $(a, b, 1)$ of these lines, (a, b) is obtained by rotating (α, β) through $\pm\frac{\pi}{2}$ about the origin.

In Exercises 40 and 41, let $\mathcal{C}$ be a curve in $\mathbf{R}^3$ not passing through the origin. The cone on $\mathcal{C}$ is the surface consisting of all lines passing through the origin and a point on $\mathcal{C}$ [Figure 17(A)].

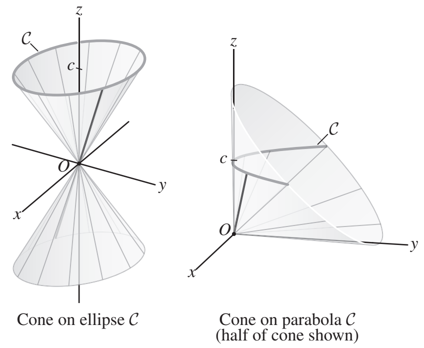


FIGURE 17

40. Show that the elliptic cone $\left(\frac{z}{c}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ is, in fact, a cone on the ellipse $\mathcal{C}$ consisting of all points (x, y, c) such that $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION

Step 1. We verify that the lines $\overline{OP}$ where P is a point (α, β, c) such that $\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2 = 1$ are contained in the elliptic cone $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$. The parametric equations of the line $\overline{OP}$ are

$$x = t\alpha, \quad y = t\beta, \quad z = tc$$

Substituting in the left hand side of the equation of the cone $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$ gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{t\alpha}{a}\right)^2 + \left(\frac{t\beta}{b}\right)^2 = t^2\left(\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2\right) = t^2 \cdot 1 = \left(\frac{tc}{c}\right)^2 = \left(\frac{z}{c}\right)^2$$

Therefore, the line $\overline{OP}$ is contained in the elliptic cone.

Step 2. We show that every point (x_0, y_0, z_0) on the elliptic cone is contained in a certain line $\overline{OP}$ where P is a point on $\mathcal{C}$. Since (x_0, y_0, z_0) is on the cone, we have $\left(\frac{x_0}{a}\right)^2 + \left(\frac{y_0}{b}\right)^2 = \left(\frac{z_0}{c}\right)^2$, hence,

$$\left(\frac{x_0}{a\frac{z_0}{c}}\right)^2 + \left(\frac{y_0}{b\frac{z_0}{c}}\right)^2 = 1$$

or

$$\left(\frac{x_0\frac{c}{z_0}}{a}\right)^2 + \left(\frac{y_0\frac{c}{z_0}}{b}\right)^2 = 1 \tag{1}$$

We define P as the point $P = \left(\frac{x_0 c}{z_0}, \frac{y_0 c}{z_0}, c\right)$. By (1) P is on the ellipse $\mathcal{C}$. We show that (x_0, y_0, z_0) lies on the line through the origin and P. The parametric equations of this line are

$$x = t\frac{x_0 c}{z_0}, \quad y = t\frac{y_0 c}{z_0}, \quad z = tc$$

Now, (x_0, y_0, z_0) corresponds to the parameter $t = \frac{z_0}{c}$. This proves that any point $P = (x_0, y_0, z_0)$ on the cone is included in a certain line through the origin and a point on $\mathcal{C}$. By virtue of step 1 and step 2, we conclude that the cone on the ellipse $\mathcal{C}$ is equal to the elliptic cone.

41. Let a and c be nonzero constants and let $\mathcal{C}$ be the parabola at height c consisting of all points (x, ax^2, c) [Figure 17(B)]. Let $\mathcal{S}$ be the cone consisting of all lines passing through the origin and a point on $\mathcal{C}$. This exercise shows that $\mathcal{S}$ is also an elliptic cone.

(a) Show that $\mathcal{S}$ has equation $yz = acx^2$.

(b) Show that under the change of variables $y = u + v$ and $z = u - v$, this equation becomes $acx^2 = u^2 - v^2$ or $u^2 = acx^2 + v^2$ (the equation of an elliptic cone in the variables x, v, u).

SOLUTION A point P on the parabola $\mathcal{C}$ has the form $P = \left(x_0, ax_0^2, c\right)$, hence the parametric equations of the line through the origin and P are

$$x = tx_0, \quad y = tax_0^2, \quad z = tc$$

To find a direct relation between xy and z we notice that

$$yz = tax_0^2ct = ac(tx_0)^2 = acx^2$$

Now, defining new variables $z = u - v$ and $y = u + v$. This equation becomes

$$(u+v)(u-v) = acx^2$$
$$u^2 - v^2 = acx^2 \quad \Rightarrow \quad u^2 = acx^2 + v^2$$

This is the equation of an elliptic cone in the variables x, v, u. We, thus, showed that the cone on the parabola $\mathcal{C}$ is transformed to an elliptic cone by the transformation (change of variables) $y = u + v$, $z = u - v$, $x = x$.

12.7 Cylindrical and Spherical Coordinates (LT Section 13.7)

Preliminary Questions

1. Describe the surfaces $r = R$ in cylindrical coordinates and $\rho = R$ in spherical coordinates.

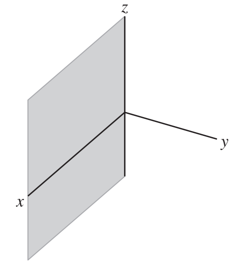
SOLUTION The surface $r = R$ consists of all points located at a distance R from the z-axis. This surface is the cylinder of radius R whose axis is the z-axis. The surface $\rho = R$ consists of all points located at a distance R from the origin. This is the sphere of radius R centered at the origin.

2. Which statement about cylindrical coordinates is correct?

(a) If $\theta = 0$, then P lies on the z-axis.

(b) If $\theta = 0$, then P lies in the xz-plane.

SOLUTION The equation $\theta = 0$ defines the half-plane of all points that project onto the ray $\theta = 0$, that is, onto the nonnegative x-axis. This half plane is part of the (x, z)-plane, therefore if $\theta = 0$, then P lies in the (x, z)-plane.



The half-plane $\theta = 0$

For instance, the point $P = (1, 0, 1)$ satisfies $\theta = 0$, but it does not lie on the z-axis. We conclude that statement (b) is correct and statement (a) is false.

3. Which statement about spherical coordinates is correct?

(a) If $\phi = 0$, then P lies on the z-axis.

(b) If $\phi = 0$, then P lies in the xy-plane.

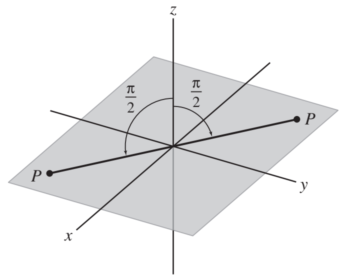
SOLUTION The equation $\phi = 0$ describes the nonnegative z-axis. Therefore, if $\phi = 0$, P lies on the z-axis as stated in (a). Statement (b) is false, since the point $(0, 0, 1)$ satisfies $\phi = 0$, but it does not lie in the (x, y)-plane.

4. The level surface $\phi = \phi_0$ in spherical coordinates, usually a cone, reduces to a half-line for two values of ϕ_0. Which two values?

SOLUTION For $\phi_0 = 0$, the level surface $\phi = 0$ is the upper part of the z-axis. For $\phi_0 = \pi$, the level surface $\phi = \pi$ is the lower part of the z-axis. These are the two values of ϕ_0 where the level surface $\phi = \phi_0$ reduces to a half-line.

5. For which value of ϕ_0 is $\phi = \phi_0$ a plane? Which plane?

SOLUTION For $\phi_0 = \frac{\pi}{2}$, the level surface $\phi = \frac{\pi}{2}$ is the xy-plane.



Exercises

In Exercises 1–4, convert from cylindrical to rectangular coordinates.

1. $(4, \pi, 4)$

SOLUTION By the given data $r = 4$, $\theta = \pi$ and $z = 4$. Hence,

$$\begin{aligned} x &= r\cos\theta = 4\cos\pi = 4\cdot(-1) = -4 \\ y &= r\sin\theta = 4\sin\pi = 4\cdot 0 \\ z &= 4 \end{aligned} \quad\Rightarrow\quad (x, y, z) = (-4, 0, 4)$$

2. $\left(2, \frac{\pi}{3}, -8\right)$

SOLUTION We are given that $(r, \theta, z) = \left(2, \frac{\pi}{3}, -8\right)$. Hence,

$$\begin{aligned} x &= r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1 \\ y &= r\sin\theta = 2\sin\frac{\pi}{3} = 2\frac{\sqrt{3}}{2} = \sqrt{3} \\ z &= -8 \end{aligned} \quad\Rightarrow\quad (x, y, z) = \left(1, \sqrt{3}, -8\right)$$

3. $\left(0, \frac{\pi}{5}, \frac{1}{2}\right)$

SOLUTION We have $r = 0$, $\theta = \frac{\pi}{5}$, $z = \frac{1}{2}$. Thus,

$$\begin{aligned} x &= r\cos\theta = 0\cdot\cos\frac{\pi}{5} = 0 \\ y &= r\sin\theta = 0\cdot\sin\frac{\pi}{5} = 0 \\ z &= \frac{1}{2} \end{aligned} \quad\Rightarrow\quad (x, y, z) = \left(0, 0, \frac{1}{2}\right)$$

4. $\left(1, \frac{\pi}{2}, -2\right)$

SOLUTION Conversion to rectangular coordinates gives

$$\begin{aligned} x &= r\cos\theta = 1\cdot\cos\frac{\pi}{2} = 1\cdot 0 = 0 \\ y &= r\sin\theta = 1\cdot\sin\frac{\pi}{2} = 1\cdot 1 = 1 \\ z &= -2 \end{aligned} \quad\Rightarrow\quad (x, y, z) = (0, 1, -2)$$

In Exercises 5–10, convert from rectangular to cylindrical coordinates.

5. $(1, -1, 1)$

SOLUTION We are given that $x = 1$, $y = -1$, $z = 1$. We find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Next we find θ. The point $(x, y) = (1, -1)$ lies in the fourth quadrant, hence,

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1, \quad \frac{3\pi}{2} \le \theta \le 2\pi \quad\Rightarrow\quad \theta = \frac{7\pi}{4}$$

We conclude that the cylindrical coordinates of the point are

$$(r, \theta, z) = \left(\sqrt{2}, \frac{7\pi}{4}, 1\right).$$

6. $(2, 2, 1)$

SOLUTION We are given that $(x, y, z) = (2, 2, 1)$. We first find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Next we find θ. The point $(x, y) = (2, 2)$ lies in the first quadrant hence $0 \le \theta \le \frac{\pi}{2}$. Therefore,

$$\tan\theta = \frac{y}{x} = \frac{2}{2} = 1, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

The cylindrical coordinates of the point are

$$(r, \theta, z) = \left(2\sqrt{2}, \frac{\pi}{4}, 1\right).$$

7. $(1, \sqrt{3}, 7)$

SOLUTION We have $x = 1$, $y = \sqrt{3}$, $z = 7$. We first find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

Since the point $(x, y) = \left(1, \sqrt{3}\right)$ lies in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$. Hence,

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

The cylindrical coordinates are thus

$$(r, \theta, z) = \left(2, \frac{\pi}{3}, 7\right).$$

8. $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 9\right)$

SOLUTION We are given that $(x, y, z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 9\right)$. Hence,

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = 3$$

Since the point $(x, y) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ is in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$. Therefore,

$$\tan\theta = \frac{y}{x} = \frac{3\sqrt{3}/2}{3/2} = \sqrt{3}, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

The cylindrical coordinates are thus

$$(r, \theta, z) = \left(3, \frac{\pi}{3}, 9\right).$$

9. $\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 2\right)$

SOLUTION We have $x = \frac{5}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$, $z = 2$. We find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2} = \sqrt{25} = 5$$

Since the point $(x, y) = \left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, therefore,

$$\tan\theta = \frac{y}{x} = \frac{5/\sqrt{2}}{5/\sqrt{2}} = 1, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

The corresponding cylindrical coordinates are

$$(r, \theta, z) = \left(5, \frac{\pi}{4}, 2\right).$$

10. $(3, 3\sqrt{3}, 2)$

SOLUTION We have $x = 3$, $y = 3\sqrt{3}$, and $z = 2$, hence,

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{36} = 6$$

The point $(x, y) = \left(3, 3\sqrt{3}\right)$ is in the first quadrant hence $0 \le \theta \le \frac{\pi}{2}$. Therefore,

$$\tan\theta = \frac{y}{x} = \frac{3\sqrt{3}}{3} = \sqrt{3}, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

The cylindrical coordinates of the given point are

$$(r, \theta, z) = \left(6, \frac{\pi}{3}, 2\right).$$

In Exercises 11–16, describe the set in cylindrical coordinates.

11. $x^2 + y^2 \le 1$

SOLUTION The inequality describes a solid cylinder of radius 1 centered on the z-axis. Since $x^2 + y^2 = r^2$, this inequality can be written as $r^2 \le 1$.

12. $x^2 + y^2 + z^2 \le 1$

SOLUTION Since $x^2 + y^2 = r^2$, this inequality can be written as

$$r^2 + z^2 \le 1 \quad \text{or} \quad r^2 \le 1 - z^2$$

13. $y^2 + z^2 \le 4, \quad x = 0$

SOLUTION The projection of the points in this set onto the xy-plane are points on the y axis, thus $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. Therefore, $y = r\sin\frac{\pi}{2} = r \cdot 1 = r$ or $y = r\sin\left(\frac{3\pi}{2}\right) = -r$. In both cases, $y^2 = r^2$, thus the inequality $y^2 + z^2 \le 4$ becomes $r^2 + z^2 \le 4$. In cylindrical coordinates, we obtain the following inequality

$$r^2 + z^2 \le 4, \quad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2}$$

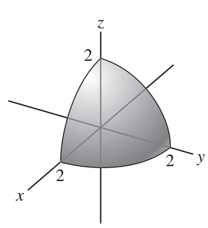
14. $x^2 + y^2 + z^2 = 4, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0$

SOLUTION We express z in terms of x and y. Since $z \ge 0$ we get

$$x^2 + y^2 + z^2 = 4 \quad \Rightarrow \quad z^2 = 4 - (x^2 + y^2) \quad \Rightarrow \quad z = \sqrt{4 - (x^2 + y^2)} \tag{1}$$

The cylindrical coordinates are (r, θ, z) where $x^2 + y^2 = r^2$. Substituting into (1) gives

$$z = \sqrt{4 - r^2} \tag{2}$$



We find the interval for θ. The given set is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant.

$$\text{Hence, the angle } \theta \text{ is changing between 0 and } \frac{\pi}{2}. \tag{3}$$

We combine (2) and (3) to obtain the following representation:

$$z = \sqrt{4 - r^2}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

15. $x^2 + y^2 \le 9, \quad x \ge y$

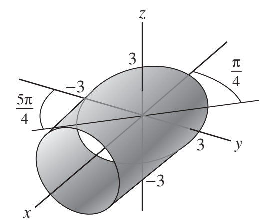
SOLUTION The equation $x^2 + y^2 \le 9$ in cylindrical coordinates becomes $r^2 \le 9$, which becomes $r \le 3$. However, we also have the restriction that $x \ge y$. This means that the projection of our set onto the xy plane is below and to the right of the line $y = x$. In other words, our θ is restricted to $-3\pi/4 \le \theta \le \pi/4$. In conclusion, the answer is:

$$r \le 3, \qquad -3\pi/4 \le \theta \le \pi/4$$

16. $y^2 + z^2 \le 9, \quad x \ge y$

SOLUTION The region $x \ge y$ in the xy-plane is determined by the inequalities $\frac{5\pi}{4} \le \theta \le 2\pi$, $0 \le \theta \le \frac{\pi}{4}$ (and the origin). Since $y = r\sin\theta$, the region $y^2 + z^2 \le 9$ can be written as

$$r^2 \sin^2\theta + z^2 \le 9$$



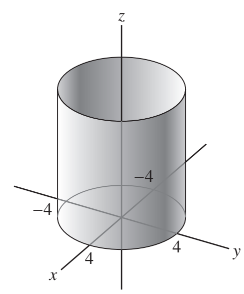
We obtain the following description in cylindrical coordinates:

$$r^2 \sin^2\theta + z^2 \le 9, \quad 0 \le \theta \le \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4} \le \theta \le 2\pi.$$

In Exercises 17–24, sketch the set (described in cylindrical coordinates).

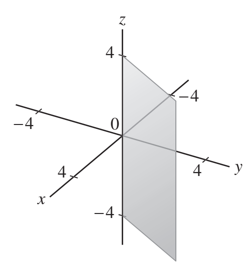
17. $r = 4$

SOLUTION The surface $r = 4$ consists of all points located at a distance 4 from the z-axis. It is a cylinder of radius 4 whose axis is the z-axis. The cylinder is shown in the following figure:



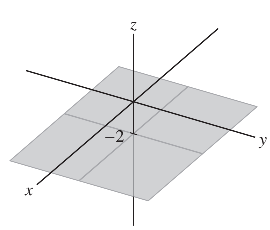
18. $\theta = \dfrac{\pi}{3}$

SOLUTION The equation $\theta = \frac{\pi}{3}$ defines the half plane of all points that project onto the ray $\theta = \frac{\pi}{3}$ in the xy-plane. This half plane is shown in the following figure:



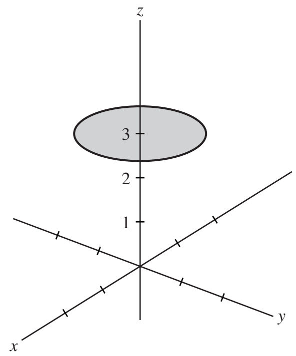
19. $z = -2$

SOLUTION $z = -2$ is the horizontal plane at height -2, shown in the following figure:



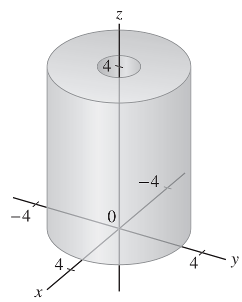
20. $r = 2, \quad z = 3$

SOLUTION This is a circle of radius 2, parallel to the xy-plane but on the plane $z = 3$, as seen in the following figure:



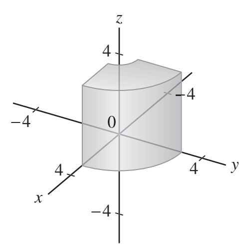
21. $1 \le r \le 3, \quad 0 \le z \le 4$

SOLUTION The region $1 \le r \le 3$, $0 \le z \le 4$ is shown in the following figure:



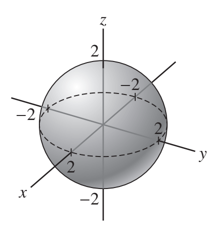
22. $1 \le r \le 3, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad 0 \le z \le 4$

SOLUTION The inequality $1 \le r \le 3$ implies that the projection of the region on the xy-plane is contained in a ring $1 \le \sqrt{x^2 + y^2} \le 3$. The inequality $0 \le \theta \le \frac{\pi}{2}$ restricts the ring to the first quadrant and $0 \le z \le 4$ determines the height. We obtain the following region:



23. $z^2 + r^2 \le 4$

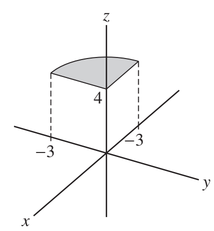
SOLUTION The region $z^2 + r^2 \le 4$ is shown in the following figure:



In rectangular coordinates the inequality is $z^2 + \left(x^2 + y^2\right) \le 4$, or $x^2 + y^2 + z^2 \le 4$, which is a ball of radius 2.

24. $r \le 3, \quad \pi \le \theta \le \dfrac{3\pi}{2}, \quad z = 4$

SOLUTION The region $r \le 3, \pi \le \theta \le \frac{3\pi}{2}, z = 4$ is shown in the following figure:



In Exercises 25–30, find an equation of the form $r = f(\theta, z)$ in cylindrical coordinates for the following surfaces.

25. $z = x + y$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ to obtain the following equation in cylindrical coordinates:

$$\begin{aligned} z &= r\cos\theta + r\sin\theta \\ z &= r(\cos\theta + \sin\theta) \end{aligned} \quad \Rightarrow \quad r = \frac{z}{\cos\theta + \sin\theta}.$$

26. $x^2 + y^2 + z^2 = 4$

SOLUTION Since $x^2 + y^2 = r^2$, we get

$$\begin{aligned} r^2 + z^2 &= 4 \\ r^2 &= 4 - z^2 \quad \Rightarrow \quad r = \sqrt{4 - z^2} \end{aligned}$$

.

27. $\dfrac{x^2}{yz} = 1$

SOLUTION We rewrite the equation in the form

$$\frac{x}{\frac{y}{x}z} = 1$$

Substituting $x = r\cos\theta$ and $\frac{y}{x} = \tan\theta$ we get

$$\begin{aligned} \frac{r\cos\theta}{(\tan\theta)\,z} &= 1 \\ r &= \frac{z\tan\theta}{\cos\theta} \end{aligned}$$

28. $x^2 - y^2 = 4$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ and use the trigonometric identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$. This gives

$$\begin{aligned} r^2\cos^2\theta - r^2\sin^2\theta &= 4 \\ r^2\left(\cos^2\theta - \sin^2\theta\right) &= 4 \\ r^2\cos 2\theta &= 4 \quad \Rightarrow \quad r = \frac{2}{\sqrt{\cos 2\theta}} \end{aligned}$$

29. $x^2 + y^2 = 4$

SOLUTION Since $x^2 + y^2 = r^2$, the equation in cylindrical coordinates is, $r^2 = 4$ or $r = 2$.

30. $z = 3xy$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ and use the trigonometric identity $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$. This gives

$$z = 3\,(r\cos\theta)\,(r\sin\theta) = 3r^2\cos\theta\sin\theta = \frac{3}{2}r^2\sin 2\theta$$

Thus,

$$r^2 = \frac{2z}{3\sin 2\theta} \quad \Rightarrow \quad r = \sqrt{\frac{2z}{3\sin 2\theta}}.$$

In Exercises 31–36, convert from spherical to rectangular coordinates.

31. $\left(3, 0, \frac{\pi}{2}\right)$

SOLUTION We are given that $\rho = 3, \theta = 0, \phi = \frac{\pi}{2}$. Using the relations between spherical and rectangular coordinates we have

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 3 \sin\frac{\pi}{2} \cos 0 = 3 \cdot 1 \cdot 1 = 3 \\ y &= \rho \sin\phi \sin\theta = 3 \sin\frac{\pi}{2} \sin 0 = 3 \cdot 1 \cdot 0 = 0 \qquad \Rightarrow \qquad (x, y, z) = (3, 0, 0) \\ z &= \rho \cos\phi = 3 \cos\frac{\pi}{2} = 3 \cdot 0 = 0 \end{aligned}$$

32. $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$

SOLUTION We are given that $\rho = 2, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{3}$. The relations between the spherical and the rectangular coordinates imply

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2} \\ y &= \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2} \qquad \Rightarrow \qquad (x, y, z) = \left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right) \\ z &= \rho \cos\phi = 2 \cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1 \end{aligned}$$

33. $(3, \pi, 0)$

SOLUTION We have $\rho = 3, \theta = \pi, \phi = 0$. Hence,

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 3 \sin 0 \cos\pi = 0 \\ y &= \rho \sin\phi \sin\theta = 3 \sin 0 \sin\pi = 0 \qquad \Rightarrow \qquad (x, y, z) = (0, 0, 3) \\ z &= \rho \cos\phi = 3 \cos 0 = 3 \end{aligned}$$

34. $\left(5, \frac{3\pi}{4}, \frac{\pi}{4}\right)$

SOLUTION We have $\rho = 5, \theta = \frac{3\pi}{4}, \phi = \frac{\pi}{4}$. Using the relations between spherical and rectangular coordinates we have

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 5 \sin\frac{\pi}{4} \cos\frac{3\pi}{4} = 5 \cdot \frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = -2.5 \\ y &= \rho \sin\phi \sin\theta = 5 \sin\frac{\pi}{4} \sin\frac{3\pi}{4} = 5 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 2.5 \qquad \Rightarrow \qquad (x, y, z) = \left(-2.5, 2.5, 2.5\sqrt{2}\right) \\ z &= \rho \cos\phi = 5 \cos\frac{\pi}{4} = 5 \cdot \frac{\sqrt{2}}{2} = 2.5\sqrt{2} \end{aligned}$$

35. $\left(6, \frac{\pi}{6}, \frac{5\pi}{6}\right)$

SOLUTION Since $\rho = 6, \theta = \frac{\pi}{6}$, and $\phi = \frac{5\pi}{6}$ we get

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 6 \sin\frac{5\pi}{6} \cos\frac{\pi}{6} = 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \\ y &= \rho \sin\phi \sin\theta = 6 \sin\frac{5\pi}{6} \sin\frac{\pi}{6} = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2} \qquad \Rightarrow \qquad (x, y, z) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, -3\sqrt{3}\right) \\ z &= \rho \cos\phi = 6 \cos\frac{5\pi}{6} = 6 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3} \end{aligned}$$

36. (0.5, 3.7, 2)

SOLUTION Using the relations between the spherical and the rectangular coordinates with $\rho = 0.5$, $\theta = 3.7$, and $\phi = 2$, we obtain

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 0.5 \sin 2 \cos 3.7 = -0.386 \\ y &= \rho \sin\phi \sin\theta = 0.5 \sin 2 \sin 3.7 = -0.241 \quad \Rightarrow \quad (x, y, z) = (-0.386, -0.241, -0.208) \\ z &= \rho \cos\phi = 0.5 \cos 2 = -0.208 \end{aligned}$$

In Exercises 37–42, convert from rectangular to spherical coordinates.

37. $(\sqrt{3}, 0, 1)$

SOLUTION By the given data $x = \sqrt{3}$, $y = 0$, and $z = 1$. We find the radial coordinate:

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\sqrt{3}\right)^2 + 0^2 + 1^2} = 2$$

The angular coordinate θ satisfies

$$\tan\theta = \frac{y}{x} = \frac{0}{\sqrt{3}} = 0 \quad \Rightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \pi$$

Since the point $(x, y) = \left(\sqrt{3}, 0\right)$ lies in the first quadrant, the correct choice is $\theta = 0$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{3}$$

The spherical coordinates of the given points are thus

$$(\rho, \theta, \phi) = \left(2, 0, \frac{\pi}{3}\right)$$

38. $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}, 1\right)$

SOLUTION We have $x = \frac{\sqrt{3}}{2}$, $y = \frac{3}{2}$, and $z = 1$. The radial coordinate is

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 1^2} = 2$$

The angular coordinate θ satisfies

$$\tan\theta = \frac{y}{x} = \frac{3/2}{\sqrt{3}/2} = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \quad \text{or} \quad \theta = \frac{4\pi}{3}$$

Since the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ is in the first quadrant, the correct choice is $\theta = \frac{\pi}{3}$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{3}$$

The spherical coordinates are thus

$$(\rho, \theta, \phi) = \left(2, \frac{\pi}{3}, \frac{\pi}{3}\right)$$

39. (1, 1, 1)

SOLUTION We have $x = y = z = 1$. The radial coordinate is

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

The angular coordinate θ is determined by $\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$ and by the quadrant of the point $(x, y) = (1, 1)$, that is, $\theta = \frac{\pi}{4}$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = 0.955$$

The spherical coordinates are thus

$$\left(\sqrt{3}, \frac{\pi}{4}, 0.955\right)$$

40. $(1, -1, 1)$

SOLUTION We have $x = 1$, $y = -1$, and $z = 1$. The radial coordinate is

$$\rho = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

The angular coordinate θ satisfies

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4} \text{ or } \theta = \frac{7\pi}{4}$$

Since $(x, y) = (1, -1)$ is in the fourth quadrant, the angle is $\theta = \frac{7\pi}{4}$. The angle of declination satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = 0.955$$

We conclude that

$$(\rho, \theta, \phi) = \left(\sqrt{3}, \frac{7\pi}{4}, 0.955\right).$$

41. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$

SOLUTION We have $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$, and $z = \sqrt{3}$. Thus

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\sqrt{3}\right)^2} = 2$$

The angular coordinate θ satisfies $0 \le \theta \le \frac{\pi}{2}$, since the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is in the first quadrant. Also $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$, hence the angle is $\theta = \frac{\pi}{3}$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{6}$$

We conclude that

$$(\rho, \theta, \phi) = \left(2, \frac{\pi}{3}, \frac{\pi}{6}\right)$$

42. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{3}\right)$

SOLUTION We are given that $x = y = \frac{\sqrt{2}}{2}$ and $z = \sqrt{3}$. Hence,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\sqrt{3}\right)^2} = 2$$

The angle θ satisfies $0 \le \theta \le \frac{\pi}{2}$ since $(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is in the first quadrant. Also $\tan\theta = \frac{y}{x} = 1$, hence $\theta = \frac{\pi}{4}$. The angle of declination satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{6}$$

We conclude that

$$(\rho, \theta, \phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{6}\right).$$

In Exercises 43 and 44, convert from cylindrical to spherical coordinates.

43. $(2, 0, 2)$

SOLUTION We are given that $r = 2$, $\theta = 0$, $z = 2$. Using the conversion formulas, we have

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2} \\
\theta &= \theta = 0 \\
\phi &= \cos^{-1}(z/\rho) = \cos^{-1}(2/(2\sqrt{2})) = \pi/4
\end{aligned}$$

44. $(3, \pi, \sqrt{3})$

SOLUTION We are given that $r = 3, \theta = \pi, z = \sqrt{3}$. Using the conversion formulas, we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{3^2 + 3} = 2\sqrt{3}$$
$$\theta = \theta = \pi$$
$$\phi = \cos^{-1}(z/\rho) = \cos^{-1}(\sqrt{3}/(2\sqrt{3})) = \pi/3$$

In Exercises 45 and 46, convert from spherical to cylindrical coordinates.

45. $\left(4, 0, \frac{\pi}{4}\right)$

SOLUTION We are given that $\rho = 4, \theta = 0$, and $\phi = \pi/4$. To find r, we use the formulas $x = r\cos\theta$ and $x = \rho\cos\theta\sin\phi$ to get $r\cos\theta = \rho\cos\theta\sin\phi$, and so

$$r = \rho\sin\phi = 4\sin\pi/4 = 2\sqrt{2}$$

Clearly $\theta = 0$, and as for z,

$$z = \rho\cos\phi = 4\cos\pi/4 = 2\sqrt{2}$$

So, in cylindrical coordinates, our point is $(2\sqrt{2}, 0, 2\sqrt{2})$

46. $\left(2, \frac{\pi}{3}, \frac{\pi}{6}\right)$

SOLUTION We are given that $\rho = 2, \theta = \pi/3$, and $\phi = \pi/6$. To find r, we use the formulas $x = r\cos\theta$ and $x = \rho\cos\theta\sin\phi$ to get $r\cos\theta = \rho\cos\theta\sin\phi$, and so

$$r = \rho\sin\phi = 2\sin\pi/6 = 1$$

Clearly $\theta = \pi/3$, and as for z,

$$z = \rho\cos\phi = 2\cos\pi/6 = \sqrt{3}$$

So, in cylindrical coordinates, our point is $(1, \pi/3, \sqrt{3})$

In Exercises 47–52, describe the given set in spherical coordinates.

47. $x^2 + y^2 + z^2 \le 1$

SOLUTION Substituting $\rho^2 = x^2 + y^2 + z^2$ we obtain $\rho^2 \le 1$ or $0 \le \rho \le 1$.

48. $x^2 + y^2 + z^2 = 1, \quad z \ge 0$

SOLUTION Since $\rho^2 = x^2 + y^2 + z^2$ the equation becomes $\rho^2 = 1$ or $\rho = 1$. The inequality $z \ge 0$ implies that $\cos\phi = \frac{z}{\rho} \ge 0$. Also $0 \le \phi \le \pi$ by definition, hence $0 \le \phi \le \frac{\pi}{2}$. The spherical description of the set is thus

$$\rho = 1, \quad 0 \le \phi \le \frac{\pi}{2}.$$

49. $x^2 + y^2 + z^2 = 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0$

SOLUTION By $\rho^2 = x^2 + y^2 + z^2$, we get $\rho^2 = 1$ or $\rho = 1$. The inequalities $x \ge 0$, $y \ge 0$ determine the first quadrant, which is also determined by $0 \le \theta \le \frac{\pi}{2}$. Finally, $z \ge 0$ gives $\cos\phi = \frac{z}{\rho} \ge 0$. Also $0 \le \phi \le \pi$, hence $0 \le \phi \le \frac{\pi}{2}$. We obtain the following description:

$$\rho = 1, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2}$$

50. $x^2 + y^2 + z^2 \le 1, \quad x = y, \quad x \ge 0, \quad y \ge 0$

SOLUTION Substituting $x^2 + y^2 + z^2 = \rho^2$ yields $\rho^2 \le 1$ or $0 \le \rho \le 1$. The inequalities $x \ge 0$, $y \ge 0$ determine the first quadrant which is also determined by

$$0 \le \theta \le \frac{\pi}{2} \tag{1}$$

The line $y = x$ is determined by $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$ (and the origin). Combining with (1) we get $\theta = \frac{\pi}{4}$. We conclude that the description of the given set in spherical coordinates is

$$\left\{(\rho, \theta, \phi) : 0 \le \rho \le 1, \theta = \frac{\pi}{4}\right\}$$

51. $y^2 + z^2 \le 4, \quad x = 0$

SOLUTION We substitute $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ in the given inequality. This gives

$$4 \ge \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\phi \tag{1}$$

The equality $x = 0$ determines that $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (and the origin). In both cases, $\sin^2\theta = 1$. Hence by (1) we get

$$\begin{aligned} \rho^2\sin^2\phi + \rho^2\cos^2\phi &\le 4 \\ \rho^2(1) &\le 4 \\ \rho &\le 2 \end{aligned}$$

We obtain the following description:

$$\left\{(\rho, \theta, \phi) : 0 \le \rho \le 2, \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}\right\}$$

52. $x^2 + y^2 = 3z^2$

SOLUTION We substitute the spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ in the given equation, and simplify. This gives

$$\begin{aligned} \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta &= 3\rho^2\cos^2\phi \\ \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) &= 3\rho^2\cos^2\phi \\ \rho^2\sin^2\phi \cdot 1 &= 3\rho^2\cos^2\phi \end{aligned}$$

One solution is $\rho = 0$ (the origin). For $\rho \ne 0$ we divide both sides by ρ to obtain

$$\sin^2\phi = 3\cos^2\phi. \tag{1}$$

When $\cos\phi = 0$, $\sin\phi \ne 0$. Hence the points where $\cos\phi = 0$ are not solutions. We, thus, can divide the two sides by $\cos^2\phi$ to obtain

$$\frac{\sin^2\phi}{\cos^2\phi} = 3 \quad \Rightarrow \quad \tan\phi = \sqrt{3} \quad \text{or} \quad \tan\phi = -\sqrt{3}.$$

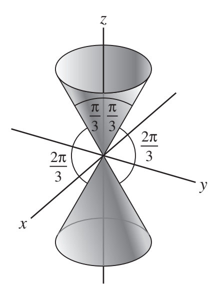
The solutions for $0 \le \phi \le \pi$ are

$$\phi = \frac{\pi}{3} \quad \text{and} \quad \phi = \frac{2\pi}{3}. \tag{2}$$

By (1) and (2) we obtain the following representation in spherical coordinates:

$$\phi = \frac{\pi}{3} \quad \text{or} \quad \phi = \frac{2\pi}{3}. \tag{3}$$

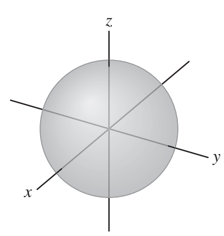
Notice that by (3) we see that the set is the surface obtained while rotating a line that makes an angle of $\frac{\pi}{3}$ with the positive z-axis, about the z-axis. In other words, a double cone.



In Exercises 53–60, sketch the set of points (described in spherical coordinates).

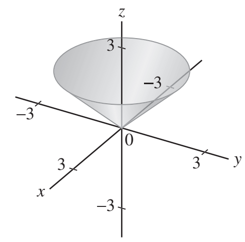
53. $\rho = 4$

SOLUTION $\rho = 4$ describes the sphere of radius 4. This is shown in the following figure:



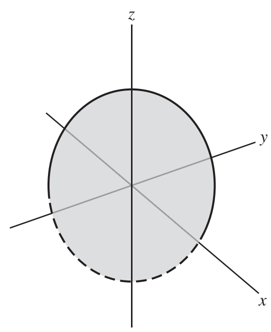
54. $\phi = \dfrac{\pi}{4}$

SOLUTION The level surface $\phi = \frac{\pi}{4}$ is the right-circular cone consisting of points P such that $\overline{OP}$ makes an angle $\frac{\pi}{4}$ with the z-axis, as shown in the following figure:



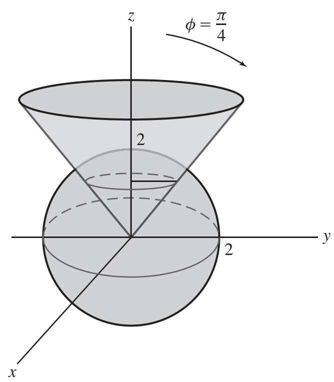
55. $\rho = 2, \quad \theta = \dfrac{\pi}{4}$

SOLUTION The equation $\rho = 2$ is a sphere of radius 2, and the equation $\theta = \frac{\pi}{4}$ is the vertical plane $y = x$. These two surfaces intersect in a (vertical) circle of radius 2, as seen here.



56. $\rho = 2, \quad \phi = \dfrac{\pi}{4}$

SOLUTION The equation $\rho = 2$ is a sphere of radius 2, and the equation $\phi = \frac{\pi}{4}$ is a right circular cone. These two surfaces intersect in a (horizontal) circle of height $\sqrt{2}$ and radius $\sqrt{2}$, as seen here.

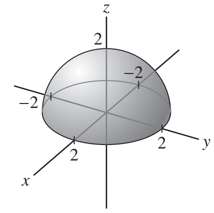


57. $\rho = 2, \quad 0 \le \phi \le \dfrac{\pi}{2}$

SOLUTION The set

$$\rho = 2, \quad 0 \le \phi \le \frac{\pi}{2}$$

is shown in the following figure:



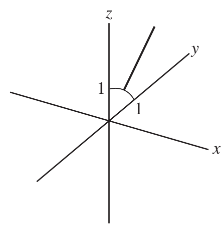
It is the upper half of the sphere with radius 2.

58. $\theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}, \quad \rho \geq 1$

SOLUTION The set

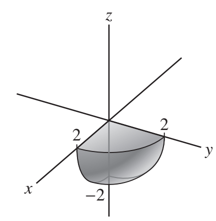
$$\theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}, \quad \rho \geq 1$$

is the line $z = y$ in the first quadrant of the yz-plane that is outside the circle $y^2 + z^2 = 1$. This set is shown in the following figure:



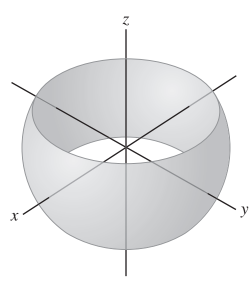
59. $\rho \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \phi \leq \pi$

SOLUTION This set is the part of the ball of radius 2 which is below the first quadrant of the xy-plane, as shown in the following figure:



60. $\rho = 1, \quad \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$

SOLUTION This set is the part of the unit sphere consisting of all the points P such that $\overline{OP}$ makes an angle $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$ with the z-axis. This set is shown in the following figure:



In Exercises 61–66, find an equation of the form $\rho = f(\theta, \phi)$ in spherical coordinates for the following surfaces.

61. $z = 2$

SOLUTION Since $z = \rho \cos \phi$, we have $\rho \cos \phi = 2$, or $\rho = \frac{2}{\cos \phi}$.

62. $z^2 = 3(x^2 + y^2)$

SOLUTION We use the formulas for x, y and z in terms of ρ, θ and ϕ. This gives

$$\rho^2 \cos^2 \phi = 3\left(\rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi\right) = 3\rho^2 \sin^2 \phi \left(\cos^2 \theta + \sin^2 \theta\right) = 3\rho^2 \sin^2 \phi$$

That is,

$$\rho^2 \cos^2 \phi = 3\rho^2 \sin^2 \phi$$

which gives either $\rho = 0$, or gives the following:

$$\cos^2 \phi = 3 \sin^2 \phi$$

$$\tan^2 \phi = \frac{1}{3}$$

$$\tan\phi = \pm\frac{1}{\sqrt{3}}$$
$$\phi = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$$

Thus, the equation is $\{\rho = 0 \text{ or } \phi = \pi/6 \text{ or } \phi = 5\pi/6\}$.

63. $x = z^2$

SOLUTION Substituting $x = \rho\cos\theta\sin\phi$ and $z = \rho\cos\phi$ we obtain

$$\rho\cos\theta\sin\phi = \rho^2\cos^2\phi$$
$$\cos\theta\sin\phi = \rho\cos^2\phi$$
$$\rho = \frac{\cos\theta\sin\phi}{\cos^2\phi} = \frac{\cos\theta\tan\phi}{\cos\phi}$$

64. $z = x^2 + y^2$

SOLUTION Using the formulas for x, y and z in terms of ρ, θ and ϕ gives

$$\rho\cos\phi = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = \rho^2\sin^2\phi$$

That is,

$$\rho\cos\phi = \rho^2\sin^2\phi$$
$$\cos\phi = \rho\sin^2\phi$$
$$\rho = \frac{\cos\phi}{\sin^2\phi} = \frac{\cot\phi}{\sin\phi}$$

65. $x^2 - y^2 = 4$

SOLUTION We substitute $x = \rho\cos\theta\sin\phi$ and $y = \rho\sin\theta\sin\phi$ to obtain

$$4 = \rho^2\cos^2\theta\sin^2\phi - \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi\left(\cos^2\theta - \sin^2\theta\right)$$

Using the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$ we get

$$4 = \rho^2\sin^2\phi\cos 2\theta$$
$$\rho^2 = \frac{4}{\sin^2\phi\cos 2\theta}$$

We take the square root of both sides. Since $0 < \phi < \pi$ we have $\sin\phi > 0$, hence,

$$\rho = \frac{2}{\sin\phi\sqrt{\cos 2\theta}}$$

66. $xy = z$

SOLUTION We substitute the formulas for x, y, and z in terms of ρ, θ, and ϕ. This gives

$$\left(\rho\cos\theta\sin\phi\right)\left(\rho\sin\theta\sin\phi\right) = \rho\cos\phi$$

Simplifying and using the identity $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$ yields

$$\rho\cos\phi = \rho^2\left(\cos\theta\sin\theta\right)\left(\sin^2\phi\right) = \frac{1}{2}\rho^2\sin 2\theta\sin^2\phi$$

Thus

$$2\cos\phi = \rho\sin 2\theta\sin^2\phi$$
$$\rho = \frac{2\cos\phi}{\sin 2\theta\sin^2\phi}$$

67. Which of (a)–(c) is the equation of the cylinder of radius R in spherical coordinates? Refer to Figure 15.

(a) $R\rho = \sin\phi$ **(b)** $\rho \sin\phi = R$ **(c)** $\rho = R\sin\phi$

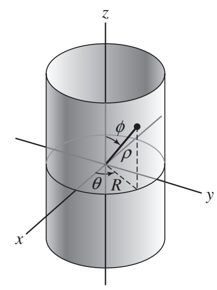


FIGURE 15

SOLUTION The equation of the cylinder of radius R in rectangular coordinates is $x^2 + y^2 = R^2$ (z is unlimited). Substituting the formulas for x and y in terms of ρ, θ and ϕ yields

$$R^2 = \rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi = \rho^2 \sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = \rho^2 \sin^2\phi$$

Hence,

$$R^2 = \rho^2 \sin^2\phi$$

We take the square root of both sides. Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$, therefore,

$$R = \rho \sin\phi$$

Equation (b) is the correct answer.

68. Let $P_1 = (1, -\sqrt{3}, 5)$ and $P_2 = (-1, \sqrt{3}, 5)$ in rectangular coordinates. In which quadrants do the projections of P_1 and P_2 onto the xy-plane lie? Find the polar angle θ of each point.

SOLUTION The projections of $P_1 = \left(1, -\sqrt{3}, 5\right)$ and $P_2 = \left(-1, \sqrt{3}, 5\right)$ on the xy-plane are the points $(x, y) = \left(1, -\sqrt{3}\right)$ and $(x, y) = \left(-1, \sqrt{3}\right)$ respectively. These points lie in the fourth and the second quadrant respectively. We find the polar angle θ of P_1. In the fourth quadrant, $\frac{3\pi}{2} \le \theta \le 2\pi$. Also $\tan\theta = \frac{y}{x} = -\sqrt{3}$. Hence $\theta = \frac{5\pi}{3}$. To find the polar angle θ of P_2, we notice that in the second quadrant (where the projection of P_2 on the xy-plane lies) we have $\frac{\pi}{2} \le \theta \le \pi$. Also $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$, hence $\theta = \frac{2\pi}{3}$.

69. Find the spherical angles (θ, ϕ) for Helsinki, Finland (60.1° N, 25.0° E) and Sao Paulo, Brazil (23.52° S, 46.52° W).

SOLUTION For Helsinki, θ is 25° and ϕ is $90 - 60.1 = 29.9°$.

For Sao Paulo, θ is $360 - 46.52 = 313.48°$ and ϕ is $90 + 23.52 = 113.52°$.

70. Find the longitude and latitude for the points on the globe with angular coordinates $(\theta, \phi) = (\pi/8, 7\pi/12)$ and $(4, 2)$.

SOLUTION For $(\theta, \phi) = (\pi/8, 7\pi/12) = (22.5°, 105°)$, we have latitude $105° - 90° = 15°$ south and longitude 22.5° east.

For $(\theta, \phi) = (4, 2) = (229.18°, 114.59°)$, we have latitude $114.59° - 90° = 24.59°$ south and longitude $360° - 229.18° = 130.82°$ west.

71. Consider a rectangular coordinate system with origin at the center of the earth, z-axis through the North Pole, and x-axis through the prime meridian. Find the rectangular coordinates of Sydney, Australia (34° S, 151° E), and Bogotá, Colombia (4° 32′ N, 74° 15′ W). A minute is $1/60°$. Assume that the earth is a sphere of radius $R = 6370$ km.

SOLUTION We first find the angle (θ, ϕ) for the two towns. For Sydney $\theta = 151°$, since its longitude lies to the east of Greenwich, that is, in the positive θ direction. Sydney's latitude is south of the equator, hence $\phi = 90 + 34 = 124°$.

For Bogota, we have $\theta = 360° - 74°15' = 285°45'$, since $74°15'W$ refers to $74°15'$ in the negative θ direction. The latitude is north of the equator hence $\phi = 90° - 4°32' = 85°28'$.

We now use the formulas of x, y and z in terms of ρ, θ, ϕ to find the rectangular coordinates of the two towns. (Notice that $285°45' = 285.75°$ and $85°28' = 85.47°$).

Sydney:

$$x = \rho\cos\theta\sin\phi = 6370\cos 151° \sin 124° = -4618.8$$

$$y = \rho\sin\theta\sin\phi = 6370\sin 151° \sin 124° = 2560$$

$$z = \rho\cos\phi = 6370\cos 124° = -3562.1$$

Bogota:

$$x = \rho\cos\theta\sin\phi = 6370\cos 285.75^\circ \sin 85.47^\circ = 1723.7$$
$$y = \rho\sin\theta\sin\phi = 6370\sin 285.75^\circ \sin 85.47^\circ = -6111.7$$
$$z = \rho\cos\phi = 6370\cos 85.47^\circ = 503.1$$

72. Find the equation in rectangular coordinates of the quadric surface consisting of the two cones $\phi = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$.

SOLUTION By $\frac{z}{\rho} = \cos\phi$, we have

$$z^2 = \rho^2\cos^2\phi \tag{1}$$

Since $\phi = \frac{\pi}{4}$ or $\phi = \frac{3\pi}{4}$, we have $\cos\phi = \frac{1}{\sqrt{2}}$ or $\cos\phi = -\frac{1}{\sqrt{2}}$, therefore, $\cos^2\phi = \frac{1}{2}$. Substituting in (1) gives

$$z^2 = \frac{\rho^2}{2} \tag{2}$$

Since $\rho^2 = x^2 + y^2 + z^2$, we get $z^2 = \frac{1}{2}(x^2 + y^2 + z^2)$, which becomes $\frac{1}{2}(z^2) = \frac{1}{2}(x^2 + y^2)$, which becomes

$$z^2 = x^2 + y^2$$

Notice that $\phi = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$ imply that z can have both positive and negative values.

73. Find an equation of the form $z = f(r, \theta)$ in cylindrical coordinates for $z^2 = x^2 - y^2$.

SOLUTION In cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Hence,

$$z^2 = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta$$

We use the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$ to obtain

$$z^2 = r^2\cos 2\theta \quad\Rightarrow\quad z = \pm r\sqrt{\cos 2\theta}$$

74. Show that $\rho = 2\cos\phi$ is the equation of a sphere with its center on the z-axis. Find its radius and center.

SOLUTION Multiplying the equation by ρ we get

$$\rho^2 = 2\rho\cos\phi$$

We now substitute $\rho^2 = x^2 + y^2 + z^2$ and $\rho\cos\phi = z$ to obtain

$$x^2 + y^2 + z^2 = 2z$$

Transferring sides and completing the square gives

$$x^2 + y^2 + z^2 - 2z = 0$$
$$x^2 + y^2 + (z-1)^2 = 1$$

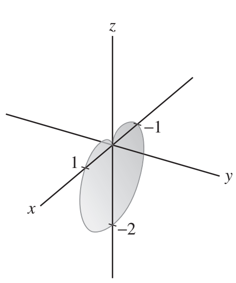
This is the rectangular equation of the sphere of radius 1, centered at the point (0, 0, 1) on the z-axis.

75. Explain the following statement: If the equation of a surface in cylindrical or spherical coordinates does not involve the coordinate θ, then the surface is rotationally symmetric with respect to the z-axis.

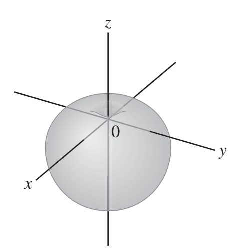
SOLUTION Since the equation of the surface does not involve the coordinate θ, then for every point P on the surface ($P = (\rho_0, \theta_0, \phi_0)$ in spherical coordinates or $P = (r_0, \theta_0, z_0)$ in cylindrical coordinates) so also all the points (ρ_0, θ, ϕ_0) or (r_0, θ, z_0) are on the surface. That is, all the points obtained by rotating P around the z-axis are on the surface. Hence, the surface is rotationally symmetric with respect to the z-axis.

76. CAS Plot the surface $\rho = 1 - \cos\phi$. Then plot the trace of S in the xz-plane and explain why S is obtained by rotating this trace.

SOLUTION The surface $\rho = 1 - \cos\phi$ and its trace in the xz-plane are shown in the following figures:



The trace of S in the xz-plane



The surface $S : \rho = 1 - \cos\phi$

Since the equation of the surface does not involve the coordinate θ, we conclude by Exercise 67 that the surface is rotationally symmetric with respect to the z-axis. Therefore, the points on the surface are obtained by rotating its trace in the xz-plane (or its trace in any other vertical plane) about the z-axis.

77. Find equations $r = g(\theta, z)$ (cylindrical) and $\rho = f(\theta, \phi)$ (spherical) for the hyperboloid $x^2 + y^2 = z^2 + 1$ (Figure 16). Do there exist points on the hyperboloid with $\phi = 0$ or π? Which values of ϕ occur for points on the hyperboloid?

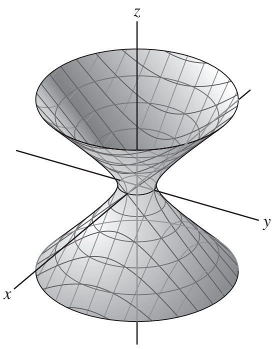


FIGURE 16 The hyperboloid $x^2 + y^2 = z^2 + 1$.

SOLUTION For the cylindrical coordinates (r, θ, z) we have $x^2 + y^2 = r^2$. Substituting into the equation $x^2 + y^2 = z^2 + 1$ gives

$$r^2 = z^2 + 1 \quad \Rightarrow \quad r = \sqrt{z^2 + 1}$$

For the spherical coordinates (ρ, θ, ϕ) we have $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$. We substitute into the equation of the hyperboloid $x^2 + y^2 = z^2 + 1$ and simplify to obtain

$$\begin{aligned} \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta &= \rho^2 \cos^2\phi + 1 \\ \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) &= \rho^2 \cos^2\phi + 1 \\ \rho^2 \left(\sin^2\phi - \cos^2\phi\right) &= 1 \end{aligned}$$

Using the trigonometric identity $\cos 2\phi = \cos^2\phi - \sin^2\phi$ we get

$$\rho^2 \cdot \left(-\cos 2\phi\right) = 1 \quad \Rightarrow \quad \rho = \sqrt{-\frac{1}{\cos 2\phi}}$$

For $\phi = 0$ and $\phi = \pi$ we have $\cos 2 \cdot 0 = 1$ and $\cos 2\pi = 1$. In both cases $-\frac{1}{\cos 2\phi} = -1 < 0$, hence there is no real value of ρ satisfying $\rho = \sqrt{-\frac{1}{\cos 2\phi}}$. We conclude that there are no points on the hyperboloid with $\phi = 0$ or π.

To obtain a real ρ such that $\rho = \sqrt{-\frac{1}{\cos 2\phi}}$, we must have $-\frac{1}{\cos 2\phi} > 0$. That is, $\cos 2\phi < 0$ (and of course $0 \le \phi \le \pi$). The corresponding values of ϕ are

$$\frac{\pi}{2} < 2\phi \le \frac{3\pi}{2} \quad \Rightarrow \quad \frac{\pi}{4} < \phi \le \frac{3\pi}{4}$$

Further Insights and Challenges

*In Exercises 78–82, a **great circle** on a sphere S with center O is a circle obtained by intersecting S with a plane that passes through O (Figure 17). If P and Q are not antipodal (on opposite sides), there is a unique great circle through P and Q on S (intersect S with the plane through O, P, and Q). The geodesic distance from P to Q is defined as the length of the smaller of the two circular arcs of this great circle.*

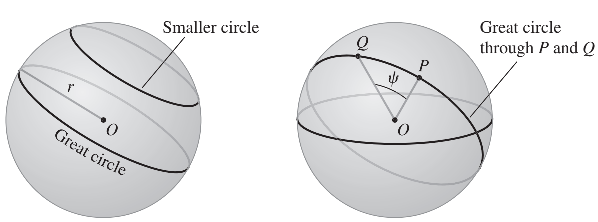


FIGURE 17

78. Show that the geodesic distance from P to Q is equal to $R\psi$, where ψ is the *central angle* between P and Q (the angle between the vectors $\mathbf{v} = \overrightarrow{OP}$ and $\mathbf{u} = \overrightarrow{OQ}$).

SOLUTION We place the xy-coordinate system in the plane of the great circle determined by P and Q, so that O is the origin and the positive x-axis is in the direction of $\overrightarrow{OP}$.

The parametric equation of the circle with the angular coordinate θ as the parameter is $x = R\cos\theta$, $y = R\sin\theta$. Also, P corresponds to $\theta = 0$ and Q corresponds to $\theta = \psi$. By the formula for the arc length we have

$$\overrightarrow{PQ} = \int_0^{\psi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \int_0^{\psi} \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}\,d\theta$$
$$= \int_0^{\psi} \sqrt{R^2\left(\sin^2\theta + \cos^2\theta\right)}\,d\theta = R\int_0^{\psi} 1\cdot d\theta = R\theta\Big|_0^{\psi} = R\psi$$

.

79. Show that the geodesic distance from $Q = (a, b, c)$ to the North Pole $P = (0, 0, R)$ is equal to $R\cos^{-1}\left(\frac{c}{R}\right)$.

SOLUTION Let ψ be the central angle between P and Q, that is, the angle between the vectors $\mathbf{v} = \overrightarrow{OP}$ and $\mathbf{u} = \overrightarrow{OQ}$. By Exercise 78 the geodesic distance from P to Q is $R\psi$. We find ψ. By the formula for the cosine of the angle between two vectors, we have

$$\cos\psi = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{1}$$

We compute the values in this quotient:

$$\mathbf{u}\cdot\mathbf{v} = \langle 0, 0, R\rangle \cdot \langle a, b, c\rangle = 0 + 0 + Rc = Rc$$
$$\|\mathbf{v}\| = \|\overrightarrow{OP}\| = R$$
$$\|\mathbf{u}\| = \|\overrightarrow{OQ}\| = \sqrt{a^2 + b^2 + c^2} = R$$

Substituting in (1) we get

$$\cos\psi = \frac{Rc}{R^2} = \frac{c}{R} \quad\Rightarrow\quad \psi = \cos^{-1}\left(\frac{c}{R}\right)$$

The geodesic distance from Q to P is thus

$$R\psi = R\cos^{-1}\left(\frac{c}{R}\right)$$

80. The coordinates of Los Angeles are 34° N and 118° W. Find the geodesic distance from the North Pole to Los Angeles, assuming that the earth is a sphere of radius $R = 6370$ km.

SOLUTION We denote by C the z-coordinate of the point Q on the surface of the earth where Los Angeles is located. Then by the previous exercise, the geodesic distance from Q to the north pole is

$$R\cos^{-1}\left(\frac{C}{R}\right) \tag{1}$$

To find C, we first must find the angles (θ, ϕ) for Los Angeles. The angle $118^\circ W$ refers to 118° in the negative θ direction, hence $\theta = 360^\circ - 118^\circ = 242^\circ$. The latitude is north hence $\phi = 90^\circ - 34^\circ = 56^\circ$. The z coordinate of Q is thus

$$C = R\cos\phi = R\cos 56^\circ = 0.56R$$

Substituting in (1) we obtain the following geodesic distance (in this formula the angle is measured in radians):

$$R\cos^{-1}\left(\frac{C}{R}\right) = 6370\cos^{-1}\left(\frac{0.56R}{R}\right) = 6370\cos^{-1}(0.56) = 6225.9km$$

81. Show that the central angle ψ between points P and Q on a sphere (of any radius) with angular coordinates (θ, ϕ) and (θ', ϕ') is equal to

$$\psi = \cos^{-1}\bigl(\sin\phi\sin\phi'\cos(\theta - \theta') + \cos\phi\cos\phi'\bigr)$$

Hint: Compute the dot product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Check this formula by computing the geodesic distance between the North and South Poles.

SOLUTION We denote the vectors $\mathbf{u} = \overrightarrow{OP}$ and $\mathbf{v} = \overrightarrow{OQ}$. By the formula for the angle between two vectors we have

$$\psi = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

Denoting by R the radius of the sphere, we have $\|\mathbf{u}\| = \|\mathbf{v}\| = R$, hence,

$$\psi = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{R^2}\right) \tag{1}$$

The rectangular coordinates of $\mathbf{u}$ and $\mathbf{v}$ are

u	v
$x = R\sin\phi\cos\theta$	$x' = R\sin\phi'\cos\theta'$
$y = R\sin\phi\sin\theta$	$y' = R\sin\phi'\sin\theta'$
$z = R\cos\phi$	$z' = R\cos\phi'$

Hence,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= R^2\sin\phi\cos\theta\sin\phi'\cos\theta' + R^2\sin\phi\sin\theta\sin\phi'\sin\theta' + R^2\cos\phi\cos\phi' \\ &= R^2\left[\sin\phi\sin\phi'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\right) + \cos\phi\cos\phi'\right]\end{aligned}$$

We use the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to obtain

$$\mathbf{u}\cdot\mathbf{v} = R^2\left(\sin\phi\sin\phi'\cos\left(\theta - \theta'\right) + \cos\phi\cos\phi'\right)$$

Substituting in (1) we obtain

$$\psi = \cos^{-1}\left(\sin\phi\sin\phi'\cos\left(\theta - \theta'\right) + \cos\phi\cos\phi'\right) \tag{2}$$

We now check this formula in the case where P and Q are the north and south poles respectively. In this case $\theta = \theta' = 0$, $\phi = 0$, $\phi' = \pi$. Substituting in (2) gives

$$\psi = \cos^{-1}(\sin 0\sin\pi\cos 0 + \cos 0\cos\pi) = \cos^{-1}(-1) = \pi$$

Using Exercise 78, the geodesic distance between the two poles is $R\psi = R\pi$, in accordance with the formula for the length of a semicircle.

82. Use Exercise 81 to find the geodesic distance between Los Angeles (34° N, 118° W) and Bombay (19° N, 72.8° E).

SOLUTION By Exercise 80 the angles (θ, ϕ) for Los Angeles are $(\theta, \phi) = \left(242^\circ, 56^\circ\right)$. The angles (θ, ϕ) for Bombay are

$$\theta' = 72.8^\circ$$
$$\phi' = 90^\circ - 19^\circ = 71^\circ$$

By Exercise 81, the central angle ψ between the two towns is

$$\psi = \cos^{-1}\left(\sin\phi\sin\phi'\cos\left(\theta - \theta'\right) + \cos\phi\cos\phi'\right)$$

Substituting the angles gives

$$\begin{aligned}\psi &= \cos^{-1}\left(\sin 56^\circ\sin 71^\circ\cos\left(242^\circ - 72.8^\circ\right) + \cos 56^\circ\cos 71^\circ\right) \\ &= \cos^{-1}(-0.77 + 0.18) = \cos^{-1}(-0.59)\end{aligned}$$

The solution for $0 \le \psi \le 180$ is $\psi = 126.2^\circ = 2.2$ rad. Using Exercise 78 with $R = 6370$ km we conclude that the geodesic distance between the two towns is

$$R\psi = 6370 \cdot 2.2 = 14{,}014 \text{ km}.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–6, let $\mathbf{v} = \langle -2, 5\rangle$ *and* $\mathbf{w} = \langle 3, -2\rangle$.

1. Calculate $5\mathbf{w} - 3\mathbf{v}$ and $5\mathbf{v} - 3\mathbf{w}$.

SOLUTION We use the definition of basic vector operations to compute the two linear combinations:

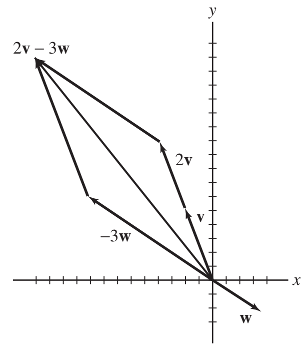
$$5\mathbf{w} - 3\mathbf{v} = 5\langle 3, -2\rangle - 3\langle -2, 5\rangle = \langle 15, -10\rangle + \langle 6, -15\rangle = \langle 21, -25\rangle$$
$$5\mathbf{v} - 3\mathbf{w} = 5\langle -2, 5\rangle - 3\langle 3, -2\rangle = \langle -10, 25\rangle + \langle -9, 6\rangle = \langle -19, 31\rangle$$

2. Sketch **v**, **w**, and 2**v** − 3**w**.

SOLUTION We have,

$$2\mathbf{v} - 3\mathbf{w} = 2\langle -2, 5\rangle - 3\langle 3, -2\rangle = \langle -4, 10\rangle + \langle -9, 6\rangle = \langle -13, 16\rangle$$

The vectors **v**, **w** and 2**v** − 3**w** are shown in the figure below:



3. Find the unit vector in the direction of **v**.

SOLUTION The unit vector in the direction of **v** is

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We compute the length of **v**:

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$$

Hence,

$$\mathbf{e_v} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{29}} = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

4. Find the length of **v** + **w**.

SOLUTION We first compute the sum **v** + **w**:

$$\mathbf{v} + \mathbf{w} = \langle -2, 5\rangle + \langle 3, -2\rangle = \langle -2 + 3, 5 - 2\rangle = \langle 1, 3\rangle$$

Using the definition of the length of a vector we obtain

$$\|\mathbf{v} + \mathbf{w}\| = \|\langle 1, 3\rangle\| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

5. Express **i** as a linear combination $r\mathbf{v} + s\mathbf{w}$.

SOLUTION We use basic properties of vector algebra to write

$$\mathbf{i} = r\mathbf{v} + s\mathbf{w} \tag{1}$$

$$\langle 1, 0\rangle = r\langle -2, 5\rangle + s\langle 3, -2\rangle = \langle -2r + 3s, 5r - 2s\rangle$$

The vector are equivalent, hence,

$$1 = -2r + 3s$$
$$0 = 5r - 2s$$

The second equation implies that $s = \frac{5}{2}r$. We substitute in the first equation and solve for r:

$$1 = -2r + 3 \cdot \frac{5}{2}r$$
$$1 = \frac{11}{2}r$$
$$r = \frac{2}{11} \quad \Rightarrow \quad s = \frac{5}{2} \cdot \frac{2}{11} = \frac{5}{11}$$

Substituting in (1) we obtain

$$\mathbf{i} = \frac{2}{11}\mathbf{v} + \frac{5}{11}\mathbf{w}.$$

6. Find a scalar α such that $\|\mathbf{v} + \alpha\mathbf{w}\| = 6$.

SOLUTION We compute the vector $\mathbf{v} + \alpha\mathbf{w}$:

$$\mathbf{v} + \alpha\mathbf{w} = \langle -2, 5\rangle + \alpha\langle 3, -2\rangle = \langle -2 + 3\alpha, 5 - 2\alpha\rangle$$

The length of $\mathbf{v} + \alpha\mathbf{w}$ is

$$\begin{aligned}\|\mathbf{v} + \alpha\mathbf{w}\| &= \sqrt{(-2+3\alpha)^2 + (5-2\alpha)^2} = \sqrt{4 - 12\alpha + 9\alpha^2 + 25 - 20\alpha + 4\alpha^2}\\ &= \sqrt{13\alpha^2 - 32\alpha + 29}\end{aligned}$$

We obtain the following equation:

$$\sqrt{13\alpha^2 - 32\alpha + 29} = 6$$

Solving for α yields

$$\begin{aligned}13\alpha^2 - 32\alpha + 29 &= 36\\ 13\alpha^2 - 32\alpha - 7 &= 0\\ \alpha_{1,2} = \frac{32 \pm \sqrt{32^2 - 4\cdot 13\cdot(-7)}}{26} &= \frac{16 \pm \sqrt{347}}{13}\end{aligned}$$

The two solutions are thus

$$\alpha = \frac{16 \pm \sqrt{347}}{13}.$$

7. If $P = (1, 4)$ and $Q = (-3, 5)$, what are the components of $\overrightarrow{PQ}$? What is the length of $\overrightarrow{PQ}$?

SOLUTION By the Definition of Components of a Vector we have

$$\overrightarrow{PQ} = \langle -3 - 1, 5 - 4\rangle = \langle -4, 1\rangle$$

The length of $\overrightarrow{PQ}$ is

$$\|\overrightarrow{PQ}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

8. Let $A = (2, -1)$, $B = (1, 4)$, and $P = (2, 3)$. Find the point Q such that $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$. Sketch $\overrightarrow{PQ}$ and $\overrightarrow{AB}$.

SOLUTION The vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent, therefore they have the same components. We denote the point Q by $Q = (a, b)$, and compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$. We get

$$\begin{aligned}\overrightarrow{AB} &= \langle 1 - 2, 4 - (-1)\rangle = \langle -1, 5\rangle\\ \overrightarrow{PQ} &= \langle a - 2, b - 3\rangle\end{aligned}$$

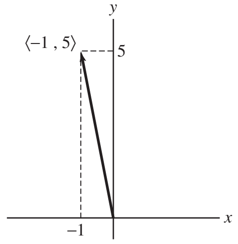
Therefore

$$-1 = a - 2 \quad \text{and} \quad 5 = b - 3$$

or

$$a = 1 \quad \text{and} \quad b = 8$$

Hence the point Q is $Q = (1, 8)$. The equivalent vectors $\overrightarrow{AB} = \overrightarrow{PQ} = \langle -1, 5\rangle$ are shown in the figure:



9. Find the vector with length 3 making an angle of $\frac{7\pi}{4}$ with the positive x-axis.

SOLUTION We denote the vector by $\mathbf{v} = \langle a, b\rangle$. $\mathbf{v}$ makes an angle $\theta = \frac{7\pi}{4}$ with the x-axis, and its length is 3, hence,

$$a = \|\mathbf{v}\| \cos\theta = 3\cos\frac{7\pi}{4} = \frac{3}{\sqrt{2}}$$
$$b = \|\mathbf{v}\| \sin\theta = 3\sin\frac{7\pi}{4} = -\frac{3}{\sqrt{2}}$$

That is,

$$\mathbf{v} = \langle a, b\rangle = \left\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right\rangle.$$

10. Calculate $3(\mathbf{i} - 2\mathbf{j}) - 6(\mathbf{i} + 6\mathbf{j})$.

SOLUTION Using basic properties of vector algebra we have

$$3(\mathbf{i} - 2\mathbf{j}) - 6(\mathbf{i} + 6\mathbf{j}) = 3\mathbf{i} - 6\mathbf{j} - 6\mathbf{i} - 36\mathbf{j} = -3\mathbf{i} - 42\mathbf{j}$$

11. Find the value of β for which $\mathbf{w} = \langle -2, \beta\rangle$ is parallel to $\mathbf{v} = \langle 4, -3\rangle$.

SOLUTION If $\mathbf{v} = \langle 4, -3\rangle$ and $\mathbf{w} = \langle -2, \beta\rangle$ are parallel, there exists a scalar λ such that $\mathbf{w} = \lambda\mathbf{v}$. That is,

$$\langle -2, \beta\rangle = \lambda\langle 4, -3\rangle = \langle 4\lambda, -3\lambda\rangle$$

yielding

$$-2 = 4\lambda \quad \text{and} \quad \beta = -3\lambda$$

These equations imply that $\lambda = -\frac{1}{2}$ and $\lambda = -\frac{\beta}{3}$. Equating the two expressions for λ gives

$$-\frac{1}{2} = -\frac{\beta}{3} \quad \text{or} \quad \beta = \frac{3}{2}.$$

12. Let $P = (1, 4, -3)$.

(a) Find the point Q such that $\overrightarrow{PQ}$ is equivalent to $\langle 3, -1, 5\rangle$.

(b) Find a unit vector $\mathbf{e}$ equivalent to $\overrightarrow{PQ}$.

SOLUTION

(a) Let $Q = (a, b, c)$. Since the vectors $\overrightarrow{PQ}$ and $\langle 3, -1, 5\rangle$ are equivalent they have the same components. That is,

$$\overrightarrow{PQ} = \langle a - 1, b - 4, c - (-3)\rangle = \langle 3, -1, 5\rangle$$

Hence,

$$\begin{array}{lcl} a - 1 = 3 & & a = 4 \\ b - 4 = -1 & \Rightarrow & b = 3 \\ c + 3 = 5 & & c = 2 \end{array}$$

The point Q is thus $Q = (4, 3, 2)$.

(b) The unit vector $\mathbf{e}$ is obtained by dividing $\overrightarrow{PQ}$ by its length:

$$\mathbf{e} = \frac{\overrightarrow{PQ}}{\left\|\overrightarrow{PQ}\right\|} = \frac{\langle 3, -1, 5\rangle}{\sqrt{3^2 + (-1)^2 + 5^2}} = \left\langle \frac{3}{\sqrt{35}}, -\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}} \right\rangle$$

Notice that the opposite vector $-\mathbf{e}$ is also a solution.

13. Let $\mathbf{w} = \langle 2, -2, 1\rangle$ and $\mathbf{v} = \langle 4, 5, -4\rangle$. Solve for $\mathbf{u}$ if $\mathbf{v} + 5\mathbf{u} = 3\mathbf{w} - \mathbf{u}$.

SOLUTION Using vector algebra we have

$$\begin{aligned} \mathbf{v} + 5\mathbf{u} &= 3\mathbf{w} - \mathbf{u} \\ 6\mathbf{u} &= 3\mathbf{w} - \mathbf{v} \\ \mathbf{u} &= \frac{1}{2}\mathbf{w} - \frac{1}{6}\mathbf{v} = \left\langle 1, -1, \frac{1}{2} \right\rangle - \left\langle \frac{4}{6}, \frac{5}{6}, -\frac{4}{6} \right\rangle = \left\langle \frac{1}{3}, -\frac{11}{6}, \frac{7}{6} \right\rangle \end{aligned}$$

14. Let $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find the length of $\mathbf{v}$ and the vector $2\mathbf{v} + 3(4\mathbf{i} - \mathbf{k})$.

SOLUTION We first compute the length of the vector $\mathbf{v}$:

$$\|\mathbf{v}\| = \|3\mathbf{i} - \mathbf{j} + 4\mathbf{k}\| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$$

We find the vector $2\mathbf{v} + 3(4\mathbf{i} - \mathbf{k})$ using properties of vector algebra:

$$\begin{aligned} 2\mathbf{v} + 3(4\mathbf{i}-\mathbf{k}) &= 2(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + 3(4\mathbf{i} - \mathbf{k}) = 6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + 12\mathbf{i} - 3\mathbf{k} \\ &= 18\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} \end{aligned}$$

15. Find a parametrization $\mathbf{r}_1(t)$ of the line passing through $(1, 4, 5)$ and $(-2, 3, -1)$. Then find a parametrization $\mathbf{r}_2(t)$ of the line parallel to $\mathbf{r}_1$ passing through $(1, 0, 0)$.

SOLUTION Since the points $P = (-2, 3, -1)$ and $Q = (1, 4, 5)$ are on the line l_1, the vector $\overrightarrow{PQ}$ is a direction vector for the line. We find this vector:

$$\overrightarrow{PQ} = \langle 1 - (-2), 4 - 3, 5 - (-1) \rangle = \langle 3, 1, 6 \rangle$$

Substituting $\mathbf{v} = \langle 3, 1, 6 \rangle$ and $P_0 = \langle 1, 4, 5 \rangle$ in the vector parametrization of the line we obtain the following equation for l_1:

$$\mathbf{r}_1(t) = \overrightarrow{OP_0} + t\mathbf{v}$$

$$\mathbf{r}_1(t) = \langle 1, 4, 5 \rangle + t\langle 3, 1, 6 \rangle = \langle 1 + 3t, 4 + t, 5 + 6t \rangle$$

The line l_2 is parallel to l_1, hence $\overrightarrow{PQ} = \langle 3, 1, 6 \rangle$ is also a direction vector for l_2. Substituting $\mathbf{v} = \langle 3, 1, 6 \rangle$ and $P_0 = (1, 0, 0)$ in the vector parametrization of the line we obtain the following equation for l_2:

$$\mathbf{r}_2(t) = \overrightarrow{OP_0} + t\mathbf{v}$$

$$\mathbf{r}_2(t) = \langle 1, 0, 0 \rangle + t\langle 3, 1, 6 \rangle = \langle 1 + 3t, t, 6t \rangle$$

16. Let $\mathbf{r}_1(t) = \mathbf{v}_1 + t\mathbf{w}_1$ and $\mathbf{r}_2(t) = \mathbf{v}_2 + t\mathbf{w}_2$ be parametrizations of lines $\mathcal{L}_1$ and $\mathcal{L}_2$. For each statement (a)–(e), provide a proof if the statement is true and a counterexample if it is false.

(a) If $\mathcal{L}_1 = \mathcal{L}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

(b) If $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$, then $\mathbf{w}_1 = \mathbf{w}_2$.

(c) If $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$.

(d) If $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$, then $\mathbf{w}_1 = \mathbf{w}_2$.

(e) If $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$, then $\mathbf{w}_1 = \lambda\mathbf{w}_2$ for some scalar λ.

SOLUTION

(a) This statement is false. Consider the following lines:

$$\mathcal{L}_1: \quad \mathbf{r}_1(t) = \langle 1, 0, 1 \rangle + t\langle 1, 1, 1 \rangle$$

$$\mathcal{L}_2: \quad \mathbf{r}_2(t) = \langle 3, 2, 3 \rangle + t\langle 2, 2, 2 \rangle$$

The line $\mathcal{L}_1$ passes through the points $P = (1, 0, 1)$ (for $t = 0$) and $Q = (2, 1, 2)$ (for $t = 1$). The line $\mathcal{L}_2$ passes through P and Q as well (for $t = -1$ and $t = -\frac{1}{2}$ respectively). Therefore, $\mathcal{L}_1 = \mathcal{L}_2$. However, $\mathbf{v}_1 = \langle 1, 0, 1 \rangle$, $\mathbf{v}_2 = \langle 3, 2, 3 \rangle$, $\mathbf{w}_1 = \langle 1, 1, 1 \rangle$, $\mathbf{w}_2 = \langle 2, 2, 2 \rangle$ hence $\mathbf{v}_1 \neq \mathbf{v}_2$ and $\mathbf{w}_1 \neq \mathbf{w}_2$.

(b) This statement is false. Consider the following lines:

$$\mathcal{L}_1: \quad \mathbf{r}_1(t) = \langle 0, 1, 0 \rangle + t\langle 1, 1, 1 \rangle$$

$$\mathcal{L}_2: \quad \mathbf{r}_2(t) = \langle 0, 1, 0 \rangle + t\langle 2, 2, 2 \rangle$$

The line $\mathcal{L}_1$ passes through the points $P = (0, 1, 0)$ (for $t = 0$) and $Q = (1, 2, 1)$ (for $t = 1$). The line $\mathcal{L}_2$ passes through P and Q as well (for $t = 0$ and $t = \frac{1}{2}$). Therefore, $\mathcal{L}_1 = \mathcal{L}_2$. Also $\mathbf{v}_1 = \mathbf{v}_2$, but $\mathbf{w}_1 \neq \mathbf{w}_2$.

(c) This statement is false. Consider the following lines:

$$\mathcal{L}_1: \quad \mathbf{r}_1(t) = \langle 1, 0, 1 \rangle + t\langle 1, 1, 1 \rangle$$

$$\mathcal{L}_2: \quad \mathbf{r}_2(t) = \langle 2, 1, 2 \rangle + t\langle 1, 1, 1 \rangle$$

The line $\mathcal{L}_1$ passes through $P = (1, 0, 1)$ and $Q = (2, 1, 2)$ (for $t = 1$). The line $\mathcal{L}_2$ passes through $P = (1, 0, 1)$ (for $t = -1$) and $Q = (2, 1, 2)$ (for $t = 0$). Therefore, $\mathcal{L}_1 = \mathcal{L}_2$. Also, $\mathbf{w}_1 = \mathbf{w}_2$ but $\mathbf{v}_1 \neq \mathbf{v}_2$.

(d) This statement is false. Consider the following lines:

$$\mathcal{L}_1: \quad \mathbf{r}_1(t) = \langle 1, 1, 1 \rangle + t\langle 1, 0, 1 \rangle$$
$$\mathcal{L}_2: \quad \mathbf{r}_2(t) = t\langle 2, 0, 2 \rangle$$

We have $\mathbf{w}_1 = \langle 1, 0, 1 \rangle$ and $\mathbf{w}_2 = \langle 2, 0, 2 \rangle$ therefore $\mathbf{w}_2 = 2\mathbf{w}_1$. We conclude that $\mathbf{w}_1$ and $\mathbf{w}_2$ are parallel vectors, hence the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel although $\mathbf{w}_1 \neq \mathbf{w}_2$.

(e) This statement is correct. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel lines, the direction vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ of these lines are parallel, hence they are scalar multiples of one another.

17. Find a and b such that the lines $\mathbf{r}_1 = \langle 1, 2, 1 \rangle + t\langle 1, -1, 1 \rangle$ and $\mathbf{r}_2 = \langle 3, -1, 1 \rangle + t\langle a, b, -2 \rangle$ are parallel.

SOLUTION The lines are parallel if and only if the direction vectors $\mathbf{v}_1 = \langle 1, -1, 1 \rangle$ and $\mathbf{v}_2 = \langle a, b, -2 \rangle$ are parallel. That is, if and only if there exists a scalar λ such that:

$$\mathbf{v}_2 = \lambda \mathbf{v}_1$$
$$\langle a, b, -2 \rangle = \lambda\langle 1, -1, 1 \rangle = \langle \lambda, -\lambda, \lambda \rangle$$

We obtain the following equations:

$$\begin{aligned} a &= \lambda \\ b &= -\lambda \quad \Rightarrow \quad a = -2, \quad b = 2 \\ -2 &= \lambda \end{aligned}$$

18. Find a such that the lines $\mathbf{r}_1 = \langle 1, 2, 1 \rangle + t\langle 1, -1, 1 \rangle$ and $\mathbf{r}_2 = \langle 3, -1, 1 \rangle + t\langle a, 4, -2 \rangle$ intersect.

SOLUTION If the lines intersect, there exists a point that lies on both lines. That is, there are unique values of t and s such that:

$$\langle 1, 2, 1 \rangle + t\langle 1, -1, 1 \rangle = \langle 3, -1, 1 \rangle + s\langle a, 4, -2 \rangle$$

or

$$\langle 1 + t, 2 - t, 1 + t \rangle = \langle 3 + as, -1 + 4s, 1 - 2s \rangle$$

yielding the following equations:

$$\begin{aligned} 1 + t &= 3 + as & \qquad & t = 2 + as \\ 2 - t &= -1 + 4s & \Rightarrow \quad & t = 3 - 4s \\ 1 + t &= 1 - 2s & & t = -2s \end{aligned}$$

Equating the expressions for t gives the following equations:

$$\begin{aligned} 2 + as &= -2s \\ 3 - 4s &= -2s \end{aligned} \quad \Rightarrow \quad \begin{aligned} (a+2)s &= -2 \\ 2s &= 3 \end{aligned} \quad \Rightarrow \quad s = \frac{3}{2}$$

Substituting $s = \frac{3}{2}$ in the first equation we obtain:

$$(a + 2) \cdot \frac{3}{2} = -2 \quad \Rightarrow \quad a = -\frac{10}{3}.$$

19. Sketch the vector sum $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ for the vectors in Figure 1(A).

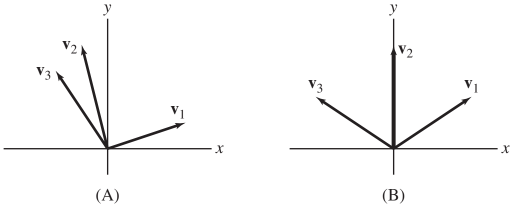
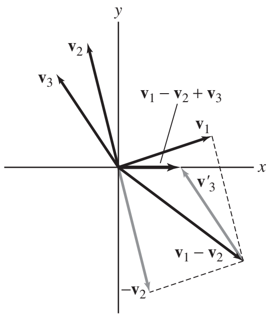


FIGURE 1

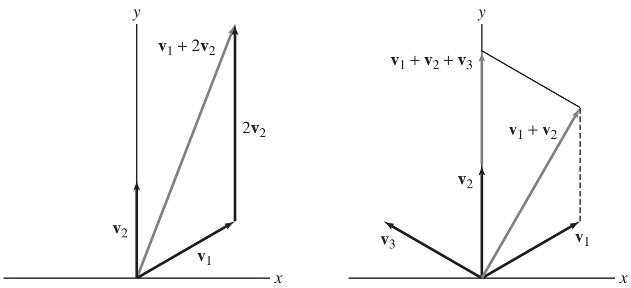
SOLUTION Using the Parallelogram Law we obtain the vector sum shown in the figure.

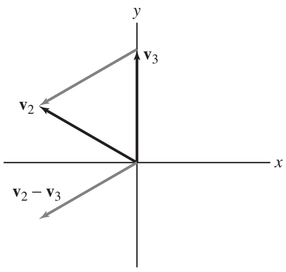


We first add $\mathbf{v}_1$ and $-\mathbf{v}_2$, then we add $\mathbf{v}_3$ to $\mathbf{v}_1 - \mathbf{v}_2$.

20. Sketch the sums $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + 2\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_3$ for the vectors in Figure 1(B).

SOLUTION We use the definition of scalar multiple of a vector and the Parallelogram Law to sketch the vectors.





To form $\mathbf{v}_2 - \mathbf{v}_3$, we draw the vector pointing from $\mathbf{v}_3$ to $\mathbf{v}_2$ and translate it back to the basepoint.

In Exercises 21–26, let $\mathbf{v} = \langle 1, 3, -2 \rangle$ *and* $\mathbf{w} = \langle 2, -1, 4 \rangle$.

21. Compute $\mathbf{v} \cdot \mathbf{w}$.

SOLUTION Using the definition of the dot product we have

$$\mathbf{v} \cdot \mathbf{w} = \langle 1, 3, -2 \rangle \cdot \langle 2, -1, 4 \rangle = 1 \cdot 2 + 3 \cdot (-1) + (-2) \cdot 4 = 2 - 3 - 8 = -9$$

22. Compute the angle between $\mathbf{v}$ and $\mathbf{w}$.

SOLUTION The cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \tag{1}$$

We compute the lengths of the vectors:

$$\|\mathbf{v}\| = \|\langle 1, 3, -2 \rangle\| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

$$\|\mathbf{w}\| = \|\langle 2, -1, 4 \rangle\| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$$

In the previous exercise we found that $\mathbf{v} \cdot \mathbf{w} = -9$. Substituting these values in (1) gives

$$\cos\theta = \frac{-9}{\sqrt{14} \cdot \sqrt{21}} = \frac{-9}{7\sqrt{6}} \approx -0.5249$$

The solution for $0 \le \theta \le \pi$ is

$$\theta = 2.123 \text{ rad}.$$

23. Compute $\mathbf{v} \times \mathbf{w}$.

SOLUTION We use the definition of the cross product as a "determinant":

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \mathbf{k}$$
$$= (12-2)\mathbf{i} - (4+4)\mathbf{j} + (-1-6)\mathbf{k} = 10\mathbf{i} - 8\mathbf{j} - 7\mathbf{k} = \langle 10, -8, -7 \rangle$$

24. Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

SOLUTION The parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ has area $\|\mathbf{v} \times \mathbf{w}\|$. In the previous exercise, we found that $\mathbf{v} \times \mathbf{w} = \langle 10, -8, -7 \rangle$. Therefore the area A of the parallelogram is

$$A = \|\mathbf{v} \times \mathbf{w}\| = \|\langle 10, -8, -7 \rangle\| = \sqrt{10^2 + (-8)^2 + (-7)^2} = \sqrt{213} \approx 14.59$$

25. Find the volume of the parallelepiped spanned by $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u} = \langle 1, 2, 6 \rangle$.

SOLUTION The volume V of the parallelepiped spanned by $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ is the following determinant:

$$V = \left| \det \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix} \right| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 6 \end{vmatrix} = \left| 1 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 6 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \right|$$
$$= |1 \cdot (-6-8) - 3(12-4) - 2(4+1)| = 48$$

26. Find all the vectors orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

SOLUTION A vector $\mathbf{u} = \langle a, b, c \rangle$ is orthogonal to $\mathbf{v}$ and to $\mathbf{w}$ if the dot products $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$ are zero. That is,

$$\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w} = 0.$$

We compute the dot products:

$$\mathbf{u} \cdot \mathbf{v} = \langle a, b, c \rangle \cdot \langle 1, 3, -2 \rangle = a + 3b - 2c$$
$$\mathbf{u} \cdot \mathbf{w} = \langle a, b, c \rangle \cdot \langle 2, -1, 4 \rangle = 2a - b + 4c$$

We obtain the following equations:

$$a + 3b - 2c = 0$$
$$2a - b + 4c = 0$$

The first equation implies $a = 2c - 3b$. Substituting in the second equation and solving for b in terms of c gives

$$2(2c - 3b) - b + 4c = 0$$
$$4c - 6b - b + 4c = 0$$
$$8c - 7b = 0 \quad \Rightarrow \quad b = \frac{8}{7}c$$

We find a in terms of c, using the relation $a = 2c - 3b$:

$$a = 2c - 3 \cdot \frac{8}{7}c = 2c - \frac{24}{7}c = -\frac{10}{7}c.$$

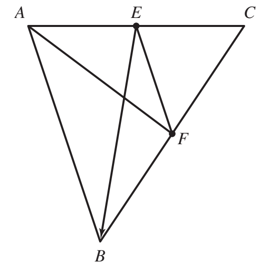
The solutions are, thus,

$$\mathbf{u} = \langle a, b, c \rangle = \left\langle -\frac{10}{7}c, \frac{8}{7}c, c \right\rangle = -\frac{c}{7}\langle 10, -8, -7 \rangle$$

We conclude that the vectors orthogonal to $\mathbf{v}$ and $\mathbf{w}$ are all the vectors parallel to $\langle 10, -8, -7 \rangle$.

27. Use vectors to prove that the line connecting the midpoints of two sides of a triangle is parallel to the third side.

SOLUTION Let E and F be the midpoints of sides AC and BC in a triangle ABC (see figure).



We must show that

$$\overrightarrow{EF} \parallel \overrightarrow{AB}$$

Using the Parallelogram Law we have

$$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF} \tag{1}$$

By the definition of the points E and F,

$$\overrightarrow{EA} = \frac{1}{2}\overrightarrow{CA}; \quad \overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$$

We substitute (1) to obtain

$$\begin{aligned}\overrightarrow{EF} &= \frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{BC}\right)\\ &= \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CA}\right) = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} = \overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AB}\end{aligned}$$

Therefore, $\overrightarrow{EF}$ is a constant multiple of $\overrightarrow{AB}$, which implies that $\overrightarrow{EF}$ and $\overrightarrow{AB}$ are parallel vectors.

28. Let $\mathbf{v} = \langle 1, -1, 3\rangle$ and $\mathbf{w} = \langle 4, -2, 1\rangle$.

(a) Find the decomposition $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ with respect to $\mathbf{w}$.

(b) Find the decomposition $\mathbf{w} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}$ with respect to $\mathbf{v}$.

SOLUTION

(a) Step 1. Compute $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{w}$. We have the following dot products:

$$\mathbf{v} \cdot \mathbf{w} = \langle 1, -1, 3\rangle \cdot \langle 4, -2, 1\rangle = 4 + 2 + 3 = 9$$
$$\mathbf{w} \cdot \mathbf{w} = \langle 4, -2, 1\rangle \cdot \langle 4, -2, 1\rangle = 16 + 4 + 1 = 21$$

Step 2. Use the formula for $\mathbf{v}_{\parallel}$. We use the following formula:

$$\mathbf{v}_{\parallel} = \text{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w}$$

Substituting the dot products from the previous step, we get:

$$\mathbf{v}_{\parallel} = \frac{9}{21}\langle 4, -2, 1\rangle = \left\langle \frac{12}{7}, -\frac{6}{7}, \frac{3}{7}\right\rangle$$

Step 3. Identifying $\mathbf{v}_{\perp}$. The orthogonal part is the following difference:

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \langle 1, -1, 3\rangle - \left\langle \frac{12}{7}, -\frac{6}{7}, \frac{3}{7}\right\rangle = \left\langle -\frac{5}{7}, -\frac{1}{7}, \frac{18}{7}\right\rangle$$

The resulting decomposition is:

$$\mathbf{v} = \langle 1, -1, 3\rangle = \underbrace{\left\langle \frac{12}{7}, -\frac{6}{7}, \frac{3}{7}\right\rangle}_{\text{projection along } \mathbf{w}} + \underbrace{\left\langle -\frac{5}{7}, -\frac{1}{7}, \frac{18}{7}\right\rangle}_{\text{orthogonal to } \mathbf{w}}$$

(b) Step 1. Compute $\mathbf{w} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{v}$. In part (a) we found that $\mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = 9$. We compute the dot product $\mathbf{v} \cdot \mathbf{v}$:

$$\mathbf{v} \cdot \mathbf{v} = \langle 1, -1, 3\rangle \cdot \langle 1, -1, 3\rangle = 1 + 1 + 9 = 11$$

Step 2. Use the formula for $\mathbf{w}_{\parallel}$. We use the following formula:

$$\mathbf{w}_{\parallel} = \text{proj}_{\mathbf{v}}(\mathbf{w}) = \left(\frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}.$$

Substituting the dot products from the previous step, we get:

$$\mathbf{w}_{\parallel} = \frac{9}{11}\langle 1, -1, 3\rangle = \left\langle \frac{9}{11}, -\frac{9}{11}, \frac{27}{11}\right\rangle.$$

Step 3. Identifying $\mathbf{w}_{\perp}$. The orthogonal part is the following difference:

$$\mathbf{w}_{\perp} = \mathbf{w} - \mathbf{w}_{\parallel} = \langle 4, -2, 1\rangle - \left\langle \frac{9}{11}, -\frac{9}{11}, \frac{27}{11}\right\rangle = \left\langle \frac{35}{11}, -\frac{13}{11}, -\frac{16}{11}\right\rangle$$

We obtain the following decomposition:

$$\mathbf{w} = \langle 4, -2, 1\rangle = \underbrace{\left\langle \frac{9}{11}, -\frac{9}{11}, \frac{27}{11}\right\rangle}_{\text{projection along } \mathbf{v}} + \underbrace{\left\langle \frac{35}{11}, -\frac{13}{11}, -\frac{16}{11}\right\rangle}_{\text{orthogonalto } \mathbf{v}}$$

29. Calculate the component of $\mathbf{v} = \left\langle -2, \frac{1}{2}, 3\right\rangle$ along $\mathbf{w} = \langle 1, 2, 2\rangle$.

SOLUTION We first compute the following dot products:

$$\mathbf{v}\cdot\mathbf{w} = \langle -2, \frac{1}{2}, 3\rangle \cdot \langle 1, 2, 2\rangle = 5$$
$$\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 = 1^2 + 2^2 + 2^2 = 9$$

The component of $\mathbf{v}$ along $\mathbf{w}$ is the following number:

$$\left\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}\right\| = \frac{5}{9}\|\mathbf{w}\| = \frac{5}{9}\cdot 3 = \frac{5}{3}$$

30. Calculate the magnitude of the forces on the two ropes in Figure 2.

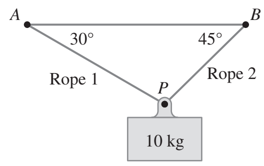
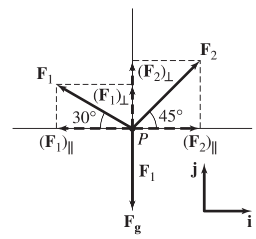


FIGURE 2

SOLUTION Gravity exerts a force $\mathbf{F}_g$ of magnitude $10g = 98$ newtons. Since the sum of $\mathbf{F}_1$ and $\mathbf{F}_2$ balance the force of gravity, we have

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_g = 0 \tag{1}$$

We resolve $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_g$ into a sum of a force along the ground and a force orthogonal to the ground.



This gives

$$\begin{aligned}
(\mathbf{F}_1)_{\parallel} &= -(\|\mathbf{F}_1\|\cos 30^\circ)\mathbf{i}, & (\mathbf{F}_1)_{\perp} &= (\|\mathbf{F}_1\|\cos 60^\circ)\mathbf{j}\\
(\mathbf{F}_2)_{\parallel} &= (\|\mathbf{F}_2\|\cos 45^\circ)\mathbf{i}, & (\mathbf{F}_2)_{\perp} &= (\|\mathbf{F}_2\|\cos 45^\circ)\mathbf{j}\\
(\mathbf{F}_g)_{\parallel} &= \mathbf{0}, & (\mathbf{F}_g)_{\perp} &= -10g\mathbf{j} \approx -98\mathbf{j}
\end{aligned}$$

Substituting these forces in (1) gives

$$\left(-\frac{\|\mathbf{F}_1\|\sqrt{3}}{2}\mathbf{i} + \frac{\|\mathbf{F}_1\|}{2}\mathbf{j}\right) + \left(\frac{\|\mathbf{F}_2\|\sqrt{2}}{2}\mathbf{i} + \frac{\|\mathbf{F}_2\|\sqrt{2}}{2}\mathbf{j}\right) - 98\mathbf{j} = 0$$

$$\frac{1}{2}\left(\sqrt{2}\|\mathbf{F}_2\| - \sqrt{3}\|\mathbf{F}_1\|\right)\mathbf{i} + \frac{1}{2}\left(\|\mathbf{F}_1\| + \sqrt{2}\|\mathbf{F}_2\| - 196\right)\mathbf{j} = 0$$

We now equate each component to zero, to obtain

$$\begin{aligned}\frac{\sqrt{2}\|\mathbf{F}_2\| - \sqrt{3}\|\mathbf{F}_1\|}{2} &= 0\\ \frac{\|\mathbf{F}_1\| + \sqrt{2}\|\mathbf{F}_2\|}{2} - 98 &= 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}\|\mathbf{F}_1\| &\approx 71.7\\ \|\mathbf{F}_2\| &\approx 87.8\end{aligned}$$

We conclude that

$$\mathbf{F}_1 = (-71.7\cos 30^\circ)\mathbf{i} + (71.7\cos 60^\circ)\mathbf{j} = -62.1\mathbf{i} + 35.9\mathbf{j}$$

$$\mathbf{F}_2 = (87.8\cos 45^\circ)\mathbf{i} + (87.8\cos 45^\circ)\mathbf{j} = 62.1\mathbf{i} + 62.1\mathbf{j}$$

(As in the statement of the problems, all units are in Newtons.)

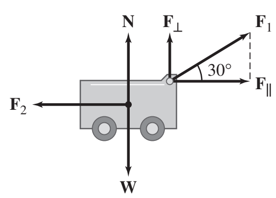
31. A 50-kg wagon is pulled to the right by a force $\mathbf{F}_1$ making an angle of 30° with the ground. At the same time the wagon is pulled to the left by a horizontal force $\mathbf{F}_2$.

(a) Find the magnitude of $\mathbf{F}_1$ in terms of the magnitude of $\mathbf{F}_2$ if the wagon does not move.

(b) What is the maximal magnitude of $\mathbf{F}_1$ that can be applied to the wagon without lifting it?

SOLUTION

(a) By Newton's Law, at equilibrium, the total force acting on the wagon is zero.



We resolve the force $\mathbf{F}_1$ into its components:

$$\mathbf{F}_1 = \mathbf{F}_{\|} + \mathbf{F}_{\perp}$$

where $\mathbf{F}_{\|}$ is the horizontal component and $\mathbf{F}_{\perp}$ is the vertical component. Since the wagon does not move, the magnitude of $\mathbf{F}_{\|}$ must be equal to the magnitude of $\mathbf{F}_2$. That is,

$$\|\mathbf{F}_{\|}\| = \|\mathbf{F}_1\|\cos 30^\circ = \|\mathbf{F}_2\|$$

The above equation gives:

$$\|\mathbf{F}_1\|\frac{\sqrt{3}}{2} = \|\mathbf{F}_2\| \quad \Rightarrow \quad \|\mathbf{F}_1\| = \frac{2\|\mathbf{F}_2\|}{\sqrt{3}}$$

(b) The maximum magnitude of force $\mathbf{F}_1$ that can be applied to the wagon without lifting the wagon is found by comparing the vertical forces:

$$\|\mathbf{F}_1\|\sin 30^\circ = 9.8 \cdot 50$$

$$\|\mathbf{F}_1\| \cdot \frac{1}{2} = 9.8 \cdot 50 \quad \Rightarrow \quad \|\mathbf{F}_1\| = 9.8 \cdot 100 = 980 \text{ N}$$

32. Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ be the vectors in $\mathbf{R}^3$. Which of the following is a scalar?

(a) $\mathbf{v} \times (\mathbf{u} + \mathbf{w})$

(b) $(\mathbf{u} + \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})$

(c) $(\mathbf{u} \times \mathbf{w}) + (\mathbf{w} - \mathbf{v})$

SOLUTION

(a) The cross product of the two vectors $\mathbf{v}$ and $\mathbf{u} + \mathbf{w}$ is a vector.

(b) The dot product of the two vectors $\mathbf{u} + \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ is a scalar.

(c) The cross product $\mathbf{u} \times \mathbf{w}$ is a vector, therefore the sum of this vector and the vector $\mathbf{w} - \mathbf{v}$ is again a vector.

In Exercises 33–36, let $\mathbf{v} = \langle 1, 2, 4 \rangle$, $\mathbf{u} = \langle 6, -1, 2 \rangle$, *and* $\mathbf{w} = \langle 1, 0, -3 \rangle$. *Calculate the given quantity.*

33. $\mathbf{v} \times \mathbf{w}$

SOLUTION We use the definition of the cross product as a determinant to compute $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 1 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{k}$$
$$= (-6 - 0)\mathbf{i} - (-3 - 4)\mathbf{j} + (0 - 2)\mathbf{k} = -6\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} = \langle -6, 7, -2 \rangle$$

34. $\mathbf{w} \times \mathbf{u}$

SOLUTION We compute the cross product as the following determinant:

$$\mathbf{w} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 6 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 6 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 6 & -1 \end{vmatrix} \mathbf{k}$$
$$= (0 - 3)\mathbf{i} - (2 - (-18))\mathbf{j} + (-1 - 0)\mathbf{k} = -3\mathbf{i} - 20\mathbf{j} - \mathbf{k} = \langle -3, -20, -1 \rangle$$

35. $\det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}$

SOLUTION We compute the determinant:

$$\det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{vmatrix} 6 & -1 & 2 \\ 1 & 2 & 4 \\ 1 & 0 & -3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$
$$= 6 \cdot (-6 - 0) + 1 \cdot (-3 - 4) + 2 \cdot (0 - 2) = -47$$

36. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$

SOLUTION We use the anticommutativity of the cross product and the cross product computed in a previous exercise to write:

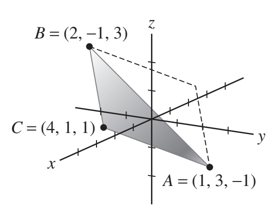
$$\mathbf{u} \times \mathbf{w} = -\mathbf{w} \times \mathbf{u} = -\langle -3, -20, -1 \rangle = \langle 3, 20, 1 \rangle$$

We now compute the dot product:

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \langle 1, 2, 4 \rangle \cdot \langle 3, 20, 1 \rangle = 1 \cdot 3 + 2 \cdot 20 + 4 \cdot 1 = 47$$

37. Use the cross product to find the area of the triangle whose vertices are $(1, 3, -1)$, $(2, -1, 3)$, and $(4, 1, 1)$.

SOLUTION Let $A = (1, 3, -1)$, $B = (2, -1, 3)$ and $C = (4, 1, 1)$.



The area S of the triangle ABC is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Using the Formula for the Area of the Parallelogram, we conclude that the area of the triangle is:

$$S = \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| \tag{1}$$

We first compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \langle 2 - 1, -1 - 3, 3 - (-1) \rangle = \langle 1, -4, 4 \rangle$$
$$\overrightarrow{AC} = \langle 4 - 1, 1 - 3, 1 - (-1) \rangle = \langle 3, -2, 2 \rangle$$

We compute the cross product of the two vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 4 \\ 3 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 4 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} \mathbf{k}$$

$$= (-8 - (-8))\mathbf{i} - (2 - 12)\mathbf{j} + (-2 - (-12))\mathbf{k}$$

$$= 10\mathbf{j} + 10\mathbf{k} = \langle 0, 10, 10 \rangle = 10\langle 0, 1, 1 \rangle$$

The length of $\overrightarrow{AB} \times \overrightarrow{AC}$ is, thus:

$$\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \| 10\langle 0, 1, 1 \rangle \| = 10 \| \langle 0, 1, 1 \rangle \| = 10\sqrt{0^2 + 1^2 + 1^2} = 10\sqrt{2}$$

Substituting in (1) gives the following area:

$$S = \frac{1}{2} \cdot 10\sqrt{2} = 5\sqrt{2}.$$

38. Calculate $\|\mathbf{v} \times \mathbf{w}\|$ if $\|\mathbf{v}\| = 2$, $\mathbf{v} \cdot \mathbf{w} = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{6}$.

SOLUTION Using the definition of the cross product we have:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta.$$

We substitute the given information, obtaining:

$$\|\mathbf{v} \times \mathbf{w}\| = 2\|\mathbf{w}\| \sin \frac{\pi}{6} = 2\|\mathbf{w}\| \cdot \frac{1}{2} = \|\mathbf{w}\| \tag{1}$$

We now must find the length of $\mathbf{w}$. By the Formula for the Cosine of the Angle between Two Vectors we have:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}.$$

Substituting the known values and solving for $\|\mathbf{w}\|$ gives:

$$\cos \frac{\pi}{6} = \frac{3}{2\|\mathbf{w}\|} \quad \Rightarrow \quad \|\mathbf{w}\| = \frac{3}{2 \cos \frac{\pi}{6}} = \frac{3}{2 \cdot \frac{\sqrt{3}}{2}} = \sqrt{3} \tag{2}$$

Combining (1) and (2) gives the following length:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{w}\| = \sqrt{3}.$$

39. Show that if the vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal, then $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

SOLUTION The vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, hence:

$$\mathbf{v} \cdot \mathbf{w} = 0 \tag{1}$$

Using the relation of the dot product with length and properties of the dot product we obtain:

$$\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w})$$

$$= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2 \tag{2}$$

Combining (1) and (2) we get:

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

40. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| = \|\mathbf{w}\|$.

SOLUTION The cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is given by:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \tag{1}$$

We denote by r the value $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = r$. To find $\mathbf{v} \cdot \mathbf{w}$ in terms of r, we evaluate $\|\mathbf{v} + \mathbf{w}\|$. Using properties of the dot product we obtain:

$$\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w})$$

$$= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$$

That is,

$$r^2 = r^2 + 2\mathbf{v} \cdot \mathbf{w} + r^2$$
$$-r^2 = 2\mathbf{v} \cdot \mathbf{w} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = -\frac{r^2}{2}$$

We now substitute $\|\mathbf{v}\| = \|\mathbf{w}\| = r$ and $\mathbf{v} \cdot \mathbf{w} = -\frac{r^2}{2}$ in (1) to obtain:

$$\cos\theta = \frac{-\frac{r^2}{2}}{r \cdot r} = -\frac{1}{2}$$

The solution for $0 \le \theta \le \pi$ is $\theta = \frac{2\pi}{3}$. That is, the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{2\pi}{3}$ rad.

41. Find $\|\mathbf{e} - 4\mathbf{f}\|$, assuming that $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e} + \mathbf{f}\| = \sqrt{3}$.

SOLUTION We use the relation of the dot product with length and properties of the dot product to write

$$3 = \|\mathbf{e} + \mathbf{f}\|^2 = (\mathbf{e} + \mathbf{f}) \cdot (\mathbf{e} + \mathbf{f}) = \mathbf{e} \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{f} + \mathbf{f} \cdot \mathbf{e} + \mathbf{f} \cdot \mathbf{f}$$
$$= \|\mathbf{e}\|^2 + 2\mathbf{e} \cdot \mathbf{f} + \|\mathbf{f}\|^2 = 1^2 + 2\mathbf{e} \cdot \mathbf{f} + 1^2 = 2 + 2\mathbf{e} \cdot \mathbf{f}$$

We now find $\mathbf{e} \cdot \mathbf{f}$:

$$3 = 2 + 2\mathbf{e} \cdot \mathbf{f} \quad \Rightarrow \quad \mathbf{e} \cdot \mathbf{f} = 1/2$$

Hence, using the same method as above, we have:

$$\|\mathbf{e} - 4\mathbf{f}\|^2 = (\mathbf{e} - 4\mathbf{f}) \cdot (\mathbf{e} - 4\mathbf{f})$$
$$= \|\mathbf{e}\|^2 - 2 \cdot \mathbf{e} \cdot 4\mathbf{f} + \|4\mathbf{f}\|^2 = 1^2 - 8\mathbf{e} \cdot \mathbf{f} + 4^2 = 17 - 4 = 13$$

Taking square roots, we get:

$$\|\mathbf{e} - 4\mathbf{f}\| = \sqrt{13}$$

42. Find the area of the parallelogram spanned by vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\|\mathbf{v}\| = \|\mathbf{w}\| = 2$ and $\mathbf{v} \cdot \mathbf{w} = 1$.

SOLUTION The area of the parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$ which equals $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, for θ the angle between the two vectors. Since

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta = 2 \cdot 2 \cdot \cos\theta = 1,$$

then $\cos\theta = 1/4$ and so $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{15/16} = \sqrt{15}/4$. Thus, the area is $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta = 2 \cdot 2 \cdot \sqrt{15}/4 = \sqrt{15}$

43. Show that the equation $\langle 1, 2, 3\rangle \times \mathbf{v} = \langle -1, 2, a\rangle$ has no solution for $a \ne -1$.

SOLUTION By properties of the cross product, the vector $\langle -1, 2, a\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$, hence the dot product of these vectors is zero. That is:

$$\langle -1, 2, a\rangle \cdot \langle 1, 2, 3\rangle = 0$$

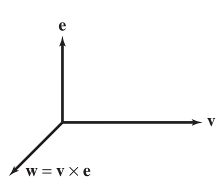
We compute the dot product and solve for a:

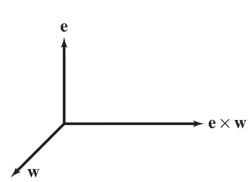
$$-1 + 4 + 3a = 0$$
$$3a = -3 \quad \Rightarrow \quad a = -1$$

We conclude that if the given equation is solvable, then $a = -1$.

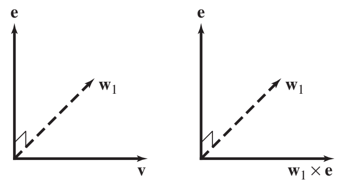
44. Prove with a diagram the following: If $\mathbf{e}$ is a unit vector orthogonal to $\mathbf{v}$, then $\mathbf{e} \times (\mathbf{v} \times \mathbf{e}) = (\mathbf{e} \times \mathbf{v}) \times \mathbf{e} = \mathbf{v}$.

SOLUTION The vectors $\mathbf{w} = \mathbf{v} \times \mathbf{e}$ and $\mathbf{e} \times \mathbf{w}$ are determined by the right-hand rule, as shown in the figure:





Similarly for the vectors $\mathbf{w_1} = \mathbf{e} \times \mathbf{v}$ and $\mathbf{w_1} \times \mathbf{e}$:



In each case, we see that the resulting vector is the vector **v**.

45. Use the identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$$

to prove that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

SOLUTION The given identity implies that:

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} \\
\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) &= (\mathbf{v} \cdot \mathbf{u})\,\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} \\
\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{w} \cdot \mathbf{v})\,\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\,\mathbf{v}
\end{aligned}$$

Adding the three equations and using the commutativity of the dot product we find that:

$$\begin{aligned}
&\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \\
&\quad = (\mathbf{u} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{u})\,\mathbf{v} + (\mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v})\,\mathbf{w} + (\mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w})\,\mathbf{u} = \mathbf{0}
\end{aligned}$$

46. Find an equation of the plane through $(1, -3, 5)$ with normal vector $\mathbf{n} = \langle 2, 1, -4\rangle$.

SOLUTION Using the scalar form of the equation of the plane we have,

$$2x + y - 4z = d = \langle 2, 1, -4\rangle \cdot \langle 1, -3, 5\rangle$$

We compute the dot product:

$$d = \langle 2, 1, -4\rangle \cdot \langle 1, -3, 5\rangle = 2 - 3 - 20 = -21$$

Therefore, the equation of the plane is,

$$2x + y - 4z = -21.$$

47. Write the equation of the plane $\mathcal{P}$ with vector equation

$$\langle 1, 4, -3\rangle \cdot \langle x, y, z\rangle = 7$$

in the form

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

Hint: You must find a point $P = (x_0, y_0, z_0)$ on $\mathcal{P}$.

SOLUTION We identify the vector $\mathbf{n} = \langle a, b, c\rangle = \langle 1, 4, -3\rangle$ that is normal to the plane, hence we may choose,

$$a = 1, \quad b = 4, \quad c = -3.$$

We now must find a point in the plane. The point $(x_0, y_0, z_0) = (0, 1, -1)$, for instance, satisfies the equation of the plane, therefore the equation may be written in the form:

$$1(x - 0) + 4(y - 1) - 3(z - (-1)) = 0$$

or

$$(x - 0) + 4(y - 1) - 3(z + 1) = 0$$

48. Find all the planes parallel to the plane passing through the points $(1, 2, 3)$, $(1, 2, 7)$, and $(1, 1, -3)$.

SOLUTION Since the points $A = (1, 2, 3)$, $B = (1, 2, 7)$, $C = (1, 1, -3)$ lie in the plane, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are in the plane. We find these vectors:

$$\overrightarrow{AB} = \langle 1-1, 2-2, 7-3\rangle = \langle 0, 0, 4\rangle$$
$$\overrightarrow{AC} = \langle 1-1, 1-2, -3-3\rangle = \langle 0, -1, -6\rangle$$

The cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is normal to the plane, therefore it is also normal to all the planes that are parallel to this plane. We could compute this cross product, but let's try a slightly more clever approach. Note that the three points A, B, C all satisfy $x = 1$. Thus, the vertical plane $x = 1$ contains these three points; the planes parallel to $x = 1$ are the planes $x = d$ for $d \neq 1$.

49. Find the plane through $P = (4, -1, 9)$ containing the line $\mathbf{r}(t) = \langle 1, 4, -3\rangle + t\langle 2, 1, 1\rangle$.

SOLUTION Since the plane contains the line, the direction vector of the line, $\mathbf{v} = \langle 2, 1, 1\rangle$, is in the plane. To find another vector in the plane, we use the points $A = (1, 4, -3)$ and $B = (4, -1, 9)$ that lie in the plane, and compute the vector $\mathbf{u} = \overrightarrow{AB}$:

$$\mathbf{u} = \overrightarrow{AB} = \langle 4-1, -1-4, 9-(-3)\rangle = \langle 3, -5, 12\rangle$$

We now compute the cross product $\mathbf{n} = \mathbf{v} \times \mathbf{u}$ that is normal to the plane:

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & -5 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -5 & 12 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & 12 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix}\mathbf{k}$$
$$= (12+5)\mathbf{i} - (24-3)\mathbf{j} + (-10-3)\mathbf{k} = 17\mathbf{i} - 21\mathbf{j} - 13\mathbf{k} = \langle 17, -21, -13\rangle$$

Finally, we use the vector form of the equation of the plane with $\mathbf{n} = \langle 17, -21, -13\rangle$ and $P_0 = (4, -1, 9)$ to obtain the following equation:

$$\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle$$
$$\langle 17, -21, -13\rangle \cdot \langle x, y, z\rangle = \langle 17, -21, -13\rangle \cdot \langle 4, -1, 9\rangle$$
$$17x - 21y - 13z = 17 \cdot 4 + 21 - 13 \cdot 9 = -28$$

The equation of the plane is, thus,

$$17x - 21y - 13z = -28.$$

50. Find the intersection of the line $\mathbf{r}(t) = \langle 3t+2, 1, -7t\rangle$ and the plane $2x - 3y + z = 5$.

SOLUTION The line has the parametric equations

$$x = 3t + 2, \quad y = 1, \quad z = -7t$$

We want to find the value of t for which the point (x, y, z) lies on the plane. We substitute the parametric equations in the equation of the plane and solve for t:

$$2(3t+2) - 3 \cdot 1 + (-7t) = 5$$
$$6t + 4 - 3 - 7t = 5$$
$$-t = 4 \quad \Rightarrow \quad t = -4$$

The point P of intersection of the line and the plane has the coordinates:

$$x = 3 \cdot (-4) + 2 = -10, \quad y = 1, \quad z = -7 \cdot (-4) = 28$$

and thus,

$$P = (-10, 1, 28).$$

51. Find the trace of the plane $3x - 2y + 5z = 4$ in the xy-plane.

SOLUTION The xy-plane has equation $z = 0$, therefore the intersection of the plane $3x - 2y + 5z = 4$ with the xy-plane must satisfy both $z = 0$ and the equation of the plane. Therefore the trace has the following equation:

$$3x - 2y + 5 \cdot 0 = 4 \quad \Rightarrow \quad 3x - 2y = 4$$

We conclude that the trace of the plane in the xy-plane is the line $3x - 2y = 4$ in the xy-plane.

52. Find the intersection of the planes $x + y + z = 1$ and $3x - 2y + z = 5$.

SOLUTION The line of intersection of the planes $x + y + z = 1$ and $3x - 2y + z = 5$ consists of all points that satisfy the equations of the two planes. Therefore, we must solve the following equations:

$$x + y + z = 1$$
$$3x - 2y + z = 5$$

The first equation implies $y = 1 - x - z$. Substituting in the second equation and solving for x in terms of z, gives:

$$3x - 2(1 - x - z) + z = 5$$
$$3x - 2 + 2x + 2z + z = 5$$
$$5x + 3z = 7 \quad \Rightarrow \quad x = \frac{7}{5} - \frac{3}{5}z$$

We find y in terms of z using the relation $y = 1 - x - z$:

$$y = 1 - \left(\frac{7}{5} - \frac{3}{5}z\right) - z = -\frac{2}{5} - \frac{2}{5}z$$

Therefore the solution is:

$$x = \frac{7}{5} - \frac{3}{5}z, \quad y = -\frac{2}{5} - \frac{2}{5}z, \quad z = z$$

We find the vector form of the equation of the line of intersection, using $z = t$ as the parameter. This gives:

$$\mathbf{r}(t) = \left\langle \frac{7}{5} - \frac{3}{5}t, -\frac{2}{5} - \frac{2}{5}t, t \right\rangle = \left\langle \frac{7}{5}, -\frac{2}{5}, 0 \right\rangle + t\left\langle -\frac{3}{5}, -\frac{2}{5}, 1 \right\rangle$$

In Exercises 53–58, determine the type of the quadric surface.

53. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + 2z^2 = 1$

SOLUTION Writing the equation in the form:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2 = 1$$

we identify the quadric surface as an ellipsoid.

54. $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 + 2z^2 = 1$

SOLUTION Writing the equation in the form:

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 + \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2 = 1$$

we identify the quadric surface as an hyperboloid of one sheet.

55. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 - 2z = 0$

SOLUTION We rewrite this equation as:

$$2z = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$$

or

$$z = \left(\frac{x}{3\sqrt{2}}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2$$

This is the equation of an elliptic paraboloid.

56. $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 - 2z = 0$

SOLUTION We rewrite this equation in the form:

$$2z = \left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2$$

or

$$z = \left(\frac{x}{3\sqrt{2}}\right)^2 - \left(\frac{y}{4\sqrt{2}}\right)^2$$

This is the equation of a hyperbolic paraboloid.

57. $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 - 2z^2 = 0$

SOLUTION This equation may be rewritten in the form

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2$$

we identify the quadric surface as an elliptic cone.

58. $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 - 2z^2 = 1$

SOLUTION We rewrite the equation in the form

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2 = 1$$

The corresponding quadric surface is a hyperboloid of two sheets.

59. Determine the type of the quadric surface $ax^2 + by^2 - z^2 = 1$ if:

(a) $a < 0, \quad b < 0$

(b) $a > 0, \quad b > 0$

(c) $a > 0, \quad b < 0$

SOLUTION

(a) If $a < 0$, $b < 0$ then for all x, y and z we have $ax^2 + by^2 - z^2 < 0$, hence there are no points that satisfy $ax^2 + by^2 - z^2 = 1$. Therefore it is the empty set.

(b) For $a > 0$ and $b > 0$ we rewrite the equation as

$$\left(\frac{x}{\frac{1}{\sqrt{a}}}\right)^2 + \left(\frac{y}{\frac{1}{\sqrt{b}}}\right)^2 - z^2 = 1$$

which is the equation of a hyperboloid of one sheet.

(c) For $a > 0$, $b < 0$ we rewrite the equation in the form

$$\left(\frac{x}{\frac{1}{\sqrt{a}}}\right)^2 - \left(\frac{y}{\frac{1}{\sqrt{|b|}}}\right)^2 - z^2 = 1$$

which is the equation of a hyperboloid of two sheets.

60. Describe the traces of the surface

$$\left(\frac{x}{2}\right)^2 - y^2 + \left(\frac{z}{2}\right)^2 = 1$$

in the three coordinate planes.

SOLUTION The xy-trace is obtained by setting $z = 0$ in the equation $\left(\frac{x}{2}\right)^2 - y^2 + \left(\frac{z}{2}\right)^2 = 1$ of the surface. This gives

$$\left(\frac{x}{2}\right)^2 - y^2 = 1$$

Therefore, the xy-trace is a hyperbola in the xy-plane. To find the xz-trace we set $y = 0$ in the equation of the surface, obtaining

$$\left(\frac{x}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

The xz-trace is a circle in the xz-plane. The yz-trace is obtained by setting $x = 0$. This gives

$$\left(\frac{z}{2}\right)^2 - y^2 = 1$$

which is a hyperbola in the yz-plane.

61. Convert $(x, y, z) = (3, 4, -1)$ from rectangular to cylindrical and spherical coordinates.

SOLUTION In cylindrical coordinates (r, θ, z) we have

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

Therefore, $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\theta = \frac{4}{3}$. The projection of the point $(3, 4, -1)$ onto the xy-plane is the point $(3, 4)$, in the first quadrant. Therefore, the corresponding value of θ is $\tan^{-1}\frac{4}{3} \approx 0.93$ rad. The cylindrical coordinates are, thus,

$$(r, \theta, z) = \left(5, \tan^{-1}\frac{4}{3}, -1\right)$$

The spherical coordinates (ρ, θ, ϕ) satisfy

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\rho}$$

Therefore,

$$\rho = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$$
$$\tan\theta = \frac{4}{3}$$
$$\cos\phi = \frac{-1}{\sqrt{26}}$$

The angle θ is the same as in the cylindrical coordinates, that is, $\theta = \tan^{-1}\frac{4}{3}$. The angle ϕ is the solution of $\cos\phi = \frac{-1}{\sqrt{26}}$ that satisfies $0 \le \phi \le \pi$, that is, $\phi = \cos^1\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77$ rad. The spherical coordinates are, thus,

$$(\rho, \theta, \phi) = \left(\sqrt{26}, \tan^{-1}\frac{4}{3}, \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right)\right).$$

62. Convert $(r, \theta, z) = \left(3, \frac{\pi}{6}, 4\right)$ from cylindrical to spherical coordinates.

SOLUTION By the given information, $r = 3$, $\theta = \frac{\pi}{6}$ and $z = 4$. We must determine the spherical coordinates (ρ, θ, ϕ). The angle θ is the same as in cylindrical coordinates. To find ρ we use the relation $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$. This gives

$$\rho = \sqrt{r^2 + z^2} = \sqrt{3^2 + 4^2} = 5$$

The angle ϕ satisfies $\cos\phi = \frac{z}{\rho} = \frac{4}{5} = 0.8$ and $0 \le \phi \le \pi$. Therefore, $\phi = \cos^{-1}0.8$. In spherical coordinates, we obtain the following description:

$$(\rho, \theta, \phi) = \left(5, \frac{\pi}{6}, \cos^{-1}0.8\right).$$

63. Convert the point $(\rho, \theta, \phi) = \left(3, \frac{\pi}{6}, \frac{\pi}{3}\right)$ from spherical to cylindrical coordinates.

SOLUTION By the given information, $\rho = 3$, $\theta = \frac{\pi}{6}$, and $\phi = \frac{\pi}{3}$. We must determine the cylindrical coordinates (r, θ, z). The angle θ is the same as in spherical coordinates. We find z using the relation $\cos\phi = \frac{z}{\rho}$, or $z = \rho\cos\phi$. We obtain

$$z = \rho\cos\phi = 3\cos\frac{\pi}{3} = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

We find r using the relation $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$, or $r = \sqrt{\rho^2 - z^2}$, we get

$$r = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4}} = \frac{3\sqrt{3}}{2}$$

Hence, in cylindrical coordinates we obtain the following description:

$$(r, \theta, z) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{6}, \frac{3}{2}\right).$$

64. Describe the set of all points $P = (x, y, z)$ satisfying $x^2 + y^2 \le 4$ in both cylindrical and spherical coordinates.

SOLUTION In cylindrical coordinates we have $x^2 + y^2 = r^2$, hence the inequality $x^2 + y^2 \le 4$ becomes

$$r^2 \le 4$$

or

$$r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

That is,

$$\{(r, \theta, z) : r \le 2,\ 0 \le \theta \le 2\pi\}$$

This is a solid cylinder of radius 2. In spherical coordinates we have $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$. Therefore,

$$x^2 + y^2 = \rho^2 - z^2 = \rho^2 - \rho^2 \cos^2 \phi = \rho^2 \left(1 - \cos^2 \phi\right) = \rho^2 \sin^2 \phi$$

The inequality $x^2 + y^2 \le 4$ in spherical coordinates is, thus,

$$\rho^2 \sin^2 \phi \le 4 \tag{1}$$

Notice that since $0 \le \phi \le \pi$, we have $\sin \phi \ge 0$. Also $\rho \ge 0$, therefore $\rho \sin \phi \ge 0$, hence inequality (1) is equivalent to

$$\rho \sin \phi \le 2$$

We obtain the following description in spherical coordinates:

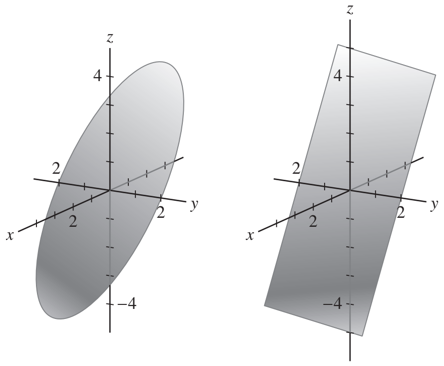
$$\{(\rho, \theta, \phi) : \rho \sin \phi \le 2,\ 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$$

65. Sketch the graph of the cylindrical equation $z = 2r \cos \theta$ and write the equation in rectangular coordinates.

SOLUTION To obtain the equation in rectangular coordinates, we substitute $x = r \cos \theta$ in the equation $z = 2r \cos \theta$:

$$z = 2r \cos \theta = 2x \quad \Rightarrow \quad z = 2x$$

This is the equation of a plane normal to the xz-plane, whose intersection with the xz-plane is the line $z = 2x$. The graph of the plane is shown in the following figure (the same plane drawn twice, using the cylindrical coordinates' equation and using the rectangular coordinates' equation):



66. Write the surface $x^2 + y^2 - z^2 = 2(x + y)$ as an equation $r = f(\theta, z)$ in cylindrical coordinates.

SOLUTION In cylindrical coordinates, we have

$$r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta$$

Substituting in the equation of the surface $x^2 + y^2 - z^2 = 2(x + y)$ gives:

$$\begin{aligned} r^2 - z^2 &= 2(r\cos\theta + r\sin\theta) \\ r^2 &= z^2 + 2r(\cos\theta + \sin\theta) \\ r^2 - 2r(\cos\theta + \sin\theta) - z^2 &= 0 \end{aligned}$$

We solve the quadratic equation for r:

$$\begin{aligned} r &= (\cos\theta + \sin\theta) \pm \sqrt{(\cos\theta + \sin\theta)^2 + z^2} \\ &= (\cos\theta + \sin\theta) \pm \sqrt{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta + z^2} \end{aligned}$$

We use the identities $\cos^2\theta + \sin^2\theta = 1$ and $2\cos\theta\sin\theta = \sin 2\theta$ to obtain the following function:

$$r = \cos\theta + \sin\theta \pm \sqrt{1 + \sin 2\theta + z^2}$$

67. Show that the cylindrical equation

$$r^2(1 - 2\sin^2\theta) + z^2 = 1$$

is a hyperboloid of one sheet.

SOLUTION We rewrite the equation in the form

$$r^2 - 2(r\sin\theta)^2 + z^2 = 1$$

To write this equation in rectangular coordinates, we substitute $r^2 = x^2 + y^2$ and $r\sin\theta = y$. This gives

$$\begin{aligned} x^2 + y^2 - 2y^2 + z^2 &= 1 \\ x^2 - y^2 + z^2 &= 1 \end{aligned}$$

We now can identify the surface as a hyperboloid of one sheet.

68. Sketch the graph of the spherical equation $\rho = 2\cos\theta\sin\phi$ and write the equation in rectangular coordinates.

SOLUTION We multiply the equation by ρ and then substitute $x = \rho\cos\theta\sin\phi$. This gives

$$\rho^2 = 2\rho\cos\theta\sin\phi = 2x$$

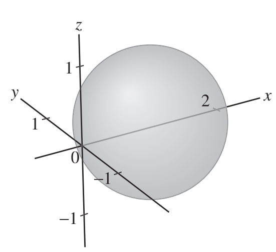
We now substitute $\rho^2 = x^2 + y^2 + z^2$ to obtain the following equation in terms of x, y, and z only:

$$x^2 + y^2 + z^2 = 2x$$

To identify the surface, we transfer sides and complete the square in x. This gives

$$\begin{aligned} x^2 - 2x + y^2 + z^2 &= 0 \\ (x-1)^2 - 1 + y^2 + z^2 &= 0 \\ (x-1)^2 + y^2 + z^2 &= 1 \end{aligned}$$

The surface is the sphere of radius 1 centered at $(1, 0, 0)$. This sphere is shown next:



$(x-1)^2 + y^2 + z^2 = 1$

69. Describe how the surface with spherical equation

$$\rho^2(1 + A\cos^2\phi) = 1$$

depends on the constant A.

SOLUTION To identify the surface we convert the equation to rectangular coordinates. We write

$$\rho^2 + A\rho^2 \cos^2\phi = 1$$

To obtain the following equation in terms of x, y, z only, we substitute $\rho^2 = x^2 + y^2 + z^2$ and $\rho\cos\phi = z$:

$$x^2 + y^2 + z^2 + Az^2 = 1$$
$$x^2 + y^2 + (1 + A)z^2 = 1 \qquad (1)$$

Case 1: $A < -1$. Then $A + 1 < 0$ and the equation can be rewritten in the form

$$x^2 + y^2 - \left(\frac{z}{|1 + A|^{-1/2}}\right)^2 = 1$$

The corresponding surface is a hyperboloid of one sheet.
Case 2: $A = -1$. Equation (1) becomes:

$$x^2 + y^2 = 1$$

In R^3, this equation describes a cylinder with the z-axis as its central axis.
Case 3: $A > -1$. Then equation (1) can be rewritten as

$$x^2 + y^2 + \left(\frac{z}{(1 + A)^{-1/2}}\right)^2 = 1$$

Then if $A = 0$ the equation $x^2 + y^2 + z^2 = 1$ describes the unit sphere in R^3. Otherwise, the surface is an ellipsoid.

70. Show that the spherical equation $\cot\phi = 2\cos\theta + \sin\theta$ defines a plane through the origin (with the origin excluded). Find a normal vector to this plane.

SOLUTION We multiply the equation by $\rho\sin\phi$, to obtain

$$\rho\sin\phi\cot\phi = 2\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta$$
$$\rho\cos\phi = 2\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta$$

We now convert to cartesian coordinates using the transition formula. We obtain

$$z = 2x + y$$
$$2x + y - z = 0$$

This is a standard representation of a plane and $n = \langle 2, 1, -1\rangle$ is orthogonal to this plane.

71. Let c be a scalar, let $\mathbf{a}$ and $\mathbf{b}$ be vectors, and let $\mathbf{X} = \langle x, y, z\rangle$. Show that the equation $(\mathbf{X} - \mathbf{a}) \cdot (\mathbf{X} - \mathbf{b}) = c^2$ defines a sphere with center $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and radius R, where $R^2 = c^2 + \left\|\frac{1}{2}(\mathbf{a} - \mathbf{b})\right\|^2$.

SOLUTION We evaluate the following length:

$$\begin{aligned}
\|\mathbf{x} - \mathbf{m}\|^2 &= \left\|\mathbf{x} - \frac{1}{2}(\mathbf{a} + \mathbf{b})\right\|^2 = \left((\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{a} - \mathbf{b})\right) \cdot \left((\mathbf{x} - \mathbf{b}) - \frac{1}{2}(\mathbf{a} - \mathbf{b})\right) \\
&= (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) - \frac{1}{2}(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{a} - \mathbf{b}) + \frac{1}{2}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{x} - \mathbf{b}) - \frac{1}{4}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) + \frac{1}{2}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{x} - \mathbf{b} - \mathbf{x} + \mathbf{a}) - \frac{1}{4}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) + \frac{1}{2}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) - \frac{1}{4}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) + \frac{1}{4}(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) + \left\|\frac{1}{2}(\mathbf{a} - \mathbf{b})\right\|^2
\end{aligned}$$

Since $R^2 = c^2 + \|\frac{1}{2}(\mathbf{a} - \mathbf{b})\|^2$ we get

$$\|\mathbf{x} - \mathbf{m}\|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) + R^2 - c^2$$

We conclude that if $(\mathbf{x} - \mathbf{a})(\mathbf{x} - \mathbf{b}) = c^2$ then $\|\mathbf{x} - \mathbf{m}\|^2 = R^2$. That is, the equation $(\mathbf{x} - \mathbf{a})(\mathbf{x} - \mathbf{b}) = c^2$ defines a sphere with center $\mathbf{m}$ and radius R.

13 CALCULUS OF VECTOR-VALUED FUNCTIONS

13.1 Vector-Valued Functions (LT Section 14.1)

Preliminary Questions

1. Which one of the following does *not* parametrize a line?

(a) $\mathbf{r}_1(t) = \langle 8-t, 2t, 3t \rangle$
(b) $\mathbf{r}_2(t) = t^3\mathbf{i} - 7t^3\mathbf{j} + t^3\mathbf{k}$
(c) $\mathbf{r}_3(t) = \left\langle 8-4t^3, 2+5t^2, 9t^3 \right\rangle$

SOLUTION

(a) This is a parametrization of the line passing through the point $(8, 0, 0)$ in the direction parallel to the vector $\langle -1, 2, 3 \rangle$, since:

$$\langle 8-t, 2t, 3t \rangle = \langle 8, 0, 0 \rangle + t \langle -1, 2, 3 \rangle$$

(b) Using the parameter $s = t^3$ we get:

$$\left\langle t^3, -7t^3, t^3 \right\rangle = \langle s, -7s, s \rangle = s \langle 1, -7, 1 \rangle$$

This is a parametrization of the line through the origin, with the direction vector $\mathbf{v} = \langle -1, 7, 1 \rangle$.
(c) The parametrization $\left\langle 8-4t^3, 2+5t^2, 9t^3 \right\rangle$ does not parametrize a line. In particular, the points $(8, 2, 0)$ (at $t = 0$), $(4, 7, 9)$ (at $t = 1$), and $(-24, 22, 72)$ (at $t = 2$) are not collinear.

2. What is the projection of $\mathbf{r}(t) = t\mathbf{i} + t^4\mathbf{j} + e^t\mathbf{k}$ onto the xz-plane?

SOLUTION The projection of the path onto the xz-plane is the curve traced by $t\mathbf{i} + e^t\mathbf{k} = \left\langle t, 0, e^t \right\rangle$. This is the curve $z = e^x$ in the xz-plane.

3. Which projection of $\langle \cos t, \cos 2t, \sin t \rangle$ is a circle?

SOLUTION The parametric equations are

$$x = \cos t, \quad y = \cos 2t, \quad z = \sin t$$

The projection onto the xz-plane is $\langle \cos t, 0, \sin t \rangle$. Since $x^2 + z^2 = \cos^2 t + \sin^2 t = 1$, the projection is a circle in the xz-plane. The projection onto the xy-plane is traced by the curve $\langle \cos t, \cos 2t, 0 \rangle$. Therefore, $x = \cos t$ and $y = \cos 2t$. We express y in terms of x:

$$y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$$

The projection onto the xy-plane is a parabola. The projection onto the yz-plane is the curve $\langle 0, \cos 2t, \sin t \rangle$. Hence $y = \cos 2t$ and $z = \sin t$. We find y as a function of z:

$$y = \cos 2t = 1 - 2\sin^2 t = 1 - 2z^2$$

The projection onto the yz-plane is again a parabola.

4. What is the center of the circle with parametrization

$$\mathbf{r}(t) = (-2 + \cos t)\mathbf{i} + 2\mathbf{j} + (3 - \sin t)\mathbf{k}?$$

SOLUTION The parametric equations are

$$x = -2 + \cos t, \quad y = 2, \quad z = 3 - \sin t$$

Therefore, the curve is contained in the plane $y = 2$, and the following holds:

$$(x+2)^2 + (z-3)^2 = \cos^2 t + \sin^2 t = 1$$

We conclude that the curve $\mathbf{r}(t)$ is the circle of radius 1 in the plane $y = 2$ centered at the point $(-2, 2, 3)$.

5. How do the paths $\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \cos t \rangle$ around the unit circle differ?

SOLUTION The two paths describe the unit circle. However, as t increases from 0 to 2π, the point on the path $\sin t\mathbf{i} + \cos t\mathbf{j}$ moves in a clockwise direction, whereas the point on the path $\cos t\mathbf{i} + \sin t\mathbf{j}$ moves in a counterclockwise direction.

6. Which three of the following vector-valued functions parametrize the same space curve?

(a) $(-2 + \cos t)\mathbf{i} + 9\mathbf{j} + (3 - \sin t)\mathbf{k}$

(b) $(2 + \cos t)\mathbf{i} - 9\mathbf{j} + (-3 - \sin t)\mathbf{k}$

(c) $(-2 + \cos 3t)\mathbf{i} + 9\mathbf{j} + (3 - \sin 3t)\mathbf{k}$

(d) $(-2 - \cos t)\mathbf{i} + 9\mathbf{j} + (3 + \sin t)\mathbf{k}$

(e) $(2 + \cos t)\mathbf{i} + 9\mathbf{j} + (3 + \sin t)\mathbf{k}$

SOLUTION All the curves except for (b) lie in the vertical plane $y = 9$. We identify each one of the curves (a), (c), (d) and (e).

(a) The parametric equations are:

$$x = -2 + \cos t, \quad y = 9, \quad z = 3 - \sin t$$

Hence,

$$(x+2)^2 + (z-3)^2 = (\cos t)^2 + (-\sin t)^2 = 1$$

This is the circle of radius 1 in the plane $y = 9$, centered at $(-2, 9, 3)$.

(c) The parametric equations are:

$$x = -2 + \cos 3t, \quad y = 9, \quad z = 3 - \sin 3t$$

Hence,

$$(x+2)^2 + (z-3)^2 = (\cos 3t)^2 + (-\sin 3t)^2 = 1$$

This is the circle of radius 1 in the plane $y = 9$, centered at $(-2, 9, 3)$.

(d) In this curve we have:

$$x = -2 - \cos t, \quad y = 9, \quad z = 3 + \sin t$$

Hence,

$$(x+2)^2 + (z-3)^2 = (-\cos t)^2 + (\sin t)^2 = 1$$

Again, the circle of radius 1 in the plane $y = 9$, centered at $(-2, 9, 3)$.

(e) In this parametrization we have:

$$x = 2 + \cos t, \quad y = 9, \quad z = 3 + \sin t$$

Hence,

$$(x-2)^2 + (z-3)^2 = (\cos t)^2 + (\sin t)^2 = 1$$

This is the circle of radius 1 in the plane $y = 9$, centered at $(2, 9, 3)$.

We conclude that (a), (c) and (d) parametrize the same circle whereas (b) and (e) are different curves.

Exercises

1. What is the domain of $\mathbf{r}(t) = e^t\mathbf{i} + \frac{1}{t}\mathbf{j} + (t+1)^{-3}\mathbf{k}$?

SOLUTION $\mathbf{r}(t)$ is defined for $t \neq 0$ and $t \neq -1$, hence the domain of $\mathbf{r}(t)$ is:

$$D = \{t \in \mathbf{R} : t \neq 0, t \neq -1\}$$

2. What is the domain of $\mathbf{r}(s) = e^s\mathbf{i} + \sqrt{s}\mathbf{j} + \cos s\mathbf{k}$?

SOLUTION $\mathbf{r}(s)$ is defined for $s \geq 0$, hence the domain of $\mathbf{r}(s)$ is:

$$D = \{s \in \mathbf{R} : s \geq 0\}.$$

3. Evaluate $\mathbf{r}(2)$ and $\mathbf{r}(-1)$ for $\mathbf{r}(t) = \left\langle \sin \frac{\pi}{2}t, t^2, (t^2+1)^{-1} \right\rangle$.

SOLUTION Since $\mathbf{r}(t) = \left\langle \sin \frac{\pi}{2}t, t^2, (t^2+1)^{-1} \right\rangle$, then

$$\mathbf{r}(2) = \left\langle \sin \pi, 4, 5^{-1} \right\rangle = \left\langle 0, 4, \frac{1}{5} \right\rangle$$

and

$$\mathbf{r}(-1) = \left\langle \sin \frac{-\pi}{2}, 1, 2^{-1} \right\rangle = \left\langle -1, 1, \frac{1}{2} \right\rangle$$

4. Does either of $P = (4, 11, 20)$ or $Q = (-1, 6, 16)$ lie on the path $\mathbf{r}(t) = \left\langle 1+t, 2+t^2, t^4 \right\rangle$?

SOLUTION The point $P = (4, 11, 20)$ lies on the path $\mathbf{r}(t) = \left\langle 1+t, 2+t^2, t^4 \right\rangle$ if there exists a value of t such that $\overrightarrow{OP} = \mathbf{r}(t)$. That is,

$$\left\langle 4, 11, 20 \right\rangle = \left\langle 1+t, 2+t^2, t^4 \right\rangle$$

Equating like components we get:

$$\begin{aligned} 1+t &= 4 \\ 2+t^2 &= 11 \\ t^4 &= 20 \end{aligned}$$

The first equation implies that $t = 3$, but this value does not satisfy the third equation. We conclude that P does not lie on the path. The point $Q = (-1, 6, 16)$ lies on the path if there exists a value of t such that:

$$\langle -1, 6, 16 \rangle = \left\langle 1+t, 2+t^2, t^4 \right\rangle$$

or equivalently:

$$\begin{aligned} 1+t &= -1 \\ 2+t^2 &= 6 \\ t^4 &= 16 \end{aligned}$$

These equations have the solution $t = -2$, hence $Q = (-1, 6, 16)$ lies on the path.

5. Find a vector parametrization of the line through $P = (3, -5, 7)$ in the direction $\mathbf{v} = \langle 3, 0, 1 \rangle$.

SOLUTION We use the vector parametrization of the line to obtain:

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 3, -5, 7 \rangle + t \langle 3, 0, 1 \rangle = \langle 3+3t, -5, 7+t \rangle$$

or in the form:

$$\mathbf{r}(t) = (3+3t)\mathbf{i} - 5\mathbf{j} + (7+t)\mathbf{k}, \quad -\infty < t < \infty$$

6. Find a direction vector for the line with parametrization $\mathbf{r}(t) = (4-t)\mathbf{i} + (2+5t)\mathbf{j} + \frac{1}{2}t\mathbf{k}$.

SOLUTION We rewrite the vector $\mathbf{r}(t)$ in the following form:

$$\mathbf{r}(t) = \left\langle 4-t, 2+5t, \frac{1}{2}t \right\rangle = \langle 4, 2, 0 \rangle + t \left\langle -1, 5, \frac{1}{2} \right\rangle$$

We identify $\mathbf{v} = \left\langle -1, 5, \frac{1}{2} \right\rangle$ as a direction vector for the line.

7. Match the space curves in Figure 8 with their projections onto the xy-plane in Figure 9.

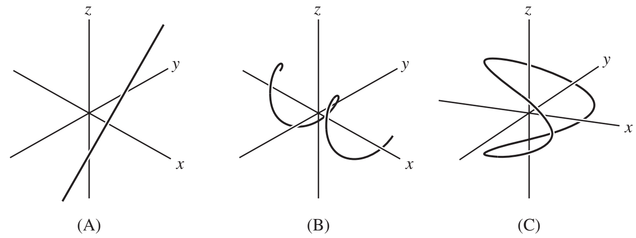


FIGURE 8

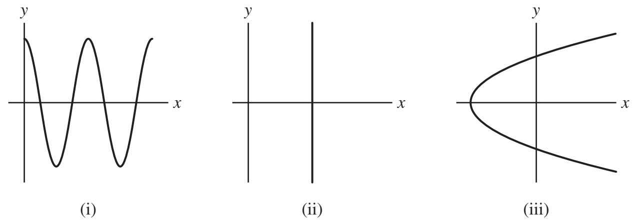


FIGURE 9

SOLUTION The projection of curve (C) onto the xy-plane is neither a segment nor a periodic wave. Hence, the correct projection is (iii), rather than the two other graphs. The projection of curve (A) onto the xy-plane is a vertical line, hence the corresponding projection is (ii). The projection of curve (B) onto the xy-plane is a periodic wave as illustrated in (i).

8. Match the space curves in Figure 8 with the following vector-valued functions:

(a) $\mathbf{r}_1(t) = \langle \cos 2t, \cos t, \sin t\rangle$

(b) $\mathbf{r}_2(t) = \langle t, \cos 2t, \sin 2t\rangle$

(c) $\mathbf{r}_3(t) = \langle 1, t, t\rangle$

SOLUTION

(a) This function traces the curve $\langle \cos 2t, \cos t\rangle$ on the xy-plane. Using the identity $\cos 2t = 2\cos^2 t - 1$, we have $x = 2y^2 - 1$. This equation corresponds to Figure 9(iii) which is the projection of curve (C), onto the xy-plane.

(b) The projection of this curve onto the xy-plane is traced by $\langle t, \cos 2t\rangle$ which is a wave moving in the x-direction as in Figure 9(i). By Exercise 7 it corresponds to curve (B).

(c) The projection of this curve onto the xy-plane is traced by $\langle 1, t\rangle$ which is a vertical line in the xy-plane, as in Figure 9(ii). By Exercise 7 it corresponds to curve (A).

9. Match the vector-valued functions (a)–(f) with the space curves (i)–(vi) in Figure 10.

(a) $\mathbf{r}(t) = \left\langle t + 15, e^{0.08t}\cos t, e^{0.08t}\sin t\right\rangle$

(b) $\mathbf{r}(t) = \left\langle \cos t, \sin t, \sin 12t\right\rangle$

(c) $\mathbf{r}(t) = \left\langle t, t, \dfrac{25t}{1+t^2}\right\rangle$

(d) $\mathbf{r}(t) = \left\langle \cos^3 t, \sin^3 t, \sin 2t\right\rangle$

(e) $\mathbf{r}(t) = \left\langle t, t^2, 2t\right\rangle$

(f) $\mathbf{r}(t) = \left\langle \cos t, \sin t, \cos t \sin 12t\right\rangle$

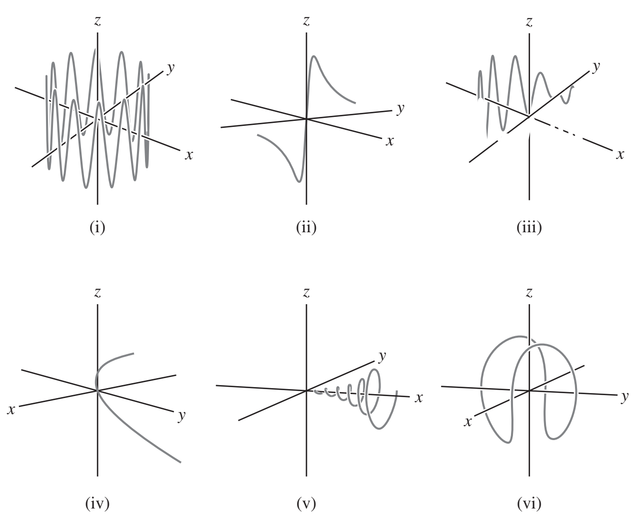


FIGURE 10

SOLUTION

(a) (v)	**(b)** (i)	**(c)** (ii)
(d) (vi)	**(e)** (iv)	**(f)** (iii)

10. Which of the following curves have the same projection onto the xy-plane?

(a) $\mathbf{r}_1(t) = \left\langle t, t^2, e^t\right\rangle$ **(b)** $\mathbf{r}_2(t) = \left\langle e^t, t^2, t\right\rangle$ **(c)** $\mathbf{r}_3(t) = \left\langle t, t^2, \cos t\right\rangle$

SOLUTION The projection onto the xy-plane is obtained by setting the z-component equal to zero. The curves $\left\langle t, t^2, e^t\right\rangle$ and $\left\langle t, t^2, \cos t\right\rangle$ have the same projection onto the xy-plane, traced by $\left\langle t, t^2, 0\right\rangle$.

11. Match the space curves (A)–(C) in Figure 11 with their projections (i)–(iii) onto the xy-plane.

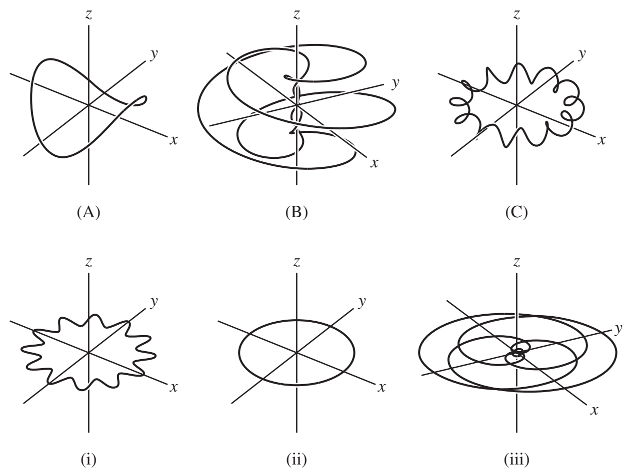


FIGURE 11

SOLUTION Observing the curves and the projections onto the xy-plane we conclude that: Projection (i) corresponds to curve (C); Projection (ii) corresponds to curve (A); Projection (iii) corresponds to curve (B).

12. Describe the projections of the circle $\mathbf{r}(t) = \langle \sin t, 0, 4 + \cos t \rangle$ onto the coordinate planes.

SOLUTION The projection onto the xy-plane is traced by $\langle \sin t, 0, 0 \rangle$, which is the segment $[-1, 1]$ on the x-axis (since $-1 \le \sin t \le 1$). The given circle is contained in the xz-plane, hence the projection on the xz-plane is the circle itself. We identify this circle. Since $x = \sin t$ and $z = 4 + \cos t$, we have:

$$x^2 + (z - 4)^2 = \sin^2 t + \cos^2 t = 1$$

This is the circle of radius 1 centered at $(0, 0, 4)$. The projection onto the yz-plane is traced by $\langle 0, 0, 4 + \cos t \rangle$, which is the segment $[3, 5]$ on the z-axis (since $3 = 4 - 1 \le 4 + \cos t \le 4 + 1 = 5$).

In Exercises 13–16, the function $\mathbf{r}(t)$ traces a circle. Determine the radius, center, and plane containing the circle.

13. $\mathbf{r}(t) = (9 \cos t)\mathbf{i} + (9 \sin t)\mathbf{j}$

SOLUTION Since $x(t) = 9 \cos t$, $y(t) = 9 \sin t$ we have:

$$x^2 + y^2 = 81 \cos^2 t + 81 \sin^2 t = 81(\cos^2 t + \sin^2 t) = 81$$

This is the equation of a circle with radius 9 centered at the origin. The circle lies in the xy-plane.

14. $\mathbf{r}(t) = 7\mathbf{i} + (12 \cos t)\mathbf{j} + (12 \sin t)\mathbf{k}$

SOLUTION We have:

$$x(t) = 7, \quad y(t) = 12 \cos t, \quad z(t) = 12 \sin t$$

Hence,

$$y(t)^2 + z(t)^2 = 144 \cos^2 t + 144 \sin^2 t = 144\left(\cos^2 t + \sin^2 t\right) = 144$$

This is the equation of a circle in the vertical plane $x = 7$. The circle is centered at the point $(7, 0, 0)$ and its radius is $\sqrt{144} = 12$.

15. $\mathbf{r}(t) = \langle \sin t, 0, 4 + \cos t \rangle$

SOLUTION $x(t) = \sin t$, $z(t) = 4 + \cos t$, hence:

$$x^2 + (z - 4)^2 = \sin^2 t + \cos^2 t = 1$$

$y = 0$ is the equation of the xz-plane. We conclude that the function traces the circle of radius 1, centered at the point $(0, 0, 4)$, and contained in the xz-plane.

16. $\mathbf{r}(t) = \langle 6 + 3\sin t, 9, 4 + 3\cos t\rangle$

SOLUTION Since $y(t) = 9$ the curve is contained in the vertical plane $y = 9$. By the given equations, $x(t) = 6 + 3\sin t$ and $z = 4 + 3\cos t$, hence:

$$\left(\frac{x-6}{3}\right)^2 + \left(\frac{z-4}{3}\right)^2 = \sin^2 t + \cos^2 t = 1$$

We conclude that the function traces a circle in the vertical plane $y = 9$, centered at the point $(6, 9, 4)$ and with radius 3.

17. Let $\mathcal{C}$ be the curve $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$.

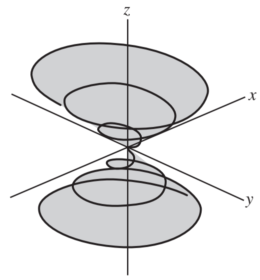
(a) Show that $\mathcal{C}$ lies on the cone $x^2 + y^2 = z^2$.

(b) Sketch the cone and make a rough sketch of $\mathcal{C}$ on the cone.

SOLUTION $x = t\cos t$, $y = t\sin t$ and $z = t$, hence:

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2\left(\cos^2 t + \sin^2 t\right) = t^2 = z^2.$$

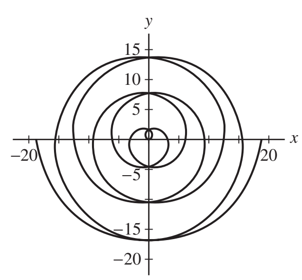
$x^2 + y^2 = z^2$ is the equation of a circular cone, hence the curve lies on a circular cone. As the height $z = t$ increases linearly with time, the x and y coordinates trace out points on the circles of increasing radius. We obtain the following curve:



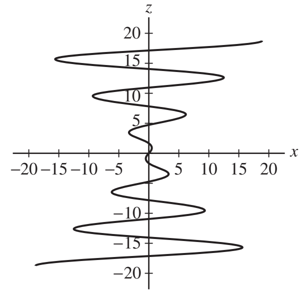
$\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$

18. CAS Use a computer algebra system to plot the projections onto the xy- and xz-planes of the curve $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$ in Exercise 17.

SOLUTION The projection of the curve onto the xy-plane is traced by the function $\langle t\cos t, t\sin t, 0\rangle$. The curve is the following spiral:



The projection of the curve onto the xz-plane is traced by $\langle t\cos t, 0, t\rangle$, which is a wave with increasing amplitude moving in the z direction as shown in the following figure:



In Exercises 19 and 20, let

$$\mathbf{r}(t) = \langle \sin t, \cos t, \sin t \cos 2t \rangle$$

as shown in Figure 12.

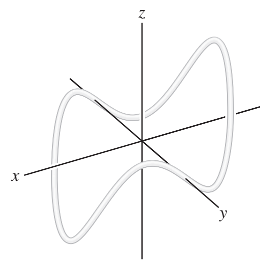


FIGURE 12

19. Find the points where $\mathbf{r}(t)$ intersects the xy-plane.

SOLUTION The curve intersects the xy-plane at the points where $z = 0$. That is, $\sin t \cos 2t = 0$ and so either $\sin t = 0$ or $\cos 2t = 0$. The solutions are, thus:

$$t = \pi k \text{ or } t = \frac{\pi}{4} + \frac{\pi k}{2}, \quad k = 0, \pm 1, \pm 2, \dots$$

The values $t = \pi k$ yield the points: $(\sin \pi k, \cos \pi k, 0) = \left(0, (-1)^k, 0\right)$. The values $t = \frac{\pi}{4} + \frac{\pi k}{2}$ yield the points:

$$k = 0 : \left(\sin \frac{\pi}{4}, \cos \frac{\pi}{4}, 0\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$k = 1 : \left(\sin \frac{3\pi}{4}, \cos \frac{3\pi}{4}, 0\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$k = 2 : \left(\sin \frac{5\pi}{4}, \cos \frac{5\pi}{4}, 0\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$k = 3 : \left(\sin \frac{7\pi}{4}, \cos \frac{7\pi}{4}, 0\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

(Other values of k do not provide new points). We conclude that the curve intersects the xy-plane at the following points: $(0, 1, 0)$, $(0, -1, 0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

20. Show that the projection of $\mathbf{r}(t)$ onto the xz-plane is the curve

$$z = x - 2x^3 \quad \text{for} \quad -1 \le x \le 1$$

SOLUTION The xz-plane projection indicates that $y = 0$. We are given $x = \sin t$, so $-1 \le x \le 1$ and note that from trigonometry, $\cos 2t = 1 - 2\sin^2 t$. Thus

$$z = \sin t \cos 2t = \sin t(1 - 2\sin^2 t) = x(1 - 2x^2) = x - 2x^3, -1 \le x \le 1$$

21. Parametrize the intersection of the surfaces

$$y^2 - z^2 = x - 2, \qquad y^2 + z^2 = 9$$

using $t = y$ as the parameter (two vector functions are needed as in Example 3).

SOLUTION We solve for z and x in terms of y. From the equation $y^2 + z^2 = 9$ we have $z^2 = 9 - y^2$ or $z = \pm\sqrt{9 - y^2}$. From the second equation we have:

$$x = y^2 - z^2 + 2 = y^2 - \left(9 - y^2\right) + 2 = 2y^2 - 7$$

Taking $t = y$ as a parameter, we have $z = \pm\sqrt{9 - t^2}$, $x = 2t^2 - 7$, yielding the following vector parametrization:

$$\mathbf{r}(t) = \left\langle 2t^2 - 7, t, \pm\sqrt{9 - t^2} \right\rangle, \text{ for } -3 \le t \le 3.$$

22. Find a parametrization of the curve in Exercise 21 using trigonometric functions.

SOLUTION The curve in Exercise 21 is the intersection of the surfaces $y^2 - z^2 = x - 2$, $y^2 + z^2 = 9$. The circle $y^2 + z^2 = 9$ is parametrized by $y = 3\cos t$, $z = 3\sin t$. Substituting in the first equation and using the identity $\cos^2 t - \sin^2 t = \cos 2t$, gives:

$$x = 2 + y^2 - z^2 = 2 + (3\cos t)^2 - (3\sin t)^2 = 2 + 9\left(\cos^2 t - \sin^2 t\right) = 2 + 9\cos 2t$$

We obtain the following trigonometric parametrization:

$$\mathbf{r}(t) = \langle 2 + 9\cos 2t, 3\cos t, 3\sin t\rangle$$

23. Viviani's Curve $\mathcal{C}$ is the intersection of the surfaces (Figure 13)

$$x^2 + y^2 = z^2, \qquad y = z^2$$

(a) Parametrize each of the two parts of $\mathcal{C}$ corresponding to $x \geq 0$ and $x \leq 0$, taking $t = z$ as parameter.
(b) Describe the projection of $\mathcal{C}$ onto the xy-plane.
(c) Show that $\mathcal{C}$ lies on the sphere of radius 1 with center (0, 1, 0). This curve looks like a figure eight lying on a sphere [Figure 13(B)].

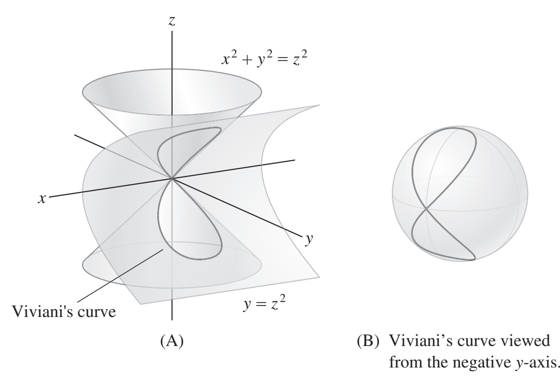


FIGURE 13 Viviani's curve is the intersection of the surfaces $x^2 + y^2 = z^2$ and $y = z^2$.

SOLUTION
(a) We must solve for y and x in terms of z (which is a parameter). We get:

$$y = z^2$$
$$x^2 = z^2 - y^2 \quad \Rightarrow \quad x = \pm\sqrt{z^2 - y^2} = \pm\sqrt{z^2 - z^4}$$

Here, the $\pm$ from $x = \pm\sqrt{z^2 - z^4}$ represents the two parts of the parametrization: $+$ for $x \geq 0$, and $-$ for $x \leq 0$. Substituting the parameter $z = t$ we get:

$$y = t^2, \quad x = \pm\sqrt{t^2 - t^4} = \pm t\sqrt{1 - t^2}.$$

We obtain the following parametrization:

$$\mathbf{r}(t) = \left\langle \pm t\sqrt{1 - t^2}, t^2, t\right\rangle \text{ for } -1 \leq t \leq 1 \tag{1}$$

(b) The projection of the curve onto the xy-plane is the curve on the xy-plane obtained by setting the z-coordinate of $\mathbf{r}(t)$ equal to zero. We obtain the following curve:

$$\left\langle \pm t\sqrt{1 - t^2}, t^2, 0\right\rangle, \quad -1 \leq t \leq 1$$

We also note that since $x = \pm t\sqrt{1 - t^2}$, then $x^2 = t^2(1 - t^2)$, but also $y = t^2$, so that gives us the equation $x^2 = y(1 - y)$ for the projection onto the xy plane. We rewrite this as follows.

$$x^2 = y(1 - y) \quad \Rightarrow \quad x^2 + y^2 - y = 0$$
$$x^2 + y^2 - y + 1/4 = 1/4$$
$$x^2 + (y - 1/2)^2 = (1/2)^2$$

We can now identify this projection as a circle in the xy plane, with radius $1/2$, centered at the xy point $(0, 1/2)$.

(c) The equation of the sphere of radius 1 with center (0, 1, 0) is:

$$x^2 + (y-1)^2 + z^2 = 1 \tag{2}$$

To show that $\mathcal{C}$ lies on this sphere, we show that the coordinates of the points on $\mathcal{C}$ (given in (1)) satisfy the equation of the sphere. Substituting the coordinates from (1) into the left side of (2) gives:

$$\begin{aligned} x^2 + (y-1)^2 + z^2 &= \left(\pm t\sqrt{1-t^2}\right)^2 + (t^2-1)^2 + t^2 = t^2(1-t^2) + (t^2-1)^2 + t^2 \\ &= (t^2-1)(t^2-1-t^2) + t^2 = 1 \end{aligned}$$

We conclude that the curve $\mathcal{C}$ lies on the sphere of radius 1 with center (0, 1, 0).

24. Show that any point on $x^2 + y^2 = z^2$ can be written in the form $(z\cos\theta, z\sin\theta, z)$ for some θ. Use this to find a parametrization of Viviani's curve (Exercise 23) with θ as parameter.

SOLUTION We first verify that $x = z\cos\theta$, $y = z\sin\theta$, and $z = z$ satisfy the equation of the surface:

$$x^2 + y^2 = z^2\cos^2\theta + z^2\sin^2\theta = z^2\left(\cos^2\theta + \sin^2\theta\right) = z^2$$

We now show that if (x, y, z) satisfies $x^2 + y^2 = z^2$, then there exists a value of θ such that $x = z\cos\theta$, $y = z\sin\theta$. Since $x^2 + y^2 = z^2$, we have $|x| \le |z|$ and $|y| \le |z|$. If $z = 0$, then also $x = y = 0$ and any value of θ is adequate. If $z \ne 0$ then $\|\frac{x}{z}\| \le 1$ and $\|\frac{y}{z}\| \le 1$, hence there exists θ_0 such that $\frac{x}{z} = \cos\theta_0$. Hence,

$$\frac{y}{z} = \pm\sqrt{\frac{z^2 - x^2}{z^2}} = \pm\sqrt{1 - \left(\frac{x}{z}\right)^2} = \pm\sqrt{1 - \cos^2\theta_0} = \pm\sin\theta_0$$

If $\frac{x}{z}$ and $\frac{y}{z}$ are both positive, we choose θ_0 such that $0 < \theta_0 < \frac{\pi}{2}$. If $\frac{x}{z} > 0$ and $\frac{y}{z} < 0$ we choose θ_0 such that $\frac{3\pi}{2} < \theta_0 < 2\pi$. If $\frac{x}{z} < 0$ and $\frac{y}{z} < 0$ we choose θ_0 such that $\pi < \theta_0 < \frac{3\pi}{2}$, and if $\frac{x}{z} < 0$ and $\frac{y}{z} > 0$ we choose θ_0 such that $\frac{\pi}{2} < \theta_0 < \pi$. In either case we can represent the points on the surface as required. Viviani's curve is the intersection of the surfaces $x^2 + y^2 = z^2$ and $x = z^2$. The points on these surfaces are of the form:

$$\begin{aligned} x^2 + y^2 = z^2: &\quad (z\cos\theta, z\sin\theta, z) \\ x = z^2: &\quad (z^2, y, z) \end{aligned} \tag{1}$$

The points (x, y, z) on the intersection curve must satisfy the following equations:

$$\begin{cases} z^2 = z\cos\theta \\ y = z\sin\theta \end{cases}$$

The first equation implies that $z = 0$ or $z = \cos\theta$. The second equation implies that $y = 0$ or $y = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$. The x coordinate is obtained by substituting $z = \cos\theta$ in $x = z\cos\theta$ (or in $x = z^2$). That is, $x = \cos^2\theta$. We obtain the following vector parametrization of the curve:

$$\mathbf{r}(t) = \left\langle \cos^2\theta, \frac{1}{2}\sin 2\theta, \cos\theta \right\rangle$$

25. Use sine and cosine to parametrize the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ (use two vector-valued functions). Then describe the projections of this curve onto the three coordinate planes.

SOLUTION The circle $x^2 + z^2 = 1$ in the xz-plane is parametrized by $x = \cos t$, $z = \sin t$, and the circle $x^2 + y^2 = 1$ in the xy-plane is parametrized by $x = \cos s$, $y = \sin s$. Hence, the points on the cylinders can be written in the form:

$$\begin{aligned} x^2 + z^2 = 1: &\quad \langle \cos t, y, \sin t\rangle, \quad 0 \le t \le 2\pi \\ x^2 + y^2 = 1: &\quad \langle \cos s, \sin s, z\rangle, \quad 0 \le t \le 2\pi \end{aligned}$$

The points (x, y, z) on the intersection of the two cylinders must satisfy the following equations:

$$\cos t = \cos s$$

$$y = \sin s$$

$$z = \sin t$$

The first equation implies that $s = \pm t + 2\pi k$. Substituting in the second equation gives $y = \sin(\pm t + 2\pi k) = \sin(\pm t) = \pm\sin t$. Hence, $x = \cos t$, $y = \pm\sin t$, $z = \sin t$. We obtain the following vector parametrization of the intersection:

$$\mathbf{r}(t) = \langle \cos t, \pm\sin t, \sin t\rangle$$

The projection of the curve on the xy-plane is traced by $\langle \cos t, \pm \sin t, 0\rangle$ which is the unit circle in this plane. The projection of the curve on the xz-plane is traced by $\langle \cos t, 0, \sin t\rangle$ which is the unit circle in the xz-plane. The projection of the curve on the yz-plane is traced by $\langle 0, \pm \sin t, \sin t\rangle$ which is the two segments $z = y$ and $z = -y$ for $-1 \le y \le 1$.

26. Use hyperbolic functions to parametrize the intersection of the surfaces $x^2 - y^2 = 4$ and $z = xy$.

SOLUTION $x = 2\cosh t$ and $y = 2\sinh t$ satisfy the equation of the (hyperbolic) cylinder since:

$$x^2 - y^2 = 4\cosh^2 t - 4\sinh^2 t = 4(\cosh^2 t - \sinh^2 t) = 4 \cdot 1 = 4$$

To find a parametrization of the curve of intersection, we substitute $x = 2\cosh t$, $y = 2\sinh t$ in the equation of the surface $z = xy$ and solve for z. This gives $z = 4\cosh t \sinh t$. We obtain the following parametrization of the curve of intersection (valid for all values of t):

$$\mathbf{r}(t) = \langle 2\cosh t, 2\sinh t, 4\sinh t \cosh t\rangle .$$

27. Use sine and cosine to parametrize the intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$ (Figure 14).

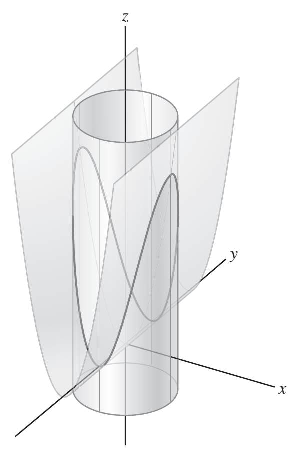


FIGURE 14 Intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$.

SOLUTION The points on the cylinder $x^2 + y^2 = 1$ and on the parabolic cylinder $z = 4x^2$ can be written in the form:

$$\begin{aligned} x^2 + y^2 = 1: &\quad \langle \cos t, \sin t, z\rangle \\ z = 4x^2: &\quad \left\langle x, y, 4x^2\right\rangle \end{aligned}$$

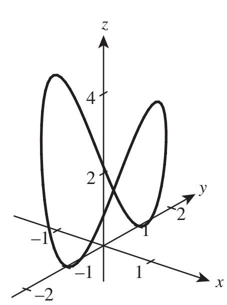
The points (x, y, z) on the intersection curve must satisfy the following equations:

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 4x^2 \end{aligned} \quad \Rightarrow \quad x = \cos t,\ y = \sin t,\ z = 4\cos^2 t$$

We obtain the vector parametrization:

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, 4\cos^2 t\right\rangle, \quad 0 \le t \le 2\pi$$

Using the CAS we obtain the following curve:



$\mathbf{r}(t) = \left\langle \cos t, \sin t, 4\cos^2 t\right\rangle$

In Exercises 28–30, two paths $\mathbf{r}_1(t)$ *and* $\mathbf{r}_2(t)$ *intersect if there is a point* P *lying on both curves. We say that* $\mathbf{r}_1(t)$ *and* $\mathbf{r}_2(t)$ collide *if* $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$ *at some time* t_0.

28. Which of the following statements are true?

(a) If $\mathbf{r}_1$ and $\mathbf{r}_2$ intersect, then they collide.

(b) If $\mathbf{r}_1$ and $\mathbf{r}_2$ collide, then they intersect.

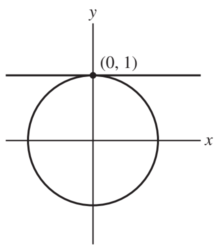
(c) Intersection depends only on the underlying curves traced by $\mathbf{r}_1$ and $\mathbf{r}_2$, but collision depends on the actual parametrizations.

SOLUTION

(a) This statement is wrong. $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ may intersect but the point of intersection may correspond to different values of the parameters in the two curves, as illustrated in the following example:

$$\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle \text{ (the unit circle)}$$

$$\mathbf{r}_2(s) = \langle s, 1 \rangle \text{ (the horizontal line } y = 1\text{)}$$



The point of intersection (0, 1) corresponds to $t = \frac{\pi}{2}$ and $s = 0$.

(b) This statement is true. If $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$, then the head of the vector $\mathbf{r}_1(t_0)$ (or $\mathbf{r}_2(t_0)$) is a point of intersection of the two curves.

(c) The statement is true. Intersection is a geometric property of the curves and it is independent of the parametrization we choose for the curves. Collision depends on the actual parametrization. Notice that if we parametrize the line $y = 1$ in the example given in part (a) by $\mathbf{r}_3(s) = \left\langle s - \frac{\pi}{2}, 1 \right\rangle$, then $\mathbf{r}_1\left(\frac{\pi}{2}\right) = \mathbf{r}_3\left(\frac{\pi}{2}\right)$ hence the two paths collide.

29. Determine whether $\mathbf{r}_1$ and $\mathbf{r}_2$ collide or intersect:

$$\mathbf{r}_1(t) = \left\langle t^2 + 3, t + 1, 6t^{-1} \right\rangle$$

$$\mathbf{r}_2(t) = \left\langle 4t, 2t - 2, t^2 - 7 \right\rangle$$

SOLUTION To determine if the paths collide, we must examine whether the following equations have a solution:

$$\begin{cases} t^2 + 3 = 4t \\ t + 1 = 2t - 2 \\ \dfrac{6}{t} = t^2 - 7 \end{cases}$$

We simplify to obtain:

$$t^2 - 4t + 3 = (t - 3)(t - 1) = 0$$

$$t = 3$$

$$t^3 - 7t - 6 = 0$$

The solution of the second equation is $t = 3$. This is also a solution of the first and the third equations. It follows that $\mathbf{r}_1(3) = \mathbf{r}_2(3)$ so the curves collide. The curves also intersect at the point where they collide. We now check if there are other points of intersection by solving the following equation:

$$\mathbf{r}_1(t) = \mathbf{r}_2(s)$$

$$\left\langle t^2 + 3, t + 1, \frac{6}{t} \right\rangle = \left\langle 4s, 2s - 2, s^2 - 7 \right\rangle$$

Equating coordinates we get:

$$\begin{cases} t^2 + 3 = 4s \\ t + 1 = 2s - 2 \\ \dfrac{6}{t} = s^2 - 7 \end{cases}$$

By the second equation, $t = 2s - 3$. Substituting into the first equation yields:

$$\begin{aligned}(2s-3)^2 + 3 &= 4s\\ 4s^2 - 12s + 9 + 3 &= 4s\\ s^2 - 4s + 3 &= 0 \quad \Rightarrow \quad s_1 = 1, \quad s_2 = 3\end{aligned}$$

Substituting $s_1 = 1$ and $s_2 = 3$ into the second equation gives:

$$\begin{aligned}t_1 + 1 = 2 \cdot 1 - 2 \quad &\Rightarrow \quad t_1 = -1\\ t_2 + 1 = 2 \cdot 3 - 2 \quad &\Rightarrow \quad t_2 = 3\end{aligned}$$

The solutions of the first two equations are:

$$t_1 = -1, \quad s_1 = 1; \qquad t_2 = 3, \quad s_2 = 3$$

We check if these solutions satisfy the third equation:

$$\begin{aligned}\frac{6}{t_1} = \frac{6}{-1} = -6, \quad s_1^2 - 7 = 1^2 - 7 = -6 \quad &\Rightarrow \quad \frac{6}{t_1} = s_1^2 - 7\\ \frac{6}{t_2} = \frac{6}{3} = 2, \quad s_2^2 - 7 = 3^2 - 7 = 2 \quad &\Rightarrow \quad \frac{6}{t^2} = s_2^2 - 7\end{aligned}$$

We conclude that the paths intersect at the endpoints of the vectors $\mathbf{r}_1(-1)$ and $\mathbf{r}_1(3)$ (or equivalently $\mathbf{r}_2(1)$ and $\mathbf{r}_2(3)$). That is, at the points $(4, 0, -6)$ and $(12, 4, 2)$.

30. Determine whether $\mathbf{r}_1$ and $\mathbf{r}_2$ collide or intersect:

$$\mathbf{r}_1(t) = \left\langle t, t^2, t^3 \right\rangle, \qquad \mathbf{r}_2(t) = \left\langle 4t + 6, 4t^2, 7 - t \right\rangle$$

SOLUTION The two paths collide if there exists a value of t such that:

$$\left\langle t, t^2, t^3 \right\rangle = \left\langle 4t + 6, 4t^2, 7 - t \right\rangle$$

Equating corresponding components we obtain the following equations:

$$\begin{aligned}t &= 4t + 6\\ t^2 &= 4t^2\\ t^3 &= 7 - t\end{aligned}$$

The second equation implies that $t = 0$, but this value does not satisfy the other equations. Therefore, the equations have no solution, which means that the paths do not collide. The two paths intersect if there exist values of t and s such that:

$$\left\langle t, t^2, t^3 \right\rangle = \left\langle 4s + 6, 4s^2, 7 - s \right\rangle$$

Or equivalently:

$$\begin{aligned}t &= 4s + 6\\ t^2 &= 4s^2\\ t^3 &= 7 - s\end{aligned} \tag{1}$$

The second equation implies that $t_1 = 2s$ or $t_2 = -2s$. Substituting $t_1 = 2s$ and $t_2 = -2s$ in the first equation gives:

$$\begin{aligned}t_1 = 2s : 2s = 4s + 6 \quad &\Rightarrow \quad 2s = -6 \quad \Rightarrow \quad s_1 = -3\\ t_2 = -2s : -2s = 4s + 6 \quad &\Rightarrow \quad 6s = -6 \quad \Rightarrow \quad s_2 = -1\end{aligned}$$

The solutions of the first two equations are thus

$$(t_1, s_1) = (-6, -3); \qquad (t_2, s_2) = (2, -1)$$

(t_1, s_1) does not satisfy the third equation whereas (t_2, s_2) does. We conclude that the equations in (1) have a solution $t = 2$, $s = -1$, hence the two paths intersect.

In Exercises 31–40, find a parametrization of the curve.

31. The vertical line passing through the point $(3, 2, 0)$

SOLUTION The points of the vertical line passing through the point $(3, 2, 0)$ can be written as $(3, 2, z)$. Using $z = t$ as parameter we get the following parametrization:

$$\mathbf{r}(t) = \langle 3, 2, t \rangle, \quad -\infty < t < \infty$$

32. The line passing through $(1, 0, 4)$ and $(4, 1, 2)$

SOLUTION We use the vector parametrization of the line passing through the point $(x_0, y_0, z_0) = (1, 0, 4)$ in the direction of $\mathbf{v} = \langle 4 - 1, 1 - 0, 2 - 4 \rangle = \langle 3, 1, -2 \rangle$. We obtain:

$$\mathbf{r}(t) = \langle 1, 0, 4 \rangle + t \langle 3, 1, -2 \rangle = \langle 1 + 3t, t, 4 - 2t \rangle$$

33. The line through the origin whose projection on the xy-plane is a line of slope 3 and whose projection on the yz-plane is a line of slope 5 (i.e., $\Delta z/\Delta y = 5$)

SOLUTION We denote by (x, y, z) the points on the line. The projection of the line on the xy-plane is the line through the origin having slope 3, that is the line $y = 3x$ in the xy-plane. The projection of the line on the yz-plane is the line through the origin with slope 5, that is the line $z = 5y$. Thus, the points on the desired line satisfy the following equalities:

$$\begin{matrix} y = 3x \\ z = 5y \end{matrix} \quad \Rightarrow \quad y = 3x, \ z = 5 \cdot 3x = 15x$$

We conclude that the points on the line are all the points in the form $(x, 3x, 15x)$. Using $x = t$ as parameter we obtain the following parametrization:

$$\mathbf{r}(t) = \langle t, 3t, 15t \rangle, \quad -\infty < t < \infty.$$

34. The horizontal circle of radius 1 with center $(2, -1, 4)$

SOLUTION The projection of the circle on the xy-plane is the circle of radius 1 centered at the point $(2, -1)$. This circle has the parametrization:

$$x = 2 + \cos t, \quad y = -1 + \sin t$$

Since the circle is contained in the horizontal plane $z = 4$, the z-coordinates of the points on the circle is $z = 4$. We obtain the following parametrization:

$$\mathbf{r}(t) = \langle 2 + \cos t, -1 + \sin t, 4 \rangle, \quad 0 \le t \le 2\pi.$$

35. The circle of radius 2 with center $(1, 2, 5)$ in a plane parallel to the yz-plane

SOLUTION The circle is parallel to the yz-plane and centered at $(1, 2, 5)$, hence the x-coordinates of the points on the circle are $x = 1$. The projection of the circle on the yz-plane is a circle of radius 2 centered at $(2, 5)$. This circle is parametrized by:

$$y = 2 + 2\cos t, \quad z = 5 + 2\sin t$$

We conclude that the points on the required circle can be written as $(1, 2 + 2\cos t, 5 + 2\sin t)$. This gives the following parametrization:

$$\mathbf{r}(t) = \langle 1, 2 + 2\cos t, 5 + 2\sin t \rangle, \quad 0 \le t \le 2\pi.$$

36. The ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ in the xy-plane, translated to have center $(9, -4, 0)$

SOLUTION We first parametrize the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ by:

$$x = 2\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi; \quad \text{or} \quad \mathbf{r}_1(t) = \langle 2\cos t, 3\sin t \rangle, \quad 0 \le t \le 2\pi$$

We verify that the above is a parametrization of the ellipse by showing that $x = 2\cos t$, $y = 3\sin t$ satisfy the equation of the ellipse:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{2\cos t}{2}\right)^2 + \left(\frac{3\sin t}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$

We now translate the ellipse so that it remains in the xy-plane, with a center at $(9, -4, 0)$. This is done by translating along the vector $\langle 9, -4, 0 \rangle$, considering the ellipse as a curve in R^3. We get:

$$\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 0 \rangle + \langle 9, -4, 0 \rangle = \langle 9 + 2\cos t, -4 + 3\sin t, 0 \rangle$$

which holds for $0 \le t \le 2\pi$.

37. The intersection of the plane $y = \frac{1}{2}$ with the sphere $x^2 + y^2 + z^2 = 1$

SOLUTION Substituting $y = \frac{1}{2}$ in the equation of the sphere gives:

$$x^2 + \left(\frac{1}{2}\right)^2 + z^2 = 1 \quad \Rightarrow \quad x^2 + z^2 = \frac{3}{4}$$

This circle in the horizontal plane $y = \frac{1}{2}$ has the parametrization $x = \frac{\sqrt{3}}{2}\cos t$, $z = \frac{\sqrt{3}}{2}\sin t$. Therefore, the points on the intersection of the plane $y = \frac{1}{2}$ and the sphere $x^2 + y^2 + z^2 = 1$, can be written in the form $\left(\frac{\sqrt{3}}{2}\cos t, \frac{1}{2}, \frac{\sqrt{3}}{2}\sin t\right)$, yielding the following parametrization:

$$\mathbf{r}(t) = \left\langle \frac{\sqrt{3}}{2}\cos t, \frac{1}{2}, \frac{\sqrt{3}}{2}\sin t \right\rangle, \quad 0 \le t \le 2\pi.$$

38. The intersection of the surfaces

$$z = x^2 - y^2 \qquad \text{and} \qquad z = x^2 + xy - 1$$

SOLUTION We solve for x and z in terms of y. Equating the two equations and solving for x gives:

$$\begin{aligned} x^2 - y^2 &= x^2 + xy - 1 \\ -y^2 &= xy - 1 \\ xy &= 1 - y^2 \end{aligned}$$

Notice that on the intersection curve, $y \ne 0$ (since $y = 0$ gives $z = x^2$ and $z = x^2 - 1$, which are not equal for any z). Dividing by y we get:

$$x = \frac{1 - y^2}{y} = -y + \frac{1}{y}$$

We now substitute x in terms of y in the equation $z = x^2 - y^2$, to obtain:

$$z = \left(\frac{1 - y^2}{y}\right)^2 - y^2 = \frac{(1 - y^2)^2 - y^4}{y^2} = \frac{1 - 2y^2}{y^2} = -2 + \frac{1}{y^2}$$

Therefore the points of the intersection can be written in the form $\left(-y + \frac{1}{y}, y, -2 + \frac{1}{y^2}\right)$. Choosing $y = t$ as parameter we get the following parametrization:

$$\mathbf{r}(t) = \left\langle -t + \frac{1}{t}, t, -2 + \frac{1}{t^2} \right\rangle.$$

39. The ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = 1$ in the xz-plane, translated to have center $(3, 1, 5)$ [Figure 15(A)]

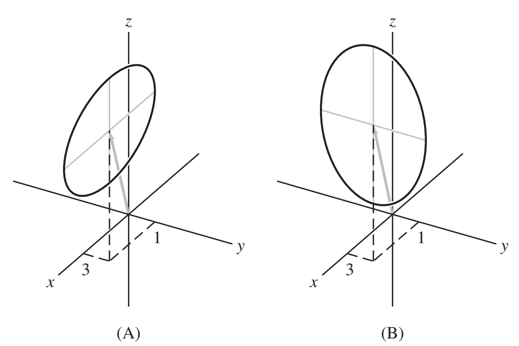


FIGURE 15 The ellipses described in Exercises 39 and 40.

SOLUTION The translated ellipse is in the vertical plane $y = 1$, hence the y-coordinate of the points on this ellipse is $y = 1$. The x and z coordinates satisfy the equation of the ellipse:

$$\left(\frac{x - 3}{2}\right)^2 + \left(\frac{z - 5}{3}\right)^2 = 1.$$

This ellipse is parametrized by the following equations:

$$x = 3 + 2\cos t, \quad z = 5 + 3\sin t.$$

Therefore, the points on the translated ellipse can be written as $(3 + 2\cos t, 1, 5 + 3\sin t)$. This gives the following parametrization:

$$\mathbf{r}(t) = \langle 3 + 2\cos t, 1, 5 + 3\sin t \rangle, \quad 0 \le t \le 2\pi.$$

40. The ellipse $\left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = 1$, translated to have center (3, 1, 5) [Figure 15(B)]

SOLUTION The translated ellipse is contained in the plane $x = 3$, hence the x-coordinate of the points on the ellipse is $x = 3$. The y and z coordinates satisfy the equation of the ellipse:

$$\left(\frac{y-1}{2}\right)^2 + \left(\frac{z-5}{3}\right)^2 = 1$$

This ellipse is parametrized by the equations:

$$y = 1 + 2\cos t, \quad z = 5 + 3\sin t$$

We conclude that the points on the translated ellipse can be written as $(3,\, 1 + 2\cos t,\, 5 + 3\sin t)$, which gives the following parametrization:

$$\mathbf{r}(t) = \langle 3, 1 + 2\cos t, 5 + 3\sin t\rangle, \quad 0 \le t \le 2\pi.$$

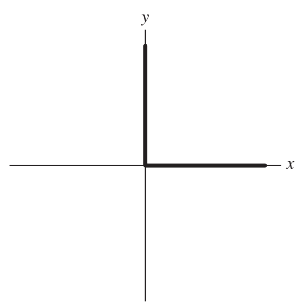
Further Insights and Challenges

41. Sketch the curve parametrized by $\mathbf{r}(t) = \langle |t| + t, |t| - t\rangle$.

SOLUTION We have:

$$|t| + t = \begin{cases} 0 & t \le 0 \\ 2t & t > 0 \end{cases}; \qquad |t| - t = \begin{cases} 2t & t \le 0 \\ 0 & t > 0 \end{cases}$$

As t increases from $-\infty$ to 0, the x-coordinate is zero and the y-coordinate is positive and decreasing to zero. As t increases from 0 to $+\infty$, the y-coordinate is zero and the x-coordinate is positive and increasing to $+\infty$. We obtain the following curve:



$\mathbf{r}(t) = \langle |t| + t, |t| - t\rangle$

42. Find the maximum height above the xy-plane of a point on $\mathbf{r}(t) = \left\langle e^t, \sin t, t(4-t)\right\rangle$.

SOLUTION The height of a point is the value of the z-coordinate of the point. Therefore we need to maximize the function $z = t\,(4 - t)$. $z(t)$ is a quadratic function having the roots $t = 0$ and $t = 4$, hence the maximum value is obtained at the midpoint of the interval $0 \le t \le 4$, that is at $t = 2$. The corresponding value of z is:

$$z_{\max} = z(2) = 2\,(4-2) = 4$$

The point of maximum height is, thus,

$$(e^2, \sin 2, 4) \approx (7.39, 0.91, 4)$$

43. Let $\mathcal{C}$ be the curve obtained by intersecting a cylinder of radius r and a plane. Insert two spheres of radius r into the cylinder above and below the plane, and let F_1 and F_2 be the points where the plane is tangent to the spheres [Figure 16(A)]. Let K be the vertical distance between the equators of the two spheres. Rediscover Archimedes's proof that $\mathcal{C}$ is an ellipse by showing that every point P on $\mathcal{C}$ satisfies

$$PF_1 + PF_2 = K \qquad \boxed{2}$$

Hint: If two lines through a point P are tangent to a sphere and intersect the sphere at Q_1 and Q_2 as in Figure 16(B), then the segments $\overline{PQ_1}$ and $\overline{PQ_2}$ have equal length. Use this to show that $PF_1 = PR_1$ and $PF_2 = PR_2$.

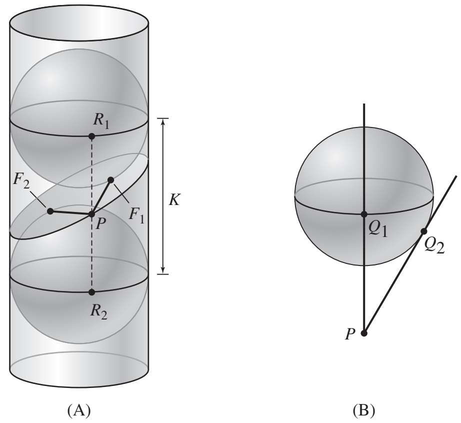
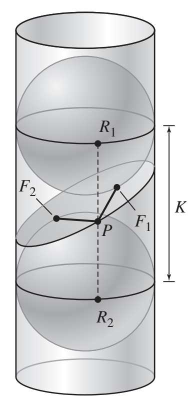


FIGURE 16

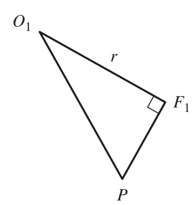
SOLUTION To show that $\mathcal{C}$ is an ellipse, we show that every point P on $\mathcal{C}$ satisfies:

$$\overline{F_1 P} + \overline{F_2 P} = K$$

We denote the points of intersection of the vertical line through P with the equators of the two spheres by R_1 and R_2 (see figure).



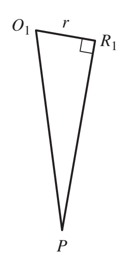
We denote by O_1 and O_2 the centers of the spheres.



Since F_1 is the tangency point, the radius $\overline{O_1 F_1}$ is perpendicular to the plane of the curve $\mathcal{C}$, and therefore it is orthogonal to the segment $\overline{PF_1}$ on this plane. Hence, $\Delta O_1 F_1 P$ is a right triangle and by Pythagoras' Theorem we have:

$$\overline{O_1 F_1}^2 + \overline{PF_1}^2 = \overline{O_1 P}^2$$

$$r^2 + \overline{PF_1}^2 = \overline{O_1 P}^2 \quad \Rightarrow \quad \overline{PF_1} = \sqrt{\overline{O_1 P}^2 - r^2} \tag{1}$$



$\Delta O_1 R_1 P$ is also a right triangle, hence by Pythagoras' Theorem we have:

$$\overline{O_1 R_1}^2 + \overline{R_1 P}^2 = \overline{O_1 P}^2$$

$$r^2 + \overline{R_1P}^2 = \overline{O_1P}^2 \quad \Rightarrow \quad \overline{PR_1} = \sqrt{\overline{O_1P}^2 - r^2} \tag{2}$$

Combining (1) and (2) we get:

$$\overline{PF_1} = \overline{PR_1} \tag{3}$$

Similarly we have:

$$\overline{PF_2} = \overline{PR_2} \tag{4}$$

We now combine (3), (4) and the equality $\overline{PR_1} + \overline{PR_2} = K$ to obtain:

$$\overline{F_1P} + \overline{F_2P} = \overline{PR_1} + \overline{PR_2} = K$$

Thus, the sum of the distances of the points P on $\mathcal{C}$ to the two fixed points F_1 and F_2 is a constant $K > 0$, hence $\mathcal{C}$ is an ellipse.

44. Assume that the cylinder in Figure 16 has equation $x^2 + y^2 = r^2$ and the plane has equation $z = ax + by$. Find a vector parametrization $\mathbf{r}(t)$ of the curve of intersection using the trigonometric functions $\cos t$ and $\sin t$.

SOLUTION Since $x^2 + y^2 = r^2$, it is convenient to make $x = r\cos t$ and $y = r\sin t$ with $0 \le t \le 2\pi$. Then the curve of intersection using these trigonometric functions will be $z = ar\cos t + br\sin t$ and one vector parameterization for this curve is

$$\mathbf{r}(t) = \langle r\cos t, r\sin t, ar\cos t + br\sin t\rangle, 0 \le t \le 2\pi.$$

45. CAS Now reprove the result of Exercise 43 using vector geometry. Assume that the cylinder has equation $x^2 + y^2 = r^2$ and the plane has equation $z = ax + by$.

(a) Show that the upper and lower spheres in Figure 16 have centers

$$C_1 = \left(0, 0, r\sqrt{a^2 + b^2 + 1}\right)$$
$$C_2 = \left(0, 0, -r\sqrt{a^2 + b^2 + 1}\right)$$

(b) Show that the points where the plane is tangent to the sphere are

$$F_1 = \frac{r}{\sqrt{a^2 + b^2 + 1}}\left(a, b, a^2 + b^2\right)$$
$$F_2 = \frac{-r}{\sqrt{a^2 + b^2 + 1}}\left(a, b, a^2 + b^2\right)$$

Hint: Show that $\overline{C_1F_1}$ and $\overline{C_2F_2}$ have length r and are orthogonal to the plane.

(c) Verify, with the aid of a computer algebra system, that Eq. (2) holds with

$$K = 2r\sqrt{a^2 + b^2 + 1}$$

To simplify the algebra, observe that since a and b are arbitrary, it suffices to verify Eq. (2) for the point $P = (r, 0, ar)$.

SOLUTION

(a) and (b) Since F_1 is the tangency point of the sphere and the plane, the radius to F_1 is orthogonal to the plane. Therefore to show that the center of the sphere is at C_1 and the tangency point is the given point we must show that:

$$\|\overrightarrow{C_1F_1}\| = r \tag{1}$$

$$\overrightarrow{C_1F_1} \text{ is orthogonal to the plane.} \tag{2}$$

We compute the vector $\overrightarrow{C_1F_1}$:

$$\overrightarrow{C_1F_1} = \left\langle \frac{ra}{\sqrt{a^2 + b^2 + 1}}, \frac{rb}{\sqrt{a^2 + b^2 + 1}}, \frac{r(a^2 + b^2)}{\sqrt{a^2 + b^2 + 1}} - r\sqrt{a^2 + b^2 + 1}\right\rangle = \frac{r}{\sqrt{a^2 + b^2 + 1}}\langle a, b, -1\rangle$$

Hence,

$$\|\overrightarrow{C_1F_1}\| = \frac{r}{\sqrt{a^2 + b^2 + 1}}\|\langle a, b, -1\rangle\| = \frac{r}{\sqrt{a^2 + b^2 + 1}}\sqrt{a^2 + b^2 + (-1)^2} = r$$

We, thus, proved that (1) is satisfied. To show (2) we must show that $\overrightarrow{C_1F_1}$ is parallel to the normal vector $\langle a, b, -1\rangle$ to the plane $z = ax + by$ (i.e., $ax + by - z = 0$). The two vectors are parallel since by (1) $\overrightarrow{C_1F_1}$ is a constant multiple of $\langle a, b, -1\rangle$. In a similar manner one can show (1) and (2) for the vector $\overrightarrow{C_2F_2}$.

(c) This is an extremely challenging problem. As suggested in the book, we use $P = (r, 0, ar)$, and we also use the expressions for F_1 and F_2 as given above. This gives us:

$$PF_1 = \sqrt{\left(1 + 2a^2 + b^2 - 2a\sqrt{1 + a^2 + b^2}\right) r^2}$$

$$PF_2 = \sqrt{\left(1 + 2a^2 + b^2 + 2a\sqrt{1 + a^2 + b^2}\right) r^2}$$

Their sum is not very inspiring:

$$PF_1 + PF_2 = \sqrt{\left(1 + 2a^2 + b^2 - 2a\sqrt{1 + a^2 + b^2}\right) r^2} + \sqrt{\left(1 + 2a^2 + b^2 + 2a\sqrt{1 + a^2 + b^2}\right) r^2}$$

Let us look, instead, at $(PF_1 + PF_2)^2$, and show that this is equal to K^2. Since everything is positive, this will imply that $PF_1 + PF_2 = K$, as desired.

$$\begin{aligned}(PF_1 + PF_2)^2 &= 2r^2 + 4a^2r^2 + 2b^2r^2 + 2\sqrt{r^4 + 2b^2r^4 + b^4r^4} \\ &= 2r^2 + 4a^2r^2 + 2b^2r^2 + 2(1 + b^2)r^2 = 4r^2(1 + a^2 + b^2) = K^2\end{aligned}$$

13.2 Calculus of Vector-Valued Functions (LT Section 14.2)

Preliminary Questions

1. State the three forms of the Product Rule for vector-valued functions.

SOLUTION The Product Rule for scalar multiple $f(t)$ of a vector-valued function $\mathbf{r}(t)$ states that:

$$\frac{d}{dt} f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

The Product Rule for dot products states that:

$$\frac{d}{dt}\mathbf{r}_1(t) \cdot \mathbf{r}_2(t) = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t)$$

Finally, the Product Rule for cross product is

$$\frac{d}{dt}\mathbf{r}_1(t) \times \mathbf{r}_2(t) = \mathbf{r}_1(t) \times \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \times \mathbf{r}_2(t).$$

In Questions 2–6, indicate whether the statement is true or false, and if it is false, provide a correct statement.

2. The derivative of a vector-valued function is defined as the limit of the difference quotient, just as in the scalar-valued case.

SOLUTION The statement is true. The derivative of a vector-valued function $\mathbf{r}(t)$ is defined a limit of the difference quotient:

$$\mathbf{r}'(t) = \lim_{t \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$

in the same way as in the scalar-valued case.

3. There are two Chain Rules for vector-valued functions: one for the composite of two vector-valued functions and one for the composite of a vector-valued and a scalar-valued function.

SOLUTION This statement is false. A vector-valued function $\mathbf{r}(t)$ is a function whose domain is a set of real numbers and whose range consists of position vectors. Therefore, if $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are vector-valued functions, the composition "$(\mathbf{r}_1 \cdot \mathbf{r}_2)(t) = \mathbf{r}_1(\mathbf{r}_2(t))$" has no meaning since $\mathbf{r}_2(t)$ is a vector and not a real number. However, for a scalar-valued function $f(t)$, the composition $\mathbf{r}(f(t))$ has a meaning, and there is a Chain Rule for differentiability of this vector-valued function.

4. The terms "velocity vector" and "tangent vector" for a path $\mathbf{r}(t)$ mean one and the same thing.

SOLUTION This statement is true.

5. The derivative of a vector-valued function is the slope of the tangent line, just as in the scalar case.

SOLUTION The statement is false. The derivative of a vector-valued function is again a vector-valued function, hence it cannot be the slope of the tangent line (which is a scalar). However, the derivative, $\mathbf{r}'(t_0)$ is the direction vector of the tangent line to the curve traced by $\mathbf{r}(t)$, at $\mathbf{r}(t_0)$.

6. The derivative of the cross product is the cross product of the derivatives.

SOLUTION The statement is false, since usually,

$$\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)\neq\mathbf{r}_1'(t)\times\mathbf{r}_2'(t)$$

The correct statement is the Product Rule for Cross Products. That is,

$$\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)=\mathbf{r}_1(t)\times\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\times\mathbf{r}_2(t)$$

7. State whether the following derivatives of vector-valued functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are scalars or vectors:

(a) $\frac{d}{dt}\mathbf{r}_1(t)$ **(b)** $\frac{d}{dt}\big(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\big)$ **(c)** $\frac{d}{dt}\big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\big)$

SOLUTION (a) vector, (b) scalar, (c) vector.

Exercises

In Exercises 1–6, evaluate the limit.

1. $\lim_{t\to 3}\left\langle t^2, 4t, \frac{1}{t}\right\rangle$

SOLUTION By the theorem on vector-valued limits we have:

$$\lim_{t\to 3}\left\langle t^2, 4t, \frac{1}{t}\right\rangle=\left\langle \lim_{t\to 3}t^2, \lim_{t\to 3}4t, \lim_{t\to 3}\frac{1}{t}\right\rangle=\left\langle 9, 12, \frac{1}{3}\right\rangle.$$

2. $\lim_{t\to\pi}\sin 2t\,\mathbf{i}+\cos t\,\mathbf{j}+\tan 4t\,\mathbf{k}$

SOLUTION We compute the limit of each component. That is:

$$\begin{aligned}\lim_{t\to\pi}(\sin 2t\,\mathbf{i}+\cos t\,\mathbf{j}+\tan 4t\,\mathbf{k})&=\left(\lim_{t\to\pi}\sin 2t\right)\mathbf{i}+\left(\lim_{t\to\pi}\cos t\right)\mathbf{j}+\left(\lim_{t\to\pi}\tan 4t\right)\mathbf{k}\\&=(\sin 2\pi)\,\mathbf{i}+(\cos\pi)\,\mathbf{j}+(\tan 4\pi)\,\mathbf{k}=-\mathbf{j}.\end{aligned}$$

3. $\lim_{t\to 0}e^{2t}\mathbf{i}+\ln(t+1)\mathbf{j}+4\mathbf{k}$

SOLUTION Computing the limit of each component, we obtain:

$$\lim_{t\to 0}\left(e^{2t}\mathbf{i}+\ln(t+1)\,\mathbf{j}+4\mathbf{k}\right)=\left(\lim_{t\to 0}e^{2t}\right)\mathbf{i}+\left(\lim_{t\to 0}\ln(t+1)\right)\mathbf{j}+\left(\lim_{t\to 0}4\right)\mathbf{k}=e^0\mathbf{i}+(\ln 1)\mathbf{j}+4\mathbf{k}=\mathbf{i}+4\mathbf{k}$$

4. $\lim_{t\to 0}\left\langle\frac{1}{t+1}, \frac{e^t-1}{t}, 4t\right\rangle$

SOLUTION We use the theorem on vector-valued limits and L'Hôpital's rule to write:

$$\lim_{t\to 0}\left\langle\frac{1}{t+1}, \frac{e^t-1}{t}, 4t\right\rangle=\left\langle\lim_{t\to 0}\frac{1}{t+1}, \lim_{t\to 0}\frac{e^t-1}{t}, \lim_{t\to 0}4t\right\rangle=\left\langle 1, \lim_{t\to 0}\frac{e^t}{1}, 0\right\rangle=\langle 1,1,0\rangle.$$

5. Evaluate $\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ for $\mathbf{r}(t)=\left\langle t^{-1}, \sin t, 4\right\rangle$.

SOLUTION This limit is the derivative $\frac{d\mathbf{r}}{dt}$. Using componentwise differentiation yields:

$$\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}=\frac{d\mathbf{r}}{dt}=\left\langle\frac{d}{dt}\left(t^{-1}\right), \frac{d}{dt}(\sin t), \frac{d}{dt}(4)\right\rangle=\left\langle-\frac{1}{t^2}, \cos t, 0\right\rangle.$$

6. Evaluate $\lim_{t\to 0} \frac{\mathbf{r}(t)}{t}$ for $\mathbf{r}(t) = \langle \sin t, 1 - \cos t, -2t \rangle$.

SOLUTION Since $\mathbf{r}(0) = \langle \sin 0, 1 - \cos 0, -2 \cdot 0 \rangle = \langle 0, 0, 0 \rangle$ we may think of the limit $\lim_{t\to 0} \frac{\mathbf{r}(t)}{t}$ as a derivative and compute it using componentwise differentiation. That is,

$$\lim_{t\to 0} \frac{\mathbf{r}(t)}{t} = \lim_{t\to 0} \frac{\mathbf{r}(t) - \mathbf{r}(0)}{t} = \lim_{h\to 0} \frac{\mathbf{r}\,(0+h) - \mathbf{r}(0)}{h} = \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \left.\left\langle \frac{d}{dt}\,(\sin t)\,, \frac{d}{dt}\,(1 - \cos t)\,, \frac{d}{dt}\,(-2t) \right\rangle\right|_{t=0}$$

$$= \left.\langle \cos t, \sin t, -2 \rangle\right|_{t=0} = \langle \cos 0, \sin 0, -2 \rangle = \langle 1, 0, -2 \rangle$$

In Exercises 7–12, compute the derivative.

7. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

SOLUTION Using componentwise differentiation we get:

$$\frac{d\mathbf{r}}{dt} = \left\langle \frac{d}{dt}(t), \frac{d}{dt}(t^2), \frac{d}{dt}(t^3) \right\rangle = \left\langle 1, 2t, 3t^2 \right\rangle$$

8. $\mathbf{r}(t) = \langle 7 - t, 4\sqrt{t}, 8 \rangle$

SOLUTION Using componentwise differentiation we get:

$$\frac{d\mathbf{r}}{dt} = \left\langle \frac{d}{dt}(7 - t), \frac{d}{dt}(4\sqrt{t}), \frac{d}{dt}(8) \right\rangle = \left\langle -1, 2t^{-1/2}, 0 \right\rangle = \left\langle -1, \frac{2}{\sqrt{t}}, 0 \right\rangle$$

9. $\mathbf{r}(s) = \langle e^{3s}, e^{-s}, s^4 \rangle$

SOLUTION Using componentwise differentiation we get:

$$\frac{d\mathbf{r}}{ds} = \left\langle \frac{d}{ds}(e^{3s}), \frac{d}{ds}(e^{-s}), \frac{d}{ds}(s^4) \right\rangle = \left\langle 3e^{3s}, -e^{-s}, 4s^3 \right\rangle$$

10. $\mathbf{b}(t) = \left\langle e^{3t-4}, e^{6-t}, (t+1)^{-1} \right\rangle$

SOLUTION Using componentwise differentiation we get:

$$\frac{d\mathbf{b}}{dt} = \left\langle \frac{d}{dt}(e^{3t-4}), \frac{d}{dt}(e^{6-t}), \frac{d}{dt}((t+1)^{-1}) \right\rangle = \left\langle 3e^{3t-4}, -e^{6-t}, \frac{-1}{(t+1)^2} \right\rangle$$

11. $\mathbf{c}(t) = t^{-1}\mathbf{i} - e^{2t}\mathbf{k}$

SOLUTION Using componentwise differentiation we get:

$$\mathbf{c}'(t) = \left(t^{-1}\right)'\mathbf{i} - \left(e^{2t}\right)'\mathbf{k} = -t^{-2}\mathbf{i} - 2e^{2t}\mathbf{k}$$

12. $\mathbf{a}(\theta) = (\cos 3\theta)\mathbf{i} + (\sin^2\theta)\mathbf{j} + (\tan\theta)\mathbf{k}$

SOLUTION Using componentwise differentiation we get:

$$\mathbf{a}'(\theta) = -3\sin 3\theta\mathbf{i} + 2\sin\theta\cos\theta\mathbf{j} + \sec^2\theta\mathbf{k}$$

13. Calculate $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

SOLUTION We perform the differentiation componentwise to obtain:

$$\mathbf{r}'(t) = \left\langle (t)', (t^2)', (t^3)' \right\rangle = \left\langle 1, 2t, 3t^2 \right\rangle$$

We now differentiate the derivative vector to find the second derivative:

$$\mathbf{r}''(t) = \frac{d}{dt}\left\langle 1, 2t, 3t^2 \right\rangle = \langle 0, 2, 6t \rangle\,.$$

14. Sketch the curve $\mathbf{r}(t) = \langle 1 - t^2, t \rangle$ for $-1 \le t \le 1$. Compute the tangent vector at $t = 1$ and add it to the sketch.

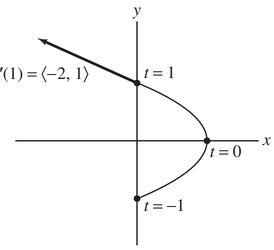
SOLUTION We find that

$$\mathbf{r}'(t) = \frac{d}{dt}\langle 1 - t^2, t \rangle = \langle -2t, 1 \rangle$$

and so at $t = 1$, we have

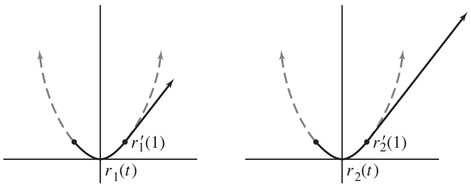
$$\mathbf{r}'(1) = \langle -2, 1 \rangle$$

To graph $\mathbf{r}(t)$, we note that it satisfies $x = 1 - y^2$. The sketch is shown here, along with the tangent vector at $t = 1$.



15. Sketch the curve $\mathbf{r}_1(t) = \langle t, t^2 \rangle$ together with its tangent vector at $t = 1$. Then do the same for $\mathbf{r}_2(t) = \langle t^3, t^6 \rangle$.

SOLUTION Note that $\mathbf{r}_1{}'(t) = \langle 1, 2t \rangle$ and so $\mathbf{r}_1{}'(1) = \langle 1, 2 \rangle$. The graph of $\mathbf{r}_1(t)$ satisfies $y = x^2$. Likewise, $\mathbf{r}_2{}'(t) = \langle 3t^2, 6t^5 \rangle$ and so $\mathbf{r}_2{}'(1) = \langle 3, 6 \rangle$. The graph of $\mathbf{r}_2(t)$ also satisfies $y = x^2$. Both graphs and tangent vectors are given here.



16. Sketch the cycloid $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ together with its tangent vectors at $t = \frac{\pi}{3}$ and $\frac{3\pi}{4}$.

SOLUTION The tangent vector $\mathbf{r}'(t)$ is the following vector:

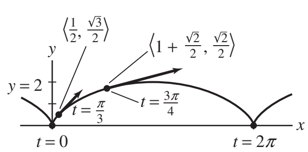
$$\mathbf{r}'(t) = \frac{d}{dt}\langle t - \sin t, 1 - \cos t \rangle = \langle 1 - \cos t, \sin t \rangle$$

Substituting the given values gives the following vectors:

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle 1 - \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$\mathbf{r}'\left(\frac{3\pi}{4}\right) = \left\langle 1 - \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4} \right\rangle = \left\langle 1 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

The cycloid $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ and the two tangent vectors are shown in the following figure:



In Exercises 17–20, evaluate the derivative by using the appropriate Product Rule, where

$$\mathbf{r}_1(t) = \langle t^2, t^3, t \rangle, \qquad \mathbf{r}_2(t) = \langle e^{3t}, e^{2t}, e^t \rangle$$

17. $\frac{d}{dt}\big(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)\big)$

SOLUTION

$$\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t)$$

$$= \left\langle t^2, t^3, t\right\rangle \cdot \left\langle 3e^{3t}, 2e^{2t}, e^t\right\rangle + \left\langle 2t, 3t^2, 1\right\rangle \cdot \left\langle e^{3t}, e^{2t}, e^t\right\rangle$$
$$= 3t^2e^{3t} + 2t^3e^{2t} + te^t + 2te^{3t} + 3t^2e^{2t} + e^t$$
$$= (3t^2+2t)e^{3t} + (2t^3+3t^2)e^{2t} + (t+1)e^t$$

18. $\dfrac{d}{dt}\left(t^4\mathbf{r}_1(t)\right)$

SOLUTION

$$\begin{aligned}\frac{d}{dt}(t^4\mathbf{r}_1(t)) &= t^4\mathbf{r}_1'(t) + \frac{d}{dt}(t^4)\mathbf{r}_1(t)\\ &= t^4\left\langle 2t, 3t^2, 1\right\rangle + (4t^3)\left\langle t^2, t^3, t\right\rangle\\ &= \left\langle 2t^5, 3t^6, t^4\right\rangle + \left\langle 4t^5, 4t^6, 4t^4\right\rangle\\ &= \left\langle 6t^5, 7t^6, 5t^4\right\rangle\end{aligned}$$

19. $\dfrac{d}{dt}\left(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right)$

SOLUTION

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}_1(t)\times\mathbf{r}_2(t)) &= \mathbf{r}_1(t)\times\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\times\mathbf{r}_2(t)\\ &= \left\langle t^2, t^3, t\right\rangle \times \left\langle 3e^{3t}, 2e^{2t}, e^t\right\rangle + \left\langle 2t, 3t^2, 1\right\rangle \times \left\langle e^{3t}, e^{2t}, e^t\right\rangle\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t^2 & t^3 & t\\ 3e^{3t} & 2e^{2t} & e^t\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2t & 3t^2 & 1\\ e^{3t} & e^{2t} & e^t\end{vmatrix}\\ &= (t^3e^t - 2te^{2t})\mathbf{i} + (3te^{3t} - t^2e^t)\mathbf{j} + (2t^2e^{2t} - 3t^3e^{3t})\mathbf{k}\\ &\quad + (3t^2e^t - e^{2t})\mathbf{i} + (e^{3t} - 2te^t)\mathbf{j} + (2te^{2t} - 3t^2e^{3t})\mathbf{k}\\ &= [(t^3+3t^2)e^t - (2t+1)e^{2t}]\mathbf{i} + [(3t+1)e^{3t} - (t^2+2t)e^t]\mathbf{j}\\ &\quad + [(2t^2+2t)e^{2t} - (3t^3+3t^2)e^{3t}]\mathbf{k}\end{aligned}$$

20. $\dfrac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}_1(t)\right)\Big|_{t=2}$, assuming that

$$\mathbf{r}(2) = \langle 2, 1, 0\rangle, \qquad \mathbf{r}'(2) = \langle 1, 4, 3\rangle$$

SOLUTION

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}(t)\cdot)\mathbf{r}_1(t))\Big|_{t=2} &= \mathbf{r}(t)\cdot\mathbf{r}_1'(t)\Big|_{t=2} + \mathbf{r}'(t)\cdot\mathbf{r}_1'(t)\Big|_{t=2}\\ &= \mathbf{r}(t)\cdot\left\langle 2t, 3t^2, 1\right\rangle\Big|_{t=2} + \mathbf{r}'(t)\cdot\left\langle t^2, t^3, t\right\rangle\Big|_{t=2}\\ &= \mathbf{r}(2)\cdot\langle 4, 12, 1\rangle + \mathbf{r}'(2)\cdot\langle 4, 8, 2\rangle\\ &= \langle 2, 1, 0\rangle\cdot\langle 4, 12, 1\rangle + \langle 1, 4, 3\rangle\cdot\langle 4, 8, 2\rangle\\ &= (8+12+0) + (4+32+6) = 62\end{aligned}$$

In Exercises 21 and 22, let

$$\mathbf{r}_1(t) = \langle t^2, 1, 2t\rangle, \qquad \mathbf{r}_2(t) = \left\langle 1, 2, e^t\right\rangle$$

21. Compute $\dfrac{d}{dt}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\Big|_{t=1}$ in two ways:

(a) Calculate $\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)$ and differentiate.

(b) Use the Product Rule.

SOLUTION

(a) First we will calculate $\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)$:

$$\begin{aligned}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t) &= \left\langle t^2, 1, 2t\right\rangle\cdot\left\langle 1, 2, e^t\right\rangle\\ &= t^2 + 2 + 2te^t\end{aligned}$$

And then differentiating we get:

$$\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)) = \frac{d}{dt}(t^2+2+2te^t) = 2t+2te^t+2e^t$$
$$\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t))\Big|_{t=1} = 2+2e+2e = 2+4e$$

(b) First we differentiate:

$$\mathbf{r}_1(t) = \left\langle t^2, 1, 2t\right\rangle, \qquad \mathbf{r}_1'(t) = \langle 2t, 0, 2\rangle$$
$$\mathbf{r}_2(t) = \left\langle 1, 2, e^t\right\rangle, \qquad \mathbf{r}_2'(t) = \left\langle 0, 0, e^t\right\rangle$$

Using the Product Rule we see:

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)) &= \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)\\ &= \left\langle t^2,1,2t\right\rangle\cdot\left\langle 0,0,e^t\right\rangle + \langle 2t,0,2\rangle\cdot\left\langle 1,2,e^t\right\rangle\\ &= 2te^t+2t+2e^t\\ \frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t))\Big|_{t=1} &= 2e+2+2e = 2+4e\end{aligned}$$

22. Compute $\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)\Big|_{t=1}$ in two ways:
(a) Calculate $\mathbf{r}_1(t)\times\mathbf{r}_2(t)$ and differentiate.
(b) Use the Product Rule.

SOLUTION
(a) First we will calculate $\mathbf{r}_1(t)\times\mathbf{r}_2(t)$:

$$\mathbf{r}_1(t)\times\mathbf{r}_2(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t^2 & 1 & 2t\\ 1 & 2 & e^t\end{vmatrix} = (e^t-4t)\mathbf{i}+(2t-t^2e^t)\mathbf{j}+(2t^2-1)\mathbf{k}$$

And then differentiating we get:

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}_1(t)\times\mathbf{r}_2(t))\Big|_{t=1} &= \frac{d}{dt}((e^t-4t)\mathbf{i}+(2t-t^2e^t)\mathbf{j}+(2t^2-1)\mathbf{k})\Big|_{t=1}\\ &= (e^t-4)\mathbf{i}+(2-t^2e^t-2te^t)\mathbf{j}+(4t)\mathbf{k}\Big|_{t=1}\\ &= (e-4)\mathbf{i}+(2-3e)\mathbf{j}+4\mathbf{k}\end{aligned}$$

(b) First we differentiate:

$$\mathbf{r}_1(t) = \left\langle t^2, 1, 2t\right\rangle \quad\Rightarrow\quad \mathbf{r}_1(1) = \langle 1,1,2\rangle$$
$$\mathbf{r}_1'(1) = \langle 2t, 0, 2\rangle\Big|_{t=1} = \langle 2,0,2\rangle$$
$$\mathbf{r}_2(t) = \left\langle 1,2,e^t\right\rangle \quad\Rightarrow\quad \mathbf{r}_2(1) = \langle 1,2,e\rangle$$
$$\mathbf{r}_2' = \left\langle 0,0,e^t\right\rangle\Big|_{t=1} = \langle 0,0,e\rangle$$

Using the Product Rule we see:

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}_1(1)\times\mathbf{r}_2(1)) &= \mathbf{r}_1(1)\times\mathbf{r}_2'(1)+\mathbf{r}_1'(1)\times\mathbf{r}_2(1)\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 2\\ 0 & 0 & e\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & 2\\ 1 & 2 & e\end{vmatrix}\\ &= [e\mathbf{i}-e\mathbf{j}+0\mathbf{k}]+[-4\mathbf{i}+(2-2e)\mathbf{j}+4\mathbf{k}]\\ &= (e-4)\mathbf{i}+(2-e-2e)\mathbf{j}+4\mathbf{k} = (e-4)\mathbf{i}+(2-3e)\mathbf{j}+4\mathbf{k}\end{aligned}$$

In Exercises 23–26, evaluate $\frac{d}{dt}\mathbf{r}(g(t))$ *using the Chain Rule.*

23. $\mathbf{r}(t) = \langle t^2, 1-t \rangle, \quad g(t) = e^t$

SOLUTION We first differentiate the two functions:

$$\mathbf{r}'(t) = \frac{d}{dt}\left\langle t^2, 1-t \right\rangle = \langle 2t, -1 \rangle$$

$$g'(t) = \frac{d}{dt}(e^t) = e^t$$

Using the Chain Rule we get:

$$\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t)) = e^t \left\langle 2e^t, -1 \right\rangle = \left\langle 2e^{2t}, -e^t \right\rangle$$

24. $\mathbf{r}(t) = \langle t^2, t^3 \rangle, \quad g(t) = \sin t$

SOLUTION We first differentiate the two functions:

$$\mathbf{r}'(t) = \frac{d}{dt}\left\langle t^2, t^3 \right\rangle = \left\langle 2t, 3t^2 \right\rangle$$

$$g'(t) = \cos t$$

Using the Chain Rule we get:

$$\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t)) = \cos t \left\langle 2\sin t, 3\sin^2 t \right\rangle = \left\langle 2\sin t\cos t, 3\sin^2 t\cos t \right\rangle$$

25. $\mathbf{r}(t) = \langle e^t, e^{2t}, 4 \rangle, \quad g(t) = 4t+9$

SOLUTION We first differentiate the two functions:

$$\mathbf{r}'(t) = \frac{d}{dt}\langle e^t, e^{2t}, 4 \rangle = \langle e^t, 2e^{2t}, 0 \rangle$$

$$g'(t) = \frac{d}{dt}(4t+9) = 4$$

Using the Chain Rule we get:

$$\frac{d}{dt}\mathbf{r}\,(g(t)) = g'(t)\mathbf{r}'(g(t)) = 4\langle e^{4t+9}, 2e^{2(4t+9)}, 0 \rangle = \langle 4e^{4t+9}, 8e^{8t+18}, 0 \rangle$$

26. $\mathbf{r}(t) = \langle 4\sin 2t, 6\cos 2t \rangle, \quad g(t) = t^2$

SOLUTION We differentiate the two functions:

$$\mathbf{r}'(t) = \frac{d}{dt}\langle 4\sin 2t, 6\cos 2t \rangle = \langle 8\cos 2t, -12\sin 2t \rangle$$

$$g'(t) = 2t$$

Using the Chain Rule we obtain:

$$\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t)) = 2t\langle 8\cos 2t^2, -12\sin 2t^2 \rangle = 8t\langle 2\cos 2t^2, -3\sin 2t^2 \rangle$$

27. Let $\mathbf{r}(t) = \langle t^2, 1-t, 4t \rangle$. Calculate the derivative of $\mathbf{r}(t)\cdot\mathbf{a}(t)$ at $t=2$, assuming that $\mathbf{a}(2) = \langle 1, 3, 3 \rangle$ and $\mathbf{a}'(2) = \langle -1, 4, 1 \rangle$.

SOLUTION By the Product Rule for dot products we have

$$\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{a}(t) = \mathbf{r}(t)\cdot\mathbf{a}'(t) + \mathbf{r}'(t)\cdot\mathbf{a}(t)$$

At $t=2$ we have

$$\left.\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{a}(t)\right|_{t=2} = \mathbf{r}(2)\cdot\mathbf{a}'(2) + \mathbf{r}'(2)\cdot\mathbf{a}(2) \tag{1}$$

We compute the derivative $\mathbf{r}'(2)$:

$$\mathbf{r}'(t) = \frac{d}{dt}\left\langle t^2, 1-t, 4t\right\rangle = \langle 2t, -1, 4\rangle \quad \Rightarrow \quad \mathbf{r}'(2) = \langle 4, -1, 4\rangle \tag{2}$$

Also, $\mathbf{r}(2) = \left\langle 2^2, 1-2, 4\cdot 2\right\rangle = \langle 4, -1, 8\rangle$. Substituting the vectors in the equation above, we obtain:

$$\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{a}(t)\bigg|_{t=2} = \langle 4, -1, 8\rangle\cdot\langle -1, 4, 1\rangle + \langle 4, -1, 4\rangle\cdot\langle 1, 3, 3\rangle = (-4-4+8)+(4-3+12) = 13$$

The derivative of $\mathbf{r}(t)\cdot\mathbf{a}(t)$ at $t=2$ is 13.

28. Let $\mathbf{v}(s) = s^2\mathbf{i} + 2s\mathbf{j} + 9s^{-2}\mathbf{k}$. Evaluate $\frac{d}{ds}\mathbf{v}(g(s))$ at $s=4$, assuming that $g(4)=3$ and $g'(4)=-9$.

SOLUTION Applying the Chain Rule we have

$$\frac{d}{ds}\mathbf{v}\left(g(s)\right) = g'(s)\mathbf{v}'\left(g(s)\right)$$

The derivative at $s=4$ is, thus,

$$\frac{d}{ds}\mathbf{v}\left(g(s)\right)\bigg|_{s=4} = g'(4)\mathbf{v}'\left(g(4)\right) = -9\mathbf{v}'(3) \tag{1}$$

We differentiate $\mathbf{v}(s)$:

$$\mathbf{v}'(s) = 2s\mathbf{i} + 2\mathbf{j} - 18s^{-3}\mathbf{k} \quad \Rightarrow \quad \mathbf{v}'(3) = 6\mathbf{i} + 2\mathbf{j} - \frac{2}{3}\mathbf{k} \tag{2}$$

Combining (1) and (2) gives:

$$\frac{d}{ds}\mathbf{v}\left(g(s)\right)\bigg|_{s=4} = -9\left(6\mathbf{i} + 2\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = -54\mathbf{i} - 18\mathbf{j} + 6\mathbf{k}$$

In Exercises 29–34, find a parametrization of the tangent line at the point indicated.

29. $\mathbf{r}(t) = \left\langle t^2, t^4\right\rangle, \quad t=-2$

SOLUTION The tangent line has the following parametrization:

$$\ell(t) = \mathbf{r}(-2) + t\mathbf{r}'(-2) \tag{1}$$

We compute the vectors $\mathbf{r}(-2)$ and $\mathbf{r}'(-2)$:

$$\mathbf{r}(-2) = \left\langle (-2)^2, (-2)^4\right\rangle = \langle 4, 16\rangle$$
$$\mathbf{r}'(t) = \frac{d}{dt}\left\langle t^2, t^4\right\rangle = \left\langle 2t, 4t^3\right\rangle \quad \Rightarrow \quad \mathbf{r}'(-2) = \langle -4, -32\rangle$$

Substituting in (1) gives:

$$\ell(t) = \langle 4, 16\rangle + t\,\langle -4, -32\rangle = \langle 4-4t, 16-32t\rangle$$

The parametrization for the tangent line is, thus,

$$x = 4-4t, \quad y = 16-32t, \quad -\infty < t < \infty.$$

To find a direct relation between y and x, we express t in terms of x and substitute in $y = 16-32t$. This gives:

$$x = 4-4t \Rightarrow t = \frac{x-4}{-4}.$$

Hence,

$$y = 16-32t = 16-32\cdot\frac{x-4}{-4} = 16+8(x-4) = 8x-16.$$

The equation of the tangent line is $y = 8x-16$.

30. $\mathbf{r}(t) = \langle \cos 2t, \sin 3t \rangle, \quad t = \frac{\pi}{4}$

SOLUTION The tangent line is parametrized by:

$$\ell(t) = \mathbf{r}\left(\frac{\pi}{4}\right) + t\mathbf{r}'\left(\frac{\pi}{4}\right) \tag{1}$$

We compute the vectors in the above parametrization:

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \langle \cos\frac{\pi}{2}, \sin\frac{3\pi}{4} \rangle = \left\langle 0, \frac{1}{\sqrt{2}} \right\rangle$$

$$\mathbf{r}'(t) = \frac{d}{dt}\langle \cos 2t, \sin 3t \rangle = \langle -2\sin 2t, 3\cos 3t \rangle$$

$$\Rightarrow \quad \mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -2\sin\frac{\pi}{2}, 3\cos\frac{3\pi}{4} \right\rangle = \left\langle -2, \frac{-3}{\sqrt{2}} \right\rangle$$

Substituting the vectors in (1) we obtain the following parametrization:

$$\ell(t) = \left\langle 0, \frac{1}{\sqrt{2}} \right\rangle + t\left\langle -2, \frac{-3}{\sqrt{2}} \right\rangle = \left\langle -2t, \frac{1}{\sqrt{2}}(1-3t) \right\rangle$$

31. $\mathbf{r}(t) = \langle 1 - t^2, 5t, 2t^3 \rangle, \quad t = 2$

SOLUTION The tangent line is parametrized by:

$$\ell(t) = \mathbf{r}(2) + t\mathbf{r}'(2) \tag{1}$$

We compute the vectors in the above parametrization:

$$\mathbf{r}(2) = \langle 1 - 2^2, 5 \cdot 2, 2 \cdot 2^3 \rangle = \langle -3, 10, 16 \rangle$$

$$\mathbf{r}'(t) = \frac{d}{dt}\langle 1 - t^2, 5t, 2t^3 \rangle = \langle -2t, 5, 6t^2 \rangle \quad \Rightarrow \quad \mathbf{r}'(2) = \langle -4, 5, 24 \rangle$$

Substituting the vectors in (1) we obtain the following parametrization:

$$\ell(t) = \langle -3, 10, 16 \rangle + t\langle -4, 5, 24 \rangle = \langle -3 - 4t, 10 + 5t, 16 + 24t \rangle$$

32. $\mathbf{r}(t) = \langle 4t, 5t, 9t \rangle, \quad t = -4$

SOLUTION The tangent line is parametrized by:

$$\ell(t) = \mathbf{r}(-4) + t\mathbf{r}'(-4) \tag{1}$$

We compute the vectors in the above parametrization:

$$\mathbf{r}(-4) = \langle 4(-4), 5(-4), 9(-4) \rangle = \langle -16, -20, -36 \rangle$$

$$\mathbf{r}'(t) = \frac{d}{dt}\langle 4t, 5t, 9t \rangle = \langle 4, 5, 9 \rangle \quad \Rightarrow \quad \mathbf{r}'(-4) = \langle 4, 5, 9 \rangle$$

Substituting the vectors in (1) we obtain the following parametrization:

$$\ell(t) = \langle -16, -20, -36 \rangle + t\langle 4, 5, 9 \rangle = \langle -16 + 4t, -20 + 5t, -36 + 9t \rangle$$

33. $\mathbf{r}(s) = 4s^{-1}\mathbf{i} - \frac{8}{3}s^{-3}\mathbf{k}, \quad s = 2$

SOLUTION The tangent line is parametrized by:

$$\ell(s) = \mathbf{r}(2) + s\mathbf{r}'(2) \tag{1}$$

We compute the vectors in the above parametrization:

$$\mathbf{r}(2) = 4(2)^{-1}\mathbf{i} - \frac{8}{3}(2)^{-3}\mathbf{k} = 2\mathbf{i} - \frac{1}{3}\mathbf{k}$$

$$\mathbf{r}'(s) = \frac{d}{ds}\left(4s^{-1}\mathbf{i} - \frac{8}{3}s^{-3}\mathbf{k}\right) = -4s^{-2}\mathbf{i} + 8s^{-4}\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(2) = -\mathbf{i} + \frac{1}{2}\mathbf{k}$$

Substituting the vectors in (1) we obtain the following parametrization:

$$\ell(t) = \left(2\mathbf{i} - \frac{1}{3}\mathbf{k}\right) + s\left(-\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = (2 - s)\mathbf{i} + \left(\frac{1}{2}s - \frac{1}{3}\right)\mathbf{k}$$

34. $\mathbf{r}(s) = (\ln s)\mathbf{i} + s^{-1}\mathbf{j} + 9s\mathbf{k}, \quad s = 1$

SOLUTION The tangent line has the following parametrization:

$$\ell(s) = \mathbf{r}(1) + s\mathbf{r}'(1) \tag{1}$$

We compute the vectors $\mathbf{r}(1)$ and $\mathbf{r}'(1)$:

$$\mathbf{r}(1) = \ln 1\mathbf{i} + 1^{-1}\mathbf{j} + 9 \cdot 1\mathbf{k} = \mathbf{j} + 9\mathbf{k}$$

$$\mathbf{r}'(s) = \frac{d}{ds}(\ln s\mathbf{i} + s^{-1}\mathbf{j} + 9s\mathbf{k}) = \frac{1}{s}\mathbf{i} - s^{-2}\mathbf{j} + 9\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(1) = \mathbf{i} - \mathbf{j} + 9\mathbf{k}$$

We substitute the vectors in (1) to obtain the following parametrization:

$$\ell(s) = \mathbf{j} + 9\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 9\mathbf{k}) = s\mathbf{i} + (1 - s)\mathbf{j} + (9 + 9s)\mathbf{k}$$

or in scalar form:

$$x = s, \quad y = 1 - s, \quad z = 9 + 9s.$$

35. Use Example 4 to calculate $\frac{d}{dt}(\mathbf{r} \times \mathbf{r}')$, where $\mathbf{r}(t) = \langle t, t^2, e^t \rangle$.

SOLUTION In Example 4 it is proved that:

$$\frac{d}{dt}\mathbf{r} \times \mathbf{r}' = \mathbf{r} \times \mathbf{r}'' \tag{1}$$

We compute the derivatives $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \frac{d}{dt}\langle t, t^2, e^t \rangle = \langle 1, 2t, e^t \rangle$$

$$\mathbf{r}''(t) = \frac{d}{dt}\langle 1, 2t, e^t \rangle = \langle 0, 2, e^t \rangle$$

Using (1) we get

$$\frac{d}{dt}\mathbf{r} \times \mathbf{r}' = \mathbf{r} \times \mathbf{r}'' = \langle t, t^2, e^t \rangle \times \langle 0, 2, e^t \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & e^t \\ 0 & 2 & e^t \end{vmatrix} = \left(t^2e^t - 2e^t\right)\mathbf{i} - \left(0 - te^t\right)\mathbf{j} + \left(2t - 0\right)\mathbf{k}$$

$$= \left(t^2 - 2\right)e^t\mathbf{i} + te^t\mathbf{j} + 2t\mathbf{k} = \left\langle \left(t^2 - 2t\right)e^t, te^t, 2t \right\rangle$$

36. Let $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t \rangle$. Show that $\|\mathbf{r}(t)\|$ is constant and conclude, using Example 7, that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Then compute $\mathbf{r}'(t)$ and verify directly that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

SOLUTION First let us compute $||\mathbf{r}(t)||$:

$$||\mathbf{r}(t)|| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = \sqrt{25(\cos^2 t + \sin^2 t)} = \sqrt{25} = 5$$

Therefore, $||\mathbf{r}(t)||$ is constant. Using Example 7, we see:

$$\frac{d}{dt}||\mathbf{r}(t)||^2 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

Since $||\mathbf{r}(t)||$ is constant, its derivative is 0, therefore we get:

$$2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \Rightarrow \quad \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

and we can conclude that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

Now, computing in a different way, we know:

$$\mathbf{r}'(t) = \langle -3\sin t, 5\cos t, -4\sin t \rangle$$

and

$$\begin{aligned} \mathbf{r}(t) \cdot \mathbf{r}'(t) &= \langle 3\cos t, 5\sin t, 4\cos t \rangle \cdot \langle -3\sin t, 5\cos t, -4\sin t \rangle \\ &= -9\cos t\sin t + 25\sin t\cos t - 16\cos t\sin t \\ &= 0 \end{aligned}$$

Hence we can conclude that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

37. Show that the *derivative of the norm* is not equal to the *norm of the derivative* by verifying that $\|\mathbf{r}(t)\|' \neq \|\mathbf{r}'(t)\|$ for $\mathbf{r}(t) = \langle t, 1, 1\rangle$.

SOLUTION First let us compute $\|\mathbf{r}(t)\|'$ for $\mathbf{r}(t) = \langle t, 1, 1\rangle$:

$$\|\mathbf{r}(t)\|' = \frac{d}{dt}(\sqrt{t^2+2}) = \frac{t}{\sqrt{t^2+2}}$$

Now, first let us compute the derivative, $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \langle 1, 0, 0\rangle$$

and then computing the norm:

$$\|\mathbf{r}'(t)\| = \|\langle 1, 0, 0\rangle\| = \sqrt{1} = 1$$

It is clear in this example, that $\|\mathbf{r}(t)\|' \neq \|\mathbf{r}'(t)\|$.

38. Show that $\frac{d}{dt}(\mathbf{a} \times \mathbf{r}) = \mathbf{a} \times \mathbf{r}'$ for any constant vector $\mathbf{a}$.

SOLUTION We use the Product Rule for cross products and the derivative $\frac{d}{dt}(\mathbf{a}) = \mathbf{0}$ of the constant vector $\mathbf{a}$, to write:

$$\frac{d}{dt}\mathbf{a} \times \mathbf{r} = \mathbf{a} \times \mathbf{r}' + \mathbf{a}' \times \mathbf{r} = \mathbf{a} \times \mathbf{r}' + \mathbf{0} \times \mathbf{r} = \mathbf{a} \times \mathbf{r}' + \mathbf{0} = \mathbf{a} \times \mathbf{r}'$$

In Exercises 39–46, evaluate the integrals.

39. $\int_{-1}^{3} \left\langle 8t^2 - t, 6t^3 + t \right\rangle dt$

SOLUTION Vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component.

$$\int_{-1}^{3} 8t^2 - t\,dt = \frac{8}{3}t^3 - \frac{t^2}{2}\bigg|_{-1}^{3} = \left(72 - \frac{9}{2}\right) - \left(-\frac{8}{3} - \frac{1}{2}\right) = \frac{212}{3}$$

$$\int_{-1}^{3} 6t^3 + t\,dt = \frac{3}{2}t^4 + \frac{t^2}{2}\bigg|_{-1}^{3} = \left(\frac{243}{2} + \frac{9}{2}\right) - \left(\frac{3}{2} + \frac{1}{2}\right) = 124$$

Therefore,

$$\int_{-1}^{3} \left\langle 8t^2 - t, 6t^3 + t \right\rangle dt = \left\langle \int_{-1}^{3} 8t^2 - t\,dt, \int_{-1}^{3} 6t^3 + t\,dt \right\rangle = \left\langle \frac{212}{3}, 124 \right\rangle$$

40. $\int_{0}^{1} \left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2} \right\rangle ds$

SOLUTION The vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component. For the second integral we use the substitution $t = 1 + s^2$, $dt = 2s\,ds$. We get:

$$\int_0^1 \frac{ds}{1+s^2} = \tan^{-1}(s)\bigg|_0^1 = \tan^{-1}(1) - \tan^1(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^1 \frac{s}{1+s^2}\,ds = \int_1^2 \frac{1}{t}\left(\frac{dt}{2}\right) = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln t\bigg|_1^2 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Therefore,

$$\int_0^1 \left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2} \right\rangle ds = \left\langle \int_0^1 \frac{ds}{1+s^2}, \int_0^1 \frac{s\,ds}{1+s^2} \right\rangle = \left\langle \frac{\pi}{4}, \frac{1}{2}\ln 2 \right\rangle$$

41. $\int_{-2}^{2} \left(u^3\mathbf{i} + u^5\mathbf{j}\right) du$

SOLUTION The vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component.

$$\int_{-2}^{2} u^3\, du = \frac{u^4}{4}\bigg|_{-2}^{2} = \frac{16}{4} - \frac{16}{4} = 0$$

$$\int_{-2}^{2} u^5\, du = \frac{u^6}{6}\bigg|_{-2}^{2} = \frac{64}{6} - \frac{64}{6} = 0$$

Therefore,

$$\int_{-2}^{2} \left(u^3\mathbf{i} + u^5\mathbf{j}\right) du = \left(\int_{-2}^{2} u^3\, du\right)\mathbf{i} + \left(\int_{-2}^{2} u^5\, du\right)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$

42. $\int_{0}^{1} \left(te^{-t^2}\mathbf{i} + t\ln(t^2+1)\mathbf{j}\right) dt$

SOLUTION We compute the integral of each component. The integral of the first component is computed using the substitution $s = -t^2$, $ds = -2t\,dt$. This gives

$$\int_0^1 te^{-t^2}\, dt = \int_0^{-1} e^s \left(-\frac{ds}{2}\right) = \frac{1}{2}\int_{-1}^{0} e^s\, ds = \frac{1}{2}e^s\bigg|_{-1}^{0} = \frac{1}{2}\left(e^0 - e^{-1}\right) = \frac{1}{2}\left(1 - e^{-1}\right)$$

For the integral of the second component we use the substitution $s = t^2 + 1$, $ds = 2t\,dt$. This gives:

$$\int_0^1 t\ln(t^2+1)\, dt = \int_1^2 \ln s \frac{ds}{2} = \frac{1}{2}\int_1^2 \ln s\, ds = \frac{1}{2}s(\ln s - 1)\bigg|_1^2 = \frac{1}{2}\left(2(\ln 2 - 1) - 1(\ln 1 - 1)\right)$$
$$= \ln 2 - 1 + \frac{1}{2} = \ln 2 - \frac{1}{2}$$

Hence,

$$\int_0^1 \left\langle te^{-t^2}, t\ln(t^2+1)\right\rangle dt = \left\langle \frac{1}{2}\left(1 - e^{-1}\right), -\frac{1}{2} + \ln 2\right\rangle.$$

43. $\int_0^1 \langle 2t, 4t, -\cos 3t\rangle\, dt$

SOLUTION The vector valued integration is defined via componentwise integration. Therefore,

$$\int_0^1 \langle 2t, 4t, -\cos 3t\rangle\, dt = \left\langle \int_0^1 2t\, dt, \int_0^1 4t\, dt, \int_0^1 -\cos 3t\, dt\right\rangle = \left\langle t^2\bigg|_0^1, 2t^2\bigg|_0^1, -\frac{\sin 3t}{3}\bigg|_0^1\right\rangle = \left\langle 1, 2, -\frac{\sin 3}{3}\right\rangle$$

44. $\int_{1/2}^{1} \left\langle \frac{1}{u^2}, \frac{1}{u^4}, \frac{1}{u^5}\right\rangle du$

SOLUTION The vector valued integration is defined via componentwise integration. Computing the integral of each component we get:

$$\int_{1/2}^{1} \frac{1}{u^2}\, du = \frac{-1}{u}\bigg|_{1/2}^{1} = -1 - (-2) = 1$$

$$\int_{1/2}^{1} \frac{1}{u^4}\, du = \frac{-1}{3u^3}\bigg|_{1/2}^{1} = \frac{-1}{3} - \frac{-8}{3} = \frac{7}{3}$$

$$\int_{1/2}^{1} \frac{1}{u^5}\, du = \frac{-1}{4u^4}\bigg|_{1/2}^{1} = \frac{-1}{4} - \frac{-16}{4} = \frac{15}{4}$$

Therefore,

$$\int_{1/2}^{1} \left\langle \frac{1}{u^2}, \frac{1}{u^4}, \frac{1}{u^5}\right\rangle du = \left\langle \int_{1/2}^{1} \frac{1}{u^2}\, du, \int_{1/2}^{1} \frac{1}{u^4}\, du, \int_{1/2}^{1} \frac{1}{u^5}\, du\right\rangle = \left\langle 1, \frac{7}{3}, \frac{15}{4}\right\rangle$$

45. $\int_1^4 \left(t^{-1}\mathbf{i} + 4\sqrt{t}\,\mathbf{j} - 8t^{3/2}\mathbf{k}\right) dt$

SOLUTION We perform the integration componentwise. Computing the integral of each component we get:

$$\int_1^4 t^{-1}\,dt = \ln t\Big|_1^4 = \ln 4 - \ln 1 = \ln 4$$

$$\int_1^4 4\sqrt{t}\,dt = 4\cdot\frac{2}{3}t^{3/2}\Big|_1^4 = \frac{8}{3}\left(4^{3/2}-1\right) = \frac{56}{3}$$

$$\int_1^4 -8t^{3/2}\,dt = -\frac{16}{5}t^{5/2}\Big|_1^4 = -\frac{16}{5}\left(4^{5/2}-1\right) = -\frac{496}{5}$$

Hence,

$$\int_1^4 \left(t^{-1}\mathbf{i} + 4\sqrt{t}\mathbf{j} - 8t^{3/2}\mathbf{k}\right) dt = (\ln 4)\,\mathbf{i} + \frac{56}{3}\mathbf{j} - \frac{496}{5}\mathbf{k}$$

46. $\int_0^t \left(3s\mathbf{i} + 6s^2\mathbf{j} + 9\mathbf{k}\right) ds$

SOLUTION We first compute the integral of each component:

$$\int_0^t 3s\,ds = \frac{3}{2}s^2\Big|_0^t = \frac{3}{2}t^2$$

$$\int_0^t 6s^2\,ds = \frac{6}{3}s^3\Big|_0^t = 2t^3$$

$$\int_0^t 9\,ds = 9s\Big|_0^t = 9t$$

Hence,

$$\int_0^t \left(3s\mathbf{i} + 6s^2\mathbf{j} + 9\mathbf{k}\right) dt = \left(\int_0^t 3s\,ds\right)\mathbf{i} + \left(\int_0^t 6s^2\,ds\right)\mathbf{j} + \left(\int_0^t 9\,ds\right)\mathbf{k} = \left(\frac{3}{2}t^2\right)\mathbf{i} + \left(2t^3\right)\mathbf{j} + (9t)\mathbf{k}$$

In Exercises 47–54, find both the general solution of the differential equation and the solution with the given initial condition.

47. $\frac{d\mathbf{r}}{dt} = \langle 1-2t, 4t\rangle, \quad \mathbf{r}(0) = \langle 3, 1\rangle$

SOLUTION We first find the general solution by integrating $\frac{d\mathbf{r}}{dt}$:

$$\mathbf{r}(t) = \int \langle 1-2t, 4t\rangle\,dt = \left\langle \int (1-2t)\,dt, \int 4t\,dt\right\rangle = \left\langle t-t^2, 2t^2\right\rangle + \mathbf{c} \tag{1}$$

Since $\mathbf{r}(0) = \langle 3, 1\rangle$, we have:

$$\mathbf{r}(0) = \left\langle 0-0^2, 2\cdot 0^2\right\rangle + \mathbf{c} = \langle 3, 1\rangle \Rightarrow \mathbf{c} = \langle 3, 1\rangle$$

Substituting in (1) gives the solution:

$$\mathbf{r}(t) = \left\langle t-t^2, 2t^2\right\rangle + \langle 3, 1\rangle = \left\langle -t^2+t+3, 2t^2+1\right\rangle$$

48. $\mathbf{r}'(t) = \mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{k}$

SOLUTION The general solution is obtained by integrating $\mathbf{r}'(t)$:

$$\mathbf{r}(t) = \int (\mathbf{i}-\mathbf{j})\,dt = \left(\int 1\,dt\right)\mathbf{i} - \left(\int 1\,dt\right)\mathbf{j} = t\mathbf{i} - t\mathbf{j} + \mathbf{c} \tag{1}$$

Hence,

$$\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} + \mathbf{c} = \mathbf{c}$$

The solution with the initial condition $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{k}$ must satisfy:

$$\mathbf{r}(0) = \mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$$

Substituting in (1) yields the solution:

$$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2\mathbf{i} + 3\mathbf{k} = (t+2)\,\mathbf{i} - t\mathbf{j} + 3\mathbf{k}$$

49. $\mathbf{r}'(t) = t^2\mathbf{i} + 5t\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{j} + 2\mathbf{k}$

SOLUTION We first find the general solution by integrating $\mathbf{r}'(t)$:

$$\mathbf{r}(t) = \int \left(t^2\mathbf{i} + 5t\mathbf{j} + \mathbf{k}\right) dt = \left(\int t^2\,dt\right)\mathbf{i} + \left(\int 5t\,dt\right)\mathbf{j} + \left(\int 1\,dt\right)\mathbf{k} = \left(\frac{1}{3}t^3\right)\mathbf{i} + \left(\frac{5}{2}t^2\right)\mathbf{j} + t\mathbf{k} + \mathbf{c} \qquad (1)$$

The solution which satisfies the initial condition must satisfy:

$$\mathbf{r}(1) = \left(\frac{1}{3}\cdot 1^3\right)\mathbf{i} + \left(\frac{5}{2}\cdot 1^2\right)\mathbf{j} + 1\cdot\mathbf{k} + \mathbf{c} = \mathbf{j} + 2\mathbf{k}$$

That is,

$$\mathbf{c} = -\frac{1}{3}\mathbf{i} - \frac{3}{2}\mathbf{j} + 1\mathbf{k}$$

Substituting in (1) gives the following solution:

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3\right)\mathbf{i} + \left(\frac{5}{2}t^2\right)\mathbf{j} + t\mathbf{k} - \frac{1}{3}\mathbf{i} - \frac{3}{2}\mathbf{j} + \mathbf{k} = \left(\frac{1}{3}t^3 - \frac{1}{3}\right)\mathbf{i} + \left(\frac{5t^2}{2} - \frac{3}{2}\right)\mathbf{j} + (t+1)\,\mathbf{k}$$

50. $\mathbf{r}'(t) = \langle \sin 3t, \sin 3t, t\rangle, \quad \mathbf{r}\left(\frac{\pi}{2}\right) = \left\langle 2, 4, \frac{\pi^2}{4}\right\rangle$

SOLUTION We first integrate the vector $\mathbf{r}'(t)$ to find the general solution:

$$\mathbf{r}(t) = \int \langle \sin 3t, \sin 3t, t\rangle\; dt = \left\langle \int \sin 3t\,dt, \int \sin 3t\,dt, \int t\,dt\right\rangle \qquad (1)$$

$$= \left\langle -\frac{1}{3}\cos 3t, -\frac{1}{3}\cos 3t, \frac{1}{2}t^2\right\rangle + \mathbf{c} \qquad (2)$$

Substituting the initial condition we obtain:

$$\mathbf{r}(\pi/2) = \left\langle -\frac{1}{3}\cos\frac{\pi}{2}, -\frac{1}{3}\cos\frac{\pi}{2}, \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2\right\rangle + \mathbf{c} = \left\langle 0, 0, \frac{\pi^2}{8}\right\rangle + \mathbf{c} = \left\langle 2, 4, \frac{\pi^2}{4}\right\rangle$$

Hence,

$$\mathbf{c} = \left\langle 2, 4, \frac{\pi^2}{4}\right\rangle - \left\langle 0, 0, \frac{\pi^2}{8}\right\rangle = \left\langle 2, 4, \frac{\pi^2}{8}\right\rangle$$

Substituting in (2) we obtain the solution:

$$\mathbf{r}(t) = \left\langle -\frac{1}{3}\cos 3t, -\frac{1}{3}\cos 3t, \frac{1}{2}t^2\right\rangle + \left\langle 2, 4, \frac{\pi^2}{8}\right\rangle = \left\langle -\frac{1}{3}\cos 3t + 2, -\frac{1}{3}\cos 3t + 4, \frac{1}{2}t^2 + \frac{\pi^2}{8}\right\rangle$$

51. $\mathbf{r}''(t) = 16\mathbf{k}, \quad \mathbf{r}(0) = \langle 1, 0, 0\rangle, \quad \mathbf{r}'(0) = \langle 0, 1, 0\rangle$

SOLUTION To find the general solution we first find $\mathbf{r}'(t)$ by integrating $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \int 16\mathbf{k}\,dt = (16t)\,\mathbf{k} + \mathbf{c}_1 \qquad (1)$$

We now integrate $\mathbf{r}'(t)$ to find the general solution $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int ((16t)\,\mathbf{k} + \mathbf{c}_1)\; dt = \left(\int 16(t)\,dt\right)\mathbf{k} + \mathbf{c}_1 t + \mathbf{c}_2 = (8t^2)\mathbf{k} + \mathbf{c}_1 t + \mathbf{c}_2 \qquad (2)$$

We substitute the initial conditions in (1) and (2). This gives:

$$\mathbf{r}'(0) = \mathbf{c}_1 = \langle 0, 1, 0\rangle = \mathbf{j}$$

$$\mathbf{r}(0) = 0\mathbf{k} + \mathbf{c}_1\cdot 0 + \mathbf{c}_2 = \langle 1, 0, 0\rangle \quad\Rightarrow\quad \mathbf{c}_2 = \langle 1, 0, 0\rangle = \mathbf{i}$$

Combining with (2) we obtain the following solution:

$$\mathbf{r}(t) = (8t^2)\mathbf{k} + t\mathbf{j} + \mathbf{i} = \mathbf{i} + t\mathbf{j} + (8t^2)\mathbf{k}$$

52. $\mathbf{r}''(t) = \left\langle e^{2t-2}, t^2-1, 1\right\rangle, \mathbf{r}(1) = \langle 0,0,1\rangle, \mathbf{r}'(1) = \langle 2,0,0\rangle$

SOLUTION To find the general solution we first find $\mathbf{r}'(t)$ by integrating $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \int \left\langle e^{2t-2}, t^2-1, 1\right\rangle\,dt = \left\langle \frac{1}{2}e^{2t-2}, \frac{t^3}{3}-t, t\right\rangle + \mathbf{c}_1 \tag{1}$$

We now integrate $\mathbf{r}'(t)$ to find the general solution $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \left(\left\langle \frac{1}{2}e^{2t-2}, \frac{t^3}{3}-t, t\right\rangle + \mathbf{c}_1\right)\,dt = \left\langle \frac{1}{4}e^{2t-2}, \frac{t^4}{12}-\frac{t^2}{2}, \frac{t^2}{2}\right\rangle + \mathbf{c}_1 t + \mathbf{c}_2 \tag{2}$$

We substitute the initial conditions in (1) and (2). This gives:

$$\begin{aligned}
\mathbf{r}'(1) &= \left\langle \frac{1}{2}, -\frac{2}{3}, 1\right\rangle + \mathbf{c}_1 = \langle 2,0,0\rangle \quad \Rightarrow \quad \mathbf{c}_1 = \left\langle \frac{3}{2}, \frac{2}{3}, -1\right\rangle \\
\mathbf{r}(1) &= \left\langle \frac{1}{4}, -\frac{5}{12}, \frac{1}{2}\right\rangle + \mathbf{c}_1(1) + \mathbf{c}_2 = < 0,0,1 > \\
&\left\langle \frac{1}{4}, -\frac{5}{12}, \frac{1}{2}\right\rangle + \left\langle \frac{3}{2}, \frac{2}{3}, -1\right\rangle + \mathbf{c}_2 = < 0,0,1 > \\
&\Rightarrow \quad \mathbf{c}_2 = \left\langle -\frac{7}{4}, -\frac{1}{4}, \frac{3}{2}\right\rangle
\end{aligned}$$

Combining with (2) we obtain the following solution:

$$\begin{aligned}
\mathbf{r}(t) &= \left\langle \frac{1}{4}e^{2t-2}, \frac{t^4}{12}-\frac{t^2}{2}, \frac{t^2}{2}\right\rangle + t\left\langle \frac{3}{2}, \frac{2}{3}, -1\right\rangle + \left\langle -\frac{7}{4}, -\frac{1}{4}, \frac{3}{2}\right\rangle \\
&= \left\langle \frac{1}{4}e^{2t-2} + \frac{3}{2}t - \frac{7}{4}, \frac{t^4}{12} - \frac{t^2}{2} + \frac{2}{3}t - \frac{1}{4}, \frac{t^2}{2} - t + \frac{3}{2}\right\rangle
\end{aligned}$$

53. $\mathbf{r}''(t) = \langle 0,2,0\rangle, \mathbf{r}(3) = \langle 1,1,0\rangle, \mathbf{r}'(3) = \langle 0,0,1\rangle$

SOLUTION To find the general solution we first find $\mathbf{r}'(t)$ by integrating $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \int \langle 0,2,0\rangle\,dt = \langle 0,2t,0\rangle + \mathbf{c}_1 \tag{1}$$

We now integrate $\mathbf{r}'(t)$ to find the general solution $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int (\langle 0,2t,0\rangle + \mathbf{c}_1)\,dt = \left\langle 0, t^2, 0\right\rangle + \mathbf{c}_1 t + \mathbf{c}_2 \tag{2}$$

We substitute the initial conditions in (1) and (2). This gives:

$$\begin{aligned}
\mathbf{r}'(3) &= \langle 0,6,0\rangle + \mathbf{c}_1 = \langle 0,0,1\rangle \quad \Rightarrow \quad \mathbf{c}_1 = \langle 0,-6,1\rangle \\
\mathbf{r}(3) &= \langle 0,9,0\rangle + \mathbf{c}_1(3) + \mathbf{c}_2 = < 1,1,0 > \\
&\langle 0,9,0\rangle + \langle 0,-18,3\rangle + \mathbf{c}_2 = < 1,1,0 > \\
&\Rightarrow \quad \mathbf{c}_2 = \langle 1,10,-3\rangle
\end{aligned}$$

Combining with (2) we obtain the following solution:

$$\begin{aligned}
\mathbf{r}(t) &= \left\langle 0, t^2, 0\right\rangle + t\,\langle 0,-6,1\rangle + \langle 1,10,-3\rangle \\
&= \left\langle 1, t^2 - 6t + 10, t - 3\right\rangle
\end{aligned}$$

54. $\mathbf{r}''(t) = \left\langle e^t, \sin t, \cos t\right\rangle, \quad \mathbf{r}(0) = \langle 1,0,1\rangle, \quad \mathbf{r}'(0) = \langle 0,2,2\rangle$

SOLUTION We perform integration componentwise on $\mathbf{r}''(t)$ to obtain:

$$\mathbf{r}'(t) = \int \left\langle e^t, \sin t, \cos t\right\rangle\,dt = \left\langle e^t, -\cos t, \sin t\right\rangle + \mathbf{c}_1 \tag{1}$$

We now integrate $\mathbf{r}'(t)$ to obtain the general solution:

$$\mathbf{r}(t) = \int \left(\left\langle e^t, -\cos t, \sin t\right\rangle + \mathbf{c}_1\right) dt = \left\langle e^t, -\sin t, -\cos t\right\rangle + \mathbf{c}_1 t + \mathbf{c}_2 \tag{2}$$

Now, we substitute the initial conditions $\mathbf{r}(0) = \langle 1, 0, 1\rangle$ and $\mathbf{r}'(0) = \langle 0, 2, 2\rangle$ into (1) and (2) and solve for the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. We obtain:

$$\mathbf{r}'(0) = \langle 1, -1, 0\rangle + \mathbf{c}_1 = \langle 0, 2, 2\rangle \quad \Rightarrow \quad \mathbf{c}_1 = \langle -1, 3, 2\rangle$$
$$\mathbf{r}(0) = \langle 1, 0, -1\rangle + \mathbf{c}_2 = \langle 1, 0, 1\rangle \quad \Rightarrow \quad \mathbf{c}_2 = \langle 0, 0, 2\rangle$$

Finally we combine the above to obtain the solution:

$$\mathbf{r}(t) = \left\langle e^t, -\sin t, -\cos t\right\rangle + \langle -1, 3, 2\rangle\, t + \langle 0, 0, 2\rangle = \left\langle e^t - t, -\sin t + 3t, -\cos t + 2t + 2\right\rangle$$

55. Find the location at $t = 3$ of a particle whose path (Figure 8) satisfies

$$\frac{d\mathbf{r}}{dt} = \left\langle 2t - \frac{1}{(t+1)^2}, 2t - 4\right\rangle, \qquad \mathbf{r}(0) = \langle 3, 8\rangle$$

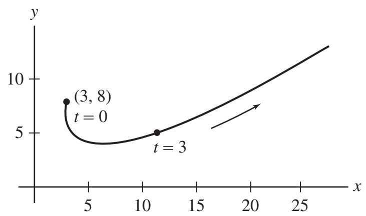


FIGURE 8 Particle path.

SOLUTION To determine the position of the particle in general, we perform integration componentwise on $\mathbf{r}'(t)$ to obtain:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\, dt \\ &= \int \left\langle 2t - \frac{1}{(t+1)^2}, 2t - 4\right\rangle dt \\ &= \left\langle t^2 + \frac{1}{t+1}, t^2 - 4t\right\rangle + \mathbf{c}_1\end{aligned}$$

Using the initial condition, observe the following:

$$\mathbf{r}(0) = \langle 1, 0\rangle + \mathbf{c}_1 = \langle 3, 8\rangle$$
$$\Rightarrow \quad \mathbf{c}_1 = \langle 2, 8\rangle$$

Therefore,

$$\mathbf{r}(t) = \left\langle t^2 + \frac{1}{t+1}, t^2 - 4t\right\rangle + \langle 2, 8\rangle = \left\langle t^2 + \frac{1}{t+1} + 2, t^2 - 4t + 8\right\rangle$$

and thus, the location of the particle at $t = 3$ is $\mathbf{r}(3) = \langle 45/4, 5\rangle = \langle 11.25, 5\rangle$

56. Find the location and velocity at $t = 4$ of a particle whose path satisfies

$$\frac{d\mathbf{r}}{dt} = \left\langle 2t^{-1/2}, 6, 8t\right\rangle, \qquad \mathbf{r}(1) = \langle 4, 9, 2\rangle$$

SOLUTION The velocity of this particle at $t = 4$ is exactly:

$$\mathbf{r}'(4) = \left\langle 2(4)^{-1/2}, 6, 8(4)\right\rangle = \langle 1, 6, 32\rangle$$

To determine the location of the particle at any general t, we will perform integration componentwise on $\mathbf{r}'(t)$:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\, dt = \int \left\langle 2t^{-1/2}, 6, 8t\right\rangle dt \\ &= \left\langle 4\sqrt{t}, 6t, 4t^2\right\rangle + \mathbf{c}_1\end{aligned}$$

Using the initial condition, observe the following:

$$\mathbf{r}(1) = \langle 4, 6, 4\rangle + \mathbf{c}_1 = \langle 4, 9, 2\rangle$$
$$\Rightarrow \quad \mathbf{c}_1 = \langle 0, 3, -2\rangle$$

Therefore,

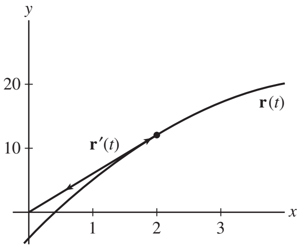
$$\mathbf{r}(t) = \left\langle 4\sqrt{t}, 6t, 4t^2\right\rangle + \langle 0, 3, -2\rangle = \left\langle 4\sqrt{t}, 6t+3, 4t^2-2\right\rangle$$

Then the location of this particle at $t = 4$ is:

$$\mathbf{r}(4) = \langle 8, 27, 62\rangle$$

57. A fighter plane, which can shoot a laser beam straight ahead, travels along the path $\mathbf{r}(t) = \left\langle 5-t, 21-t^2, 3-t^3/27\right\rangle$. Show that there is precisely one time t at which the pilot can hit a target located at the origin.

SOLUTION By the given information the laser beam travels in the direction of $\mathbf{r}'(t)$. The pilot hits a target located at the origin at the time t when $\mathbf{r}'(t)$ points towards the origin, that is, when $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel and point to opposite directions.



We find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \frac{d}{dt}\left\langle 5-t, 21-t^2, 3-\frac{t^3}{27}\right\rangle = \left\langle -1, -2t, -\frac{t^2}{9}\right\rangle$$

We first find t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel, that is, we find t such that the cross product of the two vectors is zero. We obtain:

$$\mathbf{0} = \mathbf{r}'(t) \times \mathbf{r}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2t & -\frac{t^2}{9} \\ 5-t & 21-t^2 & 3-\frac{t^3}{27} \end{vmatrix}$$
$$= \left(-2t\left(3-\frac{t^3}{27}\right) + \frac{t^2}{9}(21-t^2)\right)\mathbf{i} - \left(-\left(3-\frac{t^3}{27}\right) + \frac{t^2}{9}(5-t)\right)\mathbf{j} + \left(-\left(21-t^2\right) + 2t(5-t)\right)\mathbf{k}$$
$$= \left(\frac{-t^4}{27} + \frac{7t^2}{3} - 6t\right)\mathbf{i} - \left(-\frac{2t^3}{27} + \frac{5t^2}{9} - 3\right)\mathbf{j} + \left(-t^2 + 10t - 21\right)\mathbf{k}$$

Equating each component to zero we obtain the following equations:

$$-\frac{t^4}{27} + \frac{7}{3}t^2 - 6t = 0$$
$$-\frac{2t^3}{27} + \frac{5t^2}{9} - 3 = 0$$
$$-t^2 + 10t - 21 = -(t-7)(t-3) = 0$$

The third equation implies that $t = 3$ or $t = 7$. Only $t = 3$ satisfies the other two equations as well. We now must verify that $\mathbf{r}(3)$ and $\mathbf{r}'(3)$ point in opposite directions. We find these vectors:

$$\mathbf{r}(3) = \left\langle 5-3, 21-3^2, 3-\frac{3^3}{27}\right\rangle = \langle 2, 12, 2\rangle$$
$$\mathbf{r}'(3) = \left\langle -1, -2\cdot 3, -\frac{3^2}{9}\right\rangle = \langle -1, -6, -1\rangle$$

Since $\mathbf{r}(3) = -2\mathbf{r}'(3)$, the vectors point in opposite direction. We conclude that only at time $t = 3$ can the pilot hit a target located at the origin.

58. The fighter plane of Exercise 57 travels along the path $\mathbf{r}(t) = \left\langle t - t^3, 12 - t^2, 3 - t\right\rangle$. Show that the pilot cannot hit any target on the x-axis.

SOLUTION First we will compute the tangent line to the given path of the plane at any time t . The tangent line will be

$$\ell(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

Computing the derivatives we get:

$$\mathbf{r}'(t) = \left\langle 1 - 3t^2, -2t, -1\right\rangle$$

Therefore, the tangent line at time t is

$$\begin{aligned}\ell(t) &= \mathbf{r}(0) + t\mathbf{r}'(0)\\ &= \langle 0, 12, 3\rangle + t\,\langle 1, 0, -1\rangle\\ &= \langle t, 12, 3 - t\rangle\end{aligned}$$

The tangent line always lies on the plane $y = 12$, so it can never hit the x-axis.

59. Find all solutions to $\mathbf{r}'(t) = \mathbf{v}$ with initial condition $\mathbf{r}(1) = \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are constant vectors in $\mathbf{R}^3$.

SOLUTION We denote the components of the constant vector $\mathbf{v}$ by $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and integrate to find the general solution. This gives:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}\,dt = \int \langle v_1, v_2, v_3\rangle\,dt = \left\langle \int v_1\,dt, \int v_2\,dt, \int v_3\,dt\right\rangle\\ &= \langle v_1t + c_1, v_2t + c_2, v_3t + c_3\rangle = t\,\langle v_1, v_2, v_3\rangle + \langle c_1, c_2, c_3\rangle\end{aligned}$$

We let $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ and obtain:

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{c} = \mathbf{c} + t\mathbf{v}$$

Notice that the solutions are the vector parametrizations of all the lines with direction vector $\mathbf{v}$.

We are also given the initial condition that $\mathbf{r}(1) = \mathbf{w}$, using this information we can determine:

$$\mathbf{r}(1) = (1)\mathbf{v} + \mathbf{c} = \mathbf{w}$$

Therefore $\mathbf{c} = \mathbf{w} - \mathbf{v}$ and we get:

$$\mathbf{r}(t) = (\mathbf{w} - \mathbf{v}) + t\mathbf{v} = (t - 1)\mathbf{v} + \mathbf{w}$$

60. Let $\mathbf{u}$ be a constant vector in $\mathbf{R}^3$. Find the solution of the equation $\mathbf{r}'(t) = (\sin t)\mathbf{u}$ satisfying $\mathbf{r}'(0) = \mathbf{0}$.

SOLUTION We first integrate to find the general solution. Denoting $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ we get:

$$\begin{aligned}\mathbf{r}(t) &= \int (\sin t)\,\mathbf{u}\,dt = \int \langle u_1 \sin t, u_2 \sin t, u_3 \sin t\rangle\,dt\\ &= \left\langle \int u_1 \sin t\,dt, \int u_2 \sin t\,dt, \int u_3 \sin t\,dt\right\rangle = \left\langle u_1 \int \sin t\,dt, u_2 \int \sin t\,dt, u_3 \int \sin t\,dt\right\rangle\\ &= \langle -u_1 \cos t + c_1, -u_2 \cos t + c_2, -u_3 \cos t + c_3\rangle = -\cos t\,\langle u_1, u_2, u_3\rangle + \langle c_1, c_2, c_3\rangle\end{aligned}$$

Letting $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ we obtain the following solutions:

$$\mathbf{r}(t) = (-\cos t)\,\mathbf{u} + \mathbf{c}$$

Since $\mathbf{r}'(0) = 0$ we have $\mathbf{r}'(0) = \sin 0 \cdot \mathbf{u} = 0$.

61. Find all solutions to $\mathbf{r}'(t) = 2\mathbf{r}(t)$ where $\mathbf{r}(t)$ is a vector-valued function in three-space.

SOLUTION We denote the components of $\mathbf{r}(t)$ by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Then, $\mathbf{r}'(t) = \left\langle x'(t), y'(t), z'(t)\right\rangle$. Substituting in the differential equation we get:

$$\left\langle x'(t), y'(t), z'(t)\right\rangle = 2\,\langle x(t), y(t), z(t)\rangle$$

Equating corresponding components gives:

$$\begin{array}{lcl} x'(t) = 2x(t) & & x(t) = c_1e^{2t}\\ y'(t) = 2y(t) & \Rightarrow & y(t) = c_2e^{2t}\\ z'(t) = 2z(t) & & z(t) = c_3e^{2t}\end{array}$$

We denote the constant vector by $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ and obtain the following solutions:

$$\mathbf{r}(t) = \left\langle c_1 e^{2t}, c_2 e^{2t}, c_3 e^{2t} \right\rangle = e^{2t} \langle c_1, c_2, c_3 \rangle = e^{2t}\mathbf{c}$$

62. Show that $\mathbf{w}(t) = \langle \sin(3t+4), \sin(3t-2), \cos 3t \rangle$ satisfies the differential equation $\mathbf{w}''(t) = -9\mathbf{w}(t)$.

SOLUTION We differentiate the vector $\mathbf{w}(t)$ twice:

$$\begin{aligned}
\mathbf{w}'(t) &= \langle 3\cos(3t+4), 3\cos(3t-2), -3\sin 3t \rangle \\
\mathbf{w}''(t) &= \frac{d}{dt}\left(\mathbf{w}'(t)\right) = \langle -9\sin(3t+4), -9\sin(3t-2), -9\cos 3t \rangle \\
&= -9 \langle \sin(3t+4), \sin(3t-2), \cos 3t \rangle = -9\mathbf{w}(t)
\end{aligned}$$

We thus showed that $\mathbf{w}''(t) = -9\mathbf{w}(t)$

63. Prove that the **Bernoulli spiral** (Figure 9) with parametrization $\mathbf{r}(t) = \left\langle e^t \cos 4t, e^t \sin 4t \right\rangle$ has the property that the angle ψ between the position vector and the tangent vector is constant. Find the angle ψ in degrees.

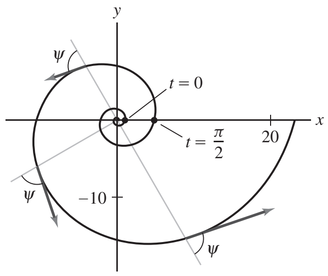


FIGURE 9 Bernoulli spiral.

SOLUTION First, let us compute the tangent vector, $\mathbf{r}'(t)$:

$$\mathbf{r}(t) = \left\langle e^t \cos 4t, e^t \sin 4t \right\rangle, \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle -4e^t \sin 4t + e^t \cos 4t, 4e^t \cos 4t + e^t \sin 4t \right\rangle$$

Then recall the identity that $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, so then,

$$\begin{aligned}
\mathbf{r}(t) \cdot \mathbf{r}'(t) &= \left\langle e^t \cos 4t, e^t \sin 4t \right\rangle \cdot \left\langle -4e^t \sin 4t + e^t \cos 4t, 4e^t \cos 4t + e^t \sin 4t \right\rangle \\
&= -4e^{2t} \sin 4t \cos 4t + e^{2t} \cos^2 4t + 4e^{2t} \sin 4t \cos 4t + e^{2t} \sin^2 4t \\
&= e^{2t}(\cos^2 4t + \sin^2 4t) \\
&= e^{2t}
\end{aligned}$$

Then, computing norms, we get:

$$\begin{aligned}
\|\mathbf{r}(t)\| &= \sqrt{e^{2t}\cos^2 4t + e^{2t}\sin^2 4t} = \sqrt{e^{2t}(\cos^2 4t + \sin^2 4t)} = e^t \\
\|\mathbf{r}'(t)\| &= \sqrt{(-4e^t \sin 4t + e^t \cos 4t)^2 + (4e^t \cos 4t + e^t \sin 4t)^2} \\
&= \sqrt{16e^{2t}\sin^2 4t - 4e^{2t}\sin 4t \cos 4t + e^{2t}\cos^2 4t + 16e^{2t}\cos^2 4t + 4e^{2t}\sin 4t\cos 4t + e^{2t}\sin^2 4t} \\
&= \sqrt{16e^{2t}(\sin^2 4t + \cos^2 4t) + e^{2t}(\cos^2 4t + \sin^2 4t)} \\
&= \sqrt{16e^{2t} + e^{2t}} \\
&= \sqrt{17}e^t
\end{aligned}$$

Then using the dot product relation listed above we get:

$$e^{2t} = e^t(\sqrt{17}e^t)\cos\theta = \sqrt{17}e^{2t}\cos\theta$$

Hence

$$\cos\theta = \frac{1}{\sqrt{17}}, \quad \Rightarrow \quad \theta \approx 75.96^\circ$$

Therefore, the angle between the position vector and the tangent vector is constant.

64. A curve in polar form $r = f(\theta)$ has parametrization

$$\mathbf{r}(\theta) = f(\theta)\langle \cos\theta, \sin\theta\rangle$$

Let ψ be the angle between the radial and tangent vectors (Figure 10). Prove that

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)}$$

Hint: Compute $\mathbf{r}(\theta) \times \mathbf{r}'(\theta)$ and $\mathbf{r}(\theta) \cdot \mathbf{r}'(\theta)$.

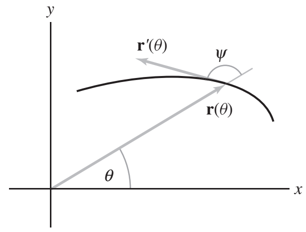


FIGURE 10 Curve with polar parametrization $\mathbf{r}(\theta) = f(\theta)\langle \cos\theta, \sin\theta\rangle$.

SOLUTION First we will compute $\mathbf{r}(\theta)$ and $\mathbf{r}'(\theta)$:

$$\mathbf{r}(\theta) = f(\theta)\langle \cos\theta, \sin\theta\rangle = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$$

$$\mathbf{r}'(\theta) = f(\theta)\langle -\sin\theta, \cos\theta\rangle + f'(\theta)\langle \cos\theta, \sin\theta\rangle$$

$$= \left\langle f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta\right\rangle$$

Now we will compute both $\mathbf{r}(\theta) \cdot \mathbf{r}'(\theta)$ and $\mathbf{r}(\theta) \times \mathbf{r}'(\theta)$.

$$\mathbf{r}(\theta) \cdot \mathbf{r}'(\theta) = f(\theta)\cos\theta[f'(\theta)\cos\theta - f(\theta)\sin\theta] + f(\theta)\sin\theta[f'(\theta)\sin\theta + f(\theta)\cos\theta]$$

$$= f(\theta)f'(\theta)\cos^2\theta - [f(\theta)]^2\cos\theta\sin\theta + f(\theta)f'(\theta)\sin\theta + [f(\theta)]^2\sin\theta\cos\theta$$

$$= f(\theta)f'(\theta)[\cos^2\theta + \sin^2\theta]$$

$$= f(\theta)f'(\theta)$$

$$\mathbf{r}(\theta) \times \mathbf{r}'(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle \times \left\langle f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta\right\rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f(\theta)\cos\theta & f(\theta)\sin\theta & 0 \\ f'(\theta)\cos\theta - f(\theta)\sin\theta & f'(\theta)\sin\theta + f(\theta)\cos\theta) & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + [f(\theta)\cos\theta(f'(\theta)\sin\theta + f(\theta)\cos\theta) - f(\theta)\sin\theta(f'(\theta)\cos\theta - f(\theta)\sin\theta)]\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + [f(\theta)]^2\mathbf{k}$$

Recall that $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\psi$ and $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\psi$ so then:

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{\frac{\|\mathbf{r}(\theta)\times\mathbf{r}'(\theta)\|}{\|\mathbf{r}(\theta)\|\|\mathbf{r}'(\theta)\|}}{\frac{\mathbf{r}(\theta)\cdot\mathbf{r}'(\theta)}{\|\mathbf{r}(\theta)\|\|\mathbf{r}'(\theta)\|}} = \frac{\|\mathbf{r}(\theta) \times \mathbf{r}'(\theta)\|}{\mathbf{r}(\theta) \cdot \mathbf{r}'(\theta)} = \frac{\sqrt{[f(\theta)]^4}}{f(\theta)f'(\theta)} = \frac{|[f(\theta)]^2|}{f(\theta)f'(\theta)} = \frac{f(\theta)}{f'(\theta)} = \frac{r}{dr/d\theta}$$

65. Prove that if $\|\mathbf{r}(t)\|$ takes on a local minimum or maximum value at t_0, then $\mathbf{r}(t_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Explain how this result is related to Figure 11. *Hint:* Observe that if $\|\mathbf{r}(t_0)\|$ is a minimum, then $\mathbf{r}(t)$ is tangent at t_0 to the sphere of radius $\|\mathbf{r}(t_0)\|$ centered at the origin.

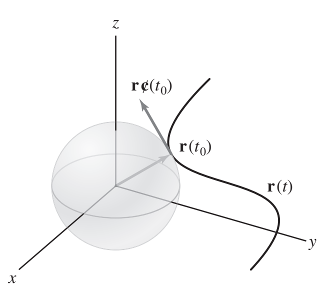


FIGURE 11

SOLUTION Suppose that $\|\mathbf{r}(t)\|$ takes on a minimum or maximum value at $t = t_0$. Hence, $\|\mathbf{r}(t)\|^2$ also takes on a minimum or maximum value at $t = t_0$, therefore $\frac{d}{dt}\|\mathbf{r}(t)\|^2\big|_{t=t_0} = 0$. Using the Product Rule for dot products we get

$$\frac{d}{dt}\|\mathbf{r}(t)\|^2\bigg|_{t=t_0} = \frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{r}(t)\bigg|_{t=t_0} = \mathbf{r}(t_0)\cdot\mathbf{r}'(t_0) + \mathbf{r}'(t_0)\cdot\mathbf{r}(t_0) = 2\mathbf{r}(t_0)\cdot\mathbf{r}'(t_0) = 0$$

Thus $\mathbf{r}(t_0)\cdot\mathbf{r}'(t_0) = 0$, which implies the orthogonality of $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$. In Figure 11, $\|\mathbf{r}(t_0)\|$ is a minimum and the path intersects the sphere of radius $\|\mathbf{r}(t_0)\|$ at a single point. Therefore, the point of intersection is a tangency point which implies that $\mathbf{r}'(t_0)$ is tangent to the sphere at t_0. We conclude that $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ are orthogonal.

66. Newton's Second Law of Motion in vector form states that $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ where $\mathbf{F}$ is the force acting on an object of mass m and $\mathbf{p} = m\mathbf{r}'(t)$ is the object's momentum. The analogs of force and momentum for rotational motion are the **torque** $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ and **angular momentum**

$$\mathbf{J} = \mathbf{r}(t)\times\mathbf{p}(t)$$

Use the Second Law to prove that $\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt}$.

SOLUTION We differentiate $\mathbf{J} = \mathbf{r}(t)\times m\mathbf{r}'(t)$ using the Product Rule for cross products. Using $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ we get:

$$\begin{aligned}\frac{d\mathbf{J}}{dt} &= \frac{d}{dt}\left(\mathbf{r}(t)\times m\mathbf{r}'(t)\right) = m\frac{d}{dt}\left(\mathbf{r}(t)\times\mathbf{r}'(t)\right) = m\left(\mathbf{r}(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'(t)\right)\\ &= m\mathbf{r}(t)\times\mathbf{r}''(t) = \mathbf{r}(t)\times m\mathbf{r}''(t) \qquad (1)\end{aligned}$$

Since $\mathbf{p} = m\mathbf{r}'(t)$, we have $\frac{d\mathbf{p}}{dt} = m\mathbf{r}''(t)$. Combining with (1) we obtain:

$$\frac{d\mathbf{J}}{dt} = \mathbf{r}(t)\times\frac{d\mathbf{p}}{dt} \qquad (2)$$

Finally, we use Newton's Second Law of Motion, $\mathbf{F} = \frac{d\mathbf{p}}{dt}$. Substituting in (2) we obtain the required equality:

$$\frac{d\mathbf{J}}{dt} = \mathbf{r}(t)\times\mathbf{F}(t) = \tau$$

Further Insights and Challenges

67. Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ trace a plane curve $\mathcal{C}$. Assume that $x'(t_0) \neq 0$. Show that the slope of the tangent vector $\mathbf{r}'(t_0)$ is equal to the slope dy/dx of the curve at $\mathbf{r}(t_0)$.

SOLUTION

(a) By the Chain Rule we have

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

Hence, at the points where $\frac{dx}{dt} \neq 0$ we have:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

(b) The line $\ell(t) = \langle a, b\rangle + t\mathbf{r}'(t_0)$ passes through (a, b) at $t = 0$. It holds that:

$$\ell(0) = \langle a, b\rangle + 0\mathbf{r}'(t_0) = \langle a, b\rangle$$

That is, (a, b) is the terminal point of the vector $\ell(0)$, hence the line passes through (a, b). The line has the direction vector $\mathbf{r}'(t_0) = \left\langle x'(t_0), y'(t_0)\right\rangle$, therefore the slope of the line is $\frac{y'(t_0)}{x'(t_0)}$ which is equal to $\frac{dy}{dx}\big|_{t=t_0}$ by part (a).

68. Prove that $\frac{d}{dt}(\mathbf{r}\cdot(\mathbf{r}'\times\mathbf{r}'')) = \mathbf{r}\cdot(\mathbf{r}'\times\mathbf{r}''')$.

SOLUTION We use the Product Rule for dot products to obtain:

$$\frac{d}{dt}\left(\mathbf{r}\cdot\left(\mathbf{r}'\times\mathbf{r}''\right)\right) = \mathbf{r}\cdot\frac{d}{dt}\left(\mathbf{r}'\times\mathbf{r}''\right) + \mathbf{r}'\cdot\left(\mathbf{r}'\times\mathbf{r}''\right) \qquad (1)$$

By the Product Rule for cross products and properties of cross products, we have:

$$\frac{d}{dt}\left(\mathbf{r}' \times \mathbf{r}''\right) = \mathbf{r}' \times \mathbf{r}''' + \mathbf{r}'' \times \mathbf{r}'' = \mathbf{r}' \times \mathbf{r}''' + \mathbf{0} = \mathbf{r}' \times \mathbf{r}''' \qquad (2)$$

Substituting (2) into (1) yields:

$$\frac{d}{dt}\left(\mathbf{r} \cdot \left(\mathbf{r}' \times \mathbf{r}''\right)\right) = \mathbf{r} \cdot \left(\mathbf{r}' \times \mathbf{r}'''\right) + \mathbf{r}' \cdot \left(\mathbf{r}' \times \mathbf{r}''\right) \qquad (3)$$

Since $\mathbf{r}' \times \mathbf{r}''$ is orthogonal to $\mathbf{r}'$, the dot product $\mathbf{r}' \cdot \left(\mathbf{r}' \times \mathbf{r}''\right) = 0$. So (3) gives:

$$\frac{d}{dt}\left(\mathbf{r} \cdot \left(\mathbf{r}' \times \mathbf{r}''\right)\right) = \mathbf{r} \cdot \left(\mathbf{r}' \times \mathbf{r}'''\right) + 0 = \mathbf{r} \cdot \left(\mathbf{r}' \times \mathbf{r}'''\right)$$

69. Verify the Sum and Product Rules for derivatives of vector-valued functions.

SOLUTION We first verify the Sum Rule stating:

$$(\mathbf{r}_1(t) + \mathbf{r}_2(t))' = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)$$

Let $\mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t)\rangle$ and $\mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t)\rangle$. Then,

$$\begin{aligned}
(\mathbf{r}_1(t) + \mathbf{r}_2(t))' &= \frac{d}{dt}\langle x_1(t) + x_2(t), y_1(t) + y_2(t), z_1(t) + z_2(t)\rangle \\
&= \left\langle (x_1(t) + x_2(t))', (y_1(t) + y_2(t))', (z_1(t) + z_2(t))'\right\rangle \\
&= \left\langle x_1'(t) + x_2'(t), y_1'(t) + y_2'(t), z_1'(t) + z_2'(t)\right\rangle \\
&= \left\langle x_1'(t), y_1'(t), z_1'(t)\right\rangle + \left\langle x_2'(t), y_2'(t), z_2'(t)\right\rangle = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)
\end{aligned}$$

The Product Rule states that for any differentiable scalar-valued function $f(t)$ and differentiable vector-valued function $\mathbf{r}(t)$, it holds that:

$$\frac{d}{dt} f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

To verify this rule, we denote $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Then,

$$\frac{d}{df} f(t)\mathbf{r}(t) = \frac{d}{dt}\langle f(t)x(t), f(t)y(t), f(t)z(t)\rangle$$

Applying the Product Rule for scalar functions for each component we get:

$$\begin{aligned}
\frac{d}{dt} f(t)\mathbf{r}(t) &= \left\langle f(t)x'(t) + f'(t)x(t), f(t)y'(t) + f'(t)y(t), f(t)z'(t) + f'(t)z(t)\right\rangle \\
&= \left\langle f(t)x'(t), f(t)y'(t), f(t)z'(t)\right\rangle + \left\langle f'(t)x(t), f'(t)y(t), f'(t)z(t)\right\rangle \\
&= f(t)\left\langle x'(t), y'(t), z'(t)\right\rangle + f'(t)\langle x(t), y(t), z(t)\rangle = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)
\end{aligned}$$

70. Verify the Chain Rule for vector-valued functions.

SOLUTION Let $g(t)$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be differentiable scalar and vector valued functions respectively. We must show that:

$$\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t)).$$

We have

$$\mathbf{r}(g(t)) = \langle x(g(t)), y(g(t)), z(g(t))\rangle$$

We differentiate the vector componentwise, using the Chain Rule for scalar functions. This gives:

$$\begin{aligned}
\frac{d}{dt}\mathbf{r}(g(t)) &= \left\langle \frac{d}{dt}(x(g(t))), \frac{d}{dt}(y(g(t))), \frac{d}{dt}(z(g(t)))\right\rangle = \left\langle g'(t)x'(g(t)), g'(t)y'(g(t)), g'(t)z'(g(t))\right\rangle \\
&= g'(t)\left\langle x'(g(t)), y'(g(t)), z'(g(t))\right\rangle = g'(t)\mathbf{r}'(g(t))
\end{aligned}$$

71. Verify the Product Rule for cross products [Eq. (5)].

SOLUTION Let $\mathbf{r}_1(t) = \langle a_1(t), a_2(t), a_3(t)\rangle$ and $\mathbf{r}_2(t) = \langle b_1(t), b_2(t), b_3(t)\rangle$. Then (we omit the independent variable t for simplicity):

$$\mathbf{r}_1(t) \times \mathbf{r}_2(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

Differentiating this vector componentwise we get:

$$\begin{aligned}\frac{d}{dt}\mathbf{r}_1 \times \mathbf{r}_2 &= \left(a_2b_3' + a_2'b_3 - a_3b_2' - a_3'b_2\right)\mathbf{i} - \left(a_1b_3' + a_1'b_3 - a_3b_1' - a_3'b_1\right)\mathbf{j} + \left(a_1b_2' + a_1'b_2 - a_2b_1' - a_2'b_1\right)\mathbf{k} \\ &= \left(\left(a_2b_3' - a_3b_2'\right)\mathbf{i} - \left(a_1b_3' - a_3b_1'\right)\mathbf{j} + \left(a_1b_2' - a_2b_1'\right)\mathbf{k}\right) \\ &\quad + \left(\left(a_2'b_3 - a_3'b_2\right)\mathbf{i} - \left(a_1'b_3 - a_3'b_1\right)\mathbf{j} + \left(a_1'b_2 - a_2'b_1\right)\mathbf{k}\right)\end{aligned}$$

Notice that the vectors in each of the two brackets can be written as the following formal determinants:

$$\begin{aligned}\frac{d}{dt}\mathbf{r}_1 \times \mathbf{r}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1' & b_2' & b_3' \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_1, a_2, a_3\rangle \times \langle b_1', b_2', b_3'\rangle + \langle a_1', a_2', a_3'\rangle \times \langle b_1, b_2, b_3\rangle \\ &= \mathbf{r}_1 \times \mathbf{r}_2' + \mathbf{r}_1' \times \mathbf{r}_2\end{aligned}$$

72. Verify the linearity properties

$$\int c\mathbf{r}(t)\,dt = c\int \mathbf{r}(t)\,dt \qquad (c \text{ any constant})$$

$$\int \left(\mathbf{r}_1(t) + \mathbf{r}_2(t)\right)dt = \int \mathbf{r}_1(t)\,dt + \int \mathbf{r}_2(t)\,dt$$

SOLUTION We denote the components of the vectors by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$; $\mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t)\rangle$; $\mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t)\rangle$. Using vector operations, componentwise integration and the linear properties for scalar functions, we obtain:

$$\begin{aligned}\int c\mathbf{r}(t)\,dt &= \int \langle cx(t), cy(t), cz(t)\rangle\,dt = \left\langle \int cx(t)\,dt, \int cy(t)\,dt, \int cz(t)\,dt\right\rangle \\ &= \left\langle c\int x(t)\,dt, c\int y(t)\,dt, c\int z(t)\,dt\right\rangle = c\left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt\right\rangle \\ &= c\int \langle x(t), y(t), z(t)\rangle\,dt = c\int \mathbf{r}(t)\,dt\end{aligned}$$

Next we prove the second linear property:

$$\begin{aligned}\int (\mathbf{r}_1(t) + \mathbf{r}_2(t))\,dt &= \int \langle x_1(t) + x_2(t), y_1(t) + y_2(t), z_1(t) + z_2(t)\rangle\,dt \\ &= \left\langle \int (x_1(t) + x_2(t))\,dt, \int (y_1(t) + y_2(t))\,dt, \int (z_1(t) + z_2(t))\,dt\right\rangle \\ &= \left\langle \int x_1(t)\,dt + \int x_2(t)\,dt, \int y_1(t)\,dt + \int y_2(t)\,dt, \int z_1(t)\,dt + \int z_2(t)\,dt\right\rangle \\ &= \left\langle \int x_1(t)\,dt, \int y_1(t)\,dt, \int z_1(t)\,dt\right\rangle + \left\langle \int x_2(t)\,dt, \int y_2(t)\,dt, \int z_2(t)\,dt\right\rangle \\ &= \int \langle x_1(t), y_1(t), z_1(t)\rangle\,dt + \int \langle x_2(t), y_2(t), z_2(t)\rangle\,dt = \int \mathbf{r}_1(t)\,dt + \int \mathbf{r}_2(t)\,dt\end{aligned}$$

73. Prove the Substitution Rule (where $g(t)$ is a differentiable scalar function):

$$\int_a^b \mathbf{r}(g(t))g'(t)\,dt = \int_{g^{-1}(a)}^{g^{-1}(b)} \mathbf{r}(u)\,du$$

SOLUTION (Note that an early edition of the textbook had the integral limits as $g(a)$ and $g(b)$; they should actually be $g^{-1}(a)$ and $g^{-1}(b)$.) We denote the components of the vector-valued function by $\mathbf{r}(t)\,dt = \langle x(t), y(t), z(t)\rangle$. Using componentwise integration we have:

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

Write $\int_a^b x(t)\,dt$ as $\int_a^b x(s)\,ds$. Let $s = g(t)$, so $ds = g'(t)\,dt$. The substitution gives us $\int_{g^{-1}(a)}^{g^{-1}(b)} x(g(t))g'(t)\,dt$. A similar procedure for the other two integrals gives us:

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,dt &= \left\langle \int_{g^{-1}(a)}^{g^{-1}(b)} x\left(g(t)\right) g'(t)\,dt, \int_{g^{-1}(a)}^{g^{-1}(b)} y\left(g(t)\right) g'(t)\,dt, \int_{g^{-1}(a)}^{g^{-1}(b)} z\left(g(t)\right) g'(t)\,dt \right\rangle \\ &= \int_{g^{-1}(a)}^{g^{-1}(b)} \left\langle x\left(g(t)\right) g'(t), y\left(g(t)\right) g'(t), z\left(g(t)\right) g'(t) \right\rangle dt \\ &= \int_{g^{-1}(a)}^{g^{-1}(b)} \left\langle x\left(g(t)\right), y\left(g(t)\right), z\left(g(t)\right) \right\rangle g'(t)\,dt = \int_{g^{-1}(a)}^{g^{-1}(b)} \mathbf{r}\left(g(t)\right) g'(t)\,dt\end{aligned}$$

74. Prove that if $\|\mathbf{r}(t)\| \le K$ for $t \in [a, b]$, then

$$\left\| \int_a^b \mathbf{r}(t)\,dt \right\| \le K(b-a)$$

SOLUTION Think of $\mathbf{r}(t)$ as a velocity vector. Then, $\int_a^b \mathbf{r}(t)\,dt$ gives the displacement vector from the location at time $t = a$ to the time $t = b$, and so $\left\| \int_a^b \mathbf{r}(t)\,dt \right\|$ gives the length of this displacement vector. But, since speed is $\|\mathbf{r}(t)\|$ which is less than or equal to K, then in the interval $a \le t \le b$, the object can move a total distance not more than $K(b-a)$. Thus, the length of the displacement vector is $\le K(b-a)$, which gives us $\left\| \int_a^b \mathbf{r}(t)\,dt \right\| \le K(b-a)$, as desired.

13.3 Arc Length and Speed (LT Section 14.3)

Preliminary Questions

1. At a given instant, a car on a roller coaster has velocity vector $\mathbf{r}' = \langle 25, -35, 10 \rangle$ (in miles per hour). What would the velocity vector be if the speed were doubled? What would it be if the car's direction were reversed but its speed remained unchanged?

SOLUTION The speed is doubled but the direction is unchanged, hence the new velocity vector has the form:

$$\lambda \mathbf{r}' = \lambda \langle 25, -35, 10 \rangle \ \text{ for } \lambda > 0$$

We use $\lambda = 2$, and so the new velocity vector is $\langle 50, -70, 20 \rangle$. If the direction is reversed but the speed is unchanged, the new velocity vector is:

$$-\mathbf{r}' = \langle -25, 35, -10 \rangle .$$

2. Two cars travel in the same direction along the same roller coaster (at different times). Which of the following statements about their velocity vectors at a given point P on the roller coaster is/are true?

(a) The velocity vectors are identical.

(b) The velocity vectors point in the same direction but may have different lengths.

(c) The velocity vectors may point in opposite directions.

SOLUTION

(a) The length of the velocity vector is the speed of the particle. Therefore, if the speeds of the cars are different the velocities are not identical. The statement is false.

(b) The velocity vector is tangent to the curve. Since the cars travel in the same direction, their velocity vectors point in the same direction. The statement is true.

(c) Since the cars travel in the same direction, the velocity vectors point in the same direction. The statement is false.

3. A mosquito flies along a parabola with speed $v(t) = t^2$. Let $s(t)$ be the total distance traveled at time t.

(a) How fast is $s(t)$ changing at $t = 2$?

(b) Is $s(t)$ equal to the mosquito's distance from the origin?

SOLUTION

(a) By the Arc Length Formula, we have:

$$s(t) = \int_{t_0}^{t} \|\mathbf{r}'(t)\|\, dt = \int_{t_0}^{t} v(t)\, dt$$

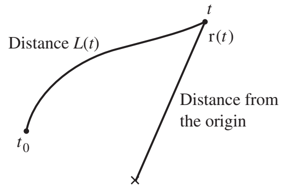
Therefore,

$$s'(t) = v(t)$$

To find the rate of change of $s(t)$ at $t = 2$ we compute the derivative of $s(t)$ at $t = 2$, that is,

$$s'(2) = v(2) = 2^2 = 4$$

(b) $s(t)$ is the distance along the path traveled by the mosquito. This distance is usually different from the mosquito's distance from the origin, which is the length of $\mathbf{r}(t)$.



4. What is the length of the path traced by $\mathbf{r}(t)$ for $4 \le t \le 10$ if $\mathbf{r}(t)$ is an arc length parametrization?

SOLUTION Since $\mathbf{r}(t)$ is an arc length parametrization, the length of the path for $4 \le t \le 10$ is equal to the length of the time interval $4 \le t \le 10$, which is 6.

Exercises

In Exercises 1–6, compute the length of the curve over the given interval.

1. $\mathbf{r}(t) = \langle 3t, 4t - 3, 6t + 1\rangle, \quad 0 \le t \le 3$

SOLUTION We have $x(t) = 3t$, $y(t) = 4t - 3$, $z(t) = 6t + 1$ hence

$$x'(t) = 3, \quad y'(t) = 4, \quad z'(t) = 6.$$

We use the Arc Length Formula to obtain:

$$s = \int_0^3 \|\mathbf{r}'(t)\|\, dt = \int_0^3 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_0^3 \sqrt{3^2 + 4^2 + 6^2}\, dt = 3\sqrt{61}$$

2. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{k}, \quad 11 \le t \le 15$

SOLUTION We have, $x(t) = 2t$, $y(t) = 0$, $z(t) = -3t$. Hence

$$x'(t) = 2, \quad y'(t) = 0, \quad z'(t) = -3$$

Using the Arc Length Formula we get:

$$\begin{aligned} s &= \int_{11}^{15} \|\mathbf{r}'(t)\|\, dt \\ &= \int_{11}^{15} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt \\ &= \int_{11}^{15} \sqrt{2^2 + 0^2 + (-3)^2}\, dt \\ &= \sqrt{13}(15 - 11) = 4\sqrt{13}. \end{aligned}$$

3. $\mathbf{r}(t) = \left\langle 2t, \ln t, t^2 \right\rangle, \quad 1 \le t \le 4$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \left\langle 2, \frac{1}{t}, 2t \right\rangle$. We use the Arc Length Formula to obtain:

$$\begin{aligned} s &= \int_1^4 \|\mathbf{r}'(t)\|\, dt = \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt = \int_1^4 \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt = \int_1^4 \sqrt{\left(2t + \frac{1}{t}\right)^2}\, dt \\ &= \int_1^4 \left(2t + \frac{1}{t}\right) dt = t^2 + \ln t \Big|_1^4 = (16 + \ln 4) - (1 + \ln 1) = 15 + \ln 4 \end{aligned}$$

4. $\mathbf{r}(t) = \left\langle 2t^2+1, 2t^2-1, t^3 \right\rangle, \quad 0 \le t \le 2$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \left\langle 4t, 4t, 3t^2 \right\rangle$. Using the Arc Length Formula we have:

$$s = \int_0^2 \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{(4t)^2+(4t)^2+(3t^2)^2}\,dt = \int_0^2 \sqrt{32t^2+9t^4}\,dt = \int_0^2 t\sqrt{32+9t^2}\,dt$$

We compute the integral using the substitution $u = 32+9t^2$, $du = 18t\,dt$. This gives:

$$s = \int_{32}^{68} \sqrt{u}\frac{du}{18} = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_{32}^{68} = \frac{1}{27}\left(68^{3/2}-32^{3/2}\right) \approx 14.063$$

5. $\mathbf{r}(t) = \langle t\cos t, t\sin t, 3t \rangle, \quad 0 \le t \le 2\pi$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 3\rangle$. The length of $\mathbf{r}'(t)$ is, thus,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 9}\\ &= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 9}\\ &= \sqrt{\left(\cos^2 t + \sin^2 t\right) + t^2\left(\sin^2 t + \cos^2 t\right) + 9} = \sqrt{t^2+10}\end{aligned}$$

Using the Arc Length Formula and the integration formula given in Exercise 6, we obtain:

$$\begin{aligned}s &= \int_0^{2\pi} \|\mathbf{r}'(t)\|\,dt = \int_0^{2\pi} \sqrt{t^2+10}\,dt = \frac{1}{2}t\sqrt{t^2+10} + \frac{1}{2}\cdot 10\ln\left(t+\sqrt{t^2+10}\right)\Big|_0^{2\pi}\\ &= \pi\sqrt{4\pi^2+10} + 5\ln\left(2\pi+\sqrt{4\pi^2+10}\right) - 5\ln\sqrt{10} = \pi\sqrt{4\pi^2+10} + 5\ln\frac{2\pi+\sqrt{4\pi^2+10}}{\sqrt{10}} \approx 29.3\end{aligned}$$

6. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (t^2-3)\mathbf{k}, \quad 0 \le t \le 2$. Use the formula:

$$\int \sqrt{t^2+a^2}\,dt = \frac{1}{2}t\sqrt{t^2+a^2} + \frac{1}{2}a^2\ln\left(t+\sqrt{t^2+a^2}\right)$$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$. Using the Arc Length Formula we get:

$$s = \int_0^2 \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{1^2+(2)^2+(2t)^2}\,dt = \int_0^2 \sqrt{4t^2+5}\,dt$$

We substitute $u = 2t$, $du = 2\,dt$ and use the given integration formula. This gives:

$$\begin{aligned}s &= \frac{1}{2}\int_0^4 \sqrt{u^2+5}\,du = \frac{1}{4}u\sqrt{u^2+5} + \frac{1}{4}\cdot 5\ln\left(u+\sqrt{u^2+5}\right)\Big|_0^4\\ &= \frac{1}{4}\cdot 4\sqrt{4^2+5} + \frac{5}{4}\ln\left(4+\sqrt{4^2+5}\right) - \frac{5}{4}\ln\sqrt{5} = \sqrt{21} + \frac{5}{4}\ln\left(4+\sqrt{21}\right) - \frac{5}{4}\ln\sqrt{5}\\ &= \sqrt{21} + \frac{5}{4}\ln\frac{4+\sqrt{21}}{\sqrt{5}} \approx 6.26\end{aligned}$$

In Exercises 7 and 8, compute the arc length function $s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du$ *for the given value of* a.

7. $\mathbf{r}(t) = \left\langle t^2, 2t^2, t^3 \right\rangle, \quad a = 0$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \left\langle 2t, 4t, 3t^2 \right\rangle$. Hence,

$$\|\mathbf{r}'(t)\| = \sqrt{(2t)^2+(4t)^2+(3t^2)^2} = \sqrt{4t^2+16t^2+9t^4} = t\sqrt{20+9t^2}$$

Hence,

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t u\sqrt{20+9u^2}\,du$$

We compute the integral using the substitution $v = 20+9u^2$, $dv = 18u\,du$. This gives:

$$s(t) = \frac{1}{18}\int_{20}^{20+9t^2} v^{1/2}\,dv = \frac{1}{18}\cdot\frac{2}{3}v^{3/2}\Big|_{20}^{20+9t^2} = \frac{1}{27}\left((20+9t^2)^{3/2} - 20^{3/2}\right).$$

8. $\mathbf{r}(t) = \langle 4t^{1/2}, \ln t, 2t\rangle, \quad a = 1$

SOLUTION The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \left\langle 2t^{-1/2}, 1/t, 2\right\rangle$ Hence,

$$\|\mathbf{r}'(t)\| = \sqrt{(2t^{-1/2})^2 + (1/t)^2 + 2^2} = \sqrt{\frac{4}{t} + \frac{1}{t^2} + 4} = \sqrt{\frac{4t^2 + 4t + 1}{t^2}} = \sqrt{\frac{(2t+1)^2}{t^2}} = \frac{2t+1}{t} = 2 + \frac{1}{t}$$

Therefore,

$$s(t) = \int_1^t \|\mathbf{r}'(u)\|\, du = \int_1^t 2 + \frac{1}{u}\, du = 2u + \ln u\Big|_1^t = 2t + \ln t - (2 + \ln 1) = 2t + \ln t - 2$$

In Exercises 9–12, find the speed at the given value of t.

9. $\mathbf{r}(t) = \langle 2t+3, 4t-3, 5-t\rangle, \quad t = 4$

SOLUTION The speed is the magnitude of the derivative $\mathbf{r}'(t) = \langle 2, 4, -1\rangle$. That is,

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{21} \approx 4.58$$

10. $\mathbf{r}(t) = \langle e^{t-3}, 12, 3t^{-1}\rangle, \quad t = 3$

SOLUTION The velocity vector is $\mathbf{r}'(t) = \langle e^{t-3}, 0, -3t^{-2}\rangle$ and at $t = 3$, $\mathbf{r}'(3) = \langle e^{3-3}, 0, -3\cdot 3^{-2}\rangle = \langle 1, 0, -\frac{1}{3}\rangle$. The speed is the magnitude of the velocity vector, that is,

$$v(3) = \|\mathbf{r}'(3)\| = \sqrt{1^2 + 0^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{10}{9}} \approx 1.05$$

11. $\mathbf{r}(t) = \langle \sin 3t, \cos 4t, \cos 5t\rangle, \quad t = \frac{\pi}{2}$

SOLUTION The velocity vector is $\mathbf{r}'(t) = \langle 3\cos 3t, -4\sin 4t, -5\sin 5t\rangle$. At $t = \frac{\pi}{2}$ the velocity vector is $\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle 3\cos\frac{3\pi}{2}, -4\sin 2\pi, -5\sin\frac{5\pi}{2}\right\rangle = \langle 0, 0, -5\rangle$. The speed is the magnitude of the velocity vector:

$$v\left(\frac{\pi}{2}\right) = \|\langle 0, 0, -5\rangle\| = 5.$$

12. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t\rangle, \quad t = 0$

SOLUTION The velocity vector is $\mathbf{r}'(t) = \langle \sinh t, \cosh t, 1\rangle$. At $t = 0$ the velocity is $\mathbf{r}'(0) = \langle \sinh(0), \cosh(0), 1\rangle = \langle 0, 1, 1\rangle$, hence the speed is

$$v(0) = \|\mathbf{r}'(0)\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}.$$

13. What is the velocity vector of a particle traveling to the right along the hyperbola $y = x^{-1}$ with constant speed 5 cm/s when the particle's location is $\left(2, \frac{1}{2}\right)$?

SOLUTION The position of the particle is given as $\mathbf{r}(t) = t^{-1}$. The magnitude of the velocity vector $\mathbf{r}'(t)$ is the speed of the particle. Hence,

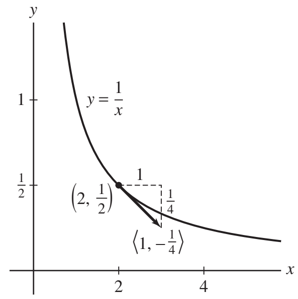
$$\|\mathbf{r}'(t)\| = 5 \tag{1}$$

The velocity vector points in the direction of motion, hence it is parallel to the tangent line to the curve $y = x^{-1}$ and points to the right. We find the slope of the tangent line at $x = 2$:

$$m = \frac{dy}{dx}\Big|_{x=2} = \frac{d}{dx}(x^{-1})\Big|_{x=2} = -x^{-2}\Big|_{x=2} = -\frac{1}{4}$$

We conclude that the vector $\left\langle 1, -\frac{1}{4}\right\rangle$ is a direction vector of the tangent line at $x = 2$, and for some $\lambda > 0$ we have at the given instance:

$$\mathbf{r}' = \lambda\left\langle 1, -\frac{1}{4}\right\rangle \tag{2}$$



To satisfy (1) we must have:

$$\|\mathbf{r}'\| = \lambda\sqrt{1^2 + \left(-\frac{1}{4}\right)^2} = \lambda\frac{\sqrt{17}}{4} = 5 \quad \Rightarrow \quad \lambda = \frac{20}{\sqrt{17}}$$

Substituting in (2) we obtain the following velocity vector at $\left(2, \frac{1}{2}\right)$:

$$\mathbf{r}' = \frac{20}{\sqrt{17}}\left\langle 1, -\frac{1}{4}\right\rangle = \left\langle \frac{20}{\sqrt{17}}, \frac{-5}{\sqrt{17}}\right\rangle$$

14. A bee with velocity vector $\mathbf{r}'(t)$ starts out at the origin at $t = 0$ and flies around for T seconds. Where is the bee located at time T if $\int_0^T \mathbf{r}'(u)\,du = \mathbf{0}$? What does the quantity $\int_0^T \|\mathbf{r}'(u)\|\,du$ represent?

SOLUTION By the Fundamental Theorem for vector-valued functions, $\int_0^T \mathbf{r}'(u)\,du = \mathbf{r}(T) - \mathbf{r}(0)$, hence by the given information $\mathbf{r}(T) = \mathbf{r}(0)$. It follows that at time T the bee is located at the starting point which is at the origin. The integral $\int_0^T \|\mathbf{r}'(u)\|\,du$ is the length of the path traveled by the bee in the time interval $0 \le t \le T$. Notice that there is a difference between the displacement and the actual length traveled.

15. Let

$$\mathbf{r}(t) = \left\langle R\cos\left(\frac{2\pi Nt}{h}\right), R\sin\left(\frac{2\pi Nt}{h}\right), t\right\rangle, \qquad 0 \le t \le h$$

(a) Show that $\mathbf{r}(t)$ parametrizes a helix of radius R and height h making N complete turns.

(b) Guess which of the two springs in Figure 5 uses more wire.

(c) Compute the lengths of the two springs and compare.

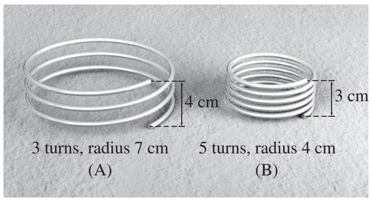


FIGURE 5 Which spring uses more wire?

SOLUTION We first verify that the projection $\mathbf{p}(t) = \left\langle R\cos\left(\frac{2\pi Nt}{h}\right), R\sin\left(\frac{2\pi Nt}{h}\right), 0\right\rangle$ onto the xy-plane describes a point moving around the circle of radius R. We have:

$$x(t)^2 + y(t)^2 = R^2\cos^2\left(\frac{2\pi Nt}{h}\right) + R^2\sin^2\left(\frac{2\pi Nt}{h}\right) = R^2\left(\cos^2\left(\frac{2\pi Nt}{h}\right) + \sin^2\left(\frac{2\pi Nt}{h}\right)\right) = R^2$$

This is the equation of the circle of radius R in the xy-plane. As t changes in the interval $0 \le t \le h$ the argument $\frac{2\pi Nt}{h}$ changes from 0 to $2\pi N$, that is, it covers N periods of the cos and sin functions. It follows that the projection onto the xy-plane describes a point moving around the circle of radius R, making N complete turns. The height of the helix is the maximum value of the z-component, which is $t = h$.

(a) The second wire seems to use more wire than the first one.

(b) Setting $R = 7$, $h = 4$ and $N = 3$ in the parametrization in Exercise 15 gives:

$$\mathbf{r}_1(t) = \left\langle 7\cos\frac{2\pi\cdot 3t}{4}, 7\sin\frac{2\pi\cdot 3t}{4}, t\right\rangle = \left\langle 7\cos\frac{3\pi t}{2}, 7\sin\frac{3\pi t}{2}, t\right\rangle, \quad 0 \le t \le 4$$

Setting $R = 4$, $h = 3$ and $N = 5$ in this parametrization we get:

$$\mathbf{r}_2(t) = \left\langle 4\cos\frac{2\pi\cdot 5t}{3}, 4\sin\frac{2\pi\cdot 5t}{3}, t\right\rangle = \left\langle 4\cos\frac{10\pi t}{3}, 4\sin\frac{10\pi t}{3}, t\right\rangle, \quad 0 \le t \le 3$$

We find the derivatives of the two vectors and their lengths:

$$\mathbf{r}_1'(t) = \left\langle -\frac{21\pi}{2}\sin\frac{3\pi t}{2}, \frac{21\pi}{2}\cos\frac{3\pi t}{2}, 1\right\rangle \quad\Rightarrow\quad \|\mathbf{r}_1'(t)\| = \sqrt{\frac{441\pi^2}{4}+1} = \frac{1}{2}\sqrt{441\pi^2+4}$$

$$\mathbf{r}_2'(t) = \left\langle -\frac{40\pi}{3}\sin\frac{10\pi t}{3}, \frac{40\pi}{3}\cos\frac{10\pi t}{3}, 1\right\rangle \quad\Rightarrow\quad \|\mathbf{r}_2'(t)\| = \sqrt{\frac{1600\pi^2}{9}+1} = \frac{1}{3}\sqrt{1600\pi^2+9}$$

Using the Arc Length Formula we obtain the following lengths:

$$s_1 = \int_0^4 \frac{1}{2}\sqrt{441\pi^2+4}\,dt = 2\sqrt{441\pi^2+4} \approx 132$$

$$s_2 = \int_0^3 \frac{1}{3}\sqrt{1600\pi^2+9}\,dt = \sqrt{1600\pi^2+9} \approx 125.7$$

We see that the first spring uses more wire than the second one.

16. Use Exercise 15 to find a general formula for the length of a helix of radius R and height h that makes N complete turns.

SOLUTION In Exercise 15 it is shown that the helix has the following parametrization:

$$\mathbf{r}(t) = \left\langle R\cos\frac{2\pi Nt}{h}, R\sin\frac{2\pi Nt}{h}, t\right\rangle; \quad 0 \le t \le h$$

We compute the derivative vector and its length:

$$\mathbf{r}'(t) = \left\langle -\frac{R\cdot 2\pi N}{h}\sin\frac{2\pi Nt}{h}, \frac{R\cdot 2\pi N}{h}\cos\frac{2\pi Nt}{h}, 1\right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{R\cdot 2\pi N}{h}\sin\frac{2\pi Nt}{h}\right)^2 + \left(\frac{R\cdot 2\pi N}{h}\cos\frac{2\pi Nt}{h}\right)^2 + 1^2}$$

$$= \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\frac{2\pi Nt}{h} + \cos^2\frac{2\pi Nt}{h}\right) + 1} = \sqrt{\frac{4\pi^2N^2R^2}{h^2}+1} = \frac{1}{h}\sqrt{4\pi^2N^2R^2+h^2}$$

We now apply the Arc Length Formula to compute the length of the helix:

$$s = \int_0^h \|\mathbf{r}'(t)\|\,dt = \int_0^h \frac{1}{h}\sqrt{4\pi^2N^2R^2+h^2}\,dt = \frac{\sqrt{4\pi^2N^2R^2+h^2}}{h}\cdot h = \sqrt{4\pi^2N^2R^2+h^2}$$

17. The cycloid generated by the unit circle has parametrization

$$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$$

(a) Find the value of t in $[0, 2\pi]$ where the speed is at a maximum.

(b) Show that one arch of the cycloid has length 8. Recall the identity $\sin^2(t/2) = (1-\cos t)/2$.

SOLUTION One arch of the cycloid is traced as $0 \le t \le 2\pi$. By the Arc Length Formula we have:

$$s = \int_0^{2\pi} \|\mathbf{r}'(t)\|\,dt \tag{1}$$

We compute the derivative and its length:

$$\mathbf{r}'(t) = \langle 1-\cos t, \sin t\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{2-2\cos t} = \sqrt{2(1-\cos t)} = \sqrt{2\cdot 2\sin^2\frac{t}{2}} = 2\left|\sin\frac{t}{2}\right|.$$

For $0 \le t \le 2\pi$, we have $0 \le \frac{t}{2} \le \pi$, so $\sin \frac{t}{2} \ge 0$. Therefore we may omit the absolute value sign and write:

$$\|\mathbf{r}'(t)\| = 2\sin\frac{t}{2}$$

Substituting in (1) and computing the integral using the substitution $u = \frac{t}{2}$, $du = \frac{1}{2}\,dt$, gives:

$$s = \int_0^{2\pi} 2\sin\frac{t}{2}\,dt = \int_0^{\pi} 2\sin u \cdot (2\,du) = 4\int_0^{\pi} \sin u\,du$$
$$= 4(-\cos u)\Big|_0^{\pi} = 4\left(-\cos\pi - (-\cos 0)\right) = 4(1+1) = 8$$

The length of one arc of the cycloid is $s = 8$. The speed is given by the function:

$$v(t) = \|\mathbf{r}'(t)\| = 2\sin\frac{t}{2}, \quad 0 \le t \le \pi$$

To find the value of t in $[0, 2\pi]$ where the speed is at maximum, we first find the critical point in this interval:

$$v'(t) = 2 \cdot \frac{1}{2}\cos\frac{t}{2} = \cos\frac{t}{2}$$
$$\cos\frac{t}{2} = 0 \quad \Rightarrow \quad \frac{t}{2} = \frac{\pi}{2} \quad \Rightarrow \quad t = \pi$$

Since $v''(t) = -\frac{1}{2}\sin\frac{t}{2}$, we have $v''(\pi) = -\frac{1}{2}\sin\frac{\pi}{2} = -\frac{1}{2} < 0$, hence the speed $v(t)$ has a maximum value at $t = \pi$.

18. Which of the following is an arc length parametrization of a circle of radius 4 centered at the origin?

(a) $\mathbf{r}_1(t) = \langle 4\sin t, 4\cos t\rangle$

(b) $\mathbf{r}_2(t) = \langle 4\sin 4t, 4\cos 4t\rangle$

(c) $\mathbf{r}_3(t) = \left\langle 4\sin\frac{t}{4}, 4\cos\frac{t}{4}\right\rangle$

SOLUTION The arc length parametrization is defined by the condition $\|\mathbf{r}'(t)\| = 1$ for all t. We thus must check whether this condition is satisfied.

(a) The derivative vector is $\mathbf{r}_1'(t) = \langle 4\cos t, -4\sin t\rangle$. We compute the length of this vector:

$$\|\mathbf{r}_1'(t)\| = \sqrt{(4\cos t)^2 + (-4\sin t)^2} = \sqrt{16\left(\cos^2 t + \sin^2 t\right)} = \sqrt{16} = 4 \ne 1$$

We conclude that this parametrization is not the arc length parametrization of the circle.

(b) We compute the derivative vector and its length:

$$\mathbf{r}_2'(t) = \langle 16\cos 4t, -16\sin 4t\rangle$$
$$\|\mathbf{r}_2'(t)\| = \sqrt{(16\cos 4t)^2 + (-16\sin 4t)^2} = \sqrt{16^2\left(\cos^2 4t + \sin^2 4t\right)} = \sqrt{16^2 \cdot 1} = 16 \ne 1$$

Hence, this parametrization is not the arc length parametrization of the circle.

(c) We find the derivative vector and its length:

$$\mathbf{r}_3'(t) = \left\langle 4 \cdot \frac{1}{4}\cos\frac{t}{4}, -4 \cdot \frac{1}{4}\sin\frac{t}{4}\right\rangle = \left\langle \cos\frac{t}{4}, -\sin\frac{t}{4}\right\rangle$$
$$\|\mathbf{r}_3'(t)\| = \sqrt{\left(\cos\frac{t}{4}\right)^2 + \left(-\sin\frac{t}{4}\right)^2} = 1$$

Hence, this parametrization is the arc length parametrization of the circle.

19. Let $\mathbf{r}(t) = \langle 3t+1, 4t-5, 2t\rangle$.

(a) Evaluate the arc length integral $s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du$.

(b) Find the inverse $g(s)$ of $s(t)$.

(c) Verify that $\mathbf{r}_1(s) = \mathbf{r}(g(s))$ is an arc length parametrization.

SOLUTION

(a) We differentiate $\mathbf{r}(t)$ componentwise and then compute the norm of the derivative vector. This gives:

$$\mathbf{r}'(t) = \langle 3, 4, 2\rangle$$
$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$$

We compute $s(t)$:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \sqrt{29}\, du = \sqrt{29}\, u\Big|_0^t = \sqrt{29}t$$

(b) We find the inverse $g(s) = t(s)$ by solving $s = \sqrt{29}t$ for t. We obtain:

$$s = \sqrt{29}t \quad \Rightarrow \quad t = g(s) = \frac{s}{\sqrt{29}}$$

We obtain the following arc length parametrization:

$$\mathbf{r}_1(s) = \mathbf{r}\left(\frac{s}{\sqrt{29}}\right) = \left\langle \frac{3s}{\sqrt{29}} + 1, \frac{4s}{\sqrt{29}} - 5, \frac{2s}{\sqrt{29}} \right\rangle$$

To verify that $\mathbf{r}_1(s)$ is an arc length parametrization we must show that $\|\mathbf{r}_1'(s)\| = 1$. We compute $\mathbf{r}_1'(s)$:

$$\mathbf{r}_1'(s) = \frac{d}{ds}\left\langle \frac{3s}{\sqrt{29}} + 1, \frac{4s}{\sqrt{29}} - 5, \frac{2s}{\sqrt{29}} \right\rangle = \left\langle \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right\rangle = \frac{1}{\sqrt{29}}\langle 3, 4, 2\rangle$$

Thus,

$$\|\mathbf{r}_1'(s)\| = \frac{1}{\sqrt{29}}\|\langle 3, 4, 2\rangle\| = \frac{1}{\sqrt{29}}\sqrt{3^2 + 4^2 + 2^2} = \frac{1}{\sqrt{29}} \cdot \sqrt{29} = 1$$

20. Find an arc length parametrization of the line $y = 4x + 9$.

SOLUTION First, let $\mathbf{r}(t) = \langle t, 4t + 9\rangle$. Then we know $\mathbf{r}'(t) = \langle 1, 4\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{17}$. We now compute $s(t)$:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \sqrt{17}\, du = t\sqrt{17}$$

We find the inverse $g(s) = t(s)$ by solving $s = t\sqrt{17}$ for t:

$$s = t\sqrt{17}, \quad \Rightarrow \quad t = \frac{1}{\sqrt{17}}s$$

We obtain the following arc length parametrization:

$$\mathbf{r}_1(s) = \mathbf{r}\left(\frac{s}{\sqrt{17}}\right) = \left\langle \frac{s}{\sqrt{17}}, \frac{4s}{\sqrt{17}} + 9 \right\rangle$$

21. Let $\mathbf{r}(t) = \mathbf{w} + t\mathbf{v}$ be the parametrization of a line.

(a) Show that the arc length function $s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du$ is given by $s(t) = t\|\mathbf{v}\|$. This shows that $\mathbf{r}(t)$ is an arc length parametrizaton if and only if $\mathbf{v}$ is a unit vector.

(b) Find an arc length parametrization of the line with $\mathbf{w} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 3, 4, 5\rangle$.

SOLUTION

(a) Since $\mathbf{r}(t) = \mathbf{w} + t\mathbf{v}$, then $\mathbf{r}'(t) = \mathbf{v}$ and $\|\mathbf{r}'(t)\| = \|\mathbf{v}\|$. Then computing $s(t)$ we get:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \|\mathbf{v}\|\, du = t\|\mathbf{v}\|$$

If we consider $s(t)$,

$$s(t) = t \text{ if and only if } \|\mathbf{v}\| = 1$$

(b) Since $\mathbf{v} = \langle 3, 4, 5\rangle$, then from part (a) we get:

$$s(t) = t\|\mathbf{v}\| = t\sqrt{3^2 + 4^2 + 5^2} = t\sqrt{50}, \quad \Rightarrow t = g(s) = \frac{s}{\sqrt{50}}$$

Therefore, since we are given $\mathbf{r}(t) = \mathbf{w} + t\mathbf{v}$, the arc length parametrization is:

$$\mathbf{r}_1(s) = \langle 1, 2, 3\rangle + \frac{s}{\sqrt{50}}\langle 3, 4, 5\rangle = \left\langle 1 + \frac{3s}{\sqrt{50}}, 2 + \frac{4s}{\sqrt{50}}, 3 + \frac{5s}{\sqrt{50}} \right\rangle$$

22. Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$.

SOLUTION We start with the following parametrization of the circle:

$$\mathbf{r}(t) = \langle 1, 4, 9\rangle + 4\langle \cos t, \sin t, 0\rangle = \langle 1 + 4\cos t, 4 + 4\sin t, 9\rangle$$

We now follow two steps:

Step 1. Find the inverse of the arc length function. The arc length function is $s(t) = \int_0^t \|\mathbf{r}'(l)\|\, dl$. We compute $\mathbf{r}'(t)$ and its length:

$$\mathbf{r}(t) = \langle -4\sin t, 4\cos t, 0\rangle$$
$$\|\mathbf{r}'(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 0^2} = \sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{16\cdot 1} = 4$$

Hence,

$$s(t) = \int_0^t 4\, dl = 4t$$

The inverse of $s = 4t$ is $t = g(s) = \frac{s}{4}$.

Step 2. Reparametrize the curve. The arc length parametrization is

$$\mathbf{r}_1(s) = \mathbf{r}\,(g(s)) = \mathbf{r}\left(\frac{s}{4}\right) = \left\langle 1 + 4\cos\frac{s}{4}, 4 + 4\sin\frac{s}{4}, 9\right\rangle$$

23. Find a path that traces the circle in the plane $y = 10$ with radius 4 and center $(2, 10, -3)$ with constant speed 8.

SOLUTION We start with the following parametrization of the circle:

$$\mathbf{r}(t) = \langle 2, 10, -3\rangle + 4\langle \cos t, 0, \sin t\rangle = \langle 2 + 4\cos t, 10, -3 + 4\sin t\rangle$$

We need to reparametrize the curve by making a substitution $t = g(s)$, so that the new parametrization $\mathbf{r}_1(s) = \mathbf{r}\,(g(s))$ satisfies $\|\mathbf{r}_1'(s)\| = 8$ for all s. We find $\mathbf{r}_1'(s)$ using the Chain Rule:

$$\mathbf{r}_1'(s) = \frac{d}{ds}\mathbf{r}\,(g(s)) = g'(s)\mathbf{r}'\,(g(s)) \qquad (1)$$

Next, we differentiate $\mathbf{r}(t)$ and then replace t by $g(s)$:

$$\mathbf{r}'(t) = \langle -4\sin t, 0, 4\cos t\rangle$$
$$\mathbf{r}'\,(g(s)) = \langle -4\sin g(s), 0, 4\cos g(s)\rangle$$

Substituting in (1) we get:

$$\mathbf{r}_1'(s) = g'(s)\langle -4\sin g(s), 0, 4\cos g(s)\rangle = -4g'(s)\langle \sin g(s), 0, -\cos g(s)\rangle$$

Hence,

$$\|\mathbf{r}_1'(s)\| = 4|g'(s)|\sqrt{(\sin g(s))^2 + (-\cos g(s))^2} = 4|g'(s)|$$

To satisfy $\|\mathbf{r}_1'(s)\| = 8$ for all s, we choose $g'(s) = 2$. We may take the antiderivative $g(s) = 2\cdot s$, and obtain the following parametrization:

$$\mathbf{r}_1(s) = \mathbf{r}\,(g(s)) = \mathbf{r}(2s) = \langle 2 + 4\cos(2s), 10, -3 + 4\sin(2s)\rangle\,.$$

This is a parametrization of the given circle, with constant speed 8.

24. Find an arc length parametrization of $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^t\rangle$.

SOLUTION An arc length parametrization is $\mathbf{r}_1(s) = \mathbf{r}\,(g(s))$ where $t = g(s)$ is the inverse of the arc length function. We compute the arc length function:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du \qquad (1)$$

Differentiating $\mathbf{r}(t)$ and computing the norm of $\mathbf{r}'(t)$ gives:

$$\mathbf{r}'(t) = \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t, e^t\rangle = e^t\langle \sin t + \cos t, \cos t - \sin t, 1\rangle$$
$$\|\mathbf{r}'(t)\| = e^t\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1^2}$$

$$= e^t\left(\sin^2 t + 2\sin t\cos t + \cos^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t + 1\right)^{1/2}$$

$$= e^t\sqrt{2(\sin^2 t + \cos^2 t) + 1} = e^t\sqrt{2\cdot 1 + 1} = \sqrt{3}\,e^t \tag{2}$$

Substituting (2) into (1) gives:

$$s(t) = \int_0^t \sqrt{3}\,e^u\,du = \sqrt{3}\,e^u\Big|_0^t = \sqrt{3}(e^t - e^0) = \sqrt{3}(e^t - 1)$$

We find the inverse function of $s(t)$ by solving $s = \sqrt{3}\left(e^t - 1\right)$ for t. We obtain:

$$s = \sqrt{3}(e^t - 1)$$

$$\frac{s}{\sqrt{3}} = e^t - 1$$

$$e^t = 1 + \frac{s}{\sqrt{3}} \quad\Rightarrow\quad t = g(s) = \ln\left(1 + \frac{s}{\sqrt{3}}\right)$$

An arc length parametrization for $\mathbf{r}_1(s) = \mathbf{r}\,(g(s))$ is:

$$\left\langle e^{\ln(1+s/(\sqrt{3}))}\sin\left(\ln\left(1+\frac{s}{\sqrt{3}}\right)\right), e^{\ln(1+s/(\sqrt{3}))}\cos\left(\ln\left(1+\frac{s}{\sqrt{3}}\right)\right), e^{\ln(1+s/(\sqrt{3}))}\right\rangle$$

$$= \left(1+\frac{s}{\sqrt{3}}\right)\left\langle \sin\left(\ln\left(1+\frac{s}{\sqrt{3}}\right)\right), \cos\left(\ln\left(1+\frac{s}{\sqrt{3}}\right)\right), 1\right\rangle$$

25. Find an arc length parametrization of $\mathbf{r}(t) = \left\langle t^2, t^3\right\rangle$.

SOLUTION We follow two steps.

Step 1. Find the inverse of the arc length function. The arc length function is the following function:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du \tag{1}$$

In our case $\mathbf{r}'(t) = \left\langle 2t, 3t^2\right\rangle$ hence $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 9t^4} = \sqrt{4 + 9t^2}\,t$. We substitute in (1) and compute the resulting integral using the substitution $v = 4 + 9u^2$, $dv = 18u\,du$. This gives:

$$s(t) = \int_0^t \sqrt{4 + 9u^2}\,u\,du = \frac{1}{18}\int_4^{4+9t^2} v^{1/2}\,dv = \frac{1}{18}\cdot\frac{2}{3}v^{3/2}\Big|_4^{4+9t^2} = \frac{1}{27}\left((4+9t^2)^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(\left(4 + 9t^2\right)^{3/2} - 8\right)$$

We find the inverse of $t = s(t)$ by solving for t in terms of s. This function is invertible for $t \ge 0$ and for $t \le 0$.

$$s = \frac{1}{27}\left((4+9t^2)^{3/2} - 8\right)$$

$$27s + 8 = (4 + 9t^2)^{3/2}$$

$$(27s + 8)^{2/3} - 4 = 9t^2$$

$$t^2 = \frac{1}{9}\left((27s+8)^{2/3} - 4\right) = \frac{1}{9}(27s+8)^{2/3} - \frac{4}{9}$$

$$t = \pm\frac{1}{3}\sqrt{(27s+8)^{2/3} - 4} \tag{2}$$

Step 2. Reparametrize the curve. The arc length parametrization is obtained by replacing t by (2) in $\mathbf{r}(t)$:

$$\mathbf{r}_1(s) = \left\langle \frac{1}{9}(27s+8)^{2/3} - \frac{4}{9}, \pm\frac{1}{27}\left((27s+8)^{2/3} - 4\right)^{3/2}\right\rangle$$

26. Find an arc length parametrization of the cycloid with parametrization $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

SOLUTION

Step 1. Find the inverse of the arc length function. We are given $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$, so then $\mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$ and

$$\|\mathbf{r}'(t)\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{2 - 2\cos t} = \sqrt{4\sin^2(t/2)} = 2\sin(t/2)$$

We then compute $s(t)$:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t 2\sin\frac{u}{2}\,du$$
$$= -4\cos\frac{u}{2}\Big|_0^t = -4\cos\frac{t}{2} + 4 = 4\left(1 - \cos\frac{t}{2}\right)$$

Then solve $s(t) = 4\left(1 - \cos\frac{t}{2}\right)$ for t:

$$s = 4\left(1 - \cos\frac{t}{2}\right) \quad \Rightarrow \frac{s}{4} = 1 - \cos\frac{t}{2} \quad \Rightarrow 1 - \frac{s}{4} = \cos\frac{t}{2} \quad \Rightarrow t = 2\arccos\left(1 - \frac{s}{4}\right)$$

Step 2. Reparametrize the curve using the t we just found. We will write this as $t = 2\arccos\left(\frac{4-s}{4}\right)$ and solving, let us first note that:

$$\sin t = 2\sin\frac{t}{2}\cos\frac{t}{2} = 2\left(\frac{\sqrt{8s - s^2}}{4}\right)\left(\frac{4-s}{4}\right) = \frac{(4-s)\sqrt{8s-s^2}}{8}$$
$$\cos t = 2\cos^2\frac{t}{2} - 1 = 2\left(\frac{4-s}{4}\right)^2 - 1$$

So then, rewriting

$$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$$
$$\mathbf{r}_1(s) = \left\langle 2\arccos\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}, 1 - \left(2\left(\frac{4-s}{4}\right)^2 - 1\right)\right\rangle$$
$$= \left\langle 2\arccos\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}, 1 - \frac{1}{8}(4-s)^2 + 1\right\rangle$$
$$= \left\langle 2\arccos\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}, 2 - \frac{1}{8}(4-s)^2\right\rangle$$

27. Find an arc length parametrization of the line $y = mx$ for an arbitrary slope m.

SOLUTION

Step 1. Find the inverse of the arc length function. We are given the line $y = mx$ and a parametrization of this line is $\mathbf{r}(t) = \langle t, mt\rangle$, thus $\mathbf{r}'(t) = \langle 1, m\rangle$ and

$$\|\mathbf{r}'(t)\| = \sqrt{1 + m^2}$$

We then compute $s(t)$:

$$s(t) = \int_0^t \sqrt{1+m^2}\,du = t\sqrt{1+m^2}$$

Solving $s = t\sqrt{1+m^2}$ for t we get:

$$t = \frac{s}{\sqrt{1+m^2}}$$

Step 2. Reparametrize the curve using the t we just found.

$$\mathbf{r}_1(s) = \left\langle \frac{s}{\sqrt{1+m^2}}, \frac{sm}{\sqrt{1+m^2}}\right\rangle$$

28. Express the arc length s of $y = x^3$ for $0 \le x \le 8$ as an integral in two ways, using the parametrizations $\mathbf{r}_1(t) = \langle t, t^3\rangle$ and $\mathbf{r}_2(t) = \langle t^3, t^9\rangle$. Do not evaluate the integrals, but use substitution to show that they yield the same result.

SOLUTION For $\mathbf{r}_1(t) = \langle t, t^3\rangle$ we have $\mathbf{r}_1'(t) = \langle 1, 3t^2\rangle$ hence $\|\mathbf{r}_1'(t)\| = \sqrt{1 + 9t^4}$. For $\mathbf{r}_2(t) = \langle t^3, t^9\rangle$ we have $\mathbf{r}_2'(t) = \langle 3t^2, 9t^8\rangle$ hence $\|\mathbf{r}_2'(t)\| = \sqrt{9t^4 + 81t^{16}}$. The length s may be computed using the two parametrizations by the following integrals (notice that in the second parametrization $0 \le t^3 \le 8$ hence $0 \le t \le 2$).

$$s = \int_0^8 \|\mathbf{r}_1'(t)\|\,dt = \int_0^8 \sqrt{1+9t^4}\,dt \tag{1}$$

$$s = \int_0^2 \|\mathbf{r}_2'(t)\|\,dt = \int_0^2 \sqrt{1+9t^{12}}\,3t^2\,dt \tag{2}$$

We use the substitution $u = t^3$, $du = 3t^2\,dt$ in the second integral to obtain:

$$\int_0^2 \sqrt{1+9t^{12}}\,3t^2\,dt = \int_0^8 \sqrt{1+9u^4}\,du$$

This integral is the same as the integral in (1), in accordance with the well known property: the arc length is independent of the parametrization we choose for the curve.

29. The curve known as the **Bernoulli spiral** (Figure 6) has parametrization $\mathbf{r}(t) = \langle e^t\cos 4t, e^t\sin 4t\rangle$.

(a) Evaluate $s(t) = \int_{-\infty}^{t} \|\mathbf{r}'(u)\|\,du$. It is convenient to take lower limit $-\infty$ because $\mathbf{r}(-\infty) = \langle 0, 0\rangle$.

(b) Use (a) to find an arc length parametrization of $\mathbf{r}(t)$.

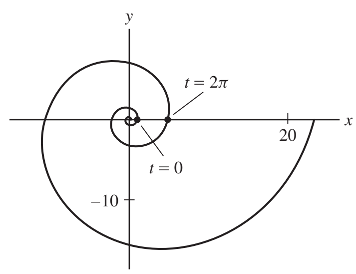


FIGURE 6 Bernoulli spiral.

SOLUTION

(a) We differentiate $\mathbf{r}(t)$ and compute the norm of the derivative vector. This gives:

$$\mathbf{r}'(t) = \langle e^t\cos 4t - 4e^t\sin 4t, e^t\sin 4t + 4e^t\cos 4t\rangle = e^t\,\langle \cos 4t - 4\sin 4t, \sin 4t + 4\cos 4t\rangle$$

$$\begin{aligned}\|\mathbf{r}'(t)\| &= e^t\sqrt{(\cos 4t - 4\sin 4t)^2 + (\sin 4t + 4\cos 4t)^2}\\ &= e^t\left(\cos^2 4t - 8\cos 4t\sin 4t + 16\sin^2 4t + \sin^2 4t + 8\sin 4t\cos 4t + 16\cos^2 4t\right)^{1/2}\\ &= e^t\sqrt{\cos^2 4t + \sin^2 4t + 16\left(\sin^2 4t + \cos^2 4t\right)} = e^t\sqrt{1+16\cdot 1} = \sqrt{17}e^t\end{aligned}$$

We now evaluate the improper integral:

$$\begin{aligned}s(t) &= \int_{-\infty}^{t} \|\mathbf{r}'(u)\|\,du = \lim_{R\to-\infty}\int_R^t \sqrt{17}e^u\,du = \lim_{R\to-\infty} \sqrt{17}e^u\Big|_R^t = \lim_{R\to-\infty}\sqrt{17}(e^t - e^R)\\ &= \sqrt{17}(e^t - 0) = \sqrt{17}e^t\end{aligned}$$

(b) An arc length parametrization of $\mathbf{r}(t)$ is $\mathbf{r}_1(s) = \mathbf{r}\,(g(s))$ where $t = g(s)$ is the inverse function of $s(t)$. We find $t = g(s)$ by solving $s = \sqrt{17}e^t$ for t:

$$s = \sqrt{17}e^t \quad\Rightarrow\quad e^t = \frac{s}{\sqrt{17}} \Rightarrow t = g(s) = \ln\frac{s}{\sqrt{17}}$$

An arc length parametrization of $\mathbf{r}(t)$ is:

$$\begin{aligned}\mathbf{r}_1(s) = \mathbf{r}\,(g(s)) &= \left\langle e^{\ln(s/(\sqrt{17}))}\cos\left(4\ln\frac{s}{\sqrt{17}}\right), e^{\ln(s/(\sqrt{17}))}\sin\left(4\ln\frac{s}{\sqrt{17}}\right)\right\rangle\\ &= \frac{s}{\sqrt{17}}\left\langle\cos\left(4\ln\frac{s}{\sqrt{17}}\right), \sin\left(4\ln\frac{s}{\sqrt{17}}\right)\right\rangle\end{aligned} \tag{1}$$

Further Insights and Challenges

30. Prove that the length of a curve as computed using the arc length integral does not depend on its parametrization. More precisely, let $\mathcal{C}$ be the curve traced by $\mathbf{r}(t)$ for $a \le t \le b$. Let $f(s)$ be a differentiable function such that $f'(s) > 0$ and that $f(c) = a$ and $f(d) = b$. Then $\mathbf{r}_1(s) = \mathbf{r}(f(s))$ parametrizes $\mathcal{C}$ for $c \le s \le d$. Verify that

$$\int_a^b \|\mathbf{r}'(t)\|\,dt = \int_c^d \|\mathbf{r}_1'(s)\|\,ds$$

SOLUTION By the Chain Rule we have:

$$\mathbf{r}_1'(t) = g'(t)\mathbf{r}'(g(t)) \quad \Rightarrow \quad \|\mathbf{r}_1'(t)\| = |g'(t)|\|\mathbf{r}'(g(t))\| = g'(t)\|\mathbf{r}'(g(t))\|$$

Hence,

$$\int_c^d \|\mathbf{r}_1'(t)\|\,dt = \int_c^d g'(t)\|\mathbf{r}'(g(t))\|dt \qquad (1)$$

We use the substitution $u = g(t)$, $du = g'(t)\,dt$. We are given that $g(c) = a$ and $g(d) = b$, hence the new limits of integration are $g(c) = a$ and $\varphi(d) = b$. Thus, by (1):

$$\int_c^d \|\mathbf{r}_1'(t)\|\,dt = \int_{g(c)}^{g(d)} \|\mathbf{r}'(u)\|\,du = \int_a^b \|\mathbf{r}'(u)\|\,du = \int_a^b \|\mathbf{r}'(t)\|\,dt$$

31. The unit circle with the point $(-1, 0)$ removed has parametrization (see Exercise 73 in Section 11.1)

$$\mathbf{r}(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle, \qquad -\infty < t < \infty$$

Use this parametrization to compute the length of the unit circle as an improper integral. *Hint:* The expression for $\|\mathbf{r}'(t)\|$ simplifies.

SOLUTION We have $x(t) = \frac{1-t^2}{1+t^2}$, $y(t) = \frac{2t}{1+t^2}$. Hence,

$$x^2(t) + y^2(t) = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{\left(1+t^2\right)^2} = \frac{1 + 2t^2 + t^4}{\left(1+t^2\right)^2} = \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2} = 1$$

It follows that the path $\mathbf{r}(t)$ lies on the unit circle. We now show that the entire circle is indeed parametrized by $\mathbf{r}(t)$ as t moves from $-\infty$ to ∞. First, note that $x'(t)$ can be written as $\left[-2t(1+t^2) - 2t(1-t^2)\right]/(1+t^2)^2$ which is $-4t/(1+t^2)^2$. So, for t negative, $x(t)$ is an increasing function, $y(t)$ is negative, and since $\lim_{t\to-\infty} x(t) = -1$ and $\lim_{t\to 0} x(t) = 1$, we conclude that $\mathbf{r}(t)$ does indeed parametrize the lower half of the circle for negative t. A similar argument proves that we get the upper half of the circle for positive t. We now compute $\mathbf{r}'(t)$ and its length:

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}, \frac{2(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} \right\rangle \\
&= \left\langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \right\rangle = \frac{1}{\left(1+t^2\right)^2}\left\langle -4t, 2(1-t^2)\right\rangle \\
\|\mathbf{r}'(t)\| &= \frac{1}{(1+t^2)^2}\sqrt{16t^2 + 4(1-t^2)^2} = \frac{2}{(1+t^2)^2}\sqrt{t^4 + 2t^2 + 1} \\
&= \frac{2}{(1+t^2)^2}\sqrt{(t^2+1)^2} = \frac{2(t^2+1)}{\left(1+t^2\right)^2} = \frac{2}{1+t^2}
\end{aligned}$$

That is,

$$\|\mathbf{r}'(t)\| = \frac{2}{1+t^2}$$

We now use the Arc Length Formula to compute the length of the circle:

$$s = \int_{-\infty}^{\infty} \|\mathbf{r}'(t)\|\,dt = 2\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = 2\left(\lim_{R\to\infty} \tan^{-1} R - \lim_{R\to-\infty} \tan^{-1} R\right) = 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi$$

32. The involute of a circle (Figure 7), traced by a point at the end of a thread unwinding from a circular spool of radius R, has parametrization (see Exercise 26 in Section 12.2)

$$\mathbf{r}(\theta) = \left\langle R(\cos\theta + \theta\sin\theta),\ R(\sin\theta - \theta\cos\theta)\right\rangle$$

Find an arc length parametrization of the involute.

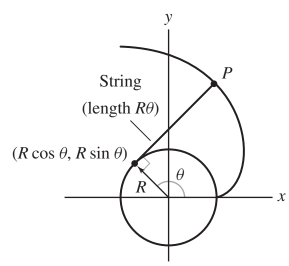
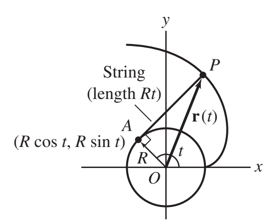


FIGURE 7 The involute of a circle.

SOLUTION We have:

$$\mathbf{r}(t) = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \tag{1}$$

We find the vectors in the right-hand side of the equation.



A is a point on the circle of radius R. Hence:

$$\overrightarrow{OA} = \langle R\cos t,\ R\sin t\rangle = R\,\langle\cos t, \sin t\rangle$$

The vector $\overrightarrow{AP}$ is orthogonal to $\overrightarrow{OA}$, hence $\overrightarrow{OA}\cdot\overrightarrow{AP} = 0$. It has length Rt and points in the opposite direction to the tangent vector, hence:

$$\overrightarrow{AP} = R\,\langle t\sin t, -t\cos t\rangle$$

Substituting in (1) gives:

$$\mathbf{r}(t) = R\,\langle\cos t + t\sin t, \sin t - t\cos t\rangle \tag{2}$$

To find the arc length function we first must compute the derivative vector $\mathbf{r}'(t)$ and its length. We get:

$$\mathbf{r}'(t) = R\,\langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t\rangle = Rt\,\langle\cos t, \sin t\rangle$$

$$\|\mathbf{r}'(t)\| = Rt\sqrt{\cos^2 t + \sin^2 t} = Rt$$

Using the Arc Length Formula we obtain:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t Ru\,du = \left.\frac{Ru^2}{2}\right|_0^t = \frac{Rt^2}{2}$$

That is:

$$s(t) = \frac{Rt^2}{2}$$

We now find the arc length parametrization. The inverse of the arc length function for $t \ge 0$ is $t = \sqrt{\frac{2s}{R}}$. Substituting in (2) we obtain the following arc length parametrization:

$$\mathbf{r}_1(s) = \mathbf{r}\left(\sqrt{\frac{2s}{R}}\right) = R\left\langle\cos\sqrt{\frac{2s}{R}} + \sqrt{\frac{2s}{R}}\sin\sqrt{\frac{2s}{R}}, \sin\sqrt{\frac{2s}{R}} - \sqrt{\frac{2s}{R}}\cos\sqrt{\frac{2s}{R}}\right\rangle$$

33. The curve $\mathbf{r}(t) = \langle t - \tanh t, \operatorname{sech} t\rangle$ is called a **tractrix** (see Exercise 92 in Section 11.1).

(a) Show that $s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du$ is equal to $s(t) = \ln(\cosh t)$.

(b) Show that $t = g(s) = \ln(e^s + \sqrt{e^{2s} - 1})$ is an inverse of $s(t)$ and verify that

$$\mathbf{r}_1(s) = \left\langle \tanh^{-1}\left(\sqrt{1 - e^{-2s}}\right) - \sqrt{1 - e^{-2s}}, e^{-s}\right\rangle$$

is an arc length parametrization of the tractrix.

SOLUTION

(a) We compute the derivative vector and its length:

$$\mathbf{r}'(t) = \left\langle 1 - \operatorname{sech}^2 t, -\operatorname{sech} t \tanh t\right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(1 - \operatorname{sech}^2 t) + \operatorname{sech}^2 t \tanh^2 t} = \sqrt{1 - 2\operatorname{sech}^2 t + \operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t}$$
$$= \sqrt{-\operatorname{sech}^2 t(2 - \tanh^2 t) + 1 + \operatorname{sech}^4 t}$$

We use the identity $1 - \tanh^2 t = \operatorname{sech}^2 t$ to write:

$$\|\mathbf{r}'(t)\| = \sqrt{-\operatorname{sech}^2 t(1 + \operatorname{sech}^2 t) + 1 + \operatorname{sech}^4 t} = \sqrt{-\operatorname{sech}^2 t - \operatorname{sech}^4 t + 1 + \operatorname{sech}^4 t}$$
$$= \sqrt{1 - \operatorname{sech}^2 t} = \sqrt{\tanh^2 t} = |\tanh t|$$

For $t \ge 0$, $\tanh t \ge 0$ hence, $\|\mathbf{r}'(t)\| = \tanh t$. We now apply the Arc Length Formula to obtain:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t (\tanh u)\, du = \ln(\cosh u)\Big|_0^t = \ln(\cosh t) - \ln(\cosh 0)$$
$$= \ln(\cosh t) - \ln 1 = \ln(\cosh t)$$

That is:

$$s(t) = \ln(\cosh t)$$

(b) We show that the function $t = g(s) = \ln\left(e^s + \sqrt{e^{2s} - 1}\right)$ is an inverse of $s(t)$. First we note that $s'(t) = \tanh t$, hence $s'(t) > 0$ for $t > 0$, which implies that $s(t)$ has an inverse function for $t \ge 0$. Therefore, it suffices to verify that $g(s(t)) = t$. We have:

$$g(s(t)) = \ln\left(e^{\ln(\cosh t)} + \sqrt{e^{2\ln(\cosh t)} - 1}\right) = \ln\left(\cosh t + \sqrt{\cosh^2 t - 1}\right)$$

Since $\cosh^2 t - 1 = \sinh^2 t$ we obtain (for $t \ge 0$):

$$g\left(s(t)\right) = \ln\left(\cosh t + \sqrt{\sinh^2 t}\right) = \ln\left(\cosh t + \sinh t\right) = \ln\left(\frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2}\right) = \ln\left(e^t\right) = t$$

We thus proved that $t = g(s)$ is an inverse of $s(t)$. Therefore, the arc length parametrization is obtained by substituting $t = g(s)$ in $\mathbf{r}(t) = \langle t - \tanh t, \operatorname{sech} t\rangle$. We compute t, $\tanh t$ and $\operatorname{sech} t$ in terms of s. We have:

$$s = \ln\left(\cosh t\right) \quad \Rightarrow \quad e^s = \cosh t \quad \Rightarrow \quad \operatorname{sech} t = e^{-s}$$

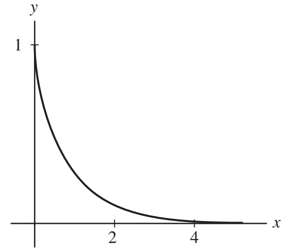
Also:

$$\tanh^2 t = 1 - \operatorname{sech}^2 t = 1 - e^{-2s} \quad \Rightarrow \quad \tanh t = \sqrt{1 - e^{-2s}} \quad \Rightarrow \quad t = \tanh^{-1}\sqrt{1 - e^{-2s}}$$

Substituting in $\mathbf{r}(t)$ gives:

$$\mathbf{r}_1(s) = \langle t - \tanh t, \operatorname{sech} t\rangle = \left\langle \tanh^{-1}\sqrt{1 - e^{-2s}} - \sqrt{1 - e^{-2s}}, e^{-s}\right\rangle$$

(c) The tractrix is shown in the following figure:



13.4 Curvature (LT Section 14.4)

Preliminary Questions

1. What is the unit tangent vector of a line with direction vector $\mathbf{v} = \langle 2, 1, -2 \rangle$?

SOLUTION A line with direction vector $\mathbf{v}$ has the parametrization:

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v}$$

hence, since $\overrightarrow{OP_0}$ and $\mathbf{v}$ are constant vectors, we have:

$$\mathbf{r}'(t) = \mathbf{v}$$

Therefore, since $\|\mathbf{v}\| = 3$, the unit tangent vector is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle 2/3, 1/3, -2/3 \rangle$$

2. What is the curvature of a circle of radius 4?

SOLUTION The curvature of a circle of radius R is $\frac{1}{R}$, hence the curvature of a circle of radius 4 is $\frac{1}{4}$.

3. Which has larger curvature, a circle of radius 2 or a circle of radius 4?

SOLUTION The curvature of a circle of radius 2 is $\frac{1}{2}$, and it is larger than the curvature of a circle of radius 4, which is $\frac{1}{4}$.

4. What is the curvature of $\mathbf{r}(t) = \langle 2 + 3t, 7t, 5 - t \rangle$?

SOLUTION $\mathbf{r}(t)$ parametrizes the line $\langle 2, 0, 5 \rangle + t \langle 3, 7, -1 \rangle$, and a line has zero curvature.

5. What is the curvature at a point where $\mathbf{T}'(s) = \langle 1, 2, 3 \rangle$ in an arc length parametrization $\mathbf{r}(s)$?

SOLUTION The curvature is given by the formula:

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

In an arc length parametrization, $\|\mathbf{r}'(t)\| = 1$ for all t, hence the curvature is $\kappa(t) = \|\mathbf{T}'(t)\|$. Using the given information we obtain the following curvature:

$$\kappa = \| \langle 1, 2, 3 \rangle \| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

6. What is the radius of curvature of a circle of radius 4?

SOLUTION The definition of the osculating circle implies that the osculating circles at the points of a circle, is the circle itself. Therefore, the radius of curvature is the radius of the circle, that is, 4.

7. What is the radius of curvature at P if $\kappa_P = 9$?

SOLUTION The radius of curvature is the reciprocal of the curvature, hence the radius of curvature at P is:

$$R = \frac{1}{\kappa_P} = \frac{1}{9}$$

Exercises

In Exercises 1–6, calculate $\mathbf{r}'(t)$ and $\mathbf{T}(t)$, and evaluate $\mathbf{T}(1)$.

1. $\mathbf{r}(t) = \left\langle 4t^2, 9t \right\rangle$

SOLUTION We differentiate $\mathbf{r}(t)$ to obtain:

$$\mathbf{r}'(t) = \langle 8t, 9 \rangle \quad \Rightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{(8t)^2 + 9^2} = \sqrt{64t^2 + 81}$$

We now find the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{64t^2 + 81}} \langle 8t, 9 \rangle$$

For $t = 1$ we obtain the vector:

$$\mathbf{T}(t) = \frac{1}{\sqrt{64 + 81}} \langle 8, 9 \rangle = \left\langle \frac{8}{\sqrt{145}}, \frac{9}{\sqrt{145}} \right\rangle.$$

2. $\mathbf{r}(t) = \langle e^t, t^2 \rangle$

SOLUTION We find $\mathbf{r}'(t)$ and its length:

$$\mathbf{r}'(t) = \langle e^t, 2t \rangle \quad \Rightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{\left(e^t\right)^2 + (2t)^2} = \sqrt{e^{2t} + 4t^2}$$

The unit tangent vector is, thus,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{e^{2t} + 4t^2}} \langle e^t, 2t \rangle$$

For $t = 1$ we get:

$$\mathbf{T}(1) = \left\langle \frac{e}{\sqrt{e^2 + 4}}, \frac{2}{\sqrt{e^2 + 4}} \right\rangle.$$

3. $\mathbf{r}(t) = \langle 3 + 4t, 3 - 5t, 9t \rangle$

SOLUTION We first find the vector $\mathbf{r}'(t)$ and its length:

$$\mathbf{r}'(t) = \langle 4, -5, 9 \rangle \quad \Rightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{4^2 + (-5)^2 + 9^2} = \sqrt{122}$$

The unit tangent vector is therefore:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{122}} \langle 4, -5, 9 \rangle = \left\langle \frac{4}{\sqrt{122}}, -\frac{5}{\sqrt{122}}, \frac{9}{\sqrt{122}} \right\rangle$$

We see that the unit tangent vector is constant, since the curve is a straight line.

4. $\mathbf{r}(t) = \langle 1 + 2t, t^2, 3 - t^2 \rangle$

SOLUTION We compute the derivative vector and its length:

$$\mathbf{r}'(t) = \langle 2, 2t, -2t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{2^2 + (2t)^2 + (-2t)^2} = \sqrt{4 + 8t^2}$$

The unit tangent vector is thus:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2\sqrt{1 + 2t^2}} \langle 2, 2t, -2t \rangle = \frac{1}{\sqrt{1 + 2t^2}} \langle 1, t, -t \rangle$$

For $t = 1$ we have:

$$\mathbf{T}(1) = \frac{1}{\sqrt{1 + 2}} \langle 1, 1, -1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle.$$

5. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$

SOLUTION We compute the derivative vector and its length:

$$\mathbf{r}'(t) = \langle -\pi \sin \pi t, \pi \cos \pi t, 1 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-\pi \sin \pi t)^2 + (\pi \cos \pi t)^2 + 1^2} = \sqrt{\pi^2(\sin^2 \pi t + \cos^2 \pi t) + 1} = \sqrt{\pi^2 + 1}$$

The unit tangent vector is thus:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{\pi^2 + 1}} \langle -\pi \sin \pi t, \pi \cos \pi t, 1 \rangle$$

For $t = 1$ we get:

$$\mathbf{T}(1) = \frac{1}{\sqrt{\pi^2 + 1}} \langle -\pi \sin \pi, \pi \cos \pi, 1 \rangle = \frac{1}{\sqrt{\pi^2 + 1}} \langle 0, -\pi, 1 \rangle = \left\langle 0, -\frac{\pi}{\sqrt{\pi^2 + 1}}, \frac{1}{\sqrt{\pi^2 + 1}} \right\rangle.$$

6. $\mathbf{r}(t) = \langle e^t, e^{-t}, t^2 \rangle$

SOLUTION We compute the tangent vector and its length:

$$\mathbf{r}'(t) = \langle e^t, -e^{-t}, 2t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(e^t)^2 + (-e^{-t})^2 + (2t)^2} = \sqrt{2e^{2t} + 4t^2}$$

The unit tangent vector is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2e^{2t} + 4t^2}} \left\langle e^t, -e^{-t}, 2t \right\rangle$$

For $t = 1$ we get:

$$\mathbf{T}(1) = \frac{1}{\sqrt{2e^2 + 4}} \left\langle e, -\frac{1}{e}, 2 \right\rangle$$

In Exercises 7–10, use Eq. (3) to calculate the curvature function $\kappa(t)$.

7. $\mathbf{r}(t) = \left\langle 1, e^t, t \right\rangle$

SOLUTION We compute the first and the second derivatives of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \left\langle 0, e^t, 1 \right\rangle, \quad \mathbf{r}''(t) = \left\langle 0, e^t, 0 \right\rangle.$$

Next, we find the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & e^t & 1 \\ 0 & e^t & 0 \end{vmatrix} = \begin{vmatrix} e^t & 1 \\ e^t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & e^t \\ 0 & e^t \end{vmatrix} \mathbf{k} = -e^t \mathbf{i} = \left\langle -e^t, 0, 0 \right\rangle$$

We need to find the lengths of the following vectors:

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left| \left\langle -e^t, 0, 0 \right\rangle \right| = e^t$$

$$\|\mathbf{r}'(t)\| = \sqrt{0^2 + (e^t)^2 + 1^2} = \sqrt{1 + e^{2t}}$$

We now use the formula for curvature to calculate $\kappa(t)$:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{e^t}{\left(\sqrt{1 + e^{2t}}\right)^3} = \frac{e^t}{\left(1 + e^{2t}\right)^{3/2}}$$

8. $\mathbf{r}(t) = \left\langle 4\cos t, t, 4\sin t \right\rangle$

SOLUTION By the formula for curvature we have:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

First we find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \langle -4\sin t, 1, 4\cos t \rangle$$

$$\mathbf{r}''(t) = \langle -4\cos t, 0, -4\sin t \rangle$$

We compute the cross product:

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin t & 1 & 4\cos t \\ -4\cos t & 0 & -4\sin t \end{vmatrix} \\ &= \begin{vmatrix} 1 & 4\cos t \\ 0 & -4\sin t \end{vmatrix} \mathbf{i} - \begin{vmatrix} -4\sin t & 4\cos t \\ -4\cos t & -4\sin t \end{vmatrix} \mathbf{j} + \begin{vmatrix} -4\sin t & 1 \\ -4\cos t & 0 \end{vmatrix} \mathbf{k} \\ &= -4\sin t\,\mathbf{i} - \left(16\sin^2 t + 16\cos^2 t\right)\mathbf{j} + 4\cos t\,\mathbf{k} \\ &= -4\sin t\,\mathbf{i} - 16\mathbf{j} + 4\cos t\,\mathbf{k} = 4\,\langle -\sin t, -4, \cos t \rangle \end{aligned}$$

We compute the lengths of the following vectors:

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 4\sqrt{(-\sin t)^2 + (-4)^2 + \cos^2 t} = 4\sqrt{\sin^2 t + 16 + \cos^2 t} = 4\sqrt{17}\|\mathbf{r}'(t)\|$$

$$= \sqrt{(-4\sin t)^2 + 1^2 + (4\cos t)^2} = \sqrt{16\sin^2 t + 1 + 16\cos^2 t} = \sqrt{17}$$

Substituting in (1) gives the following curvature:

$$\kappa(t) = \frac{4\sqrt{17}}{\left(\sqrt{17}\right)^3} = \frac{4\sqrt{17}}{17\sqrt{17}} = \frac{4}{17}$$

We see that this curve has constant curvature.

9. $\mathbf{r}(t) = \langle 4t+1, 4t-3, 2t\rangle$

SOLUTION By Formula (3) we have:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

We compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) = \langle 4, 4, 2\rangle, \quad \mathbf{r}''(t) = \langle 0, 0, 0\rangle$$

Thus $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 0, 0, 0\rangle$, $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 0$, and $\kappa(t) = 0$, as expected.

10. $\mathbf{r}(t) = \langle t^{-1}, 1, t\rangle$

SOLUTION By the formula for curvature we have:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

We now find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and their cross product. This gives:

$$\mathbf{r}'(t) = \langle -t^{-2}, 0, 1\rangle, \quad \mathbf{r}''(t) = \langle 2t^{-3}, 0, 0\rangle$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(-t^{-2}\mathbf{i} + \mathbf{k}\right) \times 2t^{-3}\mathbf{i} = 2t^{-3}\mathbf{k} \times \mathbf{i} = 2t^{-3}\mathbf{j}$$

We compute the lengths of the vector in (1):

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|2t^{-3}\mathbf{j}\| = 2|t^{-3}|$$
$$\|\mathbf{r}'(t)\| = \sqrt{\left((-t)^{-2}\right)^2 + 0^2 + 1^2} = \sqrt{t^{-4}+1}$$

Substituting in (1) we obtain the following curvature:

$$\kappa(t) = \frac{2|t|^{-3}}{\left(\sqrt{t^{-4}+1}\right)^3} = \frac{2|t|^{-3}}{\left(t^{-4}+1\right)^{3/2}}$$

We multiply through by $|t|^{4\cdot 3/2} = |t|^6$ to obtain:

$$\kappa(t) = \frac{2|t|^3}{\left(1+|t|^4\right)^{3/2}}$$

In Exercises 11–14, use Eq. (3) to evaluate the curvature at the given point.

11. $\mathbf{r}(t) = \langle 1/t, 1/t^2, t^2\rangle, \quad t = -1$

SOLUTION By the formula for curvature we know:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

We now find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and the cross product. These give:

$$\mathbf{r}'(t) = \left\langle -t^{-2}, -2t^{-3}, 2t\right\rangle, \quad \Rightarrow \mathbf{r}'(-1) = \langle -1, 2, -2\rangle$$
$$\mathbf{r}''(t) = \left\langle 2t^{-3}, 6t^{-4}, 2\right\rangle, \quad \Rightarrow \mathbf{r}''(-1) = \langle -2, 6, 2\rangle$$
$$\mathbf{r}'(-1) \times \mathbf{r}''(-1) = \langle 16, 6, -2\rangle$$

Now finding the norms, we get:

$$\|\mathbf{r}'(-1)\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$$
$$\|\mathbf{r}'(-1) \times \mathbf{r}''(-1)\| = \sqrt{16^2 + 6^2 + (-2)^2} = \sqrt{296} = 2\sqrt{74}$$

Therefore,

$$\kappa(-1) = \frac{\|\mathbf{r}'(-1) \times \mathbf{r}''(-1)\|}{\|\mathbf{r}'(-1)\|^3} = \frac{2\sqrt{74}}{3^3} = \frac{2\sqrt{74}}{27}$$

12. $\mathbf{r}(t) = \left\langle 3-t, e^{t-4}, 8t-t^2 \right\rangle, \quad t = 4$

SOLUTION By the formula for curvature we know:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

We now find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and the cross product. These give:

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle -1, e^{t-4}, 8-2t \right\rangle \quad \Rightarrow \mathbf{r}'(4) = \langle -1, 1, 0 \rangle \\ \mathbf{r}''(t) &= \left\langle 0, e^{t-4}, -2 \right\rangle \quad \Rightarrow \mathbf{r}''(4) = \langle 0, 1, -2 \rangle \\ \mathbf{r}'(4) \times \mathbf{r}''(4) &= \langle -2, -2, -1 \rangle \end{aligned}$$

Now finding norms we get:

$$\|\mathbf{r}'(4)\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\|\mathbf{r}'(4) \times \mathbf{r}''(4)\| = \sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$$

Therefore,

$$\kappa(4) = \frac{\|\mathbf{r}'(4) \times \mathbf{r}''(4)\|}{\|\mathbf{r}'(4)\|^3} = \frac{3}{(\sqrt{2})^3} = \frac{3}{2^{3/2}}$$

13. $\mathbf{r}(t) = \left\langle \cos t, \sin t, t^2 \right\rangle, \quad t = \frac{\pi}{2}$

SOLUTION By the formula for curvature we know:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

We now find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and the cross product. These give:

$$\begin{aligned} \mathbf{r}'(t) &= \langle -\sin t, \cos t, 2t \rangle \quad \Rightarrow \mathbf{r}'(\pi/2) = \langle -1, 0, \pi \rangle \\ \mathbf{r}''(t) &= \langle -\cos t, -\sin t, 2 \rangle \quad \Rightarrow \mathbf{r}''(\pi/2) = \langle 0, -1, 2 \rangle \\ \mathbf{r}'(\pi/2) \times \mathbf{r}''(\pi/2) &= \langle \pi, 2, 1 \rangle \end{aligned}$$

Now finding norms we get:

$$\|\mathbf{r}'(\pi/2)\| = \sqrt{(-1)^2 + 0^2 + \pi^2} = \sqrt{1+\pi^2}$$

$$\|\mathbf{r}'(\pi/2) \times \mathbf{r}''(\pi/2)\| = \sqrt{\pi^2 + (-1)^2 + 2^2} = \sqrt{\pi^2 + 5}$$

Therefore,

$$\kappa(\pi/2) = \frac{\|\mathbf{r}'(\pi/2) \times \mathbf{r}''(\pi/2)\|}{\|\mathbf{r}'(\pi/2)\|^3} = \frac{\sqrt{\pi^2+5}}{(\sqrt{1+\pi^2})^3} = \frac{\sqrt{\pi^2+5}}{(1+\pi^2)^{3/2}} \approx 0.108$$

14. $\mathbf{r}(t) = \left\langle \cosh t, \sinh t, t \right\rangle, \quad t = 0$

SOLUTION We compute the values needed to use the formula for curvature:

$$\mathbf{r}'(t) = \langle \sinh t, \cosh t, 1 \rangle, \quad \mathbf{r}''(t) = \langle \cosh t, \sinh t, 0 \rangle$$

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix} = \begin{vmatrix} \cosh t & 1 \\ \sinh t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \sinh t & 1 \\ \cosh t & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix} \mathbf{k} \\ &= -\sinh t\,\mathbf{i} + \cosh t\,\mathbf{j} + \left(\sinh^2 t - \cosh^2 t\right)\mathbf{k} = -\sinh t\,\mathbf{i} + \cosh t\,\mathbf{j} - \mathbf{k} \end{aligned}$$

We find the lengths of the following vectors:

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{(-\sinh t)^2 + (\cosh t)^2 + (-1)^2} = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$= \sqrt{\cosh^2 t - 1 + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1^2} = \sqrt{\sinh^2 t + 1 + \sinh^2 t + 1} = \sqrt{2 + 2\sinh^2 t} = \sqrt{2}\cosh t$$

Substituting in the formula for the curvature we get:

$$\kappa(t) = \frac{\sqrt{2}\cosh t}{\left(\sqrt{2}\cosh t\right)^3} = \frac{1}{2\cosh^2 t}$$

and

$$\kappa(0) = \frac{1}{2(1)^2} = \frac{1}{2}$$

In Exercises 15–18, find the curvature of the plane curve at the point indicated.

15. $y = e^t, \quad t = 3$

SOLUTION We use the curvature of a graph in the plane:

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + f'(t)^2\right)^{3/2}}$$

In our case $f(t) = e^t$, hence $f'(t) = f''(t) = e^t$ and we obtain:

$$\kappa(t) = \frac{e^t}{\left(1 + e^{2t}\right)^{3/2}} \quad \Rightarrow \quad \kappa(3) = \frac{e^3}{\left(1 + e^6\right)^{3/2}} \approx 0.0025$$

16. $y = \cos x, \quad x = 0$

SOLUTION We have $f(x) = \cos x$, hence $f'(x) = -\sin x$ and $f''(x) = -\cos x$. By the curvature of a graph in the plane we have:

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{|-\cos x|}{(1 + (-\sin x)^2)^{3/2}} = \frac{|\cos x|}{(1 + \sin^2 x)^{3/2}}$$

At $x = 0$ we obtain the following curvature:

$$\kappa(0) = \frac{\cos 0}{(1 + \sin^2 0)^{3/2}} = 1$$

17. $y = t^4, \quad t = 2$

SOLUTION By the curvature of a graph in the plane, we have:

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + f'(t)^2\right)^{3/2}}$$

In this case $f(t) = t^4$, $f'(t) = 4t^3$, $f''(t) = 12t^2$. Hence,

$$\kappa(t) = \frac{12t^2}{\left(1 + \left(4t^3\right)^2\right)^{3/2}} = \frac{12t^2}{\left(1 + 16t^6\right)^{3/2}}$$

At $t = 2$ we obtain the following curvature:

$$\kappa(2) = \frac{12 \cdot 2^2}{(1 + 16 \cdot 2^6)^{3/2}} = \frac{48}{(1025)^{3/2}} \approx 0.0015.$$

18. $y = t^n, \quad t = 1$

SOLUTION In this case $f(t) = t^n$, hence $f'(t) = nt^{n-1}$ and $f''(t) = n\,(n-1)\,t^{n-2}$. Using the curvature of a graph in the plane we get:

$$\kappa(t) = \frac{|f''(t)|}{\left(1 + f'(t)^2\right)^{3/2}} = \frac{|n(n-1)t^{n-2}|}{\left(1 + n^2 t^{2(n-1)}\right)^{3/2}} = \frac{n\,(n-1)\,|t|^{n-2}}{\left(1 + n^2 t^{2(n-1)}\right)^{3/2}}$$

At the point $t = 1$ we have the following curvature:

$$\kappa(1) = \frac{n(n-1)\cdot 1}{\left(1+n^2\cdot 1\right)^{3/2}} = \frac{n(n-1)}{\left(n^2+1\right)^{3/2}}$$

19. Find the curvature of $\mathbf{r}(t) = \langle 2\sin t, \cos 3t, t\rangle$ at $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$ (Figure 16).

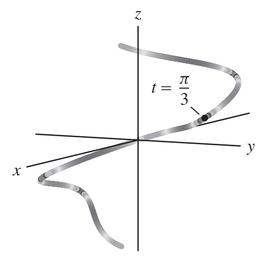


FIGURE 16 The curve $\mathbf{r}(t) = \langle 2\sin t, \cos 3t, t\rangle$.

SOLUTION By the formula for curvature we have:

$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

We compute the first and second derivatives:

$$\mathbf{r}'(t) = \langle 2\cos t, -3\sin 3t, 1\rangle, \quad \mathbf{r}''(t) = \langle -2\sin t, -9\cos 3t, 0\rangle$$

At the points $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$ we have:

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle 2\cos\frac{\pi}{3}, -3\sin\frac{3\pi}{3}, 1\right\rangle = \left\langle 2\cos\frac{\pi}{3}, -3\sin\pi, 1\right\rangle = \langle 1, 0, 1\rangle$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right) = \left\langle -2\sin\frac{\pi}{3}, -9\cos\frac{3\pi}{3}, 0\right\rangle = \left\langle -\sqrt{3}, 9, 0\right\rangle$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle 2\cos\frac{\pi}{2}, -3\sin\frac{3\pi}{2}, 1\right\rangle = \langle 0, 3, 1\rangle$$

$$\mathbf{r}''\left(\frac{\pi}{2}\right) = \left\langle -2\sin\frac{\pi}{2}, -9\cos\frac{3\pi}{2}, 0\right\rangle = \langle -2, 0, 0\rangle$$

We compute the cross products required to use (1):

$$\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -\sqrt{3} & 9 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 9 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ -\sqrt{3} & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ -\sqrt{3} & 9 \end{vmatrix}\mathbf{k} = -9\mathbf{i} - \sqrt{3}\mathbf{j} + 9\mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right)\times\mathbf{r}''\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 1 \\ -2 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix}\mathbf{k} = -2\mathbf{j} + 6\mathbf{k}$$

Hence,

$$\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\| = \sqrt{(-9)^2 + \left(-\sqrt{3}\right)^2 + 9^2} = \sqrt{165}$$

$$\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

At $t = \frac{\pi}{2}$ we have:

$$\left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\times\mathbf{r}''\left(\frac{\pi}{2}\right)\right\| = \sqrt{(-2)^2+6^2} = \sqrt{40} = 2\sqrt{10}$$

$$\left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\| = \sqrt{0^2+3^2+1^2} = \sqrt{10}$$

Substituting the values for $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$ in (1) we obtain the following curvatures:

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\sqrt{165}}{\left(\sqrt{2}\right)^3} = \frac{\sqrt{165}}{2\sqrt{2}} \approx 4.54$$

$$\kappa\left(\frac{\pi}{2}\right) = \frac{2\sqrt{10}}{\left(\sqrt{10}\right)^3} = \frac{2\sqrt{10}}{10\sqrt{10}} = 0.2$$

20. $\boxed{\text{GU}}$ Find the curvature function $\kappa(x)$ for $y = \sin x$. Use a computer algebra system to plot $\kappa(x)$ for $0 \le x \le 2\pi$. Prove that the curvature takes its maximum at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. *Hint:* As a shortcut to finding the max, observe that the maximum of the numerator and the minimum of the denominator of $\kappa(x)$ occur at the same points.

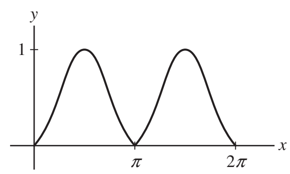
SOLUTION The curvature function is the following function:

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}} \tag{1}$$

In our case $f(x) = \sin x$, hence $f'(x) = \cos x$ and $f''(x) = -\sin x$. Substituting in (1) gives:

$$\kappa(x) = \frac{|\sin x|}{\left(1 + \cos^2 x\right)^{3/2}}$$

The graph of $\kappa(x)$ for $0 \le x \le \pi$ is shown in the following figure:



To find the points where $\kappa(x)$ takes its maximum on the interval $0 \le x \le 2\pi$, we notice that $\kappa(x)$ and $\kappa^2(x)$ take maximum values at the same points, hence we may maximize $\kappa^2(x)$. Moreover, since $\kappa^2(x) \ge 0$ and $\kappa^2(0) = \kappa^2(2\pi) = 0$, we may maximize the function in the open interval $0 < x < 2\pi$. We find the stationary points of

$$g(x) = \kappa^2(x) = \frac{\sin^2 x}{(1 + \cos^2 x)^3}$$

on $0 < x < 2\pi$:

$$g'(x) = \frac{2\sin x \cos x\left(1 + \cos^2 x\right)^3 - \sin^2 x \cdot 3\left(1 + \cos^2 x\right)^2 \cdot 2\cos x\,(-\sin x)}{\left(1 + \cos^2 x\right)^6} = 0$$

Using $2\sin x \cos x = \sin 2x$ we get:

$$\left(1 + \cos^2 x\right)^2 \sin 2x\left(1 + \cos^2 x + 3\sin^2 x\right) = 0$$

Since $1 + \cos^2 x \ne 0$ and $\sin^2 x = 1 - \cos^2 x$ we get:

$$\sin 2x\left(4 - 2\cos^2 x\right) = 0$$

We know that $0 \le \cos^2 x \le 1$, hence $4 - 2\cos^2 x \ne 0$. Therefore:

$$\sin 2x = 0 \quad \Rightarrow \quad 2x = \pi\kappa \quad \Rightarrow \quad x = \frac{\pi}{2}\kappa, \quad \kappa = 0, \pm 1, \ldots$$

The solutions in the interval $(0, 2\pi)$ are:

$$x_1 = \frac{\pi}{2}, \quad x_2 = \pi, \quad x_3 = \frac{3\pi}{2}.$$

Now, $x_2 = \pi$ is a minimum point since $g(\pi) = 0$. We compute $g\left(\frac{\pi}{2}\right)$ and $g\left(\frac{3\pi}{2}\right)$:

$$g\left(\frac{\pi}{2}\right) = \frac{\sin^2 \frac{\pi}{2}}{\left(1 + \cos^2 \frac{\pi}{2}\right)^{3/2}} = 1$$

$$g\left(\frac{3\pi}{2}\right) = \frac{\sin^2 \frac{3\pi}{2}}{\left(1 + \cos^2 \frac{3\pi}{2}\right)^{3/2}} = 1$$

Since $g(x) \le 1$ it follows that $x_1 = \frac{\pi}{2}$ and $x_2 = \frac{3\pi}{2}$ are the points where $g(x) = \kappa^2(x)$ (hence also the curvature $\kappa(x)$) takes its maximum value.

21. Show that the tractrix $\mathbf{r}(t) = \langle t - \tanh t, \operatorname{sech} t\rangle$ has the curvature function $\kappa(t) = \operatorname{sech} t$.

SOLUTION Writing $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, we have $x(t) = t - \tanh t$ and $y(t) = \operatorname{sech} t$. We compute the first and second derivatives of these functions. We use $\tanh^2 t = 1 - \operatorname{sech}^2 t$ to obtain:

$$\begin{aligned}
x'(t) &= 1 - \operatorname{sech}^2 t = \tanh^2 t \\
x''(t) &= -2 \operatorname{sech} t\,(-\operatorname{sech} t \tanh t) = 2\operatorname{sech}^2 t \tanh t \\
y'(t) &= -\operatorname{sech} t \tanh t \\
y''(t) &= -\left(-\operatorname{sech} t \tanh^2 t + \operatorname{sech}^3 t\right) = \operatorname{sech} t\left(\tanh^2 t - \operatorname{sech}^2 t\right) = \operatorname{sech} t\left(1 - 2\operatorname{sech}^2 t\right)
\end{aligned}$$

We compute the cross product $\|\mathbf{r}' \times \mathbf{r}''\|$:

$$\begin{aligned}
x'(t)y''(t) - x''(t)y'(t) &= \tanh^2 t \operatorname{sech} t(1 - 2\operatorname{sech}^2 t) + 2\operatorname{sech}^3 t \tanh^2 t \\
&= \tanh^2 t\left[\operatorname{sech} t - 2\operatorname{sech}^3 t + 2\operatorname{sech}^3 t\right] = \tanh^2 t \operatorname{sech} t
\end{aligned}$$

We compute the length of $\mathbf{r}'$:

$$x'(t)^2 + y'(t)^2 = \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t = \tanh^2 t(\tanh^2 t + \operatorname{sech}^2 t) = \tanh^2 t$$

Hence

$$\|\mathbf{r}'\|^3 = \left(\tanh^2 t\right)^{3/2} = \tanh^3 t$$

Substituting, we obtain

$$\kappa(t) = \frac{|\operatorname{sech} t \tanh^2 t|}{\tanh^3 t} = \frac{\operatorname{sech} t \tanh^2 t}{\tanh^3 t} = \frac{\operatorname{sech} t}{\tanh t}$$

22. Show that curvature at an inflection point of a plane curve $y = f(x)$ is zero.

SOLUTION The curvature of the graph $y = f(x)$ in the plane is the following function:

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}} \tag{1}$$

At an inflection point the second derivative changes its sign. Therefore, if f'' is continuous at the inflection point, it is zero at this point, hence by (1) the curvature at this point is zero.

23. Find the value of α such that the curvature of $y = e^{\alpha x}$ at $x = 0$ is as large as possible.

SOLUTION Using the curvature of a graph in the plane we have:

$$\kappa(x) = \frac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} \tag{1}$$

In our case $y'(x) = \alpha e^{\alpha x}$, $y''(x) = \alpha^2 e^{\alpha x}$. Substituting in (1) we obtain

$$\kappa(x) = \frac{\alpha^2 e^{\alpha x}}{\left(1 + \alpha^2 e^{2\alpha x}\right)^{3/2}}$$

The curvature at the origin is thus

$$\kappa(0) = \frac{\alpha^2 e^{\alpha \cdot 0}}{\left(1 + \alpha^2 e^{2\alpha \cdot 0}\right)^{3/2}} = \frac{\alpha^2}{\left(1 + \alpha^2\right)^{3/2}}$$

Since $\kappa(0)$ and $\kappa^2(0)$ have their maximum values at the same values of α, we may maximize the function:

$$g(\alpha) = \kappa^2(0) = \frac{\alpha^4}{(1 + \alpha^2)^3}$$

We find the stationary points:

$$g'(\alpha) = \frac{4\alpha^3(1 + \alpha^2)^3 - \alpha^4(3)(1 + \alpha^2)^2 2\alpha}{(1 + \alpha^2)^6} = \frac{2\alpha^3(1 + \alpha^2)^2(2 - \alpha^2)}{(1 + \alpha^2)^6} = 0$$

The stationary points are the solutions of the following equation:

$$2\alpha^3(1+\alpha^2)^2(2-\alpha^2)=0$$

$$\alpha^3 = 0 \quad \text{or} \quad 2-\alpha^2=0$$
$$\alpha = 0 \qquad\qquad \alpha = \pm\sqrt{2}$$

Since $g(\alpha) \geq 0$ and $g(0)=0$, $\alpha = 0$ is a minimum point. Also, $g'(\alpha)$ is positive immediately to the left of $\sqrt{2}$ and negative to the right. Hence, $\alpha = \sqrt{2}$ is a maximum point. Since $g(\alpha)$ is an even function, $\alpha = -\sqrt{2}$ is a maximum point as well. Conclusion: $\kappa(x)$ takes its maximum value at the origin when $\alpha = \pm\sqrt{2}$.

24. Find the point of maximum curvature on $y = e^x$.

SOLUTION We substitute $y'(x) = y''(x) = e^x$ in the curvature of a graph in the plane, to obtain the following curvature:

$$\kappa(x) = \frac{|y''(x)|}{\left(1+y'(x)^2\right)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$$

The functions $\kappa(x)$ and $g(x) = \kappa^2(x) = \frac{e^{2x}}{(1+e^{2x})^3}$ take maximum values at the same points, hence we may maximize the function $g(x)$. We find the stationary points of $g(x)$:

$$g'(x) = \frac{2e^{2x}(1+e^{2x})^3 - e^{2x}\cdot 3(1+e^{2x})^2\cdot 2e^{2x}}{(1+e^{2x})^6} = \frac{2(1+e^{2x})^2 e^{2x}(1-2e^{2x})}{(1+e^{2x})^6} = 0 \tag{1}$$

We obtain the following equation:

$$(1+e^{2x})^2 e^{2x}(1-2e^{2x}) = 0$$

Since $(1+e^{2x})^2 e^{2x} > 0$ for all x, we get:

$$1-2e^{2x} = 0 \quad\Rightarrow\quad e^{2x} = \frac{1}{2} \quad\Rightarrow\quad 2x = \ln\frac{1}{2} \quad\Rightarrow\quad x = \ln\sqrt{\frac{1}{2}}$$

If $x < \ln\sqrt{\frac{1}{2}}$, then since $1-2e^{2x}$ is decreasing we have:

$$1-2e^{2x} > 1-2e^{2\ln\sqrt{1/2}} = 1-2e^{\ln(1/2)} = 1-1 = 0$$

Likewise, if $x > \ln\sqrt{\frac{1}{2}}$ then:

$$1-2e^{2x} < 1-2e^{2\ln\sqrt{1/2}} = 1-2e^{\ln(1/2)} = 1-2\cdot\frac{1}{2} = 0$$

It follows by (1) that $g'(x) > 0$ left of $x = \ln\sqrt{\frac{1}{2}}$ and $g'(x) < 0$ right of this point. Therefore, $x = \ln\sqrt{\frac{1}{2}} \approx -0.347$ is the point where $g(x)$ (hence also $\kappa(x)$) takes its maximum value.

25. Show that the curvature function of the parametrization $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$ of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is

$$\kappa(t) = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{3/2}} \tag{9}$$

SOLUTION The curvature is the following function:

$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

We compute the derivatives and their cross product:

$$\mathbf{r}'(t) = \langle -a\sin t, b\cos t\rangle, \ \mathbf{r}''(t) = \langle -a\cos t, -b\sin t\rangle$$
$$\mathbf{r}'(t)\times\mathbf{r}''(t) = (-a\sin t\mathbf{i} + b\cos t\mathbf{j})\times(-a\cos t\mathbf{i} - b\sin t\mathbf{j})$$
$$= ab\sin^2 t\mathbf{k} + ab\cos^2 t\mathbf{k} = ab\left(\sin^2 t + \cos^2 t\right)\mathbf{k} = ab\mathbf{k}$$

Thus,

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \|ab\mathbf{k}\| = ab$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (b\cos t)^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

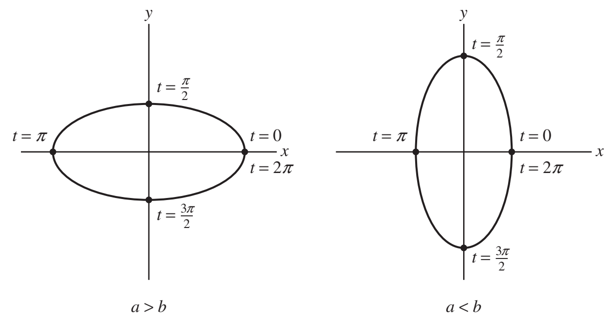
Substituting in (1) we obtain the following curvature:

$$\kappa(t) = \frac{ab}{\left(\sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)^3} = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}$$

26. Use a sketch to predict where the points of minimal and maximal curvature occur on an ellipse. Then use Eq. (9) to confirm or refute your prediction.

SOLUTION As suggested by the graphs, for $a > b$ the maximal curvature seems to occur at the points $t = 0, \pi$ and the minimal curvature at $t = \frac{\pi}{2}, \frac{3\pi}{2}$, and vice versa for $a < b$.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In Exercise 25 we showed that the curvature of the ellipse $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$ is:

$$\kappa(t) = \frac{ab}{\left(b^2\cos^2 t + a^2\sin^2 t\right)^{3/2}} \tag{1}$$

$\kappa(t)$ has minimal (maximal) value, where the denominator has maximal (minimal) value. Since the function $y = x^{3/2}$ is increasing, the minimal and maximal values of the denominator in (1) occur at the points where $g(t) = b^2\cos^2 t + a^2\sin^2 t$ has minimal and maximal values respectively. We find these points. We first find the critical points of $g(t)$:

$$g'(t) = b^2 \cdot 2\cos t(-\sin t) + a^2 \cdot 2\sin t\cos t = 2\sin t\cos t(a^2 - b^2) = (a^2 - b^2)\sin 2t$$

The critical points are the solutions of $g'(t) = 0$ in the interval $0 \le t \le 2\pi$. That is:

$$(a^2 - b^2)\sin 2t = 0$$

If $a = b$ then the ellipse is a circle, hence it has a constant curvature. For $a \ne b$ we have:

$$\sin 2t = 0 \quad \Rightarrow \quad 2t = \pi\kappa \quad \Rightarrow \quad t = \frac{\pi}{2}\kappa, \quad \kappa = 0, \pm 1, \ldots$$

The solutions in the interval $0 \le t \le 2\pi$ are

$$t_1 = 0, \quad t_2 = \frac{\pi}{2}, \quad t_3 = \pi, \quad t_4 = \frac{3\pi}{2}, \quad t_5 = 2\pi$$

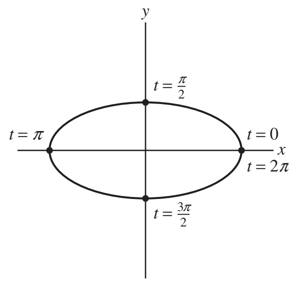
We compute the second derivative and substitute the stationary points:

$$\begin{aligned}
g''(t) &= 2(a^2 - b^2)\cdot\cos 2t \\
g''(0) &= 2(a^2 - b^2)\cdot\cos 0 = 2(a^2 - b^2) \\
g''\left(\frac{\pi}{2}\right) &= 2(a^2 - b^2)\cdot\cos\pi = 2(b^2 - a^2) \\
g''(\pi) &= 2(a^2 - b^2)\cdot\cos 2\pi = 2\left(a^2 - b^2\right) \\
g''\left(\frac{3\pi}{2}\right) &= 2(a^2 - b^2)\cdot\cos 3\pi = 2(b^2 - a^2) \\
g''(2\pi) &= 2(a^2 - b^2)\cdot\cos 4\pi = 2(a^2 - b^2)
\end{aligned}$$

We summarize the conclusions in the following table:

a > b

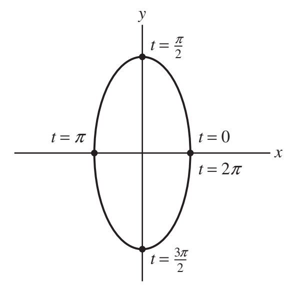
t	$g(t)$	$\kappa(t)$
0	min	max
$\frac{\pi}{2}$	max	min
π	min	max
$\frac{3\pi}{2}$	max	min
2π	min	max



$a > b$

a < b

t	$g(t)$	$\kappa(t)$
0	max	min
$\frac{\pi}{2}$	min	max
π	max	min
$\frac{3\pi}{2}$	min	max
2π	max	min



$a < b$

27. In the notation of Exercise 25, assume that $a \geq b$. Show that $b/a^2 \leq \kappa(t) \leq a/b^2$ for all t.

SOLUTION In Exercise 25 we showed that the curvature of the ellipse $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$ is the following function:

$$\kappa(t) = \frac{ab}{\left(b^2\cos^2 t + a^2\sin^2 t\right)^{3/2}}$$

Since $a \geq b > 0$ the quotient becomes greater if we replace a by b in the denominator, and it becomes smaller if we replace b by a in the denominator. We use the identity $\cos^2 t + \sin^2 t = 1$ to obtain:

$$\frac{ab}{\left(a^2\cos^2 t + a^2\sin^2 t\right)^{3/2}} \leq \kappa(t) \leq \frac{ab}{\left(b^2\cos^2 t + b^2\sin^2 t\right)^{3/2}}$$

$$\frac{ab}{\left(a^2\left(\cos^2 t + \sin^2 t\right)\right)^{3/2}} \leq \kappa(t) \leq \frac{ab}{\left(b^2\left(\cos^2 t + \sin^2 t\right)\right)^{3/2}}$$

$$\frac{ab}{a^3} = \frac{ab}{(a^2)^{3/2}} \leq \kappa(t) \leq \frac{ab}{(b^2)^{3/2}} = \frac{ab}{b^3}$$

$$\frac{b}{a^2} \leq \kappa(t) \leq \frac{a}{b^2}$$

28. Use Eq. (3) to prove that for a plane curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} \qquad \boxed{10}$$

SOLUTION By the formula for curvature we have

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

We compute the cross product of $\mathbf{r}'(t) = \left\langle x'(t), y'(t)\right\rangle$ and $\mathbf{r}''(t) = \left\langle x''(t), y''(t)\right\rangle$. Actually, since the cross product is only defined for three-dimensional vectors, we will think of these two vectors as follows: $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$ and $\mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$. Thus, the cross product is:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(x'(t)\mathbf{i} + y'(t)\mathbf{j}\right) \times \left(x''(t)\mathbf{i} + y''(t)\mathbf{j}\right) = x'(t)y''(t)\mathbf{i} \times \mathbf{j} + y'(t)x''(t)\mathbf{j} \times \mathbf{i}$$

$$= x'(t)y''(t)\mathbf{k} - y'(t)x''(t)\mathbf{k} = \left(x'(t)y''(t) - y'(t)x''(t)\right)\mathbf{k}$$

We compute the lengths of the following vectors:

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\left(x'(t)y''(t) - y'(t)x''(t)\right)\mathbf{k}\| = |x'(t)y''(t) - y'(t)x''(t)|$$

$$\|\mathbf{r}'(t)\| = \|\langle x'(t), y'(t)\rangle\| = \sqrt{x'(t)^2 + y'(t)^2}$$

Substituting in (1) we get:

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

In Exercises 29–32, use Eq. (10) to compute the curvature at the given point.

29. $\langle t^2, t^3\rangle, \quad t = 2$

SOLUTION For the given parametrization, $x(t) = t^2$, $y(t) = t^3$, hence

$$x'(t) = 2t$$
$$x''(t) = 2$$
$$y'(t) = 3t^2$$
$$y''(t) = 6t$$

At the point $t = 2$ we have

$$x'(2) = 4, \quad x''(2) = 2, \quad y'(2) = 3 \cdot 2^2 = 12, \quad y''(2) = 12$$

Substituting in Eq. (10) we get

$$\kappa(2) = \frac{|x'(2)y''(2) - x''(2)y'(2)|}{\left(x'(2)^2 + y'(2)^2\right)^{3/2}} = \frac{|4 \cdot 12 - 2 \cdot 12|}{\left(4^2 + 12^2\right)^{3/2}} = \frac{24}{160^{3/2}} \approx 0.012$$

30. $\langle \cosh s, s\rangle, \quad s = 0$

SOLUTION For this parametrization $x(s) = \cosh s$, $y(s) = s$, hence $x'(s) = \sinh s$, $x''(s) = \cosh s$, $y'(s) = 1$, $y''(s) = 0$. At the point $s = 0$ we have

$$x'(0) = \sinh 0 = 0, \quad x''(0) = \cosh 0 = 1, \quad y'(0) = 1, \quad y''(0) = 0$$

Substituting in Eq. (10) we obtain the following curvature:

$$\kappa(0) = \frac{|x'(0)y''(0) - y'(0)x''(0)|}{\left(x'(0)^2 + y'(0)^2\right)^{3/2}} = \frac{|0 \cdot 0 - 1 \cdot 1|}{\left(0^2 + 1^2\right)^{3/2}} = \frac{1}{1} = 1$$

31. $\langle t\cos t, \sin t\rangle, \quad t = \pi$

SOLUTION We have $x(t) = t\cos t$ and $y(t) = \sin t$, hence:

$$x'(t) = \cos t - t\sin t \quad \Rightarrow \quad x'(\pi) = \cos\pi - \pi\sin\pi = -1$$
$$x''(t) = -\sin t - (\sin t + t\cos t) = -2\sin t - t\cos t \quad \Rightarrow \quad x''(\pi) = -2\sin\pi - \pi\cos\pi = \pi$$
$$y'(t) = \cos t \quad \Rightarrow \quad y'(\pi) = \cos\pi = -1$$
$$y''(t) = -\sin t \quad \Rightarrow \quad y''(\pi) = -\sin\pi = 0$$

Substituting in Eq. (10) gives the following curvature:

$$\kappa(\pi) = \frac{|x'(\pi)y''(\pi) - x''(\pi)y'(\pi)|}{\left(x'(\pi)^2 + y'(\pi)^2\right)^{3/2}} = \frac{|-1 \cdot 0 - \pi \cdot (-1)|}{\left((-1)^2 + (-1)^2\right)^{3/2}} = \frac{\pi}{2\sqrt{2}} \approx 1.11$$

32. $\langle \sin 3s, 2\sin 4s\rangle, \quad s = \frac{\pi}{2}$

SOLUTION We have $x(s) = \sin 3s$, $y(s) = 2\sin 4s$. Hence

$$x'(s) = 3\cos 3s \quad \Rightarrow \quad x'\left(\frac{\pi}{2}\right) = 3\cos\frac{3\pi}{2} = 0$$

$$x''(s) = -9\sin 3s \quad \Rightarrow \quad x''\left(\frac{\pi}{2}\right) = -9\sin\frac{3\pi}{2} = 9$$

$$y'(s) = 8\cos 4s \quad \Rightarrow \quad y'\left(\frac{\pi}{2}\right) = 8\cos 2\pi = 8$$

$$y''(s) = -32\sin 4s \quad \Rightarrow \quad y''\left(\frac{\pi}{2}\right) = -32\sin 2\pi = 0$$

Substituting in Eq. (10) we get

$$\kappa\left(\frac{\pi}{2}\right) = \frac{\left|x'\left(\frac{\pi}{2}\right)y''\left(\frac{\pi}{2}\right) - x''\left(\frac{\pi}{2}\right)y'\left(\frac{\pi}{2}\right)\right|}{\left(x'\left(\frac{\pi}{2}\right)^2 + y'\left(\frac{\pi}{2}\right)^2\right)^{3/2}} = \frac{|0\cdot 0 - 9\cdot 8|}{\left(0^2+8^2\right)^{3/2}} = \frac{72}{8^3} = \frac{9}{64}$$

33. Let $s(t) = \int_{-\infty}^{t} \|\mathbf{r}'(u)\|\,du$ for the Bernoulli spiral $\mathbf{r}(t) = \langle e^t\cos 4t, e^t\sin 4t\rangle$ (see Exercise 29 in Section 13.3). Show that the radius of curvature is proportional to $s(t)$.

SOLUTION The radius of curvature is the reciprocal of the curvature:

$$R(t) = \frac{1}{\kappa(t)}$$

We compute the curvature using the equality given in Exercise 29 in Section 3:

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} \tag{1}$$

In our case, $x(t) = e^t\cos 4t$ and $y(t) = e^t\sin 4t$. Hence:

$$\begin{aligned}
x'(t) &= e^t\cos 4t - 4e^t\sin 4t = e^t(\cos 4t - 4\sin 4t)\\
x''(t) &= e^t(\cos 4t - 4\sin 4t) + e^t(-4\sin 4t - 16\cos 4t) = -e^t(15\cos 4t + 8\sin 4t)\\
y'(t) &= e^t\sin 4t + 4e^t\cos 4t = e^t(\sin 4t + 4\cos 4t)\\
y''(t) &= e^t(\sin 4t + 4\cos 4t) + e^t(4\cos 4t - 16\sin 4t) = e^t(8\cos 4t - 15\sin 4t)
\end{aligned}$$

We compute the numerator in (1):

$$\begin{aligned}
x'(t)y''(t) - x''(t)y'(t) &= e^{2t}(\cos 4t - 4\sin 4t)\cdot(8\cos 4t - 15\sin 4t)\\
&\quad + e^{2t}(15\cos 4t + 8\sin 4t)\cdot(\sin 4t + 4\cos 4t)\\
&= e^{2t}\left(68\cos^2 4t + 68\sin^2 4t\right) = 68e^{2t}
\end{aligned}$$

We compute the denominator in (1):

$$\begin{aligned}
x'(t)^2 + y'(t)^2 &= e^{2t}(\cos 4t - 4\sin 4t)^2 + e^{2t}(\sin 4t + 4\cos 4t)^2\\
&= e^{2t}\left(\cos^2 4t - 8\cos 4t\sin 4t + 16\sin^2 4t + \sin^2 4t + 8\sin 4t\cos 4t + 16\cos^2 4t\right)\\
&= e^{2t}\left(\cos^2 4t + \sin^2 4t + 16\left(\sin^2 4t + \cos^2 4t\right)\right)\\
&= e^{2t}(1 + 16\cdot 1) = 17e^{2t}
\end{aligned} \tag{2}$$

Hence

$$\left(x'(t)^2 + y'(t)^2\right)^{3/2} = 17^{3/2}e^{3t}$$

Substituting in (2) we have

$$\kappa(t) = \frac{68e^{2t}}{17^{3/2}e^{3t}} = \frac{4}{\sqrt{17}}e^{-t} \quad \Rightarrow \quad R = \frac{\sqrt{17}}{4}e^t \tag{3}$$

On the other hand, by the Fundamental Theorem and (2) we have

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{17e^{2t}} = \sqrt{17}e^t$$

We integrate to obtain

$$s(t) = \int \sqrt{17}\, e^t\, dt = \sqrt{17}\, e^t + C \tag{4}$$

Since $s(t) = \int_{-\infty}^{t} \|\mathbf{r}'(u)\|\, du$, we have $\lim_{t\to-\infty} s(t) = 0$, hence by (4):

$$0 = \lim_{t\to-\infty}\left(\sqrt{17}e^t + C\right) = 0 + C = C.$$

Substituting $C = 0$ in (4) we get:

$$s(t) = \sqrt{17}e^t \tag{5}$$

Combining (3) and (5) gives:

$$R(t) = \frac{1}{4}s(t)$$

which means that the radius of curvature is proportional to $s(t)$.

34. The **Cornu spiral** is the plane curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where

$$x(t) = \int_0^t \sin\frac{u^2}{2}\, du, \qquad y(t) = \int_0^t \cos\frac{u^2}{2}\, du$$

Verify that $\kappa(t) = |t|$. Since the curvature increases linearly, the Cornu spiral is used in highway design to create transitions between straight and curved road segments (Figure 17).

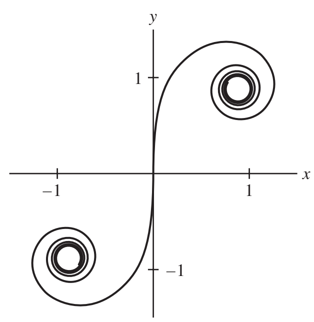


FIGURE 17 Cornu spiral.

SOLUTION We use the formula for the curvature given earlier:

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} \tag{1}$$

We compute the first and second derivatives of $x(t)$ and $y(t)$:

$$x'(t) = \frac{d}{dt}\left(\int_0^t \sin\frac{u^2}{2} du\right) = \sin\frac{t^2}{2}$$

$$x''(t) = \frac{d}{dt}\left(\sin\frac{t^2}{2}\right) = \frac{2t}{2}\cos\frac{t^2}{2} = t\cos\frac{t^2}{2}$$

$$y'(t) = \frac{d}{dt}\left(\int_0^t \cos\frac{u^2}{2} du\right) = \cos\frac{t^2}{2}$$

$$y''(t) = \frac{d}{dt}\left(\cos\frac{t^2}{2}\right) = \frac{2t}{2}\left(-\sin\frac{t^2}{2}\right) = -t\sin\frac{t^2}{2}$$

We compute the numerator in (1):

$$x'(t)y''(t) - x''(t)y'(t) = \sin\frac{t^2}{2}\left(-t\sin\frac{t^2}{2}\right) - t\cos\frac{t^2}{2}\cos\frac{t^2}{2}$$

$$= -t\left(\sin^2\left(\frac{t^2}{2}\right) + \cos^2\left(\frac{t^2}{2}\right)\right) = -t\cdot 1 = -t \tag{2}$$

We compute the denominator in (1):

$$x'(t)^2 + y'(t)^2 = \left(\sin \frac{t^2}{2}\right)^2 + \left(\cos \frac{t^2}{2}\right)^2 = 1$$

Hence:

$$\left(x'(t)^2 + y'(t)^2\right)^{3/2} = (1)^{3/2} = 1 \tag{3}$$

Substituting (2) and (3) in (1) gives the following curvature:

$$\kappa(t) = \frac{|-t|}{1} = |t|.$$

35. CAS Plot and compute the curvature $\kappa(t)$ of the **clothoid** $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where

$$x(t) = \int_0^t \sin \frac{u^3}{3}\, du, \qquad y(t) = \int_0^t \cos \frac{u^3}{3}\, du$$

SOLUTION We use the following formula for the curvature (given earlier):

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} \tag{1}$$

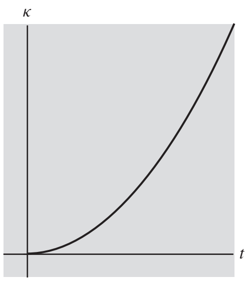
We compute the first and second derivatives of $x(t)$ and $y(t)$. Using the Fundamental Theorem and the Chain Rule we get:

$$\begin{aligned}
x'(t) &= \sin \frac{t^3}{3} \\
x''(t) &= \frac{3t^2}{3} \cos \frac{t^3}{3} = t^2 \cos \frac{t^3}{3} \\
y'(t) &= \cos \frac{t^3}{3} \\
y''(t) &= \frac{3t^2}{3}\left(-\sin \frac{t^3}{3}\right) = -t^2 \sin \frac{t^3}{3}
\end{aligned}$$

Substituting in (1) gives the following curvature function:

$$\kappa(t) = \frac{\left|\sin \frac{t^3}{3}\left(-t^2 \sin \frac{t^3}{3}\right) - t^2 \cos \frac{t^3}{3} \cos \frac{t^3}{3}\right|}{\left(\left(\sin \frac{t^3}{3}\right)^2 + \left(\cos \frac{t^3}{3}\right)^2\right)^{3/2}} = \frac{t^2\left(\sin^2 \frac{t^3}{3} + \cos^2 \frac{t^3}{3}\right)}{1^{3/2}} = t^2$$

That is, $\kappa(t) = t^2$. Here is a plot of the curvature as a function of t:



$\kappa(t) = t^2$

36. Find the unit normal vector $\mathbf{N}(\theta)$ to $\mathbf{r}(\theta) = R\langle \cos\theta, \sin\theta\rangle$, the circle of radius R. Does $\mathbf{N}(\theta)$ point inside or outside the circle? Draw $\mathbf{N}(\theta)$ at $\theta = \frac{\pi}{4}$ with $R = 4$.

SOLUTION We first find the unit tangent vector:

$$\mathbf{T}(\theta) = \frac{\mathbf{r}'(\theta)}{\|\mathbf{r}'(\theta)\|}$$

We have:

$$\mathbf{r}'(\theta) = R\langle -\sin\theta, \cos\theta\rangle \quad \Rightarrow \quad \|\mathbf{r}'(\theta)\| = R\|\langle -\sin\theta, \cos\theta\rangle\| = R\sqrt{\sin^2\theta + \cos^2\theta} = R$$

Hence:

$$\mathbf{T}(\theta) = \frac{R\langle -\sin\theta, \cos\theta\rangle}{R} = \langle -\sin\theta, \cos\theta\rangle$$

The unit normal vector is the following vector:

$$\mathbf{N}(\theta) = \frac{\mathbf{T}'(\theta)}{\|\mathbf{T}'(\theta)\|} \tag{1}$$

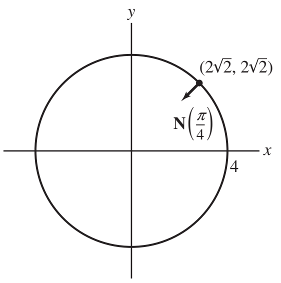
We compute $\mathbf{T}'(\theta)$ and its length:

$$\mathbf{T}'(\theta) = \langle -\cos\theta, -\sin\theta\rangle \quad \Rightarrow \quad \|\mathbf{T}'(\theta)\| = \sqrt{(-\cos\theta)^2 + (-\sin\theta)^2} = \sqrt{1} = 1$$

Substituting in (1) we get:

$$\mathbf{N}(\theta) = \langle -\cos\theta, -\sin\theta\rangle = -\langle\cos\theta, \sin\theta\rangle$$

The unit normal vector points to the "inside" of the curve, in this case it points inside the circle. For $\theta = \frac{\pi}{4}$, $\mathbf{N}\left(\frac{\pi}{4}\right) = -\left\langle\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right\rangle = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$. We plot the vector for $R = 4$:



37. Find the unit normal vector $\mathbf{N}(t)$ to $\mathbf{r}(t) = \langle 4, \sin 2t, \cos 2t\rangle$.

SOLUTION We first find the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{1}$$

We have

$$\mathbf{r}'(t) = \frac{d}{dt}\langle 4, \sin 2t, \cos 2t\rangle = \langle 0, 2\cos 2t, -2\sin 2t\rangle = 2\langle 0, \cos 2t, -\sin 2t\rangle$$

$$\|\mathbf{r}'(t)\| = 2\sqrt{0^2 + \cos^2 2t + (-\sin 2t)^2} = 2\sqrt{0+1} = 2$$

Substituting in (1) gives:

$$\mathbf{T}(t) = \frac{2\langle 0, \cos 2t, -\sin 2t\rangle}{2} = \langle 0, \cos 2t, -\sin 2t\rangle$$

The normal vector is the following vector:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{2}$$

We compute the derivative of the unit tangent vector and its length:

$$\mathbf{T}'(t) = \frac{d}{dt}\langle 0, \cos 2t, -\sin 2t\rangle = \langle 0, -2\sin 2t, -2\cos 2t\rangle = -2\langle 0, \sin 2t, \cos 2t\rangle$$

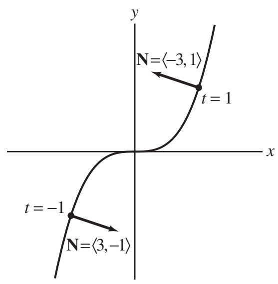
$$\|\mathbf{T}'(t)\| = 2\sqrt{0^2 + \sin^2 2t + \cos^2 2t} = 2\sqrt{0+1} = 2$$

Substituting in (2) we obtain:

$$\mathbf{N}(t) = \frac{-2\langle 0, \sin 2t, \cos 2t\rangle}{2} = \langle 0, -\sin 2t, -\cos 2t\rangle$$

38. Sketch the graph of $\mathbf{r}(t) = \langle t, t^3 \rangle$. Since $\mathbf{r}'(t) = \langle 1, 3t^2 \rangle$, the unit normal $\mathbf{N}(t)$ points in one of the two directions $\pm\langle -3t^2, 1 \rangle$. Which sign is correct at $t = 1$? Which is correct at $t = -1$?

SOLUTION The graph $\mathbf{r}(t) = \langle t, t^3 \rangle$ is shown in the following figure:



$\mathbf{r}(t) = \langle t, t^3 \rangle$

Since the unit normal vector points to the "inside" of the curve, the unit normal vector at $t = 1$ is along the direction $\langle -3, 1 \rangle$ rather than $\langle 3, -1 \rangle$. At $t = -1$ the unit normal vector is along the direction $\langle 3, -1 \rangle$ rather than $\langle -3, 1 \rangle$.

39. Find the normal vectors to $\mathbf{r}(t) = \langle t, \cos t \rangle$ at $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$.

SOLUTION The normal vector to $\mathbf{r}(t) = \langle t, \cos t \rangle$ is $\mathbf{T}'(t)$, where $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ is the unit tangent vector. We have

$$\mathbf{r}'(t) = \langle 1, -\sin t \rangle \quad \Rightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{1^2 + (\sin t)^2} = \sqrt{1 + \sin^2 t}$$

Hence,

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + \sin^2 t}} \langle 1, -\sin t \rangle$$

We compute the derivative of $\mathbf{T}(t)$ to find the normal vector.We use the Product Rule and the Chain Rule to obtain:

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{1}{\sqrt{1 + \sin^2 t}} \frac{d}{dt} \langle 1, -\sin t \rangle + \left(\frac{1}{\sqrt{1 + \sin^2 t}} \right)' \langle 1, -\sin t \rangle \\
&= \frac{1}{\sqrt{1 + \sin^2 t}} \langle 0, -\cos t \rangle - \frac{1}{1 + \sin^2 t} \cdot \frac{2 \sin t \cos t}{2\sqrt{1 + \sin^2 t}} \langle 1, -\sin t \rangle \\
&= \frac{1}{\sqrt{1 + \sin^2 t}} \langle 0, -\cos t \rangle - \frac{\sin 2t}{2\left(1 + \sin^2 t\right)^{3/2}} \langle 1, -\sin t \rangle
\end{aligned}$$

At $t = \frac{\pi}{4}$ we obtain the normal vector:

$$\mathbf{T}'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{1 + \frac{1}{2}}} \left\langle 0, -\frac{1}{\sqrt{2}} \right\rangle - \frac{1}{2\left(1 + \frac{1}{2}\right)^{3/2}} \left\langle 1, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, -\frac{1}{\sqrt{3}} \right\rangle - \left\langle \frac{\sqrt{2}}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}} \right\rangle = \left\langle \frac{-\sqrt{2}}{3\sqrt{3}}, \frac{-2}{3\sqrt{3}} \right\rangle$$

At $t = \frac{3\pi}{4}$ we obtain:

$$\mathbf{T}'\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{1 + \frac{1}{2}}} \left\langle 0, \frac{1}{\sqrt{2}} \right\rangle - \frac{-1}{2\left(1 + \frac{1}{2}\right)^{3/2}} \left\langle 1, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, \frac{1}{\sqrt{3}} \right\rangle + \left\langle \frac{\sqrt{2}}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}} \right\rangle = \left\langle \frac{\sqrt{2}}{3\sqrt{3}}, \frac{2}{3\sqrt{3}} \right\rangle$$

40. Find the unit normal to the Cornu spiral (Exercise 34) at $t = \sqrt{\pi}$.

SOLUTION The Cornu Spiral is the plane curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with

$$x(t) = \int_0^t \sin \frac{u^2}{2} du, \quad y(t) = \int_0^t \cos \frac{u^2}{2} du$$

The unit normal is the following vector:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{1}$$

We find the vector $\mathbf{T}'(t)$ and its length. By the Fundamental Theorem we have

$$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \left\langle \sin\frac{t^2}{2}, \cos\frac{t^2}{2}\right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2\frac{t^2}{2} + \cos^2\frac{t^2}{2}} = \sqrt{1} = 1$$

Hence,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \sin\frac{t^2}{2}, \cos\frac{t^2}{2}\right\rangle$$

Differentiating $\mathbf{T}(t)$, using the Chain Rule, gives

$$\mathbf{T}'(t) = \left\langle \frac{2t}{2}\cos\frac{t^2}{2}, \frac{2t}{2}\left(-\sin\frac{t^2}{2}\right)\right\rangle = t\left\langle \cos\frac{t^2}{2}, -\sin\frac{t^2}{2}\right\rangle$$

$$\|\mathbf{T}'(t)\| = |t|\sqrt{\cos^2\frac{t^2}{2} + \left(-\sin\frac{t^2}{2}\right)^2} = |t|\sqrt{1} = |t|$$

Substituting in (1) gives the following unit normal:

$$\mathbf{N}(t) = \frac{t}{|t|}\left\langle \cos\frac{t^2}{2}, -\sin\frac{t^2}{2}\right\rangle \quad\Rightarrow\quad \mathbf{N}\left(\sqrt{\pi}\right) = \left\langle \cos\frac{\pi}{2}, -\sin\frac{\pi}{2}\right\rangle = \langle 0, -1\rangle$$

41. Find the unit normal to the clothoid (Exercise 35) at $t = \pi^{1/3}$.

SOLUTION The Clothoid is the plane curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with

$$x(t) = \int_0^t \sin\frac{u^3}{3}\,du, \quad y(t) = \int_0^t \cos\frac{u^3}{3}\,du$$

The unit normal is the following vector:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{1}$$

We first find the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. By the Fundamental Theorem we have

$$\mathbf{r}'(t) = \left\langle \sin\frac{t^3}{3}, \cos\frac{t^3}{3}\right\rangle \quad\Rightarrow\quad \|\mathbf{r}'(t)\| = \sqrt{\sin^2\frac{t^3}{3} + \cos^2\frac{t^3}{3}} = \sqrt{1} = 1$$

Hence,

$$\mathbf{T}(t) = \left\langle \sin\frac{t^3}{3}, \cos\frac{t^3}{3}\right\rangle$$

We now differentiate $\mathbf{T}(t)$ using the Chain Rule to obtain:

$$\mathbf{T}'(t) = \left\langle \frac{3t^2}{3}\cos\frac{t^3}{3}, \frac{-3t^2}{3}\sin\frac{t^3}{3}\right\rangle = t^2\left\langle \cos\frac{t^3}{3}, -\sin\frac{t^3}{3}\right\rangle$$

Hence,

$$\|\mathbf{T}'(t)\| = t^2\sqrt{\cos^2\frac{t^3}{3} + \left(-\sin\frac{t^3}{3}\right)^2} = t^2$$

Substituting in (1) we obtain the following unit normal:

$$\mathbf{N}(t) = \left\langle \cos\frac{t^3}{3}, -\sin\frac{t^3}{3}\right\rangle$$

At the point $T = \pi^{1/3}$ the unit normal is

$$\mathbf{N}(\pi^{1/3}) = \left\langle \cos\frac{(\pi^{1/3})^3}{3}, -\sin\frac{(\pi^{1/3})^3}{3}\right\rangle = \left\langle \cos\frac{\pi}{3}, -\sin\frac{\pi}{3}\right\rangle = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$$

42. Method for Computing N Let $v(t) = \|\mathbf{r}'(t)\|$. Show that

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|} \qquad \boxed{11}$$

Hint: $\mathbf{N}$ is the unit vector in the direction $\mathbf{T}'(t)$. Differentiate $\mathbf{T}(t) = \mathbf{r}'(t)/v(t)$ to show that $v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)$ is a positive multiple of $\mathbf{T}'(t)$.

SOLUTION Since $v(t) = \|\mathbf{r}'(t)\|$ and $\mathbf{T}(t)$ is a unit vector in the direction at $\mathbf{r}'(t)$, we may write:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{v(t)}$$

Differentiating this vector, using the Quotient Rule, we get

$$\mathbf{T}'(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{(v(t))^2}$$

thus,

$$(v(t))^2\mathbf{T}'(t) = v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)$$

Now, $\mathbf{N}(t)$ is a unit vector in the direction of $\mathbf{T}'(t)$, hence we can find it by dividing the vector $(v(t))^2\mathbf{T}'(t)$ by its length. Therefore,

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|}$$

In Exercises 43–48, use Eq. (11) to find $\mathbf{N}$ *at the point indicated.*

43. $\langle t^2, t^3 \rangle$, $\quad t = 1$

SOLUTION We use the equality

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|}$$

For $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ we have

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 2t, 3t^2 \rangle \\
\mathbf{r}''(t) &= \langle 2, 6t \rangle \\
v(t) &= \|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} \\
v'(t) &= \frac{8t + 36t^3}{2\sqrt{4t^2 + 9t^4}} = \frac{4t + 18t^3}{\sqrt{4t^2 + 9t^4}}
\end{aligned}$$

At the point $t = 1$ we get

$$\mathbf{r}''(1) = \langle 2, 6 \rangle, \quad v'(1) = \frac{4 + 18}{\sqrt{4 + 9}} = \frac{22}{\sqrt{13}},$$

and also

$$\mathbf{r}'(1) = \langle 2, 3 \rangle, \quad v(1) = \sqrt{4 + 9} = \sqrt{13}$$

Hence,

$$\begin{aligned}
v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1) &= \sqrt{13}\,\langle 2, 6 \rangle - \frac{22}{\sqrt{13}} \cdot \langle 2, 3 \rangle = \left\langle \frac{26 - 44}{\sqrt{13}}, \frac{78 - 66}{\sqrt{13}} \right\rangle = \frac{1}{\sqrt{13}}\langle -18, 12 \rangle \\
\left\|v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1)\right\| &= \left\| \frac{1}{\sqrt{13}}\langle -18, 12 \rangle \right\| = \frac{1}{\sqrt{13}}\sqrt{(-18)^2 + 12^2} = \sqrt{\frac{468}{13}} = 6
\end{aligned}$$

Substituting in (1) gives the following unit normal:

$$\mathbf{N}(1) = \frac{\frac{1}{\sqrt{13}}\langle -18, 12 \rangle}{6} = \frac{1}{\sqrt{13}}\langle -3, 2 \rangle$$

44. $\left\langle t-\sin t, 1-\cos t\right\rangle, \quad t=\pi$

SOLUTION We use the following equality:

$$\mathbf{N}(t)=\frac{v(t)\mathbf{r}''(t)-v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t)-v'(t)\mathbf{r}'(t)\|}$$

We compute the vectors in the above equality. For $\mathbf{r}(t)=\langle t-\sin t, 1-\cos t\rangle$ we have

$$\begin{aligned}
\mathbf{r}'(t)&=\langle 1-\cos t, \sin t\rangle\\
\mathbf{r}''(t)&=\langle \sin t, \cos t\rangle\\
v(t)=\|\mathbf{r}'(t)\|&=\sqrt{(1-\cos t)^2+\sin^2 t}=\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\\
&=\sqrt{1-2\cos t+1}=\sqrt{2(1-\cos t)}=\sqrt{2\cdot 2\sin^2\frac{t}{2}}=2\left|\sin\frac{t}{2}\right|
\end{aligned}$$

For $0\le t\le 2\pi$, $\sin\frac{t}{2}\ge 0$, hence $v(t)=2\sin\frac{t}{2}$. Therefore,

$$v'(t)=2\cdot\frac{1}{2}\cos\frac{t}{2}=\cos\frac{t}{2}, \quad 0\le t\le 2\pi$$

At the point $t=\pi$ we have

$$\begin{aligned}
\mathbf{r}''(\pi)&=\langle \sin\pi, \cos\pi\rangle=\langle 0,-1\rangle\\
v'(\pi)&=\cos\frac{\pi}{2}=0\\
\mathbf{r}'(\pi)&=\langle 1-\cos\pi, \sin\pi\rangle=\langle 2,0\rangle\\
v(\pi)&=2\left|\sin\frac{\pi}{2}\right|=2
\end{aligned}$$

We now substitute these values in (1) to obtain the following unit normal:

$$\mathbf{N}(\pi)=\frac{v(\pi)\mathbf{r}''(\pi)-v'(\pi)\mathbf{r}'(\pi)}{\|v(\pi)\mathbf{r}''(\pi)-v'(\pi)\mathbf{r}'(\pi)\|}=\frac{2\langle 0,-1\rangle-0\langle 2,0\rangle}{\|v(\pi)\mathbf{r}''(\pi)-v'(\pi)\mathbf{r}'(\pi)\|}=\frac{\langle 0,-2\rangle}{\sqrt{0^2+(-2)^2}}=\langle 0,-1\rangle$$

45. $\left\langle t^2/2, t^3/3, t\right\rangle, \quad t=1$

SOLUTION We use the following equality:

$$\mathbf{N}(t)=\frac{v(t)\mathbf{r}''(t)-v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t)-v'(t)\mathbf{r}'(t)\|}$$

We compute the vectors in the equality above. For $\mathbf{r}(t)=\left\langle t^2/2, t^3/3, t\right\rangle$ we get:

$$\begin{aligned}
\mathbf{r}'(t)&=\left\langle t, t^2, 1\right\rangle\\
\mathbf{r}''(t)&=\langle 1, 2t, 0\rangle\\
v(t)=\|\mathbf{r}'(t)\|&=\sqrt{t^2+t^4+1}\\
v'(t)&=\frac{1}{2}(t^2+t^4+1)^{-1/2}(4t^3+2t)=\frac{4t^3+2t}{2\sqrt{t^2+t^4+1}}
\end{aligned}$$

At the point $t=1$ we get:

$$\begin{aligned}
\mathbf{r}'(1)&=\langle 1,1,1\rangle\\
\mathbf{r}''(1)&=\langle 1,2,0\rangle\\
v'(1)&=\frac{6}{2\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}\\
v(1)&=\sqrt{3}
\end{aligned}$$

Hence,

$$v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1) = \sqrt{3}\,\langle 1, 2, 0\rangle - \sqrt{3}\,\langle 1, 1, 1\rangle = \left\langle 0, \sqrt{3}, -\sqrt{3}\right\rangle$$

$$\|v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1)\| = \sqrt{0^2 + (\sqrt{3})^2 + (-\sqrt{3})^2} = \sqrt{6}$$

We now substitute these values in (1) to obtain the following unit normal:

$$\mathbf{N}(1) = \frac{v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1)}{\|v(1)\mathbf{r}''(1) - v'(1)\mathbf{r}'(1)\|} = \frac{\left\langle 0, \sqrt{3}, -\sqrt{3}\right\rangle}{\sqrt{6}} = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$$

46. $\left\langle t^{-1}, t, t^2\right\rangle, \quad t = -1$

SOLUTION We use the equality

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|}$$

We compute the vectors in the above equality. For $\mathbf{r}(t) = \left\langle t^{-1}, t, t^2\right\rangle$ we have

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle -t^{-2}, 1, 2t\right\rangle \\
\mathbf{r}''(t) &= \left\langle 2t^{-3}, 0, 2\right\rangle \\
v(t) &= \|\mathbf{r}'(t)\| = \sqrt{t^{-4} + 1 + 4t^2} \\
v'(t) &= \frac{-4t^{-5} + 8t}{2\sqrt{t^{-4} + 1 + 4t^2}} = \frac{-2t^{-5} + 4t}{\sqrt{t^{-4} + 1 + 4t^2}}
\end{aligned}$$

At the point $t = -1$ we get

$$\mathbf{r}'(-1) = \langle -1, 1, -2\rangle, \quad \mathbf{r}''(-1) = \langle -2, 0, 2\rangle, \quad v'(-1) = \frac{2-4}{\sqrt{1+1+4}} = \frac{-2}{\sqrt{6}},$$

$$v(-1) = \sqrt{(-1)^{-4} + 1 + 4(-1)^2} = \sqrt{6}$$

Hence,

$$\begin{aligned}
v(-1)\mathbf{r}''(-1) - v'(-1)\mathbf{r}'(-1) &= \sqrt{6}\,\langle -2, 0, 2\rangle + \frac{2}{\sqrt{6}}\,\langle -1, 1, -2\rangle \\
&= \left\langle \frac{-12-2}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{12-4}{\sqrt{6}}\right\rangle \\
&= \left\langle -\frac{14}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{8}{\sqrt{6}}\right\rangle \\
&= \sqrt{\frac{2}{3}}\,\langle -7, 1, 4\rangle \\
\|v(-1)\mathbf{r}''(-1) - v'(-1)\mathbf{r}'(-1)\| &= \sqrt{\frac{2}{3}}\sqrt{49 + 1 + 16} = \sqrt{44}
\end{aligned}$$

Substituting in (1) gives the following unit normal:

$$\mathbf{N}(-1) = \frac{\sqrt{\frac{2}{3}}\,\langle -7, 1, 4\rangle}{\sqrt{44}} = \frac{1}{\sqrt{66}}\,\langle -7, 1, 4\rangle$$

47. $\left\langle t, e^t, t\right\rangle, \quad t = 0$

SOLUTION We use the equality

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|}$$

For $\mathbf{r}(t) = \left\langle t, e^t, t\right\rangle$ we have

$$\mathbf{r}'(t) = \left\langle 1, e^t, 1\right\rangle$$

$$\mathbf{r}''(t) = \left\langle 0, e^t, 0 \right\rangle$$
$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{1^2 + (e^t)^2 + 1^2} = \sqrt{e^{2t} + 2}$$
$$v'(t) = \frac{2e^{2t}}{2\sqrt{e^{2t}+2}} = \frac{e^{2t}}{\sqrt{e^{2t}+2}}$$

At the point $t = 0$ we have

$$\mathbf{r}'(0 = \langle 1, 1, 1 \rangle, \quad \mathbf{r}''(0) = \left\langle 0, 1, 0 \right\rangle, \quad v(0) = \sqrt{3}, \quad v'(0) = \frac{1}{\sqrt{3}},$$

Hence,

$$v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0) = \sqrt{3}\,\langle 0, 1, 0\rangle - \frac{1}{\sqrt{3}}\,\langle 1, 1, 1\rangle$$
$$= \left\langle -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$
$$= \frac{1}{\sqrt{3}}\,\langle -1, 2, -1\rangle$$
$$\|v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0)\| = \frac{1}{\sqrt{3}}\sqrt{1+4+1} = \sqrt{2}$$

Substituting in (1) we obtain the following unit normal:

$$\mathbf{N}(0) = \frac{\frac{1}{\sqrt{3}}\,\langle -1, 2, -1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\,\langle -1, 2, -1\rangle$$

48. $\left\langle \cosh t, \sinh t, t^2 \right\rangle, \quad t = 0$

SOLUTION We use the following equality:

$$\mathbf{N}(t) = \frac{v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)}{\|v(t)\mathbf{r}''(t) - v'(t)\mathbf{r}'(t)\|}$$

For $\mathbf{r}(t) = \left\langle \cosh t, \sinh t, t^2 \right\rangle$ we have

$$\mathbf{r}'(t) = \langle \sinh t, \cosh t, 2t\rangle$$
$$\mathbf{r}''(t) = \langle \cosh t, \sinh t, 2\rangle$$
$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + (2t)^2}$$
$$= \sqrt{\left(\cosh^2 t - 1\right) + \cosh^2 t + 4t^2}$$
$$= \sqrt{2\cosh^2 t + 4t^2 - 1}$$
$$v'(t) = \frac{1}{2}(2\cosh^2 t + 4t^2 - 1)^{-1/2}(4\cosh t \sinh t + 8t) = \frac{4\cosh t \sinh t}{2\sqrt{2\cosh^2 t + 4t^2 - 1}}$$

At the point $t = 0$ we get

$$\mathbf{r}'(0) = \langle \sinh 0, \cosh 0, 0\rangle = \langle 0, 1, 0\rangle$$
$$\mathbf{r}''(0) = \langle \cosh 0, \sinh 0, 2\rangle = \langle 1, 0, 2\rangle$$
$$v(0) = \sqrt{2(1)^2 + 4(0)^2 - 1} = 1$$
$$v'(0) = \frac{0}{2(1)} = 0$$

Hence,

$$v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0) = 1\,\langle 1, 0, 2\rangle - \mathbf{0} = \langle 1, 0, 2\rangle$$
$$\|v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0)\| = \|\,\langle 1, 0, 2\rangle\,\| = \sqrt{5}$$

Substituting in (1) gives the following unit normal:

$$\mathbf{N}(0) = \frac{v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0)}{\|v(0)\mathbf{r}''(0) - v'(0)\mathbf{r}'(0)\|} = \frac{1}{\sqrt{5}}\langle 1, 0, 2\rangle = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle$$

49. Let $f(x) = x^2$. Show that the center of the osculating circle at (x_0, x_0^2) is given by $\left(-4x_0^3, \frac{1}{2} + 3x_0^2\right)$.

SOLUTION We parametrize the curve by $\mathbf{r}(x) = \langle x, x^2 \rangle$. The center Q of the osculating circle at $x = x_0$ has the position vector

$$\overrightarrow{OQ} = \mathbf{r}(x_0) + \kappa(x_0)^{-1}\mathbf{N}(x_0) \tag{1}$$

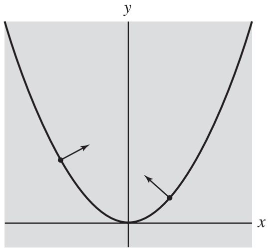
We first find the curvature, using the formula for the curvature of a graph in the plane. We have $f'(x) = 2x$ and $f''(x) = 2$, hence,

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}} \quad \Rightarrow \quad \kappa(x_0)^{-1} = \frac{1}{2}(1 + 4x_0^2)^{3/2}$$

To find the unit normal vector $\mathbf{N}(x_0)$ we use the following considerations:

- The tangent vector is $\mathbf{r}'(x_0) = \langle 1, 2x_0 \rangle$, hence the vector $\langle -2x_0, 1 \rangle$ is orthogonal to $\mathbf{r}'(x_0)$ (since their dot product is zero). Hence $\mathbf{N}(x_0)$ is one of the two unit vectors $\pm \frac{1}{\sqrt{1+4x_0^2}} \langle -2x_0, 1 \rangle$.
- The graph of $f(x) = x^2$ shows that the unit normal vector points in the positive y-direction, hence, the appropriate choice is:

$$\mathbf{N}(x_0) = \frac{1}{\sqrt{1 + 4x_0^2}}\langle -2x_0, 1 \rangle \tag{2}$$



$f(x) = x^2$

We now substitute (2), (3), and $\mathbf{r}(x_0) = \langle x_0, x_0^2 \rangle$ in (1) to obtain

$$\overrightarrow{OQ} = \langle x_0, x_0^2 \rangle + \frac{1}{2}\left(1 + 4x_0^2\right)^{3/2} \cdot \frac{1}{\sqrt{1 + 4x_0^2}}\langle -2x_0, 1 \rangle = \langle x_0, x_0^2 \rangle + \frac{1}{2}\left(1 + 4x_0^2\right)\langle -2x_0, 1 \rangle$$

$$= \langle x_0, x_0^2 \rangle + \left\langle -x_0 - 4x_0^3, \frac{1}{2}\left(1 + 4x_0^2\right) \right\rangle = \left\langle -4x_0^3, \frac{1}{2} + 3x_0^2 \right\rangle$$

The center of the osculating circle is the terminal point of $\overrightarrow{OQ}$, that is,

$$Q = \left(-4x_0^3, \frac{1}{2} + 3x_0^2\right)$$

50. Use Eq. (8) to find the center of curvature to $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ at $t = 1$.

SOLUTION The center Q of curvature has the position vector:

$$\overrightarrow{OQ} = \mathbf{r}(1) + \kappa(1)^{-1}\mathbf{N}(1) \tag{1}$$

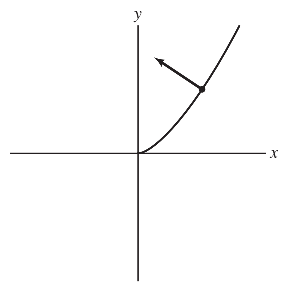
We find the curvature $\kappa(x)$, using the formula for the curvature of a graph in the plane. Since $x = t^2$ and $y = t^3$ we have $y = x^{3/2}$, hence $y'(x) = \frac{3}{2}x^{1/2}$ and $y''(x) = \frac{3}{4}x^{-1/2}$. Thus:

$$\kappa(x) = \frac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} = \frac{\frac{3}{4}x^{-1/2}}{\left(1 + \frac{9}{4}x\right)^{3/2}}$$

The point $t = 1$ corresponds to $x = 1^2 = 1$, hence the curvature at this point is:

$$\kappa(1) = \frac{\frac{3}{4} \cdot 1^{-1/2}}{\left(1 + \frac{9}{4} \cdot 1\right)^{3/2}} = \frac{\frac{3}{4}}{\left(\frac{13}{4}\right)^{3/2}} = \frac{6}{13^{3/2}} \tag{2}$$

We now must find the unit vector $\mathbf{N}$. The tangent vector is $\mathbf{r}'(t) = \left\langle 2t, 3t^2 \right\rangle$, and we observe that the vector $\langle -3t, 2\rangle$ is orthogonal to $\mathbf{r}'(t)$ (since the dot product is zero). Therefore $\mathbf{N}(t)$ is the unit vector in the direction of $\langle -3t, 2\rangle$ or $-\langle -3t, 2\rangle$. Recall that the vector $\mathbf{N}$ points to the "inside" of the curve, hence as shown in the graph the unit normal points in the positive y-direction.



$y = x^{3/2}$

Therefore we must take the positive sign. That is,

$$\mathbf{N}(t) = \frac{\langle -3t, 2\rangle}{\sqrt{9t^2 + 4}} \quad \Rightarrow \quad \mathbf{N}(1) = \frac{1}{\sqrt{13}} \langle -3, 2\rangle \tag{3}$$

We now substitute (2), (3), and $\mathbf{r}(1) = \left\langle 1^2, 1^3 \right\rangle = \langle 1, 1\rangle$ in (1) to obtain

$$\overrightarrow{OQ} = \langle 1, 1\rangle + \frac{13^{3/2}}{6} \cdot \frac{1}{13^{1/2}} \langle -3, 2\rangle = \langle 1, 1\rangle + \frac{13}{6} \langle -3, 2\rangle$$

The center of the curvature is the endpoint $Q = \left(-\frac{11}{2}, \frac{16}{3}\right)$.

In Exercises 51–58, find a parametrization of the osculating circle at the point indicated.

51. $\mathbf{r}(t) = \left\langle \cos t, \sin t \right\rangle, \quad t = \frac{\pi}{4}$

SOLUTION The curve $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ is the unit circle. By the definition of the osculating circle, it follows that the osculating circle at each point of the circle is the circle itself. Therefore the osculating circle to the unit circle at $t = \frac{\pi}{4}$ is the unit circle itself.

52. $\mathbf{r}(t) = \left\langle \sin t, \cos t \right\rangle, \quad t = 0$

SOLUTION The parametrization $\mathbf{r}(t) = \langle \sin t, \cos t\rangle$ parametrizes the unit circle. We can see this by using the parametrization $s = \frac{\pi}{2} - t$ to obtain:

$$\mathbf{r}(s) = \mathbf{r}\left(\frac{\pi}{2} - t\right) = \left\langle \sin\left(\frac{\pi}{2} - t\right), \cos\left(\frac{\pi}{2} - t\right)\right\rangle = \langle \cos t, \sin t\rangle = \mathbf{r}_1(t)$$

which is a parametrization of the unit circle. Since the osculating circle at each point of a circle is the circle itself, the osculating circle at $t = 0$ is the unit circle itself.

53. $y = x^2, \quad x = 1$

SOLUTION Let $f(x) = x^2$. We use the parametrization $\mathbf{r}(x) = \langle x, x^2\rangle$ and proceed by the following steps.

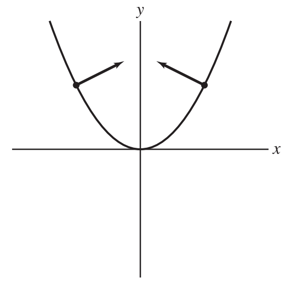
Step 1. Find κ and $\mathbf{N}$. We compute κ using the curvature of a graph in the plane:

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}$$

We have $f'(x) = 2x$, $f''(x) = 2$, therefore,

$$\kappa(x) = \frac{2}{(1 + (2x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}} \quad \Rightarrow \quad \kappa(1) = \frac{2}{5^{3/2}} \tag{1}$$

To find $\mathbf{N}(x)$ we notice that the tangent vector is $\mathbf{r}'(x) = \langle 1, 2x\rangle$ hence $\langle -2x, 1\rangle$ is orthogonal to $\mathbf{r}'(x)$ (their dot product is zero). Therefore, $\mathbf{N}(x)$ is the unit vector in the direction of $\langle -2x, 1\rangle$ or $-\langle -2x, 1\rangle$ that points to the "inside" of the curve.



$y = x^2$

As shown in the figure, the unit normal vector points in the positive y-direction, hence:

$$\mathbf{N}(x) = \frac{\langle -2x, 1\rangle}{\sqrt{4x^2+1}} \quad \Rightarrow \quad \mathbf{N}(1) = \frac{1}{\sqrt{5}}\langle -2, 1\rangle \tag{2}$$

Step 2. Find the center of the osculating circle. The center Q at $\mathbf{r}(1)$ has the position vector

$$\overrightarrow{OQ} = \mathbf{r}(1) + \kappa(1)^{-1}\mathbf{N}(1)$$

Substituting (1), (2) and $\mathbf{r}(1) = \langle 1, 1\rangle$ we get:

$$\overrightarrow{OQ} = \langle 1, 1\rangle + \frac{5^{3/2}}{2}\cdot\frac{1}{5^{1/2}}\langle -2, 1\rangle = \langle 1, 1\rangle + \frac{5}{2}\langle -2, 1\rangle = \left\langle -4, \frac{7}{2}\right\rangle$$

Step 3. Parametrize the osculating circle. The osculating circle has radius $R = \frac{1}{\kappa(1)} = \frac{5^{3/2}}{2}$ and it is centered at the point $\left(-4, \frac{7}{2}\right)$, therefore it has the following parametrization:

$$\mathbf{c}(t) = \left\langle -4, \frac{7}{2}\right\rangle + \frac{5^{3/2}}{2}\langle \cos t, \sin t\rangle$$

54. $y = \sin x, \quad x = \frac{\pi}{2}$

SOLUTION We use the parametrization $\mathbf{r}(x) = \langle x, \sin x\rangle$. The radius of the osculating circle is the radius of curvature $R = \frac{1}{\kappa\left(\frac{\pi}{2}\right)}$ and the center is the terminal point of the following vector:

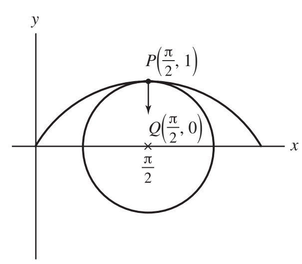
$$\overrightarrow{OQ} = \mathbf{r}\left(\frac{\pi}{2}\right) + R\mathbf{N}\left(\frac{\pi}{2}\right)$$

We first compute the curvature. Since $y'(x) = \cos x$ and $y''(x) = -\sin x$, we have:

$$\kappa(x) = \frac{|y''(x)|}{\left(1+y'(x)^2\right)^{3/2}} = \frac{|-\sin x|}{\left(1+\cos^2 x\right)^{3/2}} \quad \Rightarrow \quad \kappa\left(\frac{\pi}{2}\right) = \frac{\sin\frac{\pi}{2}}{\left(1+\cos^2\frac{\pi}{2}\right)^{3/2}} = \frac{1}{1} = 1$$

We compute the unit normal vector $\mathbf{N}(x)$. $\mathbf{N}(x)$ is a unit vector orthogonal to the tangent vector $\mathbf{r}'(x) = \langle 1, \cos x\rangle$. We observe that $\langle -\cos x, 1\rangle$ is orthogonal to $\mathbf{r}'(x)$, since their dot product is zero. Therefore, $\mathbf{N}(x)$ is the unit vector in the direction of either $\langle -\cos x, 1\rangle$ or $-\langle -\cos x, 1\rangle$, depending on the graph. Considering the accompanying figure, we see that the unit normal vector at $x = \pi/2$ points to the negative y-direction. Thus,

$$\mathbf{N}(x) = \frac{\langle \cos x, -1\rangle}{\|\langle \cos x, -1\rangle\|} = \frac{\langle \cos x, -1\rangle}{\sqrt{\cos^2 x + (-1)^2}} \quad \Rightarrow \quad \mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$$



We now find the center of the osculating circle. We substitute $R = \frac{1}{\kappa\left(\frac{\pi}{2}\right)} = 1$, $\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$, and $\mathbf{r}\left(\frac{\pi}{2}\right) = \left\langle \frac{\pi}{2}, \sin\frac{\pi}{2}\right\rangle = \left\langle \frac{\pi}{2}, 1\right\rangle$ into (1) to obtain

$$\overrightarrow{OQ} = \left\langle \frac{\pi}{2}, 1\right\rangle + 1\cdot\langle 0, -1\rangle = \left\langle \frac{\pi}{2}, 0\right\rangle$$

The osculating circle is the circle with center at the point $\left(\frac{\pi}{2}, 0\right)$ and radius 1, so it has the following parametrization:

$$\mathbf{c}(t) = \left\langle \frac{\pi}{2}, 0 \right\rangle + 1 \cdot \langle \cos t, \sin t \rangle = \left\langle \frac{\pi}{2}, 0 \right\rangle + \langle \cos t, \sin t \rangle$$

55. $\langle t - \sin t, 1 - \cos t \rangle, \quad t = \pi$

SOLUTION

Step 1. Find κ and $\mathbf{N}$. In Exercise 44 we found that:

$$\mathbf{N}(\pi) = \langle 0, -1 \rangle \tag{1}$$

To find κ we use the formula for curvature:

$$\kappa(\pi) = \frac{\|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)\|}{\|\mathbf{r}'(\pi)\|^3} \tag{2}$$

For $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ we have:

$$\mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle \quad \Rightarrow \quad \mathbf{r}'(\pi) = \langle 1 - \cos \pi, \sin \pi \rangle = \langle 2, 0 \rangle$$
$$\mathbf{r}''(t) = \langle \sin t, \cos t \rangle \quad \Rightarrow \quad \mathbf{r}''(\pi) = \langle \sin \pi, \cos \pi \rangle = \langle 0, -1 \rangle$$

Hence,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = 2\mathbf{i} \times (-\mathbf{j}) = -2\mathbf{k}$$
$$\|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)\| = \|-2\mathbf{k}\| = 2 \quad \text{and} \quad \|\mathbf{r}'(\pi)\| = \|\langle 2, 0 \rangle\| = 2$$

Substituting in (2) we get:

$$\kappa(\pi) = \frac{2}{2^3} = \frac{1}{4} \tag{3}$$

Step 2. Find the center of the osculating circle. The center Q of the osculating circle at $\mathbf{r}(\pi) = \langle \pi, 2 \rangle$ has position vector

$$\overrightarrow{OQ} = \mathbf{r}(\pi) + \kappa(\pi)^{-1} N(\pi)$$

Substituting (1), (3) and $\mathbf{r}(\pi) = \langle \pi, 2 \rangle$ we get:

$$\overrightarrow{OQ} = \langle \pi, 2 \rangle + \left(\frac{1}{4}\right)^{-1} \langle 0, -1 \rangle = \langle \pi, 2 \rangle + \langle 0, -4 \rangle = \langle \pi, -2 \rangle$$

Step 3. Parametrize the osculating circle. The osculating circle has radius $R = \frac{1}{\kappa(\pi)}$ and it is centered at $(\pi, -2)$, hence it has the following parametrization:

$$\mathbf{c}(t) = \langle \pi, -2 \rangle + 4 \langle \cos t, \sin t \rangle$$

56. $\mathbf{r}(t) = \langle t^2/2, t^3/3, t \rangle, \quad t = 0$

SOLUTION

Step 1. Find κ and $\mathbf{N}$. The unit normal is:

$$\mathbf{N}(t) = \frac{\mathbf{r}''(t) - v'(t)\mathbf{T}(t)}{\|\mathbf{r}''(t) - v'(t)\mathbf{T}(t)\|}$$

and the curvature is

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

For $\mathbf{r}(t) = \left\langle t^2/2, t^3/3, t \right\rangle$, we have:

$$\mathbf{r}'(t) = \left\langle t, t^2, 1 \right\rangle$$
$$\mathbf{r}''(t) = \langle 1, 2t, 0 \rangle$$
$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{t^2 + t^4 + 1}$$
$$v'(t) = \frac{1}{2}(t^2 + t^4 + 1)^{-1/2}(4t^3 + 2t) = \frac{4t^3 + 2t}{2\sqrt{t^2 + t^4 + 1}}$$

At the point $t = 0$ we have:

$$\begin{aligned}\mathbf{r}'(0) &= \langle 0, 0, 1\rangle\\ \mathbf{r}''(0) &= \langle 1, 0, 0\rangle\\ \mathbf{r}'(0)\times\mathbf{r}''(0) &= \langle 0, 1, 0\rangle\\ v'(0) &= 0\end{aligned}$$

And calculating norms we get:

$$\begin{aligned}\|\mathbf{r}'(0)\| &= \sqrt{0^2+0^2+1^2} = 1\\ \|\mathbf{r}'(0)\times\mathbf{r}''(0)\| &= \sqrt{0^2+1^2+0^2} = 1\\ \mathbf{T}(0) &= \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \langle 0, 0, 1\rangle\end{aligned}$$

Also,

$$\mathbf{r}''(0) - v'(0)\mathbf{T}(0) = \langle 1, 0, 0\rangle \quad \Rightarrow \quad \|\mathbf{r}''(0) - v'(0)\mathbf{T}(0)\| = 1$$

Therefore,

$$\mathbf{N}(0) = \frac{\mathbf{r}''(0) - v'(0)\mathbf{T}(0)}{\|\mathbf{r}''(0) - v'(0)\mathbf{T}(0)\|} = \langle 1, 0, 0\rangle$$

and

$$\kappa(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|^3} = \frac{1}{1} = 1$$

Step 2. Find the center of the osculating circle. In this case, the center Q of the osculating circle at $\mathbf{r}(0) = \langle 0, 0, 0\rangle$ has position vector:

$$\overrightarrow{OQ} = \mathbf{r}(0) + \kappa^{-1}(0)\mathbf{N}(0) = \langle 0, 0, 0\rangle + 1\,\langle 1, 0, 0\rangle = \langle 1, 0, 0\rangle$$

Step 3. Parametrize the osculating circle. The circle has radius $R = 1/\kappa(0) = 1$ and it is centered at $(1, 0, 0)$. Therefore, it has the following parametrization:

$$\begin{aligned}\mathbf{c}(t) &= \overrightarrow{OQ} + R\mathbf{N}(0)\cos t + R\mathbf{T}(0)\sin t\\ &= \langle 1, 0, 0\rangle + \cos t\,\langle 1, 0, 0\rangle + \sin t\,\langle 0, 0, 1\rangle\\ &= \langle 1+\cos t, 0, \sin t\rangle\end{aligned}$$

57. $\mathbf{r}(t) = \left\langle \cos t, \sin t, t\right\rangle, \quad t = 0$

SOLUTION The curvature is the following quotient:

$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

We compute the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}\langle \cos t, \sin t, t\rangle = \langle -\sin t, \cos t, 1\rangle \quad (2)\\ \mathbf{r}''(t) &= \frac{d}{dt}\langle -\sin t, \cos t, 1\rangle = \langle -\cos t, -\sin t, 0\rangle\end{aligned}$$

We now compute the following cross product:

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 1\\ -\cos t & -\sin t & 0\end{vmatrix} = \begin{vmatrix}\cos t & 1\\ -\sin t & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}-\sin t & 1\\ -\cos t & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}-\sin t & \cos t\\ -\cos t & -\sin t\end{vmatrix}\mathbf{k}\\ &= (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + 1\cdot\mathbf{k} \quad (3)\end{aligned}$$

We calculate the norms of the vectors in (1). By (2) and (3) we have:

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \sqrt{\sin^2 t + (-\cos t)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{1+1} = \sqrt{2} \tag{4}$$

Substituting (4) in (1) yields the following curvature:

$$\kappa(t) = \frac{\sqrt{2}}{\left(\sqrt{2}\right)^3} = \frac{1}{2} \quad \Rightarrow \quad \kappa(0) = \frac{1}{2} \tag{5}$$

The unit normal vector is the following vector:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{6}$$

By (2) and (4) we have:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle \quad \Rightarrow \quad \mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle \tag{7}$$

$$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{(-\cos t)^2 + (-\sin t)^2 + 0^2} = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

Combining (6) and (7) gives:

$$\mathbf{N}(t) = \langle -\cos t, -\sin t, 0\rangle \quad \Rightarrow \quad \mathbf{N}(0) = \langle -1, 0, 0\rangle \tag{8}$$

The center of curvature at $t = 0$ is:

$$\overrightarrow{OQ} = \mathbf{r}(0) + \kappa(0)^{-1}\mathbf{N}(0)$$

By (5), (8) and $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ we get:

$$\overrightarrow{OQ} = \langle 1, 0, 0\rangle + 2\langle -1, 0, 0\rangle = \langle 1, 0, 0\rangle + \langle -2, 0, 0\rangle = \langle -1, 0, 0\rangle$$

Finally, we find a parametrization of the osculating circle at $t = 0$. The osculating circle has radius $R = \frac{1}{\kappa(0)} = 2$ and center $\langle -1, 0, 0\rangle$, hence it has the following parametrization:

$$\mathbf{c}(t) = \langle -1, 0, 0\rangle + 2\mathbf{N}(0)\cos t + 2\mathbf{T}(0)\sin t = \langle -1, 0, 0\rangle + 2\langle -1, 0, 0\rangle \cos t + \frac{2}{\sqrt{2}}\langle 0, 1, 1\rangle \sin t$$

$$\mathbf{c}(t) = \left\langle -1 - 2\cos t, \frac{2\sin t}{\sqrt{2}}, \frac{2\sin t}{\sqrt{2}}\right\rangle$$

58. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t\rangle, \quad t = 0$

SOLUTION

Step 1. Find κ and $\mathbf{N}$. In Exercise 14 we found that:

$$\kappa(t) = \frac{1}{2\cosh^2 t} \quad \Rightarrow \quad \kappa(0) = \frac{1}{2\cosh^2 0} = \frac{1}{2} \tag{1}$$

We now must find the unit normal $\mathbf{N}$. We have:

$$\mathbf{r}'(t) = \langle \sinh t, \cosh t, 1\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{\cosh^2 t - 1 + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}\cosh t}\langle \sinh t, \cosh t, 1\rangle = \frac{1}{\sqrt{2}}\langle \tanh t, 1, \operatorname{sech} t\rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\left\langle \operatorname{sech}^2 t, 0, -\operatorname{sech} t \tanh t\right\rangle$$

We compute the length of $\mathbf{T}'(t)$. Using the identity $\tanh^2 t + \operatorname{sech}^2 t = 1$ we get:

$$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{\operatorname{sech}^4 t + 0 + \operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}}\sqrt{\operatorname{sech}^2 t\left(\tanh^2 t + \operatorname{sech}^2 t\right)} = \frac{1}{\sqrt{2}}\sqrt{\operatorname{sech}^2 t \cdot 1} = \frac{\operatorname{sech} t}{\sqrt{2}}$$

Hence,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\sqrt{2}}{\operatorname{sech} t}\frac{1}{\sqrt{2}} \cdot \left\langle \operatorname{sech}^2 t, 0, -\operatorname{sech} t \tanh t\right\rangle = \langle \operatorname{sech} t, 0, -\tanh t\rangle$$

At the point $t = 0$ we have $\operatorname{sech} 0 = 1$, $\tanh 0 = 0$, hence

$$\mathbf{N}(0) = \langle 1, 0, 0\rangle \tag{2}$$

Step 2. Find the center of the osculating circle. The center Q of the osculating circle at $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ has position vector:

$$\overrightarrow{OQ} = \mathbf{r}(0) + \kappa(0)^{-1}\mathbf{N}(0)$$

Substituting (1), (2) and $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ we get:

$$\overrightarrow{OQ} = \langle 1, 0, 0\rangle + 2 \cdot \langle 1, 0, 0\rangle = \langle 3, 0, 0\rangle$$

Step 3. Parametrize the osculating circle. The osculating circle is centered at $Q = (3, 0, 0)$ and has radius $R = \frac{1}{\kappa(0)} = 2$, hence it has the following parametrization:

$$\mathbf{c}(t) = \langle 3, 0, 0\rangle + 2\mathbf{N}\cos t + 2\mathbf{T}\sin t = \langle 3, 0, 0\rangle + 2\langle 1, 0, 0\rangle \cos t + \frac{2}{\sqrt{2}}\langle 0, 1, 1\rangle \sin t$$

59. Figure 18 shows the graph of the half-ellipse $y = \pm\sqrt{2rx - px^2}$, where r and p are positive constants. Show that the radius of curvature at the origin is equal to r. *Hint:* One way of proceeding is to write the ellipse in the form of Exercise 25 and apply Eq. (9).

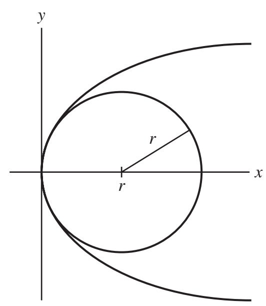


FIGURE 18 The curve $y = \pm\sqrt{2rx - px^2}$ and the osculating circle at the origin.

SOLUTION The radius of curvature is the reciprocal of the curvature. We thus must find the curvature at the origin. We use the following simple variant of the formula for the curvature of a graph in the plane:

$$\kappa(y) = \frac{|x''(y)|}{\left(1 + x'(y)^2\right)^{3/2}} \tag{1}$$

(The traditional formula of $\kappa(x) = \frac{|y''(x)|}{\left(1+y'(x)^2\right)^{3/2}}$ is inappropriate for this problem, as $y'(x)$ is undefined at $x = 0$.) We find x in terms of y:

$$\begin{aligned} y &= \sqrt{2rx - px^2} \\ y^2 &= 2rx - px^2 \\ px^2 - 2rx + y^2 &= 0 \end{aligned}$$

We solve for x and obtain:

$$x = \pm\frac{1}{p}\sqrt{r^2 - py^2} + \frac{r}{p}, \quad y \geq 0.$$

We find x' and x'':

$$x' = \pm\frac{-2py}{2p\sqrt{r^2 - py^2}} = \pm\frac{y}{\sqrt{r^2 - py^2}}$$

$$x'' = \pm\frac{1 \cdot \sqrt{r^2 - py^2} - y \cdot \frac{-py}{\sqrt{r^2 - py^2}}}{r^2 - py^2} = \pm\frac{r^2 - py^2 + py^2}{\left(r^2 - py^2\right)^{3/2}} = \pm\frac{r^2}{\left(r^2 - py^2\right)^{3/2}}$$

At the origin we get:

$$x'(0) = 0, \quad x''(0) = \frac{\pm r^2}{(r^2)^{3/2}} = \frac{\pm 1}{r}$$

Substituting in (1) gives the following curvature at the origin:

$$\kappa(0) = \frac{|x''(0)|}{(1 + x'(0)^2)^{3/2}} = \frac{|\frac{\pm 1}{r}|}{(1+0)^{3/2}} = \frac{1}{|r|} = \frac{1}{r}$$

We conclude that the radius of curvature at the origin is

$$R = \frac{1}{\kappa(0)} = r$$

60. In a recent study of laser eye surgery by Gatinel, Hoang-Xuan, and Azar, a vertical cross section of the cornea is modeled by the half-ellipse of Exercise 59. Show that the half-ellipse can be written in the form $x = f(y)$, where $f(y) = p^{-1}\left(r - \sqrt{r^2 - py^2}\right)$. During surgery, tissue is removed to a depth $t(y)$ at height y for $-S \le y \le S$, where $t(y)$ is given by Munnerlyn's equation (for some $R > r$):

$$t(y) = \sqrt{R^2 - S^2} - \sqrt{R^2 - y^2} - \sqrt{r^2 - S^2} + \sqrt{r^2 - y^2}$$

After surgery, the cross section of the cornea has the shape $x = f(y) + t(y)$ (Figure 19). Show that after surgery, the radius of curvature at the point P (where $y = 0$) is R.

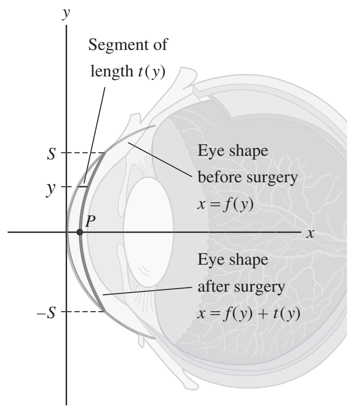


FIGURE 19 Contour of cornea before and after surgery.

SOLUTION We consider the half-ellipse:

$$r > 0, \quad p > 0, \quad y = \sqrt{2rx - px^2}$$

As in Exercise 59, squaring the two sides and solving for x gives:

$$x - \frac{r}{p} = \pm\frac{1}{p}\sqrt{r^2 - py^2} \quad \Rightarrow \quad x = \frac{1}{p}\left(r \pm \sqrt{r^2 - py^2}\right)$$

The negative sign must be taken, since this is the half in which we are interested. That is,

$$f(y) = x = \frac{1}{p}\left(r - \sqrt{r^2 - py^2}\right) \tag{1}$$

We are given the following curve:

$$x = f(y) + t(y)$$

where

$$t(y) = \sqrt{R^2 - S^2} - \sqrt{R^2 - y^2} - \sqrt{r^2 - S^2} + \sqrt{r^2 - y^2} \tag{2}$$

We must show that the radius of curvature at O is R. We use the curvature of a graph in the plane:

$$\kappa(y) = \frac{|x''(y)|}{\left(1 + x'(y)^2\right)^{3/2}} \quad \Rightarrow \quad \kappa(0) = \frac{|x''(0)|}{\left(1 + x'(0)^2\right)^{3/2}} \tag{3}$$

We differentiate $f(y)$ in (1) and $t(y)$ in (2) twice. This gives:

$$f'(y) = -\frac{1}{p} \cdot \frac{-py}{\sqrt{r^2 - py^2}} = \frac{y}{\sqrt{r^2 - py^2}}$$

$$f''(y) = \frac{\sqrt{r^2 - py^2} - y \cdot \frac{-py}{\sqrt{r^2-py^2}}}{r^2 - py^2} = \frac{r^2 - py^2 + py^2}{\left(r^2 - py^2\right)^{3/2}} = \frac{r^2}{\left(r^2 - py^2\right)^{3/2}}$$

$$t'(y) = -\frac{-2y}{2\sqrt{R^2 - y^2}} + \frac{-2y}{2\sqrt{r^2 - y^2}} = \frac{y}{\sqrt{R^2 - y^2}} - \frac{y}{\sqrt{r^2 - y^2}}$$

$$t''(y) = \frac{\sqrt{R^2 - y^2} - y \cdot \frac{-y}{\sqrt{R^2-y^2}}}{R^2 - y^2} - \frac{\sqrt{r^2 - y^2} - y \cdot \frac{-y}{\sqrt{r^2-y^2}}}{r^2 - y^2} = \frac{R^2 - y^2 + y^2}{\left(R^2 - y^2\right)^{3/2}} - \frac{r^2 - y^2 + y^2}{\left(r^2 - y^2\right)^{3/2}}$$

$$= \frac{R^2}{\left(R^2 - y^2\right)^{3/2}} - \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

At $y = 0$ we obtain:

$$f'(0) = 0; \quad t'(0) = 0$$

$$f''(0) = \frac{r^2}{(r^2)^{3/2}} = \frac{1}{r}; \quad t''(0) = \frac{R^2}{(R^2)^{3/2}} - \frac{r^2}{(r^2)^{3/2}} = \frac{1}{R} - \frac{1}{r}$$

Hence,

$$x''(0) = f''(0) + t''(0) = \frac{1}{r} + \frac{1}{R} - \frac{1}{r} = \frac{1}{R}$$

$$x'(0) = f'(0) + t'(0) = 0$$

Substituting in (3) gives the following curvature:

$$\kappa(0) = \frac{\frac{1}{R}}{\left(1 + 0^2\right)^{3/2}} = \frac{1}{R}.$$

The radius of curvature at the origin is therefore,

$$\frac{1}{\kappa(0)} = \frac{1}{\frac{1}{R}} = R.$$

61. The **angle of inclination** at a point P on a plane curve is the angle θ between the unit tangent vector $\mathbf{T}$ and the x-axis (Figure 20). Assume that $\mathbf{r}(s)$ is a arc length parametrization, and let $\theta = \theta(s)$ be the angle of inclination at $\mathbf{r}(s)$. Prove that

$$\kappa(s) = \left|\frac{d\theta}{ds}\right| \qquad \boxed{12}$$

Hint: Observe that $\mathbf{T}(s) = \langle \cos\theta(s), \sin\theta(s)\rangle$.

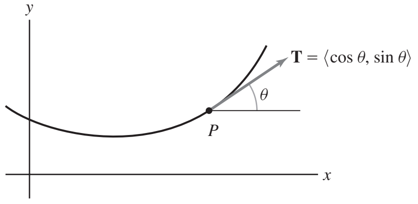


FIGURE 20 The curvature at P is the quantity $|d\theta/ds|$.

SOLUTION Since $\mathbf{T}(t)$ is a unit vector that makes an angle $\theta(t)$ with the positive x-axis, we have

$$\mathbf{T}(t) = \langle \cos\theta(t), \sin\theta(t)\rangle .$$

Differentiating this vector using the Chain Rule gives:

$$\mathbf{T}'(t) = \left\langle -\theta'(t)\sin\theta(t), \theta'(t)\cos\theta(t)\right\rangle = \theta'(t)\left\langle -\sin\theta(t), \cos\theta(t)\right\rangle$$

We compute the norm of the vector $\mathbf{T}'(t)$:

$$\|\mathbf{T}'(t)\| = \|\theta'(t)\left\langle -\sin\theta(t), \cos\theta(t)\right\rangle\| = |\theta'(t)|\sqrt{(-\sin\theta(t))^2 + (\cos\theta(t))^2} = |\theta'(t)| \cdot 1 = |\theta'(t)|$$

When $\mathbf{r}(s)$ is a parametrization by arc length we have:

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}}{dt} \right\| \left| \frac{dt}{d\theta} \frac{d\theta}{ds} \right| = \left|\theta'(t)\right| \frac{1}{|\theta'(t)|} \left| \frac{d\theta}{ds} \right| = \left| \frac{d\theta}{ds} \right|$$

as desired.

62. A particle moves along the path $y = x^3$ with unit speed. How fast is the tangent turning (i.e., how fast is the angle of inclination changing) when the particle passes through the point (2, 8)?

SOLUTION The particle has unit speed hence the parametrization is in arc length parametrization. Therefore, the change in the angle of inclination is, by Exercise 61:

$$\left| \frac{d\theta}{ds} \right| = \kappa(s)$$

In particular at the point (2, 8) we have:

$$\left| \frac{d\theta}{ds} \right|_{(2,8)} = \kappa(s)\big|_{(2,8)} \tag{1}$$

We, thus, must find the curvature at the given point. We use the formula for the curvature of a graph in the plane:

$$\kappa(x) = \frac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} \quad \Rightarrow \quad \kappa(2) = \frac{|y''(2)|}{\left(1 + y'(2)^2\right)^{3/2}} \tag{2}$$

For $y = x^3$ we have $y' = 3x^2$ and $y'' = 6x$, hence $y'(2) = 12$ and $y''(2) = 12$. Substituting in (2), we get:

$$\kappa(2) = \frac{12}{\left(1 + 12^2\right)^{3/2}} = \frac{12}{145^{3/2}} \approx 0.0069$$

Combining with (1) we conclude that when the particle passes through the point (2, 8) the tangent is turning at a rate of 0.0069.

63. Let $\theta(x)$ be the angle of inclination at a point on the graph $y = f(x)$ (see Exercise 61).

(a) Use the relation $f'(x) = \tan\theta$ to prove that $\dfrac{d\theta}{dx} = \dfrac{f''(x)}{(1 + f'(x)^2)}$.

(b) Use the arc length integral to show that $\dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$.

(c) Now give a proof of Eq. (5) using Eq. (12).

SOLUTION

(a) By the relation $f'(x) = \tan\theta$ we have $\theta = \tan^{-1} f'(x)$. Differentiating using the Chain Rule we get:

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1} f'(x)\right) = \frac{1}{1 + f'(x)^2} \frac{d}{dx}\left(f'(x)\right) = \frac{f''(x)}{1 + f'(x)^2}$$

(b) We use the parametrization $\mathbf{r}(x) = \langle x, f(x)\rangle$. Hence, $\mathbf{r}'(x) = \left\langle 1, f'(x)\right\rangle$ and we obtain the following arc length function:

$$S(x) = \int_0^x \|\mathbf{r}'(u)\|\, du = \int_0^x \left\|\left\langle 1, f'(u)\right\rangle\right\| du = \int_0^x \sqrt{1 + f'(u)^2}\, du$$

Differentiating using the Fundamental Theorem gives:

$$\frac{ds}{dx} = \frac{d}{dx}\left(\int_0^x \sqrt{1 + f'(u)^2}\, du\right) = \sqrt{1 + f'(x)^2}$$

(c) By Eq. (12),

$$\kappa(s) = \left| \frac{d\theta}{ds} \right| \tag{1}$$

Using the Chain Rule and the equalities in part (a) and part (b), we obtain:

$$\frac{d\theta}{ds} = \frac{d\theta}{dx} \cdot \frac{dx}{ds} = \frac{d\theta}{dx} \cdot \frac{1}{\frac{ds}{dx}} = \frac{f''(x)}{1 + f'(x)^2} \cdot \frac{1}{\sqrt{1 + f'(x)^2}} = \frac{f''(x)}{\left(1 + f'(x)^2\right)^{3/2}}$$

Combining with (1) we obtain the curvature as the following function of x:

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}$$

which proves Eq. (5).

64. Use the parametrization $\mathbf{r}(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$ to show that a curve $r = f(\theta)$ in polar coordinates has curvature

$$\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - 2f(\theta)f''(\theta)|}{\left(f(\theta)^2 + f'(\theta)^2\right)^{3/2}} \qquad \boxed{13}$$

SOLUTION By the formula for curvature we have

$$\kappa(\theta) = \frac{\|\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)\|}{\|\mathbf{r}'(\theta)\|^3} \tag{1}$$

We differentiate $\mathbf{r}(\theta)$ and $\mathbf{r}'(\theta)$:

$$\begin{aligned}
\mathbf{r}'(\theta) &= \left\langle f'(\theta)\cos\theta - f(\theta)\sin\theta,\ f'(\theta)\sin\theta + f(\theta)\cos\theta\right\rangle\\
\mathbf{r}''(\theta) &= \left\langle f''(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta - f(\theta)\cos\theta,\right.\\
&\qquad \left. f''(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta - f(\theta)\sin\theta\right\rangle\\
&= \left\langle \left(f''(\theta) - f(\theta)\right)\cos\theta - 2f'(\theta)\sin\theta,\ \left(f''(\theta) - f(\theta)\right)\sin\theta + 2f'(\theta)\cos\theta\right\rangle
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{r}'(\theta) \times \mathbf{r}''(\theta) &= \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)\cdot\left(\left(f''(\theta) - f(\theta)\right)\sin\theta + 2f'(\theta)\cos\theta\right)\mathbf{k}\\
&\quad - \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)\cdot\left(\left(f''(\theta) - f(\theta)\right)\cos\theta - 2f'(\theta)\sin\theta\right)\mathbf{k}\\
&= \Big\{ f'(\theta)\left(f''(\theta) - f(\theta)\right)\cos\theta\sin\theta - f(\theta)\left(f''(\theta) - f(\theta)\right)\sin^2\theta + 2f'^2(\theta)\cos^2\theta\\
&\quad - 2f(\theta)f'(\theta)\sin\theta\cos\theta\Big(-f'(\theta)\left(f''(\theta) - f(\theta)\right)\sin\theta\cos\theta - f(\theta)\left(f''(\theta) - f(\theta)\right)\cos^2\theta\\
&\quad + 2f'(\theta)^2\sin^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta\Big)\Big\}\mathbf{k}\\
&= \Big(-f(\theta)\left(f''(\theta) - f(\theta)\right)\left(\sin^2\theta + \cos^2\theta\right) + 2f'^2(\theta)\left(\cos^2\theta + \sin^2\theta\right)\Big)\mathbf{k}\\
&= \Big(-f(\theta)\left(f''(\theta) - f(\theta)\right) + 2f'^2(\theta)\Big)\mathbf{k}\\
&= \Big(-f(\theta)f''(\theta) + f^2(\theta) + 2f'^2(\theta)\Big)\mathbf{k}
\end{aligned}$$

The length of the cross product is:

$$\|\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)\| = |f^2(\theta) + 2f'^2(\theta) - f(\theta)f''(\theta)| \tag{2}$$

We compute the length of $\mathbf{r}'(\theta)$:

$$\begin{aligned}
\|\mathbf{r}'(\theta)\|^2 &= \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2\\
&= f'^2(\theta)\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + f^2(\theta)\sin^2\theta + f'^2(\theta)\sin^2\theta\\
&\quad + 2f'(\theta)f(\theta)\sin\theta\cos\theta + f^2(\theta)\cos^2\theta\\
&= f'^2(\theta)\left(\cos^2\theta + \sin^2\theta\right) + f^2(\theta)\left(\sin^2\theta + \cos^2\theta\right) = f'^2(\theta) + f^2(\theta)
\end{aligned}$$

Hence,

$$\|\mathbf{r}'(\theta)\| = \sqrt{f'^2(\theta) + f^2(\theta)} \tag{3}$$

Substituting (2) and (3) in (1) gives:

$$\kappa(\theta) = \frac{|f^2(\theta) + 2f'^2(\theta) - f(\theta)f''(\theta)|}{\left(f'^2(\theta) + f^2(\theta)\right)^{3/2}}$$

In Exercises 65–67, use Eq. (13) to find the curvature of the curve given in polar form.

65. $f(\theta) = 2\cos\theta$

SOLUTION By Eq. (13):,

$$\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{\left(f(\theta)^2 + f'^2(\theta)\right)^{3/2}}$$

We compute the derivatives $f'(\theta)$ and $f''(\theta)$ and evaluate the numerator of $\kappa(\theta)$. This gives:

$$\begin{aligned} f'(\theta) &= -2\sin\theta \\ f''(\theta) &= -2\cos\theta \\ f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta) &= 4\cos^2\theta + 2\cdot 4\sin^2\theta - 2\cos\theta(-2\cos\theta) \\ &= 8\cos^2\theta + 8\sin^2\theta = 8 \end{aligned}$$

We compute the denominator of $\kappa(\theta)$:

$$\left(f(\theta)^2 + f'(\theta)^2\right)^{3/2} = \left(4\cos^2\theta + 4\sin^2\theta\right)^{3/2} = 4^{3/2} = 8$$

Hence,

$$\kappa(\theta) = \frac{8}{8} = 1$$

66. $f(\theta) = \theta$

SOLUTION We have $f'(\theta) = 1$, $f''(\theta) = 0$. The numerator and denominator in Eq. (13) are thus:

$$\begin{aligned} f(\theta)^2 + 2f'(\theta) - f(\theta)f''(\theta) &= \theta^2 + 2\cdot 1 - 0 = \theta^2 + 2 \\ \left(f(\theta)^2 + f'(\theta)^2\right)^{3/2} &= \left(\theta^2 + 1\right)^{3/2} \end{aligned}$$

Hence,

$$\kappa(\theta) = \frac{\theta^2 + 2}{\left(\theta^2 + 1\right)^{3/2}}$$

67. $f(\theta) = e^\theta$

SOLUTION By Eq. (13) we have the following curvature:

$$\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{\left(f(\theta)^2 + f'^2(\theta)\right)^{3/2}}$$

Since $f(\theta) = e^\theta$ also $f'(\theta) = f''(\theta) = e^\theta$. We compute the numerator and denominator of $\kappa(\theta)$:

$$\begin{aligned} f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta) &= e^{2\theta} + 2e^{2\theta} - e^\theta\cdot e^\theta = 2e^{2\theta} \\ \left(f(\theta)^2 + f'(\theta)^2\right)^{3/2} &= \left(e^{2\theta} + e^{2\theta}\right)^{3/2} = \left(2e^{2\theta}\right)^{3/2} = 2\sqrt{2}e^{3\theta} \end{aligned}$$

Substituting in the formula for $\kappa(\theta)$ we obtain:

$$\kappa(\theta) = \frac{2e^{2\theta}}{2\sqrt{2}e^{3\theta}} = \frac{1}{\sqrt{2}}e^{-\theta}$$

68. Use Eq. (13) to find the curvature of the general Bernoulli spiral $r = ae^{b\theta}$ in polar form (a and b are constants).

SOLUTION By Eq. (13):

$$\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{\left(f(\theta)^2 + f'^2(\theta)\right)^{3/2}}$$

In our case $f(\theta) = ae^{b\theta}$ hence $f'(\theta) = abe^{b\theta}$ and $f''(\theta) = ab^2e^{b\theta}$. We compute the numerator of $\kappa(\theta)$:

$$f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta) = a^2e^{2b\theta} + 2a^2b^2e^{2b\theta} - ae^{b\theta}\cdot ab^2e^{b\theta} = a^2e^{2b\theta} + 2a^2b^2e^{2b\theta} - a^2b^2e^{2b\theta}$$

$$= a^2e^{2b\theta} + a^2b^2e^{2b\theta} = a^2(1+b^2)e^{2b\theta}$$

We compute the denominator of $\kappa(\theta)$:

$$\left(f(\theta)^2 + f'(\theta)^2\right)^{3/2} = \left(a^2e^{2b\theta} + a^2b^2e^{2b\theta}\right)^{3/2} = \left(a^2e^{2b\theta}\left(1+b^2\right)\right)^{3/2} = a^3e^{3b\theta}\left(1+b^2\right)^{3/2}$$

Therefore:

$$\kappa(\theta) = \frac{a^2(1+b^2)e^{2b\theta}}{a^3(1+b^2)^{3/2}e^{3b\theta}} = \frac{1}{a\sqrt{1+b^2}}e^{-b\theta}$$

69. Show that both $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ lie in the osculating plane for a vector function $\mathbf{r}(t)$. *Hint:* Differentiate $\mathbf{r}'(t) = v(t)\mathbf{T}(t)$.

SOLUTION The osculating plane at P is the plane through P determined by the unit tangent $\mathbf{T}$ and the unit normal $\mathbf{N}$ at P. Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ we have $\mathbf{r}'(t) = v(t)\mathbf{T}(t)$ where $v(t) = \|\mathbf{r}'(t)\|$. That is, $\mathbf{r}'(t)$ is a scalar multiple of $\mathbf{T}(t)$, hence it lies in every plane containing $\mathbf{T}(t)$, in particular in the osculating plane. We now differentiate $\mathbf{r}'(t) = v(t)\mathbf{T}(t)$ using the Product Rule:

$$\mathbf{r}''(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \tag{1}$$

By $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ we have $\mathbf{T}'(t) = b(t)\mathbf{N}(t)$ for $b(t) = \|\mathbf{T}'(t)\|$. Substituting in (1) gives:

$$\mathbf{r}''(t) = v'(t)\mathbf{T}(t) + v(t)b(t)\mathbf{N}(t)$$

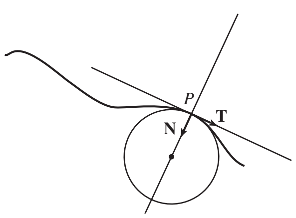
We see that $\mathbf{r}''(t)$ is a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$, hence $\mathbf{r}''(t)$ lies in the plane determined by these two vectors, that is, $\mathbf{r}''(t)$ lies in the osculating plane.

70. Show that

$$\gamma(s) = \mathbf{r}(t_0) + \frac{1}{\kappa}\mathbf{N} + \frac{1}{\kappa}\left((\sin\kappa s)\mathbf{T} - (\cos\kappa s)\mathbf{N}\right)$$

is an arc length parametrization of the osculating circle at $\mathbf{r}(t_0)$.

SOLUTION Let P be a fixed point on the curve $\mathcal{C}$, $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and the unit normal to the curve at P. We place the xy-coordinate system so that the origin is at P and the x and y axes are in the directions of $\mathbf{T}$ and $\mathbf{N}$, respectively. We next show that $\gamma(s)$ is an arc length parametrization of the osculating circle at P.



We compute the following expression:

$$\left\|\gamma(s) - \frac{1}{\kappa}\mathbf{N}\right\|^2 = \frac{1}{\kappa^2}\|(\sin\kappa s)\,\mathbf{T} - (\cos\kappa s)\,\mathbf{N}\|^2 = \frac{1}{\kappa^2}\left((\sin\kappa s)\,\mathbf{T} - (\cos\kappa s)\,\mathbf{N}\right)\cdot\left((\sin\kappa s)\,\mathbf{T} - (\cos\kappa s)\,\mathbf{N}\right)$$

$$= \frac{1}{\kappa^2}\left(\sin^2\kappa s\mathbf{T}\cdot\mathbf{T} - (\sin\kappa s\cos\kappa s)\,\mathbf{T}\cdot\mathbf{N} - (\cos\kappa s\sin\kappa s)\,\mathbf{N}\cdot\mathbf{T} + \left(\cos^2\kappa s\right)\mathbf{N}\cdot\mathbf{N}\right)$$

The vectors $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, hence $\mathbf{T}\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{T} = 0$ and $\mathbf{T}\cdot\mathbf{T} = \|\mathbf{T}\|^2 = 1$, $\mathbf{N}\cdot\mathbf{N} = \|\mathbf{N}\|^2 = 1$. We use the identity $\sin^2(\kappa s) + \cos^2(\kappa s) = 1$ to obtain

$$\left\|\gamma(s) - \frac{1}{\kappa}\mathbf{N}\right\|^2 = \frac{1}{\kappa^2}\left(\sin^2\kappa s + \cos^2\kappa s\right) = \frac{1}{\kappa^2}$$

That is,

$$\left\|\gamma(s) - \frac{1}{\kappa}\mathbf{N}\right\| = \frac{1}{\kappa} \tag{1}$$

Notice that κ, $\mathbf{N}$, and $\mathbf{T}$ are fixed and only s is changing in $\gamma(s)$. It follows by (1) that $\gamma(s)$ is a circle of radius $\frac{1}{\kappa}$ centered at $\frac{1}{\kappa}\mathbf{N}$. The curvature of the circle is the reciprocal of the radius, which is κ (the curvature of $\mathcal{C}$ at the point P). We thus showed that the circle $\gamma(s)$ satisfies the second condition in the definition of the osculating circle. We now show that the first condition is satisfied as well.

The center of the circle is the terminal point of the vector $\frac{1}{\kappa}\mathbf{N}$, which is in the direction of $\mathbf{N}$ and orthogonal to $\mathbf{T}$. This shows that $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and unit normal to the circle at P. Finally, we verify that the given parametrization is the arc length parametrization, by showing that $\|\gamma'(s)\| = 1$. Differentiating $\gamma(s)$ with respect to s gives (notice that κ, $\mathbf{T}$, and $\mathbf{N}$ are fixed):

$$\gamma'(s) = \frac{1}{\kappa}\big((\kappa\cos\kappa s)\,\mathbf{T} + (\kappa\sin\kappa s)\,\mathbf{N}\big) = (\cos\kappa s)\mathbf{T} + (\sin\kappa s)\mathbf{N}$$

Hence, since $\mathbf{T}\cdot\mathbf{T} = \mathbf{N}\cdot\mathbf{N} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{T} = 0$ we get:

$$\begin{aligned}\|\gamma'(s)\|^2 &= \big((\cos\kappa s)\mathbf{T} + (\sin\kappa s)\mathbf{N}\big)\cdot\big((\cos\kappa s)\mathbf{T} + (\sin\kappa s)\mathbf{N}\big)\\ &= \big(\cos^2\kappa s\big)\mathbf{T}\cdot\mathbf{T} + (\cos\kappa s)(\sin\kappa s)\mathbf{T}\cdot\mathbf{N} + (\sin\kappa s\cos\kappa s)\mathbf{N}\cdot\mathbf{T} + \big(\sin^2\kappa s\big)\mathbf{N}\cdot\mathbf{N}\\ &= \cos^2\kappa s + \sin^2\kappa s = 1\end{aligned}$$

Hence

$$\|\gamma'(s)\| = 1$$

71. Two vector-valued functions $\mathbf{r}_1(s)$ and $\mathbf{r}_2(s)$ are said to *agree to order 2* at s_0 if

$$\mathbf{r}_1(s_0) = \mathbf{r}_2(s_0), \quad \mathbf{r}_1'(s_0) = \mathbf{r}_2'(s_0), \quad \mathbf{r}_1''(s_0) = \mathbf{r}_2''(s_0)$$

Let $\mathbf{r}(s)$ be an arc length parametrization of a path $\mathcal{C}$, and let P be the terminal point of $\mathbf{r}(0)$. Let $\gamma(s)$ be the arc length parametrization of the osculating circle given in Exercise 70. Show that $\mathbf{r}(s)$ and $\gamma(s)$ agree to order 2 at $s = 0$ (in fact, the osculating circle is the unique circle that approximates $\mathcal{C}$ to order 2 at P).

SOLUTION The arc length parametrization of the osculating circle at P, described in the xy-coordinate system with P at the origin and the x and y axes in the directions of $\mathbf{T}$ and $\mathbf{N}$ respectively, is given in Exercise 70 by:

$$\gamma(s) = \frac{1}{\kappa}\mathbf{N} + \frac{1}{\kappa}\big((\sin\kappa s)\mathbf{T} - (\cos\kappa s)\mathbf{N}\big)$$

Hence

$$\begin{aligned}\gamma(0) &= \frac{1}{\kappa}\mathbf{N} + \frac{1}{\kappa}\big((\sin 0)\mathbf{T} - (\cos 0)\mathbf{N}\big) = \frac{1}{\kappa}\mathbf{N} + \frac{1}{\kappa}(0 - 1\cdot\mathbf{N}) = \frac{1}{\kappa}\mathbf{N} - \frac{1}{\kappa}\mathbf{N} = \mathbf{0}\\ \mathbf{r}(0) &= \overrightarrow{OP} = \mathbf{0}\end{aligned}$$

We get:

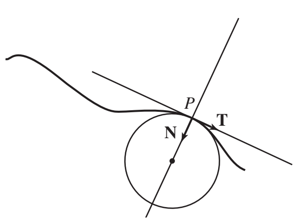
$$\gamma(0) = \mathbf{r}(0) \tag{1}$$

Differentiating $\gamma(s)$ gives (notice that $\mathbf{N}$, $\mathbf{T}$, and κ are fixed):

$$\gamma'(s) = \frac{1}{\kappa}\big((\kappa\cos\kappa s)\mathbf{T} + (\kappa\sin\kappa s)\mathbf{N}\big) = (\cos\kappa s)\mathbf{T} + (\sin\kappa s)\mathbf{N}$$

Hence:

$$\gamma'(0) = (\cos\kappa\cdot 0)\,\mathbf{T} + (\sin\kappa\cdot 0)\,\mathbf{N} = 1\cdot\mathbf{T} + 0\cdot\mathbf{N} = \mathbf{T}$$



Also, since $\mathbf{r}(s)$ is the arc length parametrization, $\|\mathbf{r}'(s)\| = 1$, hence:

$$\mathbf{T} = \mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \mathbf{r}'(0)$$

We conclude that:

$$\gamma'(0) = \mathbf{r}'(0) \tag{2}$$

We differentiate $\gamma'(s)$ to obtain:

$$\gamma''(s) = (-\kappa \sin \kappa s)\,\mathbf{T} + (\kappa \cos \kappa s)\,\mathbf{N}$$

Hence:

$$\gamma''(0) = (-\kappa \sin 0)\,\mathbf{T} + (\kappa \cos 0)\,\mathbf{N} = 0\mathbf{T} + \kappa\mathbf{N} = \kappa\mathbf{N}$$

For the arc length parametrization $\mathbf{r}(s)$ we have:

$$\mathbf{r}''(s) = \mathbf{T}'(s) = \|\mathbf{T}'(s)\|\mathbf{N}(s) = \|\mathbf{r}'(s)\|\kappa(s)\mathbf{N}(s) = 1 \cdot \kappa(s)\mathbf{N}(s)$$

Hence:

$$\mathbf{r}''(0) = \kappa(0)\mathbf{N}(0) = \kappa\mathbf{N}$$

We conclude that:

$$\gamma''(0) = \mathbf{r}''(0) \tag{3}$$

(1), (2), and (3) show that $\mathbf{r}(s)$ and $\gamma(s)$ agree to order two at $s = 0$.

72. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a path with curvature $\kappa(t)$ and define the scaled path $\mathbf{r}_1(t) = \langle \lambda x(t), \lambda y(t), \lambda z(t)\rangle$, where $\lambda \neq 0$ is a constant. Prove that curvature varies inversely with the scale factor. That is, prove that the curvature $\kappa_1(t)$ of $\mathbf{r}_1(t)$ is $\kappa_1(t) = \lambda^{-1}\kappa(t)$. This explains why the curvature of a circle of radius R is proportional to $1/R$ (in fact, it is equal to $1/R$). *Hint:* Use Eq. (3).

SOLUTION The resulting curvature k_1 and the original curvature κ are:

$$\kappa_1(t) = \frac{\|\mathbf{r}_1'(t) \times \mathbf{r}_1''(t)\|}{\|\mathbf{r}_1'(t)\|^3}, \quad \kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

We have

$$\mathbf{r}_1'(t) = \frac{d}{dt}\left(\lambda\mathbf{r}(t)\right) = \lambda\mathbf{r}'(t)$$

$$\mathbf{r}_1''(t) = \frac{d}{dt}\left(r_1'(t)\right) = \frac{d}{dt}\left(\lambda\mathbf{r}'(t)\right) = \lambda\mathbf{r}''(t)$$

Hence,

$$\|\mathbf{r}_1'(t) \times \mathbf{r}_1''(t)\| = \|\lambda\mathbf{r}'(t) \times \lambda\mathbf{r}''(t)\| = \lambda^2\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$$

$$\|\mathbf{r}_1'(t)\| = \|\lambda\mathbf{r}'(t)\| = |\lambda|\|\mathbf{r}'(t)\|$$

Substituting in (1) we get:

$$\kappa_1(t) = \frac{\lambda^2\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{|\lambda|^3\|\mathbf{r}'(t)\|^3} = \frac{1}{|\lambda|}\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{1}{|\lambda|}\kappa(t)$$

We conclude that the resulting curvature is:

$$\kappa_1(t) = \frac{1}{|\lambda|}\kappa(t)$$

Multiplying the vector by λ causes the curvature to be divided by $|\lambda|$.

Further Insights and Challenges

73. Show that the curvature of Viviani's curve, given by $\mathbf{r}(t) = \langle 1 + \cos t, \sin t, 2\sin(t/2)\rangle$, is

$$\kappa(t) = \frac{\sqrt{13 + 3\cos t}}{(3 + \cos t)^{3/2}}$$

SOLUTION We use the formula for curvature:

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{1}$$

Differentiating $\mathbf{r}(t)$ gives

$$\mathbf{r}'(t) = \left\langle -\sin t, \cos t, 2 \cdot \frac{1}{2}\cos\frac{t}{2}\right\rangle = \left\langle -\sin t, \cos t, \cos\frac{t}{2}\right\rangle$$

$$\mathbf{r}''(t) = \left\langle -\cos t, -\sin t, -\frac{1}{2}\sin\frac{t}{2}\right\rangle$$

We compute the cross product in (1):

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & \cos\frac{t}{2} \\ -\cos t & -\sin t & -\frac{1}{2}\sin\frac{t}{2} \end{vmatrix}$$

$$= \left(-\frac{1}{2}\cos t \sin\frac{t}{2} + \sin t\cos\frac{t}{2}\right)\mathbf{i} - \left(\frac{1}{2}\sin t\sin\frac{t}{2} + \cos t\cos\frac{t}{2}\right)\mathbf{j} + \mathbf{k}$$

We find the length of the cross product:

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2 = \left(-\frac{1}{2}\cos t\sin\frac{t}{2} + \sin t\cos\frac{t}{2}\right)^2 + \left(\frac{1}{2}\sin t\sin\frac{t}{2} + \cos t\cos\frac{t}{2}\right)^2 + 1$$

$$= \frac{1}{4}\sin^2\frac{t}{2}\left(\cos^2 t + \sin^2 t\right) + \cos^2\frac{t}{2}\left(\sin^2 t + \cos^2 t\right) + 1$$

$$= \frac{1}{4}\sin^2\frac{t}{2} + \cos^2\frac{t}{2} + 1$$

We use the identities $\sin^2\frac{t}{2} + \cos^2\frac{t}{2} = 1$ and $\cos^2\frac{t}{2} = \frac{1}{2} + \frac{1}{2}\cos t$ to write:

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2 = \frac{1}{4}\sin^2\frac{t}{2} + \cos^2\frac{t}{2} + 1 = \frac{1}{4}\left(\sin^2\frac{t}{2} + \cos^2\frac{t}{2}\right) + \frac{3}{4}\cos^2\frac{t}{2} + 1$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{1}{2} + \frac{1}{2}\cos t\right) + 1 = \frac{3}{8}\cos t + \frac{13}{8}$$

Hence:

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \frac{1}{\sqrt{8}}\sqrt{13 + 3\cos t} \tag{2}$$

We compute the length of $\mathbf{r}'(t)$:

$$\|\mathbf{r}'(t)\|^2 = (-\sin t)^2 + \cos^2 t + \cos^2\frac{t}{2} = 1 + \cos^2\frac{t}{2} = 1 + \left(\frac{1}{2} + \frac{1}{2}\cos t\right) = \frac{3}{2} + \frac{1}{2}\cos t$$

Hence,

$$\|\mathbf{r}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{3 + \cos t} \tag{3}$$

Substituting (2) and (3) in (1) gives:

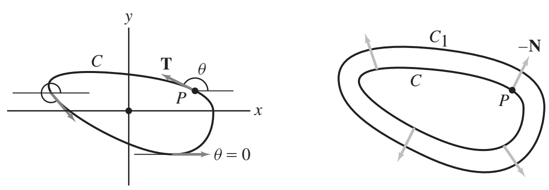
$$\kappa(t) = \frac{\frac{1}{\sqrt{8}}\sqrt{13 + 3\cos t}}{\left(\frac{1}{\sqrt{2}}\sqrt{3 + \cos t}\right)^3} = \frac{\frac{1}{\sqrt{8}}\sqrt{13 + 3\cos t}}{\frac{1}{2}\frac{1}{\sqrt{2}}(3 + \cos t)^{3/2}} = \frac{\sqrt{13 + 3\cos t}}{(3 + \cos t)^{3/2}}$$

74. Let $\mathbf{r}(s)$ be an arc length parametrization of a closed curve $\mathcal{C}$ of length L. We call $\mathcal{C}$ an **oval** if $d\theta/ds > 0$ (see Exercise 61). Observe that $-\mathbf{N}$ points to the *outside* of $\mathcal{C}$. For $k > 0$, the curve $\mathcal{C}_1$ defined by $\mathbf{r}_1(s) = \mathbf{r}(s) - k\mathbf{N}$ is called the expansion of $c(s)$ in the normal direction.

(a) Show that $\|\mathbf{r}_1'(s)\| = \|\mathbf{r}'(s)\| + k\kappa(s)$.

(b) As P moves around the oval counterclockwise, θ increases by 2π [Figure 21(A)]. Use this and a change of variables to prove that $\int_0^L \kappa(s)\,ds = 2\pi$.

(c) Show that $\mathcal{C}_1$ has length $L + 2\pi k$.



(A) An oval

(B) C_1 is the expansion of C in the normal direction.

FIGURE 21 As P moves around the oval, θ increases by 2π.

SOLUTION

(a) Since $\mathbf{r}_1(s) = \mathbf{r}(s) - k\mathbf{N}$ we have

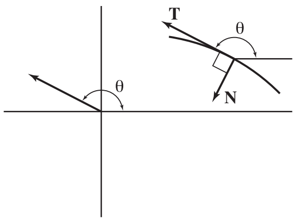
$$\mathbf{r}_1'(s) = \mathbf{r}'(s) - k\frac{d\mathbf{N}}{ds} \tag{1}$$

We compute $\frac{d\mathbf{N}}{ds}$ using the Chain Rule:

$$\frac{d\mathbf{N}}{ds} = \frac{d\mathbf{N}}{d\theta} \cdot \frac{d\theta}{ds} \tag{2}$$

By Exercise 61 and since $\mathcal{C}$ is oval we have:

$$\kappa(s) = \left|\frac{d\theta}{ds}\right| = \frac{d\theta}{ds} \tag{3}$$



Also, as illustrated in the figure, the following holds:

$$\mathbf{N} = \left\langle \cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right)\right\rangle = \langle -\sin\theta, \cos\theta\rangle$$

Hence:

$$\frac{d\mathbf{N}}{d\theta} = \langle -\cos\theta, -\sin\theta\rangle = -\langle\cos\theta, \sin\theta\rangle = -\mathbf{T} \tag{4}$$

Substituting (3) and (4) in (2) yields:

$$\frac{d\mathbf{N}}{ds} = -\kappa(s)\mathbf{T}(s)$$

Substituting in (1) we obtain:

$$\mathbf{r}_1'(s) = \mathbf{r}'(s) + k\kappa(s)\mathbf{T}(s)$$

In the arc length parametrization, $\mathbf{T}(s) = \mathbf{r}'(s)$, therefore:

$$\mathbf{r}_1'(s) = \mathbf{r}'(s) + k\kappa(s)\mathbf{r}'(s) = \mathbf{r}'(s)\,(1 + k\kappa(s))$$

Computing the length and using $\|\mathbf{r}'(s)\| = 1$ we obtain:

$$\|\mathbf{r}_1'(s)\| = \|\mathbf{r}'(s)\|\,(1 + k\kappa(s)) = \|\mathbf{r}'(s)\| + \|\mathbf{r}'(s)\| \cdot k\kappa(s) = \|\mathbf{r}'(s)\| + k\kappa(s)$$

(b) In Exercise 61 we showed that:

$$\kappa(s) = \left|\frac{d\theta}{ds}\right|$$

Since $\frac{d\theta}{ds} > 0$ we have $\kappa(s) = \frac{d\theta}{ds}$. As P moves around the oval, θ increases by 2π, hence $\theta\,(s = L) - \theta\,(s = 0) = 2\pi$. Using these considerations we get:

$$\int_0^L \kappa(s)\,ds = \int_{\theta(0)}^{\theta(L)} \frac{d\theta}{ds}\,ds = \int_{\theta(0)}^{\theta(L)} d\theta = \theta(L) - \theta(0) = 2\pi.$$

(c) We use the Arc Length Formula and the equality in part (a) to write the length L_1 of C_1 as the following integral:

$$L_1 = \int_0^L \|\mathbf{r}_1'(s)\|\,ds = \int_0^L \|\mathbf{r}'(s)\|\,ds + k\int_0^L \kappa(s)\,ds$$

By the Arc Length Formula, the first integral is the length L of $\mathcal{C}$. The second integral was computed in part (b). Therefore we get:

$$L_1 = L + k \cdot 2\pi = L + 2\pi k.$$

In Exercises 75–82, let **B** *denote the* **binormal vector** *at a point on a space curve $\mathcal{C}$, defined by* $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

75. Show that **B** is a unit vector.

SOLUTION **T** and **N** are orthogonal unit vectors, therefore the length of their cross product is:

$$\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\|\|\mathbf{N}\| \sin \frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1$$

Therefore **B** is a unit vector.

76. Follow steps (a)–(c) to prove that there is a number τ (lowercase Greek "tau") called the **torsion** such that

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \tag{14}$$

(a) Show that $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$ and conclude that $d\mathbf{B}/ds$ is orthogonal to **T**.
(b) Differentiate $\mathbf{B} \cdot \mathbf{B} = 1$ with respect to s to show that $d\mathbf{B}/ds$ is orthogonal to **B**.
(c) Conclude that $d\mathbf{B}/ds$ is a multiple of **N**.

SOLUTION

(a) Using the Product Rule for cross product we have:

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

N is a unit vector in the direction of $\frac{d\mathbf{T}}{ds}$, hence $\frac{d\mathbf{T}}{ds} \times \mathbf{N} = \mathbf{0}$, so we obtain:

$$\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

By properties of cross products we conclude that $\frac{d\mathbf{B}}{ds}$ is orthogonal to **T**.

(b) We differentiate $\mathbf{B} \cdot \mathbf{B} = 1$ using the Product Rule for dot products:

$$\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} + \frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0$$

$$2\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0 \quad \Rightarrow \quad \mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0$$

Since the dot product of **B** and $\frac{d\mathbf{B}}{ds}$ is zero, the two vectors are orthogonal.

(c) In parts (a) and (b) we showed that $\frac{d\mathbf{B}}{ds}$ is orthogonal to **B** and **T**. It follows that $\frac{d\mathbf{B}}{ds}$ is parallel to any other vector that is orthogonal to **B** and **T**. We show that **N** is such a vector.

Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, the vectors **N** and **B** are orthogonal. The unit normal **N** is also orthogonal to the unit tangent **T**. We conclude that $\frac{d\mathbf{B}}{ds}$ and **N** are parallel, hence there exists a number $(-\tau)$ such that:

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}.$$

77. Show that if $\mathcal{C}$ is contained in a plane $\mathcal{P}$, then **B** is a unit vector normal to $\mathcal{P}$. Conclude that $\tau = 0$ for a plane curve.

SOLUTION If $\mathcal{C}$ is contained in a plane $\mathcal{P}$, then the unit normal **N** and the unit tangent **T** are in $\mathcal{P}$. The cross product $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is orthogonal to **T** and **N** which are in the plane, hence **B** is normal to the plane. Thus, **B** is a unit vector normal to the plane. There are only two different unit normal vectors to a plane, one pointing "up" and the other pointing "down". Thus, we can assume (due to continuity) that **B** is a constant vector, therefore

$$\frac{d\mathbf{B}}{ds} = 0 \quad \text{or} \quad \tau = 0.$$

78. Torsion means "twisting." Is this an appropriate name for τ? Explain by interpreting τ geometrically.

SOLUTION **B** is the unit normal to the osculating plane at a point P on the curve. As P moves along the curve, the unit normal **B** is changing by $\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$. Geometrically the osculating plane is "twisted" and τ is a measure for this twisting.

79. Use the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

to prove

$$\mathbf{N} \times \mathbf{B} = \mathbf{T}, \qquad \mathbf{B} \times \mathbf{T} = \mathbf{N} \tag{15}$$

SOLUTION We use the given equality and the definition $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ to write:

$$\mathbf{N} \times \mathbf{B} = \mathbf{N} \times (\mathbf{T} \times \mathbf{N}) = (\mathbf{N} \cdot \mathbf{N})\,\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\,\mathbf{N} \tag{1}$$

The unit normal $\mathbf{N}$ and the unit tangent $\mathbf{T}$ are orthogonal unit vectors, hence $\mathbf{N} \cdot \mathbf{N} = \|\mathbf{N}\|^2 = 1$ and $\mathbf{N} \cdot \mathbf{T} = 0$. Therefore, (1) gives:

$$\mathbf{N} \times \mathbf{B} = 1 \cdot \mathbf{T} - 0\mathbf{N} = \mathbf{T}$$

To prove the second equality, we substitute $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and then use the given equality. We obtain:

$$\mathbf{B} \times \mathbf{T} = \mathbf{B} \times (\mathbf{N} \times \mathbf{B}) = (\mathbf{B} \cdot \mathbf{B})\,\mathbf{N} - (\mathbf{B} \cdot \mathbf{N})\,\mathbf{B} \tag{2}$$

Now, $\mathbf{B}$ is a unit vector, hence $\mathbf{B} \cdot \mathbf{B} = \|\mathbf{B}\|^2 = 1$. Also, since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, $\mathbf{B}$ is orthogonal to $\mathbf{N}$ which implies that $\mathbf{B} \cdot \mathbf{N} = 0$. Substituting in (2) we get:

$$\mathbf{B} \times \mathbf{T} = 1\mathbf{N} - 0\mathbf{B} = \mathbf{N}.$$

80. Follow steps (a)–(b) to prove

$$\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B} \tag{16}$$

(a) Show that $d\mathbf{N}/ds$ is orthogonal to $\mathbf{N}$. Conclude that $d\mathbf{N}/ds$ lies in the plane spanned by $\mathbf{T}$ and $\mathbf{B}$, and hence, $d\mathbf{N}/ds = a\mathbf{T} + b\mathbf{B}$ for some scalars a, b.

(b) Use $\mathbf{N} \cdot \mathbf{T} = 0$ to show that $\mathbf{T} \cdot \dfrac{d\mathbf{N}}{ds} = -\mathbf{N} \cdot \dfrac{d\mathbf{T}}{ds}$ and compute a. Compute b similarly. Equations (14) and (16) together with $d\mathbf{T}/dt = \kappa \mathbf{N}$ are called the **Frenet formulas** and were discovered by the French geometer Jean Frenet (1816–1900).

SOLUTION

(a) We first show that $\frac{d\mathbf{N}}{ds}$ is orthogonal to $\mathbf{N}$. Earlier we showed that $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$ and $\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$, hence:

$$-\tau \mathbf{N} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

By properties of the cross product, this equality implies that $\frac{d\mathbf{N}}{ds}$ is orthogonal to $-\tau\mathbf{N}$, hence it is orthogonal to $\mathbf{N}$. Now, $\mathbf{N}$ is orthogonal to $\mathbf{T}$ and $\mathbf{B}$, hence $\mathbf{N}$ is normal to the plane spanned by $\mathbf{T}$ and $\mathbf{B}$. Therefore, since $\mathbf{N}$ is orthogonal to $\frac{d\mathbf{N}}{ds}$, this last vector lies in the plane spanned by $\mathbf{T}$ and $\mathbf{B}$, that is, there exist scalars a and b such that:

$$\frac{d\mathbf{N}}{ds} = a\mathbf{T} + b\mathbf{B}$$

(b) By the orthogonality of $\mathbf{N}$ and $\mathbf{T}$ we have:

$$\mathbf{N} \cdot \mathbf{T} = 0$$

Differentiating this equality, using the product rule for dot product we get:

$$\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{N}}{ds} \cdot \mathbf{T} = 0 \quad \Rightarrow \quad \mathbf{T} \cdot \frac{d\mathbf{N}}{ds} = -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds}$$

To compute a, we substitute $\frac{d\mathbf{N}}{ds} = a\mathbf{T} + b\mathbf{B}$ and use $\mathbf{T} \cdot \mathbf{T} = \|\mathbf{T}\|^2 = 1$ and $\mathbf{T} \cdot \mathbf{B} = 0$. This gives:

$$\begin{aligned} \mathbf{T} \cdot (a\mathbf{T} + b\mathbf{B}) &= -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} \\ a\mathbf{T} \cdot \mathbf{T} + b\mathbf{T} \cdot \mathbf{B} &= -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} \\ a \cdot 1 + b \cdot 0 &= -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} \quad \Rightarrow \quad a = -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} \end{aligned} \tag{1}$$

To find b we differentiate the equality $\mathbf{N} \cdot \mathbf{B} = 0$ (notice that by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ follows the orthogonality of $\mathbf{N}$ and $\mathbf{B}$). We get:

$$\mathbf{N} \cdot \frac{d\mathbf{B}}{ds} + \frac{d\mathbf{N}}{ds} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \frac{d\mathbf{N}}{ds} \cdot \mathbf{B} = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}$$

We now substitute $\frac{d\mathbf{N}}{ds} = a\mathbf{T} + b\mathbf{B}$ and we use $\mathbf{B} \cdot \mathbf{B} = \|\mathbf{B}\|^2 = 1$ and $\mathbf{T} \cdot \mathbf{B} = 0$ to obtain:

$$(a\mathbf{T} + b\mathbf{B}) \cdot \mathbf{B} = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}$$

$$a\mathbf{T}\cdot\mathbf{B}+b\mathbf{B}\cdot\mathbf{B}=-\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}$$

$$a\cdot 0+b\cdot 1=-\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}\quad\Rightarrow\quad b=-\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}$$

Since $\frac{d\mathbf{B}}{ds}=-\tau\mathbf{N}$ we may write:

$$b=-\mathbf{N}\cdot(-\tau\mathbf{N})=\tau\mathbf{N}\cdot\mathbf{N}=\tau\|\mathbf{N}\|^2=\tau \tag{2}$$

Also for the arc length parametrization $\frac{d\mathbf{T}}{ds}=\kappa(s)\mathbf{N}$, hence by (1):

$$a=-\mathbf{N}\cdot\kappa(s)\mathbf{N}=-\kappa(s)\mathbf{N}\cdot\mathbf{N}=-\kappa(s)\|\mathbf{N}\|^2=-\kappa(s) \tag{3}$$

We combine (2), (3), and part (a) to conclude:

$$\frac{d\mathbf{N}}{ds}=-\kappa\mathbf{T}+\tau\mathbf{B}.$$

81. Show that $\mathbf{r}'\times\mathbf{r}''$ is a multiple of $\mathbf{B}$. Conclude that

$$\mathbf{B}=\frac{\mathbf{r}'\times\mathbf{r}''}{\|\mathbf{r}'\times\mathbf{r}''\|} \qquad \boxed{17}$$

SOLUTION By the definition of the binormal vector, $\mathbf{B}=\mathbf{T}\times\mathbf{N}$. We denote $a(t)=\frac{1}{\|\mathbf{r}'(t)\|}$ and write:

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=a(t)\mathbf{r}'(t) \tag{1}$$

We differentiate $\mathbf{T}(t)$ using the Product Rule:

$$\mathbf{T}'(t)=a(t)\mathbf{r}''(t)+a'(t)\mathbf{r}'(t)$$

We denote $b(t)=\|\mathbf{T}'(t)\|$ and obtain:

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}=\frac{a(t)}{b(t)}\mathbf{r}''(t)+\frac{a'(t)}{b(t)}\mathbf{r}'(t)$$

For $c_1=\frac{a(t)}{b(t)}$ and $c_2=\frac{a'(t)}{b(t)}$ we have:

$$\mathbf{N}(t)=c_1(t)\mathbf{r}''(t)+c_2(t)\mathbf{r}'(t) \tag{2}$$

We now find $\mathbf{B}$ as the cross product of $\mathbf{T}(t)$ in (1) and $\mathbf{N}(t)$ in (2). This gives:

$$\begin{aligned}\mathbf{B}(t)&=a(t)\mathbf{r}'(t)\times\big(c_1(t)\mathbf{r}''(t)+c_2(t)\mathbf{r}'(t)\big)=a(t)c_1(t)\mathbf{r}'(t)\times\mathbf{r}''(t)+a(t)c_2(t)\mathbf{r}'(t)\times\mathbf{r}'(t)\\&=a(t)c_1(t)\mathbf{r}'(t)\times\mathbf{r}''(t)+\mathbf{0}=a(t)c_1(t)\mathbf{r}'(t)\times\mathbf{r}''(t)\end{aligned}$$

We see that $\mathbf{B}$ is parallel to $\mathbf{r}'\times\mathbf{r}''$. Since $\mathbf{B}$ is a unit vector we have:

$$\mathbf{B}=\frac{\mathbf{r}'\times\mathbf{r}''}{\|\mathbf{r}'\times\mathbf{r}''\|}.$$

82. The vector $\mathbf{N}$ can be computed using $\mathbf{N}=\mathbf{B}\times\mathbf{T}$ [Eq. (15)] with $\mathbf{B}$, as in Eq. (17). Use this method to find $\mathbf{N}$ in the following cases:

(a) $\mathbf{r}(t)=\left\langle\cos t,t,t^2\right\rangle$ at $t=0$

(b) $\mathbf{r}(t)=\left\langle t^2,t^{-1},t\right\rangle$ at $t=1$

SOLUTION

(a) We first compute the vector $\mathbf{B}$ using Eq. (17):

$$\mathbf{B}=\frac{\mathbf{r}'\times\mathbf{r}''}{\|\mathbf{r}'\times\mathbf{r}''\|} \tag{1}$$

Differentiating $\mathbf{r}(t)=\left\langle\cos t,t,t^2\right\rangle$ gives

$$\begin{matrix}\mathbf{r}'(t)=\langle-\sin t,1,2t\rangle\\ \mathbf{r}''(t)=\langle-\cos t,0,2\rangle\end{matrix}\quad\Rightarrow\quad\begin{matrix}\mathbf{r}'(0)=\langle 0,1,0\rangle\\ \mathbf{r}''(0)=\langle-1,0,2\rangle\end{matrix}$$

We compute the cross product:

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \mathbf{j} \times (-\mathbf{i} + 2\mathbf{k}) = -\mathbf{j} \times \mathbf{i} + 2\mathbf{j} \times \mathbf{k} = \mathbf{k} + 2\mathbf{i} = \langle 2, 0, 1 \rangle$$
$$\|\mathbf{r}'(0) \times \mathbf{r}''(0)\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$$

Substituting in (1) we obtain:

$$\mathbf{B}(0) = \frac{\langle 2, 0, 1 \rangle}{\sqrt{5}} = \frac{1}{\sqrt{5}} \langle 2, 0, 1 \rangle$$

We now compute $\mathbf{T}(0)$:

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\langle 0, 1, 0 \rangle}{\| \langle 0, 1, 0 \rangle \|} = \langle 0, 1, 0 \rangle$$

Finally we find $\mathbf{N} = \mathbf{B} \times \mathbf{T}$:

$$\mathbf{N}(0) = \frac{1}{\sqrt{5}} \langle 2, 0, 1 \rangle \times \langle 0, 1, 0 \rangle = \frac{1}{\sqrt{5}} (2\mathbf{i} + \mathbf{k}) \times \mathbf{j} = \frac{1}{\sqrt{5}} (2\mathbf{i} \times \mathbf{j} + \mathbf{k} \times \mathbf{j}) = \frac{1}{\sqrt{5}} (2\mathbf{k} - \mathbf{i}) = \frac{1}{\sqrt{5}} \langle -1, 0, 2 \rangle$$

(b) Differentiating $\mathbf{r}(t) = \langle t^2, t^{-1}, t \rangle$ gives

$$\begin{array}{l} \mathbf{r}'(t) = \langle 2t, -t^{-2}, 1 \rangle \\ \mathbf{r}''(t) = \langle 2, 2t^{-3}, 0 \rangle \end{array} \Rightarrow \begin{array}{l} \mathbf{r}'(1) = \langle 2, -1, 1 \rangle \\ \mathbf{r}''(1) = \langle 2, 2, 0 \rangle \end{array}$$

We compute the cross product:

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} = \langle -2, 2, 6 \rangle$$

$$\|\mathbf{r}'(1) \times \mathbf{r}''(1)\| = \sqrt{(-2)^2 + 2^2 + 6^2} = \sqrt{44} = 2\sqrt{11}$$

Substituting in (1) gives:

$$\mathbf{B}(1) = \frac{\langle -2, 2, 6 \rangle}{2\sqrt{11}} = \frac{1}{\sqrt{11}} \langle -1, 1, 3 \rangle$$

We now find $\mathbf{T}(1)$:

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 2, -1, 1 \rangle}{\sqrt{4 + 1 + 1}} = \frac{1}{\sqrt{6}} \langle 2, -1, 1 \rangle$$

Finally we find $\mathbf{N}(1)$ by computing the following cross product:

$$\mathbf{N}(1) = \mathbf{B}(1) \times \mathbf{T}(1) = \frac{1}{\sqrt{11}} \langle -1, 1, 3 \rangle \times \frac{1}{\sqrt{6}} \langle 2, -1, 1 \rangle = \frac{1}{\sqrt{66}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= \frac{1}{\sqrt{66}} \left\{ \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{k} \right\} = \frac{1}{\sqrt{66}} (4\mathbf{i} + 7\mathbf{j} - \mathbf{k}) = \frac{1}{\sqrt{66}} \langle 4, 7, -1 \rangle$$

13.5 Motion in Three-Space (LT Section 14.5)

Preliminary Questions

1. If a particle travels with constant speed, must its acceleration vector be zero? Explain.

SOLUTION If the speed of the particle is constant, the tangential component, $a_T(t) = v'(t)$, of the acceleration is zero. However, the normal component, $a_N(t) = \kappa(t)v(t)^2$ is not necessarily zero, since the particle may change its direction.

2. For a particle in uniform circular motion around a circle, which of the vectors $\mathbf{v}(t)$ or $\mathbf{a}(t)$ always points toward the center of the circle?

SOLUTION For a particle in uniform circular motion around a circle, the acceleration vector $\mathbf{a}(t)$ points towards the center of the circle, whereas $\mathbf{v}(t)$ is tangent to the circle.

3. Two objects travel to the right along the parabola $y = x^2$ with nonzero speed. Which of the following statements must be true?

(a) Their velocity vectors point in the same direction.
(b) Their velocity vectors have the same length.
(c) Their acceleration vectors point in the same direction.

SOLUTION

(a) The velocity vector points in the direction of motion, hence the velocities of the two objects point in the same direction.

(b) The length of the velocity vector is the speed. Since the speeds are not necessarily equal, the velocity vectors may have different lengths.

(c) The acceleration is determined by the tangential component $v'(t)$ and the normal component $\kappa(t)v(t)^2$. Since v and v' may be different for the two objects, the acceleration vectors may have different directions.

4. Use the decomposition of acceleration into tangential and normal components to explain the following statement: If the speed is constant, then the acceleration and velocity vectors are orthogonal.

SOLUTION If the speed is constant, $v'(t) = 0$. Therefore, the acceleration vector has only the normal component:

$$\mathbf{a}(t) = a_N(t)\mathbf{N}(t)$$

The velocity vector always points in the direction of motion. Since the vector $\mathbf{N}(t)$ is orthogonal to the direction of motion, the vectors $\mathbf{a}(t)$ and $\mathbf{v}(t)$ are orthogonal.

5. If a particle travels along a straight line, then the acceleration and velocity vectors are (choose the correct description):

(a) Orthogonal **(b)** Parallel

SOLUTION Since a line has zero curvature, the normal component of the acceleration is zero, hence $\mathbf{a}(t)$ has only the tangential component. The velocity vector is always in the direction of motion, hence the acceleration and the velocity vectors are parallel to the line. We conclude that (b) is the correct statement.

6. What is the length of the acceleration vector of a particle traveling around a circle of radius 2 cm with constant velocity 4 cm/s?

SOLUTION The acceleration vector is given by the following decomposition:

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \kappa(t)v(t)^2\mathbf{N}(t) \tag{1}$$

In our case $v(t) = 4$ is constant hence $v'(t) = 0$. In addition, the curvature of a circle of radius 2 is $\kappa(t) = \frac{1}{2}$. Substituting $v(t) = 4$, $v'(t) = 0$ and $\kappa(t) = \frac{1}{2}$ in (1) gives:

$$\mathbf{a}(t) = \frac{1}{2} \cdot 4^2 N(t) = 8N(t)$$

The length of the acceleration vector is, thus,

$$\|\mathbf{a}(t)\| = 8 \text{ cm/s}^2$$

7. Two cars are racing around a circular track. If, at a certain moment, both of their speedometers read 110 mph. then the two cars have the same (choose one):

(a) $a_{\mathbf{T}}$ **(b)** $a_{\mathbf{N}}$

SOLUTION The tangential acceleration a_T and the normal acceleration a_N are the following values:

$$a_T(t) = v'(t); \quad a_N(t) = \kappa(t)v(t)^2$$

At the moment where both speedometers read 110 mph, the speeds of the two cars are $v = 110$ mph. Since the track is circular, the curvature $\kappa(t)$ is constant, hence the normal accelerations of the two cars are equal at this moment. Statement (b) is correct.

Exercises

1. Use the table below to calculate the difference quotients $\dfrac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}$ for $h = -0.2, -0.1, 0.1, 0.2$. Then estimate the velocity and speed at $t = 1$.

$\mathbf{r}(0.8)$	$\langle 1.557, 2.459, -1.970 \rangle$
$\mathbf{r}(0.9)$	$\langle 1.559, 2.634, -1.740 \rangle$
$\mathbf{r}(1)$	$\langle 1.540, 2.841, -1.443 \rangle$
$\mathbf{r}(1.1)$	$\langle 1.499, 3.078, -1.035 \rangle$
$\mathbf{r}(1.2)$	$\langle 1.435, 3.342, -0.428 \rangle$

SOLUTION

$$(h = -0.2)$$
$$\frac{\mathbf{r}(1-0.2)-\mathbf{r}(1)}{-0.2} = \frac{\mathbf{r}(0.8)-\mathbf{r}(1)}{-0.2} = \frac{\langle 1.557, 2.459, -1.970\rangle - \langle 1.540, 2.841, -1.443\rangle}{-0.2}$$
$$= \frac{\langle 0.017, -0.382, -0.527\rangle}{-0.2} = \langle -0.085, 1.91, 2.635\rangle$$

$$(h = -0.1)$$
$$\frac{\mathbf{r}(1-0.1)-\mathbf{r}(1)}{-0.1} = \frac{\mathbf{r}(0.9)-\mathbf{r}(1)}{-0.1} = \frac{\langle 1.559, 2.634, -1.740\rangle - \langle 1.540, 2.841, -1.443\rangle}{-0.1}$$
$$= \frac{\langle 0.019, -0.207, -0.297\rangle}{-0.1} = \langle -0.19, 2.07, 2.97\rangle$$

$$(h = 0.1)$$
$$\frac{\mathbf{r}(1+0.1)-\mathbf{r}(1)}{0.1} = \frac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1} = \frac{\langle 1.499, 3.078, -1.035\rangle - \langle 1.540, 2.841, -1.443\rangle}{0.1}$$
$$= \frac{\langle -0.041, 0.237, 0.408\rangle}{0.1} = \langle -0.41, 2.37, 4.08\rangle$$

$$(h = 0.2)$$
$$\frac{\mathbf{r}(1+0.2)-\mathbf{r}(1)}{0.2} = \frac{\mathbf{r}(1.2)-\mathbf{r}(1)}{0.2} = \frac{\langle 1.435, 3.342, -0.428\rangle - \langle 1.540, 2.841, -1.443\rangle}{0.2}$$
$$= \frac{\langle -0.105, 0.501, 1.015\rangle}{0.2} = \langle -0.525, 2.505, 5.075\rangle$$

The velocity vector is defined by:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

We may estimate the velocity at $t = 1$ by:

$$\mathbf{v}(1) \approx \langle -0.3, 2.2, 3.5\rangle$$

and the speed by:

$$v(1) = \|\mathbf{v}(1)\| \approx \sqrt{0.3^2 + 2.2^2 + 3.5^2} \cong 4.1$$

2. Draw the vectors $\mathbf{r}(2+h) - \mathbf{r}(2)$ and $\dfrac{\mathbf{r}(2+h)-\mathbf{r}(2)}{h}$ for $h = 0.5$ for the path in Figure 10. Draw $\mathbf{v}(2)$ (using a rough estimate for its length).

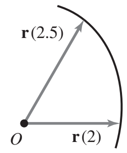
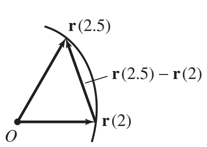
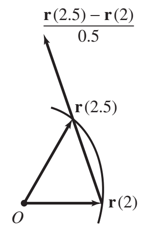


FIGURE 10

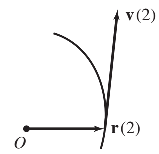
SOLUTION The difference $\mathbf{r}(2+h) - \mathbf{r}(2) = \mathbf{r}(2.5) - \mathbf{r}(2)$ is the following vector:



The vector $\frac{\mathbf{r}(2.5)-\mathbf{r}(2)}{0.5}$ points in the direction of $\mathbf{r}(2.5) - \mathbf{r}(2)$, and its length is twice the length of this vector, as shown in the following figure:



By definition, $\mathbf{v}(2) = \lim_{h\to 0} \dfrac{\mathbf{r}(2+h)-\mathbf{r}(2)}{h}$. This vector is tangent to the path at the endpoint of $\mathbf{r}(2)$. The vector $\mathbf{v}(2)$ is shown in the following figure:



In Exercises 3–6, calculate the velocity and acceleration vectors and the speed at the time indicated.

3. $\mathbf{r}(t) = \left\langle t^3, 1-t, 4t^2\right\rangle, \quad t = 1$

SOLUTION In this case $\mathbf{r}(t) = \left\langle t^3, 1-t, 4t^2\right\rangle$ hence:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 3t^2, -1, 8t\right\rangle \quad \Rightarrow \quad \mathbf{v}(1) = \langle 3, -1, 8\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 6t, 0, 8\rangle \quad \Rightarrow \quad \mathbf{a}(1) = \langle 6, 0, 8\rangle$$

The speed is the magnitude of the velocity vector, that is,

$$v(1) = \|\mathbf{v}(1)\| = \sqrt{3^2 + (-1)^2 + 8^2} = \sqrt{74}$$

4. $\mathbf{r}(t) = e^t\mathbf{j} - \cos(2t)\mathbf{k}, \quad t = 0$

SOLUTION Since $\mathbf{r}(t) = e^t\mathbf{j} - \cos(2t)\mathbf{k}$ we have:

$$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\mathbf{j} + 2\sin(2t)\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(0) = e^0\mathbf{j} + (2\sin 0)\mathbf{k} = \mathbf{j}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = e^t\mathbf{j} + 4\cos(2t)\mathbf{k} \quad \Rightarrow \quad \mathbf{a}(0) = e^0\mathbf{j} + (4\cos 0)\mathbf{k} = \mathbf{j} + 4\mathbf{k}$$

The speed is the magnitude of the velocity vectors, that is,

$$v(0) = \|\mathbf{v}(0)\| = \|\mathbf{j}\| = 1$$

5. $\mathbf{r}(\theta) = \langle \sin\theta, \cos\theta, \cos 3\theta\rangle, \quad \theta = \frac{\pi}{3}$

SOLUTION Differentiating $\mathbf{r}(\theta) = \langle \sin\theta, \cos\theta, \cos 3\theta\rangle$ gives:

$$\begin{aligned}
\mathbf{v}(\theta) &= \mathbf{r}'(\theta) = \langle \cos\theta, -\sin\theta, -3\sin 3\theta\rangle \\
&\Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = \left\langle \cos\frac{\pi}{3}, -\sin\frac{\pi}{3}, -3\sin\pi\right\rangle = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right\rangle \\
\mathbf{a}(\theta) &= \mathbf{r}''(\theta) = \langle -\sin\theta, -\cos\theta, -9\cos 3\theta\rangle \\
&\Rightarrow \mathbf{a}\left(\frac{\pi}{3}\right) = \left\langle -\sin\frac{\pi}{3}, -\cos\frac{\pi}{3}, -9\cos\pi\right\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 9\right\rangle
\end{aligned}$$

The speed is the magnitude of the velocity vector, that is:

$$v\left(\frac{\pi}{3}\right) = \left\|\mathbf{v}\left(\frac{\pi}{3}\right)\right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + 0^2} = 1$$

6. $\mathbf{r}(s) = \left\langle \dfrac{1}{1+s^2}, \dfrac{s}{1+s^2} \right\rangle, \quad s = 2$

SOLUTION The velocity and acceleration vectors are the first and second derivatives of $\mathbf{r}(s) = \left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2} \right\rangle$. We compute these vectors:

$$\mathbf{v}(s) = \mathbf{r}'(s) = \left\langle \frac{-2s}{(1+s^2)^2}, \frac{1+s^2-s\cdot 2s}{(1+s^2)^2} \right\rangle = \left\langle -\frac{2s}{(1+s^2)^2}, \frac{1-s^2}{(1+s^2)^2} \right\rangle$$

$$\mathbf{a}(s) = \mathbf{r}''(s) = \left\langle \frac{-2(1+s^2)^2 + 2s\cdot 2(1+s^2)\cdot 2s}{(1+s^2)^4}, \frac{-2s(1+s^2)^2 - (1-s^2)\cdot 2(1+s^2)\cdot 2s}{(1+s^2)^4} \right\rangle$$

$$= \left\langle \frac{6s^2-2}{(1+s^2)^3}, \frac{s(2s^2-6)}{(1+s^2)^3} \right\rangle$$

At the point $s = 2$ we obtain:

$$\mathbf{v}(2) = \left\langle -\frac{4}{(1+2^2)^2}, \frac{1-4}{(1+2^2)^2} \right\rangle = -\frac{1}{25}\langle 4, 3\rangle$$

$$\mathbf{a}(2) = \left\langle \frac{6\cdot 4-2}{(1+2^2)^3}, \frac{2(2\cdot 4-6)}{(1+2^2)^3} \right\rangle = \frac{2}{125}\langle 11, 2\rangle$$

The speed is the magnitude of the velocity vector:

$$v(2) = \frac{1}{25}\sqrt{4^2+3^2} = \frac{1}{5}$$

7. Find $\mathbf{a}(t)$ for a particle moving around a circle of radius 8 cm at a constant speed of $v = 4$ cm/s (see Example 4). Draw the path and acceleration vector at $t = \frac{\pi}{4}$.

SOLUTION The position vector is:

$$\mathbf{r}(t) = 8\langle \cos\omega t, \sin\omega t\rangle$$

Hence,

$$\mathbf{v}(t) = \mathbf{r}'(t) = 8\langle -\omega\sin\omega t, \omega\cos\omega t\rangle = 8\omega\langle -\sin\omega t, \cos\omega t\rangle \tag{1}$$

We are given that the speed of the particle is $v = 4$ cm/s. The speed is the magnitude of the velocity vector, hence:

$$v = 8\omega\sqrt{(-\sin\omega \mathrm{t})^2 + \cos^2\omega \mathrm{t}} = 8\omega = 4 \quad\Rightarrow\quad \omega = \frac{1}{2}\ \text{rad/s}$$

Substituting in (2) we get:

$$\mathbf{v}(t) = 4\left\langle -\sin\frac{t}{2}, \cos\frac{t}{2} \right\rangle$$

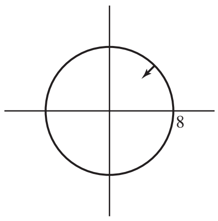
We now find $\mathbf{a}(t)$ by differentiating the velocity vector. This gives

$$\mathbf{a}(t) = \mathbf{v}'(t) = 4\left\langle -\frac{1}{2}\cos\frac{t}{2}, -\frac{1}{2}\sin\frac{t}{2} \right\rangle = -2\left\langle \cos\frac{t}{2}, \sin\frac{t}{2} \right\rangle$$

The path of the particle is $\mathbf{r}(t) = 8\left\langle \cos\frac{t}{2}, \sin\frac{t}{2}\right\rangle$ and the acceleration vector at $t = \frac{\pi}{4}$ is:

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -2\left\langle \cos\frac{\pi}{8}, \sin\frac{\pi}{8} \right\rangle \approx \langle -1.85, -0.77\rangle$$

The path $\mathbf{r}(t)$ and the acceleration vector at $t = \frac{\pi}{4}$ are shown in the following figure:



$\mathbf{r}(t) = 8\left\langle \cos\frac{t}{2}, \sin\frac{t}{2}\right\rangle$

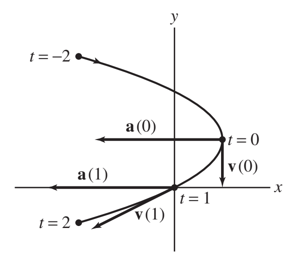
8. Sketch the path $\mathbf{r}(t) = \langle 1 - t^2, 1 - t\rangle$ for $-2 \le t \le 2$, indicating the direction of motion. Draw the velocity and acceleration vectors at $t = 0$ and $t = 1$.

SOLUTION We compute the velocity and acceleration vectors at $t = 0$ and $t = 1$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2t, -1\rangle \quad \Rightarrow \quad \begin{array}{l} \mathbf{v}(0) = \langle 0, -1\rangle \\ \mathbf{v}(1) = \langle -2, -1\rangle \end{array}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -2, 0\rangle \quad \Rightarrow \quad \mathbf{a}(0) = \mathbf{a}(1) = \langle -2, 0\rangle$$

Below is a sketch of the path and the velocity and acceleration vectors at $t = 0$ and $t = 1$:



$\mathbf{r}(t) = \langle 1 - t^2, 1 - t\rangle, -2 \le t \le 2$

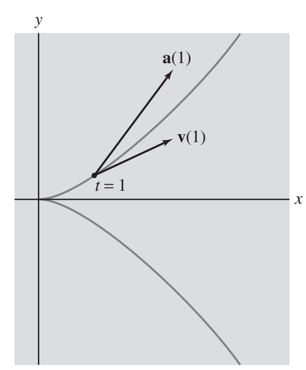
9. Sketch the path $\mathbf{r}(t) = \langle t^2, t^3\rangle$ together with the velocity and acceleration vectors at $t = 1$.

SOLUTION We compute the velocity and acceleration vectors at $t = 1$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2\rangle \quad \Rightarrow \quad \mathbf{v}(1) = \langle 2, 3\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t\rangle \quad \Rightarrow \quad \mathbf{a}(1) = \langle 2, 6\rangle$$

The following figure shows the path $\mathbf{r}(t) = \langle t^2, t^3\rangle$ and the vectors $\mathbf{v}(1)$ and $\mathbf{a}(1)$:



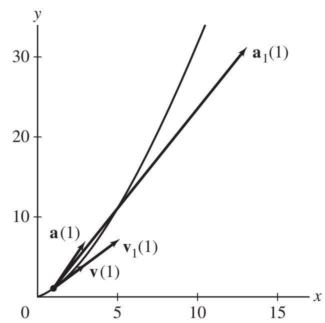
$\mathbf{r}(t) = (t^2, t^3)$

10. The paths $\mathbf{r}(t) = \langle t^2, t^3\rangle$ and $\mathbf{r}_1(t) = \langle t^4, t^6\rangle$ trace the same curve, and $\mathbf{r}_1(1) = \mathbf{r}(1)$. Do you expect either the velocity vectors or the acceleration vectors of these paths at $t = 1$ to point in the same direction? Compute these vectors and draw them on a single plot of the curve.

SOLUTION The paths $\mathbf{r}_1(t) = \langle t^4, t^6\rangle$ and $\mathbf{r}(t) = \langle t^2, t^3\rangle$ trace the same curve and $\mathbf{r}_1(1) = \mathbf{r}(1)$. However, the velocity and acceleration vectors determined by the two parametrizations may differ at the same point $*$. Notice that a parametrization of a curve includes information on the velocity and acceleration of the particle, hence different parametrizations correspond to different velocity and acceleration vectors. We compute these vectors at $t = 1$ for each parametrization:

$$\begin{array}{ll}
\mathbf{r}_1(t) = \langle t^4, t^6\rangle & \mathbf{r}(t) = \langle t^2, t^3\rangle \\
\mathbf{v}_1(t) = \mathbf{r}_1'(t) = \langle 4t^3, 6t^5\rangle & \mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2\rangle \\
\mathbf{a}_1(1) = \mathbf{r}_1''(t) = \langle 12t^2, 30t^4\rangle & \mathbf{a}(1) = \mathbf{r}''(t) = \langle 2, 6t\rangle \\
\text{Hence,} & \text{Hence,} \\
\mathbf{v}_1(1) = \langle 4, 6\rangle, \mathbf{a}_1(1) = \langle 12, 30\rangle & \mathbf{v}(1) = \langle 2, 3\rangle, \mathbf{a}(1) = \langle 2, 6\rangle
\end{array}$$

We see that the vectors in the two parametrizations, are different. We draw the vectors on the following plot:



The velocity vectors should point in the same direction but may be of different size. The acceleration vectors may point to different directions.

In Exercises 11–14, find $\mathbf{v}(t)$ *given* $\mathbf{a}(t)$ *and the initial velocity.*

11. $\mathbf{a}(t) = \langle t, 4\rangle, \quad \mathbf{v}(0) = \left\langle \frac{1}{3}, -2\right\rangle$

SOLUTION We find $\mathbf{v}(t)$ by integrating $\mathbf{a}(t)$:

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(u)du = \int_0^t \langle u, 4\rangle\, du = \left\langle \frac{1}{2}u^2, 4u\right\rangle\Big|_0^t + \mathbf{v}_0 = \left\langle \frac{t^2}{2}, 4t\right\rangle + \mathbf{v}_0$$

The initial condition gives:

$$\mathbf{v}(0) = \langle 0, 0\rangle + \mathbf{v}_0 = \left\langle \frac{1}{3}, -2\right\rangle \quad \Rightarrow \quad \mathbf{v}_0 = \left\langle \frac{1}{3}, -2\right\rangle$$

Hence,

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2}, 4t\right\rangle + \left\langle \frac{1}{3}, -2\right\rangle = \left\langle \frac{3t^2+2}{6}, 4t-2\right\rangle$$

12. $\mathbf{a}(t) = \langle e^t, 0, t+1\rangle, \quad \mathbf{v}(0) = \langle 1, -3, \sqrt{2}\rangle$

SOLUTION Integrating $\mathbf{a}(t)$ gives:

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(u)du = \int_0^t \langle e^u, 0, u+1\rangle du = \left\langle e^u, 0, \frac{u^2}{2} + u\right\rangle\Big|_0^t + \mathbf{v}_0 = \left\langle e^t - 1, 0, \frac{t^2}{2} + t\right\rangle + \mathbf{v}_0 \tag{1}$$

The initial condition gives:

$$\mathbf{v}(0) = \left\langle e^0 - 1, 0, \frac{0^2}{2} + 0\right\rangle + \mathbf{v}_0 = \langle 1, -3, \sqrt{2}\rangle$$

$$\langle 0, 0, 0\rangle + \mathbf{v}_0 = \langle 1, -3, \sqrt{2}\rangle$$

$$\Rightarrow \quad \mathbf{v}_0 = \langle 1, -3, \sqrt{2}\rangle$$

Substituting in (1) we obtain:

$$\mathbf{v}(t) = \left\langle e^t - 1, 0, \frac{t^2}{2} + t\right\rangle + \langle 1, -3, \sqrt{2}\rangle = \left\langle e^t, -3, \frac{t^2}{2} + t + \sqrt{2}\right\rangle$$

13. $\mathbf{a}(t) = \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}$

SOLUTION We compute $\mathbf{v}(t)$ by integrating the acceleration vector:

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(u)\, du = \int_0^t \mathbf{k}\, du = \mathbf{k}u\Big|_0^t + \mathbf{v}_0 = t\mathbf{k} + \mathbf{v}_0 \tag{1}$$

Substituting the initial condition gives:

$$\mathbf{v}(0) = 0\mathbf{k} + \mathbf{v}_0 = \mathbf{i} \quad \Rightarrow \quad \mathbf{v}_0 = \mathbf{i}$$

Combining with (1) we obtain:

$$\mathbf{v}(t) = \mathbf{i} + t\mathbf{k}$$

14. $\mathbf{a}(t) = t^2\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j}$

SOLUTION We integrate the acceleration vector to find the velocity vector $\mathbf{v}(t)$. This gives

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(u)\,du = \int_0^t u^2\mathbf{k}\,du = \frac{u^3}{3}\mathbf{k}\Big|_0^t + \mathbf{v}_0 = \frac{t^3}{3}\mathbf{k} + \mathbf{v}_0 \tag{1}$$

The initial condition gives:

$$\mathbf{v}(0) = \frac{0^3}{3}\mathbf{k} + \mathbf{v}_0 = \mathbf{i} - \mathbf{j} \quad \Rightarrow \quad \mathbf{v}_0 = \mathbf{i} - \mathbf{j}$$

Substituting in (1) we obtain:

$$\mathbf{v}(t) = \frac{t^3}{3}\mathbf{k} + \mathbf{i} - \mathbf{j} = \mathbf{i} - \mathbf{j} + \frac{t^3}{3}\mathbf{k}$$

In Exercises 15–18, find $\mathbf{r}(t)$ and $\mathbf{v}(t)$ given $\mathbf{a}(t)$ and the initial velocity and position.

15. $\mathbf{a}(t) = \langle t, 4\rangle, \quad \mathbf{v}(0) = \langle 3, -2\rangle, \quad \mathbf{r}(0) = \langle 0, 0\rangle$

SOLUTION We first integrate $\mathbf{a}(t)$ to find the velocity vector:

$$\mathbf{v}(t) = \int_0^t \langle u, 4\rangle\; du = \left\langle \frac{u^2}{2}, 4u \right\rangle\Big|_0^t + \mathbf{v}_0 = \left\langle \frac{t^2}{2}, 4t \right\rangle + \mathbf{v}_0 \tag{1}$$

The initial condition $\mathbf{v}(0) = \langle 3, -2\rangle$ gives:

$$\mathbf{v}(0) = \langle 0, 0\rangle + \mathbf{v}_0 = \langle 3, -2\rangle \quad \Rightarrow \quad \mathbf{v}_0 = \langle 3, -2\rangle$$

Substituting in (1) we get:

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2}, 4t \right\rangle + \langle 3, -2\rangle = \left\langle \frac{t^2}{2} + 3, 4t - 2 \right\rangle$$

We now integrate the velocity vector to find $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int_0^t \left\langle \frac{u^2}{2} + 3, 4u - 2 \right\rangle du = \left\langle \frac{u^3}{6} + 3u, 2u^2 - 2u \right\rangle\Big|_0^t + \mathbf{r}_0 = \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t \right\rangle + \mathbf{r}_0$$

The initial condition $\mathbf{r}(0) = \langle 0, 0\rangle$ gives:

$$\mathbf{r}(0) = \langle 0, 0\rangle + \mathbf{r}_0 = \langle 0, 0\rangle \quad \Rightarrow \quad \mathbf{r}_0 = \langle 0, 0\rangle$$

Hence,

$$\mathbf{r}(t) = \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t \right\rangle$$

16. $\mathbf{a}(t) = \langle e^t, 2t, t+1\rangle, \quad \mathbf{v}(0) = \langle 1, 0, 1\rangle, \quad \mathbf{r}(0) = \langle 2, 1, 1\rangle$

SOLUTION We integrate $\mathbf{a}(t)$ to find $\mathbf{v}(t)$:

$$\mathbf{v}(t) = \int_0^t \langle e^u, 2u, u+1\rangle\, du = \left\langle e^t - 1, t^2, \frac{t^2}{2} + t \right\rangle + \mathbf{v}_0$$

The initial condition for the velocity vector gives:

$$\begin{aligned} \mathbf{v}(0) &= \langle e^0 - 1, 0, 0\rangle + \mathbf{v}_0 = \langle 1, 0, 1\rangle \\ &\langle 0, 0, 0\rangle + \mathbf{v}_0 = \langle 1, 0, 1\rangle \quad \Rightarrow \quad \mathbf{v}_0 = \langle 1, 0, 1\rangle \end{aligned}$$

Hence,

$$\mathbf{v}(t) = \left\langle e^t, t^2, \frac{t^2}{2} + t + 1 \right\rangle$$

We now integrate $\mathbf{v}(t)$ to find $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int_0^t \left\langle e^u, u^2, \frac{u^2}{2} + u + 1 \right\rangle du = \left\langle e^t - 1, \frac{t^3}{3}, \frac{t^3}{6} + \frac{t^2}{2} + t \right\rangle + \mathbf{r}_0 \tag{1}$$

The initial condition for $\mathbf{r}(t)$ gives:

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle + \mathbf{r}_0 = \langle 2, 1, 1 \rangle$$
$$\langle 0, 0, 0 \rangle + \mathbf{r}_0 = \langle 2, 1, 1 \rangle \quad \Rightarrow \quad \mathbf{r}_0 = \langle 2, 1, 1 \rangle$$

Substituting in (1) we get:

$$\mathbf{r}(t) = \left\langle e^t + 1, \frac{t^3}{3} + 1, \frac{t^3}{6} + \frac{t^2}{2} + t + 1 \right\rangle$$

17. $\mathbf{a}(t) = t\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

SOLUTION Integrating the acceleration vector gives:

$$\mathbf{v}(t) = \int_0^t u\mathbf{k}\, du = \frac{u^2}{2}\mathbf{k}\Big|_0^t + \mathbf{v}_0 = \frac{t^2}{2}\mathbf{k} + \mathbf{v}_0 \tag{1}$$

The initial condition for $\mathbf{v}(t)$ gives:

$$\mathbf{v}(0) = \frac{0^2}{2}\mathbf{k} + \mathbf{v}_0 = \mathbf{i} \quad \Rightarrow \quad \mathbf{v}_0 = \mathbf{i}$$

We substitute in (1):

$$\mathbf{v}(t) = \frac{t^2}{2}\mathbf{k} + \mathbf{i} = \mathbf{i} + \frac{t^2}{2}\mathbf{k}$$

We now integrate $\mathbf{v}(t)$ to find $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int_0^t \left(\mathbf{i} + \frac{u^2}{2}\mathbf{k}\right) du = u\mathbf{i} + \frac{u^3}{6}\mathbf{k}\Big|_0^t + \mathbf{r}_0 = t\mathbf{i} + \frac{t^3}{6}\mathbf{k} + \mathbf{r}_0 \tag{2}$$

The initial condition for $\mathbf{r}(t)$ gives:

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{k} + \mathbf{r}_0 = \mathbf{j} \quad \Rightarrow \quad \mathbf{r}_0 = \mathbf{j}$$

Combining with (2) gives the position vector:

$$\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \frac{t^3}{6}\mathbf{k}$$

18. $\mathbf{a}(t) = \cos t\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{i}$

SOLUTION We integrate $\mathbf{a}(t)$ to find the velocity vector $\mathbf{v}(t)$:

$$\mathbf{v}(t) = \int_0^t \cos u\mathbf{k}\, du = \sin t\mathbf{k} + \mathbf{v}_0 \tag{1}$$

The initial condition for $\mathbf{v}(t)$ gives:

$$\mathbf{v}(0) = \sin 0\mathbf{k} + \mathbf{v}_0 = \mathbf{0} + \mathbf{v}_0 = \mathbf{i} - \mathbf{j} \quad \Rightarrow \quad \mathbf{v}_0 = \mathbf{i} - \mathbf{j}$$

Substituting in (1) we obtain:

$$\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + \sin t\mathbf{k}$$

We now integrate $\mathbf{v}(t)$ to find the position vector $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int_0^t (\mathbf{i} - \mathbf{j} + \sin u\mathbf{k})\, du = t\mathbf{i} - t\mathbf{j} - (\cos t - 1)\mathbf{k} + \mathbf{r}_0$$

The initial condition for $\mathbf{r}(t)$ gives:

$$\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{r}_0 = \mathbf{i}$$

$$\Rightarrow \quad \mathbf{r}_0 = \mathbf{i}$$

Substituting in (2) we obtain:

$$\mathbf{r}(t) = (t+1)\mathbf{i} - t\mathbf{j} + (1 - \cos t)\mathbf{k}$$

In Exercises 19–24, recall that $g = 9.8$ m/s^2 is the acceleration due to gravity on the earth's surface.

19. A bullet is fired from the ground at an angle of 45°. What initial speed must the bullet have in order to hit the top of a 120-m tower located 180 m away?

SOLUTION We place the gun at the origin and let $\mathbf{r}(t)$ be the bullet's position vector.

Step 1. Use Newton's Law. The net force vector acting on the bullet is the force of gravity $\mathbf{F} = \langle 0, -gm \rangle = m\langle 0, -g \rangle$. By Newton's Second Law, $\mathbf{F} = m\mathbf{r}''(t)$, hence:

$$m\langle 0, -g \rangle = m\mathbf{r}''(t) \quad \Rightarrow \quad \mathbf{r}''(t) = \langle 0, -g \rangle$$

We compute the position vector by integrating twice:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -g \rangle\,du = \langle 0, -gt \rangle + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u)\,du = \int_0^t (\langle 0, -gu \rangle + \mathbf{v}_0)\,du = \left\langle 0, -g\frac{t^2}{2} \right\rangle + \mathbf{v}_0 t + \mathbf{r}_0$$

That is,

$$\mathbf{r}(t) = \left\langle 0, \frac{-g}{2}t^2 \right\rangle + \mathbf{v}_0 t + \mathbf{r}_0 \tag{1}$$

Since the gun is at the origin, $\mathbf{r}_0 = \mathbf{0}$. The bullet is fired at an angle of 45°, hence the initial velocity $\mathbf{v}_0$ points in the direction of the unit vector $\langle \cos 45°, \sin 45° \rangle = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ therefore, $\mathbf{v}_0 = v_0 \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$. Substituting these initial values in (1) gives:

$$\mathbf{r}(t) = \left\langle 0, \frac{-g}{2}t^2 \right\rangle + t v_0 \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

Step 2. Solve for v_0. The position vector of the top of the tower is $\langle 180, 120 \rangle$, hence at the moment of hitting the tower we have,

$$\mathbf{r}(t) = \left\langle 0, \frac{-g}{2}t^2 \right\rangle + t v_0 \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = \langle 180, 120 \rangle$$

$$\left\langle t v_0 \frac{\sqrt{2}}{2}, \frac{-g}{2}t^2 + \frac{\sqrt{2}}{2} t v_0 \right\rangle = \langle 180, 120 \rangle$$

Equating components, we get the equations:

$$\begin{cases} t v_0 \dfrac{\sqrt{2}}{2} = 180 \\[2mm] -\dfrac{g}{2}t^2 + \dfrac{\sqrt{2}}{2} t v_0 = 120 \end{cases}$$

The first equation implies that $t = \frac{360}{\sqrt{2}v_0}$. We substitute in the second equation and solve for v_0 (we use $g = 9.8$ m/s^2):

$$-\frac{9.8}{2}\left(\frac{360}{\sqrt{2}v_0}\right)^2 + \frac{\sqrt{2}}{2}\left(\frac{360}{\sqrt{2}v_0}\right)v_0 = 120$$

$$-2.45\left(\frac{360}{v_0}\right)^2 + 180 = 120$$

$$\left(\frac{360}{v_0}\right)^2 = \frac{1200}{49} \quad \Rightarrow \quad \frac{360}{v_0} = \sqrt{\frac{1200}{49}} \quad \Rightarrow \quad v_0 = 42\sqrt{3} \approx 72.746 \text{ m/s}$$

The initial speed of the bullet must be $v_0 = 42\sqrt{3}$ m/s ≈ 72.746 m/s.

20. Find the initial velocity vector $\mathbf{v}_0$ of a projectile released with initial speed 100 m/s that reaches a maximum height of 300 m.

SOLUTION If $\mathbf{v} = \langle a, b\rangle$, then in the text we found that the maximum height is $\frac{b^2}{2g} = 300$ and

$$b = \sqrt{2(9.8)(300)} = \sqrt{5880} = 14\sqrt{30} \approx 76.68$$

and using the Pythagorean Theorem for right triangles, $a = \sqrt{100^2 - (\sqrt{5880})^2} = 2\sqrt{1030} \approx 64.19$. Therefore, the initial velocity vector is

$$\mathbf{v}_0 = \left\langle 2\sqrt{1030}, 14\sqrt{30}\right\rangle \approx \langle 64.19, 76.68\rangle$$

21. Show that a projectile fired at an angle θ with initial speed v_0 travels a total distance $(v_0^2/g)\sin 2\theta$ before hitting the ground. Conclude that the maximum distance (for a given v_0) is attained for $\theta = 45°$.

SOLUTION We place the gun at the origin and let $\mathbf{r}(t)$ be the projectile's position vector. The net force acting on the projectile is $\mathbf{F} = \langle 0, -mg\rangle = m\langle 0, -g\rangle$. By Newton's Second Law, $\mathbf{F} = m\mathbf{r}''(t)$, hence:

$$m\langle 0, -g\rangle = m\mathbf{r}''(t) \quad \Rightarrow \quad \mathbf{r}''(t) = \langle 0, -g\rangle$$

Integrating twice we get:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -g\rangle\,du = \langle 0, -\text{gt}\rangle + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u)\,du = \int_0^t (\langle 0, -g\cdot u\rangle + \mathbf{v}_0)\,du = \left\langle 0, -\frac{g}{2}t^2\right\rangle + \mathbf{v}_0 t + \mathbf{r}_0 \tag{1}$$

Since the gun is at the origin, $\mathbf{r}_0 = 0$. The firing was at an angle θ, hence the initial velocity points in the direction of the unit vector $\langle\cos\theta, \sin\theta\rangle$. Hence, $\mathbf{v}_0 = v_0\langle\cos\theta, \sin\theta\rangle$. We substitute the initial vectors in (1) to obtain:

$$\mathbf{r}(t) = \left\langle 0, -\frac{g}{2}t^2\right\rangle + v_0 t\langle\cos\theta, \sin\theta\rangle \tag{2}$$

The total distance is obtained when the y-component of $\mathbf{r}(t)$ is zero (besides the original moment, that is,

$$-\frac{g}{2}t^2 + (v_0\sin\theta)\,t = 0$$

$$t\left(-\frac{g}{2}t + v_0\sin\theta\right) = 0 \quad \Rightarrow \quad t = 0 \quad \text{or} \quad t = \frac{2v_0\sin\theta}{g}$$

The appropriate choice is $t = \frac{2v_0\sin\theta}{g}$. We now find the total distance x_T by substituting this value of t in the x-component of $\mathbf{r}(t)$ in (2). We obtain:

$$x(t) = v_0 t\cos\theta$$

$$x_T = v_0\cos\theta\cdot\frac{2v_0\sin\theta}{g} = \frac{2v_0^2\cos\theta\sin\theta}{g} = \frac{v_0^2\sin 2\theta}{g}$$

The maximum distance is attained when $\sin 2\theta = 1$, that is $2\theta = 90°$ or $\theta = 45°$.

22. One player throws a baseball to another player standing 25 m away with initial speed 18 m/s. Use the result of Exercise 21 to find two angles θ at which the ball can be released. Which angle gets the ball there faster?

SOLUTION We suppose that the baseball is thrown from the origin, and that $\mathbf{r}(t)$ is the baseball's position vector. By Exercise 21 the total distance travelled by the ball is $\frac{v_0^2}{g}\sin 2\theta$. Using the given information we obtain the following equation:

$$\frac{18^2}{9.8}\sin 2\theta = 25$$

$$\sin 2\theta = \frac{9.8\cdot 25}{18^2} \approx 0.756$$

The solutions for $0 \le \theta \le 90°$ are:

$$\begin{array}{c} 2\theta \approx 49.13° \\ \theta \approx 24.56° \end{array} \quad \text{or} \quad \begin{array}{c} 2\theta \approx 139.12° \\ \theta \approx 69.56° \end{array}$$

By Newton's Second Law we have:

$$\mathbf{F} = m\langle 0, -g\rangle = m\mathbf{r}''(t) \quad\Rightarrow\quad \mathbf{r}''(t) = \langle 0, -g\rangle = \langle 0, -9.8\rangle$$

Integrating gives:

$$\mathbf{v}(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -9.8\rangle\,du = \langle 0, -9.8t\rangle + \mathbf{v}_0 \tag{1}$$

The initial velocity points in the direction of the unit vector $\langle\cos\theta, \sin\theta\rangle$ and its magnitude is the initial speed $v_0 = 18$. Hence, $\mathbf{v}_0 = 18\langle\cos\theta, \sin\theta\rangle$. Substituting in (1) we get:

$$\mathbf{v}(t) = \langle 0, -9.8t\rangle + 18\langle\cos\theta, \sin\theta\rangle \tag{2}$$

Integrating this vector with respect to t and using $\mathbf{r}_0 = \mathbf{0}$ we obtain:

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(u)\,du = \int_0^t (\langle 0, -9.8u\rangle + 18\langle\cos\theta, \sin\theta\rangle)\,du = \left\langle 0, -4.9t^2\right\rangle + 18t\langle\cos\theta, \sin\theta\rangle$$

At the final time $x(t) = 25$. This gives:

$$x(t) = 18t\cos\theta = 25 \quad\Rightarrow\quad t = \frac{25}{18\cos\theta}$$

Since we want to minimize t we need to maximize $\cos\theta$, hence, to minimize θ. Therefore, $\theta = 24.56^\circ$ will get the ball faster to the other player.

23. A bullet is fired at an angle $\theta = \frac{\pi}{4}$ at a tower located $d = 600$ m away, with initial speed $v_0 = 120$ m/s. Find the height H at which the bullet hits the tower.

SOLUTION We place the gun at the origin and let $\mathbf{r}(t)$ be the bullet's position vector.

Step 1. Use Newton's Law. The net force vector acting on the bullet is the force of gravity $\mathbf{F} = \langle 0, -gm\rangle = m\langle 0, -g\rangle$. By Newton's Second Law, $\mathbf{F} = m\mathbf{r}''(t)$, hence:

$$m\langle 0, -g\rangle = m\mathbf{r}''(t) \quad\Rightarrow\quad \mathbf{r}''(t) = \langle 0, -g\rangle$$

We compute the position vector by integrating twice:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -g\rangle\,du = \langle 0, -gt\rangle + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u)\,du = \int_0^t (\langle 0, -gu\rangle + \mathbf{v}_0)\,du = \left\langle 0, -g\frac{t^2}{2}\right\rangle + \mathbf{v}_0 t + \mathbf{r}_0$$

That is,

$$\mathbf{r}(t) = \left\langle 0, \frac{-g}{2}t^2\right\rangle + \mathbf{v}_0 t + \mathbf{r}_0 \tag{1}$$

Since the gun is at the origin, $\mathbf{r}_0 = \mathbf{0}$. The bullet is fired at an angle of $\pi/4$ radians, hence the initial velocity $\mathbf{v}_0$ points in the direction of the unit vector $\langle\cos\pi/4, \sin\pi/4\rangle = \left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$ therefore, $\mathbf{v}_0 = v_0\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$. Substituting these initial values in (1) gives:

$$\mathbf{r}(t) = \left\langle 0, \frac{-g}{2}t^2\right\rangle + tv_0\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

Step 2. Solve for H.

The position vector for the point at which the bullet hits the tower, 600 meters away, is $\langle 600, H\rangle$, hence at the moment of hitting the tower we have,

$$\left\langle 0, \frac{-g}{2}t^2\right\rangle + tv_0\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \langle 600, H\rangle$$

Therefore, for $v_0 = 120$:

$$\frac{tv_0}{\sqrt{2}} = 600 \quad\Rightarrow\quad t = \frac{600\sqrt{2}}{120} = 5\sqrt{2}$$

and

$$-\frac{gt^2}{2} + \frac{tv_0}{\sqrt{2}} = \frac{-9.8(50)}{2} + \frac{5(\sqrt{2})(120)}{\sqrt{2}} = H$$

Hence, $H = 355$ meters. The bullet hits the tower at 355 meters high.

24. Show that a bullet fired at an angle θ will hit the top of an h-meter tower located d meters away if its initial speed is

$$v_0 = \frac{\sqrt{g/2}\, d \sec\theta}{\sqrt{d\tan\theta - h}}$$

SOLUTION We place the gun at the origin and let $\mathbf{r}(t)$ be the position vector of the bullet. The net force acting on the bullet is $\mathbf{F} = \langle 0, -mg \rangle = m\langle 0, -g \rangle$. By Newton's Second Law, $\mathbf{F} = m\mathbf{r}''(t)$, hence:

$$m\langle 0, -g \rangle = m\mathbf{r}''(t) \quad \Rightarrow \quad \mathbf{r}''(t) = \langle 0, -g \rangle$$

We integrate twice and use the initial value $\mathbf{r}(0) = \mathbf{0}$, to obtain:

$$\mathbf{v}(t) = \int_0^t \mathbf{r}''(u)\,du = \int_0^t \langle 0, -g \rangle\,du = \langle 0, -gt \rangle + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(u)\,du = \int_0^t (\langle 0, -gu \rangle + \mathbf{v}_0)\,du = \left\langle 0, -\frac{g}{2}t^2 \right\rangle + \mathbf{v}_0 t \tag{1}$$

To determine $\mathbf{v}_0$ we notice that the initial velocity vector points in the direction of the unit vector $\langle \cos\theta, \sin\theta \rangle$ and its magnitude is the initial speed v_0. Hence $\mathbf{v}_0 = v_0\langle \cos\theta, \sin\theta \rangle$. Substituting in (1), we get:

$$\mathbf{r}(t) = \left\langle 0, -\frac{g}{2}t^2 \right\rangle + v_0 t\langle \cos\theta, \sin\theta \rangle$$

For the bullet to hit the top of the tower, there must be a value of t such that $\mathbf{r}(t) = \langle d, h \rangle$. That is:

$$\left\langle 0, -\frac{g}{2}t^2 \right\rangle + v_0 t\langle \cos\theta, \sin\theta \rangle = \langle d, h \rangle$$

Equating components we get the following equations:

$$v_0 t\cos\theta = d$$
$$-\frac{g}{2}t^2 + v_0 t\sin\theta = h$$

The first equation implies that $t = \frac{d}{v_0\cos\theta}$. We substitute in the second equation and solve for v_0:

$$-\frac{g}{2}\left(\frac{d}{v_0\cos\theta}\right)^2 + v_0\left(\frac{d}{v_0\cos\theta}\right)\sin\theta = h$$

$$-\frac{gd^2}{2v_0^2\cos^2\theta} + d\tan\theta = h$$

$$\frac{gd^2}{2v_0^2\cos^2\theta} = d\tan\theta - h$$

$$v_0{}^2 = \frac{gd^2}{2\cos^2\theta\,(d\tan\theta - h)} = \frac{gd^2\sec^2\theta}{2\,(d\tan\theta - h)}$$

Therefore,

$$v_0 = \frac{\sqrt{g/2}\, d \sec\theta}{\sqrt{d\tan\theta - h}}$$

25. A constant force $\mathbf{F} = \langle 5, 2 \rangle$ (in newtons) acts on a 10-kg mass. Find the position of the mass at $t = 10$ s if it is located at the origin at $t = 0$ and has initial velocity $\mathbf{v}_0 = \langle 2, -3 \rangle$ (in meters per second).

SOLUTION We know that $\mathbf{F} = m\mathbf{a}$ and thus $\langle 5, 2 \rangle = 10\mathbf{a}$ so then $\mathbf{a} = \langle 0.5, 0.2 \rangle$. Using integration we know

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = t\mathbf{a} + \mathbf{c}$$

and we know $\mathbf{v}(0) = \langle 2, -3 \rangle = \mathbf{c}$. Therefore,

$$\mathbf{v}(t) = t\mathbf{a} + \mathbf{v}_0 = t\langle 0.5, 0.2 \rangle + \langle 2, -3 \rangle = \langle 0.5t + 2, 0.2t - 3 \rangle$$

Again, integrating,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$$

$$= \int t\mathbf{a} + \mathbf{v}_0\, dt$$

$$= \frac{t^2}{2}\mathbf{a} + t\mathbf{v}_0 + \mathbf{c}$$

$$= \frac{t^2}{2}\langle 0.5, 0.2\rangle + t\langle 2, -3\rangle$$

$$= \left\langle 0.25t^2 + 2t, 0.1t^2 - 3t\right\rangle + \mathbf{r}_0$$

Using the initial condition $\mathbf{r}(0) = \langle 0, 0\rangle = \mathbf{c}$, we conclude

$$\mathbf{r}(t) = \left\langle 0.25t^2 + 2t, 0.1t^2 - 3t\right\rangle$$

and hence the position of the mass at $t = 10$ is $\mathbf{r}(10) = \langle 45, -20\rangle$.

26. A force $\mathbf{F} = \langle 24t, 16 - 8t\rangle$ (in newtons) acts on a 4-kg mass. Find the position of the mass at $t = 3$ s if it is located at (10, 12) at $t = 0$ and has zero initial velocity.

SOLUTION We know $\mathbf{F} = m\mathbf{a}$ and thus $\langle 24t, 16 - 8t\rangle = 4\mathbf{a}$, therefore, $\mathbf{a} = \langle 6t, 4 - 2t\rangle$, and

$$\mathbf{v}(t) = \int \mathbf{a}\, dt = \int \langle 6t, 4 - 2t\rangle\, dt = \left\langle 3t^2, 4t - t^2\right\rangle + \mathbf{c}$$

using the initial velocity condition, $\mathbf{v}(0) = \langle 0, 0\rangle$, then $\mathbf{c} = \langle 0, 0\rangle$ and

$$\mathbf{v}(t) = \left\langle 3t^2, 4t - t^2\right\rangle$$

Now integrating again, we get:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \left\langle 3t^2, 4t - t^2\right\rangle dt = \left\langle t^3, 2t^2 - \frac{t^3}{3}\right\rangle + \mathbf{c}$$

Using the initial position condition $\mathbf{r}(0) = \langle 10, 12\rangle$, we get $\mathbf{r}(0) = \langle 10, 12\rangle = \mathbf{c}$ and therefore,

$$\mathbf{r} = \left\langle t^3 + 10, 2t^2 - \frac{t^3}{3} + 12\right\rangle$$

and at $t = 3$ the position of the mass is $\mathbf{r}(3) = \left\langle 3^3 + 10, 2(9) - 9 + 12\right\rangle = \langle 37, 21\rangle$.

27. A particle follows a path $\mathbf{r}(t)$ for $0 \le t \le T$, beginning at the origin O. The vector $\overline{\mathbf{v}} = \frac{1}{T}\int_0^T \mathbf{r}'(t)\, dt$ is called the **average velocity** vector. Suppose that $\overline{\mathbf{v}} = \mathbf{0}$. Answer and explain the following:

(a) Where is the particle located at time T if $\overline{\mathbf{v}} = \mathbf{0}$?

(b) Is the particle's average speed necessarily equal to zero?

SOLUTION

(a) If the average velocity is 0, then the particle must be back at its original position at time $t = T$. This is perhaps best seen by noting that $\overline{\mathbf{v}} = \frac{1}{T}\int_0^T \mathbf{r}'(t)\, dt = \mathbf{r}(t)\Big|_0^T$.

(b) The average speed need not be zero! Consider a particle moving at constant speed around a circle, with position vector $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. From 0 to 2π, this has average velocity of 0, but constant average speed of 1.

28. At a certain moment, a moving particle has velocity $\mathbf{v} = \langle 2, 2, -1\rangle$ and $\mathbf{a} = \langle 0, 4, 3\rangle$. Find $\mathbf{T}$, $\mathbf{N}$, and the decomposition of $\mathbf{a}$ into tangential and normal components.

SOLUTION We go through the following steps:

Step 1. Compute $\mathbf{T}$ and $a_{\mathbf{T}}$. The unit tangent is the following vector:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, 2, -1\rangle}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{1}{3}\langle 2, 2, -1\rangle \tag{1}$$

The tangential component of $\mathbf{a} = \langle 0, 4, 3\rangle$ is:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \langle 0, 4, 3\rangle \cdot \frac{1}{3}\langle 2, 2, -1\rangle = \frac{1}{3}(0 + 8 - 3) = \frac{5}{3}$$

Step 2. Compute $a_{\mathbf{N}}$ and $\mathbf{N}$. Since $a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T}$, we have:

$$a_{\mathbf{N}}\mathbf{N} = \langle 0, 4, 3\rangle - \frac{5}{3}\cdot\frac{1}{3}\langle 2, 2, -1\rangle = \langle 0, 4, 3\rangle - \left\langle \frac{10}{9}, \frac{10}{9}, -\frac{5}{9}\right\rangle = \frac{1}{9}\langle -10, 26, 32\rangle \tag{3}$$

The unit normal $\mathbf{N}$ is a unit vector, therefore:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \frac{1}{9}\sqrt{(-10)^2 + 26^2 + 32^2} = \frac{1}{9}\cdot 30\sqrt{2} = \frac{10\sqrt{2}}{3} \tag{4}$$

We compute $\mathbf{N}$, using (3) and (4):

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\frac{1}{9}\langle -10, 26, 32\rangle}{\frac{10\sqrt{2}}{3}} = \frac{1}{15\sqrt{2}}\langle -5, 13, 16\rangle$$

Step 3. Write the decomposition. Using (1)–(4) we obtain the following decomposition:

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$
$$\langle 0, 4, 3\rangle = \frac{5}{3}\mathbf{T} + \frac{10\sqrt{2}}{3}\mathbf{N},$$

where $\mathbf{T} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle$ and $\mathbf{N} = \frac{1}{15\sqrt{2}}\langle -5, 13, 16\rangle$.

29. At a certain moment, a particle moving along a path has velocity $\mathbf{v} = \langle 12, 20, 20\rangle$ and acceleration $\mathbf{a} = \langle 2, 1, -3\rangle$. Is the particle speeding up or slowing down?

SOLUTION We are asked if the particle is speeding up or slowing down, that is if $\|\mathbf{v}\|$ or $\|\mathbf{v}\|^2$ is increasing or decreasing. We check $\left(\|\mathbf{v}\|^2\right)'$:

$$\left(\|\mathbf{v}\|^2\right)' = (\mathbf{v}\cdot\mathbf{v})' = 2\mathbf{v}'\cdot\mathbf{v} = 2\cdot\mathbf{a}\cdot\mathbf{v} = 2\langle 2, 1, -3\rangle\cdot\langle 12, 20, 20\rangle = 2\cdot(24 + 20 - 60) = -32 < 0$$

So the speed is decreasing.

In Exercises 30–33, use Eq. (3) to find the coefficients $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as a function of t (or at the specified value of t).

30. $\mathbf{r}(t) = \langle t^2, t^3\rangle$

SOLUTION We find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ by differentiating $\mathbf{r}(t)$ twice:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, 6t\rangle$$

The unit tangent vector $\mathbf{T}$ is:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2t, 3t^2\rangle}{\sqrt{(2t)^2 + \left(3t^2\right)^2}} = \frac{\langle 2t, 3t^2\rangle}{\sqrt{4t^2 + 9t^4}}$$

We compute $a_{\mathbf{T}}$:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{\langle 2, 6t\rangle\cdot\langle 2t, 3t^2\rangle}{\sqrt{4t^2 + 9t^4}} = \frac{4t + 18t^3}{\sqrt{4t^2 + 9t^4}}$$

To find $a_{\mathbf{N}}$ we first compute the cross product $\mathbf{a}\times\mathbf{v}$:

$$\mathbf{a}\times\mathbf{v} = (2\mathbf{i} + 6t\mathbf{j})\times\left(2t\mathbf{i} + 3t^2\mathbf{j}\right) = 6t^2\mathbf{i}\times\mathbf{j} + 12t^2\mathbf{j}\times\mathbf{i} = 6t^2\mathbf{k} - 12t^2\mathbf{k} = -6t^2\mathbf{k}$$

Hence,

$$a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\|-6t^2\mathbf{k}\|}{\sqrt{4t^2 + 9t^4}} = \frac{6t^2}{\sqrt{4t^2 + 9t^4}}$$

31. $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$

SOLUTION We find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ using the following equalities:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T},\ a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|}.$$

We compute $\mathbf{v}$ and $\mathbf{a}$ by differentiating $\mathbf{r}$ twice:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle \quad \Rightarrow \quad \|\mathbf{v}(t)\| = \sqrt{1 + (-\sin t)^2 + \cos^2 t} = \sqrt{2}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -\cos t, -\sin t\rangle$$

The unit tangent vector $\mathbf{T}$ is, thus:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2}}\langle 1, -\sin t, \cos t\rangle$$

Since the speed is constant ($v = \|\mathbf{v}(t)\| = \sqrt{2}$), the tangential component of the acceleration is zero, that is:

$$a_{\mathbf{T}} = 0$$

To find $a_{\mathbf{N}}$ we first compute the following cross product:

$$\mathbf{a}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\cos t & -\sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -\sin t \\ 1 & \cos t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -\cos t \\ 1 & -\sin t \end{vmatrix}\mathbf{k}$$
$$= -\left(\cos^2 t + \sin^2 t\right)\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k} = -\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k} = \langle -1, -\sin t, \cos t\rangle$$

Hence,

$$a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{(-1)^2 + (-\sin t)^2 + \cos^2 t}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

32. $\mathbf{r}(t) = \left\langle t^{-1}, \ln t, t^2\right\rangle, \quad t = 1$

SOLUTION We use the following equalities:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T}, \quad a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|}.$$

We first find $\mathbf{a}$ and $\mathbf{v}$ by twice differentiating $\mathbf{r}$. We get:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle -\frac{1}{t^2}, \frac{1}{t}, 2t\right\rangle \quad \Rightarrow \quad \|\mathbf{v}(t)\| = \sqrt{\frac{1}{t^4} + \frac{1}{t^2} + 4t^2} = \frac{1}{t^2}\sqrt{1 + t^2 + 4t^6}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 2t^{-3}, -t^{-2}, 2\right\rangle$$

At the point $t = 1$ we have:

$$\mathbf{v}(1) = \langle -1, 1, 2\rangle, \quad \|\mathbf{v}(1)\| = \sqrt{6}, \quad \mathbf{a}(1) = \langle 2, -1, 2\rangle$$

Hence, $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}\langle -1, 1, 2\rangle$ and we obtain:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \langle 2, -1, 2\rangle\cdot\frac{1}{\sqrt{6}}\langle -1, 1, 2\rangle = \frac{1}{\sqrt{6}}(-2 - 1 + 4) = \frac{1}{\sqrt{6}}$$

To find $a_{\mathbf{N}}$ we first compute the following cross product:

$$\mathbf{a}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{k} = -4\mathbf{i} - 6\mathbf{j} + \mathbf{k} = \langle -4, -6, 1\rangle$$

Therefore:

$$a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{(-4)^2 + (-6)^2 + 1^2}}{\sqrt{6}} = \sqrt{\frac{53}{6}}$$

33. $\mathbf{r}(t) = \left\langle e^{2t}, t, e^{-t}\right\rangle, \quad t = 0$

SOLUTION We will use the following equalities:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T}, \quad a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|}.$$

We first find **a** and **v** by twice differentiating **r**. We get:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2e^{2t}, 1, -e^{-t} \right\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 4e^{2t}, 0, e^{-t} \right\rangle$$

Then evaluating at $t = 0$ we get:

$$\mathbf{v}(0) = \langle 2, 1, -1 \rangle, \quad \Rightarrow \|\mathbf{v}(0)\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$
$$\mathbf{a}(0) = \langle 4, 0, 1 \rangle$$

Hence, $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}} \langle 2, 1, -1 \rangle$ and we obtain:

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \langle 4, 0, 1 \rangle \cdot \frac{1}{\sqrt{6}} \langle 2, 1, -1 \rangle = \frac{1}{\sqrt{6}}(8 + 0 - 1) = \frac{7}{\sqrt{6}}$$

To find $a_{\mathbf{N}}$ we first compute the following cross product:

$$\mathbf{a} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -1, 6, 4 \rangle$$

Therefore,

$$a_{\mathbf{N}} = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{(-1)^2 + 6^2 + 4^2}}{\sqrt{6}} = \sqrt{\frac{53}{6}}$$

In Exercise 34–41, find the decomposition of $\mathbf{a}(t)$ *into tangential and normal components at the point indicated, as in Example 6.*

34. $\mathbf{r}(t) = \left\langle e^t, 1 - t \right\rangle, \quad t = 0$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle e^t, -1 \right\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle e^t, 0 \right\rangle$$

At $t = 0$ we have:

$$\mathbf{v} = \mathbf{r}'(0) = \langle 1, -1 \rangle$$
$$\mathbf{a} = \mathbf{r}''(0) = \langle 1, 0 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle 1, 0 \rangle \cdot \langle 1, -1 \rangle = 1$$
$$\|\mathbf{v}\| = \sqrt{1 + 1} = \sqrt{2}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1, -1 \rangle}{\sqrt{1 + 1}} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle 1, 0 \rangle - \frac{1}{\sqrt{2}} \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \langle 1, 0 \rangle - \left\langle \frac{1}{2}, -\frac{1}{2} \right\rangle = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle}{1/\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Finally, we obtain the decomposition,

$$\mathbf{a} = \langle 1, 0 \rangle = \frac{1}{\sqrt{2}}\mathbf{T} + \frac{1}{\sqrt{2}}\mathbf{N}$$

where $\mathbf{T} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ and $\mathbf{N} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

35. $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, 1 - 3t \right\rangle, \quad t = -2$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle t^2, -3 \right\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2t, 0 \rangle$$

At $t = -2$ we have:

$$\mathbf{v} = \mathbf{r}'(-2) = \langle 4, -3 \rangle$$
$$\mathbf{a} = \mathbf{r}''(-2) = \langle -4, 0 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle -4, 0 \rangle \cdot \langle 4, -3 \rangle = -16$$
$$\|\mathbf{v}\| = \sqrt{16 + 9} = 5$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 4, -3 \rangle}{5} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = -\frac{16}{5}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle -4, 0 \rangle + \frac{16}{5}\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle = \left\langle -\frac{36}{25}, -\frac{48}{25} \right\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \sqrt{\left(-\frac{36}{25}\right)^2 + \left(-\frac{48}{25}\right)^2} = \frac{60}{25} = \frac{12}{5}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\left\langle -\frac{36}{25}, -\frac{48}{25} \right\rangle}{12/5} = \left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle -4, 0 \rangle = -\frac{16}{5}\mathbf{T} + \frac{12}{5}\mathbf{N}$$

where $\mathbf{T} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$ and $\mathbf{N} = \left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$.

36. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3 \right\rangle, \quad t = 1$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 1, t, \frac{1}{2}t^2 \right\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 1, t \rangle$$

At $t = 1$ we have:

$$\mathbf{v} = \mathbf{r}'(1) = \left\langle 1, 1, \frac{1}{2} \right\rangle$$
$$\mathbf{a} = \mathbf{r}''(1) = \langle 0, 1, 1 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle 0, 1, 1 \rangle \cdot \left\langle 1, 1, \frac{1}{2} \right\rangle = \frac{3}{2}$$
$$\|\mathbf{v}\| = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\left\langle 1, 1, \frac{1}{2} \right\rangle}{3/2} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{3/2}{3/2} = 1$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle 0, 1, 1 \rangle - 1 \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle}{1} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle 0, 1, 1 \rangle = (1)\mathbf{T} + (1)\mathbf{N}$$

where $\mathbf{T} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ and $\mathbf{N} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$.

37. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3 \right\rangle, \quad t = 4$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 1, t, \frac{1}{2}t^2 \right\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 1, t \rangle$$

At $t = 4$ we have:

$$\mathbf{v} = \mathbf{r}'(4) = \langle 1, 4, 8 \rangle$$
$$\mathbf{a} = \mathbf{r}''(4) = \langle 0, 1, 4 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle 0, 1, 4 \rangle \cdot \langle 1, 4, 8 \rangle = 36$$
$$\|\mathbf{v}\| = \sqrt{1 + 16 + 64} = \sqrt{81} = 9$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1, 4, 8 \rangle}{9} = \left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9} \right\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{36}{9} = 4$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle 0, 1, 4\rangle - 4\left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right\rangle = \left\langle -\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}\right\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \sqrt{\frac{16}{81} + \frac{49}{81} + \frac{16}{81}} = 1$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\left\langle -\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}\right\rangle}{1} = \left\langle -\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}\right\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle 0, 1, 4\rangle = 4\mathbf{T} + (1)\mathbf{N}$$

where $\mathbf{T} = \left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right\rangle$ and $\mathbf{N} = \left\langle -\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}\right\rangle$.

38. $\mathbf{r}(t) = \langle 4 - t, t + 1, t^2\rangle, \quad t = 2$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -1, 1, 2t\rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0, 2\rangle$$

At $t = 2$ we have:

$$\mathbf{v} = \mathbf{r}'(2) = \langle -1, 1, 4\rangle$$
$$\mathbf{a} = \mathbf{r}''(2) = \langle 0, 0, 2\rangle$$

Thus,

$$\mathbf{a}\cdot\mathbf{v} = \langle 0, 0, 2\rangle \cdot \langle -1, 1, 4\rangle = 8$$
$$\|\mathbf{v}\| = \sqrt{1 + 1 + 16} = 3\sqrt{2}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -1, 1, 4\rangle}{3\sqrt{2}} = \left\langle \frac{-1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{8}{3\sqrt{2}}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle 0, 0, 2\rangle - \frac{8}{3\sqrt{2}}\left\langle \frac{-1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right\rangle$$
$$= \langle 0, 0, 2\rangle + \left\langle \frac{4}{9}, -\frac{4}{9}, -\frac{16}{9}\right\rangle = \left\langle \frac{4}{9}, -\frac{4}{9}, \frac{2}{9}\right\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \left\|\frac{2}{9}\langle 2, -2, 1\rangle\right\| = \frac{2}{9}\sqrt{4 + 4 + 1} = \frac{2}{3}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\frac{2}{9}\langle 2, -2, 1\rangle}{2/3} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle 0, 0, 2\rangle = \frac{8}{3\sqrt{2}}\mathbf{T} + \frac{2}{3}\mathbf{N}$$

where $\mathbf{T} = \left\langle \frac{-1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right\rangle$ and $\mathbf{N} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$.

39. $\mathbf{r}(t) = \langle t, e^t, te^t \rangle, \quad t = 0$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, e^t, (t+1)e^t \rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, e^t, (t+2)e^t \rangle$$

At $t = 0$ we have:

$$\mathbf{v} = \mathbf{r}'(0) = \langle 1, 1, 1 \rangle$$
$$\mathbf{a} = \mathbf{r}''(0) = \langle 0, 1, 2 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle 0, 1, 2 \rangle \cdot \langle 1, 1, 1 \rangle = 3$$
$$\|\mathbf{v}\| = \sqrt{1+1+1} = \sqrt{3}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle 0, 1, 2 \rangle - \sqrt{3}\frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \langle -1, 0, 1 \rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \sqrt{1+1} = \sqrt{2}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\langle -1, 0, 1 \rangle}{\sqrt{2}} = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle 0, 1, 2 \rangle = \sqrt{3}\mathbf{T} + \sqrt{2}\mathbf{N}$$

where $\mathbf{T} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$ and $\mathbf{N} = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$.

40. $\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, \theta \rangle, \quad \theta = 0$

SOLUTION First note here that:

$$\mathbf{v}(\theta) = \mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta, 1 \rangle$$
$$\mathbf{a}(\theta) = \mathbf{r}''(\theta) = \langle -\cos\theta, -\sin\theta, 0 \rangle$$

At $\theta = 0$ we have:

$$\mathbf{v} = \mathbf{r}'(0) = \langle 0, 1, 1 \rangle$$
$$\mathbf{a} = \mathbf{r}''(0) = \langle -1, 0, 0 \rangle$$

Thus,

$$\mathbf{a} \cdot \mathbf{v} = \langle -1, 0, 0 \rangle \cdot \langle 0, 1, 1 \rangle = 0$$
$$\|\mathbf{v}\| = \sqrt{1+1} = \sqrt{2}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{0}{\sqrt{2}} = 0$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \langle -1, 0, 0\rangle - \mathbf{0} = \langle -1, 0, 0\rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = 1$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\langle -1, 0, 0\rangle}{1} = \langle -1, 0, 0\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \langle -1, 0, 0\rangle = 0\mathbf{T} + (1)\mathbf{N}$$

where $\mathbf{T} = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle$ and $\mathbf{N} = \langle -1, 0, 0\rangle$.

Notice that the vector $\mathbf{a}(0) = \langle -1, 0, 0\rangle$ is already a unit vector, hence (1) implies that $\mathbf{N}(0) = \mathbf{a}(0)$ and $a_{\mathbf{N}}(0) = 1$. Hence the required decomposition reduces to:

$$\mathbf{a}(0) = 1 \cdot \mathbf{N}(0) = 1 \cdot \mathbf{N} \quad \text{where} \quad \mathbf{N} = \mathbf{a}(0) = \langle -1, 0, 0\rangle$$

41. $\mathbf{r}(t) = \langle t, \cos t, t\sin t\rangle, \quad t = \frac{\pi}{2}$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -\sin t, t\cos t + \sin t, \rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -\cos t, -t\sin t + 2\cos t\rangle$$

At $t = \frac{\pi}{2}$ we have:

$$\mathbf{v} = \mathbf{r}'(\pi/2) = \langle 1, -1, 1\rangle$$
$$\mathbf{a} = \mathbf{r}''(-2) = \left\langle 0, 0, -\frac{\pi}{2}\right\rangle$$

Thus,

$$\mathbf{a}\cdot\mathbf{v} = \left\langle 0, 0, -\frac{\pi}{2}\right\rangle \cdot \langle 1, -1, 1\rangle = -\frac{\pi}{2}$$
$$\|\mathbf{v}\| = \sqrt{1+1+1} = \sqrt{3}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{-\pi/2}{\sqrt{3}} = -\frac{\pi}{2\sqrt{3}}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$\begin{aligned} a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} &= \left\langle 0, 0, -\frac{\pi}{2}\right\rangle + \frac{\pi}{2\sqrt{3}}\frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle \\ &= \left\langle 0, 0, -\frac{\pi}{2}\right\rangle + \frac{\pi}{6}\langle 1, -1, 1\rangle \\ &= \left\langle \frac{\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{3}\right\rangle = \frac{\pi}{6}\langle 1, -1, -2\rangle \end{aligned}$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \left\|\frac{\pi}{6}\langle 1, -1, -2\rangle\right\| = \frac{\pi}{6}\sqrt{1+1+4} = \frac{\pi\sqrt{6}}{6} = \frac{\pi}{\sqrt{6}}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\frac{\pi}{6}\langle 1, -1, -2\rangle}{\frac{\pi}{\sqrt{6}}} = \frac{1}{\sqrt{6}}\langle 1, -1, -2\rangle$$

Finally we obtain the decomposition,

$$\mathbf{a} = \left\langle 0, 0, -\frac{\pi}{2} \right\rangle = \frac{\pi}{2\sqrt{3}}\mathbf{T} + \frac{\pi}{\sqrt{6}}\mathbf{N}$$

where $\mathbf{T} = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$ and $\mathbf{N} = \frac{1}{\sqrt{6}}\langle 1, -1, -2\rangle$.

42. Let $\mathbf{r}(t) = \left\langle t^2, 4t - 3\right\rangle$. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$, and show that the decomposition of $\mathbf{a}(t)$ into tangential and normal components is

$$\mathbf{a}(t) = \left(\frac{2t}{\sqrt{t^2+4}}\right)\mathbf{T} + \left(\frac{4}{\sqrt{t^2+4}}\right)\mathbf{N}$$

SOLUTION

(a) We differentiate $\mathbf{r}(t)$ twice to obtain:

$$\mathbf{r}'(t) = \langle 2t, 4\rangle \tag{1}$$
$$\mathbf{r}''(t) = \langle 2, 0\rangle$$

By the formula for the tangential component of $\mathbf{a}(t)$, we have:

$$a_{\mathbf{T}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{a}\cdot\mathbf{r}'}{\|\mathbf{r}'\|} \tag{2}$$

By (1) we get:

$$\|\mathbf{r}'\| = \|\langle 2t, 4\rangle\| = \sqrt{(2t)^2 + 4^2} = \sqrt{4t^2+16} = 2\sqrt{t^2+4} \tag{3}$$
$$\mathbf{a} = \mathbf{r}'' = \langle 2, 0\rangle$$

Substituting (3) in (2) yields:

$$a_{\mathbf{T}} = \frac{\langle 2, 0\rangle \cdot \langle 2t, 4\rangle}{2\sqrt{t^2+4}} = \frac{2\cdot 2t + 0\cdot 4}{2\sqrt{t^2+4}} = \frac{2t}{\sqrt{t^2+4}} \tag{4}$$

(b) By the formula for the normal component of $\mathbf{a}(t)$, we have:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{r}'}{\mathbf{r}'\cdot\mathbf{r}'}\right)\mathbf{r}' \tag{5}$$

We use (1) and (3) to compute the values in (5):

$$\mathbf{a}\cdot\mathbf{r}' = \langle 2, 0\rangle \cdot \langle 2t, 4\rangle = 2\cdot 2t + 0\cdot 4 = 4t$$
$$\mathbf{r}'\cdot\mathbf{r}' = \langle 2t, 4\rangle\cdot\langle 2t, 4\rangle = 2t\cdot 2t + 4\cdot 4 = 4t^2 + 16 = 4(t^2+4) \tag{6}$$

Substituting (1), (3) and (6) into (5) gives:

$$a_{\mathbf{N}}\mathbf{N} = \langle 2, 0\rangle - \frac{4t}{4\left(t^2+4\right)}\langle 2t, 4\rangle = \langle 2, 0\rangle - \frac{t}{t^2+4}\langle 2t, 4\rangle$$
$$= \left\langle 2 - \frac{2t^2}{t^2+4}, \frac{-4t}{t^2+4}\right\rangle = \left\langle \frac{8}{t^2+4}, \frac{-4t}{t^2+4}\right\rangle = \frac{1}{t^2+4}\langle 8, -4t\rangle \tag{7}$$

Since $a_{\mathbf{N}} = \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|}$, $a_{\mathbf{N}}$ is positive, hence $\mathbf{N}$ is a unit vector in the direction of $a_{\mathbf{N}}\mathbf{N}$. Hence, by (7):

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{\|a_{\mathbf{N}}\mathbf{N}\|} = \frac{\frac{1}{t^2+4}\langle 8, -4t\rangle}{\frac{1}{t^2+4}\sqrt{8^2+(-4t)^2}} = \frac{\langle 8, -4t\rangle}{\sqrt{64+16t^2}} = \frac{4\langle 2, -t\rangle}{4\sqrt{4+t^2}} = \frac{1}{\sqrt{4+t^2}}\langle 2, -t\rangle \tag{8}$$

Combining (7) and (8) yields:

$$a_{\mathbf{N}}\mathbf{N} = \frac{4}{t^2+4}\langle 2, -t\rangle = \frac{4}{\sqrt{t^2+4}}\mathbf{N} \quad \Rightarrow \quad a_{\mathbf{N}} = \frac{4}{\sqrt{t^2+4}} \tag{9}$$

(c) By (4) and (9) we obtain the following decomposition:

$$\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T} + a_{\mathbf{N}}(t)\mathbf{N} = \left(\frac{2t}{\sqrt{t^2+4}}\right)\mathbf{T} + \left(\frac{4}{\sqrt{t^2+4}}\right)\mathbf{N}$$

43. Find the components $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ of the acceleration vector of a particle moving along a circular path of radius $R = 100$ cm with constant velocity $v_0 = 5$ cm/s.

SOLUTION Since the particle moves with constant speed, we have $v'(t) = 0$, hence:

$$a_{\mathbf{T}} = v'(t) = 0$$

The normal component of the acceleration is $a_{\mathbf{N}} = \kappa(t)v(t)^2$. The curvature of a circular path of radius $R = 100$ is $\kappa(t) = \frac{1}{R} = \frac{1}{100}$, and the velocity is the constant value $v(t) = v_0 = 5$. Hence,

$$a_{\mathbf{N}} = \frac{1}{R}v_0^2 = \frac{25}{100} = 0.25 \text{ cm/s}^2$$

44. In the notation of Example 5, find the acceleration vector for a person seated in a car at (a) the highest point of the Ferris wheel and (b) the two points level with the center of the wheel.

SOLUTION In Example 6 we are given that the ferris wheel has radius $R = 30$ m. At time $t = t_0$, the wheel rotates counterclockwise with speed of 40 meters per minute and is slowing at a rate of 15 m/min^2. The decomposition of $\mathbf{a}(t)$ into tangential and normal direction at time t_0 is:

$$\mathbf{a}(t_0) = a_{\mathbf{T}}(t_0)\mathbf{T}(t_0) + a_{\mathbf{N}}(t_0)\mathbf{N}(t_0) \tag{1}$$

where

$$a_{\mathbf{T}}(t_0) = v'(t_0) \quad \text{and} \quad a_{\mathbf{N}}(t_0) = \kappa(t_0)v(t_0)^2 \tag{2}$$

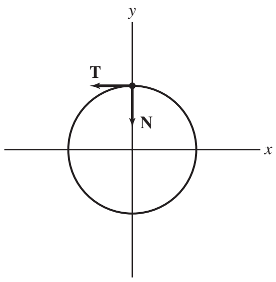
By the given information, $v(t_0) = 40$ and $v'(t_0) = -15$. Also, the curvature of the wheel is $\kappa = \frac{1}{R} = \frac{1}{30}$. Substituting in (2) we have:

$$a_{\mathbf{T}}(t_0) = -15, \quad a_{\mathbf{N}}(t_0) = \frac{40^2}{30} = \frac{160}{3}$$

Combining with (1) we get:

$$\mathbf{a}(t_0) = -15\mathbf{T}(t_0) + \frac{160}{3}\mathbf{N}(t_0) \tag{3}$$

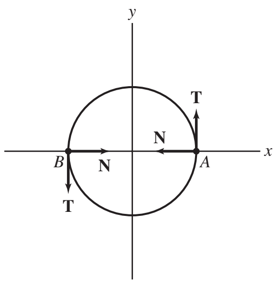
(a)



At the highest point of the wheel, $\mathbf{T} = \langle -1, 0\rangle$ and $\mathbf{N} = \langle 0, -1\rangle$, therefore by (3) the acceleration vector at this point is:

$$\mathbf{a}(t_0) = -15\,\langle -1, 0\rangle + \frac{160}{3}\,\langle 0, -1\rangle \approx \langle 15, -53.3\rangle$$

(b)



At the point A (see figure) we have $\mathbf{T} = \langle 0, 1\rangle$ and $\mathbf{N} = \langle -1, 0\rangle$, and at the point B, $\mathbf{T} = \langle 0, -1\rangle$ and $\mathbf{N} = \langle 1, 0\rangle$. Substituting in (3) we obtain the following accelerations at these points, at $t = t_0$:

At the point A:

$$\mathbf{a}(t_0) = -15\,\langle 0, 1\rangle + \frac{160}{3}\,\langle -1, 0\rangle = \langle -160/3, -15\rangle$$

At the point B:

$$\mathbf{a}(t_0) = -15\,\langle 0, -1\rangle + \frac{160}{3}\,\langle 1, 0\rangle = \langle 160/3, 15\rangle$$

45. Suppose that the Ferris wheel in Example 5 is rotating clockwise and that the point P at angle 45° has acceleration vector $\mathbf{a} = \langle 0, -50\rangle$ m/min^2 pointing down, as in Figure 11. Determine the speed and tangential acceleration of the Ferris wheel.

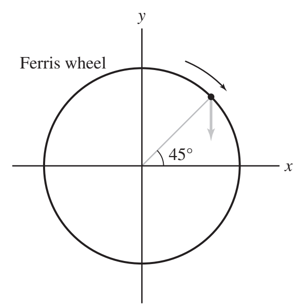


FIGURE 11

SOLUTION The normal and tangential accelerations are both $50/\sqrt{2} \approx 35$ m/min^2. The normal acceleration is $v^2/R = v^2/30 = 35$, so the speed is

$$v = \sqrt{35(28)} \approx 31.3$$

46. At time t_0, a moving particle has velocity vector $\mathbf{v} = 2\mathbf{i}$ and acceleration vector $\mathbf{a} = 3\mathbf{i} + 18\mathbf{k}$. Determine the curvature $\kappa(t_0)$ of the particle's path at time t_0.

SOLUTION The curvature is the following value:

$$\kappa(t_0) = \frac{\|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)\|}{\|\mathbf{r}'(t_0)\|^3} \tag{1}$$

Since $\mathbf{r}'(t_0) = \mathbf{v}(t) = 2\mathbf{i}$ and $\mathbf{r}''(t_0) = \mathbf{a}(t_0) = 3\mathbf{i} + 18\mathbf{k}$ we have:

$$\mathbf{r}'(t_0) \times \mathbf{r}''(t_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 3 & 0 & 18 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 18 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 3 & 18 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}\mathbf{k} = -36\mathbf{j} \tag{2}$$

$$\|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)\| = \|-36\mathbf{j}\| = 36 \tag{3}$$

$$\|\mathbf{r}'(t_0)\| = \|2\mathbf{i}\| = 2$$

Substituting (2) into (1) gives the following curvature:

$$\kappa(t_0) = \frac{36}{2^3} = 4.5$$

47. A space shuttle orbits the earth at an altitude 400 km above the earth's surface, with constant speed $v = 28{,}000$ km/h. Find the magnitude of the shuttle's acceleration (in km/h^2), assuming that the radius of the earth is 6378 km (Figure 12).

FIGURE 12 Space shuttle orbit.

SOLUTION The shuttle is in a uniform circular motion, therefore the tangential component of its acceleration is zero, and the acceleration can be written as:

$$\mathbf{a} = \kappa v^2 \mathbf{N} \tag{1}$$

The radius of motion is $6378 + 400 = 6778$ km hence the curvature is $\kappa = \frac{1}{6778}$. Also by the given information the constant speed is $v = 28000$ km/h. Substituting these values in (1) we get:

$$\mathbf{a} = \left(\frac{1}{6778} \cdot 28000^2\right)\mathbf{N} = (11.5668 \cdot 10^4 \text{ km/h}^2)\mathbf{N}$$

The magnitude of the shuttle's acceleration is thus:

$$\|\mathbf{a}\| = 11.5668 \cdot 10^4 \text{ km/h}^2$$

In units of m/s^2 we obtain

$$\|\mathbf{a}\| = \frac{11.5668 \cdot 10^4 \cdot 1000}{3600^2} = 8.925 \text{ m/s}^2$$

48. A car proceeds along a circular path of radius $R = 300$ m centered at the origin. Starting at rest, its speed increases at a rate of t m/s^2. Find the acceleration vector $\mathbf{a}$ at time $t = 3$ s and determine its decomposition into normal and tangential components.

SOLUTION The acceleration vector can be decomposed into tangential and normal directions as follows:

$$\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t) \tag{1}$$

where

$$a_{\mathbf{T}}(t) = v'(t) \quad \text{and} \quad a_{\mathbf{N}}(t) = \kappa(t)v(t)^2 \tag{2}$$

Since the speed $v(t)$ is increasing at a rate of t m/s^2, we have $v'(t) = t$. The car starts at rest hence the initial speed is $v_0 = 0$. We now integrate to find $v(t)$:

$$v(t) = \int_0^t v'(u)\,du = \int_0^t u\,du = \frac{1}{2}t^2 + v_0 = \frac{1}{2}t^2 + 0 = \frac{1}{2}t^2$$

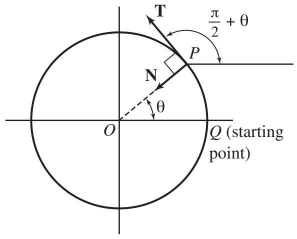
The curvature of the circular path is $\kappa(t) = \frac{1}{R} = \frac{1}{300}$. Substituting $v'(t) = t$, $\kappa = \frac{1}{300}$, and $v(t) = \frac{1}{2}t^2$ in (2) gives:

$$a_{\mathbf{T}}(t) = t, \quad a_{\mathbf{N}}(t) = \frac{1}{300}\left(\frac{1}{2}t^2\right)^2 = \frac{1}{1200}t^4$$

Combining with (1) gives the following decomposition:

$$a(t) = t\mathbf{T}(t) + \frac{1}{1200}t^4\mathbf{N}(t) \tag{3}$$

We now find the unit tangent $\mathbf{T}(t)$ and the unit normal $\mathbf{N}(t)$.



We have (see figure):

$$\mathbf{T} = \left\langle \cos\left(\frac{\pi}{2} + \theta\right), \sin\left(\frac{\pi}{2} + \theta\right) \right\rangle = \langle -\sin\theta, \cos\theta \rangle \tag{4}$$

$$\mathbf{N} = \langle \cos(\pi + \theta), \sin(\pi + \theta) \rangle = \langle -\cos\theta, -\sin\theta \rangle \tag{5}$$

We use the arc length formula to find θ:

$$\widehat{PQ} = \int_0^t \|r'(u)\|\,du = \int_0^t v(u)\,du = \int_0^t \frac{1}{2}u^2\,du = \frac{t^3}{6}$$

In addition, $\widehat{PQ} = R\theta = 300\theta$. Hence,

$$300\theta = \frac{t^3}{6} \quad \Rightarrow \quad \theta = \frac{t^3}{1800}$$

Substituting in (4) and (5) yields:

$$\mathbf{T} = \left\langle -\sin\frac{t^3}{1800}, \cos\frac{t^3}{1800} \right\rangle; \quad \mathbf{N} = \left\langle -\cos\frac{t^3}{1800}, -\sin\frac{t^3}{1800} \right\rangle \tag{6}$$

We now combine (3) and (6) to obtain the following decomposition:

$$\mathbf{a}(t) = t\left\langle -\sin\frac{t^3}{1800}, \cos\frac{t^3}{1800} \right\rangle + \frac{1}{1200}t^4\left\langle -\cos\frac{t^3}{1800}, -\sin\frac{t^3}{1800} \right\rangle$$

At $t = 3$ we get:

$$a_{\mathbf{T}} = 3a_{\mathbf{N}} = \frac{3^4}{1200} \approx 0.0675$$

$$\mathbf{T} = \left\langle -\sin\frac{3^3}{1800}, \cos\frac{3^3}{1800} \right\rangle \approx \langle -0.0150, 0.9999 \rangle$$

$$\mathbf{N} = \left\langle -\cos\frac{3^3}{1800}, -\sin\frac{3^3}{1800} \right\rangle \approx \langle -0.9999, -0.0150 \rangle$$

49. A runner runs along the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. When he is at position $\mathbf{r}\left(\frac{\pi}{2}\right)$, his speed is 3 m/s and he is accelerating at a rate of $\frac{1}{2}$ m/s^2. Find his acceleration vector $\mathbf{a}$ at this moment. *Note:* The runner's acceleration vector does not coincide with the acceleration vector of $\mathbf{r}(t)$.

SOLUTION We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2},$$

$$\Rightarrow \quad \mathbf{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

By definition, $\mathbf{N}$ is the unit vector in the direction of

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \qquad \Rightarrow \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle$$

Therefore $\mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle$. At $t = \pi/2$, we have

$$\mathbf{T} = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle, \qquad \mathbf{N} = \langle 0, -1, 0 \rangle$$

The acceleration vector is

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

We need to find the curvature, which happens to be constant:

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\frac{d\mathbf{T}}{dt}\|}{\|\mathbf{r}'\|} = \frac{\|\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle\|}{\sqrt{2}} = \frac{1}{2}$$

Now we have

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N} = \left(\frac{1}{2}\right)\mathbf{T} + \left(\frac{1}{2}\right)(3^2)\mathbf{N} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\langle -1, 0, 1 \rangle + \frac{9}{2}\langle 0, -1, 0 \rangle$$

$$= \left\langle -\frac{1}{2\sqrt{2}}, -\frac{9}{2}, \frac{1}{2\sqrt{2}} \right\rangle$$

50. Explain why the vector **w** in Figure 13 cannot be the acceleration vector of a particle moving along the circle. *Hint:* Consider the sign of $\mathbf{w} \cdot \mathbf{N}$.

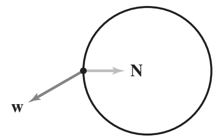


FIGURE 13

SOLUTION If we consider the sign of $\mathbf{w} \cdot \mathbf{N}$, recall that:

$$\mathbf{w} \cdot \mathbf{N} = \|\mathbf{w}\|\|\mathbf{N}\| \cos\theta$$

where θ is the angle between them. Since θ in the figure is larger than $\pi/2$ we know that $\cos\theta < 0$. Therefore, $\mathbf{w} \cdot \mathbf{N} < 0$ as well.

If we assume that the particle follows this circular path, and **w** is the acceleration vector ($\mathbf{w} = \mathbf{a}$), we will compute $\mathbf{a} \cdot \mathbf{N}$ to see:

$$\mathbf{w} \cdot \mathbf{N} = \mathbf{a} \cdot \mathbf{N} = \mathbf{r}'' \cdot \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{1}{\|\mathbf{T}'\|}(\mathbf{r}'' \cdot \mathbf{T}')$$

Now we know $\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$ so then differentiating we get:

$$\mathbf{T}' = \frac{1}{\|\mathbf{r}'\|}\mathbf{r}''$$

Substituting this fact in we see:

$$\begin{aligned} \mathbf{w} \cdot \mathbf{N} = \mathbf{a} \cdot \mathbf{N} &= \mathbf{r}'' \cdot \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{1}{\|\mathbf{T}'\|}(\mathbf{r}'' \cdot \mathbf{T}') \\ &= \frac{1}{\|\mathbf{T}'\|}\left(\mathbf{r}'' \cdot \frac{\mathbf{r}''}{\|\mathbf{r}'\|}\right) = \frac{1}{\|\mathbf{T}'\|}\frac{1}{\|\mathbf{r}'\|}(\mathbf{r}'' \cdot \mathbf{r}'') = \frac{\|\mathbf{r}''\|^2}{\|\mathbf{T}'\|\|\mathbf{r}'\|} > 0 \end{aligned}$$

This is a contradiction, **w** cannot be the acceleration vector of the particle moving along this circle.

51. Figure 14 shows acceleration vectors of a particle moving clockwise around a circle. In each case, state whether the particle is speeding up, slowing down, or momentarily at constant speed. Explain.

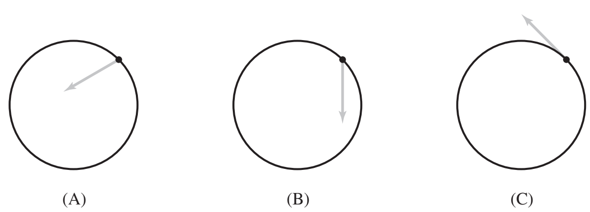


FIGURE 14

SOLUTION In (A) and (B) the acceleration vector has a nonzero tangential and normal components; these are both possible acceleration vectors. In (C) the normal component of the acceleration toward the inside of the curve is zero, that is, **a** is parallel to **T**, so $\kappa \cdot v(t)^2 = 0$, so either $\kappa = 0$ (meaning our curve is not a circle) or $v(t) = 0$ (meaning our particle isn't moving). Either way, (C) is not a possible acceleration vector.

52. Prove that $a_{\mathbf{N}} = \dfrac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|}$.

SOLUTION We have $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ and therefore

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{T}$$

Using this information, consider the following:

$$\begin{aligned} \mathbf{a} \times \mathbf{v} &= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \times (\|v\|)\mathbf{T} \\ &= (a_{\mathbf{T}}\mathbf{T} \times \|\mathbf{v}\|\mathbf{T}) + (a_{\mathbf{N}}\mathbf{N} \times \|\mathbf{v}\|\mathbf{T}) \\ &= a_{\mathbf{T}}\|\mathbf{v}\|(\mathbf{T} \times \mathbf{T}) + a_{\mathbf{N}}\|\mathbf{v}\|(\mathbf{N} \times \mathbf{T}) \\ &= \mathbf{0} + a_{\mathbf{N}}\|\mathbf{v}\|(\mathbf{N} \times \mathbf{T}) \end{aligned}$$

$$= a_{\mathbf{N}}\|\mathbf{v}\|(\mathbf{N}\times\mathbf{T})$$

Both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors and they are orthogonal to each other. It follows that $\mathbf{N}\times\mathbf{T}$ is a unit vector and

$$\|\mathbf{a}\times\mathbf{v}\| = a_{\mathbf{N}}\|\mathbf{v}\|(\mathbf{N}\times\mathbf{T}) = a_{\mathbf{N}}\|\mathbf{v}\|\cdot 1 \quad \Rightarrow \frac{\|\mathbf{a}\times\mathbf{v}\|}{\|\mathbf{v}\|} = a_{\mathbf{N}}$$

53. Suppose that $\mathbf{r} = \mathbf{r}(t)$ lies on a sphere of radius R for all t. Let $\mathbf{J} = \mathbf{r}\times\mathbf{r}'$. Show that $\mathbf{r}' = (\mathbf{J}\times\mathbf{r})/\|\mathbf{r}\|^2$. *Hint:* Observe that $\mathbf{r}$ and $\mathbf{r}'$ are perpendicular.

SOLUTION

(a) Solution 1. Since $\mathbf{r} = \mathbf{r}(t)$ lies on the sphere, the vectors $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{r}' = \mathbf{r}'(t)$ are orthogonal, therefore:

$$\mathbf{r}\cdot\mathbf{r}' = 0 \tag{1}$$

We use the following well-known equality:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$$

Using this equality and (1) we obtain:

$$\begin{aligned}\mathbf{J}\times\mathbf{r} &= \left(\mathbf{r}\times\mathbf{r}'\right)\times\mathbf{r} = -\mathbf{r}\times\left(\mathbf{r}\times\mathbf{r}'\right) = -\left(\left(\mathbf{r}\cdot\mathbf{r}'\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\,\mathbf{r}'\right)\\ &= -\left(\mathbf{r}\cdot\mathbf{r}'\right)\mathbf{r} + \|\mathbf{r}\|^2\mathbf{r}' = 0\mathbf{r} + \|\mathbf{r}\|^2\mathbf{r}' = \|\mathbf{r}\|^2\mathbf{r}'\end{aligned}$$

Divided by the scalar $\|\mathbf{r}\|^2$ we obtain:

$$\mathbf{r}' = \frac{\mathbf{J}\times\mathbf{r}}{\|\mathbf{r}\|^2}$$

(b) Solution 2. The cross product $\mathbf{J} = \mathbf{r}\times\mathbf{r}'$ is orthogonal to $\mathbf{r}$ and $\mathbf{r}'$. Also, $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal, hence the vectors $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{J}$ are mutually orthogonal. Now, since $\mathbf{r}'$ is orthogonal to $\mathbf{r}$ and $\mathbf{J}$, the right-hand rule implies that $\mathbf{r}'$ points in the direction of $\mathbf{J}\times\mathbf{r}$. Therefore, for some $\alpha > 0$ we have:

$$\mathbf{r}' = \alpha\mathbf{J}\times\mathbf{r} = \|\mathbf{r}'\|\cdot\frac{\mathbf{J}\times\mathbf{r}}{\|\mathbf{J}\times\mathbf{r}\|} \tag{2}$$

By properties of the cross product and since $\mathbf{J}$, $\mathbf{r}$, and $\mathbf{r}'$ are mutually orthogonal we have:

$$\|\mathbf{J}\times\mathbf{r}\| = \|\mathbf{J}\|\|\mathbf{r}\| = \|\mathbf{r}\times\mathbf{r}'\|\|\mathbf{r}\| = \|\mathbf{r}\|\|\mathbf{r}'\|\|\mathbf{r}\| = \|\mathbf{r}\|^2\|\mathbf{r}'\|$$

Substituting in (2) we get:

$$\mathbf{r}' = \|\mathbf{r}'\|\frac{\mathbf{J}\times\mathbf{r}}{\|\mathbf{r}\|^2\|\mathbf{r}'\|} = \frac{\mathbf{J}\times\mathbf{r}}{\|\mathbf{r}\|^2}$$

Further Insights and Challenges

54. The orbit of a planet is an ellipse with the sun at one focus. The sun's gravitational force acts along the radial line from the planet to the sun (the dashed lines in Figure 15), and by Newton's Second Law, the acceleration vector points in the same direction. Assuming that the orbit has positive eccentricity (the orbit is not a circle), explain why the planet must slow down in the upper half of the orbit (as it moves away from the sun) and speed up in the lower half. Kepler's Second Law, discussed in the next section, is a precise version of this qualitative conclusion. *Hint:* Consider the decomposition of $\mathbf{a}$ into normal and tangential components.

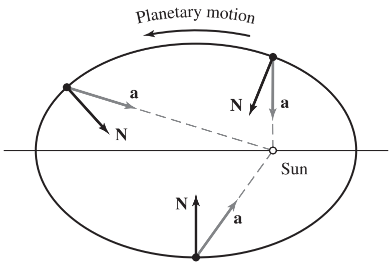
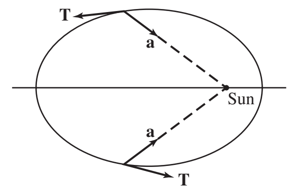


FIGURE 15 Elliptical orbit of a planet around the sun.

SOLUTION In the upper half of the orbit, as the planet moves away from the sun the acceleration vector has a negative component in the tangential direction $\mathbf{T}$, so the particle's velocity is decreasing (since $a_{\mathbf{T}}(t) = v'(t) < 0$).



However, in the lower half of the orbit, as the planet gets closer to the sun, the acceleration has a positive component in the tangential direction, that is, $a_{\mathbf{T}}(t) = v'(t) > 0$. Therefore the velocity $v(t)$ is increasing.

In Exercises 55–59, we consider an automobile of mass m traveling along a curved but level road. To avoid skidding, the road must supply a frictional force $\mathbf{F} = m\mathbf{a}$*, where* $\mathbf{a}$ *is the car's acceleration vector. The maximum magnitude of the frictional force is* μmg*, where* μ *is the coefficient of friction and* $g = 9.8$ m/s^2*. Let* v *be the car's speed in meters per second.*

55. Show that the car will not skid if the curvature κ of the road is such that (with $R = 1/\kappa$)

$$(v')^2 + \left(\frac{v^2}{R}\right)^2 \le (\mu g)^2 \qquad \boxed{5}$$

Note that braking ($v' < 0$) and speeding up ($v' > 0$) contribute equally to skidding.

SOLUTION To avoid skidding, the frictional force the road must supply is:

$$\mathbf{F} = m\mathbf{a}$$

where **a** is the acceleration of the car. We consider the decomposition of the acceleration **a** into normal and tangential directions:

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \kappa v^2(t)\mathbf{N}(t)$$

Since **N** and **T** are orthogonal unit vectors, $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} \cdot \mathbf{T} = \mathbf{N} \cdot \mathbf{N} = 1$. Thus:

$$\begin{aligned}\|\mathbf{a}\|^2 &= \left(v'\mathbf{T} + \kappa v^2\mathbf{N}\right) \cdot \left(v'\mathbf{T} + \kappa v^2\mathbf{N}\right) = v'^2\mathbf{T} \cdot \mathbf{T} + 2\kappa v^2 v'\mathbf{N} \cdot \mathbf{T} + \kappa^2 v^4 \mathbf{N} \cdot \mathbf{N} \\ &= v'^2 + \kappa^2 v^4 = v'^2 + \frac{v^4}{R^2}\end{aligned}$$

Therefore:

$$\|\mathbf{a}\| = \sqrt{\left(v'\right)^2 + \frac{v^4}{R^2}}$$

Since the maximal fractional force is μmg we obtain that to avoid skidding the curvature must satisfy:

$$m\sqrt{\left(v'\right)^2 + \frac{v^4}{R^2}} \le m\mu g.$$

Hence,

$$\left(v'\right)^2 + \frac{v^4}{R^2} \le (\mu g)^2,$$

which becomes:

$$\left(v'\right)^2 + \left(\frac{v^2}{R}\right)^2 \le (\mu g)^2$$

56. Suppose that the maximum radius of curvature along a curved highway is $R = 180$ m. How fast can an automobile travel (at constant speed) along the highway without skidding if the coefficient of friction is $\mu = 0.5$?

SOLUTION Recall the general result that max speed is

$$v = \sqrt{\mu g R}$$

In Exercise 55 we showed that the car will not skid if the following inequality is satisfied:

$$\left(v'\right)^2 + \frac{v^4}{R^2} < \mu^2 g^2$$

We compute the constant speed v for which the car can travel without skidding. In case of constant speed, $v' = 0$. We substitute $R = 180$, $\mu = 0.5$ and $g = 9.8$ and solve for v. This gives:

$$\frac{v^4}{180^2} < 0.5^2 \cdot 9.8^2$$

$$v^4 < 777924 \quad \Rightarrow \quad v < 29.70 \text{ m/s}$$

The maximum speed (in case of constant speed) is about 29.70 m/s.

57. Beginning at rest, an automobile drives around a circular track of radius $R = 300$ m, accelerating at a rate of 0.3 m/s^2. After how many seconds will the car begin to skid if the coefficient of friction is $\mu = 0.6$?

SOLUTION By Exercise 55 the car will begin to skid when:

$$(v')^2 + \frac{v^4}{R^2} = \mu^2 g^2 \tag{1}$$

We are given that $v' = 0.3$ and $v_0 = 0$. Integrating gives:

$$v = \int_0^t v'\,dt = \int_0^t 0.3\,dt = 0.3t + v_0 = 0.3t$$

We substitute $v = t$, $v' = 0.3$, $R = 300$, $\mu = 0.6$ and $g = 9.8$ in (1) and solve for t. This gives:

$$(0.3)^2 + \frac{0.3^4 t^4}{300^2} = 0.6^2 \cdot 9.8^2$$

$$t^4 = \frac{300^2\left(0.6^2 \cdot 9.8^2 - 0.3^2\right)}{0.3^4} = 383{,}160{,}000$$

$$t = 139.91 \text{ s}$$

After 139.91 s or 2.33 minutes, the car will begin to skid.

58. You want to reverse your direction in the shortest possible time by driving around a semicircular bend (Figure 16). If you travel at the maximum possible *constant speed* v that will not cause skidding, is it faster to hug the inside curve (radius r) or the outside curb (radius R)? *Hint:* Use Eq. (5) to show that at maximum speed, the time required to drive around the semicircle is proportional to the square root of the radius.

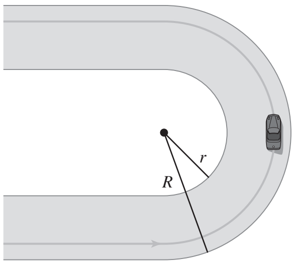


FIGURE 16 Car going around the bend.

SOLUTION In Exercise 55 we showed that the car will not skid if the following inequality is satisfied:

$$\left(v'\right)^2 + \frac{v^4}{R^2} < \mu^2 g^2$$

In case of constant speed, $v' = 0$, so the inequality becomes:

$$\frac{v^4}{R^2} < \mu^2 g^2$$

We solve for v:

$$v^4 < (\mu g R)^2 \quad \Rightarrow \quad v < \sqrt{\mu g R}$$

The maximum speed in which skidding does not occur is, thus,

$$v \approx \sqrt{\mu g R} \tag{1}$$

If T is the time required to drive around the semicircle of radius R at the constant speed v, then the length of the semicircle can be written as:

$$\pi R = \int_0^T \|\mathbf{r}'(t)\|\,dt = \int_0^T v\,dt = vT$$

Hence,

$$T = \frac{\pi R}{v} \tag{2}$$

Combining (1) and (2) gives:

$$T \approx \frac{\pi R}{\sqrt{\mu g R}} \approx \frac{\pi}{\sqrt{\mu g}}\sqrt{R}$$

We conclude that it is faster to hug the inside curve of radius r $(r < R)$, rather than the outside curve of radius R.

59. What is the smallest radius R about which an automobile can turn without skidding at 100 km/h if $\mu = 0.75$ (a typical value)?

SOLUTION In Exercise 55 we showed that the car will not skid if the following inequality holds:

$$\left(v'\right)^2 + \frac{v^4}{R^2} < \mu^2 g^2$$

In case of constant speed, $v' = 0$, so the inequality becomes:

$$\frac{v^4}{R^2} < \mu^2 g^2$$

Solving for R we get:

$$v^4 < \mu^2 g^2 R^2$$

$$\frac{v^4}{\mu^2 g^2} < R^2 \quad \Rightarrow \quad R > \frac{v^2}{\mu g}$$

The smallest radius R in which skidding does not occur is, thus,

$$R \approx \frac{v^2}{\mu g}$$

We substitute $v = 100$ km/h, $\mu = 0.75$, and $g \approx 127{,}008$ km/h^2 to obtain:

$$R \approx \frac{100^2}{0.75 \cdot 127{,}008} = 0.105 \text{ km}.$$

13.6 Planetary Motion According to Kepler and Newton (LT Section 14.6)

Preliminary Questions

1. Describe the relation between the vector $\mathbf{J} = \mathbf{r} \times \mathbf{r}'$ and the rate at which the radial vector sweeps out area.

SOLUTION The rate at which the radial vector sweeps out area equals half the magnitude of the vector $\mathbf{J}$. This relation is expressed in the formula:

$$\frac{dA}{dt} = \frac{1}{2}\|\mathbf{J}\|.$$

2. Equation (1) shows that $\mathbf{r}''$ is proportional to $\mathbf{r}$. Explain how this fact is used to prove Kepler's Second Law.

SOLUTION In the proof of Kepler's Second Law it is shown that the rate at which area is swept out is

$$\frac{dA}{dt} = \frac{1}{2}\|\mathbf{J}\|, \quad \text{where} \quad \mathbf{J} = \mathbf{r}(t) \times \mathbf{r}'(t)$$

To show that $\|\mathbf{J}\|$ is constant, show that $\mathbf{J}$ is constant. This is done using the proportionality of $\mathbf{r}''$ and $\mathbf{r}$ which implies that $\mathbf{r}(t) \times \mathbf{r}''(t) = 0$. Using this we get:

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times \mathbf{r}'\right) = \mathbf{r} \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}' = \mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{J} = \text{const}$$

3. How is the period T affected if the semimajor axis a is increased four-fold?

SOLUTION Kepler's Third Law states that the period T of the orbit is given by:

$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

or

$$T = \frac{2\pi}{\sqrt{GM}}a^{3/2}$$

If a is increased four-fold the period becomes:

$$\frac{2\pi}{\sqrt{GM}}(4a)^{3/2} = 8 \cdot \frac{2\pi}{\sqrt{GM}}a^{3/2}$$

That is, the period is increased eight-fold.

Exercises

1. Kepler's Third Law states that T^2/a^3 has the same value for each planetary orbit. Do the data in the following table support this conclusion? Estimate the length of Jupiter's period, assuming that $a = 77.8 \times 10^{10}$ m.

Planet	Mercury	Venus	Earth	Mars
a (10^{10} m)	5.79	10.8	15.0	22.8
T (years)	0.241	0.615	1.00	1.88

SOLUTION Using the given data we obtain the following values of T^2/a^3, where a, as always, is measured not in meters but in 10^{10} m:

Planet	Mercury	Venus	Earth	Mars
T^2/a^3	$2.99 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$2.96 \cdot 10^{-4}$	$2.98 \cdot 10^{-4}$

The data on the planets supports Kepler's prediction. We estimate Jupiter's period (using the given a) as $T \approx \sqrt{a^3 \cdot 3 \cdot 10^{-4}} \approx 11.9$ years.

2. Finding the Mass of a Star Using Kepler's Third Law, show that if a planet revolves around a star with period T and semimajor axis a, then the mass of the star is $M = \left(\frac{4\pi^2}{G}\right)\left(\frac{a^3}{T^2}\right)$.

SOLUTION By Kepler's Third Law with the star replacing the sun we have:

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

Solving for M we obtain:

$$T^2GM = 4\pi^2a^3 \Rightarrow M = \frac{4\pi^2}{G} \cdot \frac{a^3}{T^2}$$

3. Ganymede, one of Jupiter's moons discovered by Galileo, has an orbital period of 7.154 days and a semimajor axis of 1.07×10^9 m. Use Exercise 2 to estimate the mass of Jupiter.

SOLUTION By Exercise 2, the mass of Jupiter can be computed using the following equality:

$$M = \frac{4\pi^2}{G}\frac{a^3}{T^2}$$

We substitute the given data $T = 7.154 \cdot 24 \cdot 60^2 = 618{,}105.6$ $a = 1.07 \times 10^9$ m and $G = 6.67300 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-1}$, to obtain:

$$M = \frac{4\pi^2 \cdot \left(1.07 \times 10^9\right)^3}{6.67300 \times 10^{-11} \cdot (618{,}105.6)^2} \approx 1.897 \times 10^{27} \text{ kg.}$$

4. An astronomer observes a planet orbiting a star with a period of 9.5 years and a semimajor axis of 3×10^8 km. Find the mass of the star using Exercise 2.

SOLUTION By Exercise 2 the mass of the star can be computed using the following equality:

$$M = \frac{4\pi^2}{G}\frac{a^3}{T^2} \tag{1}$$

Using units of m, kg, and s we have the following data:

$$T = 9.5 \text{ years} = 9.5 \cdot 365 \cdot 24 \cdot 3600 = 2.99 \cdot 10^8 \text{ s}$$
$$a = 3 \times 10^{11} \text{ m}$$
$$G = 6.673 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-1}$$

Substituting in (1) we get:

$$M = \frac{4\pi^2 \cdot (3 \times 10^{11})^3}{6.673 \times 10^{-11} \cdot (2.99 \times 10^8)^2} \approx 17.87 \times 10^{28} \text{ kg}.$$

5. Mass of the Milky Way The sun revolves around the center of mass of the Milky Way galaxy in an orbit that is approximately circular, of radius $a \approx 2.8 \times 10^{17}$ km and velocity $v \approx 250$ km/s. Use the result of Exercise 2 to estimate the mass of the portion of the Milky Way inside the sun's orbit (place all of this mass at the center of the orbit).

SOLUTION Write $a = 2.8 \times 10^{20}$ m and $v = 250 \times 10^3$ m/s. The circumference of the sun's orbit (which is assumed circular) is $2\pi a$ m; since the sun's speed is a constant v m/s, its period is $T = \frac{2\pi a}{v}$ s. By Exercise 2, the mass of the portion of the Milky Way inside the sun's orbit is

$$M = \left(\frac{4\pi^2}{G}\right)\left(\frac{a^3}{T^2}\right)$$

Substituting the values of a and T from above, $G = 6.673 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} gives

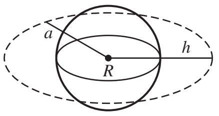
$$M = \frac{4\pi^2 a^3}{G\left(\frac{4\pi^2 a^2}{v^3}\right)} = \frac{av^2}{G} = \frac{2.8 \cdot 10^{20} \cdot (250 \times 10^3)^2}{6.673 \times 10^{-11}} = 2.6225 \times 10^{41} \text{ kg}.$$

The mass of the sun is 1.989×10^{30} kg, hence M is 1.32×10^{11} times the mass of the sun (132 billions times the mass of the sun).

6. A satellite orbiting above the equator of the earth is **geosynchronous** if the period is $T = 24$ hours (in this case, the satellite stays over a fixed point on the equator). Use Kepler's Third Law to show that in a circular geosynchronous orbit, the distance from the center of the earth is $R \approx 42{,}246$ km. Then compute the altitude h of the orbit above the earth's surface. The earth has mass $M \approx 5.974 \times 10^{24}$ kg and radius $R \approx 6371$ km.

SOLUTION By Kepler's Third Law,

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$



We substitute $T = 24 \cdot 3600 = 86{,}400$ s, $M = 5.974 \times 10^{24}$ kg, and $G = 6.673 \times 10^{-11}$ m^{3}kg^{-1}s^{-1} and solve for a. We obtain:

$$86{,}400^2 = \frac{4\pi^2}{6.673 \cdot 10^{-11} \cdot 5.974 \cdot 10^{24}} a^3$$
$$7.465 \cdot 10^9 = 0.99 \cdot 10^{-13} a^3$$
$$a^3 = 75.4 \cdot 10^{21} \Rightarrow a = 4.2246 \cdot 10^7 \text{ m} = 42{,}246 \text{ km}$$

The altitude h is thus

$$h = a - R_{\text{earth}} = 42{,}246 - 6{,}371 = 35{,}875 \text{ km}.$$

7. Show that a planet in a circular orbit travels at constant speed. *Hint:* Use that $\mathbf{J}$ is constant and that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for a circular orbit.

SOLUTION It is shown in the proof of Kepler's Second Law that the vector $\mathbf{J} = \mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, hence its length is constant:

$$\|\mathbf{J}\| = \|\mathbf{r}(t) \times \mathbf{r}'(t)\| = \text{const} \tag{1}$$

We consider the orbit as a circle of radius R, therefore, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal and $\|\mathbf{r}(t)\| = R$. By (1) and using properties of the cross product we obtain:

$$\|\mathbf{r}(t) \times \mathbf{r}'(t)\| = \|\mathbf{r}(t)\|\|\mathbf{r}'(t)\| \sin \frac{\pi}{2} = R \cdot \|\mathbf{r}'(t)\| = \text{const}$$

We conclude that $\|\mathbf{r}'(t)\|$ is constant, that is the speed $v = \|\mathbf{r}'(t)\|$ of the planet is constant.

8. Verify that the circular orbit

$$\mathbf{r}(t) = \langle R \cos \omega t,\, R \sin \omega t \rangle$$

satisfies the differential equation, Eq. (1), provided that $\omega^2 = kR^{-3}$. Then deduce Kepler's Third Law $T^2 = \left(\frac{4\pi^2}{k}\right) R^3$ for this orbit.

SOLUTION Note that $\|\mathbf{r}\| = R$, and note that

$$\mathbf{r}' = \langle -R\omega \sin \omega t,\, R\omega \cos \omega t \rangle \quad \text{and} \quad \mathbf{r}'' = \left\langle -R\omega^2 \cos \omega t,\, -R\omega^2 \sin \omega t \right\rangle$$

We rewrite this as:

$$\mathbf{r}'' = -\omega^2 \langle R \cos \omega t,\, R \sin \omega t \rangle = -\omega^2 \mathbf{r}$$

Since $\omega^2 = k/R^3$ and $R = \|\mathbf{r}\|$, we get $\mathbf{r}'' = \frac{-k}{\|\mathbf{r}\|^3}\mathbf{r}$, as desired. Since $T = \frac{2\pi}{\omega}$ then $T^2 = \frac{4\pi^2}{\omega^2} = \frac{4\pi^2 R^3}{k}$, as desired.

9. Prove that if a planetary orbit is circular of radius R, then $vT = 2\pi R$, where v is the planet's speed (constant by Exercise 7) and T is the period. Then use Kepler's Third Law to prove that $v = \sqrt{\frac{k}{R}}$.

SOLUTION By the Arc Length Formula and since the speed $v = \|\mathbf{r}'(t)\|$ is constant, the length L of the circular orbit can be computed by the following integral:

$$L = \int_0^T \|\mathbf{r}'(t)\|\, dt = \int_0^T v\, dt = vt\Big|_0^T = vT$$

On the other hand, the length of a circular orbit of radius R is $2\pi R$, so we obtain:

$$vT = 2\pi R \Rightarrow T = \frac{2\pi R}{v} \tag{1}$$

In a circular orbit of radius R, $a = R$, hence by Kepler's Third Law we have:

$$T^2 = \frac{4\pi^2}{GM} R^3 \tag{2}$$

We now substitute (1) in (2) and solve for v. This gives:

$$\left(\frac{2\pi R}{v}\right)^2 = \frac{4\pi^2 R^3}{GM}$$
$$\frac{4\pi^2 R^2}{v^2} = \frac{4\pi^2 R^3}{GM}$$
$$\frac{1}{v^2} = \frac{R}{GM} \Rightarrow v = \sqrt{\frac{GM}{R}}$$

10. Find the velocity of a satellite in geosynchronous orbit about the earth. *Hint:* Use Exercises 6 and 9.

SOLUTION In Exercise 9 we showed that the velocity of a planet in a circular orbit of radius a is:

$$v = \frac{2\pi a}{T} \tag{1}$$

A geosynchronous orbit has period $T = 24$ hours and in Exercise 6 we found that $a = 42{,}246$ km. Substituting in (1) we get:

$$v = \frac{2\pi \cdot 42{,}246}{24} = 11{,}060 \text{ km/h}$$

11. A communications satellite orbiting the earth has initial position $\mathbf{r} = \langle 29{,}000, 20{,}000, 0\rangle$ (in km) and initial velocity $\mathbf{r}' = \langle 1, 1, 1\rangle$ (in km/s), where the origin is the earth's center. Find the equation of the plane containing the satellite's orbit. *Hint:* This plane is orthogonal to $\mathbf{J}$.

SOLUTION The vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ lie in the plane containing the satellite's orbit, in particular the initial position $\mathbf{r} = \langle 29{,}000, 20{,}000, 0\rangle$ and the initial velocity $\mathbf{r}' = \langle 1, 1, 1\rangle$. Therefore, the cross product $\mathbf{J} = \mathbf{r} \times \mathbf{r}'$ is perpendicular to the plane. We compute $\mathbf{J}$:

$$\mathbf{J} = \mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 29{,}000 & 20{,}000 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 20{,}000 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 29{,}000 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 29{,}000 & 20{,}000 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$

$$= 20{,}000\mathbf{i} - 29{,}000\mathbf{j} + 9000\mathbf{k} = \langle 20{,}000, -29{,}000, 9000\rangle$$

We now use the vector form of the equation of the plane with $\mathbf{n} = \mathbf{J} = \langle 20{,}000, -29{,}000, 9000\rangle$ and $\langle x_0, y_0, z_0\rangle = \mathbf{r} = \langle 29{,}000, 20{,}000, 0\rangle$, to obtain the following equation:

$$\langle 29{,}000, -20{,}000, 9000\rangle \cdot \langle x, y, z\rangle = \langle 29{,}000, -20{,}000, 9000\rangle \cdot \langle 29{,}000, 20{,}000, 9000\rangle$$

$$1000\,\langle 29, -20, 9\rangle \cdot \langle x, y, z\rangle = 1000\,\langle 29, -20, 9\rangle \cdot \langle 29{,}000, 20{,}000, 9000\rangle$$

$$29x - 20y + 9z = 841{,}000 - 400{,}000 + 81{,}000 = 0$$

$$29x - 20y + 9z - 522{,}000 = 0$$

The plane containing the satellite's orbit is, thus:

$$\mathcal{P} = \{(x, y, z) : 29x - 20y + 9z - 522{,}000 = 0\}$$

12. Assume that the earth's orbit is circular of radius $R = 150 \times 10^6$ km (it is nearly circular with eccentricity $e = 0.017$). Find the rate at which the earth's radial vector sweeps out area in units of km^2/s. What is the magnitude of the vector $\mathbf{J} = \mathbf{r} \times \mathbf{r}'$ for the earth (in units of km^2 per second)?

SOLUTION The rate at which the earth's radial vector sweeps out area is

$$\frac{dA}{dt} = \frac{1}{2}\|\mathbf{J}\|; \quad \mathbf{J} = \mathbf{r}(t) \times \mathbf{r}'(t) \tag{1}$$

Since $\mathbf{J}$ is a constant vector, its length is constant. Moreover, if we assume that the orbit is circular then $\mathbf{r}(t)$ lies on a circle, and therefore $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Using properties of the cross product we get:

$$\|\mathbf{J}\| = \|\mathbf{r}(t) \times \mathbf{r}'(t)\| = \|\mathbf{r}(t)\|\|\mathbf{r}'(t)\| = R\|\mathbf{r}'(t)\| = \text{const}$$

We conclude that the speed $v = \|\mathbf{r}'(t)\|$ is constant. We find the speed using the following equality:

$$2\pi R = vT \Rightarrow v = \frac{2\pi R}{T}.$$

Therefore,

$$\|\mathbf{J}\| = R \cdot \frac{2\pi R}{T} = \frac{2\pi R^2}{T}.$$

Substituting in (1) we get:

$$\frac{dA}{dt} = \frac{1}{2} \cdot \frac{2\pi R^2}{T} = \frac{\pi R^2}{T}.$$

For $R = 150 \times 10^6$ km and $T = 365 \times 24 \times 3600 = 31{,}536{,}000$ s we obtain:

$$\|\mathbf{J}\| = \frac{2\pi \cdot (150 \cdot 10^6)^2}{31{,}536{,}000} = 4.483 \times 10^9 \text{ km}^2/\text{s}$$

$$\frac{dA}{dt} = 2.241 \times 10^9 \text{ km}^2/\text{s}$$

Exercises 13–19: The perihelion and aphelion are the points on the orbit closest to and farthest from the sun, respectively (Figure 8). The distance from the sun at the perihelion is denoted r_{per} and the speed at this point is denoted v_{per}. Similarly, we write r_{ap} and v_{ap} for the distance and speed at the aphelion. The semimajor axis is denoted a.

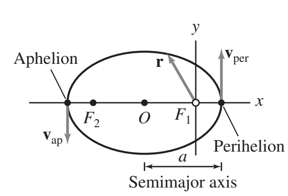


FIGURE 8 $\mathbf{r}$ and $\mathbf{v} = \mathbf{r}'$ are perpendicular at the perihelion and aphelion.

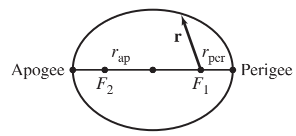
13. Use the polar equation of an ellipse

$$r = \frac{p}{1 + e\cos\theta}$$

to show that $r_{\text{per}} = a(1-e)$ and $r_{\text{ap}} = a(1+e)$. *Hint:* Use the fact that $r_{\text{per}} + r_{\text{ap}} = 2a$.

SOLUTION We use the polar equation of the elliptic orbit:

$$r = \frac{p}{1 + e\cos\theta} \tag{1}$$



At the perigee, $\theta = 0$ and at the apogee $\theta = \pi$. Substituting these values in (1) gives the distances r_{per} and r_{ap} respectively. That is,

$$r_{\text{per}} = \frac{p}{1 + e\cos\theta} = \frac{p}{1+e} \tag{2}$$

$$r_{\text{ap}} = \frac{p}{1 + e\cos\pi} = \frac{p}{1-e} \tag{3}$$

To obtain the solutions in terms of a rather than p, we notice that:

$$r_{\text{per}} + r_{\text{ap}} = 2a$$

Hence:

$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{p(1-e) + p(1+e)}{(1+e)(1-e)} = \frac{2p}{(1+e)(1-e)}$$

yielding

$$p = a(1+e)(1-e)$$

Substituting in (2) and (3) we obtain:

$$r_{\text{per}} = \frac{a(1+e)(1-e)}{1+e} = a(1-e)$$

$$r_{\text{ap}} = \frac{a(1+e)(1-e)}{1-e} = a(1+e)$$

14. Use the result of Exercise 13 to prove the formulas

$$e = \frac{r_{\text{ap}} - r_{\text{per}}}{r_{\text{ap}} + r_{\text{per}}}, \qquad p = \frac{2r_{\text{ap}}r_{\text{per}}}{r_{\text{ap}} + r_{\text{per}}}$$

SOLUTION In Exercise 13 we showed that:

$$r_{\text{per}} = a(1-e)\ ,\ r_{\text{ap}} = a(1+e)$$

Solving for a we get:

$$a = \frac{r_{\text{per}}}{1-e}\ ,\ a = \frac{r_{\text{ap}}}{1+e}$$

We equate the two expressions and solve for e to obtain:

$$\frac{r_{\text{per}}}{1-e} = \frac{r_{\text{ap}}}{1+e}$$

$$(1+e)r_{\text{per}} = (1-e)r_{\text{ap}}$$

$$r_{\text{per}} + er_{\text{per}} = r_{\text{ap}} - er_{\text{ap}}$$

$$e(r_{\text{per}} + r_{\text{ap}}) = r_{\text{ap}} - r_{\text{per}} \quad \Rightarrow \quad e = \frac{r_{\text{ap}} - r_{\text{per}}}{r_{\text{ap}} + r_{\text{per}}}$$

To show the equality for p we use the polar equation $r = \frac{p}{1+e\cos\theta}$. At the perigee $\theta = 0$ and at the apogee $\theta = \pi$, hence,

$$r_{\text{per}} = \frac{p}{1 + e\cos\theta} = \frac{p}{1+e}\ ,\ r_{\text{ap}} = \frac{p}{1 + e\cos\pi} = \frac{p}{1-e}$$

By these equalities we get:

$$r_{\text{per}} = \frac{p}{1+e} \quad \Rightarrow \quad 1+e = \frac{p}{r_{\text{per}}} \quad \Rightarrow \quad e = \frac{p}{r_{\text{per}}} - 1$$
$$r_{\text{ap}} = \frac{p}{1-e} \quad \Rightarrow \quad 1-e = \frac{p}{r_{\text{ap}}} \quad \Rightarrow \quad e = 1 - \frac{p}{r_{\text{ap}}}$$

We equate the two expressions for e and solve for p. This gives:

$$\frac{p}{r_{\text{per}}} - 1 = 1 - \frac{p}{r_{\text{ap}}}$$
$$p\left(\frac{1}{r_{\text{per}}} + \frac{1}{r_{\text{ap}}}\right) = 2$$
$$p\frac{r_{\text{ap}} + r_{\text{per}}}{r_{\text{per}} \cdot r_{\text{ap}}} = 2 \quad \Rightarrow \quad p = \frac{2r_{\text{ap}}r_{\text{per}}}{r_{\text{ap}} + r_{\text{per}}}$$

15. Use the fact that $\mathbf{J} = \mathbf{r} \times \mathbf{r}'$ is constant to prove

$$v_{\text{per}}(1-e) = v_{\text{ap}}(1+e)$$

Hint: $\mathbf{r}$ is perpendicular to $\mathbf{r}'$ at the perihelion and aphelion.

SOLUTION Since the vector $\mathbf{J}(t) = \mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, it is the same vector at the perigee and at the apogee, hence we may equate the length of $\mathbf{J}(t)$ at these two points. Since at the perigee and at the apogee $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal we have by properties of the cross product:

$$\|\mathbf{r}_{\text{ap}} \times \mathbf{r}'_{\text{ap}}\| = \|\mathbf{r}_{\text{ap}}\|\|\mathbf{r}'_{\text{ap}}\| = r_{\text{ap}}v_{\text{ap}}$$
$$\|\mathbf{r}_{\text{per}} \times \mathbf{r}'_{\text{per}}\| = \|\mathbf{r}_{\text{per}}\|\|\mathbf{r}'_{\text{per}}\| = r_{\text{per}}v_{\text{per}}$$

Equating the two values gives:

$$r_{\text{ap}}v_{\text{ap}} = r_{\text{per}}v_{\text{per}} \tag{1}$$

In Exercise 13 we showed that $r_{\text{per}} = a(1-e)$ and $r_{\text{ap}} = a(1+e)$. Substituting in (1) we obtain:

$$a(1+e)v_{\text{ap}} = a(1-e)v_{\text{per}}$$
$$(1+e)v_{\text{ap}} = (1-e)v_{\text{per}}$$

16. Compute r_{per} and r_{ap} for the orbit of Mercury, which has eccentricity $e = 0.244$ (see the table in Exercise 1 for the semimajor axis).

SOLUTION The length of the semi-major axis of the orbit of mercury is $a = 5.79 \cdot 10^7$ km. We substitute a and $e = 0.244$ in the formulas for r_{per} and r_{ap} obtained in Exercise 13, to obtain the shortest and longest distances respectively. This gives:

$$r_{\text{per}} = a(1-e) = 5.79 \cdot 10^7(1 - 0.244) = 4.377 \cdot 10^7 \text{ km}$$
$$r_{\text{ap}} = a(1+e) = 5.79 \cdot 10^7(1 + 0.244) = 7.203 \cdot 10^7 \text{ km}.$$

17. Conservation of Energy The total mechanical energy (kinetic energy plus potential energy) of a planet of mass m orbiting a sun of mass M with position $\mathbf{r}$ and speed $v = \|\mathbf{r}'\|$ is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{\|\mathbf{r}\|} \tag*{$\boxed{8}$}$$

(a) Prove the equations

$$\frac{d}{dt}\frac{1}{2}mv^2 = \mathbf{v} \cdot (m\mathbf{a}), \qquad \frac{d}{dt}\frac{GMm}{\|\mathbf{r}\|} = \mathbf{v} \cdot \left(-\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right)$$

(b) Then use Newton's Law to show that E is conserved—that is, $\frac{dE}{dt} = 0$.

SOLUTION We start by observing that since $\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{r}$, we have (using Eq. (4) in Theorem 3, Section 13.2)

$$\frac{d}{dt}\|\mathbf{r}\|^2 = 2\|\mathbf{r}\|\frac{d}{dt}\|\mathbf{r}\|, \quad \text{and} \quad \frac{d}{dt}\|\mathbf{r}\|^2 = \frac{d}{dt}\mathbf{r} \cdot \mathbf{r} = 2\mathbf{r} \cdot \mathbf{r}'$$

Equating these two expressions gives

$$\frac{d}{dt}\|\mathbf{r}\| = \frac{\mathbf{r}\cdot\mathbf{r}'}{\|\mathbf{r}\|} \tag{1}$$

(a) Applying (1) to $\mathbf{r}'$, we have

$$\frac{d}{dt}\frac{1}{2}mv^2 = \frac{d}{dt}\frac{1}{2}m\|\mathbf{r}'\|^2 = m\|\mathbf{r}'\|\frac{d}{dt}\|\mathbf{r}'\| = m\|\mathbf{r}'\|\frac{\mathbf{r}'\cdot\mathbf{r}''}{\|\mathbf{r}'\|} = \mathbf{r}'\cdot(m\mathbf{r}'') = \mathbf{v}\cdot(m\mathbf{a})$$

proving half of formula 2. For the other half, note that again by (1),

$$\begin{aligned}\frac{d}{dt}\frac{GMm}{\|\mathbf{r}\|} &= GMm\frac{d}{dt}\|\mathbf{r}\|^{-1} = -GMm\|\mathbf{r}\|^{-2}\frac{d}{dt}\|\mathbf{r}\| = -GMm\|\mathbf{r}\|^{-2}\cdot\frac{\mathbf{r}\cdot\mathbf{r}'}{\|r\|}\\ &= \mathbf{r}'\cdot\left(-\frac{GMm}{\|\mathbf{r}\|^3}\right)\mathbf{r} = \mathbf{v}\cdot\left(-\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right)\end{aligned}$$

(b) We have by part (a)

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) - \frac{d}{dt}\left(\frac{GMm}{\|\mathbf{r}\|}\right) = \mathbf{v}\cdot(m\mathbf{a}) + \mathbf{v}\cdot\left(\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right) = \mathbf{v}\cdot\left(m\mathbf{a} + \frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right) \tag{2}$$

By Newton's Law, formula (1) in the text,

$$\mathbf{r}'' = -\frac{GM}{\|\mathbf{r}\|^2}\mathbf{e}_r = -\frac{GM}{\|\mathbf{r}\|^3}\mathbf{r} \tag{3}$$

Substituting (3) into (2), and noting that $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$ gives

$$\frac{dE}{dt} = \mathbf{r}'\cdot\left(m\mathbf{r}'' + \frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right) = \mathbf{r}'\cdot\left(-\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r}\right) = 0$$

18. Show that the total energy [Eq. (8)] of a planet in a circular orbit of radius R is $E = -\frac{GMm}{2R}$. *Hint:* Use Exercise 9.

SOLUTION The total energy of a planet in a circular orbit of radius R is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{\|\mathbf{r}\|} = \frac{1}{2}mv^2 - \frac{GMm}{R} \tag{1}$$

In Exercise 9 we showed that

$$v^2 = \frac{GM}{R} \tag{2}$$

Substituting (2) in (1) we obtain:

$$E = \frac{1}{2}m\frac{GM}{R} - \frac{GMm}{R} = -\frac{1}{2}\frac{GMm}{R} = -\frac{GMm}{2R}.$$

19. Prove that $v_{\text{per}} = \sqrt{\left(\frac{GM}{a}\right)\frac{1+e}{1-e}}$ as follows:

(a) Use Conservation of Energy (Exercise 17) to show that

$$v_{\text{per}}^2 - v_{\text{ap}}^2 = 2GM\left(r_{\text{per}}^{-1} - r_{\text{ap}}^{-1}\right)$$

(b) Show that $r_{\text{per}}^{-1} - r_{\text{ap}}^{-1} = \frac{2e}{a(1-e^2)}$ using Exercise 13.

(c) Show that $v_{\text{per}}^2 - v_{\text{ap}}^2 = 4\frac{e}{(1+e)^2}v_{\text{per}}^2$ using Exercise 15. Then solve for v_{per} using (a) and (b).

SOLUTION

(a) The total mechanical energy of a planet is constant. That is,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{\|\mathbf{r}\|} = \text{const.}$$

Therefore, E has equal values at the perigee and apogee. Hence,

$$\begin{aligned}\frac{1}{2}mv_{\text{per}}^2 - \frac{GMm}{r_{\text{per}}} &= \frac{1}{2}mv_{\text{ap}}^2 - \frac{GMm}{r_{\text{ap}}}\\ \frac{1}{2}m\left(v_{\text{per}}^2 - v_{\text{ap}}^2\right) &= GMm\left(\frac{1}{r_{\text{per}}} - \frac{1}{r_{\text{ap}}}\right)\\ v_{\text{per}}^2 - v_{\text{ap}}^2 &= 2GM\left(r_{\text{per}}^{-1} - r_{\text{ap}}^{-1}\right)\end{aligned}$$

(b) In Exercise 13 we showed that $r_{\text{per}} = a(1-e)$ and $r_{\text{ap}} = a(1+e)$. Therefore,

$$r_{\text{per}}^{-1} - r_{\text{ap}}^{-1} = \frac{1}{a(1-e)} - \frac{1}{a(1+e)} = \frac{1+e-(1-e)}{a(1-e)(1+e)} = \frac{2e}{a(1-e^2)}$$

(c) In Exercise 15 we showed that

$$v_{\text{per}}(1-e) = v_{\text{ap}}(1+e)$$

Hence,

$$v_{\text{ap}} = \frac{1-e}{1+e} v_{\text{per}}$$

We compute the following difference,

$$v_{\text{per}}^2 - v_{\text{ap}}^2 = v_{\text{per}}^2 - \left(\frac{1-e}{1+e} v_{\text{per}}\right)^2 = v_{\text{per}}^2 \left(1 - \left(\frac{1-e}{1+e}\right)^2\right)$$

$$= v_{\text{per}}^2 \frac{(1+e)^2 - (1-e)^2}{(1+e)^2} = v_{\text{per}}^2 \frac{1+2e+e^2-(1-2e+e^2)}{(1+e)^2} = 4\frac{e}{(1+e)^2} v_{\text{per}}^2$$

We combine this equality with the equality in part (a) to write

$$\frac{4e}{(1+e)^2} v_{\text{per}}^2 = 2GM\left(r_{\text{per}}^{-1} - r_{\text{ap}}^{-1}\right)$$

Replacing the difference in the right-hand side by $\frac{2e}{a(1-e^2)}$ (from part (b)) and solving for v_{per} we obtain:

$$\frac{4e}{(1+e)^2} v_{\text{per}}^2 = 2GM \cdot \frac{2e}{a(1-e^2)}$$

$$v_{\text{per}}^2 = \frac{4GMe}{a(1-e)(1+e)} \cdot \frac{(1+e)^2}{4e} = \frac{GM(1+e)}{a(1-e)}$$

or,

$$v_{\text{per}} = \sqrt{\frac{GM}{a} \frac{1+e}{1-e}}$$

20. Show that a planet in an elliptical orbit has total mechanical energy $E = -\dfrac{GMm}{2a}$, where a is the semimajor axis. *Hint:* Use Exercise 19 to compute the total energy at the perihelion.

SOLUTION The total energy of a planet of mass m orbiting a sun of mass M with position $\mathbf{r}$ and speed $v = \|\mathbf{r}'\|$ is (given in Exercise 17):

$$E = \frac{1}{2}mv^2 - \frac{GMm}{\|\mathbf{r}\|} \tag{1}$$

The energy E is conserved, so we can compute it using any point on the elliptical orbit, for instance the perihelion. By Exercise 13 and Exercise 19 we have:

$$r_{\text{per}} = a(1-e)$$

$$v_{\text{per}} = \sqrt{\frac{GM}{a} \frac{1+e}{1-e}} \tag{2}$$

Substituting (2) into (1) gives:

$$E = \frac{1}{2}m \cdot \frac{GM}{a} \frac{1+e}{1-e} - \frac{GMm}{a(1-e)} = \frac{GMm}{a(1-e)}\left(\frac{1+e}{2} - 1\right) = \frac{GMm}{a(1-e)} \frac{1+e-2}{2}$$

$$= \frac{GMm}{a(1-e)} \frac{e-1}{2} = -\frac{GMm}{2a}$$

21. Prove that $v^2 = GM\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$ at any point on an elliptical orbit, where $r = \|\mathbf{r}\|$, v is the velocity, and a is the semimajor axis of the orbit.

SOLUTION The total energy $E = \frac{1}{2}mv^2 - \frac{GMm}{\|\mathbf{r}\|}$ is conserved, and in Exercise 20 we showed that its constant value is $-\frac{GMm}{2a}$. We obtain the following equality:

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

Algebraic manipulations yield:

$$v^2 = \frac{2GM}{r} - \frac{GM}{a} = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

22. Two space shuttles A and B orbit the earth along the solid trajectory in Figure 9. Hoping to catch up to B, the pilot of A applies a forward thrust to increase her shuttle's kinetic energy. Use Exercise 20 to show that shuttle A will move off into a larger orbit as shown in the figure. Then use Kepler's Third Law to show that A's orbital period T will increase (and she will fall farther and farther behind B)!

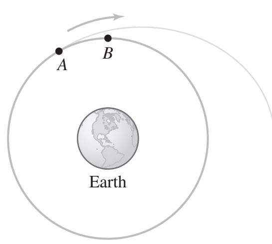


FIGURE 9

SOLUTION In Exercise 20 we showed that the total mechanical energy E of a planet in an elliptical orbit with semimajor axis a is

$$E = \frac{-GMm}{2a} \tag{1}$$

Since E is increased, a is increased, resulting in moving to an elliptic orbit as the dashed orbit in the figure. Now, by Kepler's Third Law,

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

We conclude that the orbital period T of shuttle A is also increasing, which means that A will get further and further behind B.

Further Insights and Challenges

Exercises 23 and 24 prove Kepler's Third Law. Figure 10 shows an elliptical orbit with polar equation

$$r = \frac{p}{1 + e\cos\theta}$$

where $p = J^2/k$. The origin of the polar coordinates is at F_1. Let a and b be the semimajor and semiminor axes, respectively.

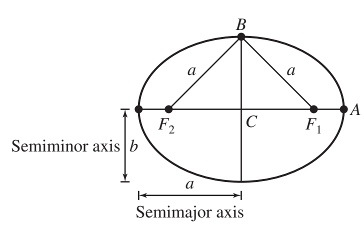


FIGURE 10

23. This exercise shows that $b = \sqrt{pa}$.

(a) Show that $CF_1 = ae$. *Hint:* $r_{\text{per}} = a(1 - e)$ by Exercise 13.

(b) Show that $a = \dfrac{p}{1 - e^2}$.

(c) Show that $F_1A + F_2A = 2a$. Conclude that $F_1B + F_2B = 2a$ and hence $F_1B = F_2B = a$.

(d) Use the Pythagorean Theorem to prove that $b = \sqrt{pa}$.

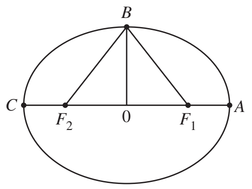
SOLUTION

(a) Since $CF_2 = AF_1$, we have:

$$F_2A = CA - CF_2 = 2a - F_1A$$

Therefore,

$$F_1A + F_2A = 2a \tag{1}$$



The ellipse is the set of all points such that the sum of the distances to the two foci F_1 and F_2 is constant. Therefore,

$$F_1A + F_2A = F_1B + F_2B \tag{2}$$

Combining (1) and (2), we obtain:

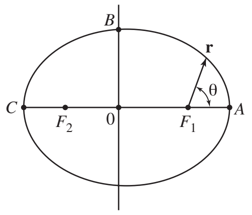
$$F_1B + F_2B = 2a \tag{3}$$

The triangle F_2BF_1 is isosceles, hence $F_2B = F_1B$ and so we conclude that

$$F_1B = F_2B = a$$

(b) The polar equation of the ellipse, where the focus F_1 is at the origin is

$$r = \frac{p}{1 + e\cos\theta}$$



The point A corresponds to $\theta = 0$, hence,

$$F_1A = \frac{p}{1 + e\cos 0} = \frac{p}{1 + e} \tag{4}$$

The point C corresponds to $\theta = \pi$ hence,

$$F_1C = \frac{p}{1 + e\cos\pi} = \frac{p}{1 - e}$$

We now find F_2A. Using the equality $CF_2 = AF_1$ we get:

$$F_2A = F_2F_1 + F_1A = F_2F_1 + F_2C = F_1C = \frac{p}{1 - e}$$

That is,

$$F_2A = \frac{p}{1 - e} \tag{5}$$

Combining (1), (4), and (5) we obtain:

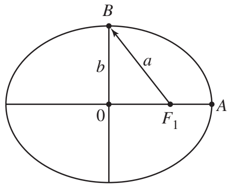
$$\frac{p}{1 + e} + \frac{p}{1 - e} = 2a$$

Hence,

$$a = \frac{1}{2}\left(\frac{p}{1 + e} + \frac{p}{1 - e}\right) = \frac{p(1 - e) + p(1 + e)}{2(1 + e)(1 - e)} = \frac{2p}{2\left(1 - e^2\right)} = \frac{p}{1 - e^2}$$

(c) We use Pythagoras' Theorem for the triangle OBF_1:

$$OB^2 + OF_1^2 = BF_1^2 \tag{6}$$



Using (4) we have

$$OF_1 = a - F_1A = a - \frac{p}{1+e}$$

Also $OB = b$ and $BF_1 = a$, hence (6) gives:

$$b^2 + \left(a - \frac{p}{1+e}\right)^2 = a^2$$

We solve for b:

$$b^2 + a^2 - \frac{2ap}{1+e} + \frac{p^2}{(1+e)^2} = a^2$$

$$b^2 - \frac{2ap}{1+e} + \frac{p^2}{(1+e)^2} = 0$$

In part (b) we showed that $a = \frac{p}{1-e^2}$. We substitute to obtain:

$$b^2 - \frac{2p}{1+e} \cdot \frac{p}{1-e^2} + \frac{p^2}{(1+e)^2} = 0$$

$$b^2 = \frac{2p^2}{(1+e)^2(1-e)} - \frac{p^2}{(1+e)^2} = \frac{2p^2 - p^2(1-e)}{(1+e)^2(1-e)}$$

$$= \frac{p^2(1+e)}{(1+e)^2(1-e)} = \frac{p^2}{1-e^2}$$

Hence,

$$b = \frac{p}{\sqrt{1-e^2}}$$

Since $1 - e^2 = \frac{p}{a}$ we also have

$$b = \frac{p}{\sqrt{\frac{p}{a}}} = \sqrt{ap}$$

24. The area A of the ellipse is $A = \pi ab$.
(a) Prove, using Kepler's First Law, that $A = \frac{1}{2}JT$, where T is the period of the orbit.
(b) Use Exercise 23 to show that $A = (\pi\sqrt{p})a^{3/2}$.
(c) Deduce Kepler's Third Law: $T^2 = \dfrac{4\pi^2}{GM}a^3$.

SOLUTION
(a) The area of an ellipse with semimajor and semiminor axes a , b respectively is,

$$A = \pi ab \tag{1}$$

In Exercise 23 we showed that $b = \sqrt{pa}$. Substituting in (1) gives:

$$A = \pi a\sqrt{pa} = \left(\pi\sqrt{p}\right)a^{3/2}$$

(b) The magnitude $\frac{1}{2}\|\mathbf{J}\|$ is the rate at which the position vector $\mathbf{r}(t)$ sweeps out areas. Since this rate is constant, the total area is obtained by multiplying the rate by the period T. That is,

$$A = \frac{1}{2}\|\mathbf{J}\|T$$

(c) Equating the expressions for A obtained in parts (a) and (b), (recall that $p = \frac{\|\mathbf{J}\|^2}{GM}$) we obtain:

$$\left(\pi\sqrt{p}\right)a^{3/2} = \frac{1}{2}\|\mathbf{J}\|T$$

$$\frac{\pi\|\mathbf{J}\|}{\sqrt{GM}}a^{3/2} = \frac{1}{2}\|\mathbf{J}\|T$$

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM}a^3$$

25. According to Eq. (7) the velocity vector of a planet as a function of the angle θ is

$$\mathbf{v}(\theta) = \frac{k}{J}\mathbf{e}_\theta + \mathbf{c}$$

Use this to explain the following statement: As a planet revolves around the sun, its velocity vector traces out a circle of radius k/J with center $\mathbf{c}$ (Figure 11). This beautiful but hidden property of orbits was discovered by William Rowan Hamilton in 1847.

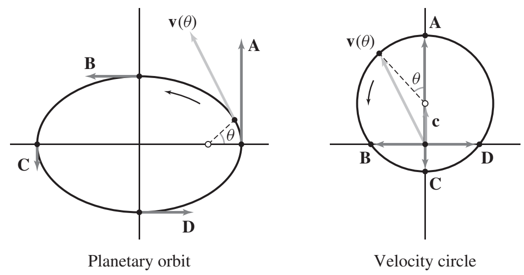


FIGURE 11 The velocity vector traces out a circle as the planet travels along its orbit.

SOLUTION Recall that $\mathbf{e}_\theta = \langle -\sin\theta, \cos\theta\rangle$, so that

$$\mathbf{v}(\theta) = \frac{k}{J}\langle -\sin\theta, \cos\theta\rangle + \mathbf{c} = \frac{k}{J}\langle \sin(-\theta), \cos(-\theta)\rangle + \mathbf{c}$$

The first term is obviously a clockwise (due to having $-\theta$ instead of θ) parametrization of a circle of radius k/J centered at the origin. It follows that $\mathbf{v}(\theta)$ is a clockwise parametrization of a circle of radius k/J and center $\mathbf{c}$.

CHAPTER REVIEW EXERCISES

1. Determine the domains of the vector-valued functions.

(a) $\mathbf{r}_1(t) = \left\langle t^{-1}, (t+1)^{-1}, \sin^{-1} t\right\rangle$ **(b)** $\mathbf{r}_2(t) = \left\langle \sqrt{8-t^3}, \ln t, e^{\sqrt{t}}\right\rangle$

SOLUTION

(a) We find the domain of $\mathbf{r}_1(t) = \left\langle t^{-1}, (t+1)^{-1}, \sin^{-1} t\right\rangle$. The function t^{-1} is defined for $t \neq 0$. $(t+1)^{-1}$ is defined for $t \neq -1$ and $\sin^{-1} t$ is defined for $-1 \le t \le 1$. Hence, the domain of $\mathbf{r}_1(t)$ is defined by the following inequalities:

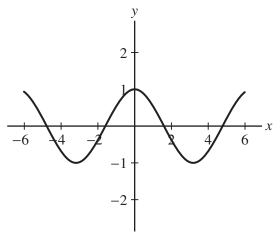
$$\begin{aligned} t &\neq 0 \\ t &\neq -1 \quad \Rightarrow \quad -1 < t < 0 \quad \text{or} \quad 0 < t \le 1 \\ -1 \le t &\le 1 \end{aligned}$$

(b) We find the domain of $\mathbf{r}_2(t) = \left\langle \sqrt{8-t^3}, \ln t, e^{\sqrt{t}}\right\rangle$. The domain of $\sqrt{8-t^3}$ is $8 - t^3 \ge 0$. The domain of $\ln t$ is $t > 0$ and $e^{\sqrt{t}}$ is defined for $t \ge 0$. Hence, the domain of $\mathbf{r}_2(t)$ is defined by the following inequalities:

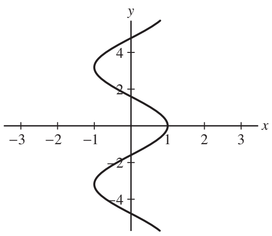
$$\begin{aligned} 8 - t^3 &\ge 0 \\ t &> 0 \\ t &\ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} t^3 &\le 8 \\ t &> 0 \end{aligned} \quad \Rightarrow \quad 0 < t \le 2$$

2. Sketch the paths $\mathbf{r}_1(\theta) = \langle \theta, \cos\theta\rangle$ and $\mathbf{r}_2(\theta) = \langle \cos\theta, \theta\rangle$ in the xy-plane.

SOLUTION The parametric equations of $\mathbf{r}_1(\theta) = (\theta, \cos\theta)$ are $x = \theta$, $y = \cos\theta$. Therefore, $y = \cos x$. The parametric equations of $\mathbf{r}_2(\theta) = (\cos\theta, \theta)$ are $x = \cos\theta$, $y = \theta$. Therefore, $x = \cos y$. We can sketch the graphs of $\mathbf{r}_1(\theta)$ and $\mathbf{r}_2(\theta)$ in the xy-plane, using the explicit relations between y and x for the two parametric representations. We obtain:



$\mathbf{r}_1(\theta) = (\theta, \cos\theta),\ y = \cos x$



$\mathbf{r}_2(\theta) = (\cos\theta, \theta),\ x = \cos y$

As seen in the graph, although $\mathbf{r}_2(\theta) = (\cos\theta, \theta)$ is a function of θ, y is not a function of x.

3. Find a vector parametrization of the intersection of the surfaces $x^2 + y^4 + 2z^3 = 6$ and $x = y^2$ in $\mathbf{R}^3$.

SOLUTION We need to find a vector parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for the intersection curve. Using $t = y$ as a parameter, we have $x = t^2$ and $y = t$. We substitute in the equation of the surface $x^2 + y^4 + 2z^3 = 6$ and solve for z in terms of t. This gives:

$$\begin{aligned} t^4 + t^4 + 2z^3 &= 6 \\ 2t^4 + 2z^3 &= 6 \\ z^3 &= 3 - t^4 \quad \Rightarrow \quad z = \sqrt[3]{3 - t^4} \end{aligned}$$

We obtain the following parametrization of the intersection curve:

$$\mathbf{r}(t) = \left\langle t^2, t, \sqrt[3]{3 - t^4}\right\rangle.$$

4. Find a vector parametrization using trigonometric functions of the intersection of the plane $x + y + z = 1$ and the elliptical cylinder $\left(\frac{y}{3}\right)^2 + \left(\frac{z}{8}\right)^2 = 1$ in $\mathbf{R}^3$.

SOLUTION We need to find a vector parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for the intersection curve. We parametrize the elliptical cylinder by:

$$x = x, \quad y = 3\sin t, \quad z = 8\cos t$$

We substitute in the equation of the plane $x + y + z = 1$ and solve for x in terms of t. This gives:

$$x + 3\sin t + 8\cos t = 1 \quad \Rightarrow \quad x = 1 - 3\sin t - 8\cos t$$

We obtain the following parametrization of the intersection curve:

$$\mathbf{r}(t) = \langle 1 - 3\sin t - 8\cos t, 3\sin t, 8\cos t\rangle$$

In Exercises 5–10, calculate the derivative indicated.

5. $\mathbf{r}'(t), \quad \mathbf{r}(t) = \left\langle 1 - t, t^{-2}, \ln t\right\rangle$

SOLUTION We use the Theorem on Componentwise Differentiation to compute the derivative $\mathbf{r}'(t)$. We get

$$\mathbf{r}'(t) = \left\langle (1-t)', (t^{-2})', (\ln t)'\right\rangle = \left\langle -1, -2t^{-3}, \frac{1}{t}\right\rangle$$

6. $\mathbf{r}'''(t), \quad \mathbf{r}(t) = \left\langle t^3, 4t^2, 7t\right\rangle$

SOLUTION We use the Theorem on Componentwise Differentiation to find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \left\langle (t^3)', (4t^2)', (7t)'\right\rangle = \left\langle 3t^2, 8t, 7\right\rangle$$

We differentiate $\mathbf{r}'(t)$ componentwise to find $\mathbf{r}''(t)$:

$$\mathbf{r}''(t) = \langle 6t, 8, 0\rangle$$

Differentiating $\mathbf{r}''(t)$ componentwise gives $\mathbf{r}'''(t)$:

$$\mathbf{r}'''(t) = \langle 6, 0, 0\rangle$$

7. $\mathbf{r}'(0)$, $\quad \mathbf{r}(t) = \left\langle e^{2t}, e^{-4t^2}, e^{6t} \right\rangle$

SOLUTION We differentiate $\mathbf{r}(t)$ componentwise to find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \left\langle (e^{2t})', \left(e^{-4t^2}\right)', (e^{6t})' \right\rangle = \left\langle 2e^{2t}, -8te^{-4t^2}, 6e^{6t} \right\rangle$$

The derivative $\mathbf{r}'(0)$ is obtained by setting $t = 0$ in $\mathbf{r}'(t)$. This gives

$$\mathbf{r}'(0) = \left\langle 2e^{2\cdot 0}, -8 \cdot 0e^{-4\cdot 0^2}, 6e^{6\cdot 0} \right\rangle = \langle 2, 0, 6 \rangle$$

8. $\mathbf{r}''(-3)$, $\quad \mathbf{r}(t) = \left\langle t^{-2}, (t+1)^{-1}, t^3 - t \right\rangle$

SOLUTION We differentiate componentwise to find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \left\langle -2t^{-3}, -(t+1)^{-2}, 3t^2 - 1 \right\rangle$$

We differentiate componentwise to find $\mathbf{r}''(t)$ and evaluate at $t = -3$:

$$\mathbf{r}''(t) = \left\langle 6t^{-4}, 2(t+1)^{-3}, 6t \right\rangle, \quad \Rightarrow \mathbf{r}''(-3) = \left\langle \frac{2}{27}, -\frac{1}{4}, -18 \right\rangle$$

9. $\dfrac{d}{dt} e^t \left\langle 1, t, t^2 \right\rangle$

SOLUTION Using the Product Rule for differentiation gives

$$\begin{aligned}\frac{d}{dt} e^t \left\langle 1, t, t^2 \right\rangle &= e^t \frac{d}{dt} \left\langle 1, t, t^2 \right\rangle + \left(e^t\right)' \left\langle 1, t, t^2 \right\rangle = e^t \langle 0, 1, 2t \rangle + e^t \left\langle 1, t, t^2 \right\rangle \\ &= e^t \left(\langle 0, 1, 2t \rangle + \left\langle 1, t, t^2 \right\rangle \right) = e^t \left\langle 1, 1+t, 2t + t^2 \right\rangle\end{aligned}$$

10. $\dfrac{d}{d\theta} \mathbf{r}(\cos\theta)$, $\quad \mathbf{r}(s) = \left\langle s, 2s, s^2 \right\rangle$

SOLUTION We use the Chain Rule to compute the derivative. That is,

$$\begin{aligned}\frac{d}{d\theta} \mathbf{r}(\cos\theta) &= \left(\left.\frac{d\mathbf{r}}{ds}\right|_{s=\cos\theta} \right) \cdot \frac{d}{d\theta}(\cos\theta) = -\sin\theta \cdot \langle 1, 2, 2s \rangle \Big|_{s=\cos\theta} \\ &= -\sin\theta \cdot \langle 1, 2, 2\cos\theta \rangle = \langle -\sin\theta, -2\sin\theta, -2\sin\theta\cos\theta \rangle \\ &= -\langle \sin\theta, 2\sin\theta, \sin 2\theta \rangle\end{aligned}$$

In Exercises 11–14, calculate the derivative at $t = 3$, assuming that

$$\begin{aligned}\mathbf{r}_1(3) &= \langle 1, 1, 0 \rangle, \qquad \mathbf{r}_2(3) = \langle 1, 1, 0 \rangle \\ \mathbf{r}_1'(3) &= \langle 0, 0, 1 \rangle, \qquad \mathbf{r}_2'(3) = \langle 0, 2, 4 \rangle\end{aligned}$$

11. $\dfrac{d}{dt}(6\mathbf{r}_1(t) - 4 \cdot \mathbf{r}_2(t))$

SOLUTION Using Differentiation Rules we obtain:

$$\begin{aligned}\left.\frac{d}{dt}(6\mathbf{r}_1(t) - 4\mathbf{r}_2(t))\right|_{t=3} &= 6\mathbf{r}_1'(3) - 4\mathbf{r}_2'(3) = 6 \cdot \langle 0, 0, 1 \rangle - 4 \cdot \langle 0, 2, 4 \rangle \\ &= \langle 0, 0, 6 \rangle - \langle 0, 8, 16 \rangle = \langle 0, -8, -10 \rangle\end{aligned}$$

12. $\dfrac{d}{dt}\left(e^t \mathbf{r}_2(t)\right)$

SOLUTION Using the Product Rule gives:

$$\frac{d}{dt}\left(e^t \mathbf{r}_2(t)\right) = e^t \mathbf{r}_2'(t) + \left(e^t\right)' \mathbf{r}_2(t) = e^t \left(\mathbf{r}_2'(t) + \mathbf{r}_2(t)\right)$$

Setting $t = 3$ we get:

$$\left.\frac{d}{dt}\left(e^t \mathbf{r}_2(t)\right)\right|_{t=3} = e^3 \left(\mathbf{r}_2'(3) + \mathbf{r}_2(3)\right) = e^3 (\langle 0, 2, 4 \rangle + \langle 1, 1, 0 \rangle) = e^3 \langle 1, 3, 4 \rangle$$

13. $\frac{d}{dt}\left(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right)$

SOLUTION Using Product Rule for Dot Products we obtain:

$$\frac{d}{dt}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)=\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)$$

Setting $t=3$ gives:

$$\left.\frac{d}{dt}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right|_{t=3}=\mathbf{r}_1(3)\cdot\mathbf{r}_2'(3)+\mathbf{r}_1'(3)\cdot\mathbf{r}_2(3)=\langle 1,1,0\rangle\cdot\langle 0,2,4\rangle+\langle 0,0,1\rangle\cdot\langle 1,1,0\rangle=2+0=2$$

14. $\frac{d}{dt}\left(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right)$

SOLUTION We use the Product Rule for Cross Product to write:

$$\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)=\mathbf{r}_1(t)\times\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\times\mathbf{r}_2(t)$$

Setting $t=3$ we obtain:

$$\begin{aligned}\left.\frac{d}{dt}\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right|_{t=3}&=\mathbf{r}_1(3)\times\mathbf{r}_2'(3)+\mathbf{r}_1'(3)\times\mathbf{r}_2(3)=\langle 1,1,0\rangle\times\langle 0,2,4\rangle+\langle 0,0,1\rangle\times\langle 1,1,0\rangle\\&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&2&4\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&1&0\end{vmatrix}=(4\mathbf{i}-4\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\\&=3\mathbf{i}-3\mathbf{j}+2\mathbf{k}=\langle 3,-3,2\rangle\end{aligned}$$

15. Calculate $\int_0^3\left\langle 4t+3,t^2,-4t^3\right\rangle dt$.

SOLUTION By the definition of vector-valued integration, we have

$$\int_0^3\left\langle 4t+3,t^2,-4t^3\right\rangle dt=\left\langle\int_0^3(4t+3)\,dt,\int_0^3 t^2\,dt,\int_0^3-4t^3\,dt\right\rangle \tag{1}$$

We compute the integrals on the right-hand side:

$$\int_0^3(4t+3)\,dt=2t^2+3t\Big|_0^3=2\cdot 9+3\cdot 3-0=27$$

$$\int_0^3 t^2\,dt=\frac{t^3}{3}\Big|_0^3=\frac{3^3}{3}=9$$

$$\int_0^3-4t^3\,dt=-t^4\Big|_0^3=-3^4=-81$$

Substituting in (1) gives the following integral:

$$\int_0^3\left\langle 4t+3,t^2,-4t^3\right\rangle dt=\langle 27,9,-81\rangle$$

16. Calculate $\int_0^\pi\left\langle\sin\theta,\theta,\cos 2\theta\right\rangle d\theta$.

SOLUTION By the definition of vector-valued integration, we have

$$\int_0^\pi\langle\sin\theta,\theta,\cos 2\theta\rangle\,d\theta=\left\langle\int_0^\pi\sin\theta\,d\theta,\int_0^\pi\theta\,d\theta,\int_0^\pi\cos 2\theta\,d\theta\right\rangle \tag{1}$$

We compute the integrals on the right hand-side:

$$\int_0^\pi\sin\theta\,d\theta=-\cos\theta\Big|_0^\pi=-(\cos\pi-\cos 0)=-(-1-1)=2$$

$$\int_0^\pi\theta\,d\theta=\frac{1}{2}\theta^2\Big|_0^\pi=\frac{\pi^2}{2}$$

$$\int_0^{\pi} \cos 2\theta \, d\theta = \frac{1}{2}\sin 2\theta \Big|_0^{\pi} = \frac{1}{2}(\sin 2\pi - \sin 0) = 0$$

Substituting in (1) gives the following integral:

$$\int_0^{\pi} \langle \sin\theta, \theta, \cos 2\theta \rangle \, d\theta = \left\langle 2, \frac{\pi^2}{2}, 0 \right\rangle$$

17. A particle located at $(1, 1, 0)$ at time $t = 0$ follows a path whose velocity vector is $\mathbf{v}(t) = \langle 1, t, 2t^2 \rangle$. Find the particle's location at $t = 2$.

SOLUTION We first find the path $\mathbf{r}(t)$ by integrating the velocity vector $\mathbf{v}(t)$:

$$\mathbf{r}(t) = \int \left\langle 1, t, 2t^2 \right\rangle dt = \left\langle \int 1\, dt, \int t\, dt, \int 2t^2\, dt \right\rangle = \left\langle t + c_1, \frac{1}{2}t^2 + c_2, \frac{2}{3}t^3 + c_3 \right\rangle$$

Denoting by $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ the constant vector, we obtain:

$$\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{2}{3}t^3 \right\rangle + \mathbf{c} \tag{1}$$

To find the constant vector $\mathbf{c}$, we use the given information on the initial position of the particle. At time $t = 0$ it is at the point $(1, 1, 0)$. That is, by (1):

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle + \mathbf{c} = \langle 1, 1, 0 \rangle$$

or,

$$\mathbf{c} = \langle 1, 1, 0 \rangle$$

We substitute in (1) to obtain:

$$\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{2}{3}t^3 \right\rangle + \langle 1, 1, 0 \rangle = \left\langle t + 1, \frac{1}{2}t^2 + 1, \frac{2}{3}t^3 \right\rangle$$

Finally, we substitute $t = 2$ to obtain the particle's location at $t = 2$:

$$\mathbf{r}(2) = \left\langle 2 + 1, \frac{1}{2} \cdot 2^2 + 1, \frac{2}{3} \cdot 2^3 \right\rangle = \left\langle 3, 3, \frac{16}{3} \right\rangle$$

At time $t = 2$ the particle is located at the point

$$\left(3, 3, \frac{16}{3}\right)$$

18. Find the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ in $\mathbf{R}^2$ satisfying $\mathbf{r}'(t) = -\mathbf{r}(t)$ with initial conditions $\mathbf{r}(0) = \langle 1, 2 \rangle$.

SOLUTION We rewrite the differential equation by components as:

$$\langle x'(t), y'(t) \rangle = -\langle x(t), y(t) \rangle \quad \text{or} \quad \langle x'(t), y'(t) \rangle = \langle -x(t), -y(t) \rangle$$

Equating corresponding components, we obtain:

$$\begin{aligned} x'(t) &= -x(t) \\ y'(t) &= -y(t) \end{aligned} \quad \Rightarrow \quad \frac{x'(t)}{x(t)} = -1, \quad \frac{y'(t)}{y(t)} = -1$$

By integration we get $\ln(x(t)) = -t + A$, $\ln(y(t)) = -t + B$ or:

$$\begin{aligned} x(t) &= ae^{-t} \\ y(t) &= be^{-t} \end{aligned} \quad \text{where} \quad a = e^A, \quad b = e^B$$

By the given information $\mathbf{r}(0) = \langle 1, 2 \rangle$. Therefore,

$$\begin{aligned} x(0) &= ae^{-0} = a = 1 \\ y(0) &= be^{-0} = b = 2 \end{aligned} \quad \Rightarrow \quad a = 1, \quad b = 2$$

We obtain the following vector:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle ae^{-t}, be^{-t} \right\rangle = \left\langle e^{-t}, 2e^{-t} \right\rangle.$$

19. Calculate $\mathbf{r}(t)$ assuming that

$$\mathbf{r}''(t) = \left\langle 4 - 16t, 12t^2 - t \right\rangle, \qquad \mathbf{r}'(0) = \langle 1, 0 \rangle, \qquad \mathbf{r}(0) = \langle 0, 1 \rangle$$

SOLUTION Using componentwise integration we get:

$$\begin{aligned} \mathbf{r}'(t) &= \int \left\langle 4 - 16t, 12t^2 - t \right\rangle dt \\ &= \left\langle \int 4 - 16t\, dt, \int 12t^2 - t\, dt \right\rangle \\ &= \left\langle 4t - 8t^2, 4t^3 - \frac{t^2}{2} \right\rangle + \mathbf{c}_1 \end{aligned}$$

Then using the initial condition $\mathbf{r}'(0) = \langle 1, 0 \rangle$ we get:

$$\mathbf{r}'(0) = \langle 1, 0 \rangle = \mathbf{c}_1$$

so then

$$\mathbf{r}'(t) = \left\langle 4t - 8t^2, 4t^3 - \frac{t^2}{2} \right\rangle + \langle 1, 0 \rangle = \left\langle 4t - 8t^2 + 1, 4t^3 - \frac{t^2}{2} \right\rangle$$

Then integrating componentwise once more we get:

$$\begin{aligned} \mathbf{r}(t) &= \int \left\langle 4t - 8t^2 + 1, 4t^3 - \frac{t^2}{2} \right\rangle dt \\ &= \left\langle \int 4t - 8t^2 + 1\, dt, \int 4t^3 - \frac{t^2}{2}\, dt \right\rangle \\ &= \left\langle 2t^2 - \frac{8}{3}t^3 + t, t^4 - \frac{t^3}{6} \right\rangle + \mathbf{c}_2 \end{aligned}$$

Using the initial condition $\mathbf{r}(0) = \langle 0, 1 \rangle$ we have:

$$\mathbf{r}(0) = \langle 0, 1 \rangle = \mathbf{c}_2$$

Therefore,

$$\mathbf{r}(t) = \left\langle 2t^2 - \frac{8}{3}t^3 + t, t^4 - \frac{t^3}{6} \right\rangle + \langle 0, 1 \rangle = \left\langle 2t^2 - \frac{8}{3}t^3 + t, t^4 - \frac{t^3}{6} + 1 \right\rangle$$

20. Solve $\mathbf{r}''(t) = \left\langle t^2 - 1, t + 1, t^3 \right\rangle$ subject to the initial conditions $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ and $\mathbf{r}'(0) = \langle -1, 1, 0 \rangle$

SOLUTION Using integration componentwise we get:

$$\begin{aligned} \mathbf{r}'(t) &= \int \left\langle t^2 - 1, t + 1, t^3 \right\rangle dt \\ &= \left\langle \int t^2 - 1\, dt, \int t + 1\, dt, \int t^3\, dt \right\rangle \\ &= \left\langle \frac{t^3}{3} - t, \frac{t^2}{2} + t, \frac{t^4}{4} \right\rangle + \mathbf{c}_1 \end{aligned}$$

Using the initial condition $\mathbf{r}'(1) = \langle -1, 1, 0 \rangle$ we get:

$$\mathbf{r}'(1) = \langle -1, 1, 0 \rangle = \left\langle -\frac{2}{3}, \frac{3}{2}, \frac{1}{4} \right\rangle + \mathbf{c}_1$$

so then, $\mathbf{c_1} = \left\langle -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{4} \right\rangle$ and

$$\mathbf{r}'(t) = \left\langle \frac{t^3}{3} - t, \frac{t^2}{2} + t, \frac{t^4}{4} \right\rangle + \left\langle -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{4} \right\rangle = \left\langle \frac{t^3}{3} - t - \frac{1}{3}, \frac{t^2}{2} + t - \frac{1}{2}, \frac{t^4}{4} - \frac{1}{4} \right\rangle$$

Using integration componentwise once more we get:

$$\begin{aligned}\mathbf{r}(t) &= \int \left\langle \frac{t^3}{3} - t - \frac{1}{3}, \frac{t^2}{2} + t - \frac{1}{2}, \frac{t^4}{4} - \frac{1}{4} \right\rangle dt \\ &= \left\langle \int \frac{t^3}{3} - t - \frac{1}{3}\, dt, \int \frac{t^2}{2} + t - \frac{1}{2}\, dt, \int \frac{t^4}{4} - \frac{1}{4}\, dt \right\rangle \\ &= \left\langle \frac{t^4}{12} - \frac{t^2}{2} - \frac{t}{3}, \frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{2}, \frac{t^5}{20} - \frac{t}{4} \right\rangle + \mathbf{c}_2\end{aligned}$$

Using the initial condition, $\mathbf{r}(1) = \langle 1, 0, 0 \rangle$ we get:

$$\mathbf{r}(1) = \langle 1, 0, 0 \rangle = \left\langle -\frac{3}{4}, \frac{1}{6}, -\frac{1}{5} \right\rangle + \mathbf{c}_2$$

and

$$\mathbf{c}_2 = \left\langle \frac{7}{4}, -\frac{1}{6}, \frac{1}{5} \right\rangle$$

Therefore,

$$\begin{aligned}\mathbf{r}(t) &= \left\langle \frac{t^4}{12} - \frac{t^2}{2} - \frac{t}{3}, \frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{2}, \frac{t^5}{20} - \frac{t}{4} \right\rangle + \left\langle \frac{7}{4}, -\frac{1}{6}, \frac{1}{5} \right\rangle \\ &= \left\langle \frac{t^4}{12} - \frac{t^2}{2} - \frac{t}{3} + \frac{7}{4}, \frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{2} - \frac{1}{6}, \frac{t^5}{20} - \frac{t}{4} + \frac{1}{5} \right\rangle\end{aligned}$$

21. Compute the length of the path

$$\mathbf{r}(t) = \left\langle \sin 2t, \cos 2t, 3t - 1 \right\rangle \quad \text{for } 1 \le t \le 3$$

SOLUTION We use the formula for the arc length:

$$s = \int_1^3 \|\mathbf{r}'(t)\|\, dt \tag{1}$$

We compute the derivative vector $\mathbf{r}'(t)$ and its length:

$$\begin{aligned}\mathbf{r}'(t) &= \langle 2\cos 2t, -2\sin 2t, 3 \rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2 + 3^2} = \sqrt{4\cos^2 2t + 4\sin^2 2t + 9} \\ &= \sqrt{4\left(\cos^2 2t + \sin^2 2t\right) + 9} = \sqrt{4 \cdot 1 + 9} = \sqrt{13}\end{aligned}$$

We substitute in (1) and compute the integral to obtain the following length:

$$s = \int_1^3 \sqrt{13}\, dt = \sqrt{13}t \Big|_1^3 = 2\sqrt{13}.$$

22. CAS Express the length of the path $\mathbf{r}(t) = \left\langle \ln t, t, e^t \right\rangle$ for $1 \le t \le 2$ as a definite integral, and use a computer algebra system to find its value to two decimal places.

SOLUTION By the arc length formula we have

$$s = \int_1^2 \|\mathbf{r}'(t)\|\, dt \tag{1}$$

We compute the vector $\mathbf{r}'(t)$ and its length:

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle \frac{1}{t}, 1, e^t \right\rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{\left(\frac{1}{t}\right)^2 + 1^2 + \left(e^t\right)^2} = \sqrt{t^{-2} + 1 + e^{2t}}\end{aligned}$$

Combining with (1) we get:

$$s = \int_1^2 \sqrt{t^{-2} + 1 + e^{2t}}\, dt$$

Using a CAS we obtain the following approximation:

$$s \approx 4.84677$$

23. Find an arc length parametrization of a helix of height 20 cm that makes four full rotations over a circle of radius 5 cm.

SOLUTION Since the radius is 5 cm and the height is 20 cm, the helix is traced by a parametrization of the form:

$$\mathbf{r}(t) = \langle 5\cos at, 5\sin at, t\rangle, \quad 0 \le t \le 20$$

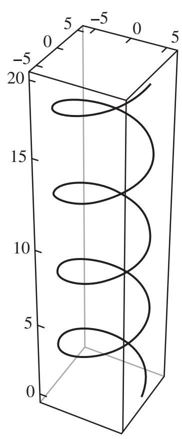
Since the helix makes exactly 4 full rotations, we have:

$$a \cdot 20 = 4 \cdot 2\pi \quad \Rightarrow \quad a = \frac{2\pi}{5}$$

The parametrization of the helix is, thus:

$$\mathbf{r}(t) = \left\langle 5\cos\frac{2\pi t}{5}, 5\sin\frac{2\pi t}{5}, t\right\rangle, \quad 0 \le t \le 20$$

The helix is shown in the following figure:



To find the arc length parametrization for the helix, we use:

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du \tag{1}$$

We find $\mathbf{r}'(t)$ and its length:

$$\mathbf{r}'(t) = \left\langle -5 \cdot \frac{2\pi}{5}\sin\frac{2\pi t}{5}, 5 \cdot \frac{2\pi}{5}\cos\frac{2\pi t}{5}, 1\right\rangle = \left\langle -2\pi\sin\frac{2\pi t}{5}, 2\pi\cos\frac{2\pi t}{5}, 1\right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{4\pi^2\sin^2\frac{2\pi t}{5} + 4\pi^2\cos^2\frac{2\pi t}{5} + 1} = \sqrt{4\pi^2\left(\sin^2\frac{2\pi t}{5} + \cos^2\frac{2\pi t}{5}\right) + 1} = \sqrt{1 + 4\pi^2}$$

Substituting in (1) we get:

$$s(t) = \int_0^t \sqrt{1 + 4\pi^2}\, du = t\sqrt{1 + 4\pi^2}$$

Therefore, we let $s = t\sqrt{1 + 4\pi^2}$ and thus,

$$t = \frac{s}{\sqrt{1 + 4\pi^2}} = g(s)$$

Thus, we can write

$$\mathbf{r}(s) = \left\langle 5\cos\frac{sa}{\sqrt{1 + 4\pi^2}}, 5\sin\frac{sa}{\sqrt{1 + 4\pi^2}}, \frac{s}{\sqrt{1 + 4\pi^2}}\right\rangle, \quad 0 \le s \le 20\sqrt{1 + 4\pi^2} \approx 127.245$$

24. Find the minimum speed of a particle with trajectory $\mathbf{r}(t) = \langle t, e^{t-3}, e^{4-t} \rangle$.

SOLUTION The speed of the particle is the following function:

$$v(t) = \|\mathbf{r}'(t)\|$$

We compute the derivative vector $\mathbf{r}'(t)$ and its length:

$$\mathbf{r}'(t) = \frac{d}{dt}\langle t, e^{t-3}, e^{4-t} \rangle = \langle 1, e^{t-3}, -e^{4-t} \rangle$$
$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (e^{t-3})^2 + (-e^{4-t})^2} = \sqrt{1 + e^{2t-6} + e^{8-2t}}$$

Therefore:

$$v(t) = \sqrt{1 + e^{2t-6} + e^{8-2t}}$$

Since the function $f(x) = x^2$ is increasing for $x \geq 0$, $v(t)$ and $v^2(t)$ assume their minimum values at the same value of t. Thus, we minimize the function:

$$F(t) = v^2(t) = 1 + e^{2t-6} + e^{8-2t}$$

We compute the critical point by solving $F'(t) = 0$. This gives:

$$F'(t) = 2e^{2t-6} - 2e^{8-2t} = 0$$
$$e^{2t-6} = e^{8-2t}$$

Therefore:

$$2t - 6 = 8 - 2t$$
$$4t = 14 \quad \Rightarrow \quad t = \frac{7}{2}$$

We compute the second derivative and substitute $t = \frac{7}{2}$:

$$F''(t) = 4e^{2t-6} + 4e^{8-2t}$$
$$F''\left(\frac{7}{2}\right) = 4e^{2\cdot(7/2)-6} + 4e^{8-2\cdot(7/2)} = 4e + 4e = 8e > 0$$

The Second Derivative Test implies that $F(t)$, hence $v(t)$ as well, have a minimum at $t = \frac{7}{2}$. The minimum speed is:

$$v\left(\frac{7}{2}\right) = \sqrt{1 + e^{2\cdot(7/2)-6} + e^{8-2\cdot(7/2)}} = \sqrt{1 + 2e}$$

25. A projectile fired at an angle of 60° lands 400 m away. What was its initial speed?

SOLUTION Place the projectile at the origin, and let $\mathbf{r}(t)$ be the position vector of the projectile.

Step 1. Use Newton's Law

Gravity exerts a downward force of magnitude mg, where m is the mass of the bullet and $g = 9.8$ m/s^2. In vector form,

$$\mathbf{F} = \langle 0, -mg \rangle = m \langle 0, -g \rangle$$

Newton's Second Law $\mathbf{F} = m\mathbf{r}'(t)$ yields $m \langle 0, -g \rangle = m\mathbf{r}''(t)$ or $\mathbf{r}''(t) = \langle 0, -g \rangle$. We determine $\mathbf{r}(t)$ by integrating twice:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u)\, du = \int_0^t \langle 0, -g \rangle \, du = \langle 0, -gt \rangle + \mathbf{v}_0$$
$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u)\, du = \int_0^t (\langle 0, -gu \rangle + \mathbf{v}_0)\, du = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + t\mathbf{v}_0 + \mathbf{r}_0$$

Step 2. Use the initial conditions

By our choice of coordinates, $\mathbf{r}_0 = \mathbf{0}$. The initial velocity $\mathbf{v}_0$ has unknown magnitude v_0, but we know that it points in the direction of the unit vector $\langle \cos 60°, \sin 60° \rangle$. Therefore,

$$\mathbf{v}_0 = v_0 \langle \cos 60°, \sin 60° \rangle = v_0 \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$
$$\mathbf{r}(t) = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + t v_0 \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

Step 3. Solve for v_0.

The projectile hits the point $\langle 400, 0\rangle$ on the ground if there exists a time t such that $\mathbf{r}(t) = \langle 400, 0\rangle$; that is,

$$\left\langle 0, -\frac{1}{2}gt^2\right\rangle + tv_0\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \langle 400, 0\rangle$$

Equating components, we obtain

$$\frac{1}{2}tv_0 = 400, \quad -\frac{1}{2}gt^2 + \frac{\sqrt{3}}{2}tv_0 = 0$$

The first equation yields $t = \frac{800}{v_0}$. Now substitute in the second equation and solve, using $g = 9.8\text{m/s}^2$:

$$-4.9\left(\frac{800}{v_0}\right)^2 + \frac{\sqrt{3}}{2}\left(\frac{800}{v_0}\right)v_0 = 0$$

$$\left(\frac{800}{v_0}\right)^2 = \frac{400\sqrt{3}}{4.9}$$

$$\left(\frac{v_0}{800}\right)^2 = \frac{4.9}{400\sqrt{3}} \approx 0.00707$$

$$v_0^2 = 4526.42611, \quad v_0 \approx 67.279 \text{ m/s}$$

We obtain $v_0 \approx 67.279$ m/s.

26. A specially trained mouse runs counterclockwise in a circle of radius 0.6 m on the floor of an elevator with speed 0.3 m/s while the elevator ascends from ground level (along the z-axis) at a speed of 12 m/s. Find the mouse's acceleration vector as a function of time. Assume that the circle is centered at the origin of the xy-plane and the mouse is at $(2, 0, 0)$ at $t = 0$.

SOLUTION The x and y coordinates must trace out a circle of radius 0.6 at speed 0.3, starting at $x = 2$ and $y = 0$, so it seems reasonable to choose $x(t) = 0.6\cos\frac{t}{\sqrt{1.2}}$ and $y(t) = 0.6\sin\frac{t}{\sqrt{1.2}}$. Notice that $\left[x'(t)\right]^2 + \left[y'(t)\right]^2 = 0.3$, so this choice of $x(t)$ and $y(t)$ really does trace out a circle with speed 0.3. The z coordinate must give a (vertical) speed of 12, so $z(t) = 12t$. Thus, we have

$$\mathbf{r}(t) = \left\langle 0.6\cos\frac{t}{\sqrt{1.2}}, 0.6\sin\frac{t}{\sqrt{1.2}}, 12t\right\rangle$$

so

$$\mathbf{r}'(t) = \left\langle -\frac{0.6}{\sqrt{1.2}}\sin\frac{t}{\sqrt{1.2}}, \frac{0.6}{\sqrt{1.2}}\cos\frac{t}{\sqrt{1.2}}, 12\right\rangle$$

and

$$\mathbf{r}''(t) = \left\langle -\frac{1}{2}\cos\frac{t}{\sqrt{1.2}}, -\frac{1}{2}\sin\frac{t}{\sqrt{1.2}}, 0\right\rangle,$$

which is the acceleration vector.

27. During a short time interval $[0.5, 1.5]$, the path of an unmanned spy plane is described by

$$\mathbf{r}(t) = \left\langle -\frac{100}{t^2}, 7 - t, 40 - t^2\right\rangle$$

A laser is fired (in the tangential direction) toward the yz-plane at time $t = 1$. Which point in the yz-plane does the laser beam hit?

SOLUTION Notice first that by differentiating we get the tangent vector:

$$\mathbf{r}'(t) = \left\langle \frac{200}{t^3}, -1, -2t\right\rangle, \quad \Rightarrow \quad \mathbf{r}'(1) = \langle 200, -1, -2\rangle$$

and the tangent line to the path would be:

$$\ell(s) = \mathbf{r}(1) + s\mathbf{r}'(1) = \langle -100, 6, 39\rangle + s\langle 200, -1, -2\rangle = \langle -100 + 200s, 6 - s, 39 - 2s\rangle$$

If the laser is fired in the tangential direction toward the yz-plane means that the x-coordinate will be zero - this is when $s = 1/2$. Therefore,

$$\ell(1/2) = \langle 0, 11/2, 38\rangle$$

Hence, the laser beam will hit the point $(0, 11/2, 38)$.

28. A force $\mathbf{F} = \langle 12t + 4, 8 - 24t \rangle$ (in newtons) acts on a 2-kg mass. Find the position of the mass at $t = 2$ s if it is located at $(4, 6)$ at $t = 0$ and has initial velocity $\langle 2, 3 \rangle$ in m/s.

SOLUTION Recall the formula $\mathbf{F} = m\mathbf{a}$ then using $\mathbf{F} = \langle 12t + 4, 8 - 24t \rangle$ and $m = 2$ we get:

$$\langle 12t + 4, 8 - 24t \rangle = 2\mathbf{a}, \quad \Rightarrow \quad \mathbf{a}(t) = \mathbf{r}''(t) = \langle 6t + 2, 4 - 12t \rangle$$

Then using componentwise integration,

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 6t + 2, 4 - 12t \rangle\,dt = \left\langle 3t^2 + 2t, 4t - 6t^2 \right\rangle + \mathbf{c}_1$$

Using the initial condition $\mathbf{v}_0 = \mathbf{v}(0) = \langle 2, 3 \rangle$, we get:

$$\mathbf{v}(0) = \langle 2, 3 \rangle = \mathbf{c}_1$$

and therefore,

$$\mathbf{v}(t) = \left\langle 3t^2 + 2t + 2, 4t - 6t^2 + 3 \right\rangle$$

Using componentwise integration once more,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left\langle 3t^2 + 2t + 2, 4t - 6t^2 + 3 \right\rangle dt = \left\langle t^3 + t^2 + 2t, 2t^2 - 2t^3 + 3t \right\rangle + \mathbf{c}_2$$

Using the initial condition $\mathbf{r}(0) = \langle 4, 6 \rangle$ we get:

$$\mathbf{r}(0) = \langle 4, 6 \rangle = \mathbf{c}_2$$

Therefore,

$$\mathbf{r}(t) = \left\langle t^3 + t^2 + 2t + 4, 2t^2 - 2t^3 + 3t + 6 \right\rangle$$

and the position of the mass at $t = 2$ is $\mathbf{r}(2) = \langle 20, 4 \rangle$.

29. Find the unit tangent vector to $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$ at $t = \pi$.

SOLUTION The unit tangent vector at $t = \pi$ is

$$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{\|\mathbf{r}'(\pi)\|} \tag{1}$$

We differentiate $\mathbf{r}(t)$ componentwise to obtain:

$$\mathbf{r}'(t) = \langle \cos t, 1, -\sin t \rangle$$

Therefore,

$$\mathbf{r}'(\pi) = \langle \cos \pi, 1, -\sin \pi \rangle = \langle -1, 1, 0 \rangle$$

We compute the length of $\mathbf{r}'(\pi)$:

$$\|\mathbf{r}'(\pi)\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}$$

Substituting in (1) gives:

$$\mathbf{T}(\pi) = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

30. Find the unit tangent vector to $\mathbf{r}(t) = \langle t^2, \tan^{-1} t, t \rangle$ at $t = 1$.

SOLUTION The unit tangent vector at $t = 1$ is the following vector:

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} \tag{1}$$

We differentiate $\mathbf{r}(t) = \langle t^2, \tan^{-1} t, t \rangle$ componentwise:

$$\mathbf{r}'(t) = \left\langle 2t, \frac{1}{1 + t^2}, 1 \right\rangle$$

Setting $t = 1$, we get:

$$\mathbf{r}'(1) = \left\langle 2 \cdot 1, \frac{1}{1+1^2}, 1 \right\rangle = \left\langle 2, \frac{1}{2}, 1 \right\rangle$$

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + \left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{21}}{2}$$

Substituting in (1) we obtain the following unit tangent vector:

$$\mathbf{T}(1) = \frac{2}{\sqrt{21}} \left\langle 2, \frac{1}{2}, 1 \right\rangle = \left\langle \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$$

31. Calculate $\kappa(1)$ for $\mathbf{r}(t) = \langle \ln t, t \rangle$.

SOLUTION Recall,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Computing derivatives we get:

$$\mathbf{r}'(t) = \left\langle \frac{1}{t}, 1 \right\rangle, \quad \Rightarrow \quad \mathbf{r}'(1) = \langle 1, 1 \rangle, \quad \Rightarrow \quad \|\mathbf{r}'(1)\| = \sqrt{2}$$

$$\mathbf{r}''(t) = \left\langle -\frac{1}{t^2}, 0 \right\rangle, \quad \Rightarrow \quad \mathbf{r}''(1) = \langle -1, 0 \rangle$$

Computing the cross product we get:

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle$$

and $\|\mathbf{r}'(1) \times \mathbf{r}''(1)\| = 1$. Therefore,

$$\kappa(1) = \frac{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{1}{(\sqrt{2})^3} = \frac{1}{2^{3/2}}$$

32. Calculate $\kappa\left(\frac{\pi}{4}\right)$ for $\mathbf{r}(t) = \langle \tan t, \sec t, \cos t \rangle$.

SOLUTION Recall,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Computing derivatives we get:

$$\mathbf{r}'(t) = \left\langle \sec^2 t, \sec t \tan t, -\sin t \right\rangle, \quad \Rightarrow \quad \mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle 2, \sqrt{2}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$\mathbf{r}''(t) = \left\langle 2\sec^2 t \tan t, \sec^3 t + \tan^2 t \sec t, -\cos t \right\rangle, \quad \Rightarrow \quad \mathbf{r}''\left(\frac{\pi}{4}\right) = \left\langle 4, 3\sqrt{2}, -\frac{1}{\sqrt{2}} \right\rangle$$

Note here that $\|\mathbf{r}'(\pi/4)\| = \sqrt{4+2+1/2} = \sqrt{\frac{13}{2}}$. Computing cross products we get:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 4 & 3\sqrt{2} & -\frac{1}{\sqrt{2}} \end{vmatrix} = \left\langle 2, -\sqrt{2}, 2\sqrt{2} \right\rangle$$

where $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{4+2+8} = \sqrt{14}$. Therefore,

$$\kappa(\pi/4) = \frac{\|\mathbf{r}'(\pi/4) \times \mathbf{r}''(\pi/4)\|}{\|\mathbf{r}'(\pi/4)\|^3} = \frac{\sqrt{14}}{(\sqrt{13/2})^3} = \frac{2\sqrt{28}}{13^{3/2}}$$

In Exercises 33 and 34, write the acceleration vector **a** *at the point indicated as a sum of tangential and normal components.*

33. $\mathbf{r}(\theta) = \langle \cos\theta, \sin 2\theta \rangle, \quad \theta = \frac{\pi}{4}$

SOLUTION First note here that:

$$\mathbf{v}(\theta) = \mathbf{r}'(\theta) = \langle -\sin\theta, 2\cos 2\theta \rangle$$
$$\mathbf{a}(\theta) = \mathbf{r}''(\theta) = \langle -\cos\theta, -4\sin 2\theta \rangle$$

At $t = \pi/4$ we have:

$$\mathbf{v} = \mathbf{r}'(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, 0 \right\rangle$$
$$\mathbf{a} = \mathbf{r}''(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -4 \right\rangle$$

Thus,

$$\mathbf{a}\cdot\mathbf{v} = \left\langle -\frac{1}{\sqrt{2}}, -4 \right\rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, 0 \right\rangle = \frac{1}{2}$$
$$\|\mathbf{v}\| = \sqrt{\frac{1}{2} + 0} = \frac{1}{\sqrt{2}}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\left\langle -\frac{1}{\sqrt{2}}, 0 \right\rangle}{1/\sqrt{2}} = \langle -1, 0 \rangle$$
$$a_{\mathbf{T}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{1/2}{1/\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \left\langle -\frac{1}{\sqrt{2}}, -4 \right\rangle - \frac{1}{\sqrt{2}}\langle -1, 0 \rangle = \langle 0, -4 \rangle$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = 4$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\langle 0, -4 \rangle}{4} = \langle 0, -1 \rangle$$

Finally, we obtain the decomposition,

$$\mathbf{a} = \left\langle -\frac{1}{\sqrt{2}}, -4 \right\rangle = \frac{1}{\sqrt{2}}\mathbf{T} + 4\mathbf{N}$$

where $\mathbf{T} = \langle -1, 0 \rangle$ and $\mathbf{N} = \langle 0, -1 \rangle$.

34. $\mathbf{r}(t) = \langle t^2, 2t - t^2, t \rangle, \quad t = 2$

SOLUTION First note here that:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 2 - 2t, 1 \rangle$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, -2, 0 \rangle$$

At $t = 2$ we have:

$$\mathbf{v} = \mathbf{r}'(2) = \langle 4, -2, 1 \rangle$$
$$\mathbf{a} = \mathbf{r}''(2) = \langle 2, -2, 0 \rangle$$

Thus,

$$\mathbf{a}\cdot\mathbf{v} = \langle 4, -2, 1 \rangle \cdot \langle 2, -2, 0 \rangle = 12$$
$$\|\mathbf{v}\| = \sqrt{16 + 4 + 1} = \sqrt{21}$$

Recall that we have:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 4, -2, 1\rangle}{\sqrt{21}} = \frac{1}{\sqrt{21}}\langle 4, -2, 1\rangle$$

$$a_{\mathbf{T}} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{12}{\sqrt{21}} = \frac{4\sqrt{21}}{7}$$

Next, we compute $a_{\mathbf{N}}$ and $\mathbf{N}$:

$$\begin{aligned} a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} &= \langle 2, -2, 0\rangle - \frac{12}{\sqrt{21}}\frac{1}{\sqrt{21}}\langle 4, -2, 1\rangle \\ &= \langle 2, -2, 0\rangle - \frac{4}{7}\langle 4, -2, 1\rangle = \left\langle -\frac{2}{7}, -\frac{6}{7}, -\frac{4}{7}\right\rangle \\ &= \frac{2}{7}\langle -1, -3, -2\rangle \end{aligned}$$

This vector has length:

$$a_{\mathbf{N}} = \|a_{\mathbf{N}}\mathbf{N}\| = \frac{2}{7}\sqrt{1+9+4} = \frac{2\sqrt{14}}{7}$$

and thus,

$$\mathbf{N} = \frac{a_{\mathbf{N}}\mathbf{N}}{a_{\mathbf{N}}} = \frac{\frac{2}{7}\langle -1, -3, -2\rangle}{\frac{2\sqrt{14}}{7}} = \frac{1}{\sqrt{14}}\langle -1, -3, -2\rangle$$

Finally, we obtain the decomposition,

$$\mathbf{a} = \langle 2, -2, 0\rangle = \frac{4\sqrt{21}}{7}\mathbf{T} + \frac{2\sqrt{14}}{7}\mathbf{N}$$

where $\mathbf{T} = \frac{1}{\sqrt{21}}\langle 4, -2, 1\rangle$ and $\mathbf{N} = \frac{1}{\sqrt{14}}\langle -1, -3, -2\rangle$.

35. At a certain time t_0, the path of a moving particle is tangent to the y-axis in the positive direction. The particle's speed at time t_0 is 4 m/s, and its acceleration vector is $\mathbf{a} = \langle 5, 4, 12\rangle$. Determine the curvature of the path at t_0.

SOLUTION We are given that the particle is moving tangent to the y-axis with speed 4 m/s, so then:

$$\mathbf{r}' = \langle 0, 4, 0\rangle$$

and $\mathbf{a} = \mathbf{r}'' = \langle 5, 4, 12\rangle$. Recall the formula for curvature:

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}$$

First calculate the cross product:

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 0 \\ 5 & 4 & 12 \end{vmatrix} = \langle 48, 0, -20\rangle$$

Then the length of $\mathbf{r}'$ and $\mathbf{r}' \times \mathbf{r}''$:

$$\|\mathbf{r}'\| = 4, \quad \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{48^2 + 20^2} = \sqrt{2704} = 52$$

so then for curvature we get:

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{52}{4^3} = \frac{13}{16}$$

36. Parametrize the osculating circle to $y = x^2 - x^3$ at $x = 1$.

SOLUTION First differentiate twice:

$$f'(x) = 2x - 3x^2, \quad f''(x) = 2 - 6x$$

and at the point $x = 1$ we get:

$$f'(1) = -1, \quad f''(1) = -4$$

Step 1. Find the radius Recall the formula for curvature:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

and evaluating at $x = 1$ we have:

$$\kappa(1) = \frac{4}{[1 + (-1)^2]^{3/2}} = \frac{4}{2^{3/2}} = \sqrt{2}$$

Therefore, the radius of the osculating circle is $R = \frac{1}{\sqrt{2}}$.

Step 2. Find N at $x = 1$.

First we will parametrize the curve $f(x) = x^2 - x^3$ as:

$$\mathbf{r}(x) = \left\langle x, x^2 - x^3 \right\rangle, \quad \mathbf{r}(1) = \langle 1, 0 \rangle$$

and differentiate:

$$\mathbf{r}'(x) = \left\langle 1, 2x - 3x^2 \right\rangle$$

Note here that the vector $\left\langle 2x - 3x^2, -1 \right\rangle$ is orthogonal to $\mathbf{r}'(x)$ for all values of x and points in the direction of the bending of the curve $y = x^2 - x^3$.

Computing the unit normal to the curve, using the vector orthogonal to $\mathbf{r}'(x)$, we get:

$$\mathbf{N}(x) = \frac{\left\langle 2x - 3x^2, -1 \right\rangle}{\sqrt{(2x - 3x^2)^2 + 1}}, \quad \mathbf{N}(1) = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$$

Step 3. Find the center Q

Now to find the center Q of the osculating circle:

$$\begin{aligned} \overrightarrow{OQ} &= \mathbf{r}(1) + \kappa^{-1}\mathbf{N}(1) \\ &= \langle 1, 0 \rangle + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle -1, -1 \rangle \\ &= \left\langle \frac{1}{2}, -\frac{1}{2} \right\rangle \end{aligned}$$

The center of the osculating circle is $Q = (1/2, -1/2)$.

Step 4. Parametrize the osculating circle.

Then parametrizing the osculating circle we get:

$$\mathbf{c}(t) = \left\langle \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{1}{\sqrt{2}} \langle \cos t, \sin t \rangle$$

37. Parametrize the osculating circle to $y = \sqrt{x}$ at $x = 4$.

SOLUTION First differentiate twice:

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4x^{3/2}}$$

and at the point $x = 4$ we get:

$$f'(4) = \frac{1}{4}, \quad f''(4) = -\frac{1}{32}$$

Step 1. Find the radius

Then recall the formula for curvature:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

and evaluating at $x = 4$ we have:

$$\kappa(4) = \frac{\frac{1}{32}}{\left[1 + \frac{1}{16}\right]^{3/2}} = \frac{1}{32} \cdot \frac{1}{\left(\frac{17}{16}\right)^{3/2}} = \frac{1}{32} \frac{16^{3/2}}{17^{3/2}} = \frac{2}{17^{3/2}}$$

Therefore the radius of the osculating circle is $R = \frac{17^{3/2}}{2}$.

Step 2. Find N at $x = 4$

First we will parametrize the curve $f(x) = \sqrt{x}$ as:

$$\mathbf{r}(x) = \left\langle x, \sqrt{x} \right\rangle, \quad \mathbf{r}(4) = \langle 4, 2 \rangle$$

and differentiate:

$$\mathbf{r}'(x) = \left\langle 1, \frac{1}{2}x^{-1/2} \right\rangle$$

Note here that the vector $\left\langle \frac{1}{2}x^{-1/2}, -1 \right\rangle$ is orthogonal to $\mathbf{r}'(x)$ for all values of x and points in the direction of the bending of the curve $y = \sqrt{x}$.

Computing the unit normal to the curve, using the vector orthogonal to $\mathbf{r}'(x)$ we get:

$$\mathbf{N}(x) = \frac{\left\langle \frac{1}{2}x^{-1/2}, -1 \right\rangle}{\sqrt{\frac{1}{4x} + 1}}, \quad \mathbf{N}(4) = \frac{\left\langle \frac{1}{4}, -1 \right\rangle}{\sqrt{\frac{1}{16} + 1}} = \frac{4}{\sqrt{17}} \left\langle \frac{1}{4}, -1 \right\rangle$$

Step 3. Find the center Q

Now to find the center Q of the osculating circle:

$$\begin{aligned}
\overrightarrow{OQ} &= \mathbf{r}(4) + \kappa^{-1}\mathbf{N}(4) \\
&= \langle 4, 2 \rangle + \frac{17^{3/2}}{2} \frac{4}{\sqrt{17}} \left\langle \frac{1}{4}, -1 \right\rangle \\
&= \langle 4, 2 \rangle + 34 \left\langle \frac{1}{4}, -1 \right\rangle \\
&= \langle 4, 2 \rangle + \left\langle \frac{17}{2}, -34 \right\rangle \\
&= \left\langle \frac{25}{2}, -32 \right\rangle
\end{aligned}$$

The center of the osculating circle is $Q = (\frac{25}{2}, -32)$.

Step 4. Parametrize the osculating circle

Then parametrizing the osculating circle we get:

$$\mathbf{c}(t) = \left\langle \frac{25}{2}, -32 \right\rangle + \frac{17^{3/2}}{2} \langle \cos t, \sin t \rangle$$

38. If a planet has zero mass ($m = 0$), then Newton's laws of motion reduce to $\mathbf{r}''(t) = \mathbf{0}$ and the orbit is a straight line $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0$, where $\mathbf{r}_0 = \mathbf{r}(0)$ and $\mathbf{v}_0 = \mathbf{r}'(0)$ (Figure 1). Show that the area swept out by the radial vector at time t is $A(t) = \frac{1}{2}\|\mathbf{r}_0 \times \mathbf{v}_0\| t$ and thus Kepler's Second Law continues to hold (the rate is constant).

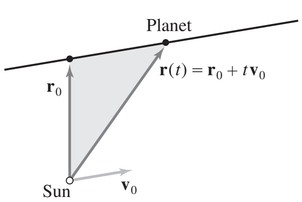


FIGURE 1

SOLUTION Integrating $\mathbf{r}''(t) = \mathbf{0}$ gives:

$$\mathbf{r}'(t) = \mathbf{c}$$

The constant $\mathbf{c}$ is $\mathbf{r}'(0) = \mathbf{v}(0)$. That is,

$$\mathbf{r}'(t) = \mathbf{v}$$

We integrate again:

$$\mathbf{r}(t) = \mathbf{v}t + \mathbf{d}$$

The constant $\mathbf{d}$ is $\mathbf{r} = \mathbf{r}(0)$. Hence, $\mathbf{r}(t) = \mathbf{r} + t\mathbf{v}$, where $\mathbf{r} = \mathbf{r}(0)$ and $\mathbf{v} = \mathbf{r}'(0)$.

39. Suppose the orbit of a planet is an ellipse of eccentricity $e = c/a$ and period T (Figure 2). Use Kepler's Second Law to show that the time required to travel from A' to B' is equal to

$$\left(\frac{1}{4} + \frac{e}{2\pi}\right)T$$

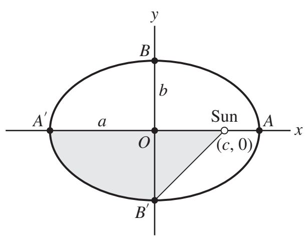


FIGURE 2

SOLUTION By the Law of Equal Areas, the position vector pointing from the sun to the planet sweeps out equal areas in equal times. We denote by S_1 the area swept by the position vector when the planet moves from A' to B', and t is the desired time. Since the position vector sweeps out the whole area of the ellipse (πab) in time T, the Law of Equal Areas implies that:

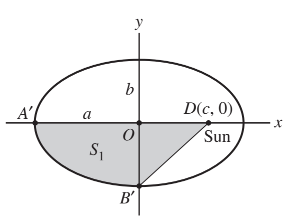
$$\frac{S_1}{\pi ab} = \frac{t}{T} \quad \Rightarrow \quad t = \frac{TS_1}{\pi ab} \tag{1}$$

We now find the area S_1 as the sum of the area of a quarter of the ellipse and the area of the triangle ODB. That is,

$$S_1 = \frac{\pi ab}{4} + \frac{\overline{OD} \cdot \overline{OB}'}{2} = \frac{\pi ab}{4} + \frac{cb}{2} = \frac{b}{4}(\pi a + 2c)$$

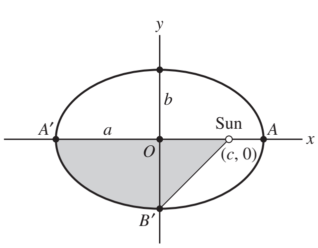
Substituting in (1) we get:

$$t = \frac{Tb(\pi a + 2c)}{4\pi ab} = \frac{T(\pi a + 2c)}{4\pi a} = T\left(\frac{1}{4} + \frac{1}{2\pi}\frac{c}{a}\right) = T\left(\frac{1}{4} + \frac{e}{2\pi}\right)$$



40. The period of Mercury is approximately 88 days, and its orbit has eccentricity 0.205. How much longer does it take Mercury to travel from A' to B' than from B' to A (Figure 2)?

SOLUTION



Let T denote the period of the orbit. By the previous exercise, the time T_1 takes the planet to travel from A' to B' is,

$$T_1 = \left(\frac{1}{4} + \frac{e}{2\pi}\right)T$$

The period of Mercury is $T = 88$ days and the eccentricity of the orbit is $e = 0.205$, hence,

$$T_1 = \left(\frac{1}{4} + \frac{0.205}{2\pi}\right) \cdot 88 \approx 24.871 \text{ days}$$

Using Kepler's Second Law, the time that takes Mercury to travel from A' to A is half a period. Therefore, the time T_2 that it takes for Mercury to travel from B' to A is the difference:

$$T_2 = \frac{1}{2}T - T_1 \approx 44 - 24.871 = 19.129 \text{ days}$$

The required time is the difference:

$$T_1 - T_2 = 24.871 - 19.129 = 5.742 \text{ days}$$

14 DIFFERENTIATION IN SEVERAL VARIABLES

14.1 Functions of Two or More Variables (LT Section 15.1)

Preliminary Questions

1. What is the difference between a horizontal trace and a level curve? How are they related?

SOLUTION A horizontal trace at height c consists of all points (x, y, c) such that $f(x, y) = c$. A level curve is the curve $f(x, y) = c$ in the xy-plane. The horizontal trace is in the $z = c$ plane. The two curves are related in the sense that the level curve is the projection of the horizontal trace on the xy-plane. The two curves have the same shape but they are located in parallel planes.

2. Describe the trace of $f(x, y) = x^2 - \sin(x^3 y)$ in the xz-plane.

SOLUTION The intersection of the graph of $f(x, y) = x^2 - \sin(x^3 y)$ with the xz-plane is obtained by setting $y = 0$ in the equation $z = x^2 - \sin(x^3 y)$. We get the equation $z = x^2 - \sin 0 = x^2$. This is the parabola $z = x^2$ in the xz-plane.

3. Is it possible for two different level curves of a function to intersect? Explain.

SOLUTION Two different level curves of $f(x, y)$ are the curves in the xy-plane defined by equations $f(x, y) = c_1$ and $f(x, y) = c_2$ for $c_1 \neq c_2$. If the curves intersect at a point (x_0, y_0), then $f(x_0, y_0) = c_1$ and $f(x_0, y_0) = c_2$, which implies that $c_1 = c_2$. Therefore, two different level curves of a function do not intersect.

4. Describe the contour map of $f(x, y) = x$ with contour interval 1.

SOLUTION The level curves of the function $f(x, y) = x$ are the vertical lines $x = c$. Therefore, the contour map of f with contour interval 1 consists of vertical lines so that every two adjacent lines are distanced one unit from another.

5. How will the contour maps of

$$f(x, y) = x \quad \text{and} \quad g(x, y) = 2x$$

with contour interval 1 look different?

SOLUTION The level curves of $f(x, y) = x$ are the vertical lines $x = c$, and the level curves of $g(x, y) = 2x$ are the vertical lines $2x = c$ or $x = \frac{c}{2}$. Therefore, the contour map of $f(x, y) = x$ with contour interval 1 consists of vertical lines with distance one unit between adjacent lines, whereas in the contour map of $g(x, y) = 2x$ (with contour interval 1) the distance between two adjacent vertical lines is $\frac{1}{2}$.

Exercises

In Exercises 1–4, evaluate the function at the specified points.

1. $f(x, y) = x + yx^3$, $(2, 2), (-1, 4)$

SOLUTION We substitute the values for x and y in $f(x, y)$ and compute the values of f at the given points. This gives

$$f(2, 2) = 2 + 2 \cdot 2^3 = 18$$
$$f(-1, 4) = -1 + 4 \cdot (-1)^3 = -5$$

2. $g(x, y) = \dfrac{y}{x^2 + y^2}$, $(1, 3), (3, -2)$

SOLUTION We substitute $(x, y) = (1, 3)$ and $(x, y) = (3, -2)$ in the function to obtain

$$g(1, 3) = \frac{3}{1^2 + 3^2} = \frac{3}{10}; \quad g(3, -2) = \frac{-2}{3^2 + (-2)^2} = -\frac{2}{13}$$

3. $h(x, y, z) = xyz^{-2}, \quad (3, 8, 2), (3, -2, -6)$

SOLUTION Substituting $(x, y, z) = (3, 8, 2)$ and $(x, y, z) = (3, -2, -6)$ in the function, we obtain

$$h(3, 8, 2) = 3 \cdot 8 \cdot 2^{-2} = 3 \cdot 8 \cdot \frac{1}{4} = 6$$

$$h(3, -2, -6) = 3 \cdot (-2) \cdot (-6)^{-2} = -6 \cdot \frac{1}{36} = -\frac{1}{6}$$

4. $Q(y, z) = y^2 + y \sin z, \quad (y, z) = \left(2, \frac{\pi}{2}\right), \left(-2, \frac{\pi}{6}\right)$

SOLUTION We have

$$Q\left(2, \frac{\pi}{2}\right) = 2^2 + 2 \sin \frac{\pi}{2} = 4 + 2 \cdot 1 = 6$$

$$Q\left(-2, \frac{\pi}{6}\right) = (-2)^2 - 2 \sin \frac{\pi}{6} = 4 - 2 \cdot \frac{1}{2} = 3$$

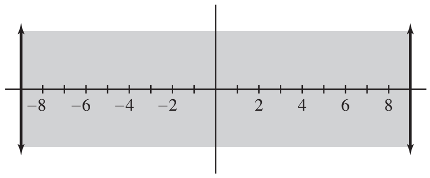
In Exercises 5–12, sketch the domain of the function.

5. $f(x, y) = 12x - 5y$

SOLUTION The function is defined for all x and y, hence the domain is the entire xy-plane.

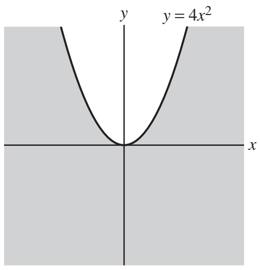
6. $f(x, y) = \sqrt{81 - x^2}$

SOLUTION The function $f(x, y) = \sqrt{81 - x^2}$ is defined if $81 - x^2 \geq 0$, that is, if $x^2 \leq 81$. In other words, $-9 \leq x \leq 9$. This region is the region enclosed by the two vertical lines $x = -9$ and $x = 9$ (including the two lines themselves).



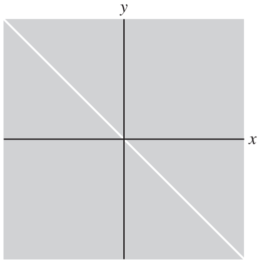
7. $f(x, y) = \ln(4x^2 - y)$

SOLUTION The function is defined if $4x^2 - y > 0$, that is, $y < 4x^2$. The domain is the region in the xy-plane that is below the parabola $y = 4x^2$.



8. $h(x, t) = \dfrac{1}{x + t}$

SOLUTION The function is defined if $x + t \neq 0$, that is, $x \neq -t$. The domain is the xt-plane with the line $x = -t$ excluded.

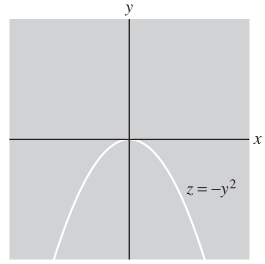


$x \neq -t$

9. $g(y, z) = \dfrac{1}{z + y^2}$

SOLUTION The function is defined if $z + y^2 \neq 0$, that is, $z \neq -y^2$. The domain is the (y, z) plane with the parabola $z = -y^2$ excluded.

$$D = \left\{(y, z) : z \neq -y^2\right\}$$

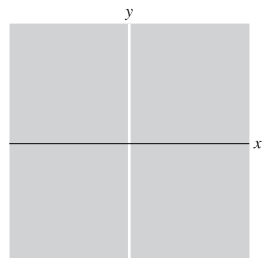


$$z + y^2 \neq 0$$

10. $f(x, y) = \sin \dfrac{y}{x}$

SOLUTION The function is defined for all $x \neq 0$. The domain is the xy-plane with the y-axis excluded.

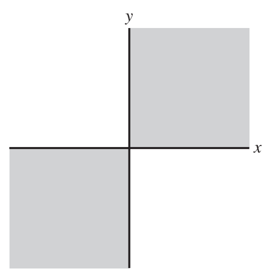
$$D = \{(x, y) : x \neq 0\}$$



$$x \neq 0$$

11. $F(I, R) = \sqrt{IR}$

SOLUTION The function is defined if $IR \geq 0$. Therefore the domain is the first and the third quadrants of the IR-plane including both axes.

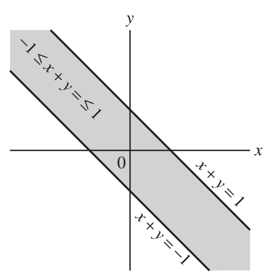


$$IR \geq 0$$

12. $f(x, y) = \cos^{-1}(x + y)$

SOLUTION Since the cosine function assume only values between -1 and 1, $x + y$ must satisfy $-1 \leq x + y \leq 1$. The domain is the region between the lines $x + y = 1$ and $x + y = -1$, including both lines.

$$D = \{(x, y) : -1 \leq x + y \leq 1\}$$



In Exercises 13–16, describe the domain and range of the function.

13. $f(x, y, z) = xz + e^y$

SOLUTION The domain of f is the entire (x, y, z)-space. Since f takes all the real values, the range is the entire real line.

14. $f(x, y, z) = x\sqrt{y + z}e^{z/x}$

SOLUTION The domain of f depends upon the term $\sqrt{y+z}$. We know that $y + z \geq 0$ so then $y \geq -z$. The domain is the region below and including the plane $y = -z$ in $\mathbb{R}^3$.

$$D = \{(x, y, z) \; : \; y \geq -z\} = \{(x, y, z) \; : \; y + z \geq 0\}$$

Since f takes all the real values, the range is the entire real line.

15. $P(r, s, t) = \sqrt{16 - r^2s^2t^2}$

SOLUTION The domain is subset of $\mathbf{R}^3$ where $rst \leq 4$ and the range is $\{w : 0 \leq w \leq 4\}$ because the minimum is 0 and the maximum of P is $\sqrt{16} = 4$.

16. $g(r, s) = \cos^{-1}(rs)$

SOLUTION Recall that the domain of the inverse cosine function is $[-1, 1]$ and the range of the inverse cosine function is $[0, \pi]$. This means that we need $|rs| \leq 1$:

$$D = \{(r, s) \; : \; |rs| \leq 1\}.$$

The range of this new function g will remain $[0, \pi]$.

17. Match graphs (A) and (B) in Figure 21 with the functions

(i) $f(x, y) = -x + y^2$ (ii) $g(x, y) = x + y^2$

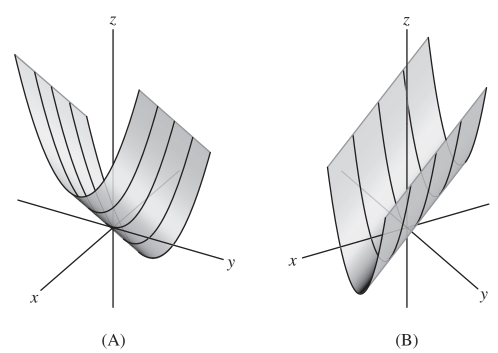


FIGURE 21

SOLUTION

(i) The vertical trace for $f(x, y) = -x + y^2$ in the xz-plane ($y = 0$) is $z = -x$. This matches the graph shown in (B).

(ii) The vertical trace for $f(x, y) = x + y^2$ in the xz-plane ($y = 0$) is $z = x$. This matches the graph show in (A).

18. Match each of graphs (A) and (B) in Figure 22 with one of the following functions:

(i) $f(x, y) = (\cos x)(\cos y)$

(ii) $g(x, y) = \cos(x^2 + y^2)$

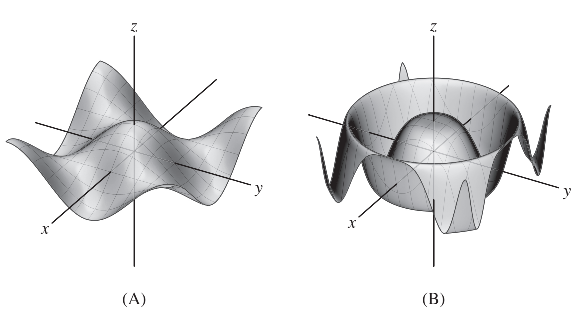


FIGURE 22

SOLUTION The level curves at $z = c$, a constant, for $g(x, y) = \cos(x^2 + y^2)$ will give

$$\cos(x^2 + y^2) = c \quad \Rightarrow \quad x^2 + y^2 = \cos^{-1}(c) \text{ which is a constant}$$

This means that the level curves are an infinite number of concentric circles centered around the z-axis whose radii differ by 2π. This means the graph in (B) is the given function in (ii).

If we consider the function $f(x, y) = (\cos x)(\cos y)$, the vertical trace if $y = 0$ will give us a graph of $\cos x$ in the xz-plane, while the vertical trace if $x = 0$ will give us a graph of $\cos y$ in the yz-plane. This means that the graph in (A) is the given function in (i).

19. Match the functions (a)–(f) with their graphs (A)–(F) in Figure 23.

(a) $f(x, y) = |x| + |y|$

(b) $f(x, y) = \cos(x - y)$

(c) $f(x, y) = \dfrac{-1}{1 + 9x^2 + y^2}$

(d) $f(x, y) = \cos(y^2)e^{-0.1(x^2+y^2)}$

(e) $f(x, y) = \dfrac{-1}{1 + 9x^2 + 9y^2}$

(f) $f(x, y) = \cos(x^2 + y^2)e^{-0.1(x^2+y^2)}$

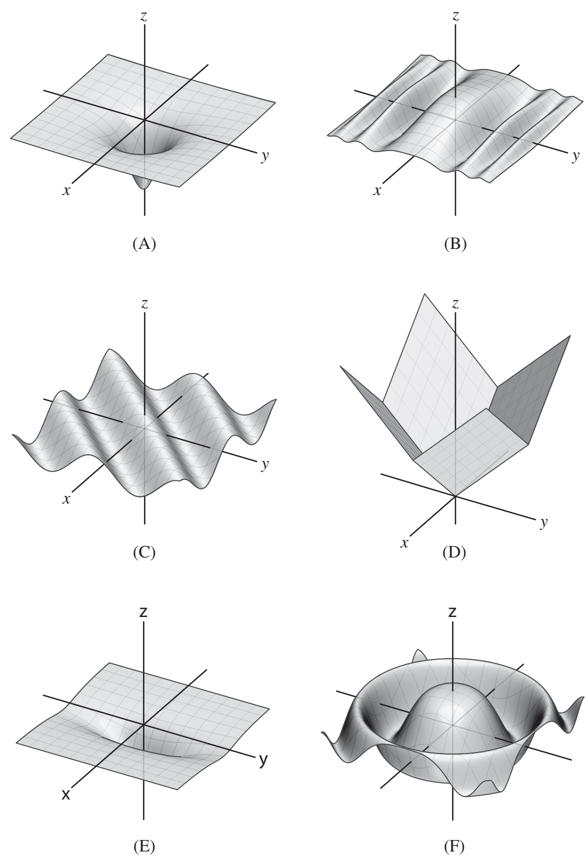


FIGURE 23

SOLUTION

(a) $|x| + |y|$. The level curves are $|x| + |y| = c$, $y = c - |x|$, or $y = -c + |x|$. The graph (D) corresponds to the function with these level curves.

(b) $\cos(x - y)$. The vertical trace in the plane $x = c$ is the curve $z = \cos(c - y)$ in the plane $x = c$. These traces correspond to the graph (C).

(c) $\dfrac{-1}{1 + 9x^2 + y^2}$ **(e)** $\dfrac{-1}{1 + 9x^2 + 9y^2}$.

The level curves of the two functions are:

$$\frac{-1}{1 + 9x^2 + y^2} = c \qquad\qquad \frac{-1}{1 + 9x^2 + 9y^2} = c$$

$$1 + 9x^2 + y^2 = -\frac{1}{c} \qquad\qquad 1 + 9x^2 + 9y^2 = -\frac{1}{c}$$

$$9x^2 + y^2 = -1 - \frac{1}{c} \qquad\qquad 9x^2 + 9y^2 = -1 - \frac{1}{c}$$

$$x^2 + y^2 = -\frac{1 + c}{9c}$$

For suitable values of c, the level curves of the function in (c) are ellipses as in (E), and the level curves of the function (e) are circles as in (A).

(d) $\cos(x^2)e^{-1/(x^2+y^2)}$ **(f)** $\cos(x^2+y^2)e^{-1/(x^2+y^2)}$.

The value of $|z|$ is decreasing to zero as x or y are decreasing, hence the possible graphs are (B) and (F).

In (f), z is constant whenever x^2+y^2 is constant, that is, z is constant whenever (x, y) varies on a circle. Hence (f) corresponds to the graph (F) and (d) corresponds to (B).

To summarize, we have the following matching:

$$\text{(a)} \leftrightarrow \text{(D)} \qquad \text{(b)} \leftrightarrow \text{(C)} \qquad \text{(c)} \leftrightarrow \text{(E)}$$
$$\text{(d)} \leftrightarrow \text{(B)} \qquad \text{(e)} \leftrightarrow \text{(A)} \qquad \text{(f)} \leftrightarrow \text{(F)}$$

20. Match the functions (a)–(d) with their contour maps (A)–(D) in Figure 24.

(a) $f(x, y) = 3x + 4y$

(b) $g(x, y) = x^3 - y$

(c) $h(x, y) = 4x - 3y$

(d) $k(x, y) = x^2 - y$

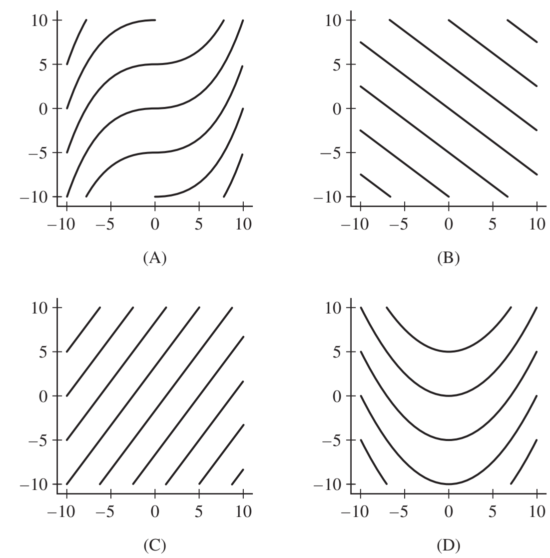


FIGURE 24

SOLUTION

(a) Computing the level curves for $f(x, y) = 3x + 4y$ we set $z = f(x, y) = c$, a constant, to see

$$3x + 4y = c \quad \Rightarrow \quad 4y = c - 3x \quad \Rightarrow \quad y = \frac{c}{4} - \frac{3}{4}x$$

This means the contour maps would be lines having slopes $-3/4$, this corresponds to the contour map shown in (B).

(b) Computing the level curves for $g(x, y) = x^3 - y$ we set $z = g(x, y) = c$, a constant, to see

$$x^3 - y = c \quad \Rightarrow \quad y = x^3 - c$$

This means the contour maps would be contours having the shape of cubic equations, this corresponds to the contour map shown in (A).

(c) Computing the level curves for $h(x, y) = 4x - 3y$ we set $z = h(x, y) = c$, a constant, to see

$$4x - 3y = c \quad \Rightarrow \quad -3y = c - 4x \quad \Rightarrow \quad y = \frac{c}{4} + \frac{4}{3}x$$

This means the contour maps would be contours that are lines having slopes $4/3$, this corresponds to the contour map shown in (C).

(d) Computing the level curves for $k(x, y) = x^2 - y$ we set $z = k(x, y) = c$, a constant, to see

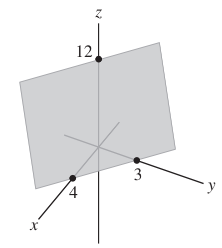
$$x^2 - y = c \quad \Rightarrow \quad y = x^2 - c$$

This means the contour maps would be contours having the shape of parabolas, this corresponds to the contour map shown in (D).

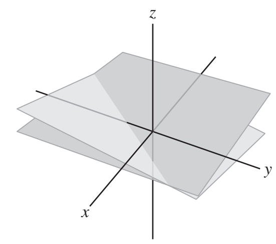
In Exercises 21–26, sketch the graph and describe the vertical and horizontal traces.

21. $f(x, y) = 12 - 3x - 4y$

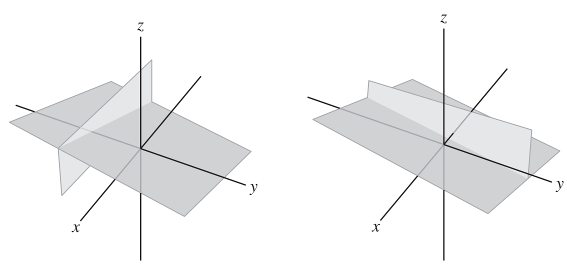
SOLUTION The graph of $f(x, y) = 12 - 3x - 4y$ is shown in the figure:



The horizontal trace at height c is the line $12 - 3x - 4y = c$ or $3x + 4y = 12 - c$ in the plane $z = c$.

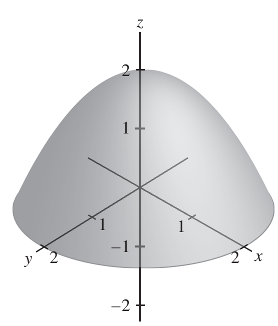


The vertical traces obtained by setting $x = a$ or $y = a$ are the lines $z = (12 - 3a) - 4y$ and $z = -3x + (12 - 4a)$ in the planes $x = a$ and $y = a$, respectively.



22. $f(x, y) = \sqrt{4 - x^2 - y^2}$

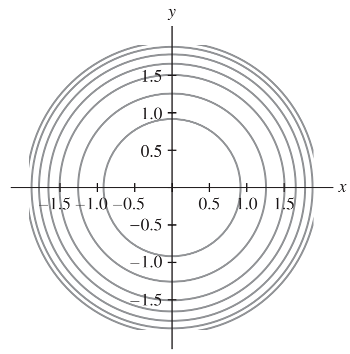
SOLUTION The graph of $f(x, y) = \sqrt{4 - x^2 - y^2}$ is shown in the figure:



The horizontal trace at height c is

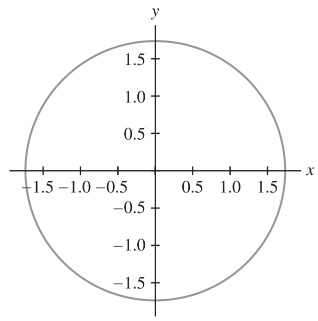
$$\sqrt{4 - x^2 - y^2} = c \quad \Rightarrow \quad 4 - x^2 - y^2 = c^2 \quad \Rightarrow \quad x^2 + y^2 = 4 - c^2$$

in the plane $z = c$ as long as $-2 \le c \le 2$. These are circles centered at the origin with radius $\sqrt{4 - c^2}$ in the plane $z = c$.

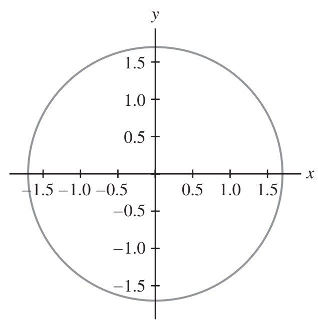


The vertical traces obtained by setting $x = a$ or $y = a$

$$\sqrt{4 - a^2 - y^2} = z \quad \Rightarrow \quad 4 - a^2 - y^2 = z^2 \quad \Rightarrow \quad y^2 + z^2 = 4 - a^2$$



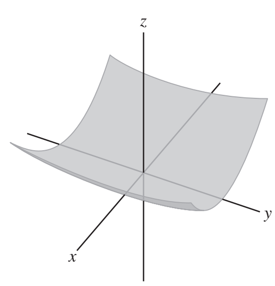
$$\sqrt{4 - x^2 - a^2} = z \quad \Rightarrow \quad 4 - x^2 - a^2 = z^2 \quad \Rightarrow \quad x^2 + z^2 = 4 - a^2$$



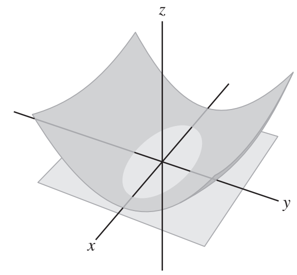
Both are the upper half circles centered at the origin with radius $\sqrt{4 - a^2}$ (in the planes $x = a$ and $y = a$) as long as $-2 \le a \le 2$. The graph is only the upper half of the sphere having radius 2, since it includes only the positive square root of z, so the vertical traces are only upper half circles.

23. $f(x, y) = x^2 + 4y^2$

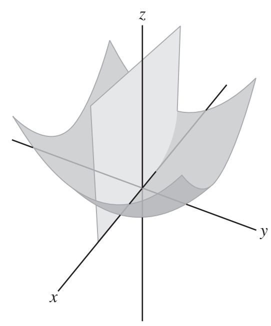
SOLUTION The graph of the function is shown in the figure:

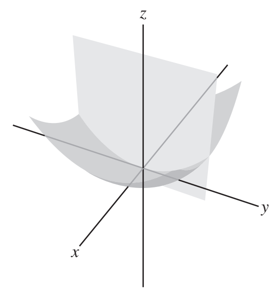


The horizontal trace at height c is the curve $x^2 + 4y^2 = c$, where $c \geq 0$ (if $c = 0$, it is the origin). The horizontal traces are ellipses for $c > 0$.



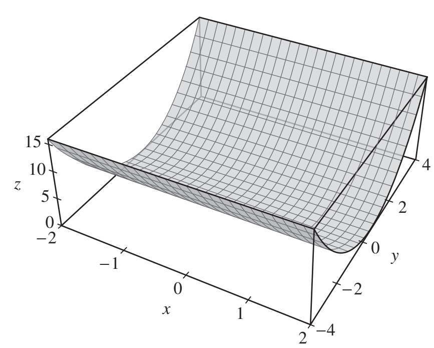
The vertical trace in the plane $x = a$ is the parabola $z = a^2 + 4y^2$, and the vertical trace in the plane $y = a$ is the parabola $z = x^2 + 4a^2$.



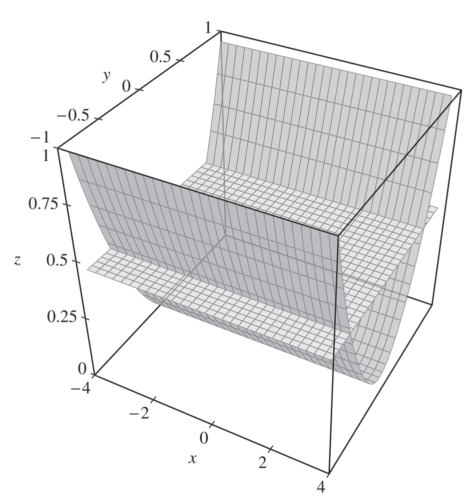


24. $f(x, y) = y^2$

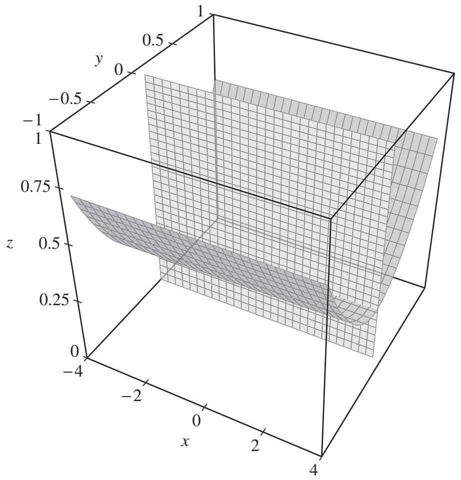
SOLUTION The graph of the function is shown in the figure:



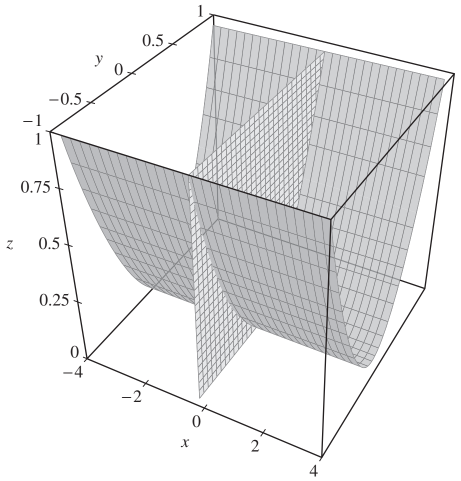
The horizontal trace at height c is $y^2 = c$. For $c > 0$ the trace consists of the two lines $y = \sqrt{c}$ and $y = -\sqrt{c}$ in the plane $z = c$, and for $c = 0$ it is the line $y = 0$.



The vertical trace in the plane $y = a$ is the line $z = a^2$.

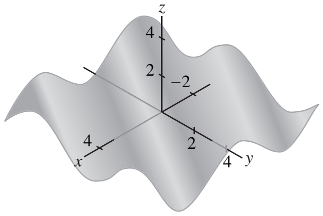


The vertical trace in the plane $x = a$ is the parabola $z = y^2$ on this plane.

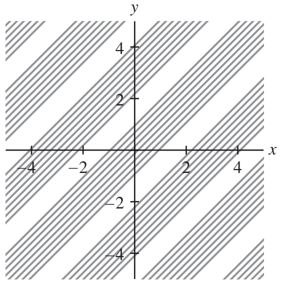


25. $f(x, y) = \sin(x - y)$

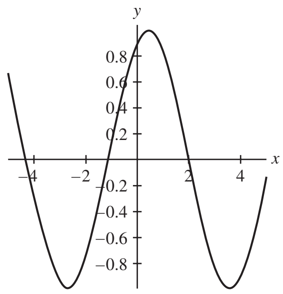
SOLUTION The graph of $f(x, y) = \sin(x - y)$ is shown in the figure:



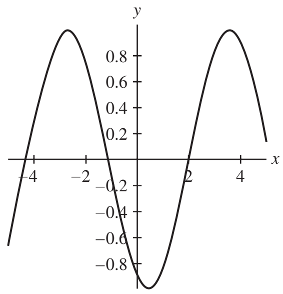
The horizontal trace at the height $z = c$ is $\sin(x - y) = c$ (we could also write $x - y = \sin^{-1}(c)$ or $y = x - \sin^{-1}(c)$). The trace consists of multiple lines all having slope 1, with y-intercepts separated by multiples of 2π.



The vertical trace in the plane $x = a$ is $\sin(a - y) = -\sin(y - a) = z$. This curve is a shifted sine curve reflected through the z-axis.

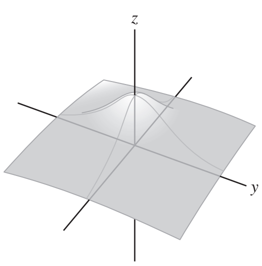


The vertical trace in the plane $y = a$ is $\sin(x - a) = z$. This curve is a shifted sine curve as well.



26. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

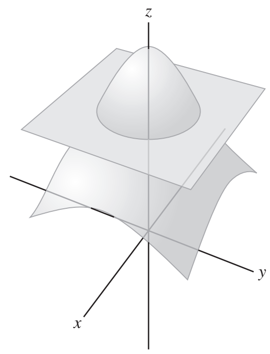
SOLUTION The graph of the function is shown in the figure:



The horizontal trace at height c is the following curve in the plane $z = c$:

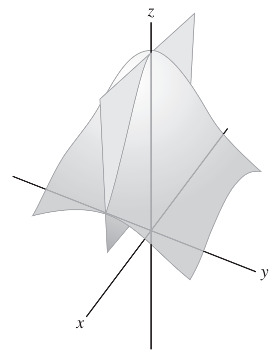
$$\frac{1}{x^2 + y^2 + 1} = c \quad \Rightarrow \quad x^2 + y^2 + 1 = \frac{1}{c} \quad \Rightarrow \quad x^2 + y^2 = \frac{1}{c} - 1$$

For $0 < c < 1$ it is a circle of radius $\sqrt{\frac{1}{c} - 1}$ centered at $(0, 0)$, and for $c = 1$ it is the origin.

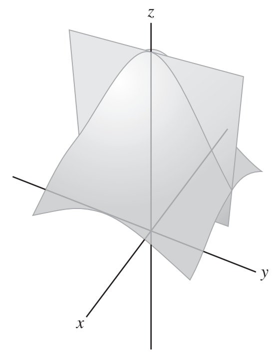


The vertical trace in the plane $x = a$ is the following curve in the plane $x = a$:

$$z = \frac{1}{a^2 + y^2 + 1} \quad \Rightarrow \quad z = \frac{1}{(1 + a^2) + y^2}$$

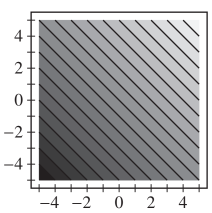


The vertical trace in the plane $y = a$ is the curve $z = \dfrac{1}{x^2 + a^2 + 1}$ in this plane.

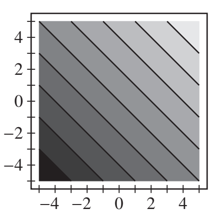


27. Sketch contour maps of $f(x, y) = x + y$ with contour intervals $m = 1$ and 2.

SOLUTION The level curves are $x + y = c$ or $y = c - x$. Using contour interval $m = 1$, we plot $y = c - x$ for various values of c.

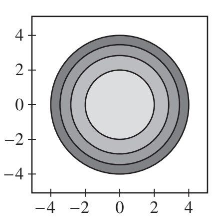


Using contour interval $m = 2$, we plot $y = c - x$ for various values of c.



28. Sketch the contour map of $f(x, y) = x^2 + y^2$ with level curves $c = 0, 4, 8, 12, 16$.

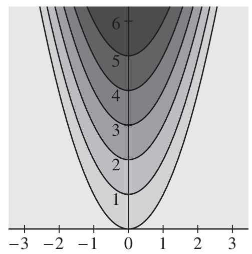
SOLUTION The level curves are $x^2 + y^2 = c$ for $c \ge 0$. We sketch the level curves $c = 0, 4, 8, 12, 16$:



In Exercises 29–36, draw a contour map of $f(x, y)$ with an appropriate contour interval, showing at least six level curves.

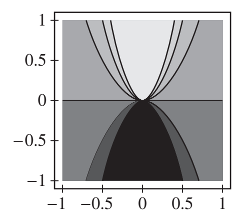
29. $f(x, y) = x^2 - y$

SOLUTION The level curves are the parabolas $y = x^2 + c$. We draw a contour plot with contour interval $m = 1$, for $c = 0, 1, 2, 3, 4, 5$:



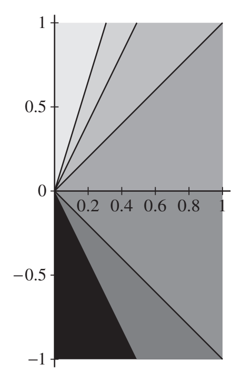
30. $f(x, y) = \dfrac{y}{x^2}$

SOLUTION The level curves are $\frac{y}{x^2} = c$ or $y = cx^2$. We use the contour interval $m = 2$ and plot $y = cx^2$ for $c = -4, -2, 0, 2, 4, 6$. For $c \neq 0$ these are parabolas.



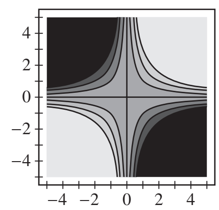
31. $f(x, y) = \dfrac{y}{x}$

SOLUTION The level curves are $\frac{y}{x} = c$ or $y = cx$. We plot $y = cx$ for $c = -2, -1, 0, 1, 2, 3$ using contour interval $m = 1$:



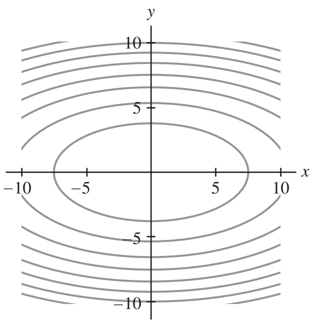
32. $f(x, y) = xy$

SOLUTION The level curves are $xy = c$ or $y = \frac{c}{x}$. These are hyperbolas in the xy-plane. We draw a contour map of the function using contour interval $m = 1$ and $c = 0, \pm 1, \pm 2, \pm 3$:



33. $f(x, y) = x^2 + 4y^2$

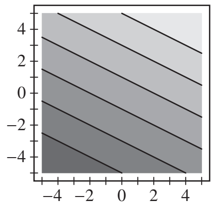
SOLUTION The level curves are $x^2 + 4y^2 = c$. These are ellipses centered at the origin in the xy-plane.



34. $f(x, y) = x + 2y - 1$

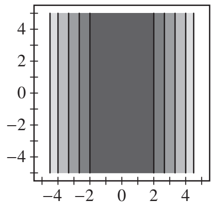
SOLUTION The level curves are the lines $x + 2y - 1 = c$ or $y = -\frac{x}{2} + \frac{c+1}{2}$. We draw a contour map using the contour interval $m = 4$ and $c = -9, -5, -1, 3, 7, 11$. The corresponding level curves are:

$$\underbrace{y = -\frac{x}{2} - 4}_{c=-9}, \quad \underbrace{y = -\frac{x}{2} - 2}_{c=-5}, \quad \underbrace{y = -\frac{x}{2}}_{c=-1}, \quad \underbrace{y = -\frac{x}{2} + 2}_{c=3},$$
$$\underbrace{y = -\frac{x}{2} + 4}_{c=7}, \quad \underbrace{y = -\frac{x}{2} + 6}_{c=11}$$



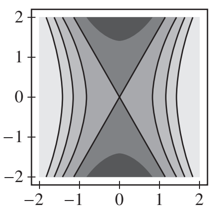
35. $f(x, y) = x^2$

SOLUTION The level curves are $x^2 = c$. For $c > 0$ these are the two vertical lines $x = \sqrt{c}$ and $x = -\sqrt{c}$ and for $c = 0$ it is the y-axis. We draw a contour map using contour interval $m = 4$ and $c = 0, 4, 8, 12, 16, 20$:



36. $f(x, y) = 3x^2 - y^2$

SOLUTION The level curves are the hyperbolas $3x^2 - y^2 = c$, $c \neq 0$, and for $c = 0$ it is the two lines $y = \pm\sqrt{3}x$. We plot a contour map with contour interval $m = 2$ using $c = -4, -2, 0, 2, 4, 6$:



37. Find the linear function whose contour map (with contour interval $m = 6$) is shown in Figure 25. What is the linear function if $m = 3$ (and the curve labeled $c = 6$ is relabeled $c = 3$)?

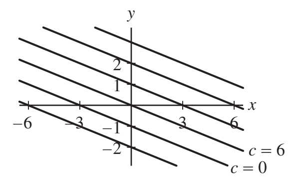


FIGURE 25 Contour map with contour interval $m = 6$

SOLUTION A linear function has the form $f(x, y) = Ax + By + C$.

Case 1: According to the contour map, the level curve through the origin (0, 0) has equation $f(x, y) = 6$. Therefore

$$f(0,0) = A(0) + B(0) + C = 6 \quad \Rightarrow \quad C = 6$$

Next, we see from the contour map that the points $(-3, 0) = 0$ and $f(0, -1)$ lie on the level curve $f(x, y) = 0$. Hence

$$f(-3,0) = A(-3) + B(0) + 6 = 0 \quad \Rightarrow \quad A = 2$$
$$f(0,-1) = A(0) + B(-1) + 6 = 0 \quad \Rightarrow \quad B = 6$$

Therefore $f(x, y) = 2x + 6y + 6$.

Case 1: If $m = 3$, then (0, 0) lies on the level curve $f(x, y) = 3$, and we proceed as before

$$f(0,0) = A(0) + B(0) + C = 3 \quad \Rightarrow \quad C = 3 f(-3,0) = A(-3) + B(0) + 3 = 0 \quad \Rightarrow \quad A = 1$$
$$f(0,-1) = A(0) + B(-1) + 3 = 0 \quad \Rightarrow \quad B = 2$$

Therefore $f(x, y) = x + 3y + 3$.

38. Use the contour map in Figure 26 to calculate the average rate of change:

(a) From A to B. **(b)** From A to C.

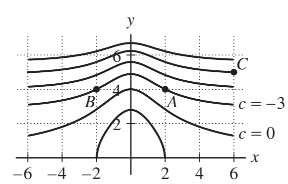


FIGURE 26

SOLUTION

(a) Using the figure to compute, we have the average rate of change from A to B:

$$\frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = 0$$

(b) Using the figure to compute, assuming that C is on the level curve $c = -9$, then we have the average rate of change from A to C

$$\frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = \frac{-9-(-3)}{\sqrt{2^2+1^2}} = -\frac{6}{\sqrt{5}}$$

39. Referring to Figure 27, answer the following questions:

(a) At which of (A)–(C) is pressure increasing in the northern direction?

(b) At which of (A)–(C) is pressure increasing in the easterly direction?

(c) In which direction at (B) is pressure increasing most rapidly?

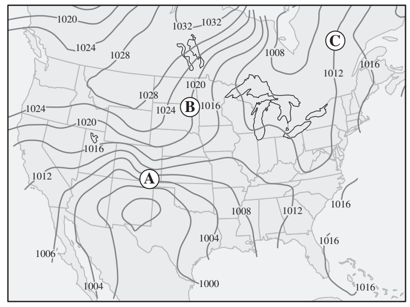


FIGURE 27 Atmospheric Pressure (in millibars) over the continental U.S. on March 26, 2009

SOLUTION

a. (A) and (B)

b. (C)

c. west

In Exercises 40–43, $\rho(S, T)$ is seawater density (kg/m^3) *as a function of salinity S* (ppt) *and temperature T* (°C). *Refer to the contour map in Figure 28.*

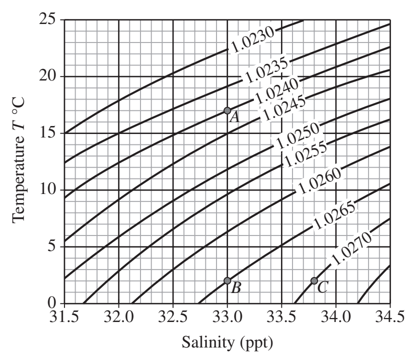


FIGURE 28 Contour map of seawater density $\rho(S, T)$ (kg/m^3).

40. Calculate the average rate of change of ρ with respect to T from B to A.

SOLUTION The segment $\overline{BA}$ spans 5 level curves and the contour interval is 0.0005. Since the density is decreasing in the direction from B to A, the change in density is $\Delta\rho = -0.0005 \cdot 5 = -0.0025$ kg/m^3. The temperature at A is 17°C and at C is 2°C, so the difference in temperature from C to A is $\Delta T = 17 - 2 = 15$°C. Hence,

$$\text{Average ROC from } B \text{ to } A = \frac{\Delta\rho}{\Delta T} = \frac{-0.0025}{15} = -0.000167 \text{ kg/m}^3\ {}^\circ\text{C}.$$

41. Calculate the average rate of change of ρ with respect to S from B to C.

SOLUTION For fixed temperature, the segment $\overline{BC}$ spans one level curve and the level curve of C is to the right of the level curve of B. Therefore, the change in density from B to C is $\Delta\rho = 0.0005$ kg/m^3. The salinity at C is greater than the salinity at B and $\Delta S = 0.8$ ppt. Therefore,

$$\text{Average ROC from } B \text{ to } C = \frac{\Delta\rho}{\Delta S} = \frac{0.0005}{0.8} = 0.000625 \text{ kg/m}^3 \cdot \text{ppt}.$$

42. At a fixed level of salinity, is seawater density an increasing or a decreasing function of temperature?

SOLUTION The level of salinity is fixed on each vertical line. The vertical lines intersect level curves with decreasing values in the direction of increasing temperature (which is the upward direction). Therefore, at a fixed level of salinity, seawater density is a decreasing function of temperature.

43. Does water density appear to be more sensitive to a change in temperature at point A or point B?

SOLUTION The two adjacent level curves are closer to the level curve of A than the corresponding two adjacent level curves are to the level curve of B. This suggests that water density is more sensitive to a change in temperature at A than at B.

In Exercises 44–47, refer to Figure 29.

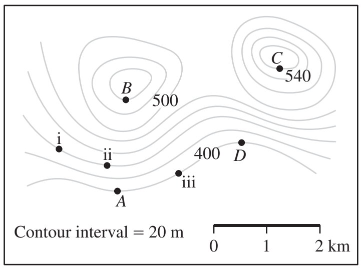


FIGURE 29

44. Find the change in elevation from A and B.

SOLUTION The segment $\overline{AB}$ spans 7 level curves and the contour interval is 20 meters. Therefore, the change in elevation from A to B is $20 \cdot 7 = 140$ m.

45. Estimate the average rate of change from A and B and from A to C.

SOLUTION The change in elevation from A to B is 140 m. The scale shows that $\overline{AB}$ is approximately 2000 m. Therefore,

$$\text{Average ROC from } A \text{ to } B = \frac{140}{2000} \approx 0.07.$$

The change in elevation from A to C is obtained by multiplying the number of level curves between A and C, which is 8, by the contour interval 20 meters, giving $8 \cdot 20 = 160$ m. Using the scale, we approximate the distance $\overline{AC}$ by 3000 m. Therefore,

$$\text{Average ROC from } A \text{ to } C = \frac{160}{3000} \approx 0.0533.$$

46. Estimate the average rate of change from A to points i, ii, and iii.

SOLUTION The points i, and ii are on a level curve two adjacent to the level curve of A, hence the change in elevation is $2 \cdot 20 = 40$ meters. The point iii is on the same level curve as A, hence the change in elevation is 0 meters. Using the scale we approximate the distances from A to the points i, ii, and iii:

From A to i: 1000 m
From A to ii: 500 m
From A to iii: 750 m

Therefore,

$$\text{Average ROC from } A \text{ to i} \approx \frac{40}{1000} = 0.04$$
$$\text{Average ROC from } P \text{ to ii} \approx \frac{40}{500} = 0.08$$
$$\text{Average ROC from } P \text{ to iii} \approx \frac{0}{750} = 0$$

47. Sketch the path of steepest ascent beginning at D.

SOLUTION Starting at D, we draw a path that everywhere along the way points on the steepest direction, that is, moves as straight as possible from one level curve to the next to end at the point C.

Further Insights and Challenges

48. The function $f(x, t) = t^{-1/2}e^{-x^2/t}$, whose graph is shown in Figure 30, models the temperature along a metal bar after an intense burst of heat is applied at its center point.

(a) Sketch the vertical traces at times $t = 1, 2, 3$. What do these traces tell us about the way heat diffuses through the bar?

(b) Sketch the vertical traces $x = c$ for $c = \pm 0.2, \pm 0.4$. Describe how temperature varies in time at points near the center.

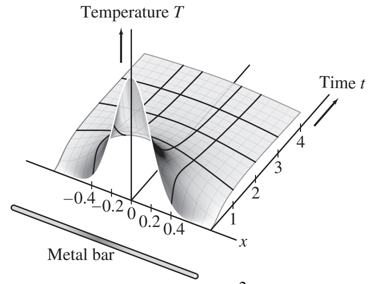


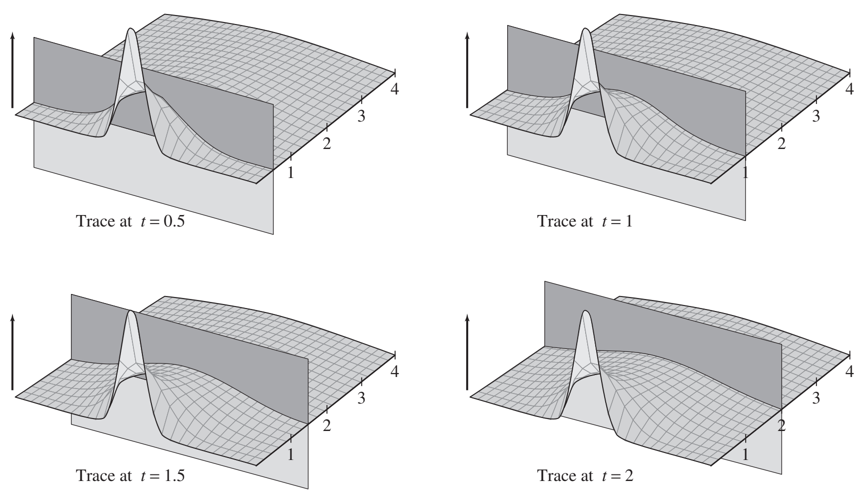
FIGURE 30 Graph of $f(x,t) = t^{-1/2}e^{-x^2/t}$ beginning shortly after $t = 0$.

SOLUTION

(a) The vertical traces at times $t = 0.5, 1, 1.5, 2$ are

$$z = \sqrt{2}e^{-2x^2} \text{ in the plane } t = 0.5$$
$$z = e^{-x^2} \text{ in the plane } t = 1$$
$$z = \frac{1}{\sqrt{3/2}}e^{-2x^2/3} \text{ in the plane } t = 1.5$$
$$z = \frac{1}{\sqrt{2}}e^{-x^2/2} \text{ in the plane } t = 2$$

These vertical traces are shown in the following figure:

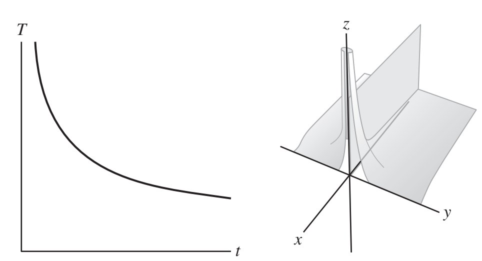


At each time the temperature decreases as we move away from the center point. Also, as t increases, the temperature at each point in the bar (except at the middle) increases and then decreases (as can be seen in Figure 30). It also shows that the temperature tends to equalize throughout the bar (because the traces become closer and closer to flat as time goes on).

(b) The vertical traces $x = c$ for the given values of c are:

$$z = \frac{1}{\sqrt{t}}e^{-\frac{0.04}{t}} \text{ in the planes } x = 0.2 \text{ and } x = -0.2$$
$$z = \frac{1}{\sqrt{t}}e^{-\frac{0.16}{t}} \text{ in the planes } x = 0.4 \text{ and } x = -0.4.$$

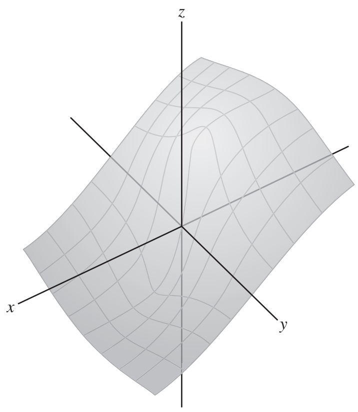
We see that for small values of t the temperature increases quickly and then slowly decreases as t increases.



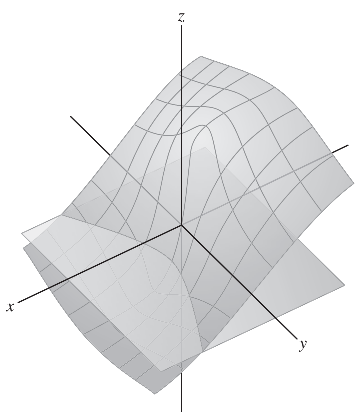
49. Let $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ for $(x,y) \neq 0$. Write f as a function $f(r,\theta)$ in polar coordinates, and use this to find the level curves of f.

SOLUTION In polar coordinates $x = r\cos\theta$ and $r = \sqrt{x^2+y^2}$. Hence,

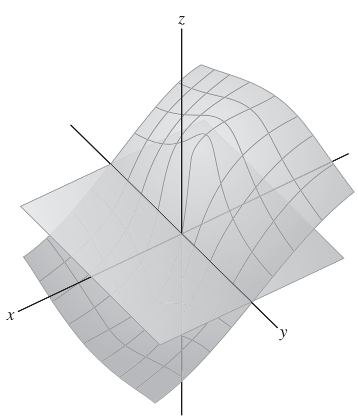
$$f(r,\theta) = \frac{r\cos\theta}{r} = \cos\theta.$$



The level curves are the curves $\cos\theta = c$ in the $r\theta$-plane, for $|c| \leq 1$. For $-1 < c < 1$, $c \neq 0$, the level curves $\cos\theta = c$ are the two rays $\theta = \cos^{-1} c$ and $\theta = -\cos^{-1} c$.



For $c = 0$, the level curve $\cos\theta = 0$ is the y-axis; for $c = 1$ the level curve $\cos\theta = 1$ is the nonnegative x-axis.



For $c = -1$, the level curve $\cos\theta = -1$ is the negative x-axis.

14.2 Limits and Continuity in Several Variables (LT Section 15.2)

Preliminary Questions

1. What is the difference between $D(P, r)$ and $D^*(P, r)$?

SOLUTION $D(P, r)$ is the open disk of radius r and center (a, b). It consists of all points distanced less than r from P, hence $D(P, r)$ includes the point P. $D^*(P, r)$ consists of all points in $D(P, r)$ other than P itself.

2. Suppose that $f(x, y)$ is continuous at $(2, 3)$ and that $f(2, y) = y^3$ for $y \neq 3$. What is the value $f(2, 3)$?

SOLUTION $f(x, y)$ is continuous at $(2, 3)$, hence the following holds:

$$f(2, 3) = \lim_{(x,y)\to(2,3)} f(x, y)$$

Since the limit exists, we may compute it by approaching $(2, 3)$ along the vertical line $x = 2$. This gives

$$f(2,3) = \lim_{(x,y)\to(2,3)} f(x,y) = \lim_{y\to 3} f(2,y) = \lim_{y\to 3} y^3 = 3^3 = 27$$

We conclude that $f(2, 3) = 27$.

3. Suppose that $Q(x, y)$ is a function such that $1/Q(x, y)$ is continuous for all (x, y). Which of the following statements are true?

(a) $Q(x, y)$ is continuous for all (x, y).
(b) $Q(x, y)$ is continuous for $(x, y) \neq (0, 0)$.
(c) $Q(x, y) \neq 0$ for all (x, y).

SOLUTION All three statements are true. Let $f(x, y) = \frac{1}{Q(x,y)}$. Hence $Q(x, y) = \frac{1}{f(x,y)}$.

(a) Since f is continuous, Q is continuous whenever $f(x, y) \neq 0$. But by the definition of f it is never zero, therefore Q is continuous at all (x, y).
(b) Q is continuous everywhere including at $(0, 0)$.
(c) Since $f(x, y) = \frac{1}{Q(x,y)}$ is continuous, the denominator is never zero, that is, $Q(x, y) \neq 0$ for all (x, y).

Moreover, there are no points where $Q(x, y) = 0$. (The equality $Q(x, y) = (0, 0)$ is meaningless since the range of Q consists of real numbers.)

4. Suppose that $f(x, 0) = 3$ for all $x \neq 0$ and $f(0, y) = 5$ for all $y \neq 0$. What can you conclude about $\lim_{(x,y)\to(0,0)} f(x, y)$?

SOLUTION We show that the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Indeed, if the limit exists, it may be computed by approaching $(0, 0)$ along the x-axis or along the y-axis. We compute these two limits:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} 3 = 3$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} 5 = 5$$

Since the limits are different, $f(x, y)$ does not approach one limit as $(x, y) \to (0, 0)$, hence the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Exercises

In Exercises 1–8, evaluate the limit using continuity

1. $\displaystyle\lim_{(x,y)\to(1,2)} (x^2 + y)$

SOLUTION Since the function $x^2 + y$ is continuous, we evaluate the limit by substitution:

$$\lim_{(x,y)\to(1,2)} (x^2 + y) = 1^2 + 2 = 3$$

2. $\displaystyle\lim_{(x,y)\to(\frac{4}{9},\frac{2}{9})} \frac{x}{y}$

SOLUTION The function $\frac{x}{y}$ is continuous at the point $\left(\frac{4}{9}, \frac{2}{9}\right)$, hence we compute the limit by substitution:

$$\lim_{(x,y)\to\left(\frac{4}{9},\frac{2}{9}\right)} \frac{x}{y} = \frac{\frac{4}{9}}{\frac{2}{9}} = 2$$

3. $\displaystyle\lim_{(x,y)\to(2,-1)} (xy - 3x^2y^3)$

SOLUTION The function $xy - 3x^2y^3$ is continuous everywhere because it is a polynomial, hence we compute the limit by substitution:

$$\lim_{(x,y)\to(2,-1)} (xy - 3x^2y^3) = 2(-1) - 3(4)(-1)^3 = -2 + 12 = 10$$

4. $\displaystyle\lim_{(x,y)\to(-2,1)} \frac{2x^2}{4x + y}$

SOLUTION We use the continuity of the function $\frac{2x^2}{4x+y}$ at the point $(-2, 1)$, hence we evaluate the limit by substitution:

$$\lim_{(x,y)\to(-2,1)} \frac{2x^2}{4x + y} = \frac{2(4)}{4(-2) + 1} = -\frac{8}{7}$$

5. $\lim_{(x,y)\to(\frac{\pi}{4},0)} \tan x \cos y$

SOLUTION We use the continuity of $\tan x \cos y$ at the point $\left(\frac{\pi}{4}, 0\right)$ to evaluate the limit by substitution:

$$\lim_{(x,y)\to\left(\frac{\pi}{4},0\right)} \tan x \cos y = \tan \frac{\pi}{4} \cos 0 = 1 \cdot 1 = 1$$

6. $\lim_{(x,y)\to(2,3)} \tan^{-1}(x^2 - y)$

SOLUTION We use the continuity of the function $\tan^{-1}(x^2 - y)$ at the point $(2, 3)$ to evaluate the limit by substitution:

$$\lim_{(x,y)\to(2,3)} \tan^{-1}(x^2 - y) = \tan^{-1}(1) = \frac{\pi}{4}$$

7. $\lim_{(x,y)\to(1,1)} \dfrac{e^{x^2} - e^{-y^2}}{x + y}$

SOLUTION The function is the quotient of two continuous functions, and the denominator is not zero at the point $(1, 1)$. Therefore, the function is continuous at this point, and we may compute the limit by substitution:

$$\lim_{(x,y)\to(1,1)} \frac{e^{x^2} - e^{-y^2}}{x + y} = \frac{e^{1^2} - e^{-1^2}}{1 + 1} = \frac{e - \frac{1}{e}}{2} = \frac{1}{2}(e - e^{-1})$$

8. $\lim_{(x,y)\to(1,0)} \ln(x - y)$

SOLUTION We use the continuity of $\ln(x - y)$ at the point $(1, 0)$ to evaluate the limit by substitution:

$$\lim_{(x,y)\to(1,0)} \ln(x - y) = \ln(1 - 0) = \ln 1 = 0$$

In Exercises 9–12, assume that

$$\lim_{(x,y)\to(2,5)} f(x, y) = 3, \qquad \lim_{(x,y)\to(2,5)} g(x, y) = 7$$

9. $\lim_{(x,y)\to(2,5)} \big(g(x, y) - 2f(x, y)\big)$

SOLUTION

$$\lim_{(x,y)\to(2,5)} \big(g(x, y) - 2f(x, y)\big) = 7 - 2(3) = 1$$

10. $\lim_{(x,y)\to(2,5)} f(x, y)^2 g(x, y)$

SOLUTION

$$\lim_{(x,y)\to(2,5)} f(x, y)^2 g(x, y) = 3^2(7) = 63$$

11. $\lim_{(x,y)\to(2,5)} e^{f(x,y)^2 - g(x,y)}$

SOLUTION

$$\lim_{(x,y)\to(2,5)} e^{f(x,y)^2 - g(x,y)} = e^{3^2 - 7} = e^2$$

12. $\lim_{(x,y)\to(2,5)} \dfrac{f(x, y)}{f(x, y) + g(x, y)}$

SOLUTION

$$\lim_{(x,y)\to(2,5)} \frac{f(x, y)}{f(x, y) + g(x, y)} = \frac{3}{3 + 7} = \frac{3}{10}$$

13. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$ exist? Explain.

SOLUTION This limit does not exist. Consider the following approaches to the point $(x, y) = (0, 0)$ - first along the line $x = 0$ and second, along the line $y = x$.

First along the line $x = 0$ we calculate:

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{y\to 0} \frac{y^2}{0^2+y^2} = \lim_{y\to 0} 1 = 1$$

Second, along the line $y = x$ we calculate:

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Since these two limits are not equal, the limit in question, $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$ does not exist.

14. Let $f(x, y) = xy/(x^2 + y^2)$. Show that $f(x, y)$ approaches zero as (x, y) approaches the origin along the x- and y-axes. Then prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist by showing that the limit along the line $y = x$ is nonzero.

SOLUTION

Case 1. Consider the limit along the x-axis ($y = 0$):

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0^2} = \lim_{x\to 0} 0 = 0$$

Case 2. Consider the limit along the y-axis ($x = 0$):

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0^2+y^2} = \lim_{y\to 0} 0 = 0$$

Case 3. Consider the limit along the line $y = x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Therefore, since the last limit we computed is not equal to zero, the limit in question, $\lim_{(x,y)\to(0,0)} xy/(x^2 + y^2)$ does not exist.

15. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

does not exist by considering the limit along the x-axis.

SOLUTION Compute this limit approaching $(x, y) = (0, 0)$ along the x-axis ($y = 0$):

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{x\to 0} \frac{x}{x^2+0^2} = \lim_{x\to 0} \frac{1}{x}$$

This limit is known not to exist (it gets arbitrarily large from the right and arbitrarily small from the left), therefore the limit in question, $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$, also does not exist.

16. Let $f(x, y) = x^3/(x^2 + y^2)$ and $g(x, y) = x^2/(x^2 + y^2)$. Using polar coordinates, prove that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

and that $\displaystyle\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist. *Hint:* Show that $g(x, y) = \cos^2\theta$ and observe that $\cos\theta$ can take on any value between -1 and 1 as $(x, y) \to (0, 0)$.

SOLUTION First we will compute $\lim_{(x,y)\to(0,0)} f(x, y)$:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{(r,\theta)\to(0,0)} \frac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{(r,\theta)\to(0,0)} r\cos^3\theta = 0$$

Now, we will compute $\lim_{(x,y)\to(0,0)} g(x, y)$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(r,\theta)\to(0,0} \frac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{(r,\theta)\to(0,0)} \cos^2\theta$$

Now $\cos\theta$ can take on any value between -1 and 1 - it depends on the angle at which (x, y) approaches the origin. (If it approaches the origin along the line with $\sin\theta$, then the limit will be $\cos\theta$.) Thus, as a result, $\cos^2\theta$ can be any value between 0 and 1. This limit does not exist, there is not just one finite value.

17. Use the Squeeze Theorem to evaluate

$$\lim_{(x,y)\to(4,0)} (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right)$$

SOLUTION Consider the following inequalities:

$$-1 \le \cos\left(\frac{1}{(x-4)^2+y^2}\right) \le 1$$

Then for x such that $x \ge 4$ then $x^2 - 16 \ge 0$ and we have:

$$(-1)(x^2-16) \le (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right) \le (x^2-16)$$

$$\lim_{(x,y)\to(4,0)} (-1)(x^2-16) \le \lim_{(x,y)\to(4,0)} (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right) \le \lim_{(x,y)\to(4,0)} (x^2-16)$$

Then the two limits at the ends of the inequality are clearly equal to 0, by the Squeeze Theorem.

Now, if $x < 4$, then $x^2 - 16 < 0$ and we have:

$$(x^2-16) \le (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right) \le (-1)(x^2-16)$$

$$\lim_{(x,y)\to(4,0)} (x^2-16) \le \lim_{(x,y)\to(4,0)} (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right) \le \lim_{(x,y)\to(4,0)} (-1)(x^2-16)$$

Then the two limits at the ends of the inequality are clearly equal to 0, by the Squeeze Theorem.

Thus we can conclude

$$\lim_{(x,y)\to(4,0)} (x^2-16)\cos\left(\frac{1}{(x-4)^2+y^2}\right) = 0$$

18. Evaluate $\lim_{(x,y)\to(0,0)} \tan x \sin\left(\frac{1}{|x|+|y|}\right)$.

SOLUTION We will try to use the Squeeze Theorem for this problem. Consider the following inequalities:

$$-1 \le \sin\left(\frac{1}{|x|+|y|}\right) \le 1$$

Then we have, if $\tan x \ge 0$:

$$(-1)\tan x \le \tan x \cdot \sin\left(\frac{1}{|x|+|y|}\right) \le \tan x$$

$$\lim_{(x,y)\to(0,0)} -\tan x \le \lim_{(x,y)\to(0,0)} \tan x \cdot \sin\left(\frac{1}{|x|+|y|}\right) \le \lim_{(x,y)\to(0,0)} \tan x$$

If we have $\tan x < 0$ then:

$$\tan x \le \tan x \cdot \sin\left(\frac{1}{|x|+|y|}\right) \le -\tan x$$

$$\lim_{(x,y)\to(0,0)} \tan x \le \lim_{(x,y)\to(0,0)} \tan x \cdot \sin\left(\frac{1}{|x|+|y|}\right) \le \lim_{(x,y)\to(0,0)} -\tan x$$

Then the two limits of the endpoints in both cases are clearly equal to 0, by the Squeeze Theorem we can conclude

$$\lim_{(x,y)\to(0,0)} \tan x \cdot \sin\left(\frac{1}{|x|+|y|}\right) = 0$$

In Exercises 19–32, evaluate the limit or determine that it does not exist.

19. $\displaystyle\lim_{(z,w)\to(-2,1)} \frac{z^4\cos(\pi w)}{e^{z+w}}$

SOLUTION This function is continuous everywhere since the denominator is never equal to 0, therefore, we will evaluate the limit by substitution:

$$\lim_{(z,w)\to(-2,1)} \frac{z^4\cos(\pi w)}{e^{z+w}} = \frac{(-2)^4\cos(\pi)}{e^{-2+1}} = \frac{16(-1)}{e^{-1}} = -16e$$

20. $\displaystyle\lim_{(z,w)\to(-1,2)} (z^2w - 9z)$

SOLUTION The function is continuous everywhere since it is a polynomial. Therefore we use substitution to evaluate the limit:

$$\lim_{(z,\omega)\to(-1,2)} (z^2\omega - 9z) = (-1)^2\cdot 2 - 9\cdot(-1) = 11.$$

21. $\displaystyle\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}}$

SOLUTION The function is continuous at the point $(4, 2)$, since it is the quotient of two continuous functions and the denominator is not zero at $(4, 2)$. We compute the limit by substitution:

$$\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}} = \frac{2-2}{\sqrt{4^2-4}} = \frac{0}{\sqrt{12}} = 0$$

22. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1+y^2}$

SOLUTION The function $\frac{x^2+y^2}{1+y^2}$ is continuous everywhere since it is a rational function whose denominator is never zero. We evaluate the limit using substitution:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1+y^2} = \frac{0^2+0^2}{1+0^2} = 0$$

23. $\displaystyle\lim_{(x,y)\to(3,4)} \frac{1}{\sqrt{x^2+y^2}}$

SOLUTION The function $\dfrac{1}{\sqrt{x^2+y^2}}$ is continuous at the point $(3, 4)$ since it is the quotient of two continuous functions and the denominator is not zero at $(3, 4)$. We compute the limit by substitution:

$$\lim_{(x,y)\to(3,4)} \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{9+16}} = \frac{1}{5}$$

24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

SOLUTION We can see that the limit along any line through $(0, 0)$ is 0, as well as along other paths through $(0, 0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. Consider the following inequalities:

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le |x|$$

since $|y| \le \sqrt{x^2+y^2}$, and $|x| \to 0$ as $(x, y) \to (0, 0)$. So then by the Squeeze Theorem, we know:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

25. $\displaystyle\lim_{(x,y)\to(1,-3)} e^{x-y}\ln(x-y)$

SOLUTION This function $e^{x-y}\ln(x-y)$ is continuous at the point $(1, -3)$ since it is the product of two continuous functions. We can compute the limit by substitution:

$$\lim_{(x,y)\to(1,-3)} e^{x-y}\ln(x-y) = e^{1+3}\ln(1+3) = e^4\ln 4$$

26. $\lim_{(x,y)\to(0,0)} \frac{|x|}{|x|+|y|}$

SOLUTION We compute the limit as (x, y) approaches the origin along the line $y = mx$, for a fixed positive value of m. Substituting $y = mx$ in the function $f(x, y) = \frac{|x|}{|x|+|y|}$, we get for $x \neq 0$:

$$f(x, mx) = \frac{|x|}{|x| + m|x|} = \frac{|x|}{|x|(1+m)} = \frac{1}{1+m}$$

As (x, y) approaches $(0, 0)$, $(x, y) \neq (0, 0)$. Therefore $x \neq 0$ on the line $y = mx$. Thus,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{x\to 0} \frac{1}{1+m} = \frac{1}{1+m}$$

We see that the limits along the lines $y = mx$ are different, hence $f(x, y)$ does not approach one limit as $(x, y) \to (0, 0)$. We conclude that the given limit does not exist.

27. $\lim_{(x,y)\to(-3,-2)} (x^2y^3 + 4xy)$

SOLUTION The function $x^2y^3 + 4xy$ is continuous everywhere because it is a polynomial. We can compute this limit by substitution:

$$\lim_{(x,y)\to(-3,-2)} (x^2y^3 + 4xy) = 9(-8) + 4(-3)(-2) = -72 + 24 = -48$$

28. $\lim_{(x,y)\to(2,1)} e^{x^2-y^2}$

SOLUTION Since $e^{x^2-y^2} = e^{x^2} \cdot e^{-y^2}$, we evaluate the limit as a product of limits:

$$\lim_{(x,y)\to(2,1)} e^{x^2-y^2} = \left(\lim_{x\to 2} e^{x^2}\right)\left(\lim_{y\to 1} e^{-y^2}\right) = e^{2^2} \cdot e^{-1^2} = e^4 \cdot e^{-1} = e^3$$

Notice that since $e^{x^2-y^2}$ is continuous everywhere, we may evaluate the limit by substitution:

$$\lim_{(x,y)\to(2,1)} e^{x^2-y^2} = e^{2^2-1^2} = e^3.$$

29. $\lim_{(x,y)\to(0,0)} \tan(x^2+y^2)\tan^{-1}\left(\frac{1}{x^2+y^2}\right)$

SOLUTION Consider the following inequalities:

$$-\frac{\pi}{2} \le \tan^{-1}\left(\frac{1}{x^2+y^2}\right) \le \frac{\pi}{2}$$

$$-\frac{\pi}{2} \cdot \tan(x^2+y^2) \le \tan(x^2+y^2) \cdot \left(\frac{1}{x^2+y^2}\right) \le \frac{\pi}{2} \tan(x^2+y^2)$$

and then taking limits:

$$\lim_{(x,y)\to(0,0)} -\frac{\pi}{2} \cdot \tan(x^2+y^2) \le \lim_{(x,y)\to(0,0)} \tan(x^2+y^2) \cdot \left(\frac{1}{x^2+y^2}\right) \le \lim_{(x,y)\to(0,0)} \frac{\pi}{2} \tan(x^2+y^2)$$

Each of the limits on the endpoints of this inequality is equal to 0, thus we can conclude:

$$\lim_{(x,y)\to(0,0)} \tan(x^2+y^2) \cdot \left(\frac{1}{x^2+y^2}\right) = 0$$

30. $\lim_{(x,y)\to(0,0)} (x+y+2)e^{-1/(x^2+y^2)}$

SOLUTION First let us recall that $\lim_{t\to 0} e^{-1/t} = 0$ since $-1/t$ gets infinitely small. Therefore we can conclude,

$$\lim_{(x,y)\to(0,0)} (x+y+2)e^{-1/(x^2+y^2)} = (0+0+2) \cdot 0 = 0$$

31. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

SOLUTION We rewrite the function by dividing and multiplying it by the conjugate of $\sqrt{x^2+y^2+1}-1$ and using the identity $(a-b)(a+b)=a^2-b^2$. This gives

$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{\left(\sqrt{x^2+y^2+1}-1\right)\left(\sqrt{x^2+y^2+1}+1\right)} = \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{\left(x^2+y^2+1\right)-1}$$

$$= \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} = \sqrt{x^2+y^2+1}+1$$

The resulting function is continuous, hence we may compute the limit by substitution. This gives

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = \sqrt{0^2+0^2+1}+1 = 2$$

32. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|}$

Hint: Rewrite the limit in terms of $u=x-1$ and $v=y-1$.

SOLUTION Taking the hint given, let us rewrite the problem, instead of $(x,y)\to(1,1)$, then if $u=x-1$ and $v=y-1$, then $(u,v)\to(0,0)$. Transforming the limit we have:

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|} = \lim_{(u,v)\to(0,0)} \frac{(u+1)^2+(v+1)^2-2}{|u|+|v|} = \lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|}$$

Now consider this limit along two different paths, one is let $v=u=|u|$ and the other $v=-u=|u|$. Examining the limit along $v=u=|u|$ we have

$$\lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|} = \lim_{u\to0} \frac{u^2+2u+u^2+2u}{u+u} = \lim_{u\to0} \frac{2u^2+4u}{2u} = \lim_{u\to0} u+2 = 2$$

whereas if $v=-u=|u|$ we get:

$$\lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|} = \lim_{u\to0} \frac{u^2+2u+u^2-2u}{-u-u} = \lim_{u\to0} \frac{2u^2}{-2u} = \lim_{u\to0} -u = 0$$

Since the limits along these two distinct paths are not equal, we conclude that the limit in question does not exist.

33. Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$.

(a) Show that

$$|x^3| \le |x|(x^2+y^2), \quad |y^3| \le |y|(x^2+y^2)$$

(b) Show that $|f(x,y)| \le |x|+|y|$.

(c) Use the Squeeze Theorem to prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

SOLUTION

(a) Since $|x|y^2 \ge 0$, we have

$$|x^3| \le |x^3| + |x|y^2 = |x|^3 + |x|y^2 = |x|(x^2+y^2)$$

Similarly, since $|y|x^2 \ge 0$, we have

$$|y^3| \le |y^3| + |y|x^2 = |y|^3 + |y|x^2 = |y|(x^2+y^2)$$

(b) We use the triangle inequality to write

$$|f(x,y)| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x^3|+|y^3|}{x^2+y^2}$$

We continue using the inequality in part (a):

$$|f(x,y)| \le \frac{|x|(x^2+y^2)+|y|(x^2+y^2)}{x^2+y^2} = \frac{(|x|+|y|)(x^2+y^2)}{x^2+y^2} = |x|+|y|$$

That is,

$$|f(x, y)| \le |x| + |y|$$

(c) In part (b) we showed that

$$|f(x, y)| \le |x| + |y| \tag{1}$$

Let $\epsilon > 0$. Then if $|x| < \frac{\epsilon}{2}$ and $|y| < \frac{\epsilon}{2}$, we have by (1)

$$|f(x, y) - 0| \le |x| + |y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{2}$$

Notice that if $x^2 + y^2 < \frac{\epsilon^2}{4}$, then $x^2 < \frac{\epsilon^2}{4}$ and $y^2 < \frac{\epsilon^2}{4}$. Hence $|x| < \frac{\epsilon}{2}$ and $|y| < \frac{\epsilon}{2}$, so (1) holds. In other words, using $D^\star\left(\frac{\epsilon}{2}\right)$ to represent the punctured disc of radius $\epsilon/2$ centered at the origin, we have

$$(x, y) \in D^\star\left(\frac{\epsilon}{2}\right) \quad \Rightarrow \quad |x| < \frac{\epsilon}{2}$$

and

$$|y| < \frac{\epsilon}{2} \quad \Rightarrow \quad |f(x, y) - 0| < \epsilon$$

We conclude by the limit definition that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

34. Let $a, b \ge 0$. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2 + y^2} = 0$ if $a + b > 2$ and that the limit does not exist if $a + b \le 2$.

SOLUTION We first show that the limit is zero if $a + b > 2$. We compute the limit using the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $(x, y) \to (0, 0)$ if and only if $x^2 + y^2 \to 0$, that is, if and only if $r \to 0+$. Therefore,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2 + y^2} &= \lim_{r\to 0+} \frac{(r\cos\theta)^a (r\sin\theta)^b}{r^2} = \lim_{r\to 0+} \frac{r^{a+b}\cos^a\theta \sin^b\theta}{r^2} \\ &= \lim_{r\to 0+} (r^{a+b-2}\cos^a\theta\sin^b\theta) \end{aligned} \tag{1}$$

The following inequality holds:

$$0 \le |r^{a+b-2}\cos^a\theta\sin^b\theta| \le r^{a+b-2} \tag{2}$$

Since $a + b > 2$, $\displaystyle\lim_{r\to 0+} r^{a+b-2} = 0$, therefore (2) and the Squeeze Theorem imply that

$$\lim_{r\to 0} (r^{a+b-2}\cos^a\theta\sin^b\theta) = 0 \tag{3}$$

We combine (1) and (3) to conclude that if $a + b > 2$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2 + y^2} = 0$$

We now consider the case $a + b < 2$. We examine the limit as (x, y) approaches the origin along the line $y = x$. Along this line, $\theta = \frac{\pi}{4}$, therefore (1) gives

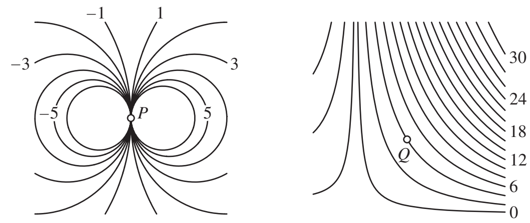
$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2 + y^2} = \lim_{r\to 0+} \left(r^{a+b-2}\cos^a\frac{\pi}{4}\sin^b\frac{\pi}{4}\right) = \lim_{r\to 0+} \left(r^{a+b-2} \cdot \left(\frac{1}{\sqrt{2}}\right)^a \cdot \left(\frac{1}{\sqrt{2}}\right)^b\right) = \lim_{r\to 0+} \frac{r^{a+b-2}}{(\sqrt{2})^{a+b}}$$

Since $a + b < 2$, we have $a + b - 2 < 0$ therefore $\displaystyle\lim_{r\to 0+} r^{a+b-2}$ does not exist. It follows that if $a + b < 2$, the given limit does not exist. Finally we examine the case $a + b = 2$. By (1) we get

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2 + y^2} = \lim_{r\to 0+} (r^0\cos^a\theta\sin^b\theta) = \lim_{r\to 0+} \cos^a\theta\sin^b\theta = \cos^a\theta\sin^b\theta$$

We see that the function does not approach one limit. For example, approaching the origin along the lines $y = x$ (i.e., $\theta = \frac{\pi}{4}$) and $y = 0$ (i.e., $\theta = 0$) gives two different limits $\cos^a\frac{\pi}{4}\sin^b\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^{a+b}$ and $\cos^a 0 \sin^b 0 = 0$. We conclude that if $a + b = 2$, the limit does not exist.

35. Figure 7 shows the contour maps of two functions. Explain why the limit $\lim_{(x,y)\to P} f(x, y)$ does not exist. Does $\lim_{(x,y)\to Q} g(x, y)$ appear to exist in (B)? If so, what is its limit?



(A) Contour map of $f(x, y)$ (B) Contour map of $g(x, y)$

FIGURE 7

SOLUTION As (x, y) approaches arbitrarily close to P, the function $f(x, y)$ takes the values ±1, ±3, and ±5. Therefore $f(x, y)$ does not approach one limit as $(x, y) \to P$. Rather, the limit depends on the contour along which (x, y) is approaching P. This implies that the limit $\lim_{(x,y)\to P} f(x, y)$ does not exist. In (B) the limit $\lim_{(x,y)\to Q} g(x, y)$ appears to exist. If it exists, it must be 4, which is the level curve of Q.

Further Insights and Challenges

36. Evaluate $\lim_{(x,y)\to(0,2)} (1 + x)^{y/x}$.

SOLUTION We denote $f(x, y) = (1 + x)^{y/x}$. Hence,

$$\ln f(x, y) = \ln(1 + x)^{y/x} = \frac{y}{x}\ln(1 + x) = y\frac{\ln(1 + x)}{x} \tag{1}$$

Using L'Hôpital's Rule we have

$$\lim_{x\to0} \frac{\ln(1 + x)}{x} = \lim_{x\to0} \frac{\frac{1}{1+x}}{1} = \lim_{x\to0} \frac{1}{1 + x} = \frac{1}{1 + 0} = 1$$

Since this limit exists, we may use the Product Rule to compute the limit of (1):

$$\lim_{(x,y)\to(0,2)} \ln f(x, y) = \left(\lim_{y\to2} y\right)\left(\lim_{x\to0} \frac{\ln(1 + x)}{x}\right) = 2 \cdot 1 = 2 \tag{2}$$

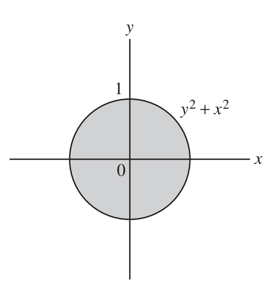
$\ln u$ approaches 2 if and only if u is approaching e^2. Therefore, the limit in (2) implies that

$$\lim_{(x,y)\to(0,2)} f(x, y) = e^2.$$

37. Is the following function continuous?

$$f(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 < 1 \\ 1 & \text{if } x^2 + y^2 \ge 1 \end{cases}$$

SOLUTION $f(x, y)$ is defined by a polynomial in the domain $x^2 + y^2 < 1$, hence f is continuous in this domain. In the domain $x^2 + y^2 > 1$, f is a constant function, hence f is continuous in this domain also. Thus, we must examine continuity at the points on the circle $x^2 + y^2 = 1$.



We express $f(x, y)$ using polar coordinates:

$$f(r,\theta) = \begin{cases} r^2 & 0 \le r < 1 \\ 1 & r \ge 1 \end{cases}$$

Since $\lim_{r\to 1-} f(r,\theta) = \lim_{r\to 1-} r^2 = 1$ and $\lim_{r\to 1+} f(r,\theta) = \lim_{r\to 1+} 1 = 1$, we have $\lim_{r\to 1} f(r,\theta) = 1$. Therefore $f(r,\theta)$ is continuous at $r = 1$, or $f(x, y)$ is continuous on $x^2 + y^2 = 1$. We conclude that f is continuous everywhere on R^2.

38. CAS The function $f(x, y) = \sin(xy)/xy$ is defined for $xy \neq 0$.

(a) Is it possible to extend the domain of $f(x, y)$ to all of $\mathbf{R}^2$ so that the result is a continuous function?

(b) Use a computer algebra system to plot $f(x, y)$. Does the result support your conclusion in (a)?

SOLUTION

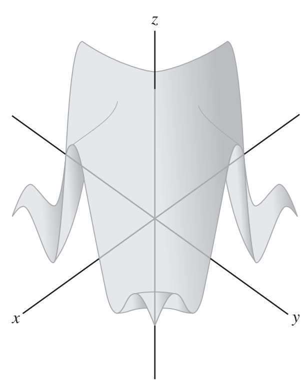
(a) We define $f(x, y)$ on the x- and y-axes by $f(x, y) = 1$ if $xy = 0$. We now show that f is continuous. f is continuous at the points where $xy \neq 0$. We next show continuity at $(x_0, 0)$ (including $x_0 = 0$). For the points $(0, y_0)$, the proof is similar and hence will be omitted. To prove continuity at $P = (x_0, 0)$ we have to show that

$$\lim_{(x,y)\to P} f(x, y) = \lim_{(x,y)\to P} \frac{\sin xy}{xy} = 1 \tag{1}$$

Let us denote $u = xy$. As $(x, y) \to (x_0, 0)$, $u = x \cdot y \to x_0 \cdot 0 = 0$. Thus,

$$\lim_{(x,y)\to P} f(x, y) = \lim_{(x,y)\to(x_0,0)} \frac{\sin xy}{xy} = \lim_{u\to 0} \frac{\sin u}{u} = 1 = f(x_0, 0).$$

(b) The following figure shows the graph of $f(x, y) = \frac{\sin xy}{xy}$:



The graph shows that, near the axes, the values of $f(x, y)$ are approaching 1, as shown in part (a).

39. Prove that the function

$$f(x, y) = \begin{cases} \dfrac{(2^x - 1)(\sin y)}{xy} & \text{if } xy \neq 0 \\ \ln 2 & \text{if } xy = 0 \end{cases}$$

is continuous at $(0, 0)$.

SOLUTION To solve this problem it is necessary to show that $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = \ln 2$. Consider the following:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{(2^x - 1)\sin y}{xy} &= \lim_{(x,y)\to(0,0)} \frac{2^x - 1}{x} \cdot \frac{\sin y}{y} \\ &= \left(\lim_{x\to 0} \frac{2^x - 1}{x}\right)\left(\lim_{y\to 0} \frac{\sin y}{y}\right) \\ &= \lim_{x\to 0} \frac{(\ln 2)2^x}{1} \cdot (1) = \ln 2 \end{aligned}$$

(Using L'Hopital's Rule on the limit in terms of x.) Thus since $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$, we see that $f(x, y)$ is continuous at $(0, 0)$.

40. Prove that if $f(x)$ is continuous at $x = a$ and $g(y)$ is continuous at $y = b$, then $F(x, y) = f(x)g(y)$ is continuous at (a, b).

SOLUTION Given that $f(x)$ is continuous at $x = a$, we know that

$$\lim_{x\to a} f(x) = f(a)$$

and given that $g(x)$ is continuous at $x = b$, we know that

$$\lim_{x\to b} g(x) = g(b).$$

Consider the limit $\lim_{(x,y)\to(a,b)} F(x, y)$. Then using the above information we have

$$\lim_{(x,y)\to(a,b)} F(x, y) = \lim_{(x,y)\to(a,b)} f(x)g(y) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{y\to b} g(y)\right) = f(a)g(b) = F(a, b)$$

Therefore, $F(x, y)$ is continuous at the point (a, b).

41. The function $f(x, y) = x^2y/(x^4 + y^2)$ provides an interesting example where the limit as $(x, y) \to (0, 0)$ does not exist, even though the limit along every line $y = mx$ exists and is zero (Figure 8).

(a) Show that the limit along any line $y = mx$ exists and is equal to 0.

(b) Calculate $f(x, y)$ at the points $(10^{-1}, 10^{-2})$, $(10^{-5}, 10^{-10})$, $(10^{-20}, 10^{-40})$. Do not use a calculator.

(c) Show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. *Hint:* Compute the limit along the parabola $y = x^2$.

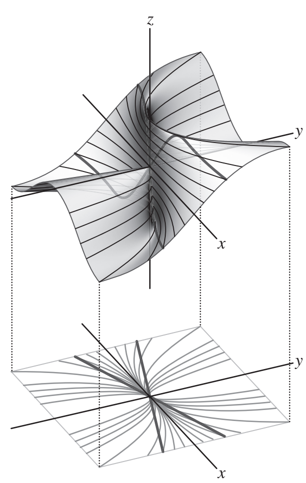


FIGURE 8 Graph of $f(x, y) = \dfrac{x^2y}{x^4 + y^2}$.

SOLUTION

(a) Substituting $y = mx$ in $f(x, y) = \frac{x^2y}{x^4+y^2}$, we get

$$f(x, mx) = \frac{x^2 \cdot mx}{x^4 + (mx)^2} = \frac{mx^3}{x^2(x^2 + m^2)} = \frac{mx}{x^2 + m^2}$$

We compute the limit as $x \to 0$ by substitution:

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = \frac{m \cdot 0}{0^2 + m^2} = 0$$

(b) We compute $f(x, y)$ at the given points:

$$f(10^{-1}, 10^{-2}) = \frac{10^{-2} \cdot 10^{-2}}{10^{-4} + 10^{-4}} = \frac{10^{-4}}{2 \cdot 10^{-4}} = \frac{1}{2}$$

$$f(10^{-5}, 10^{-10}) = \frac{10^{-10} \cdot 10^{-10}}{10^{-20} + 10^{-20}} = \frac{10^{-20}}{2 \cdot 10^{-20}} = \frac{1}{2}$$

$$f(10^{-20}, 10^{-40}) = \frac{10^{-40} \cdot 10^{-40}}{10^{-80} + 10^{-80}} = \frac{10^{-80}}{2 \cdot 10^{-80}} = \frac{1}{2}$$

(c) We compute the limit as (x, y) approaches the origin along the parabola $y = x^2$ (by part (b), the limit appears to be $\frac{1}{2}$). We substitute $y = x^2$ in the function and compute the limit as $x \to 0$. This gives

$$\lim_{\substack{(x,y)\to 0 \\ \text{along } y=x^2}} f(x, y) = \lim_{x\to 0} f(x, x^2) = \lim_{x\to 0} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

However, in part (a), we showed that the limit along the lines $y = mx$ is zero. Therefore $f(x, y)$ does not approach one limit as $(x, y) \to (0, 0)$, so the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

14.3 Partial Derivatives (LT Section 15.3)

Preliminary Questions

1. Patricia derived the following *incorrect* formula by misapplying the Product Rule:

$$\frac{\partial}{\partial x}(x^2 y^2) = x^2(2y) + y^2(2x)$$

What was her mistake and what is the correct calculation?

SOLUTION To compute the partial derivative with respect to x, we treat y as a constant. Therefore the Constant Multiple Rule must be used rather than the Product Rule. The correct calculation is:

$$\frac{\partial}{\partial x}(x^2 y^2) = y^2 \frac{\partial}{\partial x}(x^2) = y^2 \cdot 2x = 2xy^2.$$

2. Explain why it is not necessary to use the Quotient Rule to compute $\frac{\partial}{\partial x}\left(\frac{x+y}{y+1}\right)$. Should the Quotient Rule be used to compute $\frac{\partial}{\partial y}\left(\frac{x+y}{y+1}\right)$?

SOLUTION In differentiating with respect to x, y is considered a constant. Therefore in this case the Constant Multiple Rule can be used to obtain

$$\frac{\partial}{\partial x}\left(\frac{x+y}{y+1}\right) = \frac{1}{y+1}\frac{\partial}{\partial x}(x+y) = \frac{1}{y+1} \cdot 1 = \frac{1}{y+1}.$$

As for the second part, since y appears in both the numerator and the denominator, the Quotient Rule is indeed needed.

3. Which of the following partial derivatives should be evaluated without using the Quotient Rule?

(a) $\frac{\partial}{\partial x}\frac{xy}{y^2+1}$ **(b)** $\frac{\partial}{\partial y}\frac{xy}{y^2+1}$ **(c)** $\frac{\partial}{\partial x}\frac{y^2}{y^2+1}$

SOLUTION

(a) This partial derivative does not require use of the Quotient Rule, since the Constant Multiple Rule gives

$$\frac{\partial}{\partial x}\left(\frac{xy}{y^2+1}\right) = \frac{y}{y^2+1}\frac{\partial}{\partial x}(x) = \frac{y}{y^2+1} \cdot 1 = \frac{y}{y^2+1}.$$

(b) This partial derivative requires use of the Quotient Rule.

(c) Since y is considered a constant in differentiating with respect to x, we do not need the Quotient Rule to state that $\frac{\partial}{\partial x}\left(\frac{y^2}{y^2+1}\right) = 0.$

4. What is f_x, where $f(x, y, z) = (\sin yz)e^{z^3 - z^{-1}\sqrt{y}}$?

SOLUTION In differentiating with respect to x, we treat y and z as constants. Therefore, the whole expression for $f(x, y, z)$ is treated as constant, so the derivative is zero:

$$\frac{\partial}{\partial x}\left(\sin yz e^{z^3 - z^{-1}\sqrt{y}}\right) = 0.$$

5. Assuming the hypotheses of Clairaut's Theorem are satisfied, which of the following partial derivatives are equal to f_{xxy}?

(a) f_{xyx} **(b)** f_{yyx} **(c)** f_{xyy} **(d)** f_{yxx}

SOLUTION f_{xxy} involves two differentiations with respect to x and one differentiation with respect to y. Therefore, if f satisfies the assumptions of Clairaut's Theorem, then

$$f_{xxy} = f_{xyx} = f_{yxx}$$

Exercises

1. Use the limit definition of the partial derivative to verify the formulas

$$\frac{\partial}{\partial x} xy^2 = y^2, \qquad \frac{\partial}{\partial y} xy^2 = 2xy$$

SOLUTION Using the limit definition of the partial derivative, we have

$$\frac{\partial}{\partial x} xy^2 = \lim_{h\to 0} \frac{(x+h)y^2 - xy^2}{h} = \lim_{h\to 0} \frac{xy^2 + hy^2 - xy^2}{h} = \lim_{h\to 0} \frac{hy^2}{h} = \lim_{h\to 0} y^2 = y^2$$

$$\frac{\partial}{\partial y} xy^2 = \lim_{k\to 0} \frac{x(y+k)^2 - xy^2}{k} = \lim_{k\to 0} \frac{x(y^2+2yk+k^2) - xy^2}{k} = \lim_{k\to 0} \frac{xy^2 + 2xyk + xk^2 - xy^2}{k}$$

$$= \lim_{k\to 0} \frac{k(2xy+xk)}{k} = \lim_{k\to 0} (2xy+k) = 2xy + 0 = 2xy$$

2. Use the Product Rule to compute $\frac{\partial}{\partial y}(x^2+y)(x+y^4)$.

SOLUTION Using the Product Rule we obtain

$$\frac{\partial}{\partial y}(x^2+y)(x+y^4) = (x^2+y)\frac{\partial}{\partial y}(x+y^4) + (x+y^4)\frac{\partial}{\partial y}(x^2+y)$$

$$= (x^2+y)\cdot 4y^3 + (x+y^4)\cdot 1 = 4x^2y^3 + 5y^4 + x$$

3. Use the Quotient Rule to compute $\frac{\partial}{\partial y}\frac{y}{x+y}$.

SOLUTION Using the Quotient Rule we obtain

$$\frac{\partial}{\partial y}\frac{y}{x+y} = \frac{(x+y)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x+y)}{(x+y)^2} = \frac{(x+y)\cdot 1 - y\cdot 1}{(x+y)^2} = \frac{x}{(x+y)^2}$$

4. Use the Chain Rule to compute $\frac{\partial}{\partial u}\ln(u^2+uv)$.

SOLUTION By the Chain Rule $\frac{d}{du}\ln\omega = \frac{1}{\omega}\frac{d\omega}{du}$. Applying this with $\omega = u^2 + uv$ gives

$$\frac{\partial}{\partial u}\ln(u^2+uv) = \frac{1}{u^2+uv}\frac{\partial}{\partial u}(u^2+uv) = \frac{2u+v}{u^2+uv}$$

5. Calculate $f_z(2,3,1)$, where $f(x,y,z) = xyz$.

SOLUTION We first find the partial derivative $f_z(x,y,z)$:

$$f_z(x,y,z) = \frac{\partial}{\partial z}(xyz) = xy$$

Substituting the given point we get

$$f_z(2,3,1) = 2\cdot 3 = 6$$

6. Explain the relation between the following two formulas (c is a constant).

$$\frac{d}{dx}\sin(cx) = c\cos(cx), \qquad \frac{\partial}{\partial x}\sin(xy) = y\cos(xy)$$

SOLUTION $\frac{d}{dx}\sin(cx)$ is the derivative of the single-variable function $\sin(cx)$, where c is a constant. $\frac{\partial}{\partial x}\sin(xy)$ is the partial derivative of the two-variable function $\sin(xy)$ with respect to x. While differentiating, the variable y is considered constant, hence it resembles the first differentiation, and the results are the same where c is replaced by y.

7. The plane $y = 1$ intersects the surface $z = x^4 + 6xy - y^4$ in a certain curve. Find the slope of the tangent line to this curve at the point $P = (1, 1, 6)$.

SOLUTION The slope of the tangent line to the curve $z = z(x, 1) = x^4 + 6x - 1$, obtained by intersecting the surface $z = x^4 + 6xy - y^4$ with the plane $y = 1$, is the partial derivative $\frac{\partial z}{\partial x}(1, 1)$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4 + 6xy - y^4) = 4x^3 + 6y$$

$$m = \frac{\partial z}{\partial x}(1, 1) = 4 \cdot 1^3 + 6 \cdot 1 = 10$$

8. Determine whether the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are positive or negative at the point P on the graph in Figure 7.

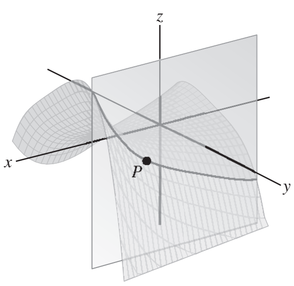


FIGURE 7

SOLUTION The graph shows that f is increasing in the direction of growing x and f is decreasing in the direction of growing y. Therefore, $\frac{\partial f}{\partial x}\big|_P > 0$ and $\frac{\partial f}{\partial y}\big|_P < 0$.

In Exercises 9–12, refer to Figure 8.

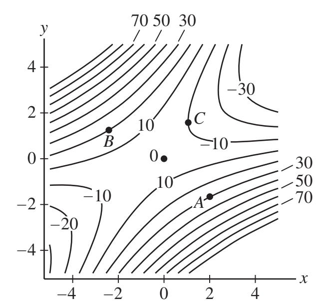


FIGURE 8 Contour map of $f(x, y)$.

9. Estimate f_x and f_y at point A.

SOLUTION To estimate f_x we move horizontally to the next level curve in the direction of growing x, to a point A'. The change in f from A to A' is the contour interval, $\Delta f = 40 - 30 = 10$. The distance between A and A' is approximately $\Delta x \approx 1.0$. Hence,

$$f_x(A) \approx \frac{\Delta f}{\Delta x} = \frac{10}{1.0} = 10$$

To estimate f_y we move vertically from A to a point A'' on the next level curve in the direction of growing y. The change in f from A to A'' is $\Delta f = 20 - 30 = -10$. The distance between A and A'' is $\Delta y \approx 0.5$. Hence,

$$f_y(A) \approx \frac{\Delta f}{\Delta y} = \frac{-10}{0.5} \approx -20.$$

10. Is f_x positive or negative at B?

SOLUTION To estimate f_x at B, we move horizontally to the next level curve in the direction of growing x, to a point B'. The change in f from B to B' is the contour interval $\Delta f = 10 - 20 = -10$ while the distance between B and B' is approximately $\Delta x \approx 1$. Hence

$$f_x(B) \approx \frac{\Delta f}{\Delta x} = \frac{-10}{1} = -10 < 0$$

Therefore $f_x(B)$ is negative.

11. Starting at point B, in which compass direction (N, NE, SW, etc.) does f increase most rapidly?

SOLUTION The distances between adjacent level curves starting at B are the smallest along the line with slope -1, upward. Therefore, f is increasing most rapidly in the direction of $\theta = 135^\circ$ or in the NW direction.

12. At which of A, B, or C is f_y smallest?

SOLUTION We consider vertical lines through A, B, and C. The distance between each point A, B, C and the intersection of the vertical line with the adjacent level curves is the largest at C. It means that f_y is smallest at C.

In Exercises 13–40, compute the first-order partial derivatives.

13. $z = x^2 + y^2$

SOLUTION We compute $z_x(x, y)$ by treating y as a constant, and we compute $z_y(x, y)$ by treating x as a constant:

$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x; \quad \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

14. $z = x^4 y^3$

SOLUTION Treating y as a constant (to find z_x) and x as a constant (to find z_y) and using Rules for Differentiation, we get,

$$\frac{\partial}{\partial x}(x^4 y^3) = y^3 \frac{\partial}{\partial x}(x^4) = y^3 \cdot 4x^3 = 4x^3 y^3$$
$$\frac{\partial}{\partial y}(x^4 y^3) = x^4 \frac{\partial}{\partial y}(y^3) = x^4 \cdot 3y^2 = 3x^4 y^2$$

15. $z = x^4 y + x y^{-2}$

SOLUTION We obtain the following partial derivatives:

$$\frac{\partial}{\partial x}(x^4 y + x y^{-2}) = 4x^3 y + y^{-2}$$
$$\frac{\partial}{\partial y}(x^4 y + x y^{-2}) = x^4 + x \cdot (-2y^{-3}) = x^4 - 2xy^{-3}$$

16. $V = \pi r^2 h$

SOLUTION We find $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial h}$:

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(\pi r^2 h) = \pi h \frac{\partial}{\partial r}(r^2) = \pi h \cdot 2r = 2\pi h r$$
$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}(\pi r^2 h) = \pi r^2$$

17. $z = \dfrac{x}{y}$

SOLUTION Treating y as a constant we have

$$\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{1}{y} \cdot 1 = \frac{1}{y}$$

We now find the derivative $z_y(x, y)$, treating x as a constant:

$$\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = x \cdot \frac{-1}{y^2} = \frac{-x}{y^2}.$$

18. $z = \dfrac{x}{x - y}$

SOLUTION We differentiate with respect to x, using the Quotient Rule. We get

$$\frac{\partial}{\partial x}\left(\frac{x}{x-y}\right) = \frac{(x-y)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x-y)}{(x-y)^2} = \frac{(x-y)\cdot 1 - x \cdot 1}{(x-y)^2} = \frac{-y}{(x-y)^2}$$

We now differentiate with respect to y, using the Chain Rule:

$$\frac{\partial}{\partial y}\left(\frac{x}{x-y}\right) = x\frac{\partial}{\partial y}\left(\frac{1}{x-y}\right) = x \cdot \frac{-1}{(x-y)^2}\frac{\partial}{\partial y}(x-y) = x \cdot \frac{-1}{(x-y)^2} \cdot (-1) = \frac{x}{(x-y)^2}$$

19. $z = \sqrt{9 - x^2 - y^2}$

SOLUTION Differentiating with respect to x, treating y as a constant, and using the Chain Rule, we obtain

$$\frac{\partial}{\partial x}\left(\sqrt{9 - x^2 - y^2}\right) = \frac{1}{2\sqrt{9 - x^2 - y^2}} \frac{\partial}{\partial x}(9 - x^2 - y^2) = \frac{-2x}{2\sqrt{9 - x^2 - y^2}} = \frac{-x}{\sqrt{9 - x^2 - y^2}}$$

We now differentiate with respect to y, treating x as a constant:

$$\frac{\partial}{\partial y}\left(\sqrt{9 - x^2 - y^2}\right) = \frac{1}{2\sqrt{9 - x^2 - y^2}} \frac{\partial}{\partial y}(9 - x^2 - y^2) = \frac{-2y}{2\sqrt{9 - x^2 - y^2}} = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

20. $z = \dfrac{x}{\sqrt{x^2 + y^2}}$

SOLUTION We compute $\frac{\partial z}{\partial x}$ using the Quotient Rule and the Chain Rule:

$$\frac{\partial z}{\partial x} = \frac{1 \cdot \sqrt{x^2 + y^2} - x \frac{\partial}{\partial x}\sqrt{x^2 + y^2}}{\left(\sqrt{x^2 + y^2}\right)^2} = \frac{\sqrt{x^2 + y^2} - x \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{x^2 + y^2} = \frac{x^2 + y^2 - x^2}{(x^2 + y^2)^{3/2}} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

We compute $\frac{\partial z}{\partial y}$ using the Chain Rule:

$$\frac{\partial z}{\partial y} = x \frac{\partial}{\partial y}(x^2 + y^2)^{-1/2} = x \cdot \left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2} \cdot 2y = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

21. $z = (\sin x)(\sin y)$

SOLUTION We obtain the following partial derivatives:

$$\frac{\partial}{\partial x}(\sin x \sin y) = \sin y \frac{\partial}{\partial x} \sin x = \sin y \cos x$$
$$\frac{\partial}{\partial y}(\sin x \sin y) = \sin x \frac{\partial}{\partial y} \sin y = \sin x \cos y$$

22. $z = \sin(u^2 v)$

SOLUTION By the Chain Rule,

$$\frac{d}{du} \sin \omega = \cos \omega \frac{d\omega}{du} \quad \text{and} \quad \frac{d}{dv} \sin \omega = \cos \omega \frac{d\omega}{dv}.$$

Applying this with $\omega = u^2 v$ gives

$$\frac{\partial}{\partial u} \sin(u^2 v) = \cos(u^2 v) \frac{\partial}{\partial u}(u^2 v) = \cos(u^2 v) \cdot 2uv = 2uv \cos(u^2 v)$$
$$\frac{\partial}{\partial v} \sin(u^2 v) = \cos(u^2 v) \frac{\partial}{\partial v}(u^2 v) = \cos(u^2 v) \cdot u^2 = u^2 \cos(u^2 v)$$

23. $z = \tan \dfrac{x}{y}$

SOLUTION By the Chain Rule,

$$\frac{d}{dx} \tan u = \frac{1}{\cos^2 u} \frac{du}{dx} \quad \text{and} \quad \frac{d}{dy} \tan u = \frac{1}{\cos^2 u} \frac{du}{dy}.$$

(We could also say that the derivative of $\tan u$ is $\sec^2 u$, but of course $\sec^2 u = 1/\cos^2 u$, so it really is the same thing.) We apply this with $u = \frac{x}{y}$ to obtain

$$\frac{\partial}{\partial x} \tan\left(\frac{x}{y}\right) = \frac{1}{\cos^2\left(\frac{x}{y}\right)} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\cos^2\left(\frac{x}{y}\right)} \cdot \frac{1}{y} = \frac{1}{y\cos^2\left(\frac{x}{y}\right)}$$
$$\frac{\partial}{\partial y} \tan\left(\frac{x}{y}\right) = \frac{1}{\cos^2\left(\frac{x}{y}\right)} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\cos^2\left(\frac{x}{y}\right)} \cdot \frac{-x}{y^2} = \frac{-x}{y^2\cos^2\left(\frac{x}{y}\right)}$$

24. $S = \tan^{-1}(wz)$

SOLUTION By the Chain Rule,

$$\frac{d}{dw}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dw} \quad \text{and} \quad \frac{d}{dz}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dz}$$

Using this rule with $u = wz$ gives

$$\frac{dS}{dw} = \frac{\partial}{\partial w}\tan^{-1}(wz) = \frac{1}{1+(wz)^2}\frac{\partial}{\partial w}(wz) = \frac{z}{1+w^2z^2}$$
$$\frac{dS}{dz} = \frac{\partial}{\partial z}\tan^{-1}(wz) = \frac{1}{1+(wz)^2}\frac{\partial}{\partial z}(wz) = \frac{w}{1+w^2z^2}$$

25. $z = \ln(x^2 + y^2)$

SOLUTION Using the Chain Rule we have

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{x^2+y^2}\cdot 2x = \frac{2x}{x^2+y^2}$$
$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2) = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

26. $A = \sin(4\theta - 9t)$

SOLUTION We use the Chain Rule to compute $\frac{\partial A}{\partial \theta}$ and $\frac{\partial A}{\partial t}$:

$$\frac{\partial A}{\partial \theta} = \cos(4\theta - 9t)\frac{\partial}{\partial \theta}(4\theta - 9t) = 4\cos(4\theta - 9t)$$
$$\frac{\partial A}{\partial t} = \cos(4\theta - 9t)\frac{\partial}{\partial t}(4\theta - 9t) = -9\cos(4\theta - 9t)$$

27. $W = e^{r+s}$

SOLUTION We use the Chain Rule to compute $\frac{\partial W}{\partial r}$ and $\frac{\partial W}{\partial s}$:

$$\frac{\partial W}{\partial r} = e^{r+s}\cdot\frac{\partial}{\partial r}(r+s) = e^{r+s}\cdot 1 = e^{r+s}$$
$$\frac{\partial W}{\partial s} = e^{r+s}\cdot\frac{\partial}{\partial s}(r+s) = e^{r+s}\cdot 1 = e^{r+s}$$

28. $Q = re^{\theta}$

SOLUTION The partial derivatives are

$$\frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}(re^{\theta}) = e^{\theta}\frac{\partial}{\partial r}(r) = e^{\theta}$$
$$\frac{\partial Q}{\partial \theta} = \frac{\partial}{\partial \theta}(re^{\theta}) = r\frac{\partial}{\partial \theta}(e^{\theta}) = re^{\theta}$$

29. $z = e^{xy}$

SOLUTION We use the Chain Rule, $\frac{d}{dx}e^u = e^u\frac{du}{dx}$; $\frac{d}{dy}e^u = e^u\frac{du}{dy}$ with $u = xy$ to obtain

$$\frac{\partial}{\partial x}e^{xy} = e^{xy}\frac{\partial}{\partial x}(xy) = e^{xy}y = ye^{xy}$$
$$\frac{\partial}{\partial y}e^{xy} = e^{xy}\frac{\partial}{\partial y}(xy) = e^{xy}x = xe^{xy}$$

30. $R = e^{-v^2/k}$

SOLUTION Using the Chain Rule gives

$$\frac{\partial R}{\partial v} = e^{-v^2/k}\frac{\partial}{\partial v}\left(-\frac{v^2}{k}\right) = e^{-v^2/k}\cdot\left(-\frac{2v}{k}\right) = -\frac{2v}{k}e^{-v^2/k}$$
$$\frac{\partial R}{\partial k} = e^{-v^2/k}\frac{\partial}{\partial k}\left(-\frac{v^2}{k}\right) = e^{-v^2/k}\cdot\left(-v^2\right)\frac{\partial}{\partial k}\left(\frac{1}{k}\right) = e^{-v^2/k}(-v^2)\cdot\frac{-1}{k^2} = \left(\frac{v}{k}\right)^2 e^{-v^2/k}$$

31. $z = e^{-x^2-y^2}$

SOLUTION We use the Chain Rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x} = e^{-x^2-y^2}\frac{\partial}{\partial x}(-x^2-y^2) = e^{-x^2-y^2}\cdot(-2x) = -2xe^{-x^2-y^2}$$
$$\frac{\partial z}{\partial y} = e^{-x^2-y^2}\frac{\partial}{\partial y}(-x^2-y^2) = e^{-x^2-y^2}\cdot(-2y) = -2ye^{-x^2-y^2}$$

32. $P = e^{\sqrt{y^2+z^2}}$

SOLUTION We use the Chain Rule to compute $\frac{\partial P}{\partial y}$ and $\frac{\partial P}{\partial z}$:

$$\frac{\partial P}{\partial y} = e^{\sqrt{y^2+z^2}}\frac{\partial}{\partial y}\sqrt{y^2+z^2} = e^{\sqrt{y^2+z^2}}\cdot\frac{2y}{2\sqrt{y^2+z^2}} = e^{\sqrt{y^2+z^2}}\cdot\frac{y}{\sqrt{y^2+z^2}}$$
$$\frac{\partial P}{\partial z} = e^{\sqrt{y^2+z^2}}\frac{\partial}{\partial z}\sqrt{y^2+z^2} = e^{\sqrt{y^2+z^2}}\cdot\frac{2z}{2\sqrt{y^2+z^2}} = e^{\sqrt{y^2+z^2}}\cdot\frac{z}{\sqrt{y^2+z^2}}$$

33. $U = \dfrac{e^{-rt}}{r}$

SOLUTION We have

$$\frac{\partial U}{\partial r} = \frac{-te^{-rt}\cdot r - e^{-rt}\cdot 1}{r^2} = \frac{-(1+rt)e^{-rt}}{r^2}$$

and also

$$\frac{\partial U}{\partial t} = \frac{-re^{-rt}}{r} = -e^{-rt}$$

34. $z = y^x$

SOLUTION To find $\frac{\partial z}{\partial y}$, we use the Power Rule for differentiation:

$$\frac{\partial z}{\partial y} = xy^{x-1}$$

To find $\frac{\partial z}{\partial x}$, we use the derivative of the exponent function:

$$\frac{\partial z}{\partial x} = y^x \ln y$$

35. $z = \sinh(x^2y)$

SOLUTION By the Chain Rule, $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$ and $\frac{d}{dy}\sinh u = \cosh u\frac{du}{dy}$. We use the Chain Rule with $u = x^2y$ to obtain

$$\frac{\partial}{\partial x}\sinh(x^2y) = \cosh(x^2y)\frac{\partial}{\partial x}(x^2y) = 2xy\cosh(x^2y)$$
$$\frac{\partial}{\partial y}\sinh(x^2y) = \cosh(x^2y)\frac{\partial}{\partial y}(x^2y) = x^2\cosh(x^2y)$$

36. $z = \cosh(t - \cos x)$

SOLUTION The partial derivatives of z are

$$\frac{\partial z}{\partial t} = \sinh(t-\cos x)$$
$$\frac{\partial z}{\partial x} = \sinh(t-\cos x)\frac{\partial}{\partial x}(t-\cos x) = \sinh(t-\cos x)\cdot\sin x$$

37. $w = xy^2z^3$

SOLUTION The partial derivatives of w are

$$\frac{\partial w}{\partial x} = y^2z^3$$
$$\frac{\partial w}{\partial y} = xz^3\frac{\partial}{\partial y}(y^2) = xz^3 \cdot 2y = 2xz^3y$$
$$\frac{\partial w}{\partial z} = xy^2\frac{\partial}{\partial z}(z^3) = xy^2 \cdot 3z^2 = 3xy^2z^2$$

38. $w = \dfrac{x}{y+z}$

SOLUTION We have

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y+z}\right) = \frac{1}{y+z}\frac{\partial}{\partial x}(x) = \frac{1}{y+z}$$

To find $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$, we use the Chain Rule:

$$\frac{\partial w}{\partial y} = x\frac{\partial}{\partial y}\left(\frac{1}{y+z}\right) = x \cdot \frac{-1}{(y+z)^2}\frac{\partial}{\partial y}(y+z) = x \cdot \frac{-1}{(y+z)^2} \cdot 1 = \frac{-x}{(y+z)^2}$$
$$\frac{\partial w}{\partial z} = x\frac{\partial}{\partial z}\left(\frac{1}{y+z}\right) = x \cdot \frac{-1}{(y+z)^2}\frac{\partial}{\partial z}(y+z) = x \cdot \frac{-1}{(y+z)^2} \cdot 1 = \frac{-x}{(y+z)^2}$$

39. $Q = \dfrac{L}{M}e^{-Lt/M}$

SOLUTION

$$\begin{aligned}
\frac{\partial Q}{\partial L} &= \frac{\partial}{\partial L}\left(\frac{L}{M}e^{-Lt/M}\right) \\
&= \frac{L}{M} \cdot e^{-Lt/M} \cdot (-t/M) + e^{-Lt/M} \cdot \frac{1}{M} \\
&= -\frac{Lt}{M^2}e^{-Lt/M} + \frac{e^{-Lt/M}}{M} \\
\frac{\partial Q}{\partial M} &= \frac{\partial}{\partial M}\left(\frac{L}{M}e^{-Lt/M}\right) \\
&= \frac{L}{M} \cdot e^{-Lt/M} \cdot \frac{Lt}{M^2} + e^{-Lt/M} \cdot \frac{-L}{M^2} \\
&= \frac{L^2t}{M^3}e^{-Lt/M} - \frac{L}{M^2}e^{-Lt/M} \\
\frac{\partial Q}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{L}{M}e^{-Lt/M}\right) \\
&= -\frac{L^2}{M^2}e^{-Lt/M}
\end{aligned}$$

40. $w = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$

SOLUTION To find $\frac{\partial w}{\partial x}$, we use the Quotient Rule and the Chain Rule:

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{1 \cdot (x^2+y^2+z^2)^{3/2} - x \cdot \frac{3}{2}(x^2+y^2+z^2)^{1/2} \cdot 2x}{(x^2+y^2+z^2)^3} = (x^2+y^2+z^2)^{1/2}\frac{(x^2+y^2+z^2) - x \cdot 3x}{(x^2+y^2+z^2)^3} \\
&= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

We now use the Chain Rule to compute $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$:

$$\frac{\partial w}{\partial y} = x\frac{\partial}{\partial y}\frac{1}{(x^2+y^2+z^2)^{3/2}} = x\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2}$$

$$= x \cdot \left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2y = -\frac{3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial w}{\partial z} = x\frac{\partial}{\partial z}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}} = x\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-3/2}$$

$$= x \cdot \left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2z = -\frac{3xz}{(x^2+y^2+z^2)^{5/2}}$$

In Exercises 41–44, compute the given partial derivatives.

41. $f(x, y) = 3x^2y + 4x^3y^2 - 7xy^5, \quad f_x(1, 2)$

SOLUTION Differentiating with respect to x gives

$$f_x(x, y) = 6xy + 12x^2y^2 - 7y^5$$

Evaluating at $(1, 2)$ gives

$$f_x(1, 2) = 6 \cdot 1 \cdot 2 + 12 \cdot 1^2 \cdot 2^2 - 7 \cdot 2^5 = -164.$$

42. $f(x, y) = \sin(x^2 - y), \quad f_y(0, \pi)$

SOLUTION We differentiate with respect to y, using the Chain Rule. This gives

$$f_y(x, y) = \cos(x^2 - y)\frac{\partial}{\partial y}(x^2 - y) = \cos(x^2 - y) \cdot (-1) = -\cos(x^2 - y)$$

Evaluating at $(0, \pi)$ we obtain

$$f_y(0, \pi) = -\cos(0^2 - \pi) = -\cos(-\pi) = -\cos\pi = 1.$$

43. $g(u, v) = u\ln(u + v), \quad g_u(1, 2)$

SOLUTION Using the Product Rule and the Chain Rule we get

$$g_u(u, v) = \frac{\partial}{\partial u}(u\ln(u + v)) = 1 \cdot \ln(u + v) + u \cdot \frac{1}{u + v} = \ln(u + v) + \frac{u}{u + v}$$

At the point $(1, 2)$ we have

$$g_u(1, 2) = \ln(1 + 2) + \frac{1}{1 + 2} = \ln 3 + \frac{1}{3}.$$

44. $h(x, z) = e^{xz - x^2z^3}, \quad h_z(3, 0)$

SOLUTION We obtain the following partial:

$$h_z(x, z) = (x - 3x^2z^2)e^{xz - x^2z^3}$$

Substituting $x = 3$, $z = 0$ we obtain the partial derivative at the point $(3, 0)$:

$$h_z(3, 0) = (3 - 0)e^{0-0} = 3.$$

Exercises 45 and 46 refer to Example 5.

45. Calculate N for $L = 0.4$, $R = 0.12$, and $d = 10$, and use the linear approximation to estimate ΔN if d is increased from 10 to 10.4.

SOLUTION From the example in the text we have

$$N = \left(\frac{2200R}{Ld}\right)^{1.9}$$

Calculating N for $L = 0.4$, $R = 0.12$, and $d = 10$ we have

$$N = \left(\frac{2200 \cdot 0.12}{0.4 \cdot 10}\right)^{1.9} \approx 2865.058$$

then we will use the derivation

$$\Delta N \approx \frac{\partial N}{\partial d}\Delta d$$

since d is increasing from 10 to 10.4. We need to compute $\partial N/\partial d$, with L and R constant:

$$\begin{aligned}\frac{\partial N}{\partial d} &= \frac{\partial}{\partial d}\left(\frac{2200R}{Ld}\right)^{1.9}\\ &= \left(\frac{2200R}{L}\right)^{1.9}\frac{\partial}{\partial d}(d^{-1.9})\\ &= -1.9\left(\frac{2200R}{L}\right)^{1.9} d^{-2.9}\end{aligned}$$

we have first

$$\left.\frac{\partial N}{\partial d}\right|_{(L,R,d)=(0.4,0.12,10)} = -1.9\left(\frac{2200\cdot 0.12}{0.4}\right)^{1.9}(10)^{-2.9} \approx -544.361$$

Therefore we can conclude:

$$\Delta N \approx \frac{\partial N}{\partial d}\Delta d \approx (-544.361)(10.4-10) = -217.744$$

46. Estimate ΔN if $(L, R, d) = (0.5, 0.15, 8)$ and R is increased from 0.15 to 0.17.

SOLUTION From the example in the text we have

$$N = \left(\frac{2200R}{Ld}\right)^{1.9}$$

then we will use the derivation,

$$\Delta N \approx \frac{\partial N}{\partial R}\Delta R$$

since R is increasing from 0.15 to 0.17. We need to compute $\partial N/\partial R$, with L and d constant:

$$\begin{aligned}\frac{\partial N}{\partial R} &= \frac{\partial}{\partial R}\left(\frac{2200R}{Ld}\right)^{1.9}\\ &= \left(\frac{2200}{Ld}\right)^{1.9}\frac{\partial}{\partial R}(R^{1.9})\\ &= 1.9\left(\frac{2200}{Ld}\right)^{1.9} R^{0.9}\end{aligned}$$

We have first

$$\left.\frac{\partial N}{\partial R}\right|_{(L,R,d)=(0.5,0.15,8)} = 1.9\left(\frac{2200}{0.5\cdot 8}\right)^{1.9}(0.15)^{0.9} \approx 55452.974$$

Therefore we can conclude:

$$\Delta N \approx \frac{\partial N}{\partial R}\Delta R \approx (55452.974)(0.17-0.15) \approx 1109.059$$

47. The **heat index** I is a measure of how hot it feels when the relative humidity is H (as a percentage) and the actual air temperature is T (in degrees Fahrenheit). An approximate formula for the heat index that is valid for (T, H) near (90, 40) is

$$\begin{aligned}I(T, H) = 45.33 + 0.6845T + 5.758H - 0.00365T^2\\ - 0.1565HT + 0.001HT^2\end{aligned}$$

(a) Calculate I at $(T, H) = (95, 50)$.

(b) Which partial derivative tells us the increase in I per degree increase in T when $(T, H) = (95, 50)$. Calculate this partial derivative.

SOLUTION

(a) Let us compute I when $T = 95$ and $H = 50$:

$$\begin{aligned} I(95, 50) &= 45.33 + 0.6845(95) + 5.758(50) - 0.00365(95)^2 - 0.1565(50)(95) + 0.001(50)(95)^2 \\ &= 73.19125 \end{aligned}$$

(b) The partial derivative we are looking for here is $\partial I/\partial T$:

$$\frac{\partial I}{\partial T} = 0.6845 - 0.00730T - 0.1565H + 0.002HT$$

and evaluating we have:

$$\frac{\partial I}{\partial T}(95, 50) = 0.6845 - 0.00730(95) - 0.1565(50) + 0.002(50)(95) = 1.666$$

48. The **wind-chill temperature** W measures how cold people feel (based on the rate of heat loss from exposed skin) when the outside temperature is T°C (with $T \le 10$) and wind velocity is v m/s (with $v \ge 2$):

$$W = 13.1267 + 0.6215T - 13.947v^{0.16} + 0.486Tv^{0.16}$$

Calculate $\partial W/\partial v$ at $(T, v) = (-10, 15)$ and use this value to estimate ΔW if $\Delta v = 2$.

SOLUTION Computing the partial derivative we get:

$$\begin{aligned} \frac{\partial W}{\partial v} &= \frac{\partial}{\partial v}\left(13.1267 + 0.6215T - 13.947v^{0.16} + 0.486Tv^{0.16}\right) \\ &= -13.947(0.16)v^{-0.84} + 0.486(0.16)Tv^{-0.84} \end{aligned}$$

$$\frac{\partial W}{\partial v}(-10, 15) = -13.947(0.16)(15)^{-0.84} + 0.486(0.16)(-10)(15)^{-0.84} \approx -0.30940$$

Now using this information we would like to estimate ΔW if $\Delta v = 2$:

$$\Delta W = \frac{\partial W}{\partial v}\Delta v \approx -0.30940 \cdot 2 \approx -0.6188$$

49. The volume of a right-circular cone of radius r and height h is $V = \frac{\pi}{3}r^2h$. Suppose that $r = h = 12$ cm. What leads to a greater increase in V, a 1-cm increase in r or a 1-cm increase in h? Argue using partial derivatives.

SOLUTION We obtain the following derivatives:

$$\begin{aligned} \frac{\partial V}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{\pi}{3}r^2h\right) = \frac{\pi h}{3}\frac{\partial}{\partial r}r^2 = \frac{\pi h}{3} \cdot 2r = \frac{2\pi hr}{3} \\ \frac{\partial V}{\partial h} &= \frac{\partial}{\partial h}\left(\frac{\pi}{3}r^2h\right) = \frac{\pi}{3}r^2 \end{aligned}$$

An increase $\Delta r = 1$ cm in r leads to an increase of $\frac{\partial V}{\partial r}(12, 12) \cdot 1$ in the volume, and an increase $\Delta h = 1$ cm in h leads to an increase of $\frac{\partial V}{\partial h}(12, 12) \cdot 1$ in V. We compute these values, using the partials computed. This gives

$$\begin{aligned} \frac{\partial V}{\partial r}(12, 12) &= \left.\frac{2\pi hr}{3}\right|_{(12,12)} = \frac{2\pi \cdot 12 \cdot 12}{3} = 301.6 \\ \frac{\partial V}{\partial h}(12, 12) &= \frac{\pi}{3} \cdot 12^2 = 150.8 \end{aligned}$$

We conclude that an increase of 1 cm in r leads to a greater increase in V than an increase of 1 cm in h.

50. Use the linear approximation to estimate the percentage change in volume of a right-circular cone of radius $r = 40$ cm if the height is increased from 40 to 41 cm.

SOLUTION First, the volume of a right-circular cone is $V = \frac{1}{3}\pi r^2h$. We obtain the following partial derivative:

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Then an increase $\Delta h = 1$ cm in h leads to an increase of $\partial V/\partial h \cdot 1$ in V.

To compute the percent change in volume of the right-circular cone we consider:

$$\frac{\Delta V}{V} \approx \frac{\partial V/\partial h \cdot \Delta h}{V} = \frac{\frac{1}{3}\pi r^2 \Delta h}{\frac{1}{3}\pi r^2 h} = \frac{\Delta h}{h} = \frac{1}{40} = 0.025$$

Therefore, the percent change is about 2.5%.

51. Calculate $\partial W/\partial E$ and $\partial W/\partial T$, where $W = e^{-E/kT}$, where k is a constant.

SOLUTION We use the Chain Rule

$$\frac{d}{dE}e^u = e^u\frac{du}{dE} \quad \text{and} \quad \frac{d}{dT}e^u = e^u\frac{du}{dT}$$

with $u = -\frac{E}{kT}$, to obtain

$$\frac{\partial W}{\partial E} = e^{-E/kT}\frac{\partial}{\partial E}\left(-\frac{E}{kT}\right) = e^{-E/kT}\left(-\frac{1}{kT}\right) = -\frac{1}{kT}e^{-E/kT}$$
$$\frac{\partial W}{\partial T} = e^{-E/kT}\frac{\partial}{\partial T}\left(-\frac{E}{kT}\right) = e^{-E/kT}\cdot\left(-\frac{E}{k}\right)\frac{\partial}{\partial T}\left(\frac{1}{T}\right) = e^{-E/kT}\left(-\frac{E}{k}\right)\left(-\frac{1}{T^2}\right) = \frac{E}{kT^2}e^{-E/kT}$$

52. Calculate $\partial P/\partial T$ and $\partial P/\partial V$, where pressure P, volume V, and temperature T are related by the ideal gas law, $PV = nRT$ (R and n are constants).

SOLUTION We differentiate the two sides of the equation $PV = nRT$ with respect to V (treating T as a constant). Using the Product Rule we obtain

$$\frac{\partial}{\partial V}PV = V\frac{\partial P}{\partial V} + P\frac{\partial V}{\partial V} = V\frac{\partial P}{\partial V} + P; \quad \frac{\partial}{\partial V}nRT = 0$$

Hence,

$$V\frac{\partial P}{\partial V} + P = 0$$

We substitute $P = \frac{nRT}{V}$ and solve for $\frac{\partial P}{\partial V}$. This gives

$$V\frac{\partial P}{\partial V} + \frac{nRT}{V} = 0 \quad \Rightarrow \quad \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

We now differentiate $PV = nRT$ with respect to T, treating V as a constant:

$$\frac{\partial}{\partial T}PV = V\frac{\partial P}{\partial T}; \quad \frac{\partial}{\partial T}nRT = nR$$

Hence,

$$V\frac{\partial P}{\partial T} = nR \quad \Rightarrow \quad \frac{\partial P}{\partial T} = \frac{nR}{V}.$$

53. Use the contour map of $f(x, y)$ in Figure 9 to explain the following statements.

(a) f_y is larger at P than at Q, and f_x is smaller (more negative) at P than at Q.

(b) $f_x(x, y)$ is decreasing as a function of y; that is, for any fixed value $x = a$, $f_x(a, y)$ is decreasing in y.

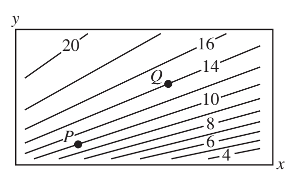


FIGURE 9 Contour interval 2.

SOLUTION

(a) A vertical segment through P meet more level curves than a vertical segment of the same size through Q, so f is increasing more rapidly in the y at P than at Q. Therefore, f_y are both larger at P than at Q.

Similarly, a horizontal segment through P meet more level curves at P than at Q, but f is *decreasing* in the positive x-direction, so f is decreasing more rapidly in the x-direction at P than at Q. Therefore, f_x is more negative at P than at Q.

(b) For any fixed value $x = a$, a horizontal segment meets fewer level curves as we move it vertically upward. This indicates that $f_x(a, y)$ in a decreasing function of y.

54. Estimate the partial derivatives at P of the function whose contour map is shown in Figure 10.

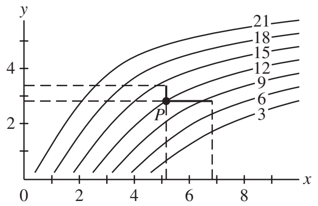
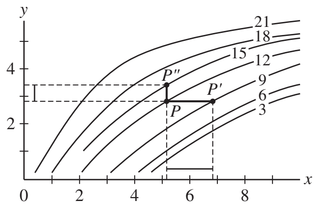


FIGURE 10

SOLUTION The contour interval is $m = 3$. To estimate the partial derivative $\frac{\partial f}{\partial x}$ at P, we estimate the change Δx between P and the point P' on the next level curve to the right, which is the length of the segment $\overline{PP'}$:

$$\Delta x \approx 2$$



The change in f between P and P' is the contour interval $\Delta f = -3$. Hence,

$$\left.\frac{\partial f}{\partial x}\right|_P \approx \frac{\Delta f}{\Delta x} = \frac{-3}{2} = -1.5$$

To estimate the partial derivative $\frac{\partial f}{\partial y}$ at P, we estimate the change Δy between P and the point P'' on the next level curve vertically above P:

$$\Delta y \approx 0.5$$

The change in f is $\Delta f = 3$ (since the level curve of P'' is to the left of the level curve of P). Hence,

$$\left.\frac{\partial f}{\partial y}\right|_P \approx \frac{\Delta f}{\Delta y} \approx \frac{3}{0.5} = 6.$$

55. Over most of the earth, a magnetic compass does not point to true (geographic) north; instead, it points at some angle east or west of true north. The angle D between magnetic north and true north is called the **magnetic declination**. Use Figure 11 to determine which of the following statements is true.

(a) $\left.\frac{\partial D}{\partial y}\right|_A > \left.\frac{\partial D}{\partial y}\right|_B$ **(b)** $\left.\frac{\partial D}{\partial x}\right|_C > 0$ **(c)** $\left.\frac{\partial D}{\partial y}\right|_C > 0$

Note that the horizontal axis increases from right to left because of the way longitude is measured.

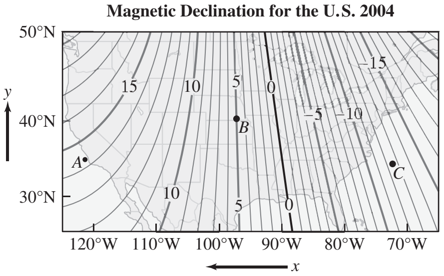


FIGURE 11 Contour interval 1°.

SOLUTION

(a) To estimate $\left.\frac{\partial D}{\partial y}\right|_A$ and $\left.\frac{\partial D}{\partial y}\right|_B$, we move vertically from A and B to the points on the next level curve in the direction of increasing y (upward). From A, we quickly come to a level curve corresponding to higher value of D; but from B, moving vertically, there is hardly any change as we move along the curve. The statement is thus true.

(b) The derivative $\frac{\partial D}{\partial x}\big|_C$ is estimated by $\frac{\Delta D}{\Delta x}$. Since x varies in the horizontal direction, we move horizontally from C to a point on the next level curve in the direction of increasing x (leftwards). Since the value of D on this level curve is greater than on the level curve of C, $\Delta D = 1$. Also $\Delta x > 0$, hence

$$\frac{\partial D}{\partial x}\bigg|_C \approx \frac{\Delta D}{\Delta x} = \frac{1}{\Delta x} > 0.$$

The statement is correct.

(c) Moving from C vertically upward (in the direction of increasing y), we come to a point on a level curve with a smaller value of D. Therefore, $\Delta D = -1$ and $\Delta y > 0$, so we obtain

$$\frac{\partial D}{\partial y}\bigg|_C \approx \frac{\Delta D}{\Delta y} = \frac{-1}{\Delta y} < 0$$

Hence, the statement is false.

56. Refer to Table 1.

(a) Estimate $\partial\rho/\partial T$ and $\partial\rho/\partial S$ at the points $(S, T) = (34, 2)$ and $(35, 10)$ by computing the average of left-hand and right-hand difference quotients.

(b) For fixed salinity $S = 33$, is ρ concave up or concave down as a function of T? *Hint:* Determine whether the quotients $\Delta\rho/\Delta T$ are increasing or decreasing. What can you conclude about the sign of $\partial^2\rho/\partial T^2$?

TABLE 1 **Seawater Density ρ as a Function of Temperature T and Salinity S**

T \ S	30	31	32	33	34	35	36
12	22.75	23.51	24.27	25.07	25.82	26.6	27.36
10	23.07	23.85	24.62	25.42	26.17	26.99	27.73
8	23.36	24.15	24.93	25.73	26.5	27.28	29.09
6	23.62	24.44	25.22	26	26.77	27.55	28.35
4	23.85	24.62	25.42	26.23	27	27.8	28.61
2	24	24.78	25.61	26.38	27.18	28.01	28.78
0	24.11	24.92	25.72	26.5	27.34	28.12	28.91

SOLUTION

(a) We estimate $\frac{\partial\rho}{\partial T}$ at the given points using the values in Table 1 and the following approximation:

$$\frac{\partial\rho}{\partial T}(34, 2) \approx \frac{\rho(34, 2+2) - \rho(34, 2)}{2} = \frac{\rho(34, 4) - \rho(34, 2)}{2} = \frac{27 - 27.18}{2} = -0.09$$

$$\frac{\partial\rho}{\partial T}(35, 10) \approx \frac{\rho(35, 10+2) - \rho(35, 10)}{2} = \frac{\rho(35, 12) - \rho(35, 10)}{2} = \frac{26.6 - 26.99}{2} = -0.195$$

Therefore, the average of the left-hand and right-hand difference quotients is:

$$\frac{1}{2}\left(\frac{\partial\rho}{\partial T}(34, 2) + \frac{\partial\rho}{\partial T}(35, 10)\right) \approx \frac{1}{2}(-0.09 - 0.195) = -0.1425$$

We estimate the partial derivative $\frac{\partial\rho}{\partial S}$ at the given points:

$$\frac{\partial\rho}{\partial S}(34, 2) \approx \frac{\rho(34+1, 2) - \rho(34, 2)}{1} = \frac{\rho(35, 2) - \rho(34, 2)}{1} = 28.01 - 27.18 = 0.83$$

$$\frac{\partial\rho}{\partial S}(35, 10) \approx \frac{\rho(35+1, 10) - \rho(35, 10)}{1} = \frac{\rho(36, 10) - \rho(35, 10)}{1} = 27.73 - 26.99 = 0.74$$

Therefore, the average of the left-hand and right-hand difference quotients is:

$$\frac{1}{2}\left(\frac{\partial\rho}{\partial S}(34, 2) + \frac{\partial\rho}{\partial S}(35, 10)\right) \approx \frac{1}{2}(0.85 + 0.74) = 0.795$$

(b) The function $\rho(33, T)$ is concave up (concave down) if $\frac{\partial\rho}{\partial T}(33, T)$ is an increasing (decreasing) function of T. We use Table 1 to estimate whether the function $\frac{\partial\rho}{\partial T}(33, T)$ is increasing or decreasing. We compute the following values:

$$\frac{\partial\rho}{\partial T}(33, 2) \approx \frac{\rho(33, 4) - \rho(33, 2)}{2} = \frac{26.23 - 26.38}{2} = -0.075$$

$$\frac{\partial \rho}{\partial T}(33,4) \approx \frac{\rho(33,6)-\rho(33,4)}{2} = \frac{26-26.23}{2} = -0.115$$
$$\frac{\partial \rho}{\partial T}(33,6) \approx \frac{\rho(33,8)-\rho(33,6)}{2} = \frac{25.73-26}{2} = -0.135$$
$$\frac{\partial \rho}{\partial T}(33,8) \approx \frac{\rho(33,10)-\rho(33,8)}{2} = \frac{25.42-25.73}{2} = -0.155$$
$$\frac{\partial \rho}{\partial T}(33,10) \approx \frac{\rho(33,12)-\rho(33,10)}{2} = \frac{25.07-25.42}{2} = -0.175$$

These values indicate that $\frac{\partial \rho}{\partial T}(33,T)$ is a decreasing function of T, which means that the second derivative is negative, i.e., $\frac{\partial^2 \rho}{\partial T^2}(33,T) < 0$ and the graph of $\rho(33,T)$ is concave down.

In Exercises 57–62, compute the derivatives indicated.

57. $f(x,y) = 3x^2y - 6xy^4$, $\quad \dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$

SOLUTION We first compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial x} = 6xy - 6y^4; \quad \frac{\partial f}{\partial y} = 3x^2 - 6x\cdot 4y^3 = 3x^2 - 24xy^3$$

We now differentiate $\frac{\partial f}{\partial x}$ with respect to x and $\frac{\partial f}{\partial y}$ with respect to y. We get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = 6y; \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y = -24x\cdot 3y^2 = -72xy^2.$$

58. $g(x,y) = \dfrac{xy}{x-y}$, $\quad \dfrac{\partial^2 g}{\partial x\,\partial y}$

SOLUTION By definition we have

$$\frac{\partial^2 g}{\partial x \partial y} = g_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)$$

Thus, we must find $\frac{\partial g}{\partial y}$:

$$\frac{\partial g}{\partial y} = x\frac{\partial}{\partial y}\left(\frac{y}{x-y}\right) = x\frac{1\cdot(x-y) - y\cdot(-1)}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

Differentiating $\frac{\partial g}{\partial y}$ with respect to x, using the Quotient Rule, we obtain

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}\frac{x^2}{(x-y)^2} = \frac{2x(x-y)^2 - x^2\cdot 2(x-y)}{(x-y)^4} = -\frac{2xy}{(x-y)^3}$$

59. $h(u,v) = \dfrac{u}{u+4v}$, $\quad h_{vv}(u,v)$

SOLUTION We first note

$$\frac{\partial h}{\partial v} = \frac{-4u}{(u+4v)^2}$$

so thus

$$\frac{\partial h^2}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{-4u}{(u+4v)^2}\right) = \frac{32u}{(u+4v)^3}$$

60. $h(x,y) = \ln(x^3+y^3)$, $\quad h_{xy}(x,y)$

SOLUTION We first note that

$$\frac{\partial h}{\partial y} = \frac{3y^2}{x^3+y^3}$$

so thus

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{3y^2}{x^3 + y^3}\right) = \frac{-9x^2y^2}{(x^3 + y^3)^2}$$

61. $f(x, y) = x \ln(y^2)$, $\quad f_{yy}(2, 3)$

SOLUTION We find f_y using the Chain Rule:

$$f_y = \frac{\partial}{\partial y}(x \ln y^2) = x \frac{\partial}{\partial y} \ln y^2 = x \frac{1}{y^2} \cdot 2y = \frac{2x}{y}$$

We now differentiate f_y with respect to y, obtaining

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y = 2x \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{-2x}{y^2}.$$

The derivative at $(2, 3)$ is thus

$$f_{yy}(2, 3) = \frac{-2 \cdot 2}{3^2} = -\frac{4}{9}.$$

62. $g(x, y) = xe^{-xy}$, $\quad g_{xy}(-3, 2)$

SOLUTION We first compute:

$$\frac{\partial g}{\partial x} = x \cdot e^{-xy} \cdot (-y) + e^{-xy} = e^{-xy}(1 - xy)$$

so thus:

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}(e^{-xy}(1 - xy)) = e^{-xy}(-x) + (1 - xy)e^{-xy} \cdot (-x) = -xe^{-xy}(2 - xy)$$

and

$$g_{xy}(-3, 2) = 3e^6(2 + 6) = 24e^6$$

63. Compute f_{xyxzy} for

$$f(x, y, z) = y \sin(xz) \sin(x + z) + (x + z^2) \tan y + x \tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)$$

Hint: Use a well-chosen order of differentiation on each term.

SOLUTION At the points where the derivatives are continuous, the partial derivative f_{xyxzy} may be performed in any order. To simplify the computation we first consider $f(x, y, z)$ as the sum of the following terms:

$$F(x, y, z) = y \sin(xz) \sin(x + z), \quad G(x, y, z) = (x + z^2) \tan y, \quad H(x, y, z) = x \tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)$$

so that

$$f(x, y, z) = F(x, y, z) + G(x, y, z) + H(x, y, z)$$

We can differentiate each in any order. First, let us work with $F(x, y, z) = y \sin(xz) \sin(x + z)$:

$$F_y(x, y, z) = \frac{\partial}{\partial y}(y \sin(xz) \sin(x + z)) = \sin(xz) \sin(x + z)$$

then

$$F_{yy}(x, y, z) = \frac{\partial}{\partial y}(F_y(x, y, z)) = 0$$

hence,

$$F_{yyxxz}(x, y, z) = 0$$

Next, let us work with $G(x, y, z) = (x + z^2) \tan y$:

$$G_x(x, y, z) = \frac{\partial}{\partial x}((x + z^2) \tan y) = \tan y$$

then

$$G_{xx}(x, y, z) = \frac{\partial}{\partial x}(G_x(x, y, z)) = 0$$

Hence

$$G_{xxyyz}(x, y, z) = 0$$

Finally, let us work with $H(x, y, z) = x \tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)$

$$H_x(x, y, z) = \frac{\partial}{\partial x}\left(x \tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)\right) = \tan\left(\frac{z + z^{-1}}{y - y^{-1}}\right)$$

then

$$H_{xx}(x, y, z) = \frac{\partial}{\partial x}(H_x(x, y, z)) = 0$$

hence,

$$H_{xxyyz}(x, y, z) = 0$$

Therefore, we can conclude that $f_{xyxzy}(x, y, z) = 0 + 0 + 0 = 0$.

64. Let

$$f(x, y, u, v) = \frac{x^2 + e^y v}{3y^2 + \ln(2 + u^2)}$$

What is the fastest way to show that $f_{uvxyvu}(x, y, u, v) = 0$ for all (x, y, u, v)?

SOLUTION We first differentiate with respect to v, obtaining

$$\begin{aligned} f_v(x, y, u, v) &= \frac{\partial}{\partial v}\left(\frac{x^2}{3y^2 + \ln(2 + u^2)}\right) + \frac{\partial}{\partial v}\left(\frac{e^y}{3y^2 + \ln(2 + u^2)} v\right) \\ &= 0 + \frac{e^y}{3y^2 + \ln(2 + u^2)} = \frac{e^y}{3y^2 + \ln(2 + u^2)} \end{aligned}$$

We now differentiate f_v with respect to x. Since f_v does not depend on x, we have

$$f_{vx}(x, y, u, v) = 0$$

Hence also,

$$f_{uvxyvu}(x, y, u, v) = \frac{\partial}{\partial u}\frac{\partial}{\partial y}\frac{\partial}{\partial v}\frac{\partial}{\partial u}(0) = 0$$

In Exercises 65–72, compute the derivative indicated.

65. $f(u, v) = \cos(u + v^2)$, $\quad f_{uuv}$

SOLUTION Using the Chain Rule, we have

$$\begin{aligned} f_u &= \frac{\partial}{\partial u}\cos(u + v^2) = -\sin(u + v^2) \cdot \frac{\partial}{\partial u}(u + v^2) = -\sin(u + v^2) \\ f_{uu} &= \frac{\partial}{\partial u}\big(-\sin(u + v^2)\big) = -\cos(u + v^2) \\ f_{uuv} &= \frac{\partial}{\partial v}\big(-\cos(u + v^2)\big) = \sin(u + v^2) \cdot \frac{\partial}{\partial v}(u + v^2) = 2v\sin(u + v^2) \end{aligned}$$

66. $g(x, y, z) = x^4 y^5 z^6$, $\quad g_{xxyz}$

SOLUTION For $g(x, y, z) = x^4 y^5 z^6$, we have

$$\begin{aligned} g_x &= y^5 z^6 \frac{\partial}{\partial x} x^4 = y^5 z^6 \cdot 4x^3 = 4x^3 y^5 z^6 \\ g_{xx} &= 4y^5 z^6 \frac{\partial}{\partial x} x^3 = 4y^5 z^6 \cdot 3x^2 = 12x^2 y^5 z^6 \end{aligned}$$

$$g_{xxy} = 12x^2z^6\frac{\partial}{\partial y}(y^5) = 12x^2z^6 \cdot 5y^4 = 60x^2y^4z^6$$

$$g_{xxyz} = 60x^2y^4\frac{\partial}{\partial z}z^6 = 60x^2y^4 \cdot 6x^5 = 360x^2y^4z^5$$

67. $F(r, s, t) = r(s^2 + t^2), \quad F_{rst}$

SOLUTION For $F(r, s, t) = r(s^2 + t^2)$, we have

$$F_r = s^2 + t^2$$
$$F_{rs} = 2s$$
$$F_{rst} = 0$$

68. $u(x, t) = t^{-1/2}e^{-(x^2/4t)}, \quad u_{xx}$

SOLUTION Using the Chain Rule we obtain

$$u_x = t^{-1/2}\frac{\partial}{\partial x}(e^{-x^2/4t}) = t^{-1/2} \cdot e^{-x^2/4t}\frac{\partial}{\partial x}\left(-\frac{x^2}{4t}\right) = t^{-1/2} \cdot e^{-x^2/4t} \cdot \frac{-2x}{4t} = -\frac{1}{2}xt^{-3/2}e^{-x^2/4t}$$

We now differentiate u_x with respect to x, using the Product Rule and the Chain Rule:

$$u_{xx} = -\frac{1}{2}t^{-3/2}\frac{\partial}{\partial x}\left(xe^{-x^2/4t}\right) = -\frac{1}{2}t^{-3/2}\left(1 \cdot e^{-x^2/4t} + x \cdot e^{-x^2/4t} \cdot \frac{-2x}{4t}\right)$$
$$= -\frac{1}{2}t^{-3/2}\left(e^{-x^2/4t} - \frac{x^2}{2t}e^{-x^2/4t}\right) = -\frac{1}{2}t^{-3/2}e^{-x^2/4t}\left(1 - \frac{x^2}{2t}\right)$$

69. $F(\theta, u, v) = \sinh(uv + \theta^2), \quad F_{uu\theta}$

SOLUTION We can compute:

$$F_u = v \cdot \cosh(uv + \theta^2)$$
$$F_{uu} = v^2 \cdot \sinh(uv + \theta^2)$$
$$F_{uu\theta} = 2\theta v^2 \cosh(uv + \theta^2)$$

70. $R(u, v, w) = \dfrac{u}{v + w}, \quad R_{uvw}$

SOLUTION We differentiate R with respect to u:

$$R_u = \frac{\partial}{\partial u}\left(\frac{u}{v + w}\right) = \frac{1}{v + w}$$

We now differentiate R_u with respect to v, using the Chain Rule:

$$R_{uv} = \frac{\partial}{\partial v}\frac{1}{v + w} = -\frac{1}{(v + w)^2}$$

Finally we differentiate R_{uv} with respect to w:

$$R_{uvw} = \frac{\partial}{\partial w}\left(-(v + w)^{-2}\right) = 2(v + w)^{-3} = \frac{2}{(v + w)^3}.$$

71. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad g_{xyz}$

SOLUTION Differentiating with respect to x, using the Chain Rule, we get

$$g_x = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

We now differentiate g_x with respect to y, using the Chain Rule. This gives

$$g_{xy} = x\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1/2} = x \cdot \left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2} \cdot 2y = \frac{-xy}{(x^2 + y^2 + z^2)^{3/2}}$$

Finally, we differentiate g_{xy} with respect to z, obtaining

$$g_{xyz} = -xy\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-3/2} = -xy\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2z = \frac{3xyz}{(x^2+y^2+z^2)^{5/2}}$$

72. $u(x,t) = \operatorname{sech}^2(x-t), \quad u_{xxx}$

SOLUTION Using the Chain Rule we have

$$u_x = \frac{\partial}{\partial x}\operatorname{sech}^2(x-t) = 2\operatorname{sech}(x-t)\cdot\big(-\operatorname{sech}(x-t)\tanh(x-t)\big)\cdot\frac{\partial}{\partial x}(x-t) = -2\operatorname{sech}^2(x-t)\tanh(x-t)$$

We now use the Product Rule and the Chain Rule to differentiate u_x with respect to x:

$$\begin{aligned} u_{xx} &= -2\big[2\operatorname{sech}(x-t)\cdot\big(-\operatorname{sech}(x-t)\tanh(x-t)\big)\tanh(x-t) + \operatorname{sech}^2(x-t)\cdot\operatorname{sech}^2(x-t)\big] \\ &= 4\operatorname{sech}^2(x-t)\tanh^2(x-t) - 2\operatorname{sech}^4(x-t) = 2\operatorname{sech}^2(x-t)\big(2\tanh^2(x-t)-\operatorname{sech}^2(x-t)\big) \end{aligned}$$

We find u_{xxx}, using the Product Rule and the Chain Rule:

$$\begin{aligned} u_{xxx} &= 4\operatorname{sech}(x-t)\big(-\operatorname{sech}(x-t)\tanh(x-t)\big)\big(2\tanh^2(x-t)-\operatorname{sech}^2(x-t)\big) \\ &\quad + 2\operatorname{sech}^2(x-t)\big[4\tanh(x-t)\cdot\operatorname{sech}^2(x-t) - 2\operatorname{sech}(x-t)\big(-\operatorname{sech}(x-t)\tanh(x-t)\big)\big] \\ &= -8\operatorname{sech}^2(x-t)\tanh^3(x-t) + 4\operatorname{sech}^4(x-t)\tanh(x-t) + 12\operatorname{sech}^4(x-t)\tanh(x-t) \\ &= 16\operatorname{sech}^4(x-t)\tanh(x-t) - 8\operatorname{sech}^2(x-t)\tanh^3(x-t) \end{aligned}$$

73. Find a function such that $\dfrac{\partial f}{\partial x} = 2xy$ and $\dfrac{\partial f}{\partial y} = x^2$.

SOLUTION The function $f(x,y) = x^2y$ satisfies $\frac{\partial f}{\partial y} = x^2$ and $\frac{\partial f}{\partial x} = 2xy$.

74. Prove that there does not exist any function $f(x,y)$ such that $\dfrac{\partial f}{\partial x} = xy$ and $\dfrac{\partial f}{\partial y} = x^2$. *Hint:* Show that f cannot satisfy Clairaut's Theorem.

SOLUTION Suppose that there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = xy$ and $\frac{\partial f}{\partial y} = x^2$. Hence,

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}xy = x \\ f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}x^2 = 2x \end{aligned}$$

The mixed partials f_{xy} and f_{yx} are continuous everywhere, but $f_{xy} \neq f_{yx}$ for $x \neq 0$. This contradicts Clairaut's Theorem on Equality of Mixed Partials. We conclude that there does not exist any function $f(x,y)$ with the given partials.

75. Assume that f_{xy} and f_{yx} are continuous and that f_{yxx} exists. Show that f_{xyx} also exists and that $f_{yxx} = f_{xyx}$.

SOLUTION Since f_{xy} and f_{yx} are continuous, Clairaut's Theorem implies that

$$f_{xy} = f_{yx} \tag{1}$$

We are given that f_{yxx} exists. Using (1) we get

$$f_{yxx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}f_y = \frac{\partial}{\partial x}f_{yx} = \frac{\partial}{\partial x}f_{xy} = f_{xyx}$$

Therefore, f_{xyx} also exists and $f_{yxx} = f_{xyx}$.

76. Show that $u(x,t) = \sin(nx)\,e^{-n^2t}$ satisfies the heat equation for any constant n:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$ $\boxed{3}$

SOLUTION We compute $\frac{\partial u}{\partial t}$ using the Chain Rule:

$$\frac{\partial u}{\partial t} = \sin(nx)\frac{\partial}{\partial t}e^{-n^2t} = \sin(nx)e^{-n^2t}\frac{\partial}{\partial t}(-n^2t) = -n^2\sin(nx)e^{-n^2t}$$

We now find u_x:

$$u_x = e^{-n^2t}\frac{\partial}{\partial x}\sin(nx) = e^{-n^2t}\cos(nx)\cdot n = n\cdot\cos(nx)e^{-n^2t}$$

Differentiating u_x with respect to x gives

$$u_{xx} = ne^{-n^2t}\frac{\partial}{\partial x}\cos(nx) = ne^{-n^2t}\left(-\sin(nx)\frac{\partial}{\partial x}(nx)\right) = ne^{-n^2t}\big(-\sin(nx)\big)\cdot n = -n^2e^{-n^2t}\sin(nx)$$

We see that $u_t = u_{xx}$, therefore u satisfies the heat equation.

77. Find all values of A and B such that $f(x,t) = e^{Ax+Bt}$ satisfies Eq. (3).

SOLUTION We compute the following partials, using the Chain Rule:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(e^{Ax+Bt}) = e^{Ax+Bt}\frac{\partial}{\partial t}(Ax+Bt) = Be^{Ax+Bt}$$
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{Ax+Bt}) = e^{Ax+Bt}\frac{\partial}{\partial x}(Ax+Bt) = Ae^{Ax+Bt}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(Ae^{Ax+Bt}) = A\frac{\partial}{\partial x}(e^{Ax+Bt}) = Ae^{Ax+Bt}\frac{\partial}{\partial x}(Ax+Bt) = A^2e^{Ax+Bt}$$

Substituting these partials in the differential equation (3), we get

$$Be^{Ax+Bt} = A^2e^{Ax+Bt}$$

We divide by the nonzero e^{Ax+Bt} to obtain

$$B = A^2$$

We conclude that $f(x,t) = e^{Ax+Bt}$ satisfies equation (5) if and only if $B = A^2$, where A is arbitrary.

78. The function

$$f(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$$

describes the temperature profile along a metal rod at time $t > 0$ when a burst of heat is applied at the origin (see Example 11). A small bug sitting on the rod at distance x from the origin feels the temperature rise and fall as heat diffuses through the bar. Show that the bug feels the maximum temperature at time $t = \frac{1}{2}x^2$.

SOLUTION From the example in the text we see that:

$$\frac{\partial f}{\partial t} = -\frac{1}{4\sqrt{\pi}}t^{-3/2}e^{-x^2/4t} + \frac{1}{8\sqrt{\pi}}x^2t^{-5/2}e^{-x^2/4t}$$

We take this expression, in order to find the maximum, and set it equal to 0 and solve for t:

$$-\frac{1}{4\sqrt{\pi}}t^{-3/2}e^{-x^2/4t} + \frac{1}{8\sqrt{\pi}}x^2t^{-5/2}e^{-x^2/4t} = 0$$
$$e^{-x^2/4t}(-2t^{-3/2} + x^2t^{-5/2}) = 0$$
$$t^{-5/2}e^{-x^2/4t}(-2t + x^2) = 0$$

Then since the exponential factor is never equal to 0 and the $t^{-5/2}$ is not either, we only consider when

$$-2t + x^2 = 0 \quad\Rightarrow\quad t = \frac{1}{2}x^2$$

Since we are told that the bug experiences the rise and then the fall of the temperature, we are assured that $t = 1/2x^2$ is the point in time when the bug experiences the maximum temperature.

In Exercises 79–82, the ***Laplace operator*** Δ *is defined by* $\Delta f = f_{xx} + f_{yy}$. *A function* $u(x,y)$ *satisfying the Laplace equation* $\Delta u = 0$ *is called* ***harmonic***.

79. Show that the following functions are harmonic:

(a) $u(x,y) = x$

(b) $u(x,y) = e^x\cos y$

(c) $u(x,y) = \tan^{-1}\dfrac{y}{x}$

(d) $u(x,y) = \ln(x^2+y^2)$

SOLUTION

(a) We compute u_{xx} and u_{yy} for $u(x, y) = x$:

$$u_x = \frac{\partial}{\partial x}(x) = 1; \quad u_{xx} = \frac{\partial}{\partial x}(1) = 0$$

$$u_y = \frac{\partial}{\partial y}(x) = 0; \quad u_{yy} = \frac{\partial}{\partial y}(0) = 0$$

Since $u_{xx} + u_{yy} = 0$, u is harmonic.

(b) We compute the partial derivatives of $u(x, y) = e^x \cos y$:

$$u_x = \frac{\partial}{\partial x}\left(e^x \cos y\right) = \cos y \frac{\partial}{\partial x} e^x = (\cos y)e^x$$

$$u_y = \frac{\partial}{\partial y}\left(e^x \cos y\right) = e^x \frac{\partial}{\partial y} \cos y = -e^x \sin y$$

$$u_{xx} = \frac{\partial}{\partial x}\left((\cos y)e^x\right) = \cos y \frac{\partial}{\partial x} e^x = (\cos y)e^x$$

$$u_{yy} = \frac{\partial}{\partial y}\left(-e^x \sin y\right) = -e^x \frac{\partial}{\partial y} \sin y = -e^x \cos y$$

Thus,

$$u_{xx} + u_{yy} = (\cos y)e^x - e^x \cos y = 0$$

Hence $u(x, y) = e^x \cos y$ is harmonic.

(c) We compute the partial derivatives of $u(x, y) = \tan^{-1} \frac{y}{x}$ using the Chain Rule and the formula

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}$$

We have

$$u_x = \frac{\partial}{\partial x} \tan^{-1} \frac{y}{x} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial x} \frac{y}{x} = \frac{1}{1+(y/x)^2} \left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$u_y = \frac{\partial}{\partial y} \tan^{-1} \frac{y}{x} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial y} \frac{y}{x} = \frac{1}{1+(y/x)^2} \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{\partial}{\partial x} \left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{\partial}{\partial y} \frac{x}{x^2+y^2} = -\frac{2xy}{(x^2+y^2)^2}$$

Therefore $u_{xx} + u_{xx} = 0$. This shows that $u(x, y) = \tan^{-1} \frac{y}{x}$ is harmonic.

(d) We compute the partial derivatives of $u(x, y) = \ln\left(x^2 + y^2\right)$ using the Chain Rule:

$$u_x = \frac{\partial}{\partial x} \ln(x^2+y^2) = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$u_y = \frac{\partial}{\partial y} \ln(x^2+y^2) = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

We now find u_{xx} and u_{yy} using the Quotient Rule:

$$u_{xx} = \frac{\partial}{\partial x} \frac{2x}{x^2+y^2} = \frac{2(x^2+y^2) - 2x \cdot 2x}{\left(x^2+y^2\right)^2} = \frac{2(y^2-x^2)}{\left(x^2+y^2\right)^2}$$

$$u_{yy} = \frac{\partial}{\partial y} \frac{2y}{x^2+y^2} = \frac{2(x^2+y^2) - 2y \cdot 2y}{\left(x^2+y^2\right)^2} = \frac{2(x^2-y^2)}{\left(x^2+y^2\right)^2}$$

Thus,

$$u_{xx} + u_{yy} = \frac{2(y^2-x^2)}{\left(x^2+y^2\right)^2} + \frac{2(x^2-y^2)}{\left(x^2+y^2\right)^2} = 0.$$

Therefore, $u(x, y) = \ln(x^2 + y^2)$ is harmonic.

80. Find all harmonic polynomials $u(x, y)$ of degree three, that is, $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

SOLUTION We compute the first-order partials u_x and u_y and the second-order partials u_{xx} and u_{yy} of the given polynomial $u(x, y)$. This gives

$$\begin{aligned} u_x &= 3ax^2 + 2bxy + cy^2 \\ u_y &= bx^2 + 2cxy + 3dy^2 \\ u_{xx} &= 6ax + 2by \\ u_{yy} &= 2cx + 6dy \end{aligned}$$

The polynomial is harmonic if $u_{xx} + u_{yy} = 0$, that is, if for all x and y

$$6ax + 2by + 2cx + 6dy = 0$$

This equality holds for all x and y if and only if the coefficients of x and y are both zero. That is, $6a + 2c = 0$ (so $c = -3a$) and $2b + 6d = 0$ (so $b = -3d$). We conclude that the harmonic polynomials in the given form are

$$u(x, y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$$

81. Show that if $u(x, y)$ is harmonic, then the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are harmonic.

SOLUTION We assume that the second-order partials are continuous, hence the partial differentiation may be performed in any order. By the given data, we have

$$u_{xx} + u_{yy} = 0 \tag{1}$$

We must show that

$$(u_x)_{xx} + (u_x)_{yy} = 0 \quad \text{and} \quad (u_y)_{xx} + (u_y)_{yy} = 0$$

We differentiate (1) with respect to x, obtaining

$$0 = (u_{xx})_x + (u_{yy})_x = u_{xxx} + u_{xyy} = (u_x)_{xx} + (u_x)_{yy} \tag{2}$$

We differentiate (1) with respect to y:

$$0 = (u_{xx})_y + (u_{yy})_y = u_{xxy} + u_{yyy} = u_{yxx} + u_{yyy} = (u_y)_{xx} + (u_y)_{yy} \tag{3}$$

Equalities (2) and (3) prove that u_x and u_y are harmonic.

82. Find all constants a, b such that $u(x, y) = \cos(ax)e^{by}$ is harmonic.

SOLUTION To determine if the functions $\cos(ax)e^{by}$ are harmonic, we compute the following derivatives:

$$\begin{aligned} (\cos ax)' = -a \sin ax \quad &\Rightarrow \quad (\cos ax)'' = -a^2 \cos ax \\ (e^{by})' = be^{by} \quad &\Rightarrow \quad (e^{by})'' = b^2 e^{by} = a^2 e^{by} \end{aligned}$$

Thus, we can conclude

$$\begin{aligned} u_{xx} &= \frac{\partial^2}{\partial x^2} \cos(ax)e^{by} = -a^2 \cos(ax)e^{by} = -a^2 u \\ u_{yy} &= \frac{\partial^2}{\partial y^2} \cos(ax)e^{by} = b^2 \cos(ax)e^{by} = b^2 u \end{aligned}$$

Thus, $u_{xx} + u_{yy} = (b^2 - a^2)u$, which equals 0 if and only if $a^2 = b^2$.

83. Show that $u(x, t) = \operatorname{sech}^2(x - t)$ satisfies the **Korteweg–deVries equation** (which arises in the study of water waves):

$$4u_t + u_{xxx} + 12uu_x = 0$$

SOLUTION In Exercise 72 we found the following derivatives:

$$\begin{aligned} u_x &= -2 \operatorname{sech}^2(x - t) \tanh(x - t) \\ u_{xxx} &= 16 \operatorname{sech}^4(x - t) \tanh(x - t) - 8 \operatorname{sech}^2(x - t) \tanh^3(x - t) \end{aligned}$$

Hence,

$$4u_t + u_{xxx} + 12uu_x = 8 \operatorname{sech}^2(x - t) \tanh(x - t) + 16 \operatorname{sech}^4(x - t) \tanh(x - t)$$

$$
\begin{aligned}
&- 8\operatorname{sech}^2(x-t)\tanh^3(x-t) - 24\operatorname{sech}^4(x-t)\tanh(x-t)\\
&= 8\operatorname{sech}^2(x-t)\left\{\tanh(x-t) - \tanh^3(x-t)\right\} - 8\operatorname{sech}^4(x-t)\tanh(x-t)\\
&= 8\operatorname{sech}^2(x-t)\tanh(x-t)\left\{1 - \tanh^2(x-t)\right\} - 8\operatorname{sech}^4(x-t)\tanh(x-t)\\
&= 8\operatorname{sech}^2(x-t)\tanh(x-t)\left\{\operatorname{sech}^2(x-t)\right\} - 8\operatorname{sech}^4(x-t)\tanh(x-t)\\
&= 0
\end{aligned}
$$

Further Insights and Challenges

84. Assumptions Matter This exercise shows that the hypotheses of Clairaut's Theorem are needed. Let

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) Verify for $(x, y) \neq (0, 0)$:

$$f_x(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

(b) Use the limit definition of the partial derivative to show that $f_x(0, 0) = f_y(0, 0) = 0$ and that $f_{yx}(0, 0)$ and $f_{xy}(0, 0)$ both exist but are not equal.

(c) Show that for $(x, y) \neq (0, 0)$:

$$f_{xy}(x, y) = f_{yx}(x, y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}$$

Show that f_{xy} is not continuous at $(0, 0)$. *Hint:* Show that $\lim_{h\to 0} f_{xy}(h, 0) \neq \lim_{h\to 0} f_{xy}(0, h)$.

(d) Explain why the result of part (b) does not contradict Clairaut's Theorem.

SOLUTION

(a) These are the partials for $(x, y) \neq (0, 0)$:

$$f_x(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

(b) Using the limit definition of the partial derivatives at the point $(0, 0)$ we have

$$f_x(0, 0) = \lim_{h\to 0}\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{h\cdot 0\frac{h^2-0^2}{h^2+0^2} - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k\to 0}\frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0}\frac{0\cdot k\frac{0^2-k^2}{0^2+k^2} - 0}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

We now use the derivatives in part (a) and the limit definition of the partial derivatives to compute $f_{yx}(0, 0)$ and $f_{xy}(0, 0)$. By the formulas in part (a), we have

$$f_y(0, 0) = 0, \quad f_y(h, 0) = \frac{h(h^4 - 0 - 0)}{(h^2 + 0)^2} = h$$

$$f_x(0, 0) = 0, \quad f_x(0, k) = \frac{k(0 + 0 - k^4)}{(0^2 + k^2)^2} = -k$$

Thus,

$$f_{yx}(0, 0) = \frac{\partial}{\partial x}f_y\bigg|_{(0,0)} = \lim_{h\to 0}\frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h\to 0}\frac{h - 0}{h} = \lim_{h\to 0} 1 = 1$$

$$f_{xy}(0,0) = \frac{\partial}{\partial y} f_x \bigg|_{(0,0)} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = \lim_{k\to 0}(-1) = -1$$

We see that the mixed partials at the point $(0,0)$ exist but are not equal.

(c) We verify that for $(x,y) \neq (0,0)$ the following derivatives hold:

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

To show that f_{xy} is not continuous at $(0,0)$, we show that the limit $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist. We compute the limit as (x,y) approaches the origin along the x-axis. Along this axis, $y = 0$; hence,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } x\text{-axis}}} f_{xy}(x,y) = \lim_{h\to 0} f_{xy}(h,0) = \lim_{h\to 0} \frac{h^6 + 9h^4\cdot 0 - 9h^2\cdot 0 - 0}{(0+h^2)^3} = \lim_{h\to 0} 1 = 1$$

We compute the limit as (x,y) approaches the origin along the y-axis. Along this axis, $x = 0$, hence,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } y\text{-axis}}} f_{xy}(x,y) = \lim_{h\to 0} f_{xy}(0,h) = \lim_{h\to 0} \frac{0+0+0-h^6}{(0+h^2)^3} = \lim_{h\to 0}(-1) = -1$$

Since the limits are not equal $f(x,y)$ does not approach one value as $(x,y) \to (0,0)$, hence the limit $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist, and $f_{xy}(x,y)$ is not continuous at the origin.

(d) The result of part (b) does not contradict Clairaut's Theorem since f_{xy} is not continuous at the origin. The continuity of the mixed derivative is essential in Clairaut's Theorem.

14.4 Differentiability and Tangent Planes (LT Section 15.4)

Preliminary Questions

1. How is the linearization of $f(x,y)$ at (a,b) defined?

SOLUTION The linearization of $f(x,y)$ at (a,b) is the linear function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This function is the equation of the tangent plane to the surface $z = f(x,y)$ at $\big(a,b,f(a,b)\big)$.

2. Define local linearity for functions of two variables.

SOLUTION $f(x,y)$ is locally linear at (a,b) if

$$f(x,y) - L(x,y) = \epsilon(x,y)\sqrt{(x-a)^2 + (y-b)^2}$$

for all (x,y) in an open disk D containing (a,b), where $\epsilon(x,y)$ satisfies $\lim_{(x,y)\to(a,b)} \epsilon(x,y) = 0$.

In Exercises 3–5, assume that

$$f(2,3) = 8, \qquad f_x(2,3) = 5, \qquad f_y(2,3) = 7$$

3. Which of (a)–(b) is the linearization of f at $(2,3)$?

(a) $L(x,y) = 8 + 5x + 7y$

(b) $L(x,y) = 8 + 5(x-2) + 7(y-3)$

SOLUTION The linearization of f at $(2,3)$ is the following linear function:

$$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$$

That is,

$$L(x,y) = 8 + 5(x-2) + 7(y-3) = -23 + 5x + 7y$$

The function in (b) is the correct answer.

4. Estimate $f(2, 3.1)$.

SOLUTION We use the linear approximation

$$f(a+h, b+k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

We let $(a, b) = (2, 3)$, $h = 0$, $k = 3.1 - 3 = 0.1$. Then,

$$f(2, 3.1) \approx f(2, 3) + f_x(2, 3) \cdot 0 + f_y(2, 3) \cdot 0.1 = 8 + 0 + 7 \cdot 0.1 = 8.7$$

We get the estimation $f(2, 3.1) \approx 8.7$.

5. Estimate Δf at $(2, 3)$ if $\Delta x = -0.3$ and $\Delta y = 0.2$.

SOLUTION The change in f can be estimated by the linear approximation as follows:

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$
$$\Delta f \approx f_x(2, 3) \cdot (-0.3) + f_y(2, 3) \cdot 0.2$$

or

$$\Delta f \approx 5 \cdot (-0.3) + 7 \cdot 0.2 = -0.1$$

The estimated change is $\Delta f \approx -0.1$.

6. Which theorem allows us to conclude that $f(x, y) = x^3y^8$ is differentiable?

SOLUTION The function $f(x, y) = x^3y^8$ is a polynomial, hence $f_x(x, y)$ and $f_y(x, y)$ exist and are continuous. Therefore the Criterion for Differentiability implies that f is differentiable everywhere.

Exercises

1. Use Eq. (2) to find an equation of the tangent plane to the graph of $f(x, y) = 2x^2 - 4xy^2$ at $(-1, 2)$.

SOLUTION The equation of the tangent plane at the point $(-1, 2, 18)$ is

$$z = f(-1, 2) + f_x(-1, 2)(x+1) + f_y(-1, 2)(y-2) \tag{1}$$

We compute the function and its partial derivatives at the point $(-1, 2)$:

$$\begin{aligned} f(x, y) &= 2x^2 - 4xy^2 & & f(-1, 2) = 18 \\ f_x(x, y) &= 4x - 4y^2 & \Rightarrow \quad & f_x(-1, 2) = -20 \\ f_y(x, y) &= -8xy & & f_y(-1, 2) = 16 \end{aligned}$$

Substituting in (1) we obtain the following equation of the tangent plane:

$$z = 18 - 20(x+1) + 16(y-2) = -34 - 20x + 16y$$

That is,

$$z = -34 - 20x + 16y$$

2. Find the equation of the plane in Figure 9, which is tangent to the graph at $(x, y) = (1, 0.8)$.

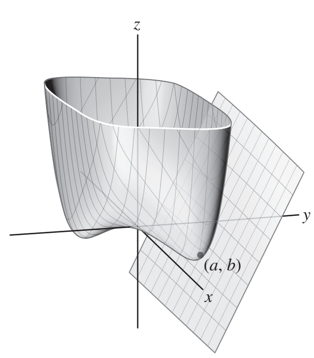


FIGURE 9 Graph of $f(x, y) = 0.2x^4 + y^6 - xy$.

SOLUTION We know that the equation of the tangent plane at the point $(1, 0.8)$ is:

$$z = f(1, 0.8) + f_x(1, 0.8)(x-1) + f_y(1, 0.8)(y-0.8)$$

We compute the function and its partial derivatives at the point $(1, 0.8)$:

$$\begin{aligned} f(x, y) &= 0.2x^4 + y^6 - xy &\Rightarrow\quad & f(1, 0.8) = -0.34 \\ f_x(x, y) &= 0.8x^3 - y &\Rightarrow\quad & f_x(1, 0.8) = 0 \\ f_y(x, y) &= 6y^5 - x &\Rightarrow\quad & f_y(1, 0.8) = 0.96608 \end{aligned}$$

Substituting in the equation of the tangent plane we obtain the following equation:

$$z = -0.34 + 0(x - 1) + 0.96608(y - 0.8)$$

That is,

$$z = 0.96608y - 1.112864$$

In Exercises 3–10, find an equation of the tangent plane at the given point.

3. $f(x, y) = x^2y + xy^3$, $(2, 1)$

SOLUTION The equation of the tangent plane at $(2, 1)$ is

$$z = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) \tag{1}$$

We compute the values of f and its partial derivatives at $(2, 1)$:

$$\begin{aligned} f(x, y) &= x^2y + xy^3 & & f(2, 1) = 6 \\ f_x(x, y) &= 2xy + y^3 &\Rightarrow\quad & f_x(2, 1) = 5 \\ f_y(x, y) &= x^2 + 3xy^2 & & f_y(2, 1) = 10 \end{aligned}$$

We now substitute these values in (1) to obtain the following equation of the tangent plane:

$$z = 6 + 5(x - 2) + 10(y - 1) = 5x + 10y - 14$$

That is,

$$z = 5x + 10y - 14.$$

4. $f(x, y) = \dfrac{x}{\sqrt{y}}$, $(4, 4)$

SOLUTION The equation of the tangent plane at $(4, 4)$ is

$$z = f(4, 4) + f_x(4, 4)(x - 4) + f_y(4, 4)(y - 4) \tag{1}$$

We compute the values of f and its partial derivatives at $(4, 4)$:

$$\begin{aligned} f(x, y) &= \frac{x}{\sqrt{y}} & & f(4, 4) = 2 \\ f_x(x, y) &= \frac{1}{\sqrt{y}} &\Rightarrow\quad & f_x(4, 4) = \frac{1}{2} \\ f_y(x, y) &= x\frac{\partial}{\partial y}y^{-1/2} = x \cdot \left(-\frac{1}{2}\right)y^{-3/2} = -\frac{x}{2y^{3/2}} & & f_y(4, 4) = -\frac{1}{4} \end{aligned}$$

Substituting these values in (1) gives

$$z = 2 + \frac{1}{2}(x - 4) - \frac{1}{4}(y - 4) = \frac{1}{2}x - \frac{1}{4}y + 1.$$

5. $f(x, y) = x^2 + y^{-2}$, $(4, 1)$

SOLUTION The equation of the tangent plane at $(4, 1)$ is

$$z = f(4, 1) + f_x(4, 1)(x - 4) + f_y(4, 1)(y - 1) \tag{1}$$

We compute the values of f and its partial derivatives at $(4, 1)$:

$$\begin{aligned} f(x, y) &= x^2 + y^{-2} & & f(4, 1) = 17 \\ f_x(x, y) &= 2x &\Rightarrow\quad & f_x(4, 1) = 8 \\ f_y(x, y) &= -2y^{-3} & & f_y(4, 1) = -2 \end{aligned}$$

Substituting in (1) we obtain the following equation of the tangent plane:

$$z = 17 + 8(x-4) - 2(y-1) = 8x - 2y - 13.$$

6. $G(u, w) = \sin(uw), \quad \left(\frac{\pi}{6}, 1\right)$

SOLUTION The equation of the tangent plane at $\left(\frac{\pi}{6}, 1\right)$ is

$$z = f\left(\frac{\pi}{6}, 1\right) + f_u\left(\frac{\pi}{6}, 1\right)\left(u - \frac{\pi}{6}\right) + f_w\left(\frac{\pi}{6}, 1\right)(w-1) \tag{1}$$

We compute the following values:

$$\begin{aligned} f(u,w) &= \sin(uw) & & f\left(\frac{\pi}{6}, 1\right) = \sin\frac{\pi}{6} = \frac{1}{2} \\ f_u(u,w) &= w\cos(uw) & \Rightarrow\quad & f_u\left(\frac{\pi}{6}, 1\right) = 1 \cdot \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ f_w(u,w) &= u\cos(uw) & & f_w\left(\frac{\pi}{6}, 1\right) = \frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}\pi}{12} \end{aligned}$$

Substituting in (1) gives the following equation of the tangent plane:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(u - \frac{\pi}{6}\right) + \frac{\sqrt{3}\pi}{12}(w-1)$$

That is,

$$z = \frac{\sqrt{3}}{2}u + \frac{\sqrt{3}\pi}{12}w + \frac{1}{2} - \frac{\sqrt{3}\pi}{6}.$$

7. $F(r, s) = r^2 s^{-1/2} + s^{-3}, \quad (2, 1)$

SOLUTION The equation of the tangent plane at (2, 1) is

$$z = f(2,1) + f_r(2,1)(r-2) + f_s(2,1)(s-1) \tag{1}$$

We compute f and its partial derivatives at (2, 1):

$$\begin{aligned} f(r,s) &= r^2 s^{-1/2} + s^{-3} & & f(2,1) = 5 \\ f_r(r,s) &= 2rs^{-1/2} & \Rightarrow\quad & f_r(2,1) = 4 \\ f_s(r,s) &= -\frac{1}{2}r^2 s^{-3/2} - 3s^{-4} & & f_s(2,1) = -5 \end{aligned}$$

We substitute these values in (1) to obtain the following equation of the tangent plane:

$$z = 5 + 4(r-2) - 5(s-1) = 4r - 5s + 2.$$

8. $g(x, y) = e^{x/y}, \quad (2, 1)$

SOLUTION The equation of the tangent plane at (2, 1) is:

$$z = g(2,1) + g_x(2,1)(x-2) + g_y(2,1)(y-1)$$

We compute g and its partial derivatives at (2, 1):

$$\begin{aligned} g(x,y) &= e^{x/y} & g(2,1) &= e^2 \\ g_x(x,y) &= \frac{1}{y}e^{x/y}, & g_x(2,1) &= e^2 \\ g_y(x,y) &= -\frac{x}{y^2}e^{x/y}, & g_y(2,1) &= -2e^2 \end{aligned}$$

We substitute these values in the tangent plane equation to obtain the following equation of the tangent plane:

$$z = e^2 + e^2(x-2) - 2e^2(y-1) = e^2x - 2e^2y + e^2 = e^2(x - 2y + 1)$$

9. $f(x, y) = \operatorname{sech}(x - y), \quad (\ln 4, \ln 2)$

SOLUTION The equation of the tangent plane at $(\ln 4, \ln 2)$ is:

$$z = f(\ln 4, \ln 2) + f_x(\ln 4, \ln 2)(x - \ln 4) + f_y(\ln 4, \ln 2)(y - \ln 2)$$

We compute f and its partial derivatives at $(\ln 4, \ln 2)$:

$$f(x, y) = \operatorname{sech}(x - y), \quad f(\ln 4, \ln 2) = \operatorname{sech}(\ln 2) = \frac{4}{5}$$

$$f_x(x, y) = -\tanh(x - y)\operatorname{sech}(x - y), \quad f_x(\ln 4, \ln 2) = -\tanh(\ln 2)\operatorname{sech}(\ln 2) = -\frac{12}{25}$$

$$f_y(x, y) = \tanh(x - y)\operatorname{sech}(x - y), \quad f_y(\ln 4, \ln 2) = \tanh(\ln 2)\operatorname{sech}(\ln 2) = \frac{12}{25}$$

We substitute these values in the tangent plane equation to obtain:

$$z = \frac{4}{5} - \frac{12}{25}(x - \ln 4) + \frac{12}{25}(x - \ln 2) = -\frac{4}{25}(3x - 3y - 5 - \ln 8)$$

10. $f(x, y) = \ln(4x^2 - y^2), \quad (1, 1)$

SOLUTION The equation of the tangent plane at $(1, 1)$ is

$$z = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

We compute the values of f and its partial derivatives at $(1, 1)$:

$$f(x, y) = \ln(4x^2 - y^2), \quad f(1, 1) = \ln 3$$

$$f_x(x, y) = \frac{8x}{4x^2 - y^2}, \quad f_x(1, 1) = \frac{8}{3}$$

$$f_y(x, y) = \frac{-2y}{4x^2 - y^2}, \quad f_y(1, 1) = -\frac{2}{3}$$

Substituting these values into the equation for the tangent plane we obtain:

$$z = \ln 3 + \frac{8}{3}(x - 1) - \frac{2}{3}(y - 1) = \frac{8}{3}x - \frac{2}{3}y + \ln 3 - 2$$

11. Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\mathbf{n} = \langle 3, 2, 2\rangle$ is normal to the tangent plane.

SOLUTION The equation of the tangent plane at the point $\big(a, b, f(a, b)\big)$ on the graph of $z = f(x, y)$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - z + f(a, b) = 0$$

Therefore, the following vector is normal to the plane:

$$\mathbf{v} = \big\langle f_x(a, b), f_y(a, b), -1\big\rangle$$

We compute the partial derivatives of the function $f(x, y) = 3x^2 - 4y^2$:

$$f_x(x, y) = 6x \quad \Rightarrow \quad f_x(a, b) = 6a$$

$$f_y(x, y) = -8y \quad \Rightarrow \quad f_y(a, b) = -8b$$

Therefore, the vector $\mathbf{v} = \langle 6a, -8b, -1\rangle$ is normal to the tangent plane at (a, b). Since we want $\mathbf{n} = \langle 3, 2, 2\rangle$ to be normal to the plane, the vectors $\mathbf{v}$ and $\mathbf{n}$ must be parallel. That is, the following must hold:

$$\frac{6a}{3} = \frac{-8b}{2} = -\frac{1}{2}$$

which implies that $a = -\frac{1}{4}$ and $b = \frac{1}{8}$. We compute the z-coordinate of the point:

$$z = 3 \cdot \left(-\frac{1}{4}\right)^2 - 4\left(\frac{1}{8}\right)^2 = \frac{1}{8}$$

The point on the graph at which the vector $\mathbf{n} = \langle 3, 2, 2\rangle$ is normal to the tangent plane is $\left(-\frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$.

12. Find the points on the graph of $z = xy^3 + 8y^{-1}$ where the tangent plane is parallel to $2x + 7y + 2z = 0$.

SOLUTION The equation of the tangent plane at the point $(a, b, f(a, b))$ on the graph of $z = f(x, y)$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - z + f(a, b) = 0$$

Therefore, the following vector is normal to the plane:

$$\mathbf{v} = \left\langle f_x(a, b), f_y(a, b), -1 \right\rangle$$

We compute the partial derivatives of the function $z = xy^3 + 8y^{-1}$:

$$f_x(x, y) = y^3, \quad f_x(a, b) = b^3$$
$$f_y(x, y) = 3xy^2 - 8y^{-2}, \quad f_y(a, b) = 3ab^2 - 8b^{-2}$$

Therefore, the vector $\mathbf{v} = \left\langle b^3, 3ab^2 - 8b^{-2}, -1 \right\rangle$ is normal to the tangent plane at (a, b). For two planes to be parallel, the vectors $\mathbf{v}$ and $\mathbf{n}$ must be parallel. The corresponding normal vector here is $\mathbf{n} = \langle 2, 7, 2 \rangle$. The following must hold:

$$\frac{b^3}{2} = \frac{3ab^2 - 8b^{-2}}{7} = -\frac{1}{2}$$

which implies that $b = -1$ and $a = 3/2$. We compute the z-coordinate of the point:

$$z = \frac{3}{2}(-1)^3 + 8(-1)^{-1} = -\frac{19}{2}$$

The point on the graph at which the tangent plane is parallel to $2x + 7y + 2z = 0$ is $\left(\frac{3}{2}, -1, -\frac{19}{2}\right)$.

13. Find the linearization $L(x, y)$ of $f(x, y) = x^2y^3$ at $(a, b) = (2, 1)$. Use it to estimate $f(2.01, 1.02)$ and $f(1.97, 1.01)$ and compare with values obtained using a calculator.

SOLUTION

(a) We compute the value of the function and its partial derivatives at $(a, b) = (2, 1)$:

$$\begin{aligned} f(x, y) &= x^2y^3 & & f(2, 1) = 4 \\ f_x(x, y) &= 2xy^3 & \Rightarrow \quad & f_x(2, 1) = 4 \\ f_y(x, y) &= 3x^2y^2 & & f_y(2, 1) = 12 \end{aligned}$$

The linear approximation is therefore

$$L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1)$$
$$L(x, y) = 4 + 4(x - 2) + 12(y - 1) = -16 + 4x + 12y$$

(b) For $h = x - 2$ and $k = y - 1$ we have the following form of the linear approximation at $(a, b) = (2, 1)$:

$$L(x, y) = f(2, 1) + f_x(2, 1)h + f_y(2, 1)k = 4 + 4h + 12k$$

To approximate $f(2.01, 1.02)$ we set $h = 2.01 - 2 = 0.01$, $k = 1.02 - 1 = 0.02$ to obtain

$$L(2.01, 1.02) = 4 + 4 \cdot 0.01 + 12 \cdot 0.02 = 4.28$$

The actual value is

$$f(2.01, 1.02) = 2.01^2 \cdot 1.02^3 = 4.2874$$

To approximate $f(1.97, 1.01)$ we set $h = 1.97 - 2 = -0.03$, $k = 1.01 - 1 = 0.01$ to obtain

$$L(1.97, 1.01) = 4 + 4 \cdot (-0.03) + 12 \cdot 0.01 = 4.$$

The actual value is

$$f(1.97, 1.01) = 1.97^2 \cdot 1.01^3 = 3.998.$$

14. Write the linear approximation to $f(x, y) = x(1+y)^{-1}$ at $(a, b) = (8, 1)$ in the form

$$f(a+h, b+k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

Use it to estimate $\frac{7.98}{2.02}$ and compare with the value obtained using a calculator.

SOLUTION We first compute the value of $f(x, y) = x(1+y)^{-1}$ and its partial derivatives at $(a, b) = (8, 1)$:

$$\begin{aligned} f(x, y) &= x(1+y)^{-1} &\Rightarrow\quad & f(8, 1) = 4 \\ f_x(x, y) &= (1+y)^{-1} &\Rightarrow\quad & f_x(8, 1) = \frac{1}{2} \\ f_y(x, y) &= -x(1+y)^{-2} &\Rightarrow\quad & f_y(8, 1) = -2 \end{aligned}$$

Hence,

$$f(8+h, 1+k) \approx 4 + \frac{1}{2}h - 2k \tag{1}$$

To estimate $\frac{7.98}{2.02} = \frac{7.98}{1+1.02}$ we set $h = 7.98 - 8 = -0.02$, $k = 1.02 - 1 = 0.02$ in the equation above to obtain

$$f(7.98, 1.02) = \frac{7.98}{2.02} \approx 4 + \frac{1}{2}(-0.02) - 2(0.02) = 3.95$$

The actual value is

$$\frac{7.98}{2.02} = 3.950495\ldots$$

15. Let $f(x, y) = x^3y^{-4}$. Use Eq. (4) to estimate the change

$$\Delta f = f(2.03, 0.9) - f(2, 1)$$

SOLUTION We compute the function and its partial derivatives at $(a, b) = (2, 1)$:

$$\begin{aligned} f(x, y) &= x^3y^{-4} & & f(2, 1) = 8 \\ f_x(x, y) &= 3x^2y^{-4} &\Rightarrow\quad & f_x(2, 1) = 12 \\ f_y(x, y) &= -4x^3y^{-5} & & f_y(2, 1) = -32 \end{aligned}$$

Also, $\Delta x = 2.03 - 2 = 0.03$ and $\Delta y = 0.9 - 1 = -0.1$. Therefore,

$$\begin{aligned} &\Delta f = f(2.03, 0.9) - f(2, 1) \approx f_x(2, 1)\Delta x + f_y\Delta y = 12 \cdot 0.03 + (-32) \cdot (-0.1) = 3.56 \\ &\Delta f \approx 3.56 \end{aligned}$$

16. Use the linear approximation to $f(x, y) = \sqrt{x/y}$ at $(9, 4)$ to estimate $\sqrt{9.1/3.9}$.

SOLUTION The linear approximation to $f(x, y) = \sqrt{\frac{x}{y}}$ at $(9, 4)$ is

$$f(9+h, 4+k) \approx f(9, 4) + f_x(9, 4)h + f_y(9, 4)k \tag{1}$$

We compute the function and its partial derivatives at $(9, 4)$:

$$\begin{aligned} f(x, y) &= x^{1/2}y^{-1/2} & & f(9, 4) = \frac{3}{2} \\ f_x(x, y) &= \frac{1}{2}x^{-1/2}y^{-1/2} &\Rightarrow\quad & f_x(9, 4) = \frac{1}{12} \\ f_y(x, y) &= -\frac{1}{2}x^{1/2}y^{-3/2} & & f_y(9, 4) = -\frac{3}{16} \end{aligned}$$

Substituting these values and $h = 0.1$, $k = -0.1$ in (1) gives the following estimation:

$$\sqrt{\frac{9.1}{3.9}} \approx \frac{3}{2} + \frac{1}{12} \cdot 0.1 - \frac{3}{16}(-0.1) \approx 1.5271$$

The value obtained by a calculator is $\sqrt{\frac{9.1}{3.9}} \approx 1.5275$. The error is 0.0004 and the percentage error is

$$\text{percentage error} \approx \frac{0.0004 \cdot 100}{1.5275} \approx 0.0262\%$$

17. Use the linear approximation of $f(x, y) = e^{x^2+y}$ at $(0, 0)$ to estimate $f(0.01, -0.02)$. Compare with the value obtained using a calculator.

SOLUTION The linear approximation of f at the point $(0, 0)$ is

$$f(h, k) \approx f(0, 0) + f_x(0, 0)h + f_y(0, 0)k \tag{1}$$

We first must compute f and its partial derivative at the point $(0, 0)$. Using the Chain Rule we obtain

$$\begin{aligned} f(x, y) &= e^{x^2+y} & & f(0, 0) = e^0 = 1 \\ f_x(x, y) &= 2xe^{x^2+y} & \Rightarrow \quad & f_x(0, 0) = 2 \cdot 0 \cdot e^0 = 0 \\ f_y(x, y) &= e^{x^2+y} & & f_y(0, 0) = e^0 = 1 \end{aligned}$$

We substitute these values and $h = 0.01$, $k = -0.02$ in (1) to obtain

$$f(0.01, -0.02) \approx 1 + 0 \cdot 0.01 + 1 \cdot (-0.02) = 0.98$$

The actual value is $f(0.01, -0.02) = e^{0.01^2-0.02} \approx 0.9803$.

18. Let $f(x, y) = x^2/(y^2 + 1)$. Use the linear approximation at an appropriate point (a, b) to estimate $f(4.01, 0.98)$.

SOLUTION We use the linear approximation at the point $(a, b) = (4, 1)$, which is the closest point with integer coordinates. That is,

$$f(4 + h, 1 + k) \approx f(4, 1) + f_x(4, 1)h + f_y(4, 1)k \tag{1}$$

We compute f and its partial derivatives at the point $(4, 1)$:

$$\begin{aligned} f(x, y) &= \frac{x^2}{y^2 + 1} & & f(4, 1) = 8 \\ f_x(x, y) &= \frac{2x}{y^2 + 1} & \Rightarrow \quad & f_x(4, 1) = 4 \\ f_y(x, y) &= x^2 \frac{\partial}{\partial y}\left(\frac{1}{y^2 + 1}\right) = x^2 \cdot \frac{-2y}{(y^2 + 1)^2} = \frac{-2x^2 y}{(y^2 + 1)^2} & & f_y(4, 1) = -8 \end{aligned}$$

Substituting these values and $h = 0.01$, $k = -0.02$ in (1) gives

$$f(4.01, 0.98) \approx 8 + 4 \cdot 0.01 + (-8)(-0.02) = 8.2$$

The actual value is

$$f(4.01, 0.98) = \frac{4.01^2}{0.98^2 + 1} = 8.202$$

19. Find the linearization of $f(x, y, z) = z\sqrt{x + y}$ at $(8, 4, 5)$.

SOLUTION The linear approximation of f at the point $(8, 4, 5)$ is:

$$f(x, y, z) \approx f(8, 4, 5) + f_x(8, 4, 5)(x - 8) + f_y(8, 4, 5)(y - 4) + f_z(8, 4, 5)(z - 5)$$

We compute the values of f and its partial derivatives at $(8, 4, 5)$:

$$\begin{aligned} f(x, y, z) &= z\sqrt{x + y}, & f(8, 4, 5) &= 5\sqrt{12} = 10\sqrt{3} \\ f_x(x, y, z) &= \frac{z}{2\sqrt{x + y}}, & f_x(8, 4, 5) &= \frac{5}{2\sqrt{12}} = \frac{5}{4\sqrt{3}} \\ f_y(x, y, z) &= \frac{z}{2\sqrt{x + y}}, & f_y(8, 4, 5) &= \frac{5}{2\sqrt{12}} = \frac{5}{4\sqrt{3}} \\ f_z(x, y, z) &= \sqrt{x + y}, & f_z(8, 4, 5) &= \sqrt{12} = 4\sqrt{3} \end{aligned}$$

Substituting these values we obtain the linearization:

$$\begin{aligned} f(x, y, z) &\approx 10\sqrt{3} + \frac{5}{4\sqrt{3}}(x - 8) + \frac{5}{4\sqrt{3}}(y - 4) + 4\sqrt{3}(z - 5) \\ &= \frac{5}{4\sqrt{3}}(x - 8) + \frac{5}{4\sqrt{3}}(y - 4) + 4\sqrt{3}z - 15\sqrt{3} \end{aligned}$$

20. Find the linearization to $f(x, y, z) = xy/z$ at the point $(2, 1, 2)$. Use it to estimate $f(2.05, 0.9, 2.01)$ and compare with the value obtained from a calculator.

SOLUTION The linear approximation to f at the point $(2, 1, 2)$ is:

$$f(x, y, z) \approx f(2, 1, 2) + f_x(2, 1, 2)(x - 2) + f_y(2, 1, 2)(y - 1) + f_z(2, 1, 2)(z - 2) \tag{1}$$

We compute the values of f and its partial derivatives at $(2, 1, 2)$:

$$\begin{array}{lcl} f(x, y, z) = \dfrac{xy}{z} & & f(2, 1, 2) = 1 \\ f_x(x, y, z) = \dfrac{y}{z} & \Rightarrow & f_x(2, 1, 2) = \dfrac{1}{2} \\ f_y(x, y, z) = \dfrac{x}{z} & & f_y(2, 1, 2) = 1 \\ f_z(x, y, z) = -\dfrac{xy}{z^2} & & f_z(2, 1, 2) = -\dfrac{1}{2} \end{array}$$

We substitute these values in (1) to obtain the following linear approximation:

$$\frac{xy}{z} \approx 1 + \frac{1}{2}(x - 2) + 1 \cdot (y - 1) - \frac{1}{2}(z - 2)$$
$$\frac{xy}{z} \approx \frac{1}{2}x + y - \frac{1}{2}z$$

To estimate $f(2.05, 0.9, 2.01)$ we will have:

$$f(2.05, 0.9, 2.01) \approx \frac{1}{2}(2.05) + 0.9 - \frac{1}{2}(2.01) = 0.92$$

Comparing this with the calculator value we get:

$$f(2.05, 0.9, 2.01) = \frac{2.05 \cdot 0.9}{2.01} \approx 0.9179$$

21. Estimate $f(2.1, 3.8)$ assuming that

$$f(2, 4) = 5, \qquad f_x(2, 4) = 0.3, \qquad f_y(2, 4) = -0.2$$

SOLUTION We use the linear approximation of f at the point $(2, 4)$, which is

$$f(2 + h, 4 + k) \approx f(2, 4) + f_x(2, 4)h + f_y(2, 4)k$$

Substituting the given values and $h = 0.1$, $k = -0.2$ we obtain the following approximation:

$$f(2.1, 3.8) \approx 5 + 0.3 \cdot 0.1 + 0.2 \cdot 0.2 = 5.07.$$

22. Estimate $f(1.02, 0.01, -0.03)$ assuming that

$$\begin{array}{ll} f(1, 0, 0) = -3, & f_x(1, 0, 0) = -2, \\ f_y(1, 0, 0) = 4, & f_z(1, 0, 0) = 2 \end{array}$$

SOLUTION The linear approximation at $(1, 0, 0)$ is

$$f(1 + h, k, l) \approx f(1, 0, 0) + f_x(1, 0, 0)h + f_y(1, 0, 0)k + f_z(1, 0, 0)l \tag{1}$$

We substitute $h = 0.02$, $k = 0.01$, $l = -0.03$ and the given values to obtain the following estimation:

$$f(1.02, 0.01, -0.03) \approx -3 + (-2) \cdot 0.02 + 4 \cdot 0.01 + 2(-0.03) = -3.06$$

That is,

$$f(1.02, 0.01, -0.03) \approx -3.06.$$

In Exercises 23–28, use the linear approximation to estimate the value. Compare with the value given by a calculator.

23. $(2.01)^3(1.02)^2$

SOLUTION The number $(2.01)^3(1.02)^2$ is a value of the function $f(x, y) = x^3y^2$. We use the li(8, near approximation at (2, 1), which is

$$f(2+h, 1+k) \approx f(2,1) + f_x(2,1)h + f_y(2,1)k \tag{1}$$

We compute the value of the function and its partial derivatives at (2, 1):

$$\begin{aligned} f(x,y) &= x^3y^2 & & f(2,1) = 8 \\ f_x(x,y) &= 3x^2y^2 & \Rightarrow \quad & f_x(2,1) = 12 \\ f_y(x,y) &= 2x^3y & & f_y(2,1) = 16 \end{aligned}$$

Substituting these values and $h = 0.01$, $k = 0.02$ in (1) gives the approximation

$$(2.01)^3(1.02)^2 \approx 8 + 12 \cdot 0.01 + 16 \cdot 0.02 = 8.44$$

The value given by a calculator is 8.4487. The error is 0.0087 and the percentage error is

$$\text{Percentage error} \approx \frac{0.0087 \cdot 100}{8.4487} = 0.103\%$$

24. $\dfrac{4.1}{7.9}$

SOLUTION The number $\frac{4.1}{7.9}$ is a value of the function $f(x, y) = xy^{-1}$. We use the linear approximation at the point (4, 8), which is

$$f(4+h, 8+k) \approx f(4,8) + f_x(4,8)h + f_y(4,8)k \tag{1}$$

We compute the values of the function and its partial derivatives at (4, 8):

$$\begin{aligned} f(x,y) &= xy^{-1} & & f(4,8) = \frac{1}{2} \\ f_x(x,y) &= y^{-1} & \Rightarrow \quad & f_x(4,8) = \frac{1}{8} \\ f_y(x,y) &= -xy^{-2} & & f_y(4,8) = -\frac{1}{16} \end{aligned}$$

We substitute these values and $h = 0.1$, $k = -0.1$ in (1) to obtain the following approximation:

$$\frac{4.1}{7.9} \approx \frac{1}{2} + \frac{1}{8} \cdot 0.1 - \frac{1}{16} \cdot (-0.1) = \frac{83}{160} = 0.51875$$

The value given by a calculator is $\frac{4.1}{7.9} \approx 0.51899$. The error is 0.00024 and the percentage error is at most

$$\text{Percentage error} \approx \frac{0.00024 \cdot 100}{0.51899} = 0.04625\%$$

25. $\sqrt{3.01^2 + 3.99^2}$

SOLUTION This is a value of the function $f(x, y) = \sqrt{x^2 + y^2}$. We use the linear approximation at the point (3, 4), which is

$$f(3+h, 4+k) \approx f(3,4) + f_x(3,4)h + f_y(3,4)k \tag{1}$$

Using the Chain Rule gives the following partial derivatives:

$$\begin{aligned} f(x,y) &= \sqrt{x^2+y^2} & & f(3,4) = 5 \\ f_x(x,y) &= \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} & \Rightarrow \quad & f_x(3,4) = \frac{3}{5} \\ f_y(x,y) &= \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} & & f_y(3,4) = \frac{4}{5} \end{aligned}$$

Substituting these values and $h = 0.01$, $k = -0.01$ in (1) gives the following approximation:

$$\sqrt{3.01^2 + 3.99^2} \approx 5 + \frac{3}{5} \cdot 0.01 + \frac{4}{5} \cdot (-0.01) = 4.998$$

The value given by a calculator is $\sqrt{3.01^2 + 3.99^2} \approx 4.99802$. The error is 0.00002 and the percentage error is at most

$$\text{Percentage error} \approx \frac{0.00002 \cdot 100}{4.99802} = 0.0004002\%$$

26. $\dfrac{0.98^2}{2.01^3 + 1}$

SOLUTION We use the linear approximation of the function $f(x, y) = \frac{x^2}{y^3+1}$ at the point $(1, 2)$, which is

$$f(1 + h, 2 + k) \approx f(1, 2) + f_x(1, 2)h + f_y(1, 2)k \tag{1}$$

We compute the values of f and its partial derivatives at $(1, 2)$. We get:

$$\begin{aligned} f(x, y) &= \frac{x^2}{y^3 + 1} & & & f(1, 2) &= \frac{1}{9} \\ f_x(x, y) &= \frac{2x}{y^3 + 1} & & \Rightarrow & f_x(1, 2) &= \frac{2}{9} \\ f_y(x, y) &= x^2 \cdot \frac{-1}{(y^3 + 1)^2} \cdot 3y^2 = \frac{-3x^2y^2}{(y^3 + 1)^2} & & & f_y(1, 2) &= -\frac{4}{27} \end{aligned}$$

Substituting these values and $h = -0.02$, $k = 0.01$ in (1) gives the following approximation:

$$\frac{0.98^2}{2.01^3 + 1} \approx \frac{1}{9} + \frac{2}{9}(-0.02) - \frac{4}{27} \cdot 0.01 \approx 0.1052$$

The value given by a calculator is $\frac{0.98^2}{2.01^3+1} \approx 0.1053$. The error is 0.0001 and the percentage error is at most

$$\text{Percentage error} \approx \frac{0.0001 \cdot 100}{0.1053} \approx 0.095\%$$

27. $\sqrt{(1.9)(2.02)(4.05)}$

SOLUTION We use the linear approximation of the function $f(x, y, z) = \sqrt{xyz}$ at the point $(2, 2, 4)$, which is

$$f(2 + h, 2 + k, 4 + l) \approx f(2, 2, 4) + f_x(2, 2, 4)h + f_y(2, 2, 4)k + f_z(2, 2, 4)l \tag{1}$$

We compute the values of the function and its partial derivatives at $(2, 2, 4)$:

$$\begin{aligned} f(x, y, z) &= \sqrt{xyz} & & & f(2, 2, 4) &= 4 \\ f_x(x, y, z) &= \frac{yz}{2\sqrt{xyz}} = \frac{1}{2}\sqrt{\frac{yz}{x}} & & \Rightarrow & f_x(2, 2, 4) &= 1 \\ f_y(x, y, z) &= \frac{xz}{2\sqrt{xyz}} = \frac{1}{2}\sqrt{\frac{xz}{y}} & & & f_y(2, 2, 4) &= 1 \\ f_z(x, y, z) &= \frac{xy}{2\sqrt{xyz}} = \frac{1}{2}\sqrt{\frac{xy}{z}} & & & f_z(2, 2, 4) &= \frac{1}{2} \end{aligned}$$

Substituting these values and $h = -0.1$, $k = 0.02$, $l = 0.05$ in (1) gives the following approximation:

$$\sqrt{(1.9)(2.02)(4.05)} = 4 + 1 \cdot (-0.1) + 1 \cdot 0.02 + \frac{1}{2}(0.05) = 3.945$$

The value given by a calculator is:

$$\sqrt{(1.9)(2.02)(4.05)} \approx 3.9426$$

28. $\dfrac{8.01}{\sqrt{(1.99)(2.01)}}$

SOLUTION We use the linear approximation of the function $f(x,y,z)=\frac{x}{\sqrt{yz}}$ at the point (8, 2, 2), which is

$$f(8+h,2+k,2+l)\approx f(8,2,2)+f_x(8,2,2)h+f_y(8,2,2)k+f_z(8,2,2)l \tag{1}$$

We compute the values of the function and its partial derivatives at (8, 2, 2). This gives

$$\begin{aligned}
f(x,y,z)&=\frac{x}{\sqrt{yz}} && f(8,2,2)=4\\
f_x(x,y,z)&=\frac{1}{\sqrt{yz}} && \Rightarrow\quad f_x(8,2,2)=\frac{1}{2}\\
f_y(x,y,z)&=x\frac{\partial}{\partial y}(yz)^{-1/2}=-\frac{1}{2}x(yz)^{-3/2}z=-\frac{1}{2}xy^{-3/2}z^{-1/2} && f_y(8,2,2)=-1\\
f_z(x,y,z)&=x\frac{\partial}{\partial z}(yz)^{-1/2}=-\frac{1}{2}x(yz)^{-3/2}y=-\frac{1}{2}xy^{-1/2}z^{-3/2} && f_z(8,2,2)=-1
\end{aligned}$$

Substituting these values and $h=0.01$, $k=-0.01$, $l=0.01$ in (1) gives the following approximation:

$$\frac{8.01}{\sqrt{(1.99)(2.01)}}=4+\frac{1}{2}\cdot 0.01-1\cdot(-0.01)-1\cdot 0.01=4.005$$

The value given by a calculator is 4.00505. The error is 0.00005 and the percentage error is at most

$$\text{Percentage error}\approx\frac{0.00005\cdot 100}{4.00505}\approx 0.00125\%$$

29. Find an equation of the tangent plane to $z=f(x,y)$ at $P=(1,2,10)$ assuming that

$$f(1,2)=10,\qquad f(1.1,2.01)=10.3,\qquad f(1.04,2.1)=9.7$$

SOLUTION The equation of the tangent plane at the point (1, 2) is

$$\begin{aligned}
z&=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)\\
z&=10+f_x(1,2)(x-1)+f_y(1,2)(y-2)
\end{aligned}\tag{1}$$

Since the values of the partial derivatives at (1, 2) are not given, we approximate them as follows:

$$\begin{aligned}
f_x(1,2)&\approx\frac{f(1.1,2)-f(1,2)}{0.1}\approx\frac{f(1.1,2.01)-f(1,2)}{0.1}=3\\
f_y(1,2)&\approx\frac{f(1,2.1)-f(1,2)}{0.1}\approx\frac{f(1.04,2.1)-f(1,2)}{0.1}=-3
\end{aligned}$$

Substituting in (1) gives the following approximation to the equation of the tangent plane:

$$z=10+3(x-1)-3(y-2)$$

That is, $z=3x-3y+13$.

30. Suppose that the plane tangent to $z=f(x,y)$ at $(-2,3,4)$ has equation $4x+2y+z=2$. Estimate $f(-2.1,3.1)$.

SOLUTION The tangent plane $z=2-4x-2y$ is also a linear approximation for f near $(-2,3)$, so we can thus calculate the following:

$$f(-2.1,3.1)\approx 2-4(-2.1)-2(3.1)=4.2$$

In Exercises 31–34, let $I=W/H^2$ denote the BMI described in Example 5.

31. A boy has weight $W=34$ kg and height $H=1.3$ m. Use the linear approximation to estimate the change in I if (W,H) changes to (36, 1.32).

SOLUTION Let $\Delta I=I(36,1.32)-I(34,1.3)$ denote the change in I. Using the linear approximation of I at the point (34, 1.3) we have

$$I(34+h,1.3+k)-I(34,1.3)\approx\frac{\partial I}{\partial W}(34,1.3)h+\frac{\partial I}{\partial H}(34,1.3)k$$

For $h=2$, $k=0.02$ we obtain

$$\Delta I\approx\frac{\partial I}{\partial W}(34,1.3)\cdot 2+\frac{\partial I}{\partial H}(34,1.3)\cdot 0.02 \tag{1}$$

We compute the partial derivatives in (1):

$$\frac{\partial I}{\partial W} = \frac{\partial}{\partial W}\frac{W}{H^2} = \frac{1}{H^2} \qquad \Rightarrow \quad \frac{\partial I}{\partial W}(34, 1.3) = 0.5917$$
$$\frac{\partial I}{\partial H} = W\frac{\partial}{\partial H}H^{-2} = W\cdot(-2H^{-3}) = \frac{-2W}{H^3} \quad \Rightarrow \quad \frac{\partial I}{\partial H}(34, 1.3) = -30.9513$$

Substituting the partial derivatives in (1) gives the following estimation of ΔI:

$$\Delta I \approx 0.5917\cdot 2 - 30.9513\cdot 0.02 = 0.5644$$

32. Suppose that $(W, H) = (34, 1.3)$. Use the linear approximation to estimate the increase in H required to keep I constant if W increases to 35.

SOLUTION The linear approximation of $I = \frac{W}{H^2}$ at the point (34, 1.3) is:

$$\Delta I = I(34+h, 1.3+k) - I(34, 1.3) \approx \frac{\partial I}{\partial W}(34, 1.3)h + \frac{\partial I}{\partial H}(34, 1.3)k \tag{1}$$

In the earlier exercise, we found that

$$\frac{\partial I}{\partial W}(34, 1.3) = 0.5917, \qquad \frac{\partial I}{\partial H}(34, 1.3) = -30.9513$$

We substitute these derivatives and $h = 1$ in (1), equate the resulting expression to zero and solve for k. This gives:

$$\Delta I \approx 0.5917\cdot 1 - 30.9513\cdot k = 0$$
$$0.5917 = 30.9513k \quad \Rightarrow \quad k = 0.0191$$

That is, for an increase in weight of 1 kg, the increase in height must be approximately 0.0191 meters (or 1.91 centimeters) in order to keep I constant.

33. (a) Show that $\Delta I \approx 0$ if $\Delta H/\Delta W \approx H/2W$.

(b) Suppose that $(W, H) = (25, 1.1)$. What increase in H will leave I (approximately) constant if W is increased by 1 kg?

SOLUTION

(a) The linear approximation implies that

$$\Delta I \approx \frac{\partial I}{\partial W}\Delta W + \frac{\partial I}{\partial H}\Delta H$$

Hence, $\Delta I \approx 0$ if

$$\frac{\partial I}{\partial W}\Delta W + \frac{\partial I}{\partial H}\Delta H = 0 \tag{1}$$

We compute the partial derivatives of $I = \frac{W}{H^2}$:

$$\frac{\partial I}{\partial W} = \frac{\partial}{\partial W}\left(\frac{W}{H^2}\right) = \frac{1}{H^2}$$
$$\frac{\partial I}{\partial H} = W\frac{\partial}{\partial H}(H^{-2}) = -2WH^{-3} = \frac{-2W}{H^3}$$

We substitute the partial derivatives in (1) to obtain

$$\frac{1}{H^2}\Delta W - \frac{2W}{H^3}\Delta H = 0$$

Hence,

$$\frac{1}{H^2}\Delta W = \frac{2W}{H^3}\Delta H$$

or

$$\frac{\Delta H}{\Delta W} = \frac{1}{H^2}\cdot\frac{H^3}{2W} = \frac{H}{2W}$$

(b) In part (a) we showed that if $\frac{\Delta H}{\Delta W} = \frac{H}{2W}$, then I remains approximately constant. We thus substitute $W = 25$, $H = 1.1$, $\Delta W = 1$, and solve for ΔH. This gives

$$\frac{\Delta H}{1} = \frac{1.1}{50} \quad \Rightarrow \quad \Delta H \approx 0.022 \text{ meters.}$$

That is, an increase of 0.022 meters in H will leave I approximately constant.

34. Estimate the change in height that will decrease I by 1 if $(W, H) = (25, 1.1)$, assuming that W remains constant.

SOLUTION If $\Delta W = 0$, then

$$\Delta I \approx -(2W/H^3)\Delta H = -1$$

This yields $\Delta H = H^3/2W = 1.1^3/50 \approx 0.027$ meters, or 2.7 cm

35. A cylinder of radius r and height h has volume $V = \pi r^2 h$.

(a) Use the linear approximation to show that

$$\frac{\Delta V}{V} \approx \frac{2\Delta r}{r} + \frac{\Delta h}{h}$$

(b) Estimate the percentage increase in V if r and h are each increased by 2%.

(c) The volume of a certain cylinder V is determined by measuring r and h. Which will lead to a greater error in V: a 1% error in r or a 1% error in h?

SOLUTION

(a) The linear approximation is

$$\Delta V \approx V_r \Delta r + V_h \Delta h \tag{1}$$

We compute the partial derivatives of $V = \pi r^2 h$:

$$V_r = \pi h \frac{\partial}{\partial r} r^2 = 2\pi h r$$

$$V_h = \pi r^2 \frac{\partial}{\partial h} h = \pi r^2$$

Substituting in (1) gives

$$\Delta V \approx 2\pi h r \Delta r + \pi r^2 \Delta h$$

We divide by $V = \pi r^2 h$ to obtain

$$\frac{\Delta V}{V} \approx \frac{2\pi h r \Delta r}{V} + \frac{\pi r^2 \Delta h}{V} = \frac{2\pi h r \Delta r}{\pi r^2 h} + \frac{\pi r^2 \Delta h}{\pi r^2 h} = \frac{2\Delta r}{r} + \frac{\Delta h}{h}$$

That is,

$$\frac{\Delta V}{V} \approx \frac{2\Delta r}{r} + \frac{\Delta h}{h}$$

(b) The percentage increase in V is, by part (a),

$$\frac{\Delta V}{V} \cdot 100 \approx 2\frac{\Delta r}{r} \cdot 100 + \frac{\Delta h}{h} \cdot 100$$

We are given that $\frac{\Delta r}{r} \cdot 100 = 2$ and $\frac{\Delta h}{h} \cdot 100 = 2$, hence the percentage increase in V is

$$\frac{\Delta V}{V} \cdot 100 = 2 \cdot 2 + 2 = 6\%$$

(c) The percentage error in V is

$$\frac{\Delta V}{V} \cdot 100 = 2\frac{\Delta r}{r} \cdot 100 + \frac{\Delta h}{h} \cdot 100$$

A 1% error in r implies that $\frac{\Delta r}{r} \cdot 100 = 1$. Assuming that there is no error in h, we get

$$\frac{\Delta V}{V} \cdot 100 = 2 \cdot 1 + 0 = 2\%$$

A 1% in h implies that $\frac{\Delta h}{h} \cdot 100 = 1$. Assuming that there is no error in r, we get

$$\frac{\Delta V}{V} \cdot 100 = 0 + 1 = 1\%$$

We conclude that a 1% error in r leads to a greater error in V than a 1% error in h.

36. Use the linear approximation to show that if $I = x^a y^b$, then

$$\frac{\Delta I}{I} \approx a\frac{\Delta x}{x} + b\frac{\Delta y}{y}$$

SOLUTION The linear approximation is

$$\Delta I \approx I_x \Delta x + I_y \Delta y \tag{1}$$

We compute the partial derivatives of $I = x^a y^b$:

$$\begin{cases} I_x = \mathrm{a}x^{a-1}y^b \\ I_y = bx^a y^{b-1} \end{cases}$$

substituting in (1) gives

$$\Delta I \approx ax^{a-1}y^b \Delta x + bx^a y^{b-1}\Delta y$$

We now divide by $I = x^a y^b$ to obtain

$$\frac{\Delta I}{I} \approx \frac{ax^{a-1}y^b\Delta x}{I} + \frac{bx^a y^{b-1}\Delta y}{I} = \frac{ax^{a-1}y^b\Delta x}{x^a y^b} + \frac{bx^a y^{b-1}\Delta y}{x^a y^b} = a\frac{\Delta x}{x} + b\frac{\Delta y}{y}$$

That is,

$$\frac{\Delta I}{I} \approx a\frac{\Delta x}{x} + b\frac{\Delta \mathrm{y}}{y}.$$

37. The monthly payment for a home loan is given by a function $f(P, r, N)$, where P is the principal (initial size of the loan), r the interest rate, and N is the length of the loan in months. Interest rates are expressed as a decimal: A 6% interest rate is denoted by $r = 0.06$. If $P = \$100{,}000$, $r = 0.06$, and $N = 240$ (a 20-year loan), then the monthly payment is $f(100{,}000, 0.06, 240) = 716.43$. Furthermore, at these values, we have

$$\frac{\partial f}{\partial P} = 0.0071, \qquad \frac{\partial f}{\partial r} = 5769, \qquad \frac{\partial f}{\partial N} = -1.5467$$

Estimate:

(a) The change in monthly payment per \$1000 increase in loan principal.
(b) The change in monthly payment if the interest rate increases to $r = 6.5\%$ and $r = 7\%$.
(c) The change in monthly payment if the length of the loan increases to 24 years.

SOLUTION

(a) The linear approximation to $f(P, r, N)$ is

$$\Delta f \approx \frac{\partial f}{\partial P}\Delta P + \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial N}\Delta N$$

We are given that $\frac{\partial f}{\partial P} = 0.0071$, $\frac{\partial f}{\partial r} = 5769$, $\frac{\partial f}{\partial N} = -1.5467$, and $\Delta P = 1000$. Assuming that $\Delta r = 0$ and $\Delta N = 0$, we get

$$\Delta f \approx 0.0071 \cdot 1000 = 7.1$$

The change in monthly payment per thousand dollar increase in loan principal is \$7.1.

(b) By the given data, we have

$$\Delta f \approx 0.0071\Delta P + 5769\Delta r - 1.5467\Delta N \tag{1}$$

The interest rate 6.5% corresponds to $r = 0.065$, and the interest rate 7% corresponds to $r = 0.07$. In the first case $\Delta r = 0.065 - 0.06 = 0.005$ and in the second case $\Delta r = 0.07 - 0.06 = 0.01$. Substituting in (1), assuming that $\Delta P = 0$ and $\Delta N = 0$, gives

$$\Delta f = 5769 \cdot 0.005 = \$28.845$$

$$\Delta f = 5769 \cdot 0.01 = \$57.69$$

(c) We substitute $\Delta N = (24 - 20) \cdot 12 = 48$ months and $\Delta r = \Delta N = 0$ in (1) to obtain

$$\Delta f \approx -1.5467 \cdot 48 = -74.2416$$

The monthly payment will be reduced by \$74.2416.

38. Automobile traffic passes a point P on a road of width w ft at an average rate of R vehicles per second. Although the arrival of automobiles is irregular, traffic engineers have found that the average waiting time T until there is a gap in traffic of at least t seconds is approximately $T = te^{Rt}$ seconds. A pedestrian walking at a speed of 3.5 ft/s (5.1 mph) requires $t = w/3.5$ s to cross the road. Therefore, the average time the pedestrian will have to wait before crossing is $f(w, R) = (w/3.5)e^{wR/3.5}$ s.

(a) What is the pedestrian's average waiting time if $w = 25$ ft and $R = 0.2$ vehicle per second?

(b) Use the linear approximation to estimate the increase in waiting time if w is increased to 27 ft.

(c) Estimate the waiting time if the width is increased to 27 ft and R decreases to 0.18.

(d) What is the rate of increase in waiting time per 1-ft increase in width when $w = 30$ ft and $R = 0.3$ vehicle per second?

SOLUTION

(a) We are given that the average time the pedestrian will have to wait for a t-second gap in traffic is

$$f(w, R) = \frac{w}{3.5}e^{wR/3.5}$$

Substituting the values $w = 25$ and $R = 0.2$, we obtain

$$f(25, 0.2) = \frac{25}{3.5}e^{(25\cdot 0.2)/3.5} \approx 29.8 \text{ seconds}$$

(b) The linear approximation at $(w, R) = (25, 0.2)$ is,

$$\Delta f \approx f_w(25, 0.2)\Delta w + f_r(25, 0.2)\Delta R \tag{1}$$

We compute the partial derivatives. Using the Product Rule and the Chain Rule we have

$$f_w(w, R) = \frac{1}{3.5}\left(e^{wR/3.5} + we^{wR/3.5}\cdot\frac{R}{3.5}\right) = \frac{e^{wR/3.5}}{3.5}\left(1 + \frac{wR}{3.5}\right)$$

By the Chain Rule we get

$$f_R(w, R) = \frac{w}{3.5}e^{wR/3.5}\cdot\frac{w}{3.5} = \left(\frac{w}{3.5}\right)^2 e^{wR/3.5}$$

At the point $(25, 0.2)$ we have

$$f_w(25, 0.2) \approx 2.9; \quad f_R(25, 0.2) \approx 212.9 \tag{2}$$

Substituting these derivatives, $\Delta w = 27 - 25 = 2$, and $\Delta r = 0$ in (1) we get

$$\Delta f = 2.9 \cdot 2 = 5.8$$

An increase of 2 ft in w causes an increase of 5.8 seconds in waiting time.

(c) We substitute the derivatives in (2) with $\Delta w = 2$ and $\Delta r = 0.18 - 0.2 = -0.02$ in the linear approximation (1) to obtain

$$\Delta f \approx 2.9 \cdot 2 - 212.9 \cdot 0.02 \approx 1.54$$

That is, the waiting time is increased by approximately 1.54 seconds. Using part (a), the estimated waiting time is

$$f(25, 0.2) + 1.54 \approx 29.8 + 1.54 = 31.34 \text{ seconds}$$

(d) The rate of increase in waiting time per one foot increase in width, when $w = 30$ and $R = 0.3$, is $\frac{\partial f}{\partial w}(30, 0.3)$. Using the derivative obtained in part (b) we have

$$\frac{\partial f}{\partial w}(30, 0.3) = \frac{e^{9/3.5}}{3.5}\left(1 + \frac{9}{3.5}\right) \approx 13.35$$

39. The volume V of a right-circular cylinder is computed using the values 3.5 m for diameter and 6.2 m for height. Use the linear approximation to estimate the maximum error in V if each of these values has a possible error of at most 5%. Recall that $V = \frac{1}{3}\pi r^2 h$.

SOLUTION We denote by d and h the diameter and height of the cylinder, respectively. By the Formula for the Volume of a Cylinder we have

$$V = \pi\left(\frac{d}{2}\right)^2 h = \frac{\pi}{4}d^2 h$$

The linear approximation is

$$\Delta V \approx \frac{\partial V}{\partial d}\Delta d + \frac{\partial V}{\partial h}\Delta h \tag{1}$$

We compute the partial derivatives at $(d, h) = (3.5, 6.2)$:

$$\begin{aligned} \frac{\partial V}{\partial d}(d,h) &= \frac{\pi}{4}h\cdot 2d = \frac{\pi}{2}hd \\ \frac{\partial V}{\partial h}(d,h) &= \frac{\pi}{4}d^2 \end{aligned} \quad\Rightarrow\quad \begin{aligned} \frac{\partial V}{\partial d}(3.5, 6.2) &\approx 34.086 \\ \frac{\partial V}{\partial h}(3.5, 6.2) &= 9.621 \end{aligned}$$

Substituting these derivatives in (1) gives

$$\Delta V \approx 34.086\Delta d + 9.621\Delta h \tag{2}$$

We are given that the errors in the measurements of d and h are at most 5%. Hence,

$$\frac{\Delta d}{3.5} = 0.05 \quad\Rightarrow\quad \Delta d = 0.175$$

$$\frac{\Delta h}{6.2} = 0.05 \quad\Rightarrow\quad \Delta h = 0.31$$

Substituting in (2) we obtain

$$\Delta V \approx 34.086 \cdot 0.175 + 9.621 \cdot 0.31 \approx 8.948$$

The error in V is approximately 8.948 meters. The percentage error is at most

$$\frac{\Delta V \cdot 100}{V} = \frac{8.948 \cdot 100}{\frac{\pi}{4}\cdot 3.5^2 \cdot 6.2} = 15\%$$

Further Insights and Challenges

40. Show that if $f(x, y)$ is differentiable at (a, b), then the function of one variable $f(x, b)$ is differentiable at $x = a$. Use this to prove that $f(x, y) = \sqrt{x^2 + y^2}$ is *not* differentiable at $(0, 0)$.

SOLUTION If $f(x, y)$ is differentiable at (a, b), then the partial derivatives f_x and f_y both exist at (a, b), which means that (in particular) $\frac{d}{dx}f(x, b)$ exists at $x = a$, which means that $f(x, b)$ is differentiable at $x = a$. In our case, for $(a, b) = (0, 0)$ and $f(x, y) = \sqrt{x^2 + y^2}$, then $f(x, b) = f(x, 0) = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$, which is not differentiable at $x = 0$. Hence the original two-variable function $f(x, y) = \sqrt{x^2 + y^2}$ is *not* differentiable at $(0, 0)$.

41. This exercise shows directly (without using Theorem 1) that the function $f(x, y) = 5x + 4y^2$ from Example 1 is locally linear at $(a, b) = (2, 1)$.

(a) Show that $f(x, y) = L(x, y) + e(x, y)$ with $e(x, y) = 4(y - 1)^2$.

(b) Show that

$$0 \le \frac{e(x, y)}{\sqrt{(x-2)^2 + (y-1)^2}} \le 4|y - 1|$$

(c) Verify that $f(x, y)$ is locally linear.

SOLUTION According to Example 1,

$$L(x, y) = -4 + 5x + 8y$$

(a) We compute the difference:

$$\begin{aligned} f(x, y) - L(x, y) &= (5x + 4y^2) - (-4 + 5x + 8y) \\ &= 4y^2 - 8y + 4 = 4(y-1)^2 \end{aligned}$$

Therefore, $f(x, y) = L(x, y) + 4(y - 1)^2$.

(b) For $(x, y) \neq (2, 1)$, we consider

$$\frac{e(x, y)}{\sqrt{(x-2)^2 + (y-1)^2}} = \frac{4(y-1)^2}{\sqrt{(x-2)^2 + (y-1)^2}}$$

The following inequality holds

$$\frac{4(y-1)^2}{\sqrt{(x-2)^2 + (y-1)^2}} \le \frac{4(y-1)^2}{\sqrt{(y-1)^2}} = 4|y - 1|$$

because we have made the denominator smaller.

(c) We have

$$f(x, y) = L(x, y) + e(x, y)$$

where

$$0 \le \frac{e(x, y)}{\sqrt{(x-2)^2 + (y-1)^2}} \le 4|y-1|$$

We have $\lim_{(x,y)\to(2,1)} 4|y-1| = 0$, and therefore

$$\lim_{(x,y)\to(2,1)} e(x, y) = 0$$

by the Squeeze Theorem. This proves that $f(x, y)$ is locally linear at (2, 1).

42. Show directly, as in Exercise 41, that $f(x, y) = xy^2$ is differentiable at (0, 2).

SOLUTION

(a) Firstly, we need to find $L(x, y)$. We know from the text that

$$L(x, y) = f(0, 2) + f_x(0, 2)(x - 0) + f_y(0, 2)(y - 2)$$

Computing with the function and the partial derivatives we see

$$\begin{aligned} f(x, y) = xy^2 &\quad\Rightarrow\quad f(0, 2) = 0 \\ f_x(x, y) = y^2 &\quad\Rightarrow\quad f_x(0, 2) = 4 \\ f_y(x, y) = 2xy &\quad\Rightarrow\quad f_y(0, 2) = 0 \end{aligned}$$

Therefore we have

$$L(x, y) = 0 + 4(x - 0) + 0(y - 2) = 4x$$

Hence, using the methods from the previous exercise we have

$$e(x, y) = f(x, y) - L(x, y) = xy^2 - 4x = x(y^2 - 4)$$

and

$$f(x, y) = L(x, y) + x(y^2 - 4)$$

(b) For $(x, y) \ne (0, 2)$, consider

$$\frac{x(y^2-4)}{\sqrt{x^2 + (y-2)^2}}$$

The following inequality holds for all x values:

$$\frac{x(y^2-4)}{\sqrt{x^2 + (y-2)^2}} \le \frac{|x|(y^2-4)}{\sqrt{x^2 + (y-2)^2}} \le \frac{|x|(y^2-4)}{\sqrt{x^2}} = \frac{|x|(y^2-4)}{|x|} = y^2 - 4$$

(c) Then we have

$$f(x, y) = L(x, y) + e(x, y)$$

where

$$0 \le \frac{e(x, y)}{\sqrt{x^2 + (y-2)^2}} \le y^2 - 4$$

and easily we know $\lim_{(x,y)\to(0,2)} (y^2 - 4) = 0$, and therefore

$$\lim_{(x,y)\to(0,2)} \frac{e(x, y)}{\sqrt{x^2 + (y-2)^2}} = 0$$

by the Squeeze Theorem. Therefore $\lim_{(x,y)\to(0,0)} e(x, y) = 0$ as well. This proves that $f(x, y)$ is locally linear at the point (0, 2), and therefore, differentiable at (0, 2).

43. Differentiability Implies Continuity Use the definition of differentiability to prove that if f is differentiable at (a, b), then f is continuous at (a, b).

SOLUTION Suppose that f is differentiable at (a, b), then we know f is locally linear at (a, b), that is

$$f(x, y) = L(x, y) + e(x, y)$$

where $e(x, y)$ satisfies

$$\lim_{(x,y)\to(a,b)} \frac{e(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = \lim_{(x,y)\to(a,b)} E(x, y) = 0$$

and

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We would like to show $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, then f would be continuous at (a, b). Consider the following computation:

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} f(x, y) &= \lim_{(x,y)\to(a,b)} L(x, y) + e(x, y) \\
&= \lim_{(x,y)\to(a,b)} L(x, y) + E(x, y)\sqrt{(x-a)^2 + (y-b)^2} \\
&= \lim_{(x,y)\to(a,b)} f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + E(x, y)\sqrt{(x-a)^2 + (y-b)^2} \\
&= f(a, b) + 0 + 0 + 0 = f(a, b)
\end{aligned}$$

Therefore we have shown that if f is differentiable at (a, b) then f is continuous at (a, b).

44. Let $f(x)$ be a function of one variable defined near $x = a$. Given a number M, set

$$L(x) = f(a) + M(x - a), \qquad e(x) = f(x) - L(x)$$

Thus $f(x) = L(x) + e(x)$. We say that f is locally linear at $x = a$ if M can be chosen so that $\lim_{x\to a} \dfrac{e(x)}{|x-a|} = 0$.

(a) Show that if $f(x)$ is differentiable at $x = a$, then $f(x)$ is locally linear with $M = f'(a)$.

(b) Show conversely that if f is locally linear at $x = a$, then $f(x)$ is differentiable and $M = f'(a)$.

SOLUTION

(a) Suppose that f is differentiable at $x = a$. From single-variable calculus we also know that f is continuous and that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Then also, using methods of linear approximation from single variable calculus, we can write

$$L(x) = f(a) + f'(a)(x - a) \text{ with } M = f'(a)$$

Now to fulfill local linearity we need to show $\lim_{x\to a} \frac{e(x)}{|x-a|} = 0$. Let us note here that $\lim_{x\to a} \frac{e(x)}{|x-a|} = 0$ if and only if $\lim_{x\to a} \frac{e(x)}{x-a} = 0$. It will be enough to show, $\lim_{x\to a} \frac{e(x)}{x-a} = 0$.

Consider the following:

$$\begin{aligned}
\lim_{x\to a} \frac{e(x)}{x-a} &= \lim_{x\to a} \frac{f(x) - L(x)}{x - a} \\
&= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - \lim_{x\to a} \frac{f'(a)(x - a)}{x - a} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - \lim_{x\to a} f'(a) \\
&= f'(a) - f'(a) = 0
\end{aligned}$$

Therefore, f is locally linear at $x = a$.

(b) Now suppose that f is locally linear at $x = a$. By definition

$$f(x) = f(a) + M(x-a) + e(x)$$

Therefore,

$$\frac{f(x) - f(a)}{x-a} = M + \frac{e(x)}{x-a}$$

If $f(x)$ is locally linear, then (by definition), $\frac{e(x)}{x-a}$ tends to zero and thus the difference quotient for $f(x)$ approaches M. Therefore, $f'(a)$ exists and equals M.

45. Assumptions Matter Define $g(x, y) = 2xy(x+y)/(x^2+y^2)$ for $(x, y) \neq 0$ and $g(0, 0) = 0$. In this exercise, we show that $g(x, y)$ is continuous at (0, 0) and that $g_x(0, 0)$ and $g_y(0, 0)$ exist, but $g(x, y)$ is not differentiable at (0, 0).

(a) Show using polar coordinates that $g(x, y)$ is continuous at (0, 0).

(b) Use the limit definitions to show that $g_x(0, 0)$ and $g_y(0, 0)$ exist and that both are equal to zero.

(c) Show that the linearization of $g(x, y)$ at (0, 0) is $L(x, y) = 0$.

(d) Show that if $g(x, y)$ were locally linear at (0, 0), we would have $\lim_{h\to 0} \frac{g(h,h)}{h} = 0$. Then observe that this is not the case because $g(h, h) = 2h$. This shows that $g(x, y)$ is not locally linear at (0, 0) and, hence, not differentiable at (0, 0).

SOLUTION

(a) We would like to show $\lim_{(x,y)\to(0,0)} g(x, y) = g(0, 0)$. Consider the following, using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2xy(x+y)}{x^2+y^2} &= \lim_{(r,\theta)\to(0,0)} \frac{2r^2\cos\theta\sin\theta(r\cos\theta + r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \lim_{(r,\theta)\to(0,0)} \frac{2r^3\cos\theta\sin\theta(\cos\theta+\sin\theta)}{r^2} \\
&= \lim_{(r,\theta)\to(0,0)} 2r\cos\theta\sin\theta(\cos\theta+\sin\theta) = 0 = g(0,0)
\end{aligned}$$

Therefore $g(x, y)$ is continuous at (0, 0).

(b) Taking partial derivatives we have:

$$g_x(x, y) = \frac{2y^2(y-x)^2}{(x^2+y^2)^2}, \quad g_y(x, y) = \frac{2x^2(x-y)^2}{(x^2+y^2)^2}$$

But we need to use limit definitions for the partial derivatives. Consider the following:

$$\begin{aligned}
g_x(0, 0) &= \lim_{h\to 0} \frac{g(h,0) - g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}(0-0) = 0
\end{aligned}$$

$$\begin{aligned}
g_y(0, 0) &= \lim_{h\to 0} \frac{g(0,h) - g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}(0-0) = 0
\end{aligned}$$

Thus both partial derivatives exist and $g_x(0, 0) = 0$ and $g_y(0, 0) = 0$.

(c) We know that the linearization of g will be:

$$g(x, y) \approx g(0, 0) + g_x(0, 0)(x-0) + g_y(0, 0)(y-0)$$

We are given that $g(0, 0) = 0$. In part (b) we know $g_x(0, 0) = 0$ and $g_y(0, 0) = 0$. Substituting in these values in the linearization we have:

$$g(x, y) \approx 0 + 0 + 0 = 0$$

(d) We know if g were locally linear at (0, 0), we would have:

$$\lim_{h\to 0} \frac{g(h,h)}{h} = 0$$

However, we know:

$$g(h, h) = \frac{2h^2(2h)}{2h^2} = 2h, \quad \frac{g(h,h)}{h} = \frac{2h}{h} = 2$$

This is a contradiction, $g(x, y)$ is not locally linear at (0, 0) and hence, is not differentiable at (0, 0).

14.5 The Gradient and Directional Derivatives (LT Section 15.5)

Preliminary Questions

1. Which of the following is a possible value of the gradient ∇f of a function $f(x, y)$ of two variables?

(a) 5 **(b)** $\langle 3, 4\rangle$ **(c)** $\langle 3, 4, 5\rangle$

SOLUTION The gradient of $f(x, y)$ is a vector with two components, hence the possible value of the gradient $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$ is (b).

2. True or false? A differentiable function increases at the rate $\|\nabla f_P\|$ in the direction of ∇f_P.

SOLUTION The statement is true. The value $\|\nabla f_P\|$ is the rate of increase of f in the direction ∇f_P.

3. Describe the two main geometric properties of the gradient ∇f.

SOLUTION The gradient of f points in the direction of maximum rate of increase of f and is normal to the level curve (or surface) of f.

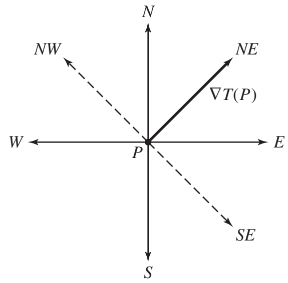
4. You are standing at a point where the temperature gradient vector is pointing in the northeast (NE) direction. In which direction(s) should you walk to avoid a change in temperature?

(a) NE **(b)** NW **(c)** SE **(d)** SW

SOLUTION The rate of change of the temperature T at a point P in the direction of a unit vector $\mathbf{u}$, is the directional derivative $D_{\mathbf{u}}T(P)$, which is given by the formula

$$D_{\mathbf{u}}T(P) = \|\nabla f_P\| \cos\theta$$

To avoid a change in temperature, we must choose the direction $\mathbf{u}$ so that $D_{\mathbf{u}}T(P) = 0$, that is, $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. Since the gradient at P is pointing NE, we should walk NW or SE to avoid a change in temperature. Thus, the answer is (b) and (c).



5. What is the rate of change of $f(x, y)$ at $(0, 0)$ in the direction making an angle of 45° with the x-axis if $\nabla f(0, 0) = \langle 2, 4\rangle$?

SOLUTION By the formula for directional derivatives, and using the unit vector $\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$, we get $\langle 2, 4\rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = 6/\sqrt{2} = 3\sqrt{2}$.

Exercises

1. Let $f(x, y) = xy^2$ and $\mathbf{c}(t) = \left(\frac{1}{2}t^2, t^3\right)$.

(a) Calculate ∇f and $\mathbf{c}'(t)$.

(b) Use the Chain Rule for Paths to evaluate $\frac{d}{dt} f(\mathbf{c}(t))$ at $t = 1$ and $t = -1$.

SOLUTION

(a) We compute the partial derivatives of $f(x, y) = xy^2$:

$$\frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = 2xy$$

The gradient vector is thus

$$\nabla f = \left\langle y^2, 2xy\right\rangle.$$

Also,

$$\mathbf{c}'(t) = \left\langle \left(\frac{1}{2}t^2\right)', \left(t^3\right)'\right\rangle = \left\langle t, 3t^2\right\rangle$$

(b) Using the Chain Rule gives

$$\frac{d}{dt} f(\mathbf{c}(t)) = \frac{d}{dt}\left(\frac{1}{2}t^2 \cdot t^6\right) = \frac{d}{dt}\left(\frac{1}{2}t^8\right) = 4t^7$$

Substituting $x = \frac{1}{2}t^2$ and $y = t^3$, we obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) = t^6 \cdot t + 2 \cdot \frac{1}{2}t^2 \cdot 3 \cdot t^3 \cdot t^2 = 4t^7$$

At the point $t = 1$ and $t = -1$, we get

$$\frac{d}{dt}(f(\mathbf{c}(t)))\bigg|_{t=1} = 4 \cdot 1^7 = 4, \quad \frac{d}{dt}(f(\mathbf{c}(t)))\bigg|_{t=-1} = 4 \cdot (-1)^7 = -4.$$

2. Let $f(x, y) = e^{xy}$ and $\mathbf{c}(t) = (t^3, 1 + t)$.

(a) Calculate ∇f and $\mathbf{c}'(t)$.

(b) Use the Chain Rule for Paths to calculate $\frac{d}{dt} f(\mathbf{c}(t))$.

(c) Write out the composite $f(\mathbf{c}(t))$ as a function of t and differentiate. Check that the result agrees with part (b).

SOLUTION

(a) We first find the partial derivatives of $f(x, y) = e^{xy}$:

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy}.$$

The gradient vector is thus

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle ye^{xy}, xe^{xy} \right\rangle$$

Differentiating $\mathbf{c}(t) = (t^3, 1 + t)$ componentwise, we obtain

$$\mathbf{c}'(t) = \left((t^3)', (1+t)'\right) = (3t^2, 1)$$

(b) We find $\frac{d}{dt} f(\mathbf{c}(t))$ using the Chain Rule and the results of part (a). This gives

$$\frac{d}{dt} f(\mathbf{c}(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (ye^{xy}) \cdot 3t^2 + (xe^{xy}) \cdot 1$$

To write the answer in terms of t only, we substitute $x = t^3$ and $y = 1 + t$. This gives

$$\frac{d}{dt} f(\mathbf{c}(t)) = (1+t)e^{t^3+t^4} \cdot 3t^2 + (t^3)e^{t^3+t^4} = (3t^2 + 4t^3)e^{t^3+t^4}$$

(c) We substitute $x = t^3$, $y = 1 + t$ in $f(x, y) = e^{xy}$ to obtain the composite function $f(\mathbf{c}(t))$:

$$f(\mathbf{c}(t)) = e^{t^3+t^4}$$

We now differentiate the composite function to obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) = \frac{d}{dt}\left(e^{t^3+t^4}\right) = (3t^2 + 4t^3)e^{t^3+t^4}$$

This result agrees with the result obtained in part (a).

3. Figure 14 shows the level curves of a function $f(x, y)$ and a path $\mathbf{c}(t)$, traversed in the direction indicated. State whether the derivative $\frac{d}{dt} f(\mathbf{c}(t))$ is positive, negative, or zero at points A–D.

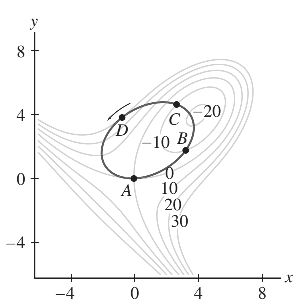


FIGURE 14

SOLUTION At points A and D, the path is (temporarily) tangent to one of the contour lines, which means that along the path $\mathbf{c}(t)$ the function $f(x, y)$ is (temporarily) constant, and so the derivative $\frac{d}{dt} f(\mathbf{c}(t))$ is zero. At point B, the path is moving from a higher contour (of -10) to a lower one (of -20), so the derivative is negative. At the point C, where the path moves from the contour of -10 towards the contour of value 0, the derivative is positive.

4. Let $f(x, y) = x^2 + y^2$ and $\mathbf{c}(t) = (\cos t, \sin t)$.

(a) Find $\dfrac{d}{dt} f(\mathbf{c}(t))$ without making any calculations. Explain.

(b) Verify your answer to (a) using the Chain Rule.

SOLUTION

(a) The level curves of $f(x, y)$ are the circles $x^2 + y^2 = \mathbf{c}^2$. Since $\mathbf{c}(t)$ is a parametrization of the unit circle, f has constant value 1 on $\mathbf{c}$. That is, $f(\mathbf{c}(t)) = 1$, which implies that $\frac{d}{dt} f(\mathbf{c}(t)) = 0$.

(b) We now find $\frac{d}{dt} f(\mathbf{c}(t))$ using the Chain Rule:

$$\frac{d}{dt} f(\mathbf{c}(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{1}$$

We compute the derivatives involved in (1):

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x, \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y$$
$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t, \quad \frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

Substituting the derivatives in (1) gives

$$\frac{d}{dt} f(\mathbf{c}(t)) = 2x(-\sin t) + 2y\cos t$$

Finally, we substitute $x = \cos t$ and $y = \sin t$ to obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) = -2\cos t\sin t + 2\sin t\cos t = 0.$$

In Exercises 5–8, calculate the gradient.

5. $f(x, y) = \cos(x^2 + y)$

SOLUTION We find the partial derivatives using the Chain Rule:

$$\frac{\partial f}{\partial x} = -\sin\left(x^2 + y\right)\frac{\partial}{\partial x}\left(x^2 + y\right) = -2x\sin\left(x^2 + y\right)$$
$$\frac{\partial f}{\partial y} = -\sin\left(x^2 + y\right)\frac{\partial}{\partial y}\left(x^2 + y\right) = -\sin\left(x^2 + y\right)$$

The gradient vector is thus

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle -2x\sin\left(x^2 + y\right), -\sin\left(x^2 + y\right) \right\rangle = -\sin\left(x^2 + y\right)\langle 2x, 1\rangle$$

6. $g(x, y) = \dfrac{x}{x^2 + y^2}$

SOLUTION We compute the partial derivatives. We first find $\frac{\partial g}{\partial x}$ using the Quotient Rule:

$$\frac{\partial g}{\partial x} = \frac{1 \cdot \left(x^2 + y^2\right) - x \cdot 2x}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

We compute $\frac{\partial g}{\partial y}$ using the Chain Rule:

$$\frac{\partial g}{\partial y} = x\frac{\partial}{\partial y}\frac{1}{x^2 + y^2} = x \cdot \frac{-1}{\left(x^2 + y^2\right)^2} \cdot 2y = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

The gradient vector is thus

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle = \left\langle \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}, \frac{-2xy}{\left(x^2 + y^2\right)^2} \right\rangle = \frac{1}{\left(x^2 + y^2\right)^2}\left\langle y^2 - x^2, -2xy \right\rangle.$$

7. $h(x, y, z) = xyz^{-3}$

SOLUTION We compute the partial derivatives of $h(x, y, z) = xyz^{-3}$, obtaining

$$\frac{\partial h}{\partial x} = yz^{-3}, \quad \frac{\partial h}{\partial y} = xz^{-3}, \quad \frac{\partial h}{\partial z} = xy \cdot \left(-3z^{-4}\right) = -3xyz^{-4}$$

The gradient vector is thus

$$\nabla h = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right\rangle = \left\langle yz^{-3}, xz^{-3}, -3xyz^{-4} \right\rangle.$$

8. $r(x, y, z, w) = xze^{yw}$

SOLUTION We find the partial derivatives of $r(x, y, z, w) = xze^{yw}$:

$$\frac{\partial r}{\partial x} = ze^{yw}, \quad \frac{\partial r}{\partial y} = xzwe^{yw}, \quad \frac{\partial r}{\partial z} = xe^{yw}, \quad \frac{\partial r}{\partial w} = xzye^{yw}$$

The gradient vector is thus

$$\nabla r = \left\langle \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}, \frac{\partial r}{\partial w} \right\rangle = \left\langle ze^{yw}, xzwe^{yw}, xe^{yw}, xzye^{yw} \right\rangle = e^{yw} \left\langle z, xzw, x, xzy \right\rangle$$

In Exercises 9–20, use the Chain Rule to calculate $\frac{d}{dt} f(\mathbf{c}(t))$.

9. $f(x, y) = 3x - 7y, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad t = 0$

SOLUTION By the Chain Rule for paths, we have

$$\frac{d}{dt} f\,(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient and the derivative $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 3, -7 \rangle, \quad \mathbf{c}'(t) = \langle -\sin t, \cos t \rangle$$

We determine these vectors at $t = 0$:

$$\mathbf{c}'(0) = \langle -\sin 0, \cos 0 \rangle = \langle 0, 1 \rangle$$

and since the gradient is a constant vector, we have

$$\nabla f_{\mathbf{c}(0)} = \nabla f_{(1,0)} = \langle 3, -7 \rangle$$

Substituting these vectors in (1) gives

$$\left. \frac{d}{dt} f\,(\mathbf{c}(t)) \right|_{t=0} = \langle 3, -7 \rangle \cdot \langle 0, 1 \rangle = 0 - 7 = -7$$

10. $f(x, y) = 3x - 7y, \quad \mathbf{c}(t) = (t^2, t^3), \quad t = 2$

SOLUTION We first compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 3, -7 \rangle, \quad \mathbf{c}'(t) = \left(2t, 3t^2\right)$$

At the point $t = 2$ we have

$$\nabla f_{\mathbf{c}(2)} = \langle 3, -7 \rangle, \quad \mathbf{c}'(2) = \langle 4, 12 \rangle$$

We now use the Chain Rule for paths to compute the following derivative:

$$\left. \frac{d}{dt} f\,(\mathbf{c}(t)) \right|_{t=2} = \nabla f_{\mathbf{c}(2)} \cdot \mathbf{c}'(2) = \langle 3, -7 \rangle \cdot \langle 4, 12 \rangle = -72$$

11. $f(x, y) = x^2 - 3xy, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad t = 0$

SOLUTION By the Chain Rule For Paths we have

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x - 3y, -3x \rangle$$
$$\mathbf{c}'(t) = \langle -\sin t, \cos t \rangle$$

At the point $t = 0$ we have

$$\mathbf{c}(0) = (\cos 0, \sin 0) = (1, 0)$$
$$\mathbf{c}'(0) = \langle -\sin 0, \cos 0 \rangle = \langle 0, 1 \rangle$$
$$\nabla f \bigg|_{\mathbf{c}(0)} = \nabla f_{(1,0)} = \langle 2 \cdot 1 - 3 \cdot 0, -3 \cdot 1 \rangle = \langle 2, -3 \rangle$$

Substituting in (1) we obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) \bigg|_{t=0} = \langle 2, -3 \rangle \cdot \langle 0, 1 \rangle = -3$$

12. $f(x, y) = x^2 - 3xy, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad t = \frac{\pi}{2}$

SOLUTION In the previous exercise we found that

$$\nabla f = \langle 2x - 3y, -3x \rangle, \quad \mathbf{c}'(t) = \langle -\sin t, \cos t \rangle$$

At the point $t = \frac{\pi}{2}$ we have

$$\mathbf{c}\left(\frac{\pi}{2}\right) = \left(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}\right) = (0, 1)$$
$$\mathbf{c}'\left(\frac{\pi}{2}\right) = \left\langle -\sin \frac{\pi}{2}, \cos \frac{\pi}{2} \right\rangle = \langle -1, 0 \rangle$$
$$\nabla f_{\mathbf{c}\left(\frac{\pi}{2}\right)} = \nabla f_{(0,1)} = \langle 2 \cdot 0 - 3 \cdot 1, -3 \cdot 0 \rangle = \langle -3, 0 \rangle$$

We now use the Chain Rule for Paths to obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) \bigg|_{t=\frac{\pi}{2}} = \nabla f_{\mathbf{c}\left(\frac{\pi}{2}\right)} \cdot \mathbf{c}'\left(\frac{\pi}{2}\right) = \langle -3, 0 \rangle \cdot \langle -1, 0 \rangle = 3 + 0 = 3$$

13. $f(x, y) = \sin(xy), \quad \mathbf{c}(t) = (e^{2t}, e^{3t}), \quad t = 0$

SOLUTION By the Chain Rule for Paths we have

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y \cos(xy), x \cos(xy) \rangle$$
$$\mathbf{c}'(t) = \left\langle 2e^{2t}, 3e^{3t} \right\rangle$$

At the point $t = 0$ we have

$$\mathbf{c}(0) = \left(e^0, e^0\right) = (1, 1)$$
$$\mathbf{c}'(0) = \left\langle 2e^0, 3e^0 \right\rangle = \langle 2, 3 \rangle$$
$$\nabla f_{\mathbf{c}(0)} = \nabla f_{(1,1)} = \langle \cos 1, \cos 1 \rangle$$

Substituting the vectors in (1) we get

$$\frac{d}{dt} f(\mathbf{c}(t)) \bigg|_{t=0} = \langle \cos 1, \cos 1 \rangle \cdot \langle 2, 3 \rangle = 5 \cos 1$$

14. $f(x, y) = \cos(y - x)$, $\quad \mathbf{c}(t) = (e^t, e^{2t})$, $\quad t = \ln 3$

SOLUTION By the Chain Rule for Paths we have

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 2xy^3, 3x^2y^2 \right\rangle$$
$$\mathbf{c}'(t) = \left\langle e^t, 2e^{2t} \right\rangle$$

At the point $t = \ln 3$ we have

$$\mathbf{c}(\ln 3) = \left(e^{\ln 3}, e^{2\ln 3}\right) = \left(3, 3^2\right) = (3, 9)$$
$$\mathbf{c}'(\ln 3) = \left\langle e^{\ln 3}, 2e^{2\ln 3}\right\rangle = \left\langle 3, 2 \cdot 3^2\right\rangle = \langle 3, 18\rangle$$
$$\nabla f_{\mathbf{c}(\ln 3)} = \nabla f_{(3,9)} = \left\langle 2 \cdot 3 \cdot 9^3, 3 \cdot 3^2 \cdot 9^2\right\rangle = 2187\,\langle 2, 1\rangle$$

Substituting the vectors in (1) we obtain

$$\left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=\ln 3} = 2187\,\langle 2, 1\rangle \cdot \langle 3, 18\rangle = 52{,}488$$

15. $f(x, y) = x - xy$, $\quad \mathbf{c}(t) = (t^2, t^2 - 4t)$, $\quad t = 4$

SOLUTION We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 1 - y, -x\rangle$$
$$\mathbf{c}'(t) = (2t, 2t - 4)$$

At the point $t = 4$ we have

$$\mathbf{c}(4) = \left(4^2, 4^2 - 4 \cdot 4\right) = (16, 0)$$
$$\mathbf{c}'(4) = \langle 2 \cdot 4, 2 \cdot 4 - 4\rangle = \langle 8, 4\rangle$$
$$\nabla f_{\mathbf{c}(4)} = \nabla f_{(16,0)} = \langle 1 - 0, -16\rangle = \langle 1, -16\rangle$$

We now use the Chain Rule for Paths to compute the following derivative:

$$\left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=4} = \nabla f_{\mathbf{c}(4)} \cdot \mathbf{c}'(4) = \langle 1, -16\rangle \cdot \langle 8, 4\rangle = 8 - 64 = -56$$

16. $f(x, y) = xe^y$, $\quad \mathbf{c}(t) = (t^2, t^2 - 4t)$, $\quad t = 0$

SOLUTION We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle e^y, xe^y\right\rangle = e^y\,\langle 1, x\rangle$$
$$\mathbf{c}'(t) = \langle 2t, 2t - 4\rangle$$

At the point $t = 0$ we have

$$\mathbf{c}(0) = (0, 0)$$
$$\mathbf{c}'(0) = \langle 0, -4\rangle$$
$$\nabla f_{\mathbf{c}(0)} = \nabla f_{(0,0)} = e^0\,\langle 1, 0\rangle = \langle 1, 0\rangle$$

Using the Chain Rule for Paths we obtain the following derivative:

$$\left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=0} = \nabla f_{\mathbf{c}(0)} \cdot \mathbf{c}'(0) = \langle 1, 0\rangle \cdot \langle 0, -4\rangle = 0$$

17. $f(x, y) = \ln x + \ln y, \quad \mathbf{c}(t) = (\cos t, t^2), \quad t = \frac{\pi}{4}$

SOLUTION We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$$
$$\mathbf{c}'(t) = \langle -\sin t, 2t \rangle$$

At the point $t = \frac{\pi}{4}$ we have

$$\mathbf{c}\left(\frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}, \left(\frac{\pi}{4}\right)^2\right) = \left(\frac{\sqrt{2}}{2}, \frac{\pi^2}{16}\right)$$
$$\mathbf{c}'\left(\frac{\pi}{4}\right) = \left\langle -\sin\frac{\pi}{4}, \frac{2\pi}{4} \right\rangle = \left\langle -\frac{\sqrt{2}}{2}, \frac{\pi}{2} \right\rangle$$
$$\nabla f_{\mathbf{c}\left(\frac{\pi}{4}\right)} = \nabla f_{\left(\frac{\sqrt{2}}{2}, \frac{\pi^2}{16}\right)} = \left\langle \sqrt{2}, \frac{16}{\pi^2} \right\rangle$$

Using the Chain Rule for Paths we obtain the following derivative:

$$\frac{d}{dt} f(\mathbf{c}(t))\Big|_{t=\frac{\pi}{4}} = \nabla f_{\mathbf{c}\left(\frac{\pi}{4}\right)} \cdot \mathbf{c}'\left(\frac{\pi}{4}\right) = \left\langle \sqrt{2}, \frac{16}{\pi^2} \right\rangle \cdot \left\langle -\frac{\sqrt{2}}{2}, \frac{\pi}{2} \right\rangle = -1 + \frac{8}{\pi} \approx 1.546$$

18. $g(x, y, z) = xye^z, \quad \mathbf{c}(t) = (t^2, t^3, t-1), \quad t = 1$

SOLUTION We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \left\langle ye^z, xe^z, xye^z \right\rangle = e^z \langle y, x, xy \rangle$$
$$\mathbf{c}'(t) = \left\langle 2t, 3t^2, 1 \right\rangle$$

At the point $t = 1$ we have

$$\mathbf{c}(1) = (1, 1, 0)$$
$$\mathbf{c}'(1) = \langle 2, 3, 1 \rangle$$
$$\nabla g_{\mathbf{c}(1)} = \nabla g_{(1,1,0)} = e^0 \langle 1, 1, 1 \rangle = \langle 1, 1, 1 \rangle$$

Using the Chain Rule for Paths we obtain the following derivative:

$$\frac{d}{dt} g(\mathbf{c}(t))\Big|_{t=1} = \nabla g_{\mathbf{c}(1)} \cdot \mathbf{c}'(1) = \langle 1, 1, 1 \rangle \cdot \langle 2, 3, 1 \rangle = 2 + 3 + 1 = 6$$

19. $g(x, y, z) = xyz^{-1}, \quad \mathbf{c}(t) = (e^t, t, t^2), \quad t = 1$

SOLUTION By the Chain Rule for Paths we have

$$\frac{d}{dt} g(\mathbf{c}(t)) = \nabla g_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \left\langle yz^{-1}, xz^{-1}, -xyz^{-2} \right\rangle$$
$$\mathbf{c}'(t) = \left\langle e^t, 1, 2t \right\rangle$$

At the point $t = 1$ we have

$$\mathbf{c}(1) = (e, 1, 1)$$
$$\mathbf{c}'(1) = \langle e, 1, 2 \rangle$$
$$\nabla g_{\mathbf{c}(1)} = \nabla g_{(e,1,1)} = \langle 1, e, -e \rangle$$

Substituting the vectors in (1) gives the following derivative:

$$\frac{d}{dt} g(\mathbf{c}(t))\Big|_{t=1} = \langle 1, e, -e \rangle \cdot \langle e, 1, 2 \rangle = e + e - 2e = 0$$

20. $g(x, y, z, w) = x + 2y + 3z + 5w, \quad \mathbf{c}(t) = (t^2, t^3, t, t-2), \quad t = 1$

SOLUTION We compute the gradient and $\mathbf{c}'(t)$:

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}, \frac{\partial g}{\partial w} \right\rangle = \langle 1, 2, 3, 5 \rangle$$
$$\mathbf{c}'(t) = \left\langle 2t, 3t^2, 1, 1 \right\rangle$$

At the point $t = 1$ we have (notice that the gradient is a constant vector)

$$\nabla g_{\mathbf{c}(1)} = \langle 1, 2, 3, 5 \rangle$$
$$\mathbf{c}'(1) = \langle 2, 3, 1, 1 \rangle$$

We now use the Chain Rule for Paths to obtain the following derivative:

$$\frac{d}{dt} g\left(\mathbf{c}(t)\right)\bigg|_{t=1} = \nabla g_{\mathbf{c}(1)} \cdot \mathbf{c}'(1) = \langle 1, 2, 3, 5 \rangle \cdot \langle 2, 3, 1, 1 \rangle = 2 + 6 + 3 + 5 = 16$$

In Exercises 21–30, calculate the directional derivative in the direction of $\mathbf{v}$ *at the given point. Remember to normalize the direction vector or use Eq. (4).*

21. $f(x, y) = x^2 + y^3, \quad \mathbf{v} = \langle 4, 3 \rangle, \quad P = (1, 2)$

SOLUTION We first normalize the direction vector $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 4, 3 \rangle}{\sqrt{4^2 + 3^2}} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

We compute the gradient of $f(x, y) = x^2 + y^3$ at the given point:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 2x, 3y^2 \right\rangle \quad \Rightarrow \quad \nabla f_{(1,2)} = \langle 2, 12 \rangle$$

Using the Theorem on Evaluating Directional Derivatives, we get

$$D_{\mathbf{u}} f(1, 2) = \nabla f_{(1,2)} \cdot \mathbf{u} = \langle 2, 12 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle = \frac{8}{5} + \frac{36}{5} = \frac{44}{5} = 8.8$$

22. $f(x, y) = x^2y^3, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}, \quad P = (-2, 1)$

SOLUTION We normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

We compute the gradient of $f(x, y) = x^2y^3$ at the point P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 2xy^3, 3x^2y^2 \right\rangle \quad \Rightarrow \quad \nabla f_{(-2,1)} = \langle -4, 12 \rangle = -4\mathbf{i} + 12\mathbf{j}$$

The directional derivative in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}} f(-2, 1) = \nabla f_{(-2,1)} \cdot \mathbf{u} = (-4\mathbf{i} + 12\mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{4}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

23. $f(x, y) = x^2y^3, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}, \quad P = \left(\frac{1}{6}, 3\right)$

SOLUTION We normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

We compute the gradient of $f(x, y) = x^2y^3$ at the point P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 2xy^3, 3x^2y^2 \right\rangle \quad \Rightarrow \quad \nabla f_{\left(\frac{1}{6},3\right)} = \left\langle 2 \cdot \frac{1}{6} \cdot 3^3, 3 \cdot \frac{1}{6^2} \cdot 3^2 \right\rangle = \left\langle 9, \frac{3}{4} \right\rangle = 9\mathbf{i} + \frac{3}{4}\mathbf{j}$$

The directional derivative in the direction $\mathbf{v}$ is thus

$$D_{\mathbf{u}} f\left(\frac{1}{6}, 3\right) = \nabla f_{\left(\frac{1}{6},3\right)} \cdot \mathbf{u} = \left(9\mathbf{i} + \frac{3}{4}\mathbf{j}\right) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = \frac{9}{\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{39}{4\sqrt{2}}$$

24. $f(x, y) = \sin(x - y)$, $\mathbf{v} = \langle 1, 1\rangle$, $P = \left(\frac{\pi}{2}, \frac{\pi}{6}\right)$

SOLUTION We normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$$

We compute the gradient of $f(x, y) = \sin(x - y)$ at the point P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \langle \cos(x - y), -\cos(x - y)\rangle \quad \Rightarrow \quad \nabla f_{\left(\frac{\pi}{2}, \frac{\pi}{6}\right)} = \left\langle \cos\frac{\pi}{3}, -\cos\frac{\pi}{3}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

The directional derivative in the direction $\mathbf{v}$ is thus

$$D_{\mathbf{u}} f(P) = \nabla f_{\left(\frac{\pi}{2}, \frac{\pi}{6}\right)} \cdot \mathbf{u} = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, 1\rangle = 0$$

25. $f(x, y) = \tan^{-1}(xy)$, $\mathbf{v} = \langle 1, 1\rangle$, $P = (3, 4)$

SOLUTION We first normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$$

We compute the gradient of $f(x, y) = \tan^{-1}(xy)$ at the point $P = (3, 4)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \left\langle \frac{y}{1 + (xy)^2}, \frac{x}{1 + (xy)^2}\right\rangle = \frac{1}{1 + x^2y^2}\langle y, x\rangle$$

$$\nabla f_{(3,4)} = \frac{1}{1 + 3^2 \cdot 4^2}\langle 4, 3\rangle = \frac{1}{145}\langle 4, 3\rangle$$

Therefore, the directional derivative in the direction $\mathbf{v}$ is

$$D_{\mathbf{u}} f(3, 4) = \nabla f_{(3,4)} \cdot \mathbf{u} = \frac{1}{145}\langle 4, 3\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, 1\rangle = \frac{1}{145\sqrt{2}}(4 + 3) = \frac{7}{145\sqrt{2}} = \frac{7\sqrt{2}}{290}$$

26. $f(x, y) = e^{xy - y^2}$, $\mathbf{v} = \langle 12, -5\rangle$, $P = (2, 2)$

SOLUTION We first normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 12, -5\rangle}{\sqrt{12^2 + (-5)^2}} = \frac{1}{13}\langle 12, -5\rangle$$

We compute the gradient of $f(x, y) = e^{xy - y^2}$ at the point $P = (2, 2)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \left\langle ye^{xy - y^2}, (x - 2y)e^{xy - y^2}\right\rangle = e^{xy - y^2}\langle y, x - 2y\rangle$$

$$\nabla f_{(2,2)} = e^0\langle 2, -2\rangle = \langle 2, -2\rangle$$

Therefore, the directional derivative in the direction $\mathbf{v}$ is thus

$$D_{\mathbf{u}} f(2, 2) = \nabla f_{(2,2)} \cdot \mathbf{u} = \langle 2, -2\rangle \cdot \frac{1}{13}\langle 12, -5\rangle = \frac{34}{13}$$

27. $f(x, y) = \ln(x^2 + y^2)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$, $P = (1, 0)$

SOLUTION We normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3^2 + (-2)^2}}(3\mathbf{i} - 2\mathbf{j}) = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

We compute the gradient of $f(x, y) = \ln\left(x^2 + y^2\right)$ at the point $P = (1, 0)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}\right\rangle = \frac{2}{x^2 + y^2}\langle x, y\rangle$$

$$\nabla f_{(1,0)} = \frac{2}{1^2 + 0^2}\langle 1, 0\rangle = \langle 2, 0\rangle = 2\mathbf{i}$$

The directional derivative in the direction **v** is thus

$$D_{\mathbf{u}}f(1,0) = \nabla f_{(1,0)} \cdot \mathbf{u} = 2\mathbf{i} \cdot \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j}) = \frac{6}{\sqrt{13}}$$

28. $g(x, y, z) = z^2 - xy^2, \quad \mathbf{v} = \langle -1, 2, 2 \rangle, \quad P = (2, 1, 3)$

SOLUTION We normalize **v** to obtain a unit vector **u** in the direction **v**:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -1, 2, 2 \rangle}{\sqrt{(-1)^2 + 2^2 + 2^2}} = \frac{1}{3}\langle -1, 2, 2 \rangle$$

We compute the gradient of $f(x, y, z) = z^2 - xy^2$ at the point $P = (2, 1, 3)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle -y^2, -2xy, 2z \right\rangle \quad \Rightarrow \quad \nabla f_{(2,1,3)} = \langle -1, -4, 6 \rangle$$

The directional derivative in the direction **v** is thus

$$D_{\mathbf{u}}f(2,1,3) = \nabla f_{(2,1,3)} \cdot \mathbf{u} = \langle -1, -4, 6 \rangle \cdot \frac{1}{3}\langle -1, 2, 2 \rangle = \frac{1}{3}(1 - 8 + 12) = \frac{5}{3}$$

29. $g(x, y, z) = xe^{-yz}, \quad \mathbf{v} = \langle 1, 1, 1 \rangle, \quad P = (1, 2, 0)$

SOLUTION We first compute a unit vector **u** in the direction **v**:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$$

We find the gradient of $f(x, y, z) = xe^{-yz}$ at the point $P = (1, 2, 0)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle e^{-yz}, -xze^{-yz}, -xye^{-yz} \right\rangle = e^{-yz}\langle 1, -xz, -xy \rangle$$

$$\nabla f_{(1,2,0)} = e^0 \langle 1, 0, -2 \rangle = \langle 1, 0, -2 \rangle$$

The directional derivative in the direction **v** is thus

$$D_{\mathbf{u}}f(1,2,0) = \nabla f_{(1,2,0)} \cdot \mathbf{u} = \langle 1, 0, -2 \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle = \frac{1}{\sqrt{3}}(1 + 0 - 2) = -\frac{1}{\sqrt{3}}$$

30. $g(x, y, z) = x\ln(y + z), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad P = (2, e, e)$

SOLUTION We first find a unit vector **u** in the direction **v**:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

We compute the gradient of $f(x, y, z) = x\ln(y + z)$ at the point $P = (2, e, e)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle \ln(y+z), \frac{x}{y+z}, \frac{x}{y+z} \right\rangle$$

$$\nabla f_{(2,e,e)} = \left\langle \ln 2e, \frac{2}{2e}, \frac{2}{2e} \right\rangle = \left\langle \ln 2e, e^{-1}, e^{-1} \right\rangle = (\ln 2e)\mathbf{i} + e^{-1}\mathbf{j} + e^{-1}\mathbf{k}$$

The directional derivative in the direction **v** is thus

$$D_{\mathbf{u}}f(2,e,e) = \nabla f_{(2,e,e)} \cdot \mathbf{u} = \left((\ln 2e)\mathbf{i} + e^{-1}\mathbf{j} + e^{-1}\mathbf{k}\right) \cdot \frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$= \frac{1}{\sqrt{6}}\left(2\ln(2e) - e^{-1} + e^{-1}\right) = \frac{2\ln 2e}{\sqrt{6}}$$

31. Find the directional derivative of $f(x, y) = x^2 + 4y^2$ at $P = (3, 2)$ in the direction pointing to the origin.

SOLUTION The direction vector is $\mathbf{v} = \overrightarrow{PO} = \langle -3, -2 \rangle$. A unit vector **u** in the direction **v** is obtained by normalizing **v**. That is,

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -3, -2 \rangle}{\sqrt{3^2 + 2^2}} = \frac{-1}{\sqrt{13}}\langle 3, 2 \rangle$$

We compute the gradient of $f(x, y) = x^2 + 4y^2$ at the point $P = (3, 2)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 8y \rangle \quad \Rightarrow \quad \nabla f_{(3,2)} = \langle 6, 16 \rangle$$

The directional derivative is thus

$$D_{\mathbf{u}} f(3, 2) = \nabla f_{(3,2)} \cdot \mathbf{u} = \langle 6, 16 \rangle \cdot \frac{-1}{\sqrt{13}} \langle 3, 2 \rangle = \frac{-50}{\sqrt{13}}$$

32. Find the directional derivative of $f(x, y, z) = xy + z^3$ at $P = (3, -2, -1)$ in the direction pointing to the origin.

SOLUTION The direction vector is $\mathbf{v} = \overrightarrow{PO} = \langle -3, 2, 1 \rangle$. We normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -3, 2, 1 \rangle}{\sqrt{9 + 4 + 1}} = \frac{1}{\sqrt{14}} \langle -3, 2, 1 \rangle$$

We compute the gradient of $f(x, y, z) = xy + z^3$ at P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle y, x, 3z^2 \right\rangle \quad \Rightarrow \quad \nabla f_{(3,-2,-1)} = \langle -2, 3, 3 \rangle$$

The directional derivative is thus

$$D_{\mathbf{u}} f_{(3,-2,-1)} = \nabla f_{(3,-2,-1)} \cdot \mathbf{u} = \langle -2, 3, 3 \rangle \cdot \frac{1}{\sqrt{14}} \langle -3, 2, 1 \rangle = \frac{1}{\sqrt{14}}(6 + 6 + 3) = \frac{15}{\sqrt{14}}$$

33. A bug located at $(3, 9, 4)$ begins walking in a straight line toward $(5, 7, 3)$. At what rate is the bug's temperature changing if the temperature is $T(x, y, z) = xe^{y-z}$? Units are in meters and degrees Celsius.

SOLUTION The bug is walking in a straight line from the point $P = (3, 9, 4)$ towards $Q = (5, 7, 3)$, hence the rate of change in the temperature is the directional derivative in the direction of $\mathbf{v} = \overrightarrow{PQ}$. We first normalize $\mathbf{v}$ to obtain

$$\mathbf{v} = \overrightarrow{PQ} = \langle 5 - 3, 7 - 9, 3 - 4 \rangle = \langle 2, -2, -1 \rangle$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, -2, -1 \rangle}{\sqrt{4 + 4 + 1}} = \frac{1}{3} \langle 2, -2, -1 \rangle$$

We compute the gradient of $T(x, y, z) = xe^{y-z}$ at $P = (3, 9, 4)$:

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \left\langle e^{y-z}, xe^{y-z}, -xe^{y-z} \right\rangle = e^{y-z} \langle 1, x, -x \rangle$$

$$\nabla T_{(3,9,4)} = e^{9-4} \langle 1, 3, -3 \rangle = e^5 \langle 1, 3, -3 \rangle$$

The rate of change of the bug's temperature at the starting point P is the directional derivative

$$D_{\mathbf{u}} f(P) = \nabla T_{(3,9,4)} \cdot \mathbf{u} = e^5 \langle 1, 3, -3 \rangle \cdot \frac{1}{3} \langle 2, -2, -1 \rangle = -\frac{e^5}{3} \approx -49.47$$

The answer is -49.47 degrees Celsius per meter.

34. The temperature at location (x, y) is $T(x, y) = 20 + 0.1(x^2 - xy)$ (degrees Celsius). Beginning at $(200, 0)$ at time $t = 0$ (seconds), a bug travels along a circle of radius 200 cm centered at the origin, at a speed of 3 cm/s. How fast is the temperature changing at time $t = \pi/3$?

SOLUTION First we should parametrize the circle the bug is walking along as:

$$\mathbf{r}(t) = \langle 200 \cos t, 200 \sin t \rangle, \ 0 \le t \le 2\pi$$

Then at $t = \pi/3$ then $x = 100$ and $y = 100\sqrt{3}$.

Next we need to calculate the velocity vector at $t = \pi/3$, using the parametrization for the circle we have

$$\mathbf{r}'(t) = \langle -200 \sin t, 200 \cos t \rangle \quad \Rightarrow \quad \mathbf{v} = \mathbf{r}'(\pi/3) = \left\langle -100\sqrt{3}, 100 \right\rangle$$

Now to normalize $\mathbf{v}$ we have

$$\mathbf{u} = \frac{1}{\sqrt{30000 + 10000}} \left\langle -100\sqrt{3}, 100 \right\rangle = \frac{1}{200} \left\langle -100\sqrt{3}, 100 \right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

We also need to compute the gradient of $T(x, y) = 20 + 0.1(x^2 - xy)$ at $t = \pi/3$ (or $x = 100$, $y = 100\sqrt{3}$):

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle = \langle 0.2x - 0.1y, -0.1x \rangle$$

$$\nabla T_{(100,100\sqrt{3})} = \left\langle 0.2(100) - 0.1(100\sqrt{3}), -0.1(100) \right\rangle = \left\langle 20 - 10\sqrt{3}, -10 \right\rangle$$

Then the rate of change of the bug's temperature at the point $t = \pi/3$ is the directional derivative:

$$D_{\mathbf{u}} f(\pi/3) = \nabla T_{(100,100\sqrt{3})} \cdot \mathbf{u} = \left\langle 20 - 10\sqrt{3}, -10 \right\rangle \cdot \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = 10 - 10\sqrt{3} \approx -7.32$$

So the temperature is changing at -7.32 degrees Celsius per second.

35. Suppose that $\nabla f_P = \langle 2, -4, 4 \rangle$. Is f increasing or decreasing at P in the direction $\mathbf{v} = \langle 2, 1, 3 \rangle$?

SOLUTION We compute the derivative of f at P with respect to $\mathbf{v}$:

$$D_{\mathbf{v}} f(P) = \nabla f_P \cdot \mathbf{v} = \langle 2, -4, 4 \rangle \cdot \langle 2, 1, 3 \rangle = 4 - 4 + 12 = 12 > 0$$

Since the derivative is positive, f is increasing at P in the direction of $\mathbf{v}$.

36. Let $f(x, y) = xe^{x^2 - y}$ and $P = (1, 1)$.

(a) Calculate $\|\nabla f_P\|$.

(b) Find the rate of change of f in the direction ∇f_P.

(c) Find the rate of change of f in the direction of a vector making an angle of 45° with ∇f_P.

SOLUTION

(a) We compute the gradient of $f(x, y) = xe^{x^2 - y}$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 1 \cdot e^{x^2 - y} + xe^{x^2 - y} \cdot 2x = e^{x^2 - y}\left(1 + 2x^2\right)$$

$$\frac{\partial f}{\partial y} = -xe^{x^2 - y}$$

The gradient vector is thus

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle e^{x^2 - y}\left(1 + 2x^2\right), -xe^{x^2 - y} \right\rangle = e^{x^2 - y}\left\langle 1 + 2x^2, -x \right\rangle$$

At the point $P = (1, 1)$ we have

$$\nabla f_P = e^0 \langle 1 + 2, -1 \rangle = \langle 3, -1 \rangle \quad \Rightarrow \quad \|\nabla f_P\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

(b) The rate of change of f in the direction of the gradient vector is the length of the gradient, that is, $\|\nabla f_P\| = \sqrt{10}$.

(c) Let $\mathbf{e_v}$ be the unit vector making an angle of 45° with ∇f_P. The rate of change of f in the direction of $\mathbf{e_v}$ is the directional derivative of f in the direction $\mathbf{e_v}$, which is the following dot product:

$$D_{\mathbf{e_v}} f(P) = \nabla f_P \cdot \mathbf{e_v} = \|\nabla f_P\| \|\mathbf{e_v}\| \cos 45^\circ = \sqrt{10} \cdot 1 \cdot \frac{1}{\sqrt{2}} = \sqrt{5} \approx 2.236$$

37. Let $f(x, y, z) = \sin(xy + z)$ and $P = (0, -1, \pi)$. Calculate $D_{\mathbf{u}} f(P)$, where $\mathbf{u}$ is a unit vector making an angle $\theta = 30^\circ$ with ∇f_P.

SOLUTION The directional derivative $D_{\mathbf{u}} f(P)$ is the following dot product:

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u}$$

Since $\mathbf{u}$ is a unit vector making an angle $\theta = 30^\circ$ with ∇f_P, we have by the properties of the dot product

$$D_{\mathbf{u}} f(P) = \|\nabla f_P\| \cdot \|\mathbf{u}\| \cos 30^\circ = \frac{\sqrt{3}}{2} \|\nabla f_P\| \tag{1}$$

We now must find the gradient at P and its length:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle y\cos(xy + z), x\cos(xy + z), \cos(xy + z) \rangle = \cos(xy + z) \langle y, x, 1 \rangle$$

$$\nabla f_{(0,-1,\pi)} = \cos \pi \langle -1, 0, 1 \rangle = -1 \langle -1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$$

Hence,

$$\|\nabla f_{(0,-1,\pi)}\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

Substituting in (1) we get

$$D_{\mathbf{u}} f(P) = \frac{\sqrt{3}}{2}\sqrt{2} = \frac{\sqrt{6}}{2}.$$

38. Let $T(x, y)$ be the temperature at location (x, y). Assume that $\nabla T = \langle y - 4, x + 2y \rangle$. Let $\mathbf{c}(t) = (t^2, t)$ be a path in the plane. Find the values of t such that

$$\frac{d}{dt} T(\mathbf{c}(t)) = 0$$

SOLUTION By the Chain Rule for Paths we have

$$\frac{d}{dt} T\,(\mathbf{c}(t)) = \nabla T_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \qquad (1)$$

We compute the gradient vector ∇T for $x = t^2$ and $y = t$:

$$\nabla T = \left\langle t - 4, t^2 + 2t \right\rangle$$

Also $\mathbf{c}'(t) = \langle 2t, 1 \rangle$. Substituting in (1) gives

$$\frac{d}{dt} T\,(\mathbf{c}(t)) = \left\langle t - 4, t^2 + 2t \right\rangle \cdot \langle 2t, 1 \rangle = (t - 4) \cdot 2t + \left(t^2 + 2t\right) \cdot 1 = 3t^2 - 6t$$

We are asked to find the values of t such that

$$\frac{d}{dt} T\,(\mathbf{c}(t)) = 3t^2 - 6t = 0$$

We solve to obtain

$$3t^2 - 6t = 3t(t - 2) = 0 \quad \Rightarrow \quad t_1 = 0, \quad t_2 = 2$$

39. Find a vector normal to the surface $x^2 + y^2 - z^2 = 6$ at $P = (3, 1, 2)$.

SOLUTION The gradient ∇f_P is normal to the level curve $f(x, y, z) = x^2 + y^2 - z^2 = 6$ at P. We compute this vector:

$$\begin{aligned} f_x(x, y, z) &= 2x \\ f_y(x, y, z) &= 2y \quad \Rightarrow \quad \nabla f_P = \nabla f_{(3,1,2)} = \langle 6, 2, -4 \rangle \\ f_z(x, y, z) &= -2z \end{aligned}$$

The vector $\langle 6, 2, -4 \rangle$ is normal to the surface $x^2 + y^2 - z^2 = 6$ at P.

40. Find a vector normal to the surface $3z^3 + x^2 y - y^2 x = 1$ at $P = (1, -1, 1)$.

SOLUTION The gradient is normal to the level surfaces, that is ∇f_P is normal to the level surface $f(x, y, z) = 3z^3 + x^2 y - y^2 x = 1$. We compute the gradient vector at $P = (1, -1, 1)$:

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle 2xy - y^2, x^2 - 2yx, 9z^2 \right\rangle \\ \nabla f_P &= \langle -3, 3, 9 \rangle \end{aligned}$$

41. Find the two points on the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

where the tangent plane is normal to $\mathbf{v} = \langle 1, 1, -2 \rangle$.

SOLUTION The gradient ∇f_P is normal to the level surface $f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$. If $\mathbf{v} = \langle 1, 1, -2 \rangle$ is also normal, then ∇f_P and $\mathbf{v}$ are parallel, that is, $\nabla f_P = k\mathbf{v}$ for some constant k. This yields the equation

$$\nabla f_P = \langle \frac{x}{2}, \frac{2y}{9}, 2z \rangle = k\,\langle 1, 1, -2 \rangle$$

Thus $x = 2k$, $y = 9k/2$, and $z = -k$. To determine k, substitute in the equation of the ellipsoid:

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = \frac{(2k)^2}{4} + \frac{(9k/2)^2}{9} + (-k)^2 = 1$$

This yields $k^2 + \frac{9}{4}k^2 + k^2 = 1$ or $k = \pm 2/\sqrt{17}$. The two points are

$$(x, y, z) = (2k, \frac{9}{2}k, -k) = \pm\left(\frac{4}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{2}{\sqrt{17}}\right)$$

In Exercises 42–45, find an equation of the tangent plane to the surface at the given point.

42. $x^2 + 3y^2 + 4z^2 = 20, \quad P = (2, 2, 1)$

SOLUTION The equation of the tangent plane is

$$\nabla f_P \cdot \langle x - 2, y - 2, z - 1\rangle = 0 \tag{1}$$

We compute the gradient of $f(x, y, z) = x^2 + 3y^2 + 4z^2$ at $P = (2, 2, 1)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \langle 2x, 6y, 8z\rangle$$

At the point P we have

$$\nabla f_P = \langle 2 \cdot 2, 6 \cdot 2, 8 \cdot 1\rangle = \langle 4, 12, 8\rangle$$

Substituting in (1) we obtain the following equation of the tangent plane:

$$\begin{aligned} \langle 4, 12, 8\rangle \cdot \langle x - 2, y - 2, z - 1\rangle &= 0 \\ 4(x - 2) + 12(y - 2) + 8(z - 1) &= 0 \\ x - 2 + 3(y - 2) + 2(z - 1) &= 0 \end{aligned}$$

or

$$x + 3y + 2z = 10$$

43. $xz + 2x^2y + y^2z^3 = 11, \quad P = (2, 1, 1)$

SOLUTION The equation of the tangent plane at P is

$$\nabla f_P \cdot \langle x - 2, y - 1, z - 1\rangle = 0 \tag{1}$$

We compute the gradient of $f(x, y, z) = xz + 2x^2y + y^2z^3$ at the point $P = (2, 1, 1)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \left\langle z + 4xy, 2x^2 + 2yz^3, x + 3y^2z^2\right\rangle$$

At the point P we have

$$\nabla f_P = \langle 9, 10, 5\rangle$$

Substituting in (1) we obtain the following equation of the tangent plane:

$$\begin{aligned} \langle 9, 10, 5\rangle \cdot \langle x - 2, y - 1, z - 1\rangle &= 0 \\ 9(x - 2) + 10(y - 1) + 5(z - 1) &= 0 \end{aligned}$$

or

$$9x + 10y + 5z = 33$$

44. $x^2 + z^2e^{y-x} = 13, \quad P = \left(2, 3, \frac{3}{\sqrt{e}}\right)$

SOLUTION We compute the gradient of $f(x, y, z) = x^2 + z^2e^{y-x}$ at the point $P = \left(2, 3, \frac{3}{\sqrt{e}}\right)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \left\langle 2x - z^2e^{y-x}, z^2e^{y-x}, 2ze^{y-x}\right\rangle$$

At the point $P = \left(2, 3, \frac{3}{\sqrt{e}}\right)$ we have

$$\nabla f_P = \left\langle 4 - \frac{9}{e} \cdot e, \frac{9}{e} \cdot e, 2 \cdot \frac{3}{\sqrt{e}} \cdot e \right\rangle = \langle -5, 9, 6\sqrt{e} \rangle$$

The equation of the tangent plane at P is

$$\nabla f_P \cdot \left\langle x - 2, y - 3, z - \frac{3}{\sqrt{e}} \right\rangle = 0$$

That is,

$$-5(x-2) + 9(y-3) + 6\sqrt{e}\left(z - \frac{3}{\sqrt{e}}\right) = 0$$

or

$$-5x + 9y + 6\sqrt{e}z = 35$$

45. $\ln[1 + 4x^2 + 9y^4] - 0.1z^2 = 0, \quad P = (3, 1, 6.1876)$

SOLUTION The equation of the tangent plane at P is

$$\nabla f_P \cdot (x - 3, y - 1, z - 6.1876) = 0 \tag{1}$$

We compute the gradient of $f(x, y, z) = \ln(1 + 4x^2 + 9y^4) - 0.1z^2$ at the point P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle \frac{8x}{1 + 4x^2 + 9y^4}, \frac{36y^3}{1 + 4x^2 + 9y^4}, -0.2z \right\rangle$$

At the point $P = (3, 1, 6.1876)$ we have

$$\nabla f_P = \left\langle \frac{24}{1 + 36 + 9}, \frac{36}{46}, -1.2375 \right\rangle = \langle 0.5217, 0.7826, -1.2375 \rangle$$

We substitute in (1) to obtain the following equation of the tangent plane:

$$0.5217(x-3) + 0.7826(y-1) - 1.2375(z - 6.1876) = 0$$

or

$$0.5217x + 0.7826y - 1.2375z = -5.309$$

46. Verify what is clear from Figure 15: Every tangent plane to the cone $x^2 + y^2 - z^2 = 0$ passes through the origin.

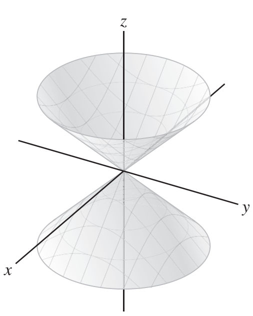


FIGURE 15 Graph of $x^2 + y^2 - z^2 = 0$.

SOLUTION The equation of the tangent plane to the surface $f(x, y, z) = x^2 + y^2 - z^2 = 0$ at the point $P = (x_0, y_0, z_0)$ on the surface is

$$\nabla f_P \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \tag{1}$$

We compute the gradient of $f(x, y, z) = x^2 + y^2 - z^2$ at P:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 2y, -2z \rangle$$

Hence,

$$\nabla f_P = \langle 2x_0, 2y_0, -2z_0 \rangle$$

Substituting in (1) we obtain the following equation of the tangent plane:

$$\langle 2x_0, 2y_0, -2z_0\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$
$$x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0) = 0$$
$$x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$$

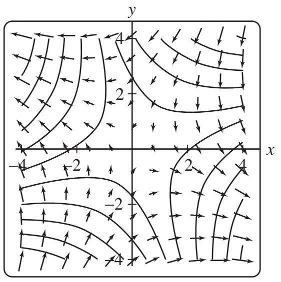
Since $P = (x_0, y_0, z_0)$ is on the surface, we have $x_0^2 + y_0^2 - z_0^2 = 0$. The equation of the tangent plane is thus

$$x_0x + y_0y - z_0z = 0$$

This plane passes through the origin.

47. CAS Use a computer algebra system to produce a contour plot of $f(x, y) = x^2 - 3xy + y - y^2$ together with its gradient vector field on the domain $[-4, 4] \times [-4, 4]$.

SOLUTION



48. Find a function $f(x, y, z)$ such that ∇f is the constant vector $\langle 1, 3, 1\rangle$.

SOLUTION The gradient of $f(x, y, z)$ must satisfy the equality

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 1, 3, 1\rangle$$

Equating corresponding components gives

$$\frac{\partial f}{\partial x} = 1$$
$$\frac{\partial f}{\partial y} = 3$$
$$\frac{\partial f}{\partial z} = 1$$

One of the functions that satisfies these equalities is

$$f(x, y, z) = x + 3y + z$$

49. Find a function $f(x, y, z)$ such that $\nabla f = \langle 2x, 1, 2\rangle$.

SOLUTION The following equality must hold:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 1, 2\rangle$$

Equating corresponding components gives

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = 1$$
$$\frac{\partial f}{\partial z} = 2$$

One of the functions that satisfies these equalities is $f(x, y, z) = x^2 + y + 2z$.

50. Find a function $f(x, y, z)$ such that $\nabla f = \langle x, y^2, z^3\rangle$.

SOLUTION The following equality must hold:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle x, y^2, z^3 \right\rangle$$

That is,

$$\frac{\partial f}{\partial x} = x$$
$$\frac{\partial f}{\partial y} = y^2$$
$$\frac{\partial f}{\partial z} = z^3$$

One of the functions that satisfies these equalities is

$$f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{4}z^4$$

51. Find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x\rangle$.

SOLUTION $f(x, y, z) = xz + y^2$ is a good choice.

52. Find a function $f(x, y)$ such that $\nabla f = \langle y, x\rangle$.

SOLUTION We must find a function $f(x, y)$ such that

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y, x\rangle$$

That is,

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

We integrate the first equation with respect to x. Since y is treated as a constant, the constant of integration is a function of y. We get

$$f(x, y) = \int y\,dx = yx + g(y) \tag{1}$$

We differentiate f with respect to y and substitute in the second equation. This gives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(yx + g(y)\right) = x + g'(y)$$

Hence,

$$x + g'(y) = x \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = C$$

Substituting in (1) gives

$$f(x, y) = yx + C$$

One of the solutions is $f(x, y) = yx$ (obtained for $C = 0$).

53. Show that there does not exist a function $f(x, y)$ such that $\nabla f = \langle y^2, x\rangle$. *Hint:* Use Clairaut's Theorem $f_{xy} = f_{yx}$.

SOLUTION Suppose that for some differentiable function $f(x, y)$,

$$\nabla f = \langle f_x, f_y\rangle = \left\langle y^2, x\right\rangle$$

That is, $f_x = y^2$ and $f_y = x$. Therefore,

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y} y^2 = 2y \quad \text{and} \quad f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x} x = 1$$

Since f_{xy} and f_{yx} are both continuous, they must be equal by Clairaut's Theorem. Since $f_{xy} \neq f_{yx}$ we conclude that such a function f does not exist.

54. Let $\Delta f = f(a+h, b+k) - f(a,b)$ be the change in f at $P = (a,b)$. Set $\Delta\mathbf{v} = \langle h, k\rangle$. Show that the linear approximation can be written

$$\Delta f \approx \nabla f_P \cdot \Delta\mathbf{v} \tag{8}$$

SOLUTION The linear approximation is

$$\Delta f \approx f_x(a,b)h + f_y(a,b)k = \left\langle f_x(a,b), f_y(a,b)\right\rangle \cdot \langle h, k\rangle = \nabla f_P \cdot \Delta\mathbf{v}$$

55. Use Eq. (8) to estimate

$$\Delta f = f(3.53, 8.98) - f(3.5, 9)$$

assuming that $\nabla f_{(3.5,9)} = \langle 2, -1\rangle$.

SOLUTION By Eq. (8),

$$\Delta f \approx \nabla f_P \cdot \Delta\mathbf{v}$$

The vector $\Delta\mathbf{v}$ is the following vector:

$$\Delta\mathbf{v} = \langle 3.53 - 3.5, 8.98 - 9\rangle = \langle 0.03, -0.02\rangle$$

Hence,

$$\Delta f \approx \nabla f_{(3,5,9)} \cdot \Delta\mathbf{v} = \langle 2, -1\rangle \cdot \langle 0.03, -0.02\rangle = 0.08$$

56. Find a unit vector $\mathbf{n}$ that is normal to the surface $z^2 - 2x^4 - y^4 = 16$ at $P = (2, 2, 8)$ that points in the direction of the xy-plane (in other words, if you travel in the direction of $\mathbf{n}$, you will eventually cross the xy-plane).

SOLUTION The gradient vector ∇f_P is normal to the surface $f(x,y,z) = z^2 - 2x^4 - y^4 = 16$ at P. We find this vector:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \left\langle -8x^3, -4y^3, 2z\right\rangle \quad \Rightarrow \quad \nabla f_{(2,2,8)} = \left\langle -8\cdot 2^3, -4\cdot 2^3, 2\cdot 8\right\rangle = \langle -64, -32, 16\rangle$$

We normalize to obtain a unit vector normal to the surface:

$$\frac{\nabla f_P}{\|\nabla f_P\|} = \frac{\langle -64, -32, 16\rangle}{\sqrt{(-64)^2 + 32^2 + 16^2}} = \frac{\langle -64, -32, 16\rangle}{16\sqrt{21}} = \frac{1}{\sqrt{21}}\langle -4, -2, 1\rangle$$

There are two unit normals to the surface at P, namely,

$$\mathbf{n} = \pm\frac{1}{\sqrt{21}}\langle -4, -2, 1\rangle$$

We need to find the normal that points in the direction of the xy-plane. Since the point $P = (2, 2, 8)$ is above the xy-plane, the normal we need has negative z-component. Hence,

$$\mathbf{n} = \frac{1}{\sqrt{21}}\langle 4, 2, -1\rangle$$

57. Suppose, in the previous exercise, that a particle located at the point $P = (2, 2, 8)$ travels toward the xy-plane in the direction normal to the surface.

(a) Through which point Q on the xy-plane will the particle pass?

(b) Suppose the axes are calibrated in centimeters. Determine the path $\mathbf{c}(t)$ of the particle if it travels at a constant speed of 8 cm/s. How long will it take the particle to reach Q?

SOLUTION

(a) The particle travels along the line through $P = (2, 2, 8)$ in the direction $\langle 4, 2, -1\rangle$. The vector parametrization of this line is

$$\mathbf{r}(t) = \langle 2, 2, 8\rangle + t\,\langle 4, 2, -1\rangle = \langle 2 + 4t, 2 + 2t, 8 - t\rangle \tag{1}$$

We must find the point where this line intersects the xy-plane. At this point the z-component is zero. Hence,

$$8 - t = 0 \quad \Rightarrow \quad t = 8$$

Substituting $t = 8$ in (1) we obtain

$$\mathbf{r}(8) = \langle 2 + 4\cdot 8, 2 + 2\cdot 8, 0\rangle = \langle 34, 18, 0\rangle$$

The particle will pass through the point $Q = (34, 18, 0)$ on the xy-plane.

(b) If $\mathbf{v}$ is a direction vector of the line PQ, so that $\|\mathbf{v}\| = 8$, the following parametrization of the line has constant speed 8:

$$\mathbf{c}(t) = \langle 2, 2, 8\rangle + t\mathbf{v}$$

(This has speed 8 because $\|\mathbf{c}'(t)\| = \|\mathbf{v}\| = 8$). In the previous exercise, we found the unit vector $\mathbf{n} = \frac{1}{\sqrt{21}}\langle 4, 2, -1\rangle$, therefore we use the direction vector $\mathbf{v} = 8\mathbf{n} = \frac{8}{\sqrt{21}}\langle 4, 2, -1\rangle$, obtaining the following parametrization of the line:

$$\mathbf{c}(t) = \langle 2, 2, 8\rangle + t \cdot \frac{8}{\sqrt{21}}\langle 4, 2, -1\rangle = \left\langle 2 + \frac{32}{\sqrt{21}}t, 2 + \frac{16}{\sqrt{21}}t, 8 - \frac{8t}{\sqrt{21}}\right\rangle$$

To find the time needed for the particle to reach Q if it travels along $\mathbf{c}(t)$, we first compute the distance $\overline{PQ}$:

$$\overline{PQ} = \sqrt{(34-2)^2 + (18-2)^2 + (0-8)^2} = \sqrt{1344} = 8\sqrt{21}$$

The time needed is thus

$$T = \frac{\overline{PQ}}{8} = \frac{8\sqrt{21}}{8} = \sqrt{21} \approx 4.58 \text{ s}$$

58. Let $f(x, y) = \tan^{-1}\frac{x}{y}$ and $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$.

(a) Calculate the gradient of f.

(b) Calculate $D_{\mathbf{u}}f(1, 1)$ and $D_{\mathbf{u}}f(\sqrt{3}, 1)$.

(c) Show that the lines $y = mx$ for $m \neq 0$ are level curves for f.

(d) Verify that ∇f_P is orthogonal to the level curve through P for $P = (x, y) \neq (0, 0)$.

SOLUTION

(a) We compute the partial derivatives of $f(x, y) = \tan^{-1}\frac{x}{y}$. Using the Chain Rule we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}$$

The gradient of f is thus

$$\nabla f = \left\langle \frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}\right\rangle = \frac{1}{x^2 + y^2}\langle y, -x\rangle$$

(b) By the Theorem on Evaluating Directional Derivatives,

$$D_{\mathbf{u}}f(a, b) = \nabla f_{(a,b)} \cdot \mathbf{u} \tag{1}$$

We find the values of the gradient at the two points:

$$\nabla f_{(1,1)} = \frac{1}{1^2 + 1^2}\langle 1, -1\rangle = \frac{1}{2}\langle 1, -1\rangle$$

$$\nabla f_{\left(\sqrt{3},1\right)} = \frac{1}{\left(\sqrt{3}\right)^2 + 1^2}\left\langle 1, -\sqrt{3}\right\rangle = \frac{1}{4}\left\langle 1, -\sqrt{3}\right\rangle$$

Substituting in (1) we obtain the following directional derivatives

$$D_{\mathbf{u}}f(1, 1) = \nabla f_{(1,1)} \cdot \mathbf{u} = \frac{1}{2}\langle 1, -1\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = 0$$

$$D_{\mathbf{u}}f\left(\sqrt{3}, 1\right) = \nabla f_{\left(\sqrt{3},1\right)} \cdot \mathbf{u} = \frac{1}{4}\left\langle 1, -\sqrt{3}\right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \frac{\sqrt{2}}{8}\left\langle 1, -\sqrt{3}\right\rangle \cdot \langle 1, 1\rangle$$

$$= \frac{\sqrt{2}}{8}\left(1 - \sqrt{3}\right) = \frac{\sqrt{2} - \sqrt{6}}{8}$$

(c) Note that f is not defined for $y = 0$. For $x = 0$, the level curve of f is the y-axis, and the gradient vector is $\langle \frac{1}{y}, 0\rangle$, which is perpendicular to the y-axis. For $y \neq 0$ and $x \neq 0$, the level curves of f are the curves where $f(x, y)$ is constant. That is,

$$\tan^{-1}\frac{x}{y} = k$$

$$\frac{x}{y} = \tan k \qquad (\text{for } k \neq 0)$$

$$y = \frac{1}{\tan k}x$$

We conclude that the lines $y = mx$, $m \neq 0$, are level curves for f.

(d) By part (c), the level curve through $P = (x_0, y_0)$ is the line $y = \frac{y_0}{x_0}x$. This line has a direction vector $\left\langle 1, \frac{y_0}{x_0}\right\rangle$. The gradient at P is, by part (a), $\nabla f_P = \frac{1}{x_0^2+y_0^2}\langle y_0, -x_0\rangle$. We verify that the two vectors are orthogonal:

$$\left(1, \frac{y_0}{x_0}\right)\cdot\nabla f_P = \left\langle 1, \frac{y_0}{x_0}\right\rangle\cdot\frac{1}{x_0^2+y_0^2}\langle y_0, -x_0\rangle = \frac{1}{x_0^2+y_0^2}\left(y_0 - \frac{x_0y_0}{x_0}\right) = 0$$

Since the dot products is zero, the two vectors are orthogonal as expected (Theorem 6).

59. Suppose that the intersection of two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ is a curve $\mathcal{C}$, and let P be a point on $\mathcal{C}$. Explain why the vector $\mathbf{v} = \nabla F_P \times \nabla G_P$ is a direction vector for the tangent line to $\mathcal{C}$ at P.

SOLUTION The gradient ∇F_P is orthogonal to all the curves in the level surface $F(x, y, z) = 0$ passing through P. Similarly, ∇G_P is orthogonal to all the curves in the level surface $G(x, y, z) = 0$ passing through P. Therefore, both ∇F_P and ∇G_P are orthogonal to the intersection curve $\mathcal{C}$ at P, hence the cross product $\nabla F_P \times \nabla G_P$ is parallel to the tangent line to $\mathcal{C}$ at P.

60. Let $\mathcal{C}$ be the curve of intersection of the spheres $x^2 + y^2 + z^2 = 3$ and $(x-2)^2 + (y-2)^2 + z^2 = 3$. Use the result of Exercise 59 to find parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 1, 1)$.

SOLUTION The parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 1, 1)$ are

$$x = 1 + at, \quad y = 1 + bt, \quad z = 1 + ct \tag{1}$$

where $\mathbf{v} = \langle a, b, c\rangle$ is a direction vector for the line. By Exercise 59 $\mathbf{v}$ may be chosen as the following cross product:

$$\mathbf{v} = \nabla F_P \times \nabla G_P \tag{2}$$

where $F(x, y, z) = x^2 + y^2 + z^2$ and $G(x, y, z) = (x-2)^2 + (y-2)^2 + z^2$. We compute ∇F_P and ∇G_P:

$$\begin{aligned} F_x(x, y, z) &= 2x \\ F_y(x, y, z) &= 2y \quad \Rightarrow \quad \nabla F_P = \langle 2\cdot 1, 2\cdot 1, 2\cdot 1\rangle = \langle 2, 2, 2\rangle \\ F_z(x, y, z) &= 2z \end{aligned}$$

$$\begin{aligned} G_x(x, y, z) &= 2(x-2) \\ G_y(x, y, z) &= 2(y-2) \quad \Rightarrow \quad \nabla G_P = \langle 2(1-2), 2(1-2), 2\cdot 1\rangle = \langle -2, -2, 2\rangle \\ G_z(x, y, z) &= 2z \end{aligned}$$

Hence,

$$\mathbf{v} = \langle 2, 2, 2\rangle \times \langle -2, -2, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ -2 & -2 & 2 \end{vmatrix} = (4+4)\mathbf{i} - (4+4)\mathbf{j} + (-4+4)\mathbf{k} = 8\mathbf{i} - 8\mathbf{j} = \langle 8, -8, 0\rangle$$

Therefore, $\mathbf{v} = \langle a, b, c\rangle = \langle 8, -8, 0\rangle$, yielding $a = 8$, $b = -8$, $c = 0$. Substituting in (1) gives the following equations of the tangent line: $x = 1 + 8t$, $y = 1 - 8t$, $z = 1$.

61. Let $\mathcal{C}$ be the curve obtained by intersecting the two surfaces $x^3 + 2xy + yz = 7$ and $3x^2 - yz = 1$. Find the parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 2, 1)$.

SOLUTION The parametric equations of the tangent line to $\mathcal{C}$ at $P = (1, 2, 1)$ are

$$x = 1 + at, \quad y = 2 + bt, \quad z = 1 + ct \tag{1}$$

where $\mathbf{v} = \langle a, b, c\rangle$ is a direction vector for the line. By Exercise 59, $\mathbf{v}$ may be chosen as the cross product:

$$\mathbf{v} = \nabla F_P \times \nabla G_P \tag{2}$$

where $F(x, y, z) = x^3 + 2xy + yz$ and $G(x, y, z) = 3x^2 - yz$. We compute the gradient vectors:

$$\begin{aligned} F_x(x, y, z) &= 3x^2 + 2y \qquad & F_x(1, 2, 1) &= 7 \\ F_y(x, y, z) &= 2x + z \quad \Rightarrow & F_y(1, 2, 1) &= 3 \quad \Rightarrow \quad \nabla F_P = \langle 7, 3, 2 \rangle \\ F_z(x, y, z) &= y & F_z(1, 2, 1) &= 2 \end{aligned}$$

$$\begin{aligned} G_x(x, y, z) &= 6x \qquad & G_x(1, 2, 1) &= 6 \\ G_y(x, y, z) &= -z \quad \Rightarrow & G_y(1, 2, 1) &= -1 \quad \Rightarrow \quad \nabla G_P = \langle 6, -1, -2 \rangle \\ G_z(x, y, z) &= -y & G_z(1, 2, 1) &= -2 \end{aligned}$$

Hence,

$$\mathbf{v} = \langle 7, 3, 2 \rangle \times \langle 6, -1, -2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 6 & -1 & -2 \end{vmatrix} = -4\mathbf{i} + 26\mathbf{j} - 25\mathbf{k} = \langle -4, 26, -25 \rangle$$

Therefore, $\mathbf{v} = \langle a, b, c \rangle = \langle -4, 26, -25 \rangle$, so we obtain

$$a = -4, \quad b = 26, \quad c = -25.$$

Substituting in (1) gives the following parametric equations of the tangent line:

$$x = 1 - 4t, \quad y = 2 + 26t, \quad z = 1 - 25t.$$

62. Verify the linearity relations for gradients:
(a) $\nabla(f + g) = \nabla f + \nabla g$
(b) $\nabla(cf) = c\nabla f$

SOLUTION
(a) We use the linearity relations for partial derivative to write

$$\begin{aligned} \nabla(f + g) &= \left\langle (f+g)_x, (f+g)_y, (f+g)_z \right\rangle = \left\langle f_x + g_x, f_y + g_y, f_z + g_z \right\rangle \\ &= \left\langle f_x, f_y, f_z \right\rangle + \left\langle g_x, g_y, g_z \right\rangle = \nabla f + \nabla g \end{aligned}$$

(b) We use the linearity properties of partial derivatives to write

$$\nabla(cf) = \left\langle (cf)_x, (cf)_y, (cf)_z \right\rangle = \left\langle cf_x, cf_y, cf_z \right\rangle = c \left\langle f_x, f_y, f_z \right\rangle = c\nabla f$$

63. Prove the Chain Rule for Gradients (Theorem 1).

SOLUTION We must show that if $F(t)$ is a differentiable function of t and $f(x, y, z)$ is differentiable, then

$$\nabla F\left(f(x, y, z)\right) = F'\left(f(x, y, z)\right) \nabla f$$

Using the Chain Rule for partial derivatives we get

$$\begin{aligned} \nabla F\left(f(x, y, z)\right) &= \left\langle \frac{\partial}{\partial x} F\left(f(x, y, z)\right), \frac{\partial}{\partial y} F\left(f(x, y, z)\right), \frac{\partial}{\partial z} F\left(f(x, y, z)\right) \right\rangle \\ &= \left\langle \frac{dF}{dt} \cdot \frac{\partial f}{\partial x}, \frac{dF}{dt} \cdot \frac{\partial f}{\partial y}, \frac{dF}{dt} \cdot \frac{\partial f}{\partial z} \right\rangle = \frac{dF}{dt} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = F'\left(f(x, y, z)\right) \nabla F \end{aligned}$$

64. Prove the Product Rule for Gradients (Theorem 1).

SOLUTION We must show that if $f(x, y, z)$ and $g(x, y, z)$ are differentiable, then

$$\nabla(fg) = f\nabla g + g\nabla f$$

Using the Product Rule for partial derivatives we get

$$\begin{aligned} \nabla(fg) &= \left\langle (fg)_x, (fg)_y, (fg)_z \right\rangle = \left\langle f_x g + f g_x, f_y g + f g_y, f_z g + f g_z \right\rangle \\ &= \left\langle f_x g, f_y g, f_z g \right\rangle + \left\langle f g_x, f g_y, f g_z \right\rangle = \left\langle f_x, f_y, f_z \right\rangle g + f \left\langle g_x, g_y, g_z \right\rangle = g\nabla f + f\nabla g \end{aligned}$$

Further Insights and Challenges

65. Let $\mathbf{u}$ be a unit vector. Show that the directional derivative $D_{\mathbf{u}} f$ is equal to the component of ∇f along $\mathbf{u}$.

SOLUTION The component of ∇f along $\mathbf{u}$ is $\nabla f \cdot \mathbf{u}$. By the Theorem on Evaluating Directional Derivatives, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$, which is the component of ∇f along $\mathbf{u}$.

66. Let $f(x, y) = (xy)^{1/3}$.

(a) Use the limit definition to show that $f_x(0, 0) = f_y(0, 0) = 0$.

(b) Use the limit definition to show that the directional derivative $D_{\mathbf{u}} f(0, 0)$ does not exist for any unit vector $\mathbf{u}$ other than $\mathbf{i}$ and $\mathbf{j}$.

(c) Is f differentiable at $(0, 0)$?

SOLUTION

(a) By the limit definition and since $f(0, 0) = 0$, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{(h\cdot 0)^{1/3} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{(0\cdot h)^{1/3} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

(b) By the limit definition of the directional derivative, and for $\mathbf{u} = \langle u_1, u_2 \rangle$ a unit vector, we have

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{\left(t^2 u_1 u_2\right)^{1/3} - 0}{t} = \lim_{t\to 0} \frac{u_1 u_2}{t^{1/3}}$$

This limit does not exist unless $u_1 = 0$ or $u_2 = 0$. $u_1 = 0$ corresponds to the unit vector $\mathbf{j}$, and $u_2 = 0$ corresponds to the unit vector $\mathbf{i}$.

(c) If f was differentiable at $(0, 0)$, then $D_{\mathbf{u}} f(0, 0)$ would exist for any vector $\mathbf{u}$. Therefore, using the result obtained in part (b), f is not differentiable at $(0, 0)$.

67. Use the definition of differentiability to show that if $f(x, y)$ is differentiable at $(0, 0)$ and

$$f(0,0) = f_x(0,0) = f_y(0,0) = 0$$

then

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0 \qquad \boxed{9}$$

SOLUTION If $f(x, y)$ is differentiable at $(0, 0)$, then there exists a function $\epsilon(x, y)$ satisfying $\lim_{(x,y)\to(0,0)} \epsilon(x, y) = 0$ such that

$$f(x,y) = L(x,y) + \epsilon(x,y)\sqrt{x^2+y^2} \qquad (1)$$

Since $f(0, 0) = 0$, the linear function $L(x, y)$ is

$$L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = f_x(0,0)x + f_y(0,0)y$$

Substituting in (1) gives

$$f(x,y) = f_x(0,0)x + f_y(0,0)y + \epsilon(x,y)\sqrt{x^2+y^2}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \epsilon(x,y) = 0$$

68. This exercise shows that there exists a function that is not differentiable at $(0, 0)$ even though all directional derivatives at $(0, 0)$ exist. Define $f(x, y) = x^2 y/(x^2 + y^2)$ for $(x, y) \neq 0$ and $f(0, 0) = 0$.

(a) Use the limit definition to show that $D_{\mathbf{v}} f(0, 0)$ exists for all vectors $\mathbf{v}$. Show that $f_x(0, 0) = f_y(0, 0) = 0$.

(b) Prove that f is *not* differentiable at $(0, 0)$ by showing that Eq. (9) does not hold.

SOLUTION

(a) Let $\mathbf{v} \neq \mathbf{0}$ be the vector $\mathbf{v} = \langle v_1, v_2 \rangle$. By the definition of the derivative $D_{\mathbf{v}} f(0, 0)$, we have

$$D_{\mathbf{v}} f(0,0) = \lim_{t \to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{(tv_1)^2 tv_2}{(tv_1)^2 + (tv_2)^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{t^3 v_1^2 v_2}{t^3 \left(v_1^2 + v_2^2\right)} = \lim_{t \to 0} \frac{v_1^2 v_2}{v_1^2 + v_2^2} = \frac{v_1^2 v_2}{v_1^2 + v_2^2} \tag{1}$$

Therefore $D_{\mathbf{v}} f(0, 0)$ exists for all vectors $\mathbf{v}$.

(b) In Exercise 67 we showed that if $f(x, y)$ is differentiable at $(0, 0)$ and $f(0, 0) = 0$, then

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2 + y^2}} = 0$$

We now show that f does not satisfy the above equation. We first compute the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$. The partial derivatives f_x and f_y are the directional derivatives in the directions of $\mathbf{v} = \langle 1, 0 \rangle$ and $\mathbf{v} = \langle 0, 1 \rangle$, respectively. Substituting $v_1 = 1$, $v_2 = 0$ in (1) gives

$$f_x(0,0) = \frac{1^2 \cdot 0}{1^2 + 0^2} = 0$$

Substituting $v_1 = 0$, $v_2 = 1$ in (1) gives

$$f_y(0,0) = \frac{0^2 \cdot 1}{0^2 + 1^2} = 0$$

Also $f(0, 0) = 0$, therefore for $(x, y) \neq (0, 0)$ we have

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{\frac{x^2 y}{x^2 + y^2} - 0x - 0y}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^2 y}{(x^2 + y^2)^{\frac{3}{2}}}$$

We compute the limit along the line $y = \sqrt{3}x$:

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = \sqrt{3}x}} \frac{x^2 y}{(x^2 + y^2)^{3/2}} = \lim_{x \to 0} \frac{x^2 \sqrt{3}x}{\left(x^2 + \left(\sqrt{3}x\right)^2\right)^{3/2}} = \lim_{x \to 0} \frac{\sqrt{3}x^3}{\left(4x^2\right)^{3/2}} = \lim_{x \to 0} \frac{\sqrt{3}x^3}{8x^3} = \frac{\sqrt{3}}{8} \neq 0$$

Since this limit is not zero, f does not satisfy Eq. (9), hence f is not differentiable at $(0, 0)$.

69. Prove that if $f(x, y)$ is differentiable and $\nabla f_{(x,y)} = \mathbf{0}$ for all (x, y), then f is constant.

SOLUTION Since $\nabla f = \langle f_x, f_y \rangle = \langle 0, 0 \rangle$ for all (x, y), we have

$$f_x(x, y) = f_y(x, y) = 0 \text{ for all } (x, y) \tag{1}$$

Let $Q_0 = (x_0, y_0)$ be a fixed point and let $P = (x_1, y_1)$ be any other point. Let $\mathbf{c}(t) = \langle x(t), y(t) \rangle$ be a parametric equation of the line joining Q_0 and P, with $P = \mathbf{c}(t_1)$ and $Q_0 = \mathbf{c}(t_0)$. We define the following function:

$$F(t) = f(x(t), y(t))$$

$F(t)$ is defined for all t, since $f(x, y)$ is defined for all (x, y). By the Chain Rule we have

$$F'(t) = f_x(x(t), y(t)) \frac{dx}{dt} + f_y(x(t), y(t)) \frac{dy}{dt}$$

Combining with (1) we get $F'(t) = 0$ for all t. We conclude that $F(t) = \text{const}$. That is, f is constant on the line $\mathbf{c}(t)$. In particular, $f(P) = f(Q_0)$. Since P is any point, it follows that $f(x, y)$ is a constant function.

70. Prove the following Quotient Rule, where f, g are differentiable:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

SOLUTION The Quotient Rule is valid for partial derivatives, therefore

$$\nabla\left(\frac{f}{g}\right) = \left\langle \frac{\partial}{\partial x}\left(\frac{f}{g}\right), \frac{\partial}{\partial y}\left(\frac{f}{g}\right), \frac{\partial}{\partial z}\left(\frac{f}{g}\right)\right\rangle = \left\langle \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}, \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}, \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right\rangle$$

$$= \left\langle \frac{g\frac{\partial f}{\partial x}}{g^2}, \frac{g\frac{\partial f}{\partial y}}{g^2}, \frac{g\frac{\partial f}{\partial z}}{g^2}\right\rangle - \left\langle \frac{f\frac{\partial g}{\partial x}}{g^2}, \frac{f\frac{\partial g}{\partial y}}{g^2}, \frac{f\frac{\partial g}{\partial z}}{g^2}\right\rangle = \frac{g}{g^2}\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle - \frac{f}{g^2}\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= \frac{g}{g^2}\nabla f - \frac{f}{g^2}\nabla g = \frac{g\nabla f - f\nabla g}{g^2}$$

In Exercises 71–73, a path $\mathbf{c}(t) = (x(t), y(t))$ *follows the gradient of a function* $f(x, y)$ *if the tangent vector* $\mathbf{c}'(t)$ *points in the direction of* ∇f *for all* t. *In other words,* $\mathbf{c}'(t) = k(t)\nabla f_{\mathbf{c}(t)}$ *for some positive function* $k(t)$. *Note that in this case,* $\mathbf{c}(t)$ *crosses each level curve of* $f(x, y)$ *at a right angle.*

71. Show that if the path $\mathbf{c}(t) = (x(t), y(t))$ follows the gradient of $f(x, y)$, then

$$\frac{y'(t)}{x'(t)} = \frac{f_y}{f_x}$$

SOLUTION Since $\mathbf{c}(t)$ follows the gradient of $f(x, y)$, we have

$$\mathbf{c}'(t) = k(t)\nabla f_{\mathbf{c}(t)} = k(t)\left\langle f_x(\mathbf{c}(t)), f_y(\mathbf{c}(t))\right\rangle$$

which implies that

$$x'(t) = k(t) f_x(\mathbf{c}(t)) \quad \text{and} \quad y'(t) = k(t) f_y(\mathbf{c}(t))$$

Hence,

$$\frac{y'(t)}{x'(t)} = \frac{k(t) f_y(\mathbf{c}(t))}{k(t) f_x(\mathbf{c}(t))} = \frac{f_y(\mathbf{c}(t))}{f_x(\mathbf{c}(t))}$$

or in short notation,

$$\frac{y'(t)}{x'(t)} = \frac{f_y}{f_x}$$

72. Find a path of the form $\mathbf{c}(t) = (t, g(t))$ passing through $(1, 2)$ that follows the gradient of $f(x, y) = 2x^2 + 8y^2$ (Figure 16). *Hint:* Use Separation of Variables.

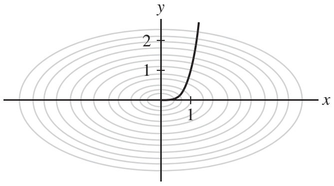


FIGURE 16 The path $\mathbf{c}(t)$ is orthogonal to the level curves of $f(x, y) = 2x^2 + 8y^2$.

SOLUTION By the previous exercise, if $\mathbf{c}(t) = (x(t), y(t))$ follows the gradient of f, then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{f_y}{f_x} \tag{1}$$

We find the partial derivatives of f:

$$f_y = \frac{\partial}{\partial y}\left(2x^2 + 8y^2\right) = 16y, \quad f_x = \frac{\partial}{\partial x}\left(2x^2 + 8y^2\right) = 4x$$

Substituting in (1) we get

$$\frac{dy}{dx} = \frac{16y}{4x} = \frac{4y}{x}$$

We solve the differential equation using separation of variables. We obtain

$$\frac{dy}{y} = 4\frac{dx}{x}$$
$$\int \frac{dy}{y} = 4\int \frac{dx}{x}$$
$$\ln y = 4\ln x + c = \ln x^4 + c$$

or

$$y = e^{\ln x^4 + c} = e^c x^4$$

Denoting $k = e^c$, we obtain the following solution:

$$y = kx^4$$

The corresponding path may be parametrized using the parameter $x = t$ as

$$\mathbf{c}(t) = \left(t, kt^4\right) \tag{2}$$

Since we want the path to pass through $(1, 2)$, there must be a solution t for the equation

$$\left(t, kt^4\right) = (1, 2)$$

or

$$\begin{aligned} t &= 1 \\ kt^4 &= 2 \end{aligned} \quad \Rightarrow \quad k \cdot 1^4 = 2 \quad \Rightarrow \quad k = 2$$

Substituting in (2) we obtain the following path:

$$\mathbf{c}(t) = \left(t, 2t^4\right)$$

We now show that $\mathbf{c}$ follows the gradient of $f(x, y) = 2x^2 + 8y^2$. We have

$$\mathbf{c}'(t) = \left(1, 8t^3\right) \quad \text{and} \quad \nabla f = \langle f_x, f_y\rangle = \langle 4x, 16y\rangle$$

Therefore, $\nabla f_{\mathbf{c}(t)} = \left\langle 4t, 16 \cdot 2t^4\right\rangle = \left\langle 4t, 32t^4\right\rangle$, so we obtain

$$\mathbf{c}'(t) = \left(1, 8t^3\right) = \frac{1}{4t}\left\langle 4t, 32t^4\right\rangle = \frac{1}{4t}\nabla f_{\mathbf{c}(t)}, \quad t \neq 0$$

For $t = 0$, $\nabla f_{\mathbf{c}(0)} = \nabla f_{(0,0)} = \langle 0, 0\rangle$ and $\mathbf{c}'(0) = \langle 1, 0\rangle$. We conclude that $\mathbf{c}$ follows the gradient of f for $t \neq 0$.

73. CAS Find the curve $y = g(x)$ passing through $(0, 1)$ that crosses each level curve of $f(x, y) = y \sin x$ at a right angle. If you have a computer algebra system, graph $y = g(x)$ together with the level curves of f.

SOLUTION Using $f_x = y\cos x$, $f_y = \sin x$, and $y(0) = 1$, we get

$$\frac{dy}{dx} = \frac{\tan x}{y} \quad \Rightarrow \quad y(0) = 1$$

We solve the differential equation using separation of variables:

$$y\,dy = \tan x\,dx$$
$$\int y\,dy = \int \tan x\,dx$$
$$\frac{1}{2}y^2 = -\ln|\cos x| + k$$
$$y^2 = -2\ln|\cos x| + k = -\ln\left(\cos^2 x\right) + k$$
$$y = \pm\sqrt{-\ln\left(\cos^2 x\right) + k}$$

Since $y(0) = 1 > 0$, the appropriate sign is the positive sign. That is,

$$y = \sqrt{-\ln\left(\cos^2 x\right) + k} \tag{1}$$

We find the constant k by substituting $x = 0$, $y = 1$ and solve for k. This gives

$$1 = \sqrt{-\ln\left(\cos^2 0\right) + k} = \sqrt{-\ln 1 + k} = \sqrt{k}$$

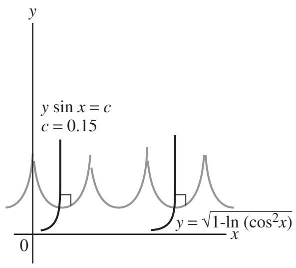
Hence,

$$k = 1$$

Substituting in (2) gives the following solution:

$$y = \sqrt{1 - \ln\left(\cos^2 x\right)} \tag{2}$$

The following figure shows the graph of the curve (3) together with some level curves of f.



14.6 The Chain Rule (LT Section 15.6)

Preliminary Questions

1. Let $f(x, y) = xy$, where $x = uv$ and $y = u + v$.

(a) What are the primary derivatives of f?

(b) What are the independent variables?

SOLUTION

(a) The primary derivatives of f are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) The independent variables are u and v, on which x and y depend.

In Questions 2 and 3, suppose that $f(u, v) = ue^v$, where $u = rs$ and $v = r + s$.

2. The composite function $f(u, v)$ is equal to:

(a) rse^{r+s} **(b)** re^s **(c)** rse^{rs}

SOLUTION The composite function $f(u, v)$ is obtained by replacing u and v in the formula for $f(u, v)$ by the corresponding functions $u = rs$ and $v = r + s$. This gives

$$f\left(u(r, s), v(r, s)\right) = u(r, s)e^{v(r,s)} = rse^{r+s}$$

Answer (a) is the correct answer.

3. What is the value of $f(u, v)$ at $(r, s) = (1, 1)$?

SOLUTION We compute $u = rs$ and $v = r + s$ at the point $(r, s) = (1, 1)$:

$$u(1, 1) = 1 \cdot 1 = 1; \quad v(1, 1) = 1 + 1 = 2$$

Substituting in $f(u, v) = ue^v$, we get

$$f(u, v)\Big|_{(r,s)=(1,1)} = 1 \cdot e^2 = e^2.$$

4. According to the Chain Rule, $\partial f/\partial r$ is equal to (choose the correct answer):

(a) $\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$ **(b)** $\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$ **(c)** $\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x}$

SOLUTION For a function $f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$, the Chain Rule states that the partial derivative $\frac{\partial f}{\partial r}$ is as given in (b). That is,

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

5. Suppose that x, y, z are functions of the independent variables u, v, w. Which of the following terms appear in the Chain Rule expression for $\partial f/\partial w$?

(a) $\frac{\partial f}{\partial v}\frac{\partial x}{\partial v}$ **(b)** $\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$ **(c)** $\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$

SOLUTION By the Chain Rule, the derivative $\frac{\partial f}{\partial w}$ is

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

Therefore (c) is the only correct answer.

6. With notation as in the previous question, does $\partial x/\partial v$ appear in the Chain Rule expression for $\partial f/\partial u$?

SOLUTION The Chain Rule expression for $\frac{\partial f}{\partial u}$ is

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

The derivative $\frac{\partial x}{\partial v}$ does not appear in differentiating f with respect to the independent variable u.

Exercises

1. Let $f(x, y, z) = x^2y^3 + z^4$ and $x = s^2$, $y = st^2$, and $z = s^2t$.

(a) Calculate the primary derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

(b) Calculate $\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}$.

(c) Compute $\frac{\partial f}{\partial s}$ using the Chain Rule:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Express the answer in terms of the independent variables s, t.

SOLUTION

(a) The primary derivatives of $f(x, y, z) = x^2y^3 + z^4$ are

$$\frac{\partial f}{\partial x} = 2xy^3, \quad \frac{\partial f}{\partial y} = 3x^2y^2, \quad \frac{\partial f}{\partial z} = 4z^3$$

(b) The partial derivatives of x, y, and z with respect to s are

$$\frac{\partial x}{\partial s} = 2s, \quad \frac{\partial y}{\partial s} = t^2, \quad \frac{\partial z}{\partial s} = 2st$$

(c) We use the Chain Rule and the partial derivatives computed in parts (a) and (b) to find the following derivative:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 2xy^3 \cdot 2s + 3x^2y^2t^2 + 4z^3 \cdot 2st = 4xy^3s + 3x^2y^2t^2 + 8z^3st$$

To express the answer in terms of the independent variables s, t we substitute $x = s^2$, $y = st^2$, $z = s^2t$. This gives

$$\frac{\partial f}{\partial s} = 4s^2\left(st^2\right)^3 s + 3\left(s^2\right)^2\left(st^2\right)^2 t^2 + 8\left(s^2t\right)^3 st = 4s^6t^6 + 3s^6t^6 + 8s^7t^4 = 7s^6t^6 + 8s^7t^4.$$

2. Let $f(x, y) = x\cos(y)$ and $x = u^2 + v^2$ and $y = u - v$.

(a) Calculate the primary derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

(b) Use the Chain Rule to calculate $\partial f/\partial v$. Leave the answer in terms of both the dependent and the independent variables.

(c) Determine (x, y) for $(u, v) = (2, 1)$ and evaluate $\partial f/\partial v$ at $(u, v) = (2, 1)$.

SOLUTION

(a) The primary derivatives of $f(x, y) = x\cos(y)$ are

$$\frac{\partial f}{\partial x} = \cos(y), \quad \frac{\partial f}{\partial y} = -x\sin(y).$$

(b) By the Chain Rule, we have

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{1}$$

We compute the partial derivatives $\frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial v}$:

$$\frac{\partial x}{\partial v}=2v, \quad \frac{\partial y}{\partial v}=-1.$$

Substituting these derivatives and the primary derivatives computed in part (a) in the Chain Rule (1) gives

$$\frac{\partial f}{\partial v}=\cos(y)\cdot 2v-x\sin(y)\cdot(-1)=2v\cos(y)+x\sin(y)$$

(c) We substitute $u=2$, $v=1$ in $x=u^2+v^2$ and $y=u-v$, and determine (x,y) for $(u,v)=(2,1)$. This gives

$$x=2^2+1^2=5, \quad y=2-1=1.$$

To find $\frac{\partial f}{\partial x}$ at $(u,v)=(2,1)$ we substitute $u=2$, $v=1$, $x=5$, and $y=1$ in $\frac{\partial f}{\partial v}$ computed in part (b). We obtain

$$\left.\frac{\partial f}{\partial v}\right|_{(u,v)=(2,1)}=2\cdot 1\cos 1+5\sin 1=2\cos 1+5\sin 1.$$

In Exercises 3–10, use the Chain Rule to calculate the partial derivatives. Express the answer in terms of the independent variables.

3. $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}$; $f(x,y,z)=xy+z^2$, $x=s^2$, $y=2rs$, $z=r^2$

SOLUTION We perform the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $f(x,y,z)=xy+z^2$ are

$$\frac{\partial f}{\partial x}=y, \quad \frac{\partial f}{\partial y}=x, \quad \frac{\partial f}{\partial z}=2z$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s} \tag{1}$$

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \tag{2}$$

We compute the partial derivatives of x, y, z with respect to s and r:

$$\frac{\partial x}{\partial s}=2s, \quad \frac{\partial y}{\partial s}=2r, \quad \frac{\partial z}{\partial s}=0.$$

$$\frac{\partial x}{\partial r}=0, \quad \frac{\partial y}{\partial r}=2s, \quad \frac{\partial z}{\partial r}=2r.$$

Substituting these derivatives and the primary derivatives computed in step 1 in (1) and (2), we get

$$\frac{\partial f}{\partial s}=y\cdot 2s+x\cdot 2r+2z\cdot 0=2ys+2xr$$

$$\frac{\partial f}{\partial r}=y\cdot 0+x\cdot 2s+2z\cdot 2r=2xs+4zr$$

Step 3. Express the answer in terms of r and s. We substitute $x=s^2$, $y=2rs$, and $z=r^2$ in $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial r}$ in step 2, to obtain

$$\frac{\partial f}{\partial s}=2rs\cdot 2s+s^2\cdot 2r=4rs^2+2rs^2=6rs^2.$$

$$\frac{\partial f}{\partial r}=2s^2\cdot s+4r^2\cdot r=2s^3+4r^3.$$

4. $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}$; $f(x,y,z)=xy+z^2$, $x=r+s-2t$, $y=3rt$, $z=s^2$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $f(x,y,z)=xy+z^2$ are

$$\frac{\partial f}{\partial x}=y, \quad \frac{\partial f}{\partial y}=x, \quad \frac{\partial f}{\partial z}=2z$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = y\frac{\partial x}{\partial r} + x\frac{\partial y}{\partial r} + 2z\frac{\partial z}{\partial r} \tag{1}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = y\frac{\partial x}{\partial t} + x\frac{\partial y}{\partial t} + 2z\frac{\partial z}{\partial t} \tag{2}$$

We compute the partial derivatives of x, y with respect to r and t:

$$\frac{\partial x}{\partial r} = 1, \quad \frac{\partial y}{\partial r} = 3t, \quad \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial t} = -2, \quad \frac{\partial y}{\partial t} = 3r, \quad \frac{\partial z}{\partial t} = 0$$

Substituting in (1) and (2), we get

$$\frac{\partial f}{\partial r} = y + 3tx + 2z \cdot 0 = y + 3xt$$

$$\frac{\partial f}{\partial t} = y \cdot (-2) + x \cdot 3r + 2z \cdot 0 = -2y + 3xr$$

Step 3. Express the answer in terms of r and t. We substitute $x = r + s - 2t$, $y = 3rt$, and $z = s^2$ in $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial t}$ obtained in step 2. This gives

$$\frac{\partial f}{\partial r} = 3rt + 3(r + s - 2t)t = 3rt + 3rt + 3st - 6t^2 = 6rt + 3st - 6t^2$$

$$\frac{\partial f}{\partial t} = -2 \cdot 3rt + 3(r + s - 2t)r = -6rt + 3r^2 + 3sr - 6tr = -12rt + 3rs + 3r^2$$

5. $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}$; $g(x, y) = \cos(x - y)$, $x = 3u - 5v$, $y = -7u + 15v$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $g(x, y) = \cos(x - y)$ are:

$$\frac{\partial g}{\partial x} = -\sin(x - y), \quad \frac{\partial g}{\partial y} = \sin(x - y)$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} = -\sin(x - y)\frac{\partial x}{\partial u} + \sin(x - y)\frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v} = -\sin(x - y)\frac{\partial x}{\partial v} + \sin(x - y)\frac{\partial y}{\partial v}$$

We compute the partial derivatives of x, y with respect to u and v:

$$\frac{\partial x}{\partial u} = 3, \quad \frac{\partial x}{\partial v} = -5$$

$$\frac{\partial y}{\partial u} = -7, \quad \frac{\partial y}{\partial v} = 15$$

substituting in the expressions above we have:

$$\frac{\partial g}{\partial u} = -\sin(x - y)(3) + \sin(x - y)(-7) = -10\sin(x - y)$$

$$\frac{\partial g}{\partial v} = -\sin(x - y)(-5) + \sin(x - y)(15) = 20\sin(x - y)$$

Step 3. Express the answer in terms of u and v. We substitute $x = 3u - 5v$ and $y = -7u + 15v$ in $\partial g/\partial u$ and $\partial g/\partial v$ found in step 2. This gives:

$$\frac{\partial g}{\partial u} = -10\sin(10u - 20v)$$

$$\frac{\partial g}{\partial v} = 20\sin(10u - 20v)$$

6. $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial v}$; $R(x, y) = (3x + 4y)^5$, $x = u^2$, $y = uv$

SOLUTION We perform the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $R(x, y) = (3x + 4y)^5$ are:

$$\frac{\partial R}{\partial x} = 15(3x + 4y)^4, \quad \frac{\partial R}{\partial y} = 20(3x + 4y)^4$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial u} = 15(3x + 4y)^4 \frac{\partial x}{\partial u} + 20(3x + 4y)^4 \frac{\partial y}{\partial u}$$

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial v} = 15(3x + 4y)^4 \frac{\partial x}{\partial v} + 20(3x + 4y)^4 \frac{\partial y}{\partial v}$$

We compute the partial derivatives of x, y with respect to u and v:

$$\frac{\partial x}{\partial u} = 2u, \quad \frac{\partial x}{\partial v} = 0$$

$$\frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = u$$

Substituting in the expressions above we get:

$$\frac{\partial R}{\partial u} = 15(3x + 4y)^4(2u) + 20(3x + 4y)^5(v) = 30(3x + 4y)^5(u) + 20v(3x + 4y)^5$$

$$\frac{\partial R}{\partial v} = 15(3x + 4y)^4(0) + 20(3x + 4y)^5(u) = 20(3x + 4y)^5(u)$$

Step 3. Express the answer in terms of u and v. We substitute $x = u^2$ and $y = uv$:

$$\frac{\partial R}{\partial u} = 30u(3u^2 + 4uv)^4 + 20v(3u^2 + 4uv)^4 = (3u^2 + 4uv)^4(30u + 20v)$$

$$\frac{\partial R}{\partial v} = 20u(3u^2 + 4uv)^4$$

7. $\dfrac{\partial F}{\partial y}$; $F(u, v) = e^{u+v}$, $u = x^2$, $v = xy$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $F(u, v) = e^{u+v}$ are

$$\frac{\partial f}{\partial u} = e^{u+v}, \quad \frac{\partial f}{\partial v} = e^{u+v}$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = e^{u+v}\frac{\partial u}{\partial y} + e^{u+v}\frac{\partial v}{\partial y} = e^{u+v}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

We compute the partial derivatives of u and v with respect to y:

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = x$$

We substitute to obtain

$$\frac{\partial F}{\partial y} = xe^{u+v} \tag{1}$$

Step 3. Express the answer in terms of x and y. We substitute $u = x^2$, $v = xy$ in (1) and (2), obtaining

$$\frac{\partial F}{\partial y} = xe^{x^2+xy}.$$

8. $\dfrac{\partial f}{\partial u}$; $f(x, y) = x^2 + y^2$, $x = e^{u+v}$, $y = u + v$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $f(x, y) = x^2 + y^2$ are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2x\frac{\partial x}{\partial u} + 2y\frac{\partial y}{\partial u}$$

We compute $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$:

$$\frac{\partial x}{\partial u} = e^{u+v}, \quad \frac{\partial y}{\partial u} = 1$$

Hence,

$$\frac{\partial f}{\partial u} = 2xe^{u+v} + 2y \tag{1}$$

Step 3. Express the answer in terms of u and v. We substitute $x = e^{u+v}$ and $y = u + v$ in (1) to obtain

$$\frac{\partial f}{\partial u} = 2e^{u+v}e^{u+v} + 2(u + v) = 2\left(e^{2(u+v)} + u + v\right)$$

9. $\dfrac{\partial h}{\partial t_2}$; $h(x, y) = \dfrac{x}{y}$, $x = t_1 t_2$, $y = t_1^2 t_2$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $h(x, y) = \frac{x}{y}$ are

$$\frac{\partial h}{\partial x} = \frac{1}{y}, \quad \frac{\partial h}{\partial y} = -\frac{x}{y^2}$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial h}{\partial t_2} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t_2} = \frac{1}{y}\frac{\partial x}{\partial t_2} - \frac{x}{y^2}\frac{\partial y}{\partial t_2}$$

We compute the partial derivatives of x and y with respect to t_2:

$$\frac{\partial x}{\partial t_2} = t_1, \quad \frac{\partial y}{\partial t_2} = t_1^2$$

Hence,

$$\frac{\partial h}{\partial t_2} = \frac{t_1}{y} - \frac{x}{y^2}t_1^2$$

Step 3. Express the answer in terms of t_1 and t_2. We substitute $x = t_1 t_2$, $y = t_1^2 t_2$ in $\frac{\partial h}{\partial t_2}$ computed in step 2, to obtain

$$\frac{\partial h}{\partial t_2} = \frac{t_1}{t_1^2 t_2} - \frac{t_1 t_2 \cdot t_1^2}{\left(t_1^2 t_2\right)^2} = \frac{1}{t_1 t_2} - \frac{1}{t_1 t_2} = 0$$

Remark: Notice that $h\left(x(t_1, t_2), y(t_1, t_2)\right) = h(t_1, t_2) = \frac{t_1 t_2}{t_1^2 t_2} = \frac{1}{t_1}$. $h(t_1, t_2)$ is independent of t_2, hence $\frac{\partial h}{\partial t_2} = 0$ (as obtained in our computations).

10. $\dfrac{\partial f}{\partial \theta}$; $f(x, y, z) = xy - z^2$, $x = r\cos\theta$, $y = \cos^2\theta$, $z = r$

SOLUTION We use the following steps:

Step 1. Compute the primary derivatives. The primary derivatives of $f(x, y, z) = xy - z^2$ are

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = -2z$$

Step 2. Apply the Chain Rule. By the Chain Rule,

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}=y\frac{\partial x}{\partial \theta}+x\frac{\partial y}{\partial \theta}-2z\frac{\partial z}{\partial \theta}$$

We compute the partial derivatives of x, y, and z with respect to θ:

$$\frac{\partial x}{\partial \theta}=-r\sin\theta,\quad \frac{\partial y}{\partial \theta}=-2\cos\theta\sin\theta=-\sin 2\theta,\quad \frac{\partial z}{\partial \theta}=0$$

Step 3. Express the answer in terms of θ and r. We substitute $x=r\cos\theta$, $y=\cos^2\theta$, and $z=r$ in (1) to obtain

$$\frac{\partial f}{\partial \theta}=-r\cos^2\theta\sin\theta-r\cos\theta\sin 2\theta=-r\cdot\frac{1}{2}\cos\theta\sin 2\theta-r\cos\theta\sin 2\theta=-\frac{3}{2}\cos\theta\sin 2\theta$$

In Exercises 11–16, use the Chain Rule to evaluate the partial derivative at the point specified.

11. $\partial f/\partial u$ and $\partial f/\partial v$ at $(u,v)=(-1,-1)$, where $f(x,y,z)=x^3+yz^2$, $x=u^2+v$, $y=u+v^2$, $z=uv$.

SOLUTION The primary derivatives of $f(x,y,z)=x^3+yz^2$ are

$$\frac{\partial f}{\partial x}=3x^2,\quad \frac{\partial f}{\partial y}=z^2,\quad \frac{\partial f}{\partial z}=2yz$$

By the Chain Rule we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}=3x^2\frac{\partial x}{\partial u}+z^2\frac{\partial y}{\partial u}+2yz\frac{\partial z}{\partial u} \tag{1}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}=3x^2\frac{\partial x}{\partial v}+z^2\frac{\partial y}{\partial v}+2yz\frac{\partial z}{\partial v} \tag{2}$$

We compute the partial derivatives of x, y, and z with respect to u and v:

$$\frac{\partial x}{\partial u}=2u,\quad \frac{\partial y}{\partial u}=1,\quad \frac{\partial z}{\partial u}=v$$

$$\frac{\partial x}{\partial v}=1,\quad \frac{\partial y}{\partial v}=2v,\quad \frac{\partial z}{\partial v}=u$$

Substituting in (1) and (2) we get

$$\frac{\partial f}{\partial u}=6x^2u+z^2+2yzv \tag{3}$$

$$\frac{\partial f}{\partial v}=3x^2+2vz^2+2yzu \tag{4}$$

We determine (x,y,z) for $(u,v)=(-1,-1)$:

$$x=(-1)^2-1=0,\quad y=-1+(-1)^2=0,\quad z=(-1)\cdot(-1)=1.$$

Finally, we substitute $(x,y,z)=(0,0,1)$ and $(u,v)=(-1,-1)$ in (3), (4) to obtain the following derivatives:

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(-1,-1)}=6\cdot 0^2\cdot(-1)+1^2+2\cdot 0\cdot 1\cdot(-1)=1$$

$$\left.\frac{\partial f}{\partial v}\right|_{(u,v)=(-1,-1)}=3\cdot 0^2+2\cdot(-1)\cdot 1^2+2\cdot 0\cdot 1\cdot(-1)=-2$$

12. $\partial f/\partial s$ at $(r,s)=(1,0)$, where $f(x,y)=\ln(xy)$, $x=3r+2s$, $y=5r+3s$.

SOLUTION The primary derivatives of $f(x,y)=\ln(xy)$ are

$$\frac{\partial f}{\partial x}=\frac{y}{xy}=\frac{1}{x},\quad \frac{\partial f}{\partial y}=\frac{x}{xy}=\frac{1}{y}$$

By the Chain Rule we have

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}=\frac{1}{x}\frac{\partial x}{\partial s}+\frac{1}{y}\frac{\partial y}{\partial s} \tag{1}$$

We compute $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$:

$$\frac{\partial x}{\partial s} = 2, \quad \frac{\partial y}{\partial s} = 3$$

Substituting in (1) we get

$$\frac{\partial f}{\partial s} = \frac{2}{x} + \frac{3}{y} \tag{2}$$

We now must determine (x, y) for $(s, r) = (1, 0)$:

$$x = 3 \cdot 0 + 2 \cdot 1 = 2, \quad y = 5 \cdot 0 + 3 \cdot 1 = 3$$

Substituting in (2) gives the following derivative:

$$\left.\frac{\partial f}{\partial s}\right|_{(s,r)=(1,0)} = \frac{2}{2} + \frac{3}{3} = 2$$

13. $\partial g/\partial\theta$ at $(r, \theta) = \left(2\sqrt{2}, \frac{\pi}{4}\right)$, where $g(x, y) = 1/(x + y^2)$, $x = r\sin\theta$, $y = r\cos\theta$.

SOLUTION We compute the primary derivatives of $g(x, y) = \frac{1}{x+y^2}$:

$$\frac{\partial g}{\partial x} = -\frac{1}{(x + y^2)^2}, \quad \frac{\partial g}{\partial y} = -\frac{2y}{(x + y^2)^2}$$

By the Chain Rule we have

$$\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{1}{(x + y^2)^2}\frac{\partial x}{\partial \theta} - \frac{2y}{(x + y^2)^2}\frac{\partial y}{\partial \theta} = -\frac{1}{(x + y^2)^2}\left(\frac{\partial x}{\partial \theta} + 2y\frac{\partial y}{\partial \theta}\right)$$

We find the partial derivatives $\frac{\partial x}{\partial \theta}$, $\frac{\partial y}{\partial \theta}$:

$$\frac{\partial x}{\partial \theta} = r\cos\theta, \quad \frac{\partial y}{\partial \theta} = -r\sin\theta$$

Hence,

$$\frac{\partial g}{\partial \theta} = -\frac{r}{(x + y^2)^2}(\cos\theta - 2y\sin\theta) \tag{1}$$

At the point $(r, \theta) = \left(2\sqrt{2}, \frac{\pi}{4}\right)$, we have $x = 2\sqrt{2}\sin\frac{\pi}{4} = 2$ and $y = 2\sqrt{2}\cos\frac{\pi}{4} = 2$. Substituting $(r, \theta) = \left(2\sqrt{2}, \frac{\pi}{4}\right)$ and $(x, y) = (2, 2)$ in (1) gives the following derivative:

$$\left.\frac{\partial g}{\partial \theta}\right|_{(r,\theta)=\left(2\sqrt{2}, \frac{\pi}{4}\right)} = \frac{-2\sqrt{2}}{(2 + 2^2)^2}\left(\cos\frac{\pi}{4} - 4\sin\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{18}\left(\frac{1}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right) = \frac{1}{6}.$$

14. $\partial g/\partial s$ at $s = 4$, where $g(x, y) = x^2 - y^2$, $x = s^2 + 1$, $y = 1 - 2s$.

SOLUTION We find the primary derivatives of $g(x, y) = x^2 - y^2$:

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = -2y$$

Applying the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial g}{\partial y} \cdot \frac{dy}{ds} = 2x\frac{dx}{ds} - 2y\frac{dy}{ds} \tag{1}$$

We compute $\frac{dx}{ds}$ and $\frac{dy}{ds}$:

$$\frac{dx}{ds} = 2s, \quad \frac{dy}{ds} = -2$$

Substituting in (1) we obtain

$$\frac{\partial g}{\partial s} = 4xs + 4y \tag{2}$$

We now determine (x, y) for $s = 4$:

$$x = 4^2 + 1 = 17, \quad y = 1 - 2 \cdot 4 = -7$$

Substituting $(x, y) = (17, -7)$ and $s = 4$ in (2) gives the following derivative:

$$\left.\frac{\partial g}{\partial s}\right|_{s=4} = 4 \cdot 17 \cdot 4 - 4 \cdot 7 = 244$$

15. $\partial g/\partial u$ at $(u, v) = (0, 1)$, where $g(x, y) = x^2 - y^2$, $x = e^u \cos v$, $y = e^u \sin v$.

SOLUTION The primary derivatives of $g(x, y) = x^2 - y^2$ are

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = -2y$$

By the Chain Rule we have

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u} = 2x\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u} \tag{1}$$

We find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$:

$$\frac{\partial x}{\partial u} = e^u \cos v, \quad \frac{\partial y}{\partial u} = e^u \sin v$$

Substituting in (1) gives

$$\frac{\partial g}{\partial u} = 2xe^u \cos v - 2ye^u \sin v = 2e^u(x \cos v - y \sin v) \tag{2}$$

We determine (x, y) for $(u, v) = (0, 1)$:

$$x = e^0 \cos 1 = \cos 1, \quad y = e^0 \sin 1 = \sin 1$$

Finally, we substitute $(u, v) = (0, 1)$ and $(x, y) = (\cos 1, \sin 1)$ in (2) and use the identity $\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$, to obtain the following derivative:

$$\left.\frac{\partial g}{\partial u}\right|_{(u,v)=(0,1)} = 2e^0\left(\cos^2 1 - \sin^2 1\right) = 2 \cdot \cos 2 \cdot 1 = 2\cos 2$$

16. $\frac{\partial h}{\partial q}$ at $(q, r) = (3, 2)$, where $h(u, v) = ue^v$, $u = q^3$, $v = qr^2$.

SOLUTION We first find the primary derivatives of $h(u, v) = ue^v$:

$$\frac{\partial h}{\partial u} = e^v, \quad \frac{\partial h}{\partial v} = ue^v$$

By the Chain Rule, we have

$$\frac{\partial h}{\partial q} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial q} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial q} = e^v\frac{\partial u}{\partial q} + ue^v\frac{\partial v}{\partial q} = e^v\left(\frac{\partial u}{\partial q} + u\frac{\partial v}{\partial q}\right) \tag{1}$$

We compute $\frac{\partial u}{\partial q}$ and $\frac{\partial v}{\partial q}$:

$$\frac{\partial u}{\partial q} = 3q^2, \quad \frac{\partial v}{\partial q} = r^2$$

Substituting in (1) gives

$$\frac{\partial h}{\partial q} = e^v\left(3q^2 + ur^2\right) \tag{2}$$

We now determine (u, v) for $(q, r) = (3, 2)$:

$$u = 3^3 = 27, \quad v = 3 \cdot 2^2 = 12$$

Substituting in (2) gives the following derivative:

$$\left.\frac{\partial h}{\partial q}\right|_{(q,r)=(3,2)} = e^{12}\left(3 \cdot 3^2 + 27 \cdot 2^2\right) = 135e^{12}$$

17. Jessica and Matthew are running toward the point P along the straight paths that make a fixed angle of θ (Figure 3). Suppose that Matthew runs with velocity v_a m/s and Jessica with velocity v_b m/s. Let $f(x, y)$ be the distance from Matthew to Jessica when Matthew is x meters from P and Jessica is y meters from P.

(a) Show that $f(x, y) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$.

(b) Assume that $\theta = \pi/3$. Use the Chain Rule to determine the rate at which the distance between Matthew and Jessica is changing when $x = 30$, $y = 20$, $v_a = 4$ m/s, and $v_b = 3$ m/s.

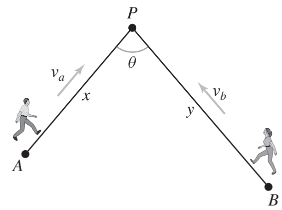


FIGURE 3

SOLUTION

(a) This is a simple application of the Law of Cosines. Connect points A and B in the diagram to form a line segment that we will call f. Then, the Law of Cosines says that $f^2 = x^2 + y^2 - 2xy\cos\theta$. By taking square roots, we find that $f = \sqrt{x^2 + y^2 - 2xy\cos\theta}$.

(b) Using the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

so we get

$$\frac{df}{dt} = \frac{(x - y\cos\theta)dx/dt}{\sqrt{x^2 + y^2 - 2xy\cos\theta}} + \frac{(y - x\cos\theta)dy/dt}{\sqrt{x^2 + y^2 - 2xy\cos\theta}}$$

and using $x = 30$, $y = 20$, and $dx/dt = 4$, $dy/dt = 3$, we get

$$\frac{df}{dt} = \frac{180 - 170\cos\theta}{\sqrt{1300 - 1200\cos\theta}}$$

18. The Law of Cosines states that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where a, b, c are the sides of a triangle and θ is the angle opposite the side of length c.

(a) Compute $\partial\theta/\partial a$, $\partial\theta/\partial b$, and $\partial\theta/\partial c$ using implicit differentiation.

(b) Suppose that $a = 10$, $b = 16$, $c = 22$. Estimate the change in θ if a and b are increased by 1 and c is increased by 2.

SOLUTION

(a) Let $F(a, b, c, \theta) = a^2 + b^2 - 2ab\cos\theta - c^2$. We use the formulas obtained by implicit differentiation (Eq. (7)) to write

$$\frac{\partial\theta}{\partial a} = -\frac{\frac{\partial F}{\partial a}}{\frac{\partial F}{\partial\theta}}, \quad \frac{\partial\theta}{\partial b} = -\frac{\frac{\partial F}{\partial b}}{\frac{\partial F}{\partial\theta}}, \quad \frac{\partial\theta}{\partial c} = -\frac{\frac{\partial F}{\partial c}}{\frac{\partial F}{\partial\theta}} \tag{1}$$

The partial derivatives of F are

$$\frac{\partial F}{\partial a} = 2a - 2b\cos\theta, \quad \frac{\partial F}{\partial b} = 2b - 2a\cos\theta, \quad \frac{\partial F}{\partial c} = -2c, \quad \frac{\partial F}{\partial\theta} = 2ab\sin\theta$$

Substituting these derivatives in (1), we obtain

$$\frac{\partial\theta}{\partial a} = -\frac{2a - 2b\cos\theta}{2ab\sin\theta} = -\frac{a - b\cos\theta}{ab\sin\theta}$$
$$\frac{\partial\theta}{\partial b} = -\frac{2b - 2a\cos\theta}{2ab\sin\theta} = -\frac{b - a\cos\theta}{ab\sin\theta}$$
$$\frac{\partial\theta}{\partial c} = -\frac{-2c}{2ab\sin\theta} = \frac{c}{ab\sin\theta}$$

(b) The linear approximation for θ is

$$\Delta\theta \approx \frac{\partial\theta}{\partial a}\Delta a + \frac{\partial\theta}{\partial b}\Delta b + \frac{\partial\theta}{\partial c}\Delta c = \frac{\partial\theta}{\partial a}\cdot 1 + \frac{\partial\theta}{\partial b}\cdot 1 + \frac{\partial\theta}{\partial c}\cdot 2 \tag{2}$$

We find the partial derivatives for $a = 10$, $b = 16$, $c = 22$. We first find θ using the relation $c^2 = a^2 + b^2 - 2ab\cos\theta$. This gives

$$22^2 = 10^2 + 16^2 - 2 \cdot 10 \cdot 16\cos\theta$$
$$484 = 356 - 320\cos\theta$$
$$\cos\theta = \frac{356 - 484}{320} = -0.4 \quad \Rightarrow \quad \theta \approx 1.98 \text{ rad}$$

We now substitute $(a, b, c, \theta) = (10, 16, 22, 1.98)$ in the partial derivatives of θ to obtain

$$\frac{\partial\theta}{\partial a} = -\frac{10 - 16\cos 1.98}{10 \cdot 16\sin 1.98} \approx -0.111$$
$$\frac{\partial\theta}{\partial b} = -\frac{16 - 10\cos 1.98}{10 \cdot 16\sin 1.98} \approx -0.136$$
$$\frac{\partial\theta}{\partial c} = \frac{22}{10 \cdot 16\sin 1.98} \approx 0.15$$

Substituting in (2) gives the following estimation for $\Delta\theta$:

$$\Delta\theta \approx -0.111 - 0.136 + 2 \cdot 0.15 = 0.053$$

We conclude that the angle θ will increase by approximately 0.053 rad.

19. Let $u = u(x, y)$, and let (r, θ) be polar coordinates. Verify the relation

$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2}u_\theta^2 \qquad \boxed{8}$$

Hint: Compute the right-hand side by expressing u_θ and u_r in terms of u_x and u_y.

SOLUTION By the Chain Rule we have

$$u_\theta = u_x x_\theta + u_y y_\theta \tag{1}$$
$$u_r = u_x x_r + u_y y_r \tag{2}$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, the partial derivatives of x and y with respect to r and θ are

$$x_\theta = -r\sin\theta, \quad y_\theta = r\cos\theta$$
$$x_r = \cos\theta, \quad y_r = \sin\theta$$

Substituting in (1) and (2) gives

$$u_\theta = (-r\sin\theta)u_x + (r\cos\theta)u_y \tag{3}$$
$$u_r = (\cos\theta)u_x + (\sin\theta)u_y \tag{4}$$

We now solve these equations for u_x and u_y in terms of u_θ and u_r. Multiplying (3) by $(-\sin\theta)$ and (4) by $r\cos\theta$ and adding the resulting equations gives

$$\begin{array}{rl} (-\sin\theta)u_\theta &= (r\sin^2\theta)u_x - (r\cos\theta\sin\theta)u_y \\ + \quad r\cos\theta u_r &= (r\cos^2\theta)u_x + (r\cos\theta\sin\theta)u_y \\ \hline (r\cos\theta)u_r - (\sin\theta)u_\theta &= ru_x \end{array}$$

or

$$u_x = (\cos\theta)u_r - \frac{\sin\theta}{r}u_\theta \tag{5}$$

Similarly, we multiply (3) by $\cos\theta$ and (4) by $r\sin\theta$ and add the resulting equations. We get

$$\begin{array}{rl} (\cos\theta)u_\theta &= (-r\sin\theta\cos\theta)u_x + \left(r\cos^2\theta\right)u_y \\ + \quad r\sin\theta u_r &= (r\sin\theta\cos\theta)u_x + (r\sin^2\theta)u_y \\ \hline (\cos\theta)u_\theta + (r\sin\theta)u_r &= ru_y \end{array}$$

or

$$u_y = (\sin\theta)u_r + \frac{\cos\theta}{r}u_\theta \tag{6}$$

We now use (5) and (6) to compute $\|\nabla u\|^2$ in terms of u_r and u_θ. We get

$$\begin{aligned}
\|\nabla u\|^2 = u_x^2 + u_y^2 &= \left((\cos\theta)u_r - \frac{\sin\theta}{r}u_\theta\right)^2 + \left((\sin\theta)u_r + \frac{\cos\theta}{r}u_\theta\right)^2 \\
&= \left(\cos^2\theta\right)u_r^2 - \frac{2\cos\theta\sin\theta}{r}u_r u_\theta + \frac{\sin^2\theta}{r^2}u_\theta^2 + \left(\sin^2\theta\right)u_r^2 + \frac{2\sin\theta\cos\theta}{r}u_r u_\theta + \frac{\cos^2\theta}{r^2}u_\theta^2 \\
&= \left(\cos^2\theta + \sin^2\theta\right)u_r^2 + \frac{1}{r^2}\left(\sin^2\theta + \cos^2\theta\right)u_\theta^2 = u_r^2 + \frac{1}{r^2}u_\theta^2
\end{aligned}$$

That is,

$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2}u_\theta^2$$

20. Let $u(r,\theta) = r^2\cos^2\theta$. Use Eq. (8) to compute $\|\nabla u\|^2$. Then compute $\|\nabla u\|^2$ directly by observing that $u(x,y) = x^2$, and compare.

SOLUTION By Eq. (8) we have

$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2}u_\theta^2$$

We compute the partial derivatives of $u(r,\theta) = r^2\cos^2\theta$:

$$u_r = 2r\cos^2\theta, \quad u_\theta = r^2 \cdot 2\cos\theta(-\sin\theta) = -2r^2\cos\theta\sin\theta$$

Substituting in Eq. (8) we get

$$\begin{aligned}
\|\nabla u\|^2 &= \left(2r\cos^2\theta\right)^2 + \frac{1}{r^2}\left(-2r^2\cos\theta\sin\theta\right)^2 = 4r^2\cos^4\theta + 4r^2\cos^2\theta\sin^2\theta \\
&= 4r^2\cos^2\theta(\cos^2\theta + \sin^2\theta) = 4r^2\cos^2\theta
\end{aligned}$$

That is,

$$\|\nabla u\|^2 = 4r^2\cos^2\theta \tag{1}$$

We now compute $\|\nabla u\|^2$ directly. We first express $u(r,\theta)$ as a function of x and y. Since $x = r\cos\theta$, we have

$$u(x,y) = x^2$$

Hence $u_x = 2x$, $u_y = 0$, so we obtain

$$\|\nabla u\|^2 = u_x^2 + u_y^2 = (2x)^2 + 0^2 = 4x^2 = 4(r\cos\theta)^2 = 4r^2\cos^2\theta$$

The answer agrees with the result in (1), as expected.

21. Let $x = s + t$ and $y = s - t$. Show that for any differentiable function $f(x,y)$,

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$$

SOLUTION By the Chain Rule we have

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot 1 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot(-1) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}
\end{aligned}$$

Hence, using the algebraic identity $(a+b)(a-b) = a^2 - b^2$, we get

$$\frac{\partial f}{\partial s}\cdot\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\cdot\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2.$$

22. Express the derivatives

$$\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \quad \text{in terms of} \quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

where (ρ, θ, ϕ) are spherical coordinates.

SOLUTION The spherical coordinates are

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi \tag{1}$$

We apply the Chain Rule to write

$$\begin{aligned}
\frac{\partial f}{\partial \rho} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho}\\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}\\
\frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}
\end{aligned} \tag{2}$$

We use (1) to compute the partial derivatives of x, y, and z with respect to ρ, θ, and ϕ. This gives

$$\begin{aligned}
&\frac{\partial x}{\partial \theta} = -\rho \sin\phi \sin\theta, \quad &&\frac{\partial y}{\partial \theta} = \rho \sin\phi \cos\theta, \quad &&\frac{\partial z}{\partial \theta} = 0\\
&\frac{\partial x}{\partial \phi} = \rho \cos\phi \cos\theta, \quad &&\frac{\partial y}{\partial \phi} = \rho \cos\phi \sin\theta, \quad &&\frac{\partial z}{\partial \phi} = -\rho \sin\phi\\
&\frac{\partial x}{\partial \rho} = \sin\phi \cos\theta, \quad &&\frac{\partial y}{\partial \rho} = \sin\phi \sin\theta, \quad &&\frac{\partial z}{\partial \rho} = \cos\phi
\end{aligned}$$

Substituting these derivatives in (2), we get

$$\begin{aligned}
\frac{\partial f}{\partial \rho} &= (\sin\phi\cos\theta)\frac{\partial f}{\partial x} + (\sin\phi\sin\theta)\frac{\partial f}{\partial y} + (\cos\phi)\frac{\partial f}{\partial z}\\
\frac{\partial f}{\partial \phi} &= (\rho\cos\phi\cos\theta)\frac{\partial f}{\partial x} + (\rho\cos\phi\sin\theta)\frac{\partial f}{\partial y} - (\rho\sin\phi)\frac{\partial f}{\partial z}\\
\frac{\partial f}{\partial \theta} &= (-\rho\sin\phi\sin\theta)\frac{\partial f}{\partial x} + (\rho\sin\phi\cos\theta)\frac{\partial f}{\partial y}
\end{aligned}$$

23. Suppose that z is defined implicitly as a function of x and y by the equation $F(x, y, z) = xz^2 + y^2z + xy - 1 = 0$.

(a) Calculate F_x, F_y, F_z.

(b) Use Eq. (7) to calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

SOLUTION

(a) The partial derivatives of F are

$$F_x = z^2 + y, \quad F_y = 2yz + x, \quad F_z = 2xz + y^2$$

(b) By Eq. (7) we have

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{z^2 + y}{2xz + y^2}\\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{2yz + x}{2xz + y^2}
\end{aligned}$$

24. Calculate $\partial z/\partial x$ and $\partial z/\partial y$ at the points $(3, 2, 1)$ and $(3, 2, -1)$, where z is defined implicitly by the equation $z^4 + z^2x^2 - y - 8 = 0$.

SOLUTION For $F(x, y, z) = z^4 + z^2x^2 - y - 8 = 0$, we use the following equalities, (Eq. (7)):

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \tag{1}$$

The partial derivatives of F are

$$F_x = 2z^2x, \quad F_y = -1, \quad F_z = 4z^3 + 2zx^2$$

Substituting in (1) gives

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{2z^2x}{4z^3 + 2zx^2} = -\frac{zx}{2z^2 + x^2}\\
\frac{\partial z}{\partial y} &= \frac{1}{4z^3 + 2zx^2}
\end{aligned}$$

At the point (3, 2, 1), we have

$$\left.\frac{\partial z}{\partial x}\right|_{(3,2,1)} = -\frac{1 \cdot 3}{2 \cdot 1^2 + 3^2} = -\frac{3}{11}, \quad \left.\frac{\partial z}{\partial y}\right|_{(3,2,1)} = \frac{1}{4 \cdot 1^3 + 2 \cdot 1 \cdot 3^2} = \frac{1}{22}$$

At the point (3, 2, −1), we have

$$\left.\frac{\partial z}{\partial x}\right|_{(3,2,-1)} = -\frac{-3}{2 \cdot (-1)^2 + 3^2} = \frac{3}{11}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(3,2,-1)} = \frac{1}{4 \cdot (-1)^3 + 2 \cdot (-1) \cdot 3^2} = -\frac{1}{22}$$

In Exercises 25–30, calculate the partial derivative using implicit differentiation.

25. $\frac{\partial z}{\partial x}$, $\quad x^2y + y^2z + xz^2 = 10$

SOLUTION For $F(x, y, z) = x^2y + y^2z + xz^2 = 10$ we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \tag{1}$$

We compute the partial derivatives of F:

$$F_x = 2xy + z^2, \quad F_z = y^2 + 2xz$$

Substituting in (1) gives the following derivative:

$$\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{2xz + y^2}$$

26. $\frac{\partial w}{\partial z}$, $\quad x^2w + w^3 + wz^2 + 3yz = 0$

SOLUTION We find the partial derivatives F_w and F_z of

$$F(x, w, z) = x^2w + w^3 + wz^2 + 3yz$$
$$F_w = x^2 + 3w^2 + z^2, \quad F_z = 2wz + 3y$$

Using Eq. (7) we get

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{2wz + 3y}{x^2 + 3w^2 + z^2}.$$

27. $\frac{\partial z}{\partial y}$, $\quad e^{xy} + \sin(xz) + y = 0$

SOLUTION We use Eq. (7):

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \tag{1}$$

The partial derivatives of $F(x, y, z) = e^{xy} + \sin(xz) + y$ are

$$F_y = xe^{xy} + 1, \quad F_z = x\cos(xz)$$

Substituting in (1), we get

$$\frac{\partial z}{\partial y} = -\frac{xe^{xy} + 1}{x\cos(xz)}$$

28. $\frac{\partial r}{\partial t}$ and $\frac{\partial t}{\partial r}$, $\quad r^2 = te^{s/r}$

SOLUTION We use the formulas obtained by implicit differentiation of $F(r, s, t) = r^2 - te^{s/r}$ (Eq. (7)):

$$\frac{\partial r}{\partial t} = -\frac{F_t}{F_r}, \quad \frac{\partial t}{\partial r} = -\frac{F_r}{F_t} \tag{1}$$

The partial derivatives of F are

$$F_r = 2r - te^{s/r}\left(-\frac{s}{r^2}\right) = 2r + \frac{st}{r^2}e^{s/r}$$

$$F_t = -e^{s/r}$$

Substituting in (1) gives

$$\frac{\partial r}{\partial t} = \frac{e^{s/r}}{2r + \frac{st}{r^2}e^{s/r}} = \frac{r^2 e^{s/r}}{2r^3 + ste^{s/r}}$$

$$\frac{\partial t}{\partial r} = \frac{2r + \frac{st}{r^2}e^{s/r}}{e^{s/r}} = \frac{2r^3 + ste^{s/r}}{r^2 e^{s/r}} = 2re^{-s/r} + \frac{st}{r^2}$$

29. $\frac{\partial w}{\partial y}$, $\quad \frac{1}{w^2 + x^2} + \frac{1}{w^2 + y^2} = 1$ at $(x, y, w) = (1, 1, 1)$

SOLUTION Using the formula obtained by implicit differentiation (Eq. (7)), we have

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} \tag{1}$$

We find the partial derivatives of $F(x, y, w) = \frac{1}{w^2+x^2} + \frac{1}{w^2+y^2} - 1$:

$$F_y = -\frac{2y}{\left(w^2 + y^2\right)^2}, \quad F_w = \frac{-2w}{\left(w^2 + x^2\right)^2} - \frac{2w}{\left(w^2 + y^2\right)^2}$$

We substitute in (1) to obtain

$$\frac{\partial w}{\partial y} = -\frac{\frac{-2y}{\left(w^2+y^2\right)^2}}{\frac{-2w}{\left(w^2+x^2\right)^2} - \frac{2w}{\left(w^2+y^2\right)^2}} = -\frac{y\left(w^2 + x^2\right)^2}{w\left(w^2 + y^2\right)^2 + w\left(w^2 + x^2\right)^2} = \frac{-y\left(w^2 + x^2\right)^2}{w\left(\left(w^2 + y^2\right)^2 + \left(w^2 + x^2\right)^2\right)}$$

30. $\partial U/\partial T$ and $\partial T/\partial U$, $\quad (TU - V)^2 \ln(W - UV) = 1$ at $(T, U, V, W) = (1, 1, 2, 4)$

SOLUTION Using the formulas obtained by implicit differentiation (Eq. (7)) we have,

$$\frac{\partial U}{\partial T} = -\frac{F_T}{F_U}, \quad \frac{\partial T}{\partial U} = -\frac{F_U}{F_T} \tag{1}$$

We compute the partial derivatives of $F(T, U, V, W) = (TU - V)^2 \ln(W - UV) - 1$:

$$F_T = 2U(TU - V)\ln(W - UV)$$

$$F_U = 2T(TU - V)\ln(W - UV) + (TU - V)^2 \cdot \frac{-V}{W - UV}$$

$$= (TU - V)\left(2T\ln(W - UV) - \frac{V(TU - V)}{W - UV}\right)$$

At the point $(T, U, V, W) = (1, 1, 2, 4)$ we have

$$F_T = 2(1 - 2)\ln(4 - 2) = -2\ln 2$$

$$F_U = (1 - 2)\left(2\ln(4 - 2) - \frac{2(1 - 2)}{4 - 2}\right) = (-2\ln 2 - 1) = -1 - 2\ln 2$$

Substituting in (1) we obtain

$$\left.\frac{\partial U}{\partial T}\right|_{(1,1,2,4)} = -\frac{2\ln 2}{1 + 2\ln 2}, \quad \left.\frac{\partial T}{\partial U}\right|_{(1,1,2,4)} = -\frac{1 + 2\ln 2}{2\ln 2}.$$

31. Let $\mathbf{r} = \langle x, y, z\rangle$ and $e_{\mathbf{r}} = \mathbf{r}/\|\mathbf{r}\|$. Show that if a function $f(x, y, z) = F(r)$ depends only on the distance from the origin $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$, then

$$\nabla f = F'(r)e_{\mathbf{r}}$$

SOLUTION The gradient of f is the following vector:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

We must express this vector in terms of $\mathbf{r}$ and r. Using the Chain Rule, we have

$$\frac{\partial f}{\partial x} = F'(r)\frac{\partial r}{\partial x} = F'(r) \cdot \frac{2x}{2\sqrt{x^2+y^2+z^2}} = F'(r) \cdot \frac{x}{r}$$
$$\frac{\partial f}{\partial y} = F'(r)\frac{\partial r}{\partial y} = F'(r) \cdot \frac{2y}{2\sqrt{x^2+y^2+z^2}} = F'(r) \cdot \frac{y}{r}$$
$$\frac{\partial f}{\partial z} = F'(r)\frac{\partial r}{\partial z} = F'(r) \cdot \frac{2z}{2\sqrt{x^2+y^2+z^2}} = F'(r) \cdot \frac{z}{r}$$

Hence,

$$\nabla f = \left\langle F'(r)\frac{x}{r}, F'(r)\frac{y}{r}, F'(r)\frac{z}{r} \right\rangle = \frac{F'(r)}{r}\langle x, y, z\rangle = F'(r)\frac{\mathbf{r}}{\|\mathbf{r}\|} = F'(r)e_{\mathbf{r}}$$

32. Let $f(x, y, z) = e^{-x^2-y^2-z^2} = e^{-r^2}$, with r as in Exercise 31. Compute ∇f directly and using Eq. (9).

SOLUTION Direct computation gives

$$\nabla f = \langle f_x, f_y, f_z\rangle = \left\langle -2xe^{-x^2-y^2-z^2}, -2ye^{-x^2-y^2-z^2}, -2ze^{-x^2-y^2-z^2} \right\rangle$$
$$= -2e^{-(x^2+y^2+z^2)}\langle x, y, z\rangle = -2e^{-r^2}\mathbf{r}$$

We now compute the gradient using Eq. (9):

$$\nabla f = F'(r)\mathbf{e_r}$$

Since $F(r) = e^{-r^2}$, we have $F'(r) = -2re^{-r^2}$. Also, $e_r = \frac{\mathbf{r}}{\|\mathbf{r}\|}$. So we obtain

$$\nabla f = -2re^{-r^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|} = -2e^{-r^2}\mathbf{r}$$

Both answers agree, as expected.

33. Use Eq. (9) to compute $\nabla\left(\frac{1}{r}\right)$.

SOLUTION To compute $\nabla\left(\frac{1}{r}\right)$ using Eq. (9), we let $F(r) = \frac{1}{r}$:

$$F'(r) = -\frac{1}{r^2}$$

We obtain

$$\nabla\left(\frac{1}{r}\right) = F'(r)\mathbf{e_r} = -\frac{1}{r^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{1}{r^3}\mathbf{r}$$

34. Use Eq. (9) to compute $\nabla(\ln r)$.

SOLUTION To compute $\nabla(\ln r)$ we let $F(r) = \ln r$, hence $F'(r) = \frac{1}{r}$. Thus,

$$\nabla(\ln r) = F'(r)\mathbf{e_r} = \frac{1}{r} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{1}{r^2}\mathbf{r}$$

35. Figure 4 shows the graph of the equation

$$F(x, y, z) = x^2 + y^2 - z^2 - 12x - 8z - 4 = 0$$

(a) Use the quadratic formula to solve for z as a function of x and y. This gives two formulas, depending on the choice of sign.

(b) Which formula defines the portion of the surface satisfying $z \geq -4$? Which formula defines the portion satisfying $z \leq -4$?

(c) Calculate $\partial z/\partial x$ using the formula $z = f(x, y)$ (for both choices of sign) and again via implicit differentiation. Verify that the two answers agree.

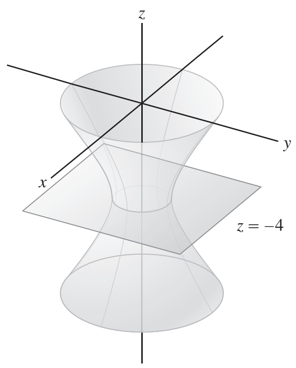


FIGURE 4 Graph of $x^2 + y^2 - z^2 - 12x - 8z - 4 = 0$.

SOLUTION

(a) We rewrite $F(x, y, z) = 0$ as a quadratic equation in the variable z:

$$z^2 + 8z + \left(4 + 12x - x^2 - y^2\right) = 0$$

We solve for z. The discriminant is

$$8^2 - 4\left(4 + 12x - x^2 - y^2\right) = 4x^2 + 4y^2 - 48x + 48 = 4\left(x^2 + y^2 - 12x + 12\right)$$

Hence,

$$z_{1,2} = \frac{-8 \pm \sqrt{4\left(x^2 + y^2 - 12x + 12\right)}}{2} = -4 \pm \sqrt{x^2 + y^2 - 12x + 12}$$

We obtain two functions:

$$z = -4 + \sqrt{x^2 + y^2 - 12x + 12}, \quad z = -4 - \sqrt{x^2 + y^2 - 12x + 12}$$

(b) The formula with the positive root defines the portion of the surface satisfying $z \ge -4$, and the formula with the negative root defines the portion satisfying $z \le -4$.

(c) Differentiating $z = -4 + \sqrt{x^2 + y^2 - 12x + 12}$ with respect to x, using the Chain Rule, gives

$$\frac{\partial z}{\partial x} = \frac{2x - 12}{2\sqrt{x^2 + y^2 - 12x + 12}} = \frac{x - 6}{\sqrt{x^2 + y^2 - 12x + 12}} \tag{1}$$

Alternatively, using the formula for $\frac{\partial z}{\partial x}$ obtained by implicit differentiation gives

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \tag{2}$$

We find the partial derivatives of $F(x, y, z) = x^2 + y^2 - z^2 - 12x - 8z - 4$:

$$F_x = 2x - 12, \quad F_z = -2z - 8$$

Substituting in (2) gives

$$\frac{\partial z}{\partial x} = -\frac{2x - 12}{-2z - 8} = \frac{x - 6}{z + 4}$$

This result is the same as the result in (1), since $z = -4 + \sqrt{x^2 + y^2 - 12x + 12}$ implies that

$$\sqrt{x^2 + y^2 - 12x + 12} = z + 4$$

For $z = -4 - \sqrt{x^2 + y^2 - 12x + 12}$, differentiating with respect to x gives

$$\frac{\partial z}{\partial x} = -\frac{2x - 12}{2\sqrt{x^2 + y^2 - 12x + 12}} = \frac{x - 6}{-\sqrt{x^2 + y^2 - 12x + 12}} = \frac{x - 6}{z + 4}$$

which is equal to $-\frac{F_x}{F_z}$ computed above.

36. For all $x > 0$, there is a unique value $y = r(x)$ that solves the equation $y^3 + 4xy = 16$.

(a) Show that $dy/dx = -4y/(3y^2 + 4x)$.

(b) Let $g(x) = f(x, r(x))$, where $f(x, y)$ is a function satisfying

$$f_x(1, 2) = 8, \quad f_y(1, 2) = 10$$

Use the Chain Rule to calculate $g'(1)$. Note that $r(1) = 2$ because $(x, y) = (1, 2)$ satisfies $y^3 + 4xy = 16$.

SOLUTION

(a) Using implicit differentiation we see:

$$3y^2 \frac{dy}{dx} + 4x \frac{dy}{dx} + 4y = 0$$

$$\frac{dy}{dx}(3y^2 + 4x) = -4y$$

$$\frac{dy}{dx} = \frac{-4y}{3y^2 + 4x}$$

(b) Note that $r'(1) = -\dfrac{4(2)}{3(2)^2 + 4(1)} = -\dfrac{1}{2}$ Therefore,

$$g'(1) = f_x(1, 2) + f_y(1, 2) \cdot r'(1) = 8 + 10\left(-\frac{1}{2}\right) = 3$$

37. The pressure P, volume V, and temperature T of a van der Waals gas with n molecules (n constant) are related by the equation

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

where a, b, and R are constant. Calculate $\partial P/\partial T$ and $\partial V/\partial P$.

SOLUTION Let F be the following function:

$$F(P, V, T) = \left(P + \frac{an^2}{V^2}\right)(V - nb) - nRT$$

By Eq. (7),

$$\frac{\partial P}{\partial T} = -\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial P}}, \quad \frac{\partial V}{\partial P} = -\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial V}} \tag{1}$$

We compute the partial derivatives of F:

$$\frac{\partial F}{\partial P} = V - nb$$

$$\frac{\partial F}{\partial T} = -nR$$

$$\frac{\partial F}{\partial V} = -2an^2V^{-3}(V - nb) + \left(P + \frac{an^2}{V^2}\right) = P + \frac{2an^3b}{V^3} - \frac{an^2}{V^2}$$

Substituting in (1) gives

$$\frac{\partial P}{\partial T} = -\frac{-nR}{V - nb} = \frac{nR}{V - nb}$$

$$\frac{\partial V}{\partial P} = -\frac{V - nb}{P + \frac{2an^3b}{V^3} - \frac{an^2}{V^2}} = \frac{nbV^3 - V^4}{PV^3 + 2an^3b - an^2V}$$

38. When x, y, and z are related by an equation $F(x, y, z) = 0$, we sometimes write $(\partial z/\partial x)_y$ in place of $\partial z/\partial x$ to indicate that in the differentiation, z is treated as a function of x with y held constant (and similarly for the other variables).

(a) Use Eq. (7) to prove the **cyclic relation**

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1 \tag{10}$$

(b) Verify Eq. (10) for $F(x, y, z) = x + y + z = 0$.

(c) Verify the cyclic relation for the variables P, V, T in the ideal gas law $PV - nRT = 0$ (n and R are constants).

SOLUTION

(a) Using implicit differentiation for $F(x, y, z) = 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

Hence,

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = -\frac{F_x}{F_z} \cdot \frac{-F_y}{F_x} \cdot \frac{-F_z}{F_y} = -1$$

(b) For $F(x, y, z) = x + y + z = 0$ we have

$$x = -y - z, \quad y = -x - z, \quad z = -x - y$$

Hence,

$$\frac{\partial z}{\partial x} = -1, \quad \frac{\partial x}{\partial y} = -1, \quad \frac{\partial y}{\partial z} = -1$$

Eq. (10) holds since

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = (-1) \cdot (-1) \cdot (-1) = -1$$

(c) If $PV - nRT = 0$, then

$$T = \frac{PV}{nR}, \quad P = \frac{nRT}{V}, \quad V = \frac{nRT}{P}$$

Hence,

$$\frac{\partial T}{\partial V} = \frac{P}{nR}, \quad \frac{\partial V}{\partial P} = -\frac{nRT}{P^2}, \quad \frac{\partial P}{\partial T} = \frac{nR}{V}$$

We have

$$\frac{\partial T}{\partial V} \cdot \frac{\partial V}{\partial P} \cdot \frac{\partial P}{\partial T} = \frac{P}{nR} \cdot -\frac{nRT}{P^2} \cdot \frac{nR}{V} = -\frac{nRT}{PV}$$

and, since $PV = nRT$, we get

$$\frac{\partial T}{\partial V} \cdot \frac{\partial V}{\partial P} \cdot \frac{\partial P}{\partial T} = -\frac{PV}{PV} = -1$$

Similarly,

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \frac{V}{nR} \cdot \left(-\frac{nRT}{V^2}\right) \cdot \frac{nR}{P} = -\frac{nRT}{VP} = -\frac{PV}{PV} = -1$$

39. Show that if $f(x)$ is differentiable and $c \neq 0$ is a constant, then $u(x, t) = f(x - ct)$ satisfies the so-called **advection equation**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

SOLUTION For $s = x - ct$, we have $u(x, t) = f(s)$. We use the Chain Rule to compute $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$:

$$\frac{\partial u}{\partial t} = f'(s)\frac{\partial s}{\partial t} = f'(s) \cdot (-c) = -cf'(s) \tag{1}$$

$$\frac{\partial u}{\partial x} = f'(s)\frac{\partial s}{\partial x} = f'(s) \cdot 1 = f'(s) \tag{2}$$

Equalities (1) and (2) imply that:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Further Insights and Challenges

In Exercises 40–43, a function $f(x, y, z)$ is called **homogeneous of degree n** *if $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ for all $\lambda \in \mathbf{R}$.*

40. Show that the following functions are homogeneous and determine their degree.

(a) $f(x, y, z) = x^2y + xyz$

(b) $f(x, y, z) = 3x + 2y - 8z$

(c) $f(x, y, z) = \ln\left(\frac{xy}{z^2}\right)$

(d) $f(x, y, z) = z^4$

SOLUTION

(a) For $f(x, y, z) = x^2y + xyz$ we have

$$f(\lambda x, \lambda y, \lambda z) = (\lambda x)^2(\lambda y) + (\lambda x)(\lambda y)(\lambda z) = \lambda^3 x^2 y + \lambda^3 xyz = \lambda^3(x^2y + xyz) = \lambda^3 f(x, y, z)$$

Hence, f is homogeneous of degree 3.

(b) For $f(x, y, z) = 3x + 2y - 8z$ we have

$$f(\lambda x, \lambda y, \lambda z) = 3(\lambda x) + 2(\lambda y) - 8(\lambda z) = \lambda(3x + 2y - 8z) = \lambda f(x, y, z)$$

Hence, f is homogeneous of degree 1.

(c) For $f(x, y, z) = \ln\left(\frac{xy}{z^2}\right)$ we have, for $\lambda \neq 0$,

$$f(\lambda x, \lambda y, \lambda z) = \ln\left(\frac{(\lambda x)(\lambda y)}{(\lambda z)^2}\right) = \ln\left(\frac{\lambda^2 xy}{\lambda^2 z^2}\right) = \ln\left(\frac{xy}{z^2}\right) = f(x, y, z) = \lambda^0 f(x, y, z)$$

Thus, f is homogeneous of degree 0.

(d) For $f(z) = z^4$ we have

$$f(\lambda z) = (\lambda z)^4 = \lambda^4 z^4 = \lambda^4 f(z)$$

Hence, f is homogeneous of degree 4.

41. Prove that if $f(x, y, z)$ is homogeneous of degree n, then $f_x(x, y, z)$ is homogeneous of degree $n - 1$. *Hint:* Either use the limit definition or apply the Chain Rule to $f(\lambda x, \lambda y, \lambda z)$.

SOLUTION We are given that $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ for all λ, and we must show that $f_x(\lambda x, \lambda y, \lambda z) = \lambda^{n-1} f_x(x, y, z)$. We use the limit definition of f_x. Since for all $\lambda \neq 0$, $\lambda h \to 0$ if and only if $h \to 0$, we get

$$\begin{aligned}
f_x(\lambda x, \lambda y, \lambda z) &= \lim_{h\to 0} \frac{f(\lambda x + \lambda h, \lambda y, \lambda z) - f(\lambda x, \lambda y, \lambda z)}{\lambda h} = \lim_{h\to 0} \frac{f(\lambda(x+h), \lambda y, \lambda z) - f(\lambda x, \lambda y, \lambda z)}{\lambda h} \\
&= \lim_{h\to 0} \frac{\lambda^n f(x+h, y, z) - \lambda^n f(x, y, z)}{\lambda h} = \lim_{h\to 0} \frac{\lambda^{n-1} f(x+h, y, z) - \lambda^{n-1} f(x, y, z)}{h} \\
&= \lambda^{n-1} \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} = \lambda^{n-1} f_x(x, y, z)
\end{aligned}$$

Alternatively, we prove this property using the Chain Rule. We use the Chain Rule to differentiate the following equality with respect to x:

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$$

We get

$$f_x(\lambda x, \lambda y, \lambda z) \cdot \frac{\partial(\lambda x)}{\partial x} + f_y(\lambda x, \lambda y, \lambda z) \cdot \frac{\partial(\lambda y)}{\partial x} + f_z(\lambda x, \lambda y, \lambda z) \cdot \frac{\partial(\lambda z)}{\partial x} = \lambda^n f_x(x, y, z)$$

Since $\frac{\partial(\lambda y)}{\partial x} = \frac{\partial(\lambda z)}{\partial x} = 0$ and $\frac{\partial(\lambda x)}{\partial x} = \lambda$, we obtain for $\lambda \neq 0$,

$$\lambda f_x(\lambda x, \lambda y, \lambda z) = \lambda^n f_x(x, y, z) \quad \text{or} \quad f_x(\lambda x, \lambda y, \lambda z) = \lambda^{n-1} f_x(x, y, z)$$

42. Prove that if $f(x, y, z)$ is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf \qquad \boxed{11}$$

Hint: Let $F(t) = f(tx, ty, tz)$ and calculate $F'(1)$ using the Chain Rule.

SOLUTION We use the Chain Rule to differentiate the function $F(t) = f(tx, ty, tz)$ with respect to t. This gives

$$F'(t) = \frac{\partial f}{\partial x} \cdot \frac{\partial (tx)}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial (ty)}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial (tz)}{\partial t} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \tag{1}$$

On the other hand, since f is homogeneous of degree n, we have

$$F(t) = f(tx, ty, tz) = t^n f(x, y, z)$$

Differentiating with respect to t we get

$$F'(t) = nt^{n-1} f(x, y, z) \tag{2}$$

By (1) and (2) we obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nt^{n-1} f(x, y, z)$$

Substituting $t = 1$ gives

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

43. Verify Eq. (11) for the functions in Exercise 40.

SOLUTION Eq. (11) states that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

(a) $f(x, y, z) = x^2y + xyz$. f is homogeneous of degree $n = 3$. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = 2xy + yz, \quad \frac{\partial f}{\partial y} = x^2 + xz, \quad \frac{\partial f}{\partial z} = xy$$

Hence,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(2xy + yz) + y(x^2 + xz) + zxy = 3x^2y + 3xyz = 3(x^2y + xyz) = 3f(x, y, z)$$

(b) $f(x, y, z) = 3x + 2y - 8z$. f is homogeneous of degree $n = 1$. We have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x \cdot 3 + y \cdot 2 + z \cdot (-8) = 3x + 2y - 8z = 1 \cdot f(x, y, z)$$

(c) $f(x, y, z) = \ln\left(\frac{xy}{z^2}\right)$. f is homogeneous of degree $n = 0$. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = \frac{\frac{y}{z^2}}{\frac{xy}{z^2}} = \frac{1}{x}, \quad \frac{\partial f}{\partial y} = \frac{\frac{x}{z^2}}{\frac{xy}{z^2}} = \frac{1}{y}, \quad \frac{\partial f}{\partial z} = \frac{-2z^{-3}xy}{xyz^{-2}} = -\frac{2}{z}$$

Hence,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x \cdot \frac{1}{x} + y \cdot \frac{1}{y} + z \cdot \left(-\frac{2}{z}\right) = 0 = 0 \cdot f(x, y, z)$$

(d) $f(x, y, z) = z^4$. f is homogeneous of degree $n = 4$. We have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x \cdot 0 + y \cdot 0 + z \cdot 4z^3 = 4z^4 = 4f(x, y, z)$$

44. Suppose that $x = g(t, s)$, $y = h(t, s)$. Show that f_{tt} is equal to

$$f_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + 2f_{xy}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + f_{yy}\left(\frac{\partial y}{\partial t}\right)^2 + f_x\frac{\partial^2 x}{\partial t^2} + f_y\frac{\partial^2 y}{\partial t^2} \qquad \boxed{12}$$

SOLUTION We are given that $x = g(t, s)$, $y = h(t, s)$. We must compute f_{tt} for a function $f(x, y)$. We first compute f_t using the Chain Rule:

$$f_t = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t}$$

To find f_{tt} we differentiate the two sides with respect to t using the Product Rule. This gives

$$f_{tt} = \frac{\partial}{\partial t}(f_x)\frac{\partial x}{\partial t} + f_x\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}(f_y)\frac{\partial y}{\partial t} + f_y\frac{\partial^2 y}{\partial t^2} \tag{1}$$

By the Chain Rule,

$$\frac{\partial}{\partial t}(f_x) = f_{xx}\frac{\partial x}{\partial t} + f_{xy}\frac{\partial y}{\partial t}$$
$$\frac{\partial}{\partial t}(f_y) = f_{yx}\frac{\partial x}{\partial t} + f_{yy}\frac{\partial y}{\partial t}$$

Substituting in (1) we obtain

$$\begin{aligned} f_{tt} &= \left(f_{xx}\frac{\partial x}{\partial t} + f_{xy}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + f_x\frac{\partial^2 x}{\partial t^2} + \left(f_{yx}\frac{\partial x}{\partial t} + f_{yy}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + f_y\frac{\partial^2 y}{\partial t^2} \\ &= f_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + f_{xy}\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial t}\right) + f_x\frac{\partial^2 x}{\partial t^2} + f_{yx}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + f_{yy}\left(\frac{\partial y}{\partial t}\right)^2 + f_y\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

If f_{xy} and f_{yx} are continuous, Clairaut's Theorem implies that $f_{xy} = f_{yx}$. Hence,

$$f_{tt} = f_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + 2f_{xy}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + f_{yy}\left(\frac{\partial y}{\partial t}\right)^2 + f_x\frac{\partial^2 x}{\partial t^2} + f_y\frac{\partial^2 y}{\partial t^2}$$

45. Let $r = \sqrt{x_1^2 + \cdots + x_n^2}$ and let $g(r)$ be a function of r. Prove the formulas

$$\frac{\partial g}{\partial x_i} = \frac{x_i}{r}g_r, \qquad \frac{\partial^2 g}{\partial x_i^2} = \frac{x_i^2}{r^2}g_{rr} + \frac{r^2 - x_i^2}{r^3}g_r$$

SOLUTION By the Chain Rule, we have

$$\frac{\partial g}{\partial x_i} = g'(r)\frac{\partial r}{\partial x_i} = g_r \cdot \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_n^2}} = g_r\frac{x_i}{r}$$

We differentiate $\frac{\partial g}{\partial x_i}$ with respect to x_i. Using the Product Rule we get

$$\frac{\partial^2 g}{\partial {x_i}^2} = \frac{\partial}{\partial x_i}(g_r) \cdot \frac{x_i}{r} + g_r\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) \tag{1}$$

We use the Chain Rule to compute $\frac{\partial}{\partial x_i}(g_r)$:

$$\frac{\partial}{\partial x_i}(g_r) = \frac{d}{dr}(g_r) \cdot \frac{\partial r}{\partial x_i} = g_{rr} \cdot \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_n^2}} = g_{rr} \cdot \frac{x_i}{r} \tag{2}$$

We compute $\frac{\partial}{\partial x_i} \cdot \left(\frac{x_i}{r}\right)$ using the Quotient Rule and the Chain Rule:

$$\frac{\partial}{\partial x_i} \cdot \left(\frac{x_i}{r}\right) = \frac{1 \cdot r - x_i \cdot \frac{\partial r}{\partial x_i}}{r^2} = \frac{r - x_i \cdot \frac{x_i}{r}}{r^2} = \frac{r^2 - x_i^2}{r^3} \tag{3}$$

Substituting (2) and (3) in (1), we obtain

$$\frac{\partial^2 g}{\partial {x_i}^2} = g_{\mathrm{rr}} \cdot \frac{x_i}{r} \cdot \frac{x_i}{r} + g_r\frac{r^2 - x_i^2}{r^3} = \frac{x_i^2}{r^2}g_{rr} + \frac{r^2 - x_i^2}{r^3}g_r$$

46. Prove that if $g(r)$ is a function of r as in Exercise 45, then

$$\frac{\partial^2 g}{\partial x_1^2} + \cdots + \frac{\partial^2 g}{\partial x_n^2} = g_{rr} + \frac{n-1}{r}g_r$$

SOLUTION In Exercise 45 we showed that

$$\frac{\partial^2 g}{\partial {x_i}^2} = \frac{x_i^2}{r^2}g_{rr} + \frac{r^2 - x_i^2}{r^3}g_r$$

Hence,

$$\begin{aligned}\frac{\partial^2 g}{\partial x_i{}^2}+\cdots+\frac{\partial^2 g}{\partial x_n{}^2}&=\left(\frac{x_1^2}{r^2}g_{rr}+\frac{r^2-x_1^2}{r^3}g_r\right)+\cdots+\left(\frac{x_n^2}{r^2}g_{rr}+\frac{r^2-x_n^2}{r^3}g_r\right)\\&=\frac{x_1^2+\cdots+x_n^2}{r^2}g_{rr}+\frac{1}{r^3}g_r\left((r^2-x_1^2)+\cdots+(r^2-x_n^2)\right)\\&=\frac{r^2}{r^2}g_{rr}+\frac{1}{r^3}g_r\left(nr^2-(x_1^2+\cdots+x_n^2)\right)\\&=g_{rr}+\frac{1}{r^3}g_r(nr^2-r^2)=g_{rr}+\frac{r^2}{r^3}g_r(n-1)=g_{rr}+\frac{n-1}{r}g_r\end{aligned}$$

In Exercises 47–51, the ***Laplace operator*** *is defined by* $\Delta f = f_{xx}+f_{yy}$. *A function* $f(x,y)$ *satisfying the Laplace equation* $\Delta f = 0$ *is called* ***harmonic***. *A function* $f(x,y)$ *is called* ***radial*** *if* $f(x,y)=g(r)$*, where* $r=\sqrt{x^2+y^2}$.

47. Use Eq. (12) to prove that in polar coordinates (r,θ),

$$\Delta f = f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r}f_r \qquad \boxed{13}$$

SOLUTION The polar coordinates are $x=r\cos\theta$, $y=r\sin\theta$. Hence,

$$\frac{\partial x}{\partial\theta}=-r\sin\theta,\quad \frac{\partial y}{\partial\theta}=r\cos\theta,\quad \frac{\partial x}{\partial r}=\cos\theta,\quad \frac{\partial y}{\partial r}=\sin\theta,$$
$$\frac{\partial^2 x}{\partial\theta^2}=-r\cos\theta,\quad \frac{\partial^2 y}{\partial\theta^2}=-r\sin\theta,\quad \frac{\partial^2 x}{\partial r^2}=\frac{\partial^2 y}{\partial r^2}=0$$

By Eq. (12) we have

$$\begin{aligned}f_{\theta\theta}&=f_{xx}\left(\frac{\partial x}{\partial\theta}\right)^2+f_{yy}\left(\frac{\partial y}{\partial\theta}\right)^2+2f_{xy}\left(\frac{\partial x}{\partial\theta}\right)\left(\frac{\partial y}{\partial\theta}\right)+f_x\frac{\partial^2 x}{\partial\theta^2}+f_y\frac{\partial^2 y}{\partial\theta^2}\\&=f_{xx}\left(r^2\sin^2\theta\right)+f_{yy}\left(r^2\cos^2\theta\right)-\left(2r^2\sin\theta\cos\theta\right)f_{xy}-(r\cos\theta)f_x-(r\sin\theta)f_y\end{aligned}\tag{1}$$

and

$$\begin{aligned}f_{rr}&=f_{xx}\left(\frac{\partial x}{\partial r}\right)^2+f_{yy}\left(\frac{\partial y}{\partial r}\right)^2+2f_{xy}\left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial y}{\partial r}\right)+f_x\frac{\partial^2 x}{\partial r^2}+f_y\frac{\partial^2 y}{\partial r^2}\\&=f_{xx}\left(\cos^2\theta\right)+f_{yy}\left(\sin^2\theta\right)+(2\cos\theta\sin\theta)f_{xy}\end{aligned}\tag{2}$$

$$f_r=f_x\frac{\partial x}{\partial r}+f_y\frac{\partial y}{\partial r}=f_x(\cos\theta)+f_y(\sin\theta)\tag{3}$$

We now compute the right-hand side of the equality we need to prove. Using (1), (2), and (3), we obtain

$$\begin{aligned}f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r}f_r&=f_{xx}\left(\cos^2\theta\right)+f_{yy}\left(\sin^2\theta\right)+(2\cos\theta\sin\theta)f_{xy}+f_{xx}\left(\sin^2\theta\right)\\&\quad+f_{yy}\left(\cos^2\theta\right)-(2\sin\theta\cos\theta)f_{xy}-\frac{\cos\theta}{r}f_x-\frac{\sin\theta}{r}f_y+f_x\frac{\cos\theta}{r}+f_y\frac{\sin\theta}{r}\\&=f_{xx}\left(\cos^2\theta+\sin^2\theta\right)+f_{yy}\left(\sin^2\theta+\cos^2\theta\right)\\&=f_{xx}+f_{yy}=\Delta f\end{aligned}$$

We thus showed that

$$\Delta f = f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r}f_r$$

48. Use Eq. (13) to show that $f(x,y)=\ln r$ is harmonic.

SOLUTION We must show that $f(r,\theta)=\ln r$ satisfies

$$\Delta f = f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r}f_r=0$$

We compute the derivatives of $f(r,\theta) = \ln r$:

$$f_r = \frac{1}{r}, \quad f_{rr} = -\frac{1}{r^2}, \quad f_\theta = 0, \quad f_{\theta\theta} = 0$$

Hence,

$$\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r = -\frac{1}{r^2} + \frac{1}{r^2} \cdot 0 + \frac{1}{r} \cdot \frac{1}{r} = -\frac{1}{r^2} + \frac{1}{r^2} = 0$$

Since $\Delta f = 0$, f is harmonic.

49. Verify that $f(x,y) = x$ and $f(x,y) = y$ are harmonic using both the rectangular and polar expressions for Δf.

SOLUTION We must show that $\Delta f = 0$.

(a) Using the rectangular expression for Δf:

$$\Delta f = f_{xx} + f_{yy}$$

For $f(x,y) = x$ we have $f_x = 1$, $f_y = 0$, hence, $f_{xx} = 0$, $f_{yy} = 0$. Therefore $\Delta f = f_{xx} + f_{yy} = 0 + 0 = 0$. For $f(x,y) = y$ we have $f_y = 1$, $f_x = 0$, hence, $f_{xx} = 0$, $f_{yy} = 0$, and again, $\Delta f = f_{xx} + f_{yy} = 0 + 0 = 0$.

(b) Using the polar expression for Δf,

$$\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r \tag{1}$$

Since $x = r\cos\theta$, we have $f(r,\theta) = x = r\cos\theta$. Hence,

$$f_r = \cos\theta, \quad f_\theta = -r\sin\theta, \quad f_{rr} = 0, \quad f_{\theta\theta} = -r\cos\theta$$

We now show that $\Delta f = 0$:

$$\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r = 0 + \frac{1}{r^2} \cdot (-r\cos\theta) + \frac{1}{r}\cos\theta = 0$$

Similarly, since $y = r\sin\theta$, we have $f(r,\theta) = y = r\sin\theta$. Hence,

$$f_r = \sin\theta, \quad f_\theta = r\cos\theta, \quad f_{rr} = 0, \quad f_{\theta\theta} = -r\sin\theta$$

Substituting in (1) gives

$$\Delta f = 0 + \frac{1}{r^2}(-r\sin\theta) + \frac{1}{r}\sin\theta = 0$$

50. Verify that $f(x,y) = \tan^{-1}\frac{y}{x}$ is harmonic using both the rectangular and polar expressions for Δf.

SOLUTION

(a) Using the rectangular expression for Δf:

$$\Delta f = f_{xx} + f_{yy}$$

We compute the partial derivatives of $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$. Using the Chain Rule we get

$$f_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

$$f_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$f_{xx} = -\frac{-y}{\left(x^2 + y^2\right)^2} \cdot 2x = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$f_{yy} = \frac{-x}{\left(x^2 + y^2\right)^2} \cdot 2y = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

Hence,

$$f_{xx} + f_{yy} = \frac{2xy}{\left(x^2 + y^2\right)^2} - \frac{2xy}{\left(x^2 + y^2\right)^2} = 0$$

(b) Using the polar expression for Δf,

$$\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r \tag{1}$$

Since $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$, we have $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(\theta) = \theta$. We compute the partial derivatives:

$$f_r = 0, \quad f_\theta = 1, \quad f_{rr} = 0, \quad f_{\theta\theta} = 0.$$

Substituting in (1), we get

$$\Delta f = 0 + \frac{1}{r^2} \cdot 0 + \frac{1}{r} \cdot 0 = 0$$

51. Use the Product Rule to show that

$$f_{rr} + \frac{1}{r} f_r = r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)$$

Use this formula to show that if f is a radial harmonic function, then $r f_r = C$ for some constant C. Conclude that $f(x, y) = C \ln r + b$ for some constant b.

SOLUTION We show that $f_{rr} + \frac{1}{r} f_r = r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)$. We use the Product Rule to compute the following derivative:

$$\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) = 1 \cdot \frac{\partial f}{\partial r} + r \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial f}{\partial r} + r \frac{\partial^2 f}{\partial r^2} = f_r + r f_{rr} = r\left(f_{rr} + \frac{1}{r} f_r\right)$$

Hence,

$$f_{rr} + \frac{1}{r} f_r = r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) \tag{1}$$

Now, suppose that $f(x, y)$ is a radial harmonic function. Since f is radial, $f(x, y) = g(r)$, therefore $f_{\theta\theta} = 0$. Substituting in the polar expressions for Δf gives

$$\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r = f_{rr} + \frac{1}{r} f_r = 0$$

Combining with (1), we get

$$r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) = 0 \quad \text{or} \quad \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) = 0$$

yielding

$$r \frac{\partial f}{\partial r} = C \quad \Rightarrow \quad f_r = \frac{C}{r}$$

We now integrate the two sides to obtain

$$\int f_r \, dr = \int \frac{C}{r} \, dr \quad \text{or} \quad f(r) = C \ln r + b.$$

14.7 Optimization in Several Variables (LT Section 15.7)

Preliminary Questions

1. The functions $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$ both have a critical point at (0, 0). How is the behavior of the two functions at the critical point different?

SOLUTION Let $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$. In the domain $\mathbf{R}^2$, the partial derivatives of f and g are

$$f_x = 2x, \quad f_{xx} = 2, \quad f_y = 2y, \quad f_{yy} = 2, \quad f_{xy} = 0$$
$$g_x = 2x, \quad g_{xx} = 2, \quad g_y = -2y, \quad g_{yy} = -2, \quad g_{xy} = 0$$

Therefore, $f_x = f_y = 0$ at $(0, 0)$ and $g_x = g_y = 0$ at $(0, 0)$. That is, the two functions have one critical point, which is the origin. Since the discriminant of f is $D = 4 > 0$, $f_{xx} > 0$, and the discriminant of g is $D = -4 < 0$, f has a local minimum (which is also a global minimum) at the origin, whereas g has a saddle point there. Moreover, since $\lim_{y\to\infty} g(0, y) = -\infty$ and $\lim_{x\to\infty} g(x, 0) = \infty$, g does not have global extrema on the plane. Similarly, f does not have a global maximum but does have a global minimum, which is $f(0, 0) = 0$.

2. Identify the points indicated in the contour maps as local minima, local maxima, saddle points, or neither (Figure 15).

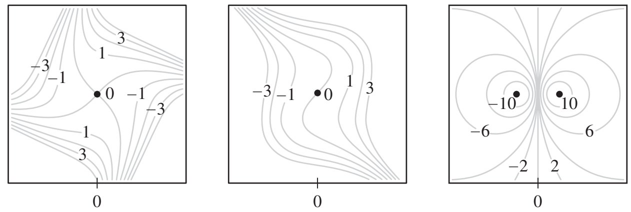
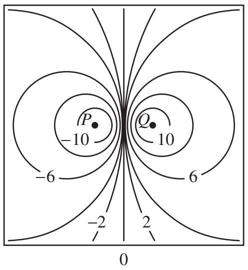


FIGURE 15

SOLUTION If $f(P)$ is a local minimum or maximum, then the nearby level curves are closed curves encircling P. In Figure (C), f increases in all directions emanating from P and decreases in all directions emanating from Q. Hence, f has a local minimum at P and local maximum at Q.



In Figure (A), the level curves through the point R consist of two intersecting lines that divide the neighborhood near R into four regions. f is decreasing in some directions and increasing in other directions. Therefore, R is a saddle point.

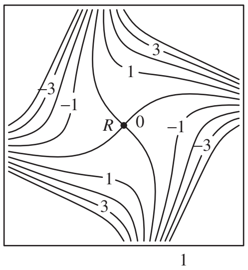


Figure (A)

Point S in Figure (B) is neither a local extremum nor a saddle point of f.

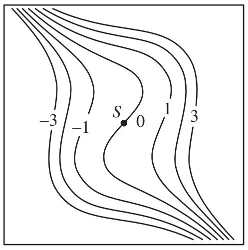


Figure (B)

3. Let $f(x, y)$ be a continuous function on a domain $\mathcal{D}$ in $\mathbf{R}^2$. Determine which of the following statements are true:

(a) If $\mathcal{D}$ is closed and bounded, then f takes on a maximum value on $\mathcal{D}$.

(b) If $\mathcal{D}$ is neither closed nor bounded, then f does not take on a maximum value of $\mathcal{D}$.

(c) $f(x, y)$ need not have a maximum value on the domain $\mathcal{D}$ defined by $0 \le x \le 1, 0 \le y \le 1$.

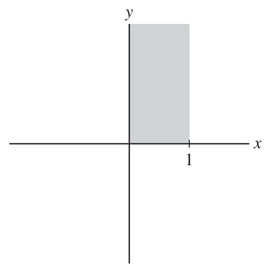
(d) A continuous function takes on neither a minimum nor a maximum value on the open quadrant

$$\{(x, y) : x > 0, y > 0\}$$

SOLUTION

(a) This statement is true. It follows by the Theorem on Existence of Global Extrema.

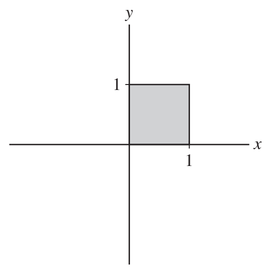
(b) The statement is false. Consider the constant function $f(x, y) = 2$ in the following domain:



$$D = \{(x, y) : 0 < x \leq 1,\ 0 \leq y < \infty\}$$

Obviously f is continuous and D is neither closed nor bounded. However, f takes on a maximum value (which is 2) on D.

(c) The domain $D = \{(x, y) : 0 \leq x, y \leq 1\}$ is the following rectangle:



$$D = \{(x, y) : 0 \leq x, y \leq 1\}$$

D is closed and bounded, hence f takes on a maximum value on D. Thus the statement is false.

(d) The statement is false. The constant function $f(x, y) = c$ takes on minimum and maximum values on the open quadrant.

Exercises

1. Let $P = (a, b)$ be a critical point of $f(x, y) = x^2 + y^4 - 4xy$.

(a) First use $f_x(x, y) = 0$ to show that $a = 2b$. Then use $f_y(x, y) = 0$ to show that $P = (0, 0)$, $(2\sqrt{2}, \sqrt{2})$, or $(-2\sqrt{2}, -\sqrt{2})$.

(b) Referring to Figure 16, determine the local minima and saddle points of $f(x, y)$ and find the absolute minimum value of $f(x, y)$.

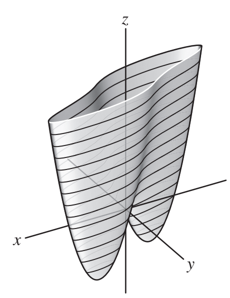


FIGURE 16

SOLUTION

(a) We find the partial derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 + y^4 - 4xy\right) = 2x - 4y$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 + y^4 - 4xy\right) = 4y^3 - 4x$$

Since $P = (a, b)$ is a critical point, $f_x(a, b) = 0$. That is,

$$2a - 4b = 0 \quad \Rightarrow \quad a = 2b$$

Also $f_y(a, b) = 0$, hence,

$$4b^3 - 4a = 0 \quad \Rightarrow \quad a = b^3$$

We obtain the following equations for the critical points (a, b):

$$\begin{cases} a = 2b \\ a = b^3 \end{cases}$$

Equating the two equations, we get

$$2b = b^3$$

$$b^3 - 2b = b(b^2 - 2) = 0 \quad \Rightarrow \quad \begin{cases} b_1 = 0 \\ b_2 = \sqrt{2} \\ b_3 = -\sqrt{2} \end{cases}$$

Since $a = 2b$, we have $a_1 = 0$, $a_2 = 2\sqrt{2}$, $a_3 = -2\sqrt{2}$. The critical points are thus

$$P_1 = (0, 0), \quad P_2 = \left(2\sqrt{2}, \sqrt{2}\right), \quad P_3 = \left(-2\sqrt{2}, -\sqrt{2}\right)$$

(b) Referring to Figure 14, we see that $P_1 = (0, 0)$ is a saddle point and $P_2 = \left(2\sqrt{2}, \sqrt{2}\right)$, $P_3 = \left(-2\sqrt{2}, -\sqrt{2}\right)$ are local minima. The absolute minimum value of f is -4.

2. Find the critical points of the functions

$$f(x, y) = x^2 + 2y^2 - 4y + 6x, \qquad g(x, y) = x^2 - 12xy + y$$

Use the Second Derivative Test to determine the local minimum, local maximum, and saddle points. Match $f(x, y)$ and $g(x, y)$ with their graphs in Figure 17.

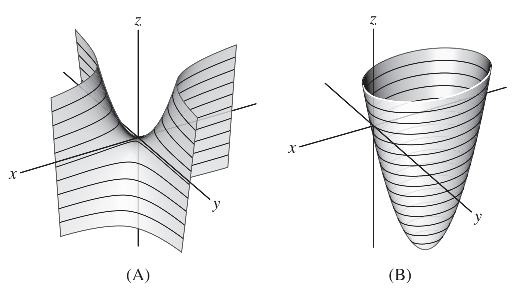


FIGURE 17

SOLUTION

Step 1. Find the critical points. We set the first partial derivatives equal to zero and solve:

$$\begin{aligned} f_x &= 2x + 6 = 0 \\ f_y &= 4y - 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -3 \\ y &= 1 \end{aligned}$$

The critical point is $(-3, 1)$.

$$\begin{aligned} g_x &= 2x - 12y = 0 \\ g_y &= -12x + 1 = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= \frac{1}{72} \\ x &= \frac{1}{12} \end{aligned}$$

The critical point is $\left(\frac{1}{12}, \frac{1}{72}\right)$.

Step 2. Compute the Discriminant. We compute the second-order partial derivatives:

$$f_{xx} = 2$$
$$f_{yy} = 4$$
$$f_{xy} = 0$$

The discriminant is $D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 4 - 0^2 = 8$.

$$g_{xx} = 2$$
$$g_{yy} = 0$$
$$g_{xy} = -12$$

The discriminant is $D(x, y) = g_{xx} g_{yy} - g_{xy}^2 = 2 \cdot 0 - 144 = -144$.

Step 3. Apply the Second Derivative Test.

For f, we have $D > 0$ and $f_{xx} > 0$, therefore $f(-3, 1)$ is a local minimum.
For g, we have $D < 0$, hence $g\left(\frac{1}{12}, \frac{1}{72}\right)$ is a saddle point.

The graph in Figure 17(A) has a saddle point, therefore it is the graph of $g(x, y)$. The graph in Figure 17(B) corresponds to $f(x, y)$, since it has a local minimum.

3. Find the critical points of

$$f(x, y) = 8y^4 + x^2 + xy - 3y^2 - y^3$$

Use the contour map in Figure 18 to determine their nature (local minimum, local maximum, or saddle point).

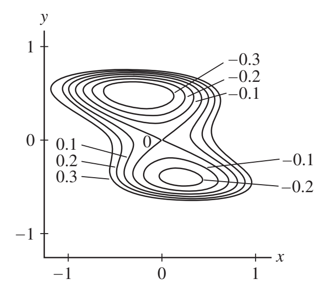


FIGURE 18 Contour map of $f(x, y) = 8y^4 + x^2 + xy - 3y^2 - y^3$.

SOLUTION The critical points are the solutions of $f_x = 0$ and $f_y = 0$. That is,

$$f_x(x, y) = 2x + y = 0$$
$$f_y(x, y) = 32y^3 + x - 6y - 3y^2 = 0$$

The first equation gives $y = -2x$. We substitute in the second equation and solve for x. This gives

$$32(-2x)^3 + x - 6(-2x) - 3(-2x)^2 = 0$$
$$-256x^3 + 13x - 12x^2 = 0$$
$$-x(256x^2 + 12x - 13) = 0$$

Hence $x = 0$ or $256x^2 + 12x - 13 = 0$. Solving the quadratic,

$$x_{1,2} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 256 \cdot (-13)}}{512} = \frac{-12 \pm 116}{512} \quad \Rightarrow \quad x = \frac{13}{64} \quad \text{or} \quad -\frac{1}{4}$$

Substituting in $y = -2x$ gives the y-coordinates of the critical points. The critical points are thus

$$(0, 0), \quad \left(\frac{13}{64}, -\frac{13}{32}\right), \quad \left(-\frac{1}{4}, \frac{1}{2}\right)$$

We now use the contour map to determine the type of each critical point. The level curves through $(0, 0)$ consist of two intersecting lines that divide the neighborhood near $(0, 0)$ into four regions. The function is decreasing in the y direction and increasing in the x-direction. Therefore, $(0, 0)$ is a saddle point. The level curves near the critical points $\left(\frac{13}{64}, -\frac{13}{32}\right)$ and $\left(-\frac{1}{4}, \frac{1}{2}\right)$ are closed curves encircling the points, hence these are local minima or maxima. The graph shows that both $\left(\frac{13}{64}, -\frac{13}{32}\right)$ and $\left(-\frac{1}{4}, \frac{1}{2}\right)$ are local minima.

4. Use the contour map in Figure 19 to determine whether the critical points A, B, C, D are local minima, local maxima, or saddle points.

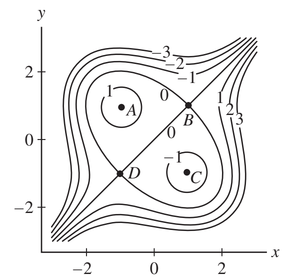


FIGURE 19

SOLUTION The nearby level curves at A and C are closed curves encircling A and C. As we move towards A the function increases in all directions, while moving towards C the function decreases in all directions. We conclude that the function has a local maximum at A and a local minimum at C. The level curves through B and D consist of two curves intersecting at these points respectively. These curves divide the neighborhoods near B and D into four regions. In some of the regions the function is increasing and in others it is decreasing as we move towards B or D. This implies that B and D are saddle points.

5. Let $f(x, y) = y^2x - yx^2 + xy$.

(a) Show that the critical points (x, y) satisfy the equations

$$y(y - 2x + 1) = 0, \qquad x(2y - x + 1) = 0$$

(b) Show that f has four critical points.

(c) Use the second derivative to determine the nature of the critical points.

SOLUTION

(a) The critical points are the solutions of the two equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$. That is,

$$\begin{aligned} f_x(x, y) &= y^2 - 2yx + y = 0 \\ f_y(x, y) &= 2yx - x^2 + x = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y(y - 2x + 1) &= 0 \\ x(2y - x + 1) &= 0 \end{aligned}$$

(b) We find the critical points by solving the equations obtained in part (a):

$$y(y - 2x + 1) = 0 \tag{1}$$

$$x(2y - x + 1) = 0 \tag{2}$$

Equation (1) implies that $y = 0$ or $y = 2x - 1$. Substituting $y = 0$ in (2) and solving for x gives

$$x(-x + 1) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 1$$

We obtain the solutions $(0, 0)$ and $(1, 0)$. We now substitute $y = 2x - 1$ in (2) and solve for x. We get

$$\begin{aligned} x(4x - 2 - x + 1) &= 0 \\ x(3x - 1) &= 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{1}{3} \end{aligned}$$

We compute the y-coordinate, using $y = 2x - 1$:

$$\begin{aligned} y &= 2 \cdot 0 - 1 = -1 \\ y &= 2 \cdot \frac{1}{3} - 1 = -\frac{1}{3} \end{aligned}$$

We obtain the solutions $(0, -1)$ and $\left(\frac{1}{3}, -\frac{1}{3}\right)$. To summarize, the critical points are $(0, 0)$, $(1, 0)$, $(0, -1)$, and $\left(\frac{1}{3}, -\frac{1}{3}\right)$. Three of the critical points have at least one zero coordinate, and one has two nonzero coordinates.

(c) We compute the second-order partial derivatives:

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}(y^2 - 2yx + y) = -2y \\ f_{yy}(x, y) &= \frac{\partial}{\partial y}(2yx - x^2 + x) = 2x \end{aligned}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(y^2 - 2yx + y) = 2y - 2x + 1$$

The discriminant is

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -2y \cdot 2x - (2y - 2x + 1)^2 = -4xy - (2y - 2x + 1)^2$$

We now apply the Second Derivative Test. We first compute the discriminants at the critical points:

$$\begin{aligned}
D(0, 0) &= -1 < 0 \\
D(1, 0) &= -1 < 0 \\
D(0, -1) &= -1 < 0 \\
D\left(\frac{1}{3}, -\frac{1}{3}\right) &= -4 \cdot \frac{1}{3}\left(-\frac{1}{3}\right) - \left(-\frac{2}{3} - \frac{2}{3} + 1\right)^2 = \frac{1}{3} > 0, \\
f_{xx}\left(\frac{1}{3}, -\frac{1}{3}\right) &= -2 \cdot \left(-\frac{1}{3}\right) = \frac{2}{3} > 0
\end{aligned}$$

The Second Derivative Test implies that the points $(0, 0)$, $(1, 0)$, and $(0, -1)$ are saddle points, and $f\left(\frac{1}{3}, -\frac{1}{3}\right)$ is a local minimum.

6. Show that $f(x, y) = \sqrt{x^2 + y^2}$ has one critical point P and that f is nondifferentiable at P. Does f take on a minimum, maximum, or saddle point at P?

SOLUTION Since $f(x, y) = \sqrt{x^2 + y^2} \geq 0$ and $f(0, 0) = 0$, $f(0, 0)$ is an absolute minimum value. To find the critical point of f we first find the first derivatives:

$$\begin{aligned}
f_x(x, y) &= \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2}\right) = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}} \\
f_y(x, y) &= \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2}\right) = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}
\end{aligned}$$

Since f_x and f_y do not exist at $(0, 0)$ and the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$ have no solutions, the only critical point is $P = (0, 0)$, a point where f is non-differentiable (and is the absolute minimum).

In Exercises 7–23, find the critical points of the function. Then use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).

7. $f(x, y) = x^2 + y^2 - xy + x$

SOLUTION

Step 1. Find the critical points. We set the first-order partial derivatives of $f(x, y) = x^2 + y^2 - xy + x$ equal to zero and solve:

$$f_x(x, y) = 2x - y + 1 = 0 \tag{1}$$

$$f_y(x, y) = 2y - x = 0 \tag{2}$$

Equation (2) implies that $x = 2y$. Substituting in (1) and solving for y gives

$$2 \cdot 2y - y + 1 = 0 \quad \Rightarrow \quad 3y = -1 \quad \Rightarrow \quad y = -\frac{1}{3}$$

The corresponding value of x is $x = 2 \cdot \left(-\frac{1}{3}\right) = -\frac{2}{3}$. The critical point is $\left(-\frac{2}{3}, -\frac{1}{3}\right)$.

Step 2. Compute the Discriminant. We find the second-order partials:

$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -1$$

The discriminant is

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)^2 = 3$$

Step 3. Applying the Second Derivative Test. We have

$$D\left(-\frac{2}{3}, -\frac{1}{3}\right) = 3 > 0 \quad \text{and} \quad f_{xx}\left(-\frac{2}{3}, -\frac{1}{3}\right) = 2 > 0$$

The Second Derivative Test implies that $f\left(-\frac{2}{3}, -\frac{1}{3}\right)$ is a local minimum.

8. $f(x, y) = x^3 - xy + y^3$

SOLUTION

Step 1. Find the critical points. We set the first-order partial derivatives of $f(x, y) = x^3 - xy + y^3$ equal to zero and solve:

$$f_x(x, y) = 3x^2 - y = 0 \qquad (1)$$
$$f_y(x, y) = -x + 3y^2 = 0 \qquad (2)$$

Equation (1) implies that $y = 3x^2$. Substituting in equation (2) and solving for x gives

$$-x + 3(3x^2)^2 = 0$$
$$-x + 27x^4 = x(-1 + 27x^3) = 0 \quad \Rightarrow \quad x = 0, \quad x = \frac{1}{3}$$

The y-coordinates are $y = 3 \cdot 0^2 = 0$ and $y = 3 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{3}$. The critical points are thus $(0, 0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

Step 2. Compute the Discriminant. We find the second-order partials:

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = -1$$

The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot 6y - (-1)^2 = 36xy - 1$$

Step 3. Apply the Second Derivative Test. We have

$$D(0, 0) = -1 < 0$$
$$D\left(\frac{1}{3}, \frac{1}{3}\right) = 36 \cdot \frac{1}{3} \cdot \frac{1}{3} - 1 = 3 > 0, \quad f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = 6 \cdot \frac{1}{3} = 2 > 0$$

Thus, $(0, 0)$ is a saddle point, whereas $f\left(\frac{1}{3}, \frac{1}{3}\right)$ is a local minimum.

9. $f(x, y) = x^3 + 2xy - 2y^2 - 10x$

SOLUTION

Step 1. Find the critical points. We set the first-order partial derivatives of $f(x, y) = x^3 + 2xy - 2y^2 - 10x$ equal to zero and solve:

$$f_x(x, y) = 3x^2 + 2y - 10 = 0 \qquad (1)$$
$$f_y(x, y) = 2x - 4y = 0 \qquad (2)$$

Equation (2) implies that $x = 2y$. We substitute in (1) and solve for y. This gives

$$3 \cdot (2y)^2 + 2y - 10 = 0$$
$$12y^2 + 2y - 10 = 0$$
$$6y^2 + y - 5 = 0$$
$$y_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 6 \cdot (-5)}}{12} = \frac{-1 \pm 11}{12} \quad \Rightarrow \quad y_1 = -1 \quad \text{and} \quad y_2 = \frac{5}{6}$$

We find the x-coordinates using $x = 2y$:

$$x_1 = 2 \cdot (-1) = -2, \quad x_2 = 2 \cdot \frac{5}{6} = \frac{5}{3}$$

The critical points are thus $(-2, -1)$ and $\left(\frac{5}{3}, \frac{5}{6}\right)$.

Step 2. Compute the Discriminant. We find the second-order partials:

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = -4, \quad f_{xy}(x, y) = 2$$

The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot (-4) - 2^2 = -24x - 4$$

Step 3. Apply the Second Derivative Test. We have

$$D(-2,-1) = -24 \cdot (-2) - 4 = 44 > 0,$$
$$f_{xx}(-2,-1) = 6 \cdot (-2) = -12 < 0$$
$$D\left(\frac{5}{3}, \frac{5}{6}\right) = -24 \cdot \frac{5}{3} - 4 = -44 < 0$$

We conclude that $f(-2,-1)$ is a local maximum and $\left(\frac{5}{3}, \frac{5}{6}\right)$ is a saddle point.

10. $f(x, y) = x^3y + 12x^2 - 8y$

SOLUTION

Step 1. Find the critical points. We set the first-order partial derivatives of $f(x, y) = x^3y + 12x^2 - 8y$ equal to zero and solve:

$$f_x(x, y) = 3x^2y + 24x = 3x(xy + 8) = 0 \qquad (1)$$
$$f_y(x, y) = x^3 - 8 = 0 \qquad (2)$$

Equation (2) implies that $x = 2$. We substitute in equation (1) and solve for y to obtain

$$6(2y + 8) = 0 \quad \text{or} \quad y = -4$$

The critical point is $(2, -4)$.

Step 2. Compute the Discriminant. We find the second-order partials:

$$f_{xx}(x, y) = 6xy + 24, \quad f_{yy} = 0, \quad f_{xy} = 3x^2$$

The discriminant is thus

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -9x^4$$

Step 3. Apply the Second Derivative Test. We have

$$D(2, -4) = -9 \cdot 2^4 < 0$$

Hence $(2, -4)$ is a saddle point.

11. $f(x, y) = 4x - 3x^3 - 2xy^2$

SOLUTION

Step 1. Find the critical points. We set the first-order derivatives of $f(x, y) = 4x - 3x^3 - 2xy^2$ equal to zero and solve:

$$f_x(x, y) = 4 - 9x^2 - 2y^2 = 0 \qquad (1)$$
$$f_y(x, y) = -4xy = 0 \qquad (2)$$

Equation (2) implies that $x = 0$ or $y = 0$. If $x = 0$, then equation (1) gives

$$4 - 2y^2 = 0 \quad \Rightarrow \quad y^2 = 2 \quad \Rightarrow \quad y = \sqrt{2}, \quad y = -\sqrt{2}$$

If $y = 0$, then equation (1) gives

$$4 - 9x^2 = 0 \quad \Rightarrow \quad 9x^2 = 4 \quad \Rightarrow \quad x = \frac{2}{3}, \quad x = -\frac{2}{3}$$

The critical points are therefore

$$\left(0, \sqrt{2}\right), \quad \left(0, -\sqrt{2}\right), \quad \left(\frac{2}{3}, 0\right), \quad \left(-\frac{2}{3}, 0\right)$$

Step 2. Compute the discriminant. The second-order partials are

$$f_{xx}(x, y) = -18x, \quad f_{yy}(x, y) = -4x, \quad f_{xy} = -4y$$

The discriminant is thus

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -18x \cdot (-4x) - (-4y)^2 = 72x^2 - 16y^2$$

Step 3. Apply the Second Derivative Test. We have

$$D\left(0,\sqrt{2}\right) = -32 < 0$$
$$D\left(0,-\sqrt{2}\right) = -32 < 0$$
$$D\left(\frac{2}{3},0\right) = 72\cdot\frac{4}{9} = 32 > 0,$$
$$f_{xx}\left(\frac{2}{3},0\right) = -18\cdot\frac{2}{3} = -12 < 0$$
$$D\left(-\frac{2}{3},0\right) = 72\cdot\frac{4}{9} = 32 > 0,$$
$$f_{xx}\left(-\frac{2}{3},0\right) = -18\cdot\left(-\frac{2}{3}\right) = 12 > 0$$

The Second Derivative Test implies that the points $\left(0,\pm\sqrt{2}\right)$ are the saddle points, $f\left(\frac{2}{3},0\right)$ is a local maximum, and $f\left(-\frac{2}{3},0\right)$ is a local minimum.

12. $f(x,y) = x^3 + y^4 - 6x - 2y^2$

SOLUTION

Step 1. Find the critical points. We set the first-order derivatives of $f(x,y) = x^3 + y^4 - 6x - 2y^2$ equal to zero and solve:

$$f_x(x,y) = 3x^2 - 6 = 0, \quad f_y(x,y) = 4y^3 - 4y = 0 \quad \text{or} \quad 4y(y^2-1) = 0$$

The first equation implies that $x = \pm\sqrt{2}$, and the second equation implies that $y = 0$ or $y = \pm 1$. The critical points are therefore

$$\left(\sqrt{2},0\right), \quad \left(\sqrt{2},1\right), \quad \left(\sqrt{2},-1\right), \quad \left(-\sqrt{2},0\right), \quad \left(-\sqrt{2},1\right), \quad \left(-\sqrt{2},-1\right)$$

Step 2. Compute the discriminant. We find the second-order partials:

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 12y^2 - 4, \quad f_{xy} = 0$$

The discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 6x\cdot 4(3y^2-1) - 0^2 = 24x(3y^2-1)$$

Step 3. Apply the Second Derivative Test. We have

$$D\left(\sqrt{2},0\right) = -24\sqrt{2} < 0$$
$$D\left(\sqrt{2},1\right) = 48\sqrt{2} > 0,\ f_{xx}\left(\sqrt{2},1\right) = 6\sqrt{2} > 0$$
$$D\left(\sqrt{2},-1\right) = 48\sqrt{2} > 0,\ f_{xx}\left(\sqrt{2},-1\right) = 6\sqrt{2} > 0$$
$$D\left(-\sqrt{2},0\right) = 24\sqrt{2} > 0,\ f_{xx}\left(-\sqrt{2},0\right) = -6\sqrt{2} < 0$$
$$D\left(-\sqrt{2},1\right) = -48\sqrt{2} < 0$$
$$D\left(-\sqrt{2},-1\right) = -48\sqrt{2} < 0$$

By the Second Derivative Test we obtain the following conclusions: $\left(\sqrt{2},0\right)$, $\left(-\sqrt{2},1\right)$, and $\left(-\sqrt{2},-1\right)$ are saddle points; $f\left(\sqrt{2},1\right)$ and $f\left(\sqrt{2},-1\right)$ are local minima; and $f\left(-\sqrt{2},0\right)$ is a local maximum.

13. $f(x,y) = x^4 + y^4 - 4xy$

SOLUTION

Step 1. Find the critical points. We set the first-order derivatives of $f(x,y) = x^4 + y^4 - 4xy$ equal to zero and solve:

$$f_x(x,y) = 4x^3 - 4y = 0, \quad f_y(x,y) = 4y^3 - 4x = 0 \tag{1}$$

Equation (1) implies that $y = x^3$. Substituting in (2) and solving for x, we obtain

$$(x^3)^3 - x = x^9 - x = x(x^8 - 1) = 0 \quad \Rightarrow \quad x = 0, \quad x = 1, \quad x = -1$$

The corresponding y coordinates are

$$y = 0^3 = 0, \quad y = 1^3 = 1, \quad y = (-1)^3 = -1$$

The critical points are therefore

$$(0,0), \quad (1,1), \quad (-1,-1)$$

Step 2. Compute the discriminant. We find the second-order partials:

$$f_{xx}(x,y) = 12x^2, \quad f_{yy}(x,y) = 12y^2, \quad f_{xy}(x,y) = -4$$

The discriminant is thus

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12x^2 \cdot 12y^2 - (-4)^2 = 144x^2y^2 - 16$$

Step 3. Apply the Second Derivative Test. We have

$$\begin{aligned} D(0,0) &= -16 < 0 \\ D(1,1) &= 144 - 16 = 128 > 0, \quad f_{xx}(1,1) = 12 > 0 \\ D(-1,-1) &= 144 - 16 = 128 > 0, \quad f_{xx}(-1,-1) = 12 > 0 \end{aligned}$$

We conclude that $(0,0)$ is a saddle point, whereas $f(1,1)$ and $f(-1,-1)$ are local minima.

14. $f(x,y) = e^{x^2-y^2+4y}$

SOLUTION

Step 1. Find the critical points. We set the first partials of $f(x,y) = e^{x^2-y^2+4y}$ equal to zero and solve:

$$f_x(x,y) = 2xe^{x^2-y^2+4y} = 0, \quad f_y(x,y) = (-2y+4)e^{x^2-y^2+4y} = 0$$

Since $e^{x^2-y^2+4y} \neq 0$, the first equation gives $x = 0$ and the second equation gives $-2y + 4 = 0$ or $y = 2$. We obtain the critical point $(0,2)$.

Step 2. Compute the discriminant. We find the second-order partials:

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}\left(2xe^{x^2-y^2+4y}\right) = 2e^{x^2-y^2+4y} + 2xe^{x^2-y^2+4y} \cdot 2x = 2e^{x^2-y^2+4y}(1+2x^2) \\ f_{yy}(x,y) &= \frac{\partial}{\partial y}\left((-2y+4)e^{x^2-y^2+4y}\right) = -2e^{x^2-y^2+4y} + (-2y+4)e^{x^2-y^2+4y} \cdot (-2y+4) \\ &= 2e^{x^2-y^2+4y}(-1 + (-y+2)(-2y+4)) = 2e^{x^2-y^2+4y}\left(2y^2 - 8y + 7\right) \\ f_{xy}(x,y) &= \frac{\partial}{\partial y}\left(2xe^{x^2-y^2+4y}\right) = 2xe^{x^2-y^2+4y}(-2y+4) = 4x(2-y)e^{x^2-y^2+4y} \end{aligned}$$

The discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4e^{2\left(x^2-y^2+4y\right)}\left(1+2x^2\right)\left(2y^2-8y+7\right) - 16x^2(2-y)^2e^{2\left(x^2-y^2+4y\right)}$$

Step 3. Apply the Second Derivative Test. We have

$$D(0,2) = -4e^8 < 0$$

Therefore, $(0,2)$ is a saddle point.

15. $f(x,y) = xye^{-x^2-y^2}$

SOLUTION

Step 1. Find the critical points. We compute the partial derivatives of $f(x,y) = xye^{-x^2-y^2}$, using the Product Rule and the Chain Rule:

$$\begin{aligned} f_x(x,y,z) &= y\left(1 \cdot e^{-x^2-y^2} + xe^{-x^2-y^2} \cdot (-2x)\right) = ye^{-x^2-y^2}\left(1 - 2x^2\right) \\ f_y(x,y,z) &= x\left(1 \cdot e^{-x^2-y^2} + ye^{-x^2-y^2} \cdot (-2y)\right) = xe^{-x^2-y^2}\left(1 - 2y^2\right) \end{aligned}$$

We set the partial derivatives equal to zero and solve to find the critical points. This gives

$$ye^{-x^2-y^2}\left(1-2x^2\right)=0$$
$$xe^{-x^2-y^2}\left(1-2y^2\right)=0$$

Since $e^{-x^2-y^2}\neq 0$, the first equation gives $y=0$ or $1-2x^2=0$, that is, $y=0$, $x=\frac{1}{\sqrt{2}}$, $x=-\frac{1}{\sqrt{2}}$. We substitute each of these values in the second equation and solve to obtain

$$y=0:\quad xe^{-x^2}=0 \quad\Rightarrow\quad x=0$$
$$x=\frac{1}{\sqrt{2}}:\quad \frac{1}{\sqrt{2}}e^{-\frac{1}{2}-y^2}\left(1-2y^2\right)=0 \quad\Rightarrow\quad 1-2y^2=0 \quad\Rightarrow\quad y=\pm\frac{1}{\sqrt{2}}$$
$$x=-\frac{1}{\sqrt{2}}:\quad -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}-y^2}\left(1-2y^2\right)=0 \quad\Rightarrow\quad 1-2y^2=0 \quad\Rightarrow\quad y=\pm\frac{1}{\sqrt{2}}$$

We obtain the following critical points: (0, 0),

$$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\quad \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\quad \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\quad \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

Step 2. Compute the second-order partials.

$$f_{xx}(x,y)=y\frac{\partial}{\partial x}\left(e^{-x^2-y^2}\left(1-2x^2\right)\right)=y\left(e^{-x^2-y^2}(-2x)\left(1-2x^2\right)+e^{-x^2-y^2}(-4x)\right)$$
$$=-2xye^{-x^2-y^2}\left(3-2x^2\right)$$
$$f_{yy}(x,y)=x\frac{\partial}{\partial y}\left(e^{-x^2-y^2}\left(1-2y^2\right)\right)=x\left(e^{-x^2-y^2}(-2y)\left(1-2y^2\right)+e^{-x^2-y^2}(-4y)\right)$$
$$=-2yxe^{-x^2-y^2}\left(3-2y^2\right)$$
$$f_{xy}(x,y)=\frac{\partial}{\partial y}f_x=\left(1-2x^2\right)\frac{\partial}{\partial y}\left(ye^{-x^2-y^2}\right)=\left(1-2x^2\right)\left(1\cdot e^{-x^2-y^2}+ye^{-x^2-y^2}(-2y)\right)$$
$$=e^{-x^2-y^2}\left(1-2x^2\right)\left(1-2y^2\right)$$

The discriminant is

$$D(x,y)=f_{xx}f_{yy}-f_{xy}^2$$

Step 3. Apply the Second Derivative Test. We construct the following table:

Critical Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(0,0)$	0	0	1	-1	$D<0$, saddle point
$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$-\frac{2}{e}$	$-\frac{2}{e}$	0	$\frac{4}{e^2}$	$D>0$, $f_{xx}<0$ local maximum
$\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	$\frac{2}{e}$	$\frac{2}{e}$	0	$\frac{4}{e^2}$	$D>0$, $f_{xx}>0$ local minimum
$\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$\frac{2}{e}$	$\frac{2}{e}$	0	$\frac{4}{e^2}$	$D>0$, $f_{xx}>0$ local minimum
$\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	$-\frac{2}{e}$	$-\frac{2}{e}$	0	$\frac{4}{e^2}$	$D>0$, $f_{xx}<0$ local maximum

16. $f(x,y)=e^x-xe^y$

SOLUTION

Step 1. Find the critical points. We set the first-order derivatives of $f(x,y)=e^x-xe^y$ equal to zero and solve:

$$f_x(x,y)=e^x-e^y=0$$
$$f_y(x,y)=-xe^y=0$$

Since $e^y\neq 0$, the second equation gives $x=0$. Substituting in the first equation, we get

$$e^0-e^y=1-e^y=0 \quad\Rightarrow\quad e^y=1 \quad\Rightarrow\quad y=0$$

The critical point is (0, 0).

Step 2. Compute the discriminant. We find the second-order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(e^x - e^y\right) = e^x$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(-xe^y\right) = -xe^y$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(e^x - e^y\right) = -e^y$$

The discriminant is

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -xe^{x+y} - e^{2y}$$

Step 3. Apply the Second Derivative Test. We have

$$D(0, 0) = 0 - e^0 = -1 < 0$$

The point (0, 0) is a saddle point.

17. $f(x, y) = \sin(x + y) - \cos x$

SOLUTION

Step 1. Find the critical points. We set the first-order derivatives of $f(x, y) = \sin(x + y) - \cos x$ equal to zero and solve:

$$f_x(x, y) = \cos(x + y) + \sin x = 0$$

$$f_y(x, y) = \cos(x + y) = 0$$

First consider the second equation, $\cos(x + y) = 0$ this is when

$$x + y = \frac{(2k+1)\pi}{2} \to y = \frac{(2k+1)\pi}{2} - x \text{ where } k \text{ is an integer}$$

Then setting the two equations equal to one another we gain $\sin x = 0$ which are the values:

$$x = 0, \pm\pi, \pm 2\pi, \cdots = \pm k\pi \text{ where } k \text{ is an integer.}$$

Thus we have:

$$x = k\pi \text{ and } y = \frac{(2n+1)\pi}{2} \text{ where } n, k \text{ are integers}$$

Step 2. Compute the discriminant. We find the second-order partial derivatives:

$$f_{xx}(x, y) = -\sin(x + y) + \cos x, \quad f_{yy}(x, y) = -\sin(x + y), \quad f_{xy}(x, y) = -\sin(x + y)$$

The discriminant is:

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (-\sin(x + y) + \cos x)(-\sin(x + y)) - \sin^2(x + y) = -\cos(x)\sin(x + y)$$

Step 3. Apply the Second Derivative Test. We have

$$D = \begin{cases} +1, & \text{if } y = \frac{4n+3}{2}\pi \\ -1, & y = \frac{4n+1}{2}\pi \end{cases}$$

Therefore, the points $\left(k\pi, \frac{4n+1}{2}\pi\right)$ are saddle points since $D < 0$.

Since $D > 0$ for the points $\left(k\pi, \frac{4n+3}{2}\pi\right)$, we need to examine f_{xx}. The results show:

$$f_{xx} > 0 \text{ if } k \text{ is even and } f_{xx} < 0 \text{ if } k \text{ is odd}$$

Thus:

$$\left(k\pi, \frac{4n+3}{2}\pi\right) \text{ are local minima if } k \text{ is even}$$

while

$$\left(k\pi, \frac{4n+3}{2}\pi\right) \text{ are local maxima if } k \text{ is odd}$$

18. $f(x, y) = x\ln(x + y)$

SOLUTION

Step 1. Find the critical points. We set the first-order partial derivatives of $f(x, y) = x\ln(x + y)$ equal to zero and solve:

$$f_x(x, y) = \ln(x + y) + x \cdot \frac{1}{x + y} = \ln(x + y) + \frac{x}{x + y} = 0$$
$$f_y(x, y) = \frac{x}{x + y} = 0$$

The second equation implies $x = 0$. Substituting in the first equation gives

$$\ln y + 0 = 0 \quad \Rightarrow \quad \ln y = 0 \quad \Rightarrow \quad y = 1.$$

We obtain the critical point $(0, 1)$. f_x and f_y do not exist at the points where $x + y = 0$, but these points are not in the domain of f, hence they are not critical points. The critical point is thus $(0, 1)$.

Step 2. Compute the discriminant. We find the second-order derivatives:

$$f_{xx} = \frac{\partial}{\partial x}\left(\ln(x + y) + \frac{x}{x + y}\right) = \frac{1}{x + y} + \frac{1 \cdot (x + y) - x \cdot 1}{(x + y)^2} = \frac{1}{x + y} + \frac{y}{(x + y)^2} = \frac{x + 2y}{(x + y)^2}$$
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{x}{x + y}\right) = -\frac{x}{(x + y)^2}$$
$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{x}{x + y}\right) = \frac{1 \cdot (x + y) - x \cdot 1}{(x + y)^2} = \frac{y}{(x + y)^2}$$

The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -\frac{x(x + 2y)}{(x + y)^4} - \frac{y^2}{(x + y)^4}$$

Step 3. Apply the Second Derivative Test. We have

$$D(0, 1) = 0 - \frac{1^2}{(0 + 1)^4} = -1 < 0$$

Therefore, $(0, 1)$ is a saddle point.

19. $f(x, y) = \ln x + 2\ln y - x - 4y$

SOLUTION

Step 1. Find the critical points. We set the first-order partials of $f(x, y) = \ln x + 2\ln y - x - 4y$ equal to zero and solve:

$$f_x(x, y) = \frac{1}{x} - 1 = 0, \quad f_y(x, y) = \frac{2}{y} - 4 = 0$$

The first equation gives $x = 1$, and the second equation gives $y = \frac{1}{2}$. We obtain the critical point $\left(1, \frac{1}{2}\right)$. Notice that f_x and f_y do not exist if $x = 0$ or $y = 0$, respectively, but these are not critical points since they are not in the domain of f. The critical point is thus $\left(1, \frac{1}{2}\right)$.

Step 2. Compute the discriminant. We find the second-order partials:

$$f_{xx}(x, y) = -\frac{1}{x^2}, \quad f_{yy}(x, y) = -\frac{2}{y^2}, \quad f_{xy}(x, y) = 0$$

The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = \frac{2}{x^2y^2}$$

Step 3. Apply the Second Derivative Test. We have

$$D\left(1, \frac{1}{2}\right) = \frac{2}{1^2 \cdot \left(\frac{1}{2}\right)^2} = 8 > 0, \quad f_{xx}\left(1, \frac{1}{2}\right) = -\frac{1}{1^2} = -1 < 0$$

We conclude that $f\left(1, \frac{1}{2}\right)$ is a local maximum.

20. $f(x, y) = (x + y)\ln(x^2 + y^2)$

SOLUTION

Step 1. Find the critical points. We set the partial derivatives of $f(x, y) = (x + y)\ln(x^2 + y^2)$ equal to zero and solve.

$$f_x(x + y) = \frac{2x(x + y)}{x^2 + y^2} + \ln(x^2 + y^2) = 0, \quad f_y(x, y) = \frac{2y(x + y)}{x^2 + y^2} + \ln(x^2 + y^2) = 0$$

and note that

$$2x(x + y) = 2y(x + y) \quad \Rightarrow \quad 2(x + y)(x - y) = 0$$

So critical points satisfy $x = \pm y$.

If $x = y$ we would have

$$\frac{2y(2y)}{2y^2} + \ln(2y^2) = 0 \quad \Rightarrow \quad \ln(2y^2) = -2 \quad \Rightarrow \quad y = \pm\frac{1}{e\sqrt{2}}$$

If $x = -y$ we would have

$$\frac{2y(0)}{2y^2} + \ln(2y^2) = 0 \quad \Rightarrow \quad \ln(2y^2) = 0 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$$

Our critical points are:

$$\left(\frac{1}{e\sqrt{2}}, \frac{1}{e\sqrt{2}}\right), \quad \left(-\frac{1}{e\sqrt{2}}, -\frac{1}{e\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Step 2. Compute the discriminant. We compute the second-order partial derivatives

$$f_{xx}(x, y) = \frac{4x}{x^2 + y^2} + \frac{2(x + y)}{x^2 + y^2} - \frac{4x^2(x + y)}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{2y}{x^2 + y^2} + \frac{2x}{x^2 + y^2} - \frac{4xy(x + y)}{(x^2 + y^2)^2}$$

$$f_{yy}(x, y) = \frac{4y}{x^2 + y^2} + \frac{2(x + y)}{x^2 + y^2} - \frac{4y^2(x + y)}{(x^2 + y^2)^2}$$

Step 3. Apply the Second Derivative Test. We can form the table

Critical point	f_{xx}	f_{yy}	f_{xy}	D	Type
$\left(\frac{1}{e\sqrt{2}}, \frac{1}{e\sqrt{2}}\right)$	$2e\sqrt{2}$	$2e\sqrt{2}$	0	$8e^2$	local minimum
$\left(-\frac{1}{e\sqrt{2}}, -\frac{1}{e\sqrt{2}}\right)$	$-2e\sqrt{2}$	$-2e\sqrt{2}$	0	$8e^2$	local maximum
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$2\sqrt{2}$	$-2\sqrt{2}$	0	-8	saddle point
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-2\sqrt{2}$	$2\sqrt{2}$	0	-8	saddle point

21. $f(x, y) = x - y^2 - \ln(x + y)$

SOLUTION

Step 1. Find the critical points. We set the partial derivatives of $f(x, y) = x - y^2 - \ln(x + y)$ equal to zero and solve.

$$f_x(x, y) = 1 - \frac{1}{x + y} = 0, \quad f_y(x, y) = -2y - \frac{1}{x + y} = 0$$

The first equation implies that $\frac{1}{x+y} = 1$. Substituting in the second equation gives

$$-2y - 1 = 0 \quad \Rightarrow \quad 2y = -1 \quad \Rightarrow \quad y = -\frac{1}{2}$$

We substitute $y = -\frac{1}{2}$ in the first equation and solve for x:

$$1 - \frac{1}{x - \frac{1}{2}} = 0 \quad \Rightarrow \quad x - \frac{1}{2} = 1 \quad \Rightarrow \quad x = \frac{3}{2}$$

We obtain the critical point $\left(\frac{3}{2}, -\frac{1}{2}\right)$. Notice that although f_x and f_y do not exist where $x + y = 0$, these are not critical points since f is not defined at these points.

Step 2. Compute the discriminant. We compute the second-order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(1 - \frac{1}{x+y}\right) = \frac{1}{(x+y)^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(-2y - \frac{1}{x+y}\right) = -2 + \frac{1}{(x+y)^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(1 - \frac{1}{x+y}\right) = \frac{1}{(x+y)^2}$$

The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = \frac{1}{(x+y)^2}\left(-2 + \frac{1}{(x+y)^2}\right) - \frac{1}{(x+y)^4} = \frac{-2}{(x+y)^2}$$

Step 3. Apply the Second Derivative Test. We have

$$D\left(\frac{3}{2}, -\frac{1}{2}\right) = \frac{-2}{\left(\frac{3}{2} - \frac{1}{2}\right)^2} = -2 < 0$$

We conclude that $\left(\frac{3}{2}, -\frac{1}{2}\right)$ is a saddle point.

22. $f(x, y) = (x - y)e^{x^2 - y^2}$

SOLUTION Find the critical points. We set the partial derivatives of $f(x, y) = (x - y)e^{x^2 - y^2}$ equal to zero and solve:

$$f_x(x, y) = e^{x^2-y^2} + (x - y)e^{x^2-y^2} \cdot 2x = e^{x^2-y^2}\left(2x^2 - 2xy + 1\right) = 0$$

$$f_y(x, y) = -e^{x^2-y^2} + (x - y)e^{x^2-y^2} \cdot (-2y) = e^{x^2-y^2}\left(2y^2 - 2xy - 1\right) = 0$$

Since $e^{x^2-y^2} \neq 0$, we have the following equations:

$$2x^2 - 2xy + 1 = 0$$
$$2y^2 - 2xy - 1 = 0$$

We add and subtract the two equations to obtain the following equations:

$$2\left(x^2 + y^2\right) - 4xy = 0$$

$$2\left(x^2 - y^2\right) + 2 = 0$$

The first equation can be rewritten as $x^2 - 2xy + y^2 = 0$ or $(x - y)^2 = 0$, yielding $x = y$. Substituting in the second equation gives $2 = 0$, we conclude that the two equations have no solutions, that is, there are no critical points (notice that f_x and f_y exist everywhere). Since local minima and local maxima can occur only at critical points, it follows that $f(x, y) = (x - y)e^{x^2-y^2}$ does not have local minima or local maxima.

23. $f(x, y) = (x + 3y)e^{y - x^2}$

SOLUTION

Step 1. Find the critical points. We compute the partial derivatives of $f(x, y) = (x + 3y)e^{y-x^2}$, using the Product Rule and the Chain Rule:

$$f_x(x, y) = 1 \cdot e^{y-x^2} + (x + 3y)e^{y-x^2} \cdot (-2x) = e^{y-x^2}\left(1 - 2x^2 - 6xy\right)$$

$$f_y(x, y) = 3e^{y-x^2} + (x + 3y)e^{y-x^2} \cdot 1 = e^{y-x^2}(3 + x + 3y)$$

We set the partial derivatives equal to zero and solve to find the critical points:

$$e^{y-x^2}\left(1 - 2x^2 - 6xy\right) = 0$$

$$e^{y-x^2}(3 + x + 3y) = 0$$

Since $e^{y-x^2} \neq 0$, we obtain the following equations:

$$1 - 2x^2 - 6xy = 0$$
$$3 + x + 3y = 0$$

The second equation gives $x = -3(1 + y)$. We substitute for x in the first equation and solve for y:

$$1 - 2 \cdot 9(1+y)^2 + 18(1+y)y = 0$$
$$1 - 18\left(1 + 2y + y^2\right) + 18\left(y + y^2\right) = 0$$
$$-17 - 18y = 0 \quad \Rightarrow \quad y = -\frac{17}{18}, \quad x = -3\left(1 - \frac{17}{18}\right) = -\frac{1}{6}$$

The critical point is $\left(-\frac{1}{6}, -\frac{17}{18}\right)$.

Step 2. Compute the second-order partials.

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x = e^{y-x^2}(-2x)\left(1 - 2x^2 - 6xy\right) + e^{y-x^2}(-4x - 6y) = 2e^{y-x^2}\left(2x^3 + 6x^2y - 3x - 3y\right)$$
$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y = e^{y-x^2}(3 + x + 3y) + e^{y-x^2} \cdot 3 = e^{y-x^2}(6 + x + 3y)$$
$$f_{xy}(x, y) = \frac{\partial}{\partial x} f_y = e^{y-x^2}(-2x)(3 + x + 3y) + e^{y-x^2} \cdot 1 = e^{y-x^2}\left(1 - 6xy - 2x^2 - 6x\right)$$

The discriminant is

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2$$

Step 3. Apply the Second Derivative Test. We obtain the following table:

Critical Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$\left(-\frac{1}{6}, -\frac{17}{18}\right)$	2.4	1.13	0.38	2.57	$D > 0$, $f_{xx} > 0$, local minimum

24. Show that $f(x, y) = x^2$ has infinitely many critical points (as a function of two variables) and that the Second Derivative Test fails for all of them. What is the minimum value of f? Does $f(x, y)$ have any local maxima?

SOLUTION First if we solve for critical points we get

$$f_x(x, y) = 2x, \quad f_y(x, y) = 0$$

Thus setting each equal to zero only yields $x = 0$ and y can be any real number. The list of critical points is

$(0, r)$ where r is any real number.

Now computing the second-order partials for the discriminant we get

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = 0, \quad f_{yy}(x, y) = 0$$

Therefore, $D = 0$. This means that the Second Derivative Test is inconclusive for every critical point, it fails.

Finally this function does have a minimum value of 0 since the smallest any square can be is 0. Since x can get arbitrarily large, this function has no maximum value, and no local maxima.

25. Prove that the function $f(x, y) = \frac{1}{3}x^3 + \frac{2}{3}y^{3/2} - xy$ satisfies $f(x, y) \geq 0$ for $x \geq 0$ and $y \geq 0$.

(a) First, verify that the set of critical points of f is the parabola $y = x^2$ and that the Second Derivative Test fails for these points.

(b) Show that for fixed b, the function $g(x) = f(x, b)$ is concave up for $x > 0$ with a critical point at $x = b^{1/2}$.

(c) Conclude that $f(a, b) \geq f(b^{1/2}, b) = 0$ for all $a, b \geq 0$.

SOLUTION

(a) To find the critical points, we need the first-order partial derivatives, set them equal to zero and solve:

$$f_x(x, y) = x^2 - y = 0, \quad f_y(x, y) = y^{1/2} - x = 0$$

This gives us:

$$y = x^2$$

as the solution set for the critical points.

Now to compute the discriminant, we need the second-order partials

$$f_{xx}(x, y) = 2x, \quad f_{yy}(x, y) = \frac{1}{2}y^{-1/2}, \quad f_{xy}(x, y) = -1$$

Thus the discriminant is

$$D(x, y) = \frac{x}{\sqrt{y}} - 1$$

Since $y = x^2$ is the solution set for the critical points we see:

$$D(x, y) = 1 - 1 = 0$$

Therefore the Second Derivative Test is inconclusive and fails us.

(b) If we fix a value b and consider $g(x) = f(x, b) = \frac{1}{3}x^3 + \frac{2}{3}b^{3/2} - bx$ to find the concavity, we see

$$g'(x) = x^2 - b, \quad g''(x) = 2x$$

Then certainly, for $x > 0$, this function is concave up. The critical point will occur at the point when $x^2 - b = 0$ or $x = b^{1/2}$.

(c) Now, since for fixed b, we know that $g(x) = f(x, b)$ is concave up if $x > 0$, and the critical point is $x = b^{1/2}$. Therefore

$$f(a, b) \geq f(b^{1/2}, b) = 0 \text{ for all } b \geq 0$$

26. Let $f(x, y) = (x^2 + y^2)e^{-x^2-y^2}$.

(a) Where does f take on its minimum value? Do not use calculus to answer this question.

(b) Verify that the set of critical points of f consists of the origin $(0, 0)$ and the unit circle $x^2 + y^2 = 1$.

(c) The Second Derivative Test fails for points on the unit circle (this can be checked by some lengthy algebra). Prove, however, that f takes on its maximum value on the unit circle by analyzing the function $g(t) = te^{-t}$ for $t > 0$.

SOLUTION

(a) We know that $e^{-(x^2+y^2)}$ is always positive and greater than 0, and $x^2 + y^2 \geq 0$, therefore the minimum is reached when $x^2 + y^2 = 0$ and the only point where this occurs is at $(0, 0)$.

(b) Find the critical points. We set the first-order derivatives equal to zero and solve:

$$f_x(x, y) = 2xe^{-x^2-y^2} + (x^2 + y^2)e^{-x^2-y^2} \cdot (-2x) = 2xe^{-x^2-y^2}(1 - x^2 - y^2) = 0$$
$$f_y(x, y) = 2ye^{-x^2-y^2} + (x^2 + y^2)e^{-x^2-y^2} \cdot (-2y) = 2ye^{-x^2-y^2}(1 - x^2 - y^2) = 0$$

Since $e^{-x^2-y^2} \neq 0$, the first equation gives $x = 0$ or $x^2 + y^2 = 1$. We substitute $x = 0$ in the second equation and solve for y:

$$2ye^{-y^2}(1 - y^2) = 0$$

Since $e^{-y^2} \neq 0$, the solutions are $y = 0$ or $y = \pm 1$. The corresponding points are $(0, 0)$, $(0, 1)$, $(0, -1)$. The solution $x^2 + y^2 = 1$ also satisfies the second equation. We conclude that there are infinitely many critical points, namely, the points on the unit circle $x^2 + y^2 = 1$ and its center $(0, 0)$.

(c) For the given function we can define $t = x^2 + y^2$ to obtain the function $g(t) = te^{-t}$. The critical point of $g(t)$ is

$$g'(t) = e^{-t} - te^{-t} = (1 - t)e^{-t} = 0 \quad \Rightarrow \quad t = 1$$

We find the second derivative at the critical point:

$$g''(t) = \frac{d}{dt}\left[(1 - t)e^{-t}\right] = -e^{-t} + (1 - t)e^{-t}(-1) = (t - 2)e^{-t}$$

Therefore, by the Second Derivative Test for functions of one variable, $t = 1$ gives a local maximum. Also, the value of $f(x, y)$ at all the points on the unit circle is the same:

$$f(x, y) = (x^2 + y^2)e^{-(x^2+y^2)} = te^{-t} = e^{-1} \text{ when } t = 1$$

It follows that at the points on the unit circle $x^2 + y^2 = 1$, $f(x, y)$ has local maxima.

27. CAS Use a computer algebra system to find a numerical approximation to the critical point of

$$f(x, y) = (1 - x + x^2)e^{y^2} + (1 - y + y^2)e^{x^2}$$

Apply the Second Derivative Test to confirm that it corresponds to a local minimum as in Figure 20.

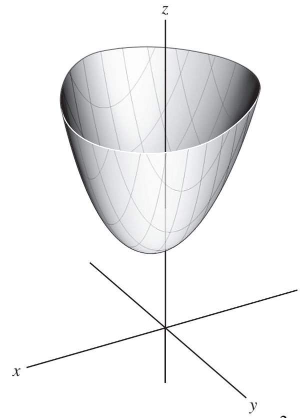


FIGURE 20 Plot of $f(x, y) = (1 - x + x^2)e^{y^2} + (1 - y + y^2)e^{x^2}$.

SOLUTION The critical points are the solutions of $f_x(x, y) = 0$ and $f_y(x, y) = 0$. We compute the partial derivatives:

$$f_x(x, y) = (-1 + 2x)e^{y^2} + \left(1 - y + y^2\right)e^{x^2} \cdot 2x$$
$$f_y(x, y) = \left(1 - x + x^2\right)e^{y^2} \cdot 2y + (-1 + 2y)e^{x^2}$$

Hence, the critical points are the solutions of the following equations:

$$(2x - 1)e^{y^2} + 2x\left(1 - y + y^2\right)e^{x^2} = 0$$
$$(2y - 1)e^{x^2} + 2y\left(1 - x + x^2\right)e^{y^2} = 0$$

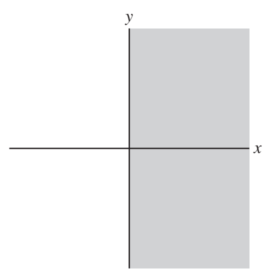
Using a CAS we obtain the following solution: $x = y = 0.27788$, which from the figure is a local minimum.

28. Which of the following domains are closed and which are bounded?

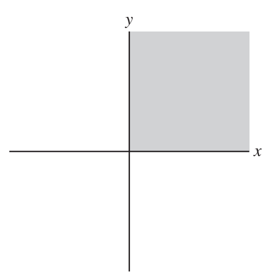
(a) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$
(b) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$
(c) $\{(x, y) \in \mathbf{R}^2 : x \ge 0\}$
(d) $\{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}$
(e) $\{(x, y) \in \mathbf{R}^2 : 1 \le x \le 4, 5 \le y \le 10\}$
(f) $\{(x, y) \in \mathbf{R}^2 : x > 0, x^2 + y^2 \le 10\}$

SOLUTION

(a) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$: This domain is bounded since it is contained, for instance, in the disk $x^2 + y^2 < 2$. The domain is also closed since it contains all of its boundary points, which are the points on the unit circle $x^2 + y^2 = 1$.
(b) $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$: The domain is contained in the disk $x^2 + y^2 < 1$, hence it is bounded. It is not closed since its boundary $x^2 + y^2 = 1$ is not contained in the domain.
(c) $\{(x, y) \in \mathbf{R}^2 : x \ge 0\}$:

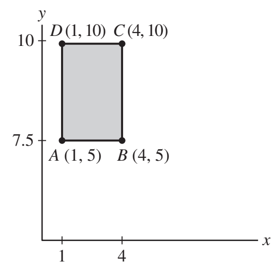


This domain is not contained in any disk, hence it is not bounded. However, the domain contains its boundary $x = 0$ (the y-axis), hence it is closed.
(d) $\{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}$:



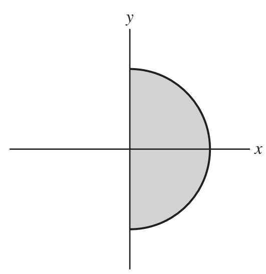
The domain is not contained in any disk, hence it is not bounded. The boundary is the positive x and y axes, and it is not contained in the domain, therefore the domain is not closed.

(e) $\{(x, y) \in \mathbf{R}^2 : 1 \leq x \leq 4, 5 \leq y \leq 10\}$:



This domain is contained in the disk $x^2 + y^2 \leq 11^2$, hence it is bounded. Moreover, the domain contains its boundary, which consists of the segments AB, BC, CD, AD shown in the figure, therefore the domain is closed.

(f) $\{(x, y) \in \mathbf{R}^2 : x > 0, x^2 + y^2 \leq 10\}$:



This domain is bounded since it is contained in the disk $x^2 + y^2 \leq 10$. It is not closed since the part $\{(0, y) \in \mathbf{R}^2 : |y| \leq \sqrt{10}\}$ of its boundary is not contained in the domain.

In Exercises 29–32, determine the global extreme values of the function on the given set without using calculus.

29. $f(x, y) = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$

SOLUTION The sum $x + y$ is maximum when $x = 1$ and $y = 1$, and it is minimum when $x = 0$ and $y = 0$. Therefore, the global maximum of f on the given set is $f(1, 1) = 1 + 1 = 2$ and the global minimum is $f(0, 0) = 0 + 0 = 0$.

30. $f(x, y) = 2x - y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 3$

SOLUTION f is maximum when x is maximum and y is minimum, that is $x = 1$ and $y = 0$. f is minimum when x is minimum and y is maximum, that is, $x = 0$, $y = 3$. Therefore, the global maximum of f in the set is $f(1, 0) = 2 \cdot 1 - 0 = 2$ and the global minimum is $f(0, 3) = 2 \cdot 0 - 3 = -3$.

31. $f(x, y) = (x^2 + y^2 + 1)^{-1}, \quad 0 \leq x \leq 3, \quad 0 \leq y \leq 5$

SOLUTION $f(x, y) = \frac{1}{x^2+y^2+1}$ is maximum when x^2 and y^2 are minimum, that is, when $x = y = 0$. f is minimum when x^2 and y^2 are maximum, that is, when $x = 3$ and $y = 5$. Therefore, the global maximum of f on the given set is $f(0, 0) = (0^2 + 0^2 + 1)^{-1} = 1$, and the global minimum is $f(3, 5) = (3^2 + 5^2 + 1)^{-1} = \frac{1}{35}$.

32. $f(x, y) = e^{-x^2-y^2}, \quad x^2 + y^2 \leq 1$

SOLUTION The function $f(x, y) = e^{-(x^2+y^2)} = \frac{1}{e^{x^2+y^2}}$ is maximum when $e^{x^2+y^2}$ is minimum, that is, when $x^2 + y^2$ is minimum. The minimum value of $x^2 + y^2$ on the given set is zero, obtained at $x = 0$ and $y = 0$. We conclude that the maximum value of f on the given set is

$$f(0, 0) = e^{-0^2-0^2} = e^0 = 1$$

f is minimum when $x^2 + y^2$ is maximum, that is, when $x^2 + y^2 = 1$. Thus, the minimum value of f on the given disk is obtained on the boundary of the disk, and it is $e^{-1} = \frac{1}{e}$.

33. Assumptions Matter Show that $f(x, y) = xy$ does not have a global minimum or a global maximum on the domain

$$\mathcal{D} = \{(x, y) : 0 < x < 1, 0 < y < 1\}$$

Explain why this does not contradict Theorem 3.

SOLUTION The largest and smallest values of f on the closed square $0 \le x, y \le 1$ are $f(1,1) = 1$ and $f(0,0) = 0$. However, on the open square $0 < x, y < 1$, f can never attain these maximum and minimum values, since the boundary (and in particular the points $(1,1)$ and $(-1,-1)$) are not included in the domain. This does not contradict Theorem 3 since the domain is open.

34. Find a continuous function that does not have a global maximum on the domain $\mathcal{D} = \{(x, y) : x + y \ge 0, x + y \le 1\}$. Explain why this does not contradict Theorem 3.

SOLUTION Consider the continuous function $f(x, y) = x$. Taking first partial derivatives we have

$$f_x = x, \quad f_y = 0$$

and second-order partials we have

$$f_{xx} = 1, \quad f_{yy} = 0, \quad f_{xy} = 0$$

Already we can see that

$$D = f_{xx} f_{yy} - f_{xy}^2 = 0$$

So the Second Derivative Test is going to be inconclusive. (In fact, there are no critical points)

Considering this function over the domain, $\mathcal{D} = \{(x, y) : x + y \ge 0, x + y \le 1\}$, we see that $f(x, y) = x$ is in the strip formed between to the two lines $y = -x$ and $y = 1 - x$. We can make $f(x, y) = x$ arbitrarily large within this region. In fact, we can see that $\lim_{x \to -\infty} f(x, y)$ is arbitrarily large. This does not contradict the theorem in the text, because the domain $\mathcal{D}$ is an bounded domain, in that for any integer n, we can see that the open interval $(-n, n + 0.5)$ is contained in this region.

35. Find the maximum of

$$f(x, y) = x + y - x^2 - y^2 - xy$$

on the square, $0 \le x \le 2$, $0 \le y \le 2$ (Figure 21).

(a) First, locate the critical point of f in the square, and evaluate f at this point.

(b) On the bottom edge of the square, $y = 0$ and $f(x, 0) = x - x^2$. Find the extreme values of f on the bottom edge.

(c) Find the extreme values of f on the remaining edges.

(d) Find the largest among the values computed in (a), (b), and (c).

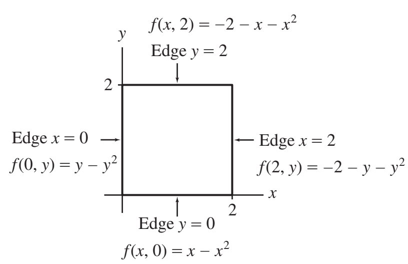


FIGURE 21 The function $f(x, y) = x + y - x^2 - y^2 - xy$ on the boundary segments of the square $0 \le x \le 2$, $0 \le y \le 2$.

SOLUTION

(a) To find the critical points, we look at the first-order partial derivatives set equal to zero and solve:

$$f_x(x, y) = 1 - 2x - y = 0, \quad f_y(x, y) = 1 - 2y - x = 0$$

This gives $y = 1 - 2x$ and $x = 1 - 2y$, solving simultaneously we see $y = 1/3$ and $x = 1/3$. The critical point is $(1/3, 1/3)$, subsequently, $f(1/3, 1/3) = 1/3$.

(b) To find the extreme points of $f(x, 0) = x - x^2$ we take the first derivative and set it equal to zero and solve:

$$f'(x, 0) = 1 - 2x = 0 \to x = 1/2$$

Thus the extreme value on the bottom edge of the square is

$$f(1/2, 0) = 1/4$$

(c) Now to find the extreme values on the other edges of the square.

First, let us use $x = 0$: $f(0, y) = y - y^2$. Taking the first derivative and setting equal to 0 gives us:

$$f'(0, y) = 1 - 2y = 0, \to y = 1/2$$

Therefore, the extreme value along $x = 0$ is $f(0, 1/2) = 1/4$.

Next, let us use $y = 2$: $f(x, 2) = -x^2 - x - 2$. Take the first derivative and setting equal to 0 gives us:

$$f'(x, 2) = -2x - 1 = 0, \to x = -1/2$$

Therefore, the extreme value along $y = 2$ is $f(-1/2, 2) = -7/4$.

Finally, let us use $x = 2$: $f(2, y) = -2 - y - y^2$. Take the first derivative and setting equal to 0 gives us:

$$f'(2, y) = -1 - 2y = 0, \to y = -1/2$$

Therefore, the extreme value along $x = 2$ is $f(2, -1/2) = -7/4$.

(d) Out of all the values we computed in parts (a), (b), and (c), $1/3$ is the largest. This value occurs at the point $(1/3, 1/3)$.

36. Find the maximum of $f(x, y) = y^2 + xy - x^2$ on the square $0 \le x \le 2$, $0 \le y \le 2$.

SOLUTION

(a) First, locate the critical point of f in the square, and evaluate f at this point.

Taking first-order partial derivatives and setting them equal to 0 to solve, we have:

$$f_x = y - 2x = 0, \quad f_y = 2y + x = 0$$

Thus $2x = y$ and we can write

$$4x + x = 0 \quad \Rightarrow \quad x = 0 \text{ and } y = 0$$

Therefore our critical point is $(0, 0)$ and note here that $f(0, 0) = 0$.

(b) Find the extreme values of f on the edges of the square, namely $x = 0, 2$ and $y = 0, 2$.

First if $x = 0$, then $f(0, y) = y^2$ and $f' = 2y$. Setting the derivative equal to 0 to solve we see $y = 0$. An extreme value occurs at the point $(0, 0)$, which was already accounted for in the step above. We also must examine the endpoints on the interval $(0, 0)$ and $(0, 2)$. Using this we have:

$$f(0, 0) = 0, \quad f(0, 2) = 4$$

Next, if $x = 2$, then $f(2, y) = y^2 + 2y - 4$ and $f' = 2y + 2$. Setting the derivative equal to 0 to solve, we see $y = -1$, but this value is not on our square, so we remove it from consideration. The endpoints along this line segment are $(2, 0)$ and $(2, 2)$. Using these we have

$$f(2, 0) = 4, \quad f(2, 2) = 4$$

Next, if $y = 0$, then $f(x, 0) = -x^2$ and $f' = -2x$. Setting the derivative equal to 0 to solve, we see $x = 0$. This value has already been accounted for in part (a). Checking the endpoints of this line segment means examining the points $(0, 0)$ and $(2, 0)$. Both have been accounted for in steps above.

Finally, if $y = 2$, then $f(x, 2) = 4 + 2x - x^2$ and $f' = 2 - 2x$. Setting this derivative equal to 0 to solve, we see $x = 1$. Evaluating at the point $(1, 2)$ we have $f(1, 2) = 5$. The endpoints of this line segment are $(0, 2)$ and $(2, 2)$, both have been accounted for in steps above.

(c) Find the largest among the values computed in (a) and (b).

The maximum occurs at critical point f on the top edge, where $f(x, 2) = 4 + 2x - x^2$. The critical point is $(x, y) = (1, 2)$ and $f(1, 2) = 5$.

In Exercises 37–43, determine the global extreme values of the function on the given domain.

37. $f(x, y) = x^3 - 2y, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

SOLUTION We use the following steps.

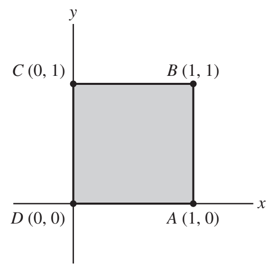
Step 1. Find the critical points. We set the first derivative equal to zero and solve:

$$f_x(x, y) = 3x^2 = 0, \quad f_y(x, y) = -2$$

The two equations have no solutions, hence there are no critical points.

Step 2. Check the boundary. The extreme values occur either at the critical points or at a point on the boundary of the domain. Since there are no critical points, the extreme values occur at boundary points. We consider each edge of the square $0 \le x, y \le 1$ separately.

The segment $\overline{OA}$: On this segment $y = 0$, $0 \le x \le 1$, and f takes the values $f(x, 0) = x^3$. The minimum value is $f(0, 0) = 0$ and the maximum value is $f(1, 0) = 1$.



The segment $\overline{AB}$: On this segment $x = 1$, $0 \le y \le 1$, and f takes the values $f(1, y) = 1 - 2y$. The minimum value is $f(1, 1) = 1 - 2 \cdot 1 = -1$ and the maximum value is $f(1, 0) = 1 - 2 \cdot 0 = 1$.

The segment $\overline{BC}$: On this segment $y = 1$, $0 \le x \le 1$, and f takes the values $f(x, 1) = x^3 - 2$. The minimum value is $f(0, 1) = 0^3 - 2 = -2$ and the maximum value is $f(1, 1) = 1^3 - 2 = -1$.

The segment $\overline{OC}$: On this segment $x = 0$, $0 \le y \le 1$, and f takes the values $f(0, y) = -2y$. The minimum value is $f(0, 1) = -2 \cdot 1 = -2$ and the maximum value is $f(0, 0) = -2 \cdot 0 = 0$.

Step 3. Conclusions. The values obtained in the previous steps are

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(1,1) = -1, \quad f(0,1) = -2$$

The smallest value is $f(0, 1) = -2$ and it is the global minimum of f on the square. The global maximum is the largest value $f(1, 0) = 1$.

38. $f(x, y) = 5x - 3y, \quad y \ge x - 2, \quad y \ge -x - 2, \quad y \le 3$

SOLUTION

Step 1. Find the critical points. We set the first partial derivatives equal to zero and solve:

$$f_x(x, y) = 5, \quad f_y(x, y) = -3$$

When we set each equal to zero, we have no solutions, hence there are no critical points.

Step 2. Check the boundary. The extreme values occur either at the critical points or at a point on the boundary of the domain. The edges of the boundary are defined by the line $y = x - 2$, the line $y = -x - 2$, and the line $y = 3$. This is the triangle with vertices $(0, -2)$, $(5, 3)$, $(-5, 3)$.

On the line $y = x - 2$ we have:

$$f(x, x - 2) = 5x - 3(x - 2) = 2x + 6 \text{ and } f' = 2$$

This means that the function is always increasing and the minimum occurs at the point $(0, -2)$ and the maximum occurs at the vertex $(5, 3)$:

$$f(0, -2) = 6, \quad f(5, 3) = 16$$

On the line $y = -x - 2$ we have:

$$f(x, -x - 2) = 5x - 3(-x - 2) = 8x + 6 \text{ and } f' = 8$$

This means that the function is always increasing and the minimum occurs at the point $(-5, 3)$ and the maximum occurs at the vertex $(0, -2)$:

$$f(-5, 3) = -34, \quad f(0, -2) = 6$$

On the line $y = 3$ we have:

$$f(x, 3) = 5x - 9 \text{ and } f' = 5$$

This means that the function is always increasing and the minimum occurs at the point $(-5, 3)$ and the maximum occurs at the vertex $(5, 3)$:

$$f(-5, 3) = -34, \quad f(5, 3) = 16$$

Step 3. Conclusions. The values obtained in the previous steps are:

$$f(0, -2) = 6, \quad f(-5, 3) = -34, \quad f(5, 3) = 16$$

The maximum value is 16 and it occurs at the point $(5, 3)$ and the minimum value is -34 and it occurs at the point $(-5, 3)$.

39. $f(x, y) = x^2 + 2y^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

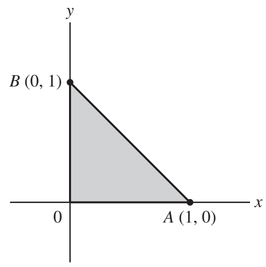
SOLUTION The sum $x^2 + 2y^2$ is maximum at the point $(1, 1)$, where x^2 and y^2 are maximum. It is minimum if $x = y = 0$, that is, at the point $(0, 0)$. Hence,

$$\text{Global maximum} = f(1, 1) = 1^2 + 2 \cdot 1^2 = 3$$
$$\text{Global minimum} = f(0, 0) = 0^2 + 2 \cdot 0^2 = 0$$

40. $f(x, y) = x^3 + x^2 y + 2y^2, \quad x, y \ge 0, \quad x + y \le 1$

SOLUTION We use the following steps.

Step 1. Examine the critical points. We find the critical points of $f(x, y) = x^3 + x^2 y + 2y^2$ in the interior of the domain (the standard region in the figure).



We set the partial derivatives of f equal to zero and solve:

$$f_x(x, y) = 3x^2 + 2xy = x(3x + 2y) = 0$$
$$f_y(x, y) = x^2 + 4y = 0$$

The first equation gives $x = 0$ or $y = -\frac{3}{2}x$. Substituting $x = 0$ in the second equation gives $4y = 0$ or $y = 0$. We obtain the critical point $(0, 0)$. We now substitute $y = -\frac{3}{2}x$ in the second equation and solve for x:

$$x^2 + 4 \cdot \left(-\frac{3}{2}x\right) = x^2 - 6x = x(x - 6) = 0 \quad \Rightarrow \quad x = 0, \quad x = 6$$

We get the critical points $(0, 0)$ and $(6, -9)$. None of the critical points $(0, 0)$ and $(6, -9)$ is in the interior of the domain.

Step 2. Check the boundary. The boundary consists of the three segments $\overline{OA}$, $\overline{OB}$, and $\overline{AB}$ shown in the figure. We consider each part of the boundary separately.

The segment $\overline{OA}$: On this segment $y = 0$, $0 \le x \le 1$, and $f(x, y) = f(x, 0) = x^3$. The minimum value is $f(0, 0) = 0^3 = 0$ and the maximum value is $f(1, 0) = 1^3 = 1$.
The segment $\overline{OB}$: On this segment $x = 0$, $0 \le y \le 1$, and $f(x, y) = f(0, y) = 2y^2$. The minimum value is $f(0, 0) = 2 \cdot 0^2 = 0$ and the maximum value is $f(0, 1) = 2 \cdot 1^2 = 2$.
The segment $\overline{AB}$: On this segment $y = 1 - x$, $0 \le x \le 1$, and

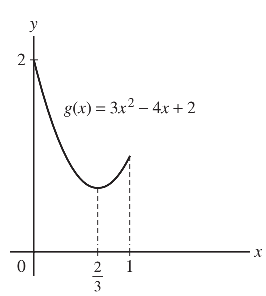
$$f(x, y) = x^3 + x^2(1 - x) + 2(1 - x)^2 = x^3 + x^2 - x^3 + 2\left(1 - 2x + x^2\right) = 3x^2 - 4x + 2$$

We find the extreme values of $g(x) = 3x^2 - 4x + 2$ in the interval $0 \le x \le 1$. With the aid of the graph of $g(x)$, and with setting the derivative g' equal to 0, we find that the minimum value is

$$g\left(\frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{1}{3}\right) = 3 \cdot \left(\frac{2}{3}\right)^2 - 4 \cdot \frac{2}{3} + 2 = \frac{2}{3}$$

and the maximum value is

$$g(0) = f(0, 1) = 3 \cdot 0^2 - 4 \cdot 0 + 2 = 2$$



Step 3. Conclusions. We compare the values of $f(x, y)$ at the points obtained in step (2), and determine the global extrema of $f(x, y)$. This gives

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(0,1) = 2, \quad f\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}$$

We conclude that the global minimum of f in the given domain is $f(0,0) = 0$ and the global maximum is $f(0,1) = 2$.

41. $f(x, y) = x^3 + y^3 - 3xy, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

SOLUTION We use the following steps.

Step 1. Examine the critical points in the interior of the domain. We set the partial derivatives equal to zero and solve:

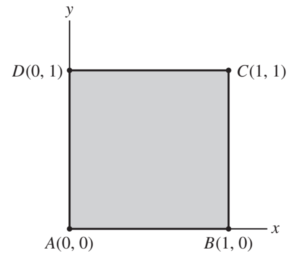
$$f_x(x, y) = 3x^2 - 3y = 0$$
$$f_y(x, y) = 3y^2 - 3x = 0$$

The first equation gives $y = x^2$. We substitute in the second equation and solve for x:

$$3\left(x^2\right)^2 - 3x = 0$$
$$3x^4 - 3x = 3x\left(x^3 - 1\right) = 0 \quad \Rightarrow \quad x = 0, \quad y = 0^2 = 0$$
$$\text{or} \quad x = 1, \quad y = 1^2 = 1$$

The critical points (0, 0) and (1, 1) are not in the interior of the domain.

Step 2. Find the extreme values on the boundary. We consider each part of the boundary separately.



The edge $\overline{AB}$: On this edge, $y = 0$, $0 \le x \le 1$, and $f(x, 0) = x^3$. The maximum value is obtained at $x = 1$ and the minimum value is obtained at $x = 0$. The corresponding extreme points are (1, 0) and (0, 0).

The edge $\overline{BC}$: On this edge $x = 1$, $0 \le y \le 1$, and $f(1, y) = y^3 - 3y + 1$. The critical points are $\frac{d}{dy}\left(y^3 - 3y + 1\right) = 3y^2 - 3 = 0$, that is, $y = \pm 1$. The point in the given domain is $y = 1$. The candidates for extreme values are thus $y = 1$ and $y = 0$, giving the points (1, 1) and (1, 0).

The edge $\overline{DC}$: On this edge $y = 1$, $0 \le x \le 1$, and $f(x, 1) = x^3 - 3x + 1$. Replacing the values of x and y in the previous solutions we get the points (1, 1) and (0, 1).

The edge $\overline{AD}$: On this edge $x = 0$, $0 \le y \le 1$, and $f(0, y) = y^3$. Replacing the values of x and y obtained for the edge $\overline{AB}$, we get (0, 1) and (0, 0).

By Theorem 3, the extreme values occur either at a critical point in the interior of the square or at a point on the boundary of the square. Since there are no critical points in the interior of the square, the candidates for extreme values are the following points:

$$(0,0), \quad (1,0), \quad (1,1), \quad (0,1)$$

We compute $f(x, y) = x^3 + y^3 - 3xy$ at these points:

$$f(0,0) = 0^3 + 0^3 - 3 \cdot 0 = 0$$
$$f(1,0) = 1^3 + 0^3 - 3 \cdot 1 \cdot 0 = 1$$
$$f(1,1) = 1^3 + 1^3 - 3 \cdot 1 \cdot 1 = -1$$
$$f(0,1) = 0^3 + 1^3 - 3 \cdot 0 \cdot 1 = 1$$

We conclude that in the given domain, the global maximum is $f(1,0) = f(0,1) = 1$ and the global minimum is $f(1,1) = -1$.

42. $f(x, y) = x^2 + y^2 - 2x - 4y, \quad x \geq 0, \quad 0 \leq y \leq 3, \quad y \geq x$

SOLUTION We use the following steps:

Step 1. Examine the critical points in the interior of the domain. We set the partial derivatives equal to zero and solve:

$$f_x(x, y) = 2x - 2, \quad f_y(x, y) = 2y - 4$$

Setting each equal to zero and solving we get: $x = 1$ and $y = 2$. Evaluating at the point $(1, 2)$ we see:

$$f(1, 2) = -5$$

Step 2. Find the extreme values on the boundary. We consider each part of the boundary separately. The region that is described is the triangle bounded by the lines $x = 0$, $y = 3$, and $y = x$ with vertices $(0, 0)$, $(3, 3)$, $(0, 3)$.

First consider the line $x = 0$:

$$f(0, y) = y^2 - 4y \quad \Rightarrow \quad f' = 2y - 4$$

Setting f' equal to zero and solving we get $y = 2$. So we must consider the point $(0, 2)$:

$$f(0, 2) = -4$$

We must also consider the endpoints of this line segment, $(0, 0)$ and $(0, 3)$:

$$f(0, 0) = 0, \quad f(0, 3) = -3$$

Next, consider the line $y = 3$:

$$f(x, 3) = x^2 + 9 - 2x - 12 = x^2 - 2x - 3 \quad \Rightarrow \quad f' = 2x - 2$$

Setting f' equal to zero and solving we get $x = 1$. So we must also consider the point $(1, 3)$:

$$f(1, 3) = -4$$

We must also consider the endpoints of this line segment, $(0, 3)$ and $(3, 3)$:

$$f(0, 3) = -3, \quad f(3, 3) = 0$$

Finally, consider the line $y = x$:

$$f(x, x) = x^2 + x^2 - 2x - 4x = 2x^2 - 6x \quad \Rightarrow \quad f' = 4x - 6$$

Setting f' equal to zero and solving, we get $x = 3/2$. So we must also consider the point $(3/2, 3/2)$:

$$f(3/2, 3/2) = -\frac{9}{2}$$

We have already examined the endpoints of this line segment in the steps above.

Step 3. Conclusions. The points that we have considered in this problem are

$$f(1, 2) = -5, \quad f(0, 2) = -4, \quad f(1, 3) = -4, \quad f(3/2, 3/2) = -\frac{9}{2}$$

$$f(0, 0) = 0, \quad f(0, 3) = -3, \quad f(3, 3) = 0$$

Therefore the minimum value is -5 and occurs at the point $(1, 2)$ and the maximum value is 0 and occurs in two places, at the points $(0, 0)$ and $(3, 3)$.

43. $f(x, y) = (4y^2 - x^2)e^{-x^2-y^2}, \quad x^2 + y^2 \leq 2$

SOLUTION We use the following steps.

Step 1. Examine the critical points. We compute the partial derivatives of $f(x, y) = \left(4y^2 - x^2\right) e^{-x^2-y^2}$, set them equal to zero and solve. This gives

$$f_x(x, y) = -2xe^{-x^2-y^2} + \left(4y^2 - x^2\right) e^{-x^2-y^2} \cdot (-2x) = -2xe^{-x^2-y^2} \left(1 + 4y^2 - x^2\right) = 0$$

$$f_y(x, y) = 8ye^{-x^2-y^2} + \left(4y^2 - x^2\right) e^{-x^2-y^2} \cdot (-2y) = -2ye^{-x^2-y^2} \left(-4 + 4y^2 - x^2\right) = 0$$

Since $e^{-x^2-y^2} \neq 0$, the first equation gives $x = 0$ or $x^2 = 1 + 4y^2$. Substituting $x = 0$ in the second equation gives

$$-2ye^{-y^2} \left(-4 + 4y^2\right) = 0.$$

Since $e^{-y^2} \neq 0$, we get

$$y\left(-1+y^2\right) = y(y-1)(y+1) = 0 \quad \Rightarrow \quad y=0, \quad y=1, \quad y=-1$$

We obtain the three points $(0,0)$, $(0,-1)$, $(0,1)$. We now substitute $x^2 = 1+4y^2$ in the second equation and solve for y:

$$-2ye^{-1-5y^2}\left(-4+4y^2-1-4y^2\right) = 0$$

$$-2ye^{-1-5y^2}\cdot(-5) = 0 \quad \Rightarrow \quad y=0$$

The corresponding values of x are obtained from

$$x^2 = 1+4\cdot 0^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

We obtain the solutions $(1,0)$ and $(-1,0)$. We conclude that the critical points are

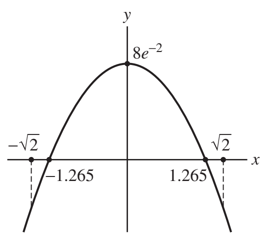
$$(0,0), \quad (0,-1), \quad (0,1), \quad (1,0), \quad \text{and} \quad (-1,0).$$

All of these points are in the interior $x^2+y^2<2$ of the given disk.

Step 2. Check the boundary. The boundary is the circle $x^2+y^2=2$. On this set $y^2 = 2-x^2$, hence the function $f(x,y)$ takes the values

$$f(x,y)\Big|_{x^2+y^2=2} = g(x) = \left(4\left(2-x^2\right)-x^2\right)e^{-2} = \left(-5x^2+8\right)e^{-2}$$

That is, $g(x) = -5e^{-2}x^2 + 8e^{-2}$. We determine the interval of x. Since $x^2+y^2=2$, we have $0 \le x^2 \le 2$ or $-\sqrt{2} \le x \le \sqrt{2}$.



We thus must find the extreme values of $g(x) = -5e^{-2}x^2 + 8e^{-2}$ on the interval $-\sqrt{2} \le x \le \sqrt{2}$. With the aid of the graph of $g(x)$, we conclude that the maximum value is $g(0) = 8e^{-2}$ and the minimum value is

$$g\left(-\sqrt{2}\right) = g\left(\sqrt{2}\right) = -5e^{-2}\left(\pm\sqrt{2}\right)^2 + 8e^{-2} = -10e^{-2} + 8e^{-2} = -2e^{-2} \approx -0.271$$

We conclude that the points on the boundary with largest and smallest values of f are

$$f\left(0, \pm\sqrt{2}\right) = 8e^{-2} \approx 1.083, \quad f\left(\pm\sqrt{2}, 0\right) = -2e^{-2} \approx -0.271$$

Step 3. Conclusions. The extreme values either occur at the critical points or at the points on the boundary, found in step 2. We compare the values of f at these points:

$$f(0,0) = 0$$
$$f(0,-1) = 4e^{-1} \approx 1.472$$
$$f(0,1) = 4e^{-1} \approx 1.472$$
$$f(1,0) = -e^{-1} \approx -0.368$$
$$f(-1,0) = -e^{-1} \approx -0.368$$
$$f\left(0, \pm\sqrt{2}\right) \approx 1.083$$
$$f\left(\pm\sqrt{2}, 0\right) \approx -0.271$$

We conclude that the global minimum is $f(1,0) = f(-1,0) = -0.368$ and the global maximum is $f(0,-1) = f(0,1) = 1.472$.

44. Find the maximum volume of a box inscribed in the tetrahedron bounded by the coordinate planes and the plane

$$x + \frac{1}{2}y + \frac{1}{3}z = 1$$

SOLUTION To maximize volume of a rectangular box we must consider the volume, $V = xyz$. But since the constraint is $x + \frac{1}{2}y + \frac{1}{3}z = 1$, we can solve this for z and get:

$$z = 3 - 3x - \frac{3}{2}y \quad \Rightarrow \quad V(x, y) = xy\left(3 - 3x - \frac{3}{2}y\right) = 3xy - 3x^2y - \frac{3}{2}xy^2$$

Now to maximize $V(x, y)$. First to find the critical points, we take the first-order partial derivatives, set them equal to zero, and solve:

$$V_x(x, y) = 3y - 6xy - \frac{3}{2}y^2 = 0, \quad V_y(x, y) = 3x - 3x^2 - 3xy = 0$$

Using the equation $V_y = 0$ we see:

$$3x - 3x^2 - 3xy = 0 \quad \Rightarrow \quad x - x^2 - xy = 0 \quad \Rightarrow \quad xy = x - x^2 \quad \Rightarrow \quad y = 1 - x \text{ or } x = 0$$

We can ignore $x = 0$, because this value would produce a box having volume 0. Using this information in the first equation, $V_x = 0$, we see

$$3y - 6xy - \frac{3}{2}y^2 = 0 \quad \Rightarrow \quad 3(1 - x) - 6x(1 - x) - \frac{3}{2}(1 - x)^2 = 0 \quad \Rightarrow \quad \frac{9}{2}x^2 - 6x + \frac{3}{2} = 0$$

Clearing this equation of fractions we have

$$3x^2 - 4x + 1 = 0 \quad \Rightarrow \quad (3x - 1)(x - 1) = 0 \quad \Rightarrow \quad x = \frac{1}{3}, 1$$

Using this information we see:

$$x = \frac{1}{3} \quad \Rightarrow \quad y = 1 - \frac{1}{3} = \frac{2}{3}$$

$$x = 1 \quad \Rightarrow \quad y = 1 - 1 = 0$$

We know that $y \neq 0$, otherwise, volume of the box will be 0 (which is not maximized). In fact, it makes no sense to use any of the coordinate plane boundaries for critical points because the resultant volume will be 0.

Therefore we examine the point where $x = \frac{1}{3}$ and $y = \frac{2}{3}$. To find z we use $z = 3 - 3x - \frac{3}{2}y$:

$$z = 3 - 3 \cdot \frac{1}{3} - \frac{3}{2} \cdot \frac{2}{3} = 1$$

Hence the maximum volume of the box is

$$V = xyz = \frac{1}{3} \cdot \frac{2}{3} \cdot 1 = \frac{2}{9} \text{ cubic units}$$

45. Find the maximum volume of the largest box of the type shown in Figure 22, with one corner at the origin and the opposite corner at a point $P = (x, y, z)$ on the paraboloid

$$z = 1 - \frac{x^2}{4} - \frac{y^2}{9} \quad \text{with } x, y, z \geq 0$$

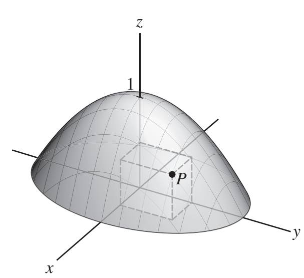


FIGURE 22

SOLUTION To maximize the volume of a rectangular box, start with the relation $V = xyz$ and using the paraboloid equation we see

$$z = 1 - \frac{x^2}{4} - \frac{y^2}{9} \quad \Rightarrow \quad V(x, y) = xy\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)$$

Therefore we will consider

$$V(x, y) = xy - \frac{1}{4}x^3y - \frac{1}{9}xy^3$$

First to find the critical points, we take the first-order partial derivatives and set them equal to zero, and solve:

$$V_x(x, y) = y - \frac{3}{4}x^2y - \frac{1}{9}y^3, \quad V_y(x, y) = x - \frac{1}{4}x^3 - \frac{1}{3}xy^2$$

Using the equation $V_y = 0$ we see

$$x - \frac{1}{4}x^3 - \frac{1}{3}xy^2 = 0 \quad \Rightarrow \quad x = 0, \quad y^2 = 3 - \frac{3}{4}x^2 \quad \Rightarrow \quad y = \sqrt{3 - \frac{3}{4}x^2}$$

(Note here, we can ignore the value $x = 0$, since it produces a box having zero volume.)

Using this relation in the first equation, $V_x = 0$, we see:

$$\sqrt{3 - \frac{3}{4}x^2} - \frac{3}{4}x^2\sqrt{3 - \frac{3}{4}x^2} - \frac{1}{9}\left(3 - \frac{3}{4}x^2\right)^{3/2} = 0$$

Factoring we see:

$$\sqrt{3 - \frac{3}{4}x^2}\left[1 - \frac{3}{4}x^2 - \frac{1}{9}\left(3 - \frac{3}{4}x^2\right)\right] = 0$$

and thus

$$3 - \frac{3}{4}x^2 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

or

$$1 - \frac{3}{4}x^2 - \frac{1}{3} + \frac{1}{12}x^2 = 0 \quad \Rightarrow \quad \frac{2}{3} - \frac{2}{3}x^2 = 0 \quad \Rightarrow \quad x = \pm 1$$

Since the governing equation $f(x, y)$ is a paraboloid, that is symmetric about the z-axis, we need only consider the point when $x = 2$ or $x = 1$.

Therefore, since $y = \sqrt{3 - \frac{3}{4}x^2}$ and $z = 1 - \frac{1}{4}x^2 - \frac{1}{9}y^2$, we have, if $x = 2$

$$y = \sqrt{3 - \frac{3}{4} \cdot 4} = 0 \quad \Rightarrow \quad z = 1 - \frac{1}{4} \cdot 4 - \frac{1}{9} \cdot 0 = 0$$

This will give a box having zero volume - not a maximum volume at all.

Using $x = 1$, and $y = \sqrt{3 - \frac{3}{4}x^2}$, $z = 1 - \frac{1}{4}x^2 - \frac{1}{9}y^2$, we have

$$y = \sqrt{3 - \frac{3}{4}} = \frac{3}{2}, \quad z = 1 - \frac{1}{4} \cdot 1^2 - \frac{1}{9} \cdot \frac{9}{4} = \frac{1}{2}$$

Therefore, the box having maximum volume has dimensions, $x = 1$, $y = 3/2$, and $z = 1/2$ and maximum value for the volume:

$$V = xyz = 1 \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

46. Find the point on the plane

$$z = x + y + 1$$

closest to the point $P = (1, 0, 0)$. *Hint:* Minimize the square of the distance.

SOLUTION Using the hint given in the text, minimize the function

$$f(x, y, z) = (x - 1)^2 + y^2 + (x + y + 1)^2$$

We get, after taking first-order partial derivatives and setting them equal to zero to solve:

$$f_x = 2(x-1) + 2(x+y+1) = 0, \quad f_y = 2y + 2(x+y+1) = 0$$

This gives $y = x - 1$ and $2(x-1) + 2(2x) = 0$ or $x = 1/3$.

Therefore, since $x = 1/3$, then $y = x - 1 = 1/3 - 1 = -2/3$ and $z = x + y + 1 = 1/3 - 2/3 + 1 = 2/3$. The point closest to the point $P(1, 0, 0)$ is the point $(1/3, -2/3, 2/3)$.

47. Show that the sum of the squares of the distances from a point $P = (c, d)$ to n fixed points $(a_1, b_1), \dots, (a_n, b_n)$ is minimized when c is the average of the x-coordinates a_i and d is the average of the y-coordinates b_i.

SOLUTION First we must form the sum of the squares of the distances from a point $P(c, d)$ to n fixed points. For instance, the square of the distance from (c, d) to (a_1, b_1) would be:

$$(c - a_1)^2 + (d - b_1)^2$$

using this pattern, the sum in question would be

$$S = \sum_{i=1}^{n} [(c - a_i)^2 + (d - b_i)^2$$

Using the methods discussed in this section of the text, we want to minimize the sum S. We will examine the first-order partial derivatives with respect to c and d and set them equal to zero and solve:

$$S_c = \sum_{i=1}^{n} 2(c - a_i) = 0, \quad S_d = \sum_{i=1}^{n} 2(d - b_i) = 0$$

Consider first the following:

$$\sum_{i=1}^{n} 2(c - a_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} (c - a_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} c - \sum_{i=1}^{n} a_i = 0$$

Therefore

$$\sum_{i=1}^{n} c = \sum_{i=1}^{n} a_i \quad \Rightarrow \quad n \cdot c = \sum_{i=1}^{n} a_i \quad \Rightarrow \quad c = \frac{1}{n} \sum_{i=1}^{n} a_i$$

Similarly we can examine $S_d = 0$ to see

$$\sum_{i=1}^{n} 2(d - b_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} (d - b_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} d - \sum_{i=1}^{n} b_i = 0$$

and

$$\sum_{i=1}^{n} d = \sum_{i=1}^{n} b_i \quad \Rightarrow \quad n \cdot d = \sum_{i=1}^{n} b_i \quad \Rightarrow \quad d = \frac{1}{n} \sum_{i=1}^{n} b_i$$

Therefore, the sum is minimized when c is the average of the x-coordinates a_i and d is the average of the y-coordinates b_i.

48. Show that the rectangular box (including the top and bottom) with fixed volume $V = 27$ m^3 and smallest possible surface area is a cube (Figure 23).

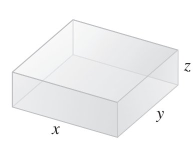
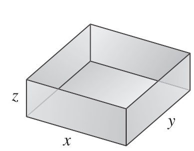


FIGURE 23 Rectangular box with sides x, y, z.

SOLUTION

Step 1. Find a function to be maximized. The surface area of the box with sides lengths x, y, z is

$$S = 2(xz + yz + xy) \tag{1}$$



We express the surface area in terms of x and y alone using the equation $V = xyz$ for the volume of the box. This equation implies that $z = \frac{V}{xy}$, hence by (1) we get

$$S = S(x, y) = 2\left(x \cdot \frac{V}{xy} + y \cdot \frac{V}{xy} + xy\right) = 2\left(\frac{V}{y} + \frac{V}{x} + xy\right) = \frac{2V}{y} + \frac{2V}{x} + 2xy$$

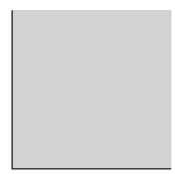
That is,

$$S = \frac{2V}{y} + \frac{2V}{x} + 2xy$$

Step 2. Determine the domain. The variables x and y express lengths, therefore, they must be nonnegative. Also, S is not defined if $x = 0$ or $y = 0$, therefore the domain is

$$D = \{(x, y) : x > 0, y > 0\}$$

We must find the minimum value of S on D. Because this domain is neither closed nor bounded, we have no guarantee that an absolute minimum exists. However, it can be proved (see later Justifications) that S has a minimum value on D, hence it must occur at a critical point in D.



Differentiating $S = \frac{2V}{y} + \frac{2V}{x} + 2xy$ and equating the partial derivatives to zero, we get

$$S_x(x, y) = -\frac{2V}{x^2} + 2y = 0, \quad S_y(x, y) = -\frac{2V}{y^2} + 2x = 0$$

The first equation gives $y = \frac{V}{x^2}$. Substituting in the second equation yields

$$2x - \frac{2V}{\frac{V^2}{x^4}} = 2x - \frac{2x^4}{V} = 2x\left(1 - \frac{x^3}{V}\right) = 0$$

The solutions are $x = 0$ and $x = \sqrt[3]{V}$. The solution $x = 0$ is not contained in D, hence the only solution in D is $x = \sqrt[3]{V}$. The corresponding value of y is obtained from $y = \frac{V}{x^2}$:

$$y = \frac{V}{\left(\sqrt[3]{V}\right)^2} = \frac{V}{V^{2/3}} = \sqrt[3]{V}$$

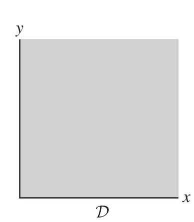
The critical point is $\left(\sqrt[3]{V}, \sqrt[3]{V}\right)$. We find the value of z, using $z = \frac{V}{xy}$:

$$z = \frac{V}{\sqrt[3]{V}\sqrt[3]{V}} = \frac{V}{V^{2/3}} = \sqrt[3]{V}$$

But how can we show that this critical point is a minumum? We provide two justifications.

Justification 1: Using the second derivative test, we have $S_{xx} = 4V/x^3$, so $S_{xx}(\sqrt[3]{V}) = 4$; $S_{yy} = 4V/y^3$, so $S_{yy}(\sqrt[3]{V}) = 4$; and $S_{xy} = 2$. Thus, $D = 4 \cdot 4 - 2^2 = 12 > 0$, and since $S_{xx} > 0$, we do indeed have a minimum surface area. This makes sense, because when x or y go to 0 or to ∞, then S (which is $2V/x + 2V/y + 2xy$) clearly goes to ∞.

Justification 2: We show that the function $S(X, Y) = \frac{2V}{y} + \frac{2V}{x} + 2xy$ has a minimum value in the domain $D = \{(x, y) : x > 0, y > 0\}$.



We denote by a_0 the value of $S(x, y)$ at the point $(2, 2)$ in D:

$$S(2, 2) = 2V + 8 = a_0 > 8$$

The following inequalities hold in D:

$$S(x, y) = \frac{V}{x} + \frac{V}{y} + 2xy \geq \frac{V}{x} \tag{2}$$

$$S(x, y) = \frac{V}{x} + \frac{V}{y} + 2xy \geq \frac{V}{y} \tag{3}$$

$$S(x, y) = \frac{V}{x} + \frac{V}{y} + 2xy \geq 2xy \tag{4}$$

Since $\lim_{x\to 0+} \frac{V}{x} = \infty$, it follows by (1) that there exists $0 < r_1 < 1$ such that, for all $0 < x < r_1$ and for all values of y,

$$S(x, y) > a_0$$

Since $\lim_{y\to 0+} \frac{V}{y} = \infty$, it follows by (2) that there exists $0 < r_2 < 1$ such that, for all $0 < y < r_1$ and for all values of x,

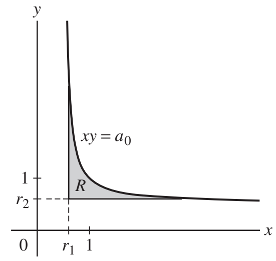
$$S(x, y) > a_0$$

By (3) it follows that if $xy > a_0$ then

$$S(x, y) > 2a_0 > a_0$$

We define the following domain:

$$R = \{(x, y) : x \geq r_1, y \geq r_2, xy \leq a_0\}$$



R is closed and bounded and $S(x, y)$ is continuous in R, therefore S has a minimum value in R.

We now show that this minimum is also the minimum value of S in D. First notice that, by the above considerations, $S(x, y) > a_0$ for all (x, y) outside R. At the point $(2, 2)$, $S(2, 2) = a_0$, and this point is in R, since $2 \geq r_1$, $2 \geq r_2$ (recall that $0 < r_1, r_2 < 1$) and $2 \cdot 2 = 4 < 8 < a_0$. Therefore, the minimum value of $S(x, y)$ in R is also the minimum value of S in D. We thus proved that S attains a minimum value on D.

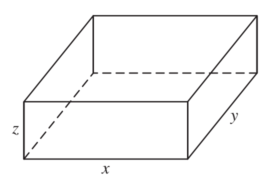
49. Consider a rectangular box B that has a bottom and sides but no top and has minimal surface area among all boxes with fixed volume V.

(a) Do you think B is a cube as in the solution to Exercise 48? If not, how would its shape differ from a cube?

(b) Find the dimensions of B and compare with your response to (a).

SOLUTION

(a) Each of the variables x and y is the length of a side of three faces (for example, x is the length of the front, back, and bottom sides), whereas z is the length of a side of four faces.

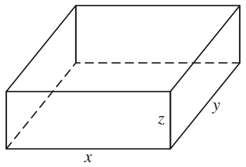


Therefore, the variables x, y, and z do not have equal influence on the surface area. We expect that in the box B with minimal surface area, z is smaller than $\sqrt[3]{V}$, which is the side of a cube with volume V (also we would expect $x = y$).

(b) We must find the dimensions of the box B, with fixed volume V and with smallest possible surface area, when the top is not included.

Step 1. Find a function to be minimized. The surface area of the box with sides lengths x, y, z when the top is not included is

$$S = 2xz + 2yz + xy \tag{1}$$



To express the surface in terms of x and y only, we use the formula for the volume of the box, $V = xyz$, giving $z = \frac{V}{xy}$. We substitute in (1) to obtain

$$S = 2x \cdot \frac{V}{xy} + 2y \cdot \frac{V}{xy} + xy = \frac{2V}{y} + \frac{2V}{x} + xy$$

That is,

$$S = \frac{2V}{y} + \frac{2V}{x} + xy.$$

Step 2. Determine the domain. The variables x, y denote lengths, hence they must be nonnegative. Moreover, S is not defined for $x = 0$ or $y = 0$. Since there are no other limitations on the variables, the domain is

$$D = \{(x, y) : x > 0, y > 0\}$$

We must find the minimum value of S on D. Because this domain is neither closed nor bounded, we are not sure that a minimum value exists. However, it can be proved (in like manner as in Exercise 48) that S does have a minimum value on D. This value occurs at a critical point in D, hence we set the partial derivatives equal to zero and solve. This gives

$$S_x(x, y) = -\frac{2V}{x^2} + y = 0$$
$$S_y(x, y) = -\frac{2V}{y^2} + x = 0$$

The first equation gives $y = \frac{2V}{x^2}$. Substituting in the second equation yields

$$x - \frac{2V}{\frac{4V^2}{x^4}} = x - \frac{x^4}{2V} = x\left(1 - \frac{x^3}{2V}\right) = 0$$

The solutions are $x = 0$ and $x = (2V)^{1/3}$. The solution $x = 0$ is not included in D, so the only solution is $x = (2V)^{1/3}$.
We find the value of y using $y = \frac{2V}{x^2}$:

$$y = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$$

We conclude that the critical point, which is the point where the minimum value of S in D occurs, is $\left((2V)^{1/3}, (2V)^{1/3}\right)$. We find the corresponding value of z using $z = \frac{V}{xy}$. We get

$$z = \frac{V}{(2V)^{1/3}(2V)^{1/3}} = \frac{V}{2^{2/3}V^{2/3}} = \frac{V^{1/3}}{2^{2/3}} = \left(\frac{V}{4}\right)^{1/3}$$

We conclude that the sizes of the box with minimum surface area are

width: $x = (2V)^{1/3}$;
length: $y = (2V)^{1/3}$;
height: $z = \left(\frac{V}{4}\right)^{1/3}$.

We see that z is smaller than x and y as predicted.

50. Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, the **linear least-squares fit** is the linear function

$$f(x) = mx + b$$

that minimizes the sum of the squares (Figure 24):

$$E(m, b) = \sum_{j=1}^{n} (y_j - f(x_j))^2$$

Show that the minimum value of E occurs for m and b satisfying the two equations

$$m\left(\sum_{j=1}^{n} x_j\right) + bn = \sum_{j=1}^{n} y_j$$

$$m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j y_j$$

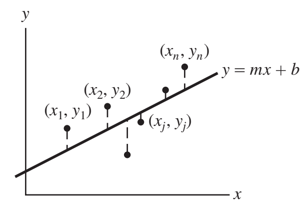


FIGURE 24 The linear least-squares fit minimizes the sum of the squares of the vertical distances from the data points to the line.

SOLUTION We first find the critical points of $E(m,b) = \sum_{j=1}^{n}\left(y_j - mx_j - b\right)^2$. Setting the partial derivatives equal to zero, we get

$$E_m(m,b) = 2\sum_{j=1}^{n}\left(y_j - mx_j - b\right)\cdot\left(-x_j\right) = -2\sum_{j=1}^{n} x_j\cdot\left(y_j - mx_j - b\right) = 0$$

$$E_b(m,b) = 2\sum_{j=1}^{n}\left(y_j - mx_j - b\right)\cdot(-1) = -2\sum_{j=1}^{n}\left(y_j - mx_j - b\right)$$

$$= -2\left(\sum_{j=1}^{n}\left(y_j - mx_j\right) - nb\right) = 0$$

We obtain the following equations:

$$\sum_{j=1}^{n} x_j\cdot y_j - m\sum_{j=1}^{n} x_j^2 - b\sum_{j=1}^{n} x_j = 0$$

$$\sum_{j=1}^{n} y_j - m\sum_{j=1}^{n} x_j - bn = 0$$

or

$$m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j\cdot y_j \tag{1}$$

$$m\sum_{j=1}^{n} x_j + bn = \sum_{j=1}^{n} y_j \tag{2}$$

By Theorem 3 the minimum value of $E(m,b)$ (if it exists) occurs at a critical point, which is the solution of equations (1) and (2). It can be shown (see justification) that $E(m,b)$ has a minimum value, hence E is minimized by the solution of (1) and (2).

Justification: We show that $E(m,b) = \sum_{j=1}^{n}\left(y_j - mx_j - b\right)^2$ has a minimum value. Let (m_0, b_0) be any point and $E(m_0, b_0) = E_0$. Since $E(m,b)$ is increasing without bound as $|m| \to \infty$ and $|b| \to \infty$, there exists a number $R > 0$ such that

$$E(m,b) > E_0 \text{ if } |m| > R \text{ and } |b| > R \tag{3}$$

The domain $D = \{(m,b) : |m| \le R \text{ and } |b| \le R\}$ is closed and bounded and $E(m,b)$ is continuous on D, hence E has a minimum value E_M on D. The point (m_0, b_0) is in D (since $E(m,b) > E_0$ for all points (m,b) that are not in D), hence

$$E_M \le E(m_0, b_0) = E_0 \tag{4}$$

It follows by (1) and (2) that E_M is the minimum value of $E(m,b)$ on the entire mb-plane.

51. The power (in microwatts) of a laser is measured as a function of current (in milliamps). Find the linear least-squares fit (Exercise 50) for the data points.

Current (mA)	1.0	1.1	1.2	1.3	1.4	1.5
Laser power (μW)	0.52	0.56	0.82	0.78	1.23	1.50

SOLUTION By Exercise 50, the coefficients of the linear least-square fit $f(x) = mx + b$ are determined by the following equations:

$$m\sum_{j=1}^{n} x_j + bn = \sum_{j=1}^{n} y_j$$

$$m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j \cdot y_j \tag{1}$$

In our case there are $n = 6$ data points:

$$(x_1, y_1) = (1, 0.52), (x_2, y_2) = (1.1, 0.56),$$
$$(x_3, y_3) = (1.2, 0.82), (x_4, y_4) = (1.3, 0.78),$$
$$(x_5, y_5) = (1.4, 1.23), (x_6, y_6) = (1.5, 1.50).$$

We compute the sums in (1):

$$\sum_{j=1}^{6} x_j = 1 + 1.1 + 1.2 + 1.3 + 1.4 + 1.5 = 7.5$$

$$\sum_{j=1}^{6} y_j = 0.52 + 0.56 + 0.82 + 0.78 + 1.23 + 1.50 = 5.41$$

$$\sum_{j=1}^{6} x_j^2 = 1^2 + 1.1^2 + 1.2^2 + 1.3^2 + 1.4^2 + 1.5^2 = 9.55$$

$$\sum_{j=1}^{6} x_j \cdot y_j = 1 \cdot 0.52 + 1.1 \cdot 0.56 + 1.2 \cdot 0.82 + 1.3 \cdot 0.78 + 1.4 \cdot 1.23 + 1.5 \cdot 1.50 = 7.106$$

Substituting in (1) gives the following equations:

$$7.5m + 6b = 5.41$$
$$9.55m + 7.5b = 7.106 \tag{2}$$

We multiply the first equation by 9.55 and the second by (-7.5), then add the resulting equations. This gives

$$\begin{array}{r} 71.625m + 57.3b = 51.6655 \\ + \ -71.625m - 56.25b = -53.295 \\ \hline 1.05b = -1.6295 \end{array} \quad \Rightarrow \quad b = -1.5519$$

We now substitute $b = -1.5519$ in the first equation in (2) and solve for m:

$$\begin{aligned} 7.5m + 6 \cdot (-1.5519) &= 5.41 \\ 7.5m &= 14.7214 \end{aligned} \quad \Rightarrow \quad m = 1.9629$$

The linear least squares fit $f(x) = mx + b$ is thus

$$f(x) = 1.9629x - 1.5519.$$

52. Let $A = (a, b)$ be a fixed point in the plane, and let $f_A(P)$ be the distance from A to the point $P = (x, y)$. For $P \neq A$, let $\mathbf{e}_{AP}$ be the unit vector pointing from A to P (Figure 25):

$$\mathbf{e}_{AP} = \frac{\overrightarrow{AP}}{\|\overrightarrow{AP}\|}$$

Show that

$$\nabla f_A(P) = \mathbf{e}_{AP}$$

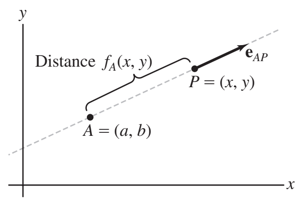


FIGURE 25 The distance from A to P increases most rapidly in the direction $\mathbf{e}_{AP}$.

SOLUTION Note that we can derive this result without calculation: Because $\nabla f_A(P)$ points in the direction of maximal increase, it must point directly away from A at P, and because the distance $f_A(x, y)$ increases at a rate of one as you move away from A along the line through A and P, $\nabla f_A(P)$ must be a unit vector.

Further Insights and Challenges

53. In this exercise, we prove that for all $x, y \geq 0$:

$$\frac{1}{\alpha}x^\alpha + \frac{1}{\beta}x^\beta \geq xy$$

where $\alpha \geq 1$ and $\beta \geq 1$ are numbers such that $\alpha^{-1} + \beta^{-1} = 1$. To do this, we prove that the function

$$f(x, y) = \alpha^{-1}x^\alpha + \beta^{-1}y^\beta - xy$$

satisfies $f(x, y) \geq 0$ for all $x, y \geq 0$.

(a) Show that the set of critical points of $f(x, y)$ is the curve $y = x^{\alpha-1}$ (Figure 26). Note that this curve can also be described as $x = y^{\beta-1}$. What is the value of $f(x, y)$ at points on this curve?

(b) Verify that the Second Derivative Test fails. Show, however, that for fixed $b > 0$, the function $g(x) = f(x, b)$ is concave up with a critical point at $x = b^{\beta-1}$.

(c) Conclude that for all $x > 0$, $f(x, b) \geq f(b^{\beta-1}, b) = 0$.

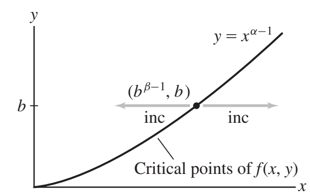


FIGURE 26 The critical points of $f(x, y) = \alpha^{-1}x^\alpha + \beta^{-1}y^\beta - xy$ form a curve $y = x^{\alpha-1}$.

SOLUTION We define the following function:

$$f(x, y) = \frac{1}{\alpha}x^\alpha + \frac{1}{\beta}y^\beta - xy$$

Notice that $f(0, 0) = 0$.

(a) Determine the critical points for $f(x, y) = f(x, y) = \alpha^{-1}x^\alpha + \beta^{-1}y^\beta - xy$. First, take the first-order partial derivatives and set them equal to zero to solve:

$$f_x = \alpha^{-1} \cdot \alpha x^{\alpha-1} - y = x^{\alpha-1} - y = 0, \quad f_y = \beta^{-1} \cdot \beta y^{\beta-1} - x = y^{\beta-1} - x = 0$$

This means that $y = x^{\alpha-1}$ and simultaneously $x = y^{\beta-1}$. Note here that we are guaranteed that the set of points satisfying both equations is nonempty because $1/\alpha + 1/\beta = 1$.

Now to compute the value of $f(x, y)$ at these points:

$$f(x, y) = f(x, x^{\alpha-1}) = \alpha^{-1}x^\alpha + \beta^{-1}(x^{\alpha-1})^\beta - x(x^{\alpha-1}) = \left(\frac{1}{\alpha} - 1\right)x^\alpha + \frac{1}{\beta}x^{\alpha\beta-\beta}$$

But remember that $\alpha^{-1} + \beta^{-1} = 1$ so we can say

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \beta + \alpha = \alpha\beta$$

Using these relations we see:

$$f(x, y) = f(x, x^{\alpha-1}) = \left(\frac{1}{\alpha} - 1\right)x^{\alpha} + \frac{1}{\beta}x^{\alpha\beta-\beta} = -\frac{1}{\beta}x^{\alpha} + \frac{1}{\beta}x^{\alpha} = 0$$

or similarly,

$$f(x, y) = f(y^{\beta-1}, y) = \frac{1}{\alpha}y^{\alpha\beta-\alpha} + \left(\frac{1}{\beta} - 1\right)y^{\beta} = \frac{1}{\alpha}y^{\beta} - \frac{1}{\alpha}y^{\beta} = 0$$

(b) Now computing the second-order partial derivatives we get

$$f_{xx} = (\alpha - 1)x^{\alpha-2}, \quad f_{yy} = (\beta - 1)y^{\beta-2}, \quad f_{xy} = -1$$

Therefore we can write the discriminant (while using the relations about α and β above):

$$D = f_{xx}f_{yy} - f_{xy}^2 = (\alpha - 1)(\beta - 1)x^{\alpha-2}y^{\beta-2} - 1 = x^{\alpha-2}y^{\beta-2} - 1$$

Evaluating this expression at the critical points when $y = x^{\alpha-1}$ we see

$$D(x, x^{\alpha-1}) = x^{\alpha-2}(x^{\alpha-1})^{\beta-2} - 1 = x^{\alpha-2}x^{\alpha\beta-\beta-2\alpha+2} - 1 = x^{\alpha-2+\alpha\beta-\beta-2\alpha+2} - 1 = x^0 - 1 = 0$$

Thus the Second Derivative Test is inconclusive and fails.

Instead, if we fix $b > 0$, consider the function

$$g(x) = f(x, b) = \frac{1}{\alpha}x^{\alpha} + \frac{1}{\beta}b^{\beta} - bx$$

Therefore, taking the first derivative and setting it equal to zero to solve, we see

$$g'(x) = x^{\alpha-1} - b = 0 \quad \Rightarrow \quad b = x^{\alpha-1}$$

In order to solve this for x, note here that $(\alpha - 1)(\beta - 1) = 1$ so then $\frac{1}{\alpha-1} = \beta - 1$ and

$$b = x^{\alpha-1} \quad \Rightarrow \quad x = b^{1/(\alpha-1)} \quad \Rightarrow \quad x = b^{\beta-1}$$

Since

$$g''(x) = (\alpha - 1)x^{\alpha-2}, \quad \alpha \geq 1$$

then $g''(x) \geq 0$ for all x. Therefore, $g(x)$ is concave up with critical point $x = b^{\beta-1}$.

(c) From our work in part (b), we can conclude, for all $x > 0$, then

$$f(x, b) \geq f(b^{\beta-1}, b) = 0$$

54. The following problem was posed by Pierre de Fermat: Given three points $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ in the plane, find the point $P = (x, y)$ that minimizes the sum of the distances

$$f(x, y) = AP + BP + CP$$

Let **e**, **f**, **g** be the unit vectors pointing from P to the points A, B, C as in Figure 27.

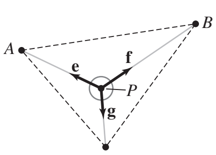
(a) Use Exercise 52 to show that the condition $\nabla f(P) = 0$ is equivalent to

$$\mathbf{e} + \mathbf{f} + \mathbf{g} = 0 \qquad \boxed{3}$$

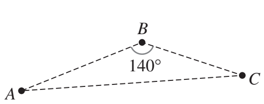
(b) Show that $f(x, y)$ is differentiable except at points A, B, C. Conclude that the minimum of $f(x, y)$ occurs either at a point P satisfying Eq. (3) or at one of the points A, B, or C.

(c) Prove that Eq. (3) holds if and only if P is the **Fermat point**, defined as the point P for which the angles between the segments $\overline{AP}$, $\overline{BP}$, $\overline{CP}$ are all 120° (Figure 27).

(d) Show that the Fermat point does not exist if one of the angles in $\triangle ABC$ is $> 120°$. Where does the minimum occur in this case?



(A) P is the Fermat point (the angles between **e**, **f**, and **g** are all 120°).

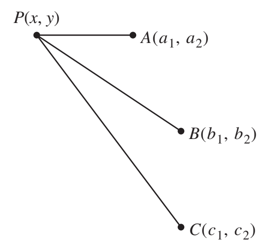


(B) Fermat point does not exist.

FIGURE 27

SOLUTION Let us examine part (b) first.

(b)



Using the formula for the length of a segment we obtain

$$f(x, y) = \sqrt{(x-a_1)^2 + (y-a_2)^2} + \sqrt{(x-b_1)^2 + (y-b_2)^2} + \sqrt{(x-c_1)^2 + (y-c_2)^2}$$

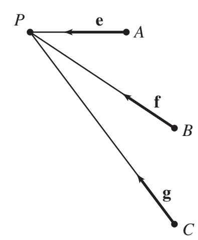
We compute the partial derivatives of f:

$$f_x(x, y) = \frac{x-a_1}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} + \frac{x-b_1}{\sqrt{(x-b_1)^2 + (y-b_2)^2}} + \frac{x-c_1}{\sqrt{(x-c_1)^2 + (y-c_2)^2}} \tag{1}$$

$$f_y(x, y) = \frac{y-a_2}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} + \frac{y-b_2}{\sqrt{(x-b_1)^2 + (y-b_2)^2}} + \frac{y-c_2}{\sqrt{(x-c_1)^2 + (y-c_2)^2}} \tag{2}$$

For all (x, y) other then (a_1, a_2), (b_1, b_2), (c_1, c_2) the partial derivatives are continuous, therefore the Criterion for Differentiability implies that f is differentiable at all points other than A, B, and C.

(a)



We compute the unit vectors $\mathbf{e}$, $\mathbf{f}$, and $\mathbf{g}$:

$$\mathbf{e} = \frac{\langle x-a_1, y-a_2\rangle}{\sqrt{(x-a_1)^2 + (y-a_2)^2}}$$

$$\mathbf{f} = \frac{\langle x-b_1, y-b_2\rangle}{\sqrt{(x-b_1)^2 + (y-b_2)^2}}$$

$$\mathbf{g} = \frac{\langle x-c_1, y-c_2\rangle}{\sqrt{(x-c_1)^2 + (y-c_2)^2}}$$

We write the condition $\mathbf{e} + \mathbf{f} + \mathbf{g} = \mathbf{0}$:

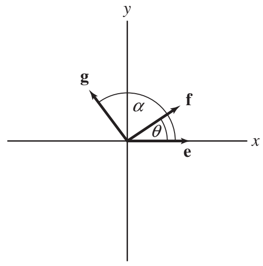
$$\begin{aligned}
\mathbf{e} + \mathbf{f} + \mathbf{g} &= \frac{\langle x-a_1, y-a_2\rangle}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} + \frac{\langle x-b_1, y-b_2\rangle}{\sqrt{(x-b_1)^2 + (y-b_2)^2}} + \frac{\langle x-c_1, y-c_2\rangle}{\sqrt{(x-c_1)^2 + (y-c_2)^2}} \\
&= \Bigg\langle \frac{x-a_1}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} + \frac{x-b_1}{\sqrt{(x-b_1)^2 + (y-b_2)^2}} + \frac{x-c_1}{\sqrt{(x-c_1)^2 + (y-c_2)^2}}, \\
&\quad \frac{y-a_2}{\sqrt{(x-a_1)^2 + (y-a_2)^2}} + \frac{y-b_2}{\sqrt{(x-b_1)^2 + (y-b_2)^2}} + \frac{y-c_2}{\sqrt{(x-c_1)^2 + (y-c_2)^2}} \Bigg\rangle
\end{aligned}$$

Combining with (1) and (2) we get

$$\mathbf{e}+\mathbf{f}+\mathbf{g}=\left\langle f_x(x,y), f_y(x,y)\right\rangle=\nabla f$$

Therefore, the condition $\nabla f=\mathbf{0}$ is equivalent to $\mathbf{e}+\mathbf{f}+\mathbf{g}=\mathbf{0}$.

(c) We now show that Eq. (3) holds if and only if the mutual angles between the unit vectors are all 120°. We place the axes so that the positive x-axis is in the direction of e.



Let θ and α be the angles that $\mathbf{f}$ and $\mathbf{g}$ make with $\mathbf{e}$, respectively. Hence,

$$\mathbf{e}=\langle 1,0\rangle, \mathbf{f}=\langle\cos\theta,\sin\theta\rangle, \mathbf{g}=\langle\cos\alpha,\sin\alpha\rangle$$

Substituting in $\mathbf{e}+\mathbf{f}+\mathbf{g}=\mathbf{0}$ we have

$$\langle\cos\theta+\cos\alpha+1, \sin\theta+\sin\alpha\rangle=\langle 0,0\rangle$$

or

$$\begin{aligned}\cos\theta+\cos\alpha+1&=0\\ \sin\theta+\sin\alpha&=0\end{aligned}$$

The second equation implies that

$$\sin\theta=-\sin\alpha=\sin(180+\alpha)$$

The solutions for $0\le\alpha,\theta\le 360$ are

$$\theta=180+\alpha,\quad \theta=360-\alpha$$

We substitute each solution in the first equation and solve for α. This gives

$$\begin{array}{ll}\underline{\theta=180+\alpha} & \underline{\theta=360^\circ-\alpha}\\ \cos(180+\alpha)+\cos\alpha+1=0 & \cos(360^\circ-\alpha)+\cos\alpha+1=0\\ -\cos\alpha+\cos\alpha+1=0 & \cos\alpha+\cos\alpha+1=0\\ 1=0 & 2\cos\alpha=-1\\ & \cos\alpha=-\frac{1}{2}\end{array}$$

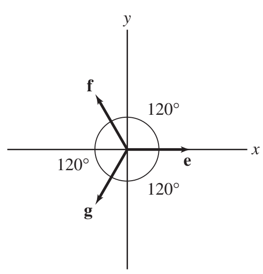
$$\Rightarrow\quad \begin{array}{ll}\alpha=120^\circ & \alpha=240^\circ\\ \theta=360^\circ-\alpha=240^\circ & \theta=360^\circ-\alpha=120^\circ\end{array}$$

We obtain the following vectors:

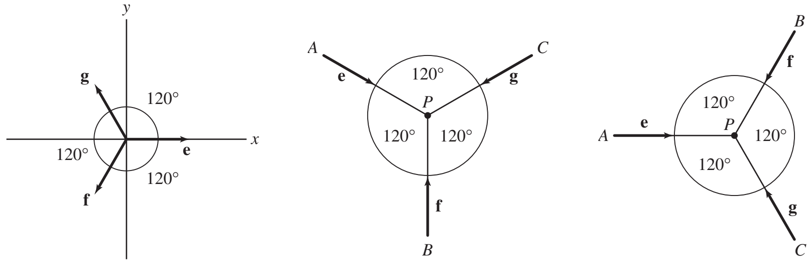
$$\mathbf{e}=\langle 1,0\rangle,\quad \mathbf{f}=\left\langle\cos 240^\circ,\sin 240^\circ\right\rangle,\quad \mathbf{g}=\left\langle\cos 120^\circ,\sin 120^\circ\right\rangle$$

or

$$\mathbf{e}=\langle 1,0\rangle,\quad \mathbf{f}=\left\langle\cos 120^\circ,\sin 120^\circ\right\rangle,\quad \mathbf{g}=\left\langle\cos 240^\circ,\sin 240^\circ\right\rangle$$



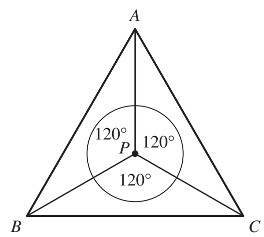
or



In either case the angles between the vectors are 120°.

Now we see $f(x, y)$ has the minimum value at a critical point:

The critical points are the points where f_x and f_y are 0 or do not exist, that is, the points A, B, C and the point where $\nabla f = \mathbf{0}$, which according to part (b) is the Fermat point. We now show that if the Fermat point P exists, then $f(P) \le f(A), f(B), f(C)$.



Suppose that the Fermat point P exists. The values of f at the critical points are

$$\begin{aligned} f(A) &= \overline{AB} + \overline{AC} \\ f(B) &= \overline{AB} + \overline{BC} \\ f(C) &= \overline{AC} + \overline{BC} \\ f(P) &= \overline{AP} + \overline{BP} + \overline{PC} \end{aligned}$$

We show that $f(P) \le f(A)$. Similarly it can be shown that also $f(P) \le f(B)$ and $f(P) \le f(C)$. By the Cosine Theorem for the triangles ABP and ACP we have

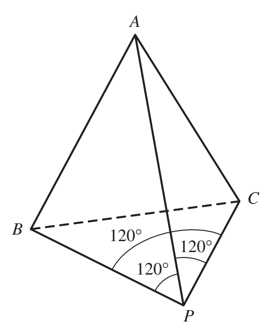
$$\overline{AB} = \sqrt{\overline{AP}^2 + \overline{BP}^2 - 2\overline{AP} \cdot \overline{BP} \cos 120^\circ} = \sqrt{\overline{AP}^2 + \overline{BP}^2 + \overline{AP} \cdot \overline{BP}}$$
$$\overline{AC} = \sqrt{\overline{AP}^2 + \overline{CP}^2 - 2\overline{AP} \cdot \overline{PC} \cos 120^\circ} = \sqrt{\overline{AP}^2 + \overline{CP}^2 + \overline{AP} \cdot \overline{PC}}$$

Hence

$$\begin{aligned} f(A) = \overline{AB} + \overline{AC} &= \sqrt{\overline{AP}^2 + \overline{BP}^2 + \overline{AP} \cdot \overline{BP}} + \sqrt{\overline{AP}^2 + \overline{CP}^2 + \overline{AP} \cdot \overline{PC}} \\ &\ge \overline{AP} + \overline{BP} + \overline{PC} = f(P) \end{aligned}$$

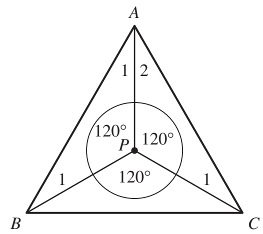
The last inequality can be verified by squaring and transferring sides. It's best to use a computer to help with the algebra; it's a daunting task to do by hand.

(d) We show that if one of the angles of ΔABC is $\ge 120°$, then the Fermat point does not exist. Notice that the Fermat point (if it exists) must fall inside the triangle ABC.



P cannot lie outside ΔABC

Suppose the Fermat point P exists.



We sum the angles in the triangles ABP and ACP, obtaining

$$\sphericalangle A_1 + \sphericalangle B_1 + 120^\circ = 180^\circ \quad \Rightarrow \quad \sphericalangle A_1 = 60^\circ - \sphericalangle B_1$$
$$\sphericalangle A_2 + \sphericalangle C_1 + 120^\circ = 180^\circ \quad \Rightarrow \quad \sphericalangle A_2 = 60^\circ - \sphericalangle C_1$$

Therefore,

$$\sphericalangle A = \sphericalangle A_1 + \sphericalangle A_2 = \left(60^\circ - \sphericalangle B_1\right) + \left(60^\circ - \sphericalangle C_1\right) = 120^\circ - (\sphericalangle B_1 + \sphericalangle C_1) < 120^\circ$$

We thus showed that if the Fermat point exists, then $\sphericalangle A < 120^\circ$. Similarly, one shows also that $\sphericalangle B$ and $\sphericalangle C$ must be smaller than 120°. We conclude that if one of the angles in ΔABC is equal or greater than 120°, then the Fermat point does not exist. In that case, the minimum value of $f(x, y)$ occurs at a point where f_x or f_y do not exist, that is, at one of the points A, B, or C.

14.8 Lagrange Multipliers: Optimizing with a Constraint (LT Section 15.8)

Preliminary Questions

1. Suppose that the maximum of $f(x, y)$ subject to the constraint $g(x, y) = 0$ occurs at a point $P = (a, b)$ such that $\nabla f_P \neq 0$. Which of the following statements is true?

(a) ∇f_P is tangent to $g(x, y) = 0$ at P.

(b) ∇f_P is orthogonal to $g(x, y) = 0$ at P.

SOLUTION

(a) Since the maximum of f subject to the constraint occurs at P, it follows by Theorem 1 that ∇f_P and ∇g_P are parallel vectors. The gradient ∇g_P is orthogonal to $g(x, y) = 0$ at P, hence ∇f_P is also orthogonal to this curve at P. We conclude that statement (b) is false (yet the statement can be true if $\nabla f_P = (0, 0)$).

(b) This statement is true by the reasoning given in the previous part.

2. Figure 9 shows a constraint $g(x, y) = 0$ and the level curves of a function f. In each case, determine whether f has a local minimum, a local maximum, or neither at the labeled point.

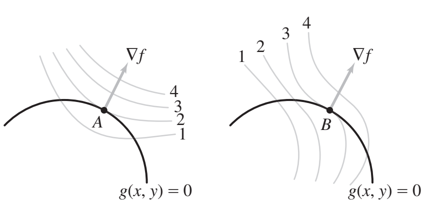


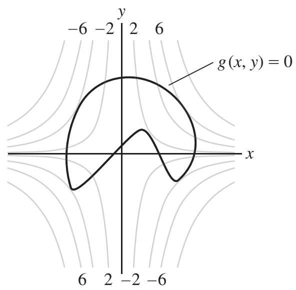
FIGURE 9

SOLUTION The level curve $f(x, y) = 2$ is tangent to the constraint curve at the point A. A close level curve that intersects the constraint curve is $f(x, y) = 1$, hence we may assume that f has a local maximum 2 under the constraint at A. The level curve $f(x, y) = 3$ is tangent to the constraint curve. However, in approaching B under the constraint, from one side f is increasing and from the other side f is decreasing. Therefore, $f(B)$ is neither local minimum nor local maximum of f under the constraint.

3. On the contour map in Figure 10:

(a) Identify the points where $\nabla f = \lambda \nabla g$ for some scalar λ.

(b) Identify the minimum and maximum values of $f(x, y)$ subject to $g(x, y) = 0$.

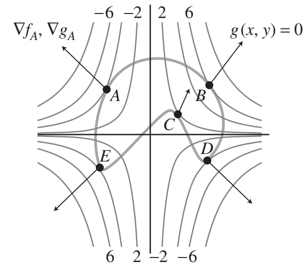


Contour plot of $f(x, y)$
(contour interval 2)

FIGURE 10 Contour map of $f(x, y)$; contour interval 2.

SOLUTION

(a) The gradient ∇g is orthogonal to the constraint curve $g(x, y) = 0$, and ∇f is orthogonal to the level curves of f. These two vectors are parallel at the points where the level curve of f is tangent to the constraint curve. These are the points A, B, C, D, E in the figure:



(b) The minimum and maximum occur where the level curve of f is tangent to the constraint curve. The level curves tangent to the constraint curve are

$$f(A) = -4, \quad f(C) = 2, \quad f(B) = 6, \quad f(D) = -4, \quad f(E) = 4$$

Therefore the global minimum of f under the constraint is -4 and the global maximum is 6.

Exercises

In this exercise set, use the method of Lagrange multipliers unless otherwise stated.

1. Find the extreme values of the function $f(x, y) = 2x + 4y$ subject to the constraint $g(x, y) = x^2 + y^2 - 5 = 0$.

(a) Show that the Lagrange equation $\nabla f = \lambda \nabla g$ gives $\lambda x = 1$ and $\lambda y = 2$.

(b) Show that these equations imply $\lambda \neq 0$ and $y = 2x$.

(c) Use the constraint equation to determine the possible critical points (x, y).

(d) Evaluate $f(x, y)$ at the critical points and determine the minimum and maximum values.

SOLUTION

(a) The Lagrange equations are determined by the equality $\nabla f = \lambda \nabla g$. We find them:

$$\nabla f = \langle f_x, f_y \rangle = \langle 2, 4 \rangle, \quad \nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle$$

Hence,

$$\langle 2, 4 \rangle = \lambda \langle 2x, 2y \rangle$$

or

$$\begin{aligned} \lambda(2x) &= 2 \\ \lambda(2y) &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda x &= 1 \\ \lambda y &= 2 \end{aligned}$$

(b) The Lagrange equations in part (a) imply that $\lambda \neq 0$. The first equation implies that $x = \frac{1}{\lambda}$ and the second equation gives $y = \frac{2}{\lambda}$. Therefore $y = 2x$.

(c) We substitute $y = 2x$ in the constraint equation $x^2 + y^2 - 5 = 0$ and solve for x and y. This gives

$$x^2 + (2x)^2 - 5 = 0$$
$$5x^2 = 5$$
$$x^2 = 1 \quad \Rightarrow \quad x_1 = -1, \quad x_2 = 1$$

Since $y = 2x$, we have $y_1 = 2x_1 = -2$, $y_2 = 2x_2 = 2$. The critical points are thus

$$(-1, -2) \quad \text{and} \quad (1, 2).$$

Extreme values can also occur at the points where $\nabla g = \langle 2x, 2y \rangle = \langle 0, 0 \rangle$. However, $(0, 0)$ is not on the constraint.

(d) We evaluate $f(x, y) = 2x + 4y$ at the critical points, obtaining

$$f(-1, -2) = 2 \cdot (-1) + 4 \cdot (-2) = -10$$
$$f(1, 2) = 2 \cdot 1 + 4 \cdot 2 = 10$$

Since f is continuous and the graph of $g = 0$ is closed and bounded, global minimum and maximum points exist. So according to Theorem 1, we conclude that the maximum of $f(x, y)$ on the constraint is 10 and the minimum is -10.

2. Find the extreme values of $f(x, y) = x^2 + 2y^2$ subject to the constraint $g(x, y) = 4x - 6y = 25$.

(a) Show that the Lagrange equations yield $2x = 4\lambda$, $4y = -6\lambda$.

(b) Show that if $x = 0$ or $y = 0$, then the Lagrange equations give $x = y = 0$. Since $(0, 0)$ does not satisfy the constraint, you may assume that x and y are nonzero.

(c) Use the Lagrange equations to show that $y = -\frac{3}{4}x$.

(d) Substitute in the constraint equation to show that there is a unique critical point P.

(e) Does P correspond to a minimum or maximum value of f? Refer to Figure 11 to justify your answer. *Hint:* Do the values of $f(x, y)$ increase or decrease as (x, y) moves away from P along the line $g(x, y) = 0$?

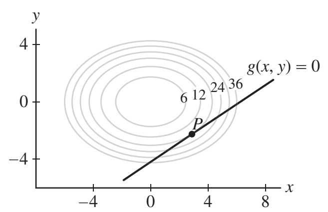


FIGURE 11 Level curves of $f(x, y) = x^2 + 2y^2$ and graph of the constraint $g(x, y) = 4x - 6y - 25 = 0$.

SOLUTION

(a) The gradients ∇f and ∇g are

$$\nabla f = \langle 2x, 4y \rangle, \quad \nabla g = \langle 4, -6 \rangle$$

The Lagrange equations are thus

$$\nabla f = \lambda \nabla g$$
$$\langle 2x, 4y \rangle = \lambda \langle 4, -6 \rangle$$

or

$$2x = 4\lambda$$
$$4y = -6\lambda$$

(b) If $x = 0$, the first equation gives $0 = 4\lambda$ or $\lambda = 0$. Substituting in the second equation gives $4y = 0$ or $y = 0$. Similarly, if $y = 0$, the second equation implies that $\lambda = 0$, hence by the first equation also $x = 0$. That is, if $x = 0$, then $y = 0$ and if $y = 0$ also $x = 0$. The point $(0, 0)$ does not satisfy the equation of the constraint, hence we may assume that $x \neq 0$ and $y \neq 0$.

(c) The first equation in part (a) gives $\lambda = \frac{x}{2}$. Substituting in the second equation we get

$$4y = -6 \cdot \frac{x}{2} = -3x \quad \Rightarrow \quad y = -\frac{3}{4}x$$

(d) We substitute $y = -\frac{3}{4}x$ in the constraint $4x - 6y = 25$ and solve for x and y. This gives

$$\begin{aligned} 4x - 6\left(-\frac{3}{4}x\right) &= 25 \\ 4x + \frac{9}{2}x &= 25 \\ 17x = 50 \quad &\Rightarrow \quad x = \frac{50}{17}, \quad y = -\frac{3}{4}\cdot\frac{50}{17} = -\frac{75}{34} \end{aligned}$$

We conclude that there is a unique critical point, which is $\left(\frac{50}{17}, -\frac{75}{34}\right)$.

(e) We now refer to Figure 11. As (x, y) moves away from P along the line $g(x, y) = 0$, the values of $f(x, y)$ increase, hence P corresponds to a minimum value of f.

3. Apply the method of Lagrange multipliers to the function $f(x, y) = (x^2 + 1)y$ subject to the constraint $x^2 + y^2 = 5$. *Hint:* First show that $y \neq 0$; then treat the cases $x = 0$ and $x \neq 0$ separately.

SOLUTION We first write out the Lagrange Equations. We have $\nabla f = \left\langle 2xy, x^2 + 1\right\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. Hence, the Lagrange Condition for $\nabla g \neq 0$ is

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ \left\langle 2xy, x^2 + 1\right\rangle &= \lambda\,\langle 2x, 2y\rangle \end{aligned}$$

We obtain the following equations:

$$\begin{aligned} 2xy &= \lambda(2x) \\ x^2 + 1 &= \lambda(2y) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x(y - \lambda) &= 0 \\ x^2 + 1 &= 2\lambda y \end{aligned} \tag{1}$$

The second equation implies that $y \neq 0$, since there is no real value of x such that $x^2 + 1 = 0$. Likewise, $\lambda \neq 0$. The solutions of the first equation are $x = 0$ and $y = \lambda$.

Case 1: $x = 0$. Substituting $x = 0$ in the second equation gives $2\lambda y = 1$, or $y = \frac{1}{2\lambda}$. We substitute $x = 0$, $y = \frac{1}{2\lambda}$ (recall that $\lambda \neq 0$) in the constraint to obtain

$$0^2 + \frac{1}{4\lambda^2} = 5 \quad \Rightarrow \quad 4\lambda^2 = \frac{1}{5} \quad \Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{20}} = \pm\frac{1}{2\sqrt{5}}$$

The corresponding values of y are

$$y = \frac{1}{2\cdot\frac{1}{2\sqrt{5}}} = \sqrt{5} \quad \text{and} \quad y = \frac{1}{2\cdot\left(-\frac{1}{2\sqrt{5}}\right)} = -\sqrt{5}$$

We obtain the critical points:

$$\left(0, \sqrt{5}\right) \quad \text{and} \quad \left(0, -\sqrt{5}\right)$$

Case 2: $x \neq 0$. Then the first equation in (1) implies $y = \lambda$. Substituting in the second equation gives

$$x^2 + 1 = 2\lambda^2 \quad \Rightarrow \quad x^2 = 2\lambda^2 - 1$$

We now substitute $y = \lambda$ and $x^2 = 2\lambda^2 - 1$ in the constraint $x^2 + y^2 = 5$ to obtain

$$\begin{aligned} 2\lambda^2 - 1 + \lambda^2 &= 5 \\ 3\lambda^2 &= 6 \\ \lambda^2 = 2 \quad &\Rightarrow \quad \lambda = \pm\sqrt{2} \end{aligned}$$

The solution (x, y) are thus

$$\begin{aligned} \lambda = \sqrt{2}: &\quad y = \sqrt{2}, \quad x = \pm\sqrt{2\cdot 2 - 1} = \pm\sqrt{3} \\ \lambda = -\sqrt{2}: &\quad y = -\sqrt{2}, \quad x = \pm\sqrt{2\cdot 2 - 1} = \pm\sqrt{3} \end{aligned}$$

We obtain the critical points:

$$\left(\sqrt{3}, \sqrt{2}\right), \quad \left(-\sqrt{3}, \sqrt{2}\right), \quad \left(\sqrt{3}, -\sqrt{2}\right), \quad \left(-\sqrt{3}, -\sqrt{2}\right)$$

We conclude that the critical points are

$$\left(0, \sqrt{5}\right), \quad \left(0, -\sqrt{5}\right), \quad \left(\sqrt{3}, \sqrt{2}\right), \quad \left(-\sqrt{3}, \sqrt{2}\right), \quad \left(\sqrt{3}, -\sqrt{2}\right), \quad \left(-\sqrt{3}, -\sqrt{2}\right).$$

We now calculate $f(x, y) = \left(x^2 + 1\right) y$ at the critical points:

$$f\left(0, \sqrt{5}\right) = \sqrt{5} \approx 2.24$$
$$f\left(0, -\sqrt{5}\right) = -\sqrt{5} \approx -2.24$$
$$f\left(\sqrt{3}, \sqrt{2}\right) = f\left(-\sqrt{3}, \sqrt{2}\right) = 4\sqrt{2} \approx 5.66$$
$$f\left(\sqrt{3}, -\sqrt{2}\right) = f\left(-\sqrt{3}, -\sqrt{2}\right) = -4\sqrt{2} \approx -5.66$$

Since the constraint gives a closed and bounded curve, f achieves a minimum and a maximum under it. We conclude that the maximum of $f(x, y)$ on the constraint is $4\sqrt{2}$ and the minimum is $-4\sqrt{2}$.

In Exercises 4–13, find the minimum and maximum values of the function subject to the given constraint.

4. $f(x, y) = 2x + 3y, \quad x^2 + y^2 = 4$

SOLUTION We find the extreme values of $f(x, y) = 2x + 3y$ under the constraint $g(x, y) = x^2 + y^2 - 4 = 0$.

Step 1. Write the Lagrange Equations. We have $\nabla f = \langle 2, 3\rangle$ and $\nabla g = \langle 2x, 2y\rangle$, hence the Lagrange Condition is

$$\nabla f = \lambda \nabla g$$
$$\langle 2, 3\rangle = \lambda \langle 2x, 2y\rangle$$

The corresponding equations are

$$2 = \lambda(2x)$$
$$3 = \lambda(2y)$$

Step 2. Solve for x and y using the constraint. The two equations imply that $x \neq 0$ and $y \neq 0$, hence

$$\lambda = \frac{1}{x} \quad \text{and} \quad \lambda = \frac{3}{2y}$$

The two expressions for λ must be equal, so we obtain

$$\frac{1}{x} = \frac{3}{2y} \quad \Rightarrow \quad y = \frac{3}{2}x$$

We now substitute $y = \frac{3}{2}x$ in the constraint equation $x^2 + y^2 = 4$ and solve for x and y:

$$x^2 + \left(\frac{3}{2}x\right)^2 = 4$$
$$x^2 + \frac{9}{4}x^2 = 4$$
$$13x^2 = 16 \quad \Rightarrow \quad x_1 = \frac{4}{\sqrt{13}}, \quad x_2 = -\frac{4}{\sqrt{13}}$$

Since $y = \frac{3}{2}x$, the corresponding values of y are

$$y_1 = \frac{3}{2} \cdot \frac{4}{\sqrt{13}} = \frac{6}{\sqrt{13}}, \quad y_2 = \frac{3}{2} \cdot \left(-\frac{4}{\sqrt{13}}\right) = -\frac{6}{\sqrt{13}}$$

We obtain the critical points:

$$\left(\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right), \quad \left(-\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}\right)$$

Extreme points may occur also where $\nabla g = \langle 2x, 2y\rangle = \langle 0, 0\rangle$. However, the point $(0, 0)$ is not on the constraint.

Step 3. Calculate f at the critical points. We evaluate $f(x, y) = 2x + 3y$ at the critical points:

$$f\left(\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right) = \frac{8}{\sqrt{13}} + \frac{18}{\sqrt{13}} = \frac{26}{\sqrt{13}} \approx 7.21$$
$$f\left(-\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}\right) = -\frac{8}{\sqrt{13}} - \frac{18}{\sqrt{13}} = -\frac{26}{\sqrt{13}} \approx -7.21$$

We conclude that the maximum of f on the constraint is about 7.21 and the minimum is about -7.21.

5. $f(x, y) = x^2 + y^2, \quad 2x + 3y = 6$

SOLUTION We find the extreme values of $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = 2x + 3y - 6 = 0$.
Step 1. Write out the Lagrange Equations. The gradients of f and g are $\nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle 2, 3\rangle$. The Lagrange Condition is

$$\nabla f = \lambda \nabla g$$
$$\langle 2x, 2y\rangle = \lambda \langle 2, 3\rangle$$

We obtain the following equations:

$$2x = \lambda \cdot 2$$
$$2y = \lambda \cdot 3$$

Step 2. Solve for λ in terms of x and y. Notice that if $x = 0$, then the first equation gives $\lambda = 0$, therefore by the second equation also $y = 0$. The point $(0, 0)$ does not satisfy the constraint. Similarly, if $y = 0$ also $x = 0$. We therefore may assume that $x \neq 0$ and $y \neq 0$ and obtain by the two equations:

$$\lambda = x \quad \text{and} \quad \lambda = \frac{2}{3}y.$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ gives

$$x = \frac{2}{3}y \quad \Rightarrow \quad y = \frac{3}{2}x$$

We substitute $y = \frac{3}{2}x$ in the constraint $2x + 3y = 6$ and solve for x and y:

$$2x + 3 \cdot \frac{3}{2}x = 6$$
$$13x = 12 \quad \Rightarrow \quad x = \frac{12}{13}, \quad y = \frac{3}{2} \cdot \frac{12}{13} = \frac{18}{13}$$

We obtain the critical point $\left(\frac{12}{13}, \frac{18}{13}\right)$.
Step 4. Calculate f at the critical point. We evaluate $f(x, y) = x^2 + y^2$ at the critical point:

$$f\left(\frac{12}{13}, \frac{18}{13}\right) = \left(\frac{12}{13}\right)^2 + \left(\frac{18}{13}\right)^2 = \frac{468}{169} \approx 2.77$$

Rewriting the constraint as $y = -\frac{2}{3}x + 2$, we see that as $|x| \to +\infty$ then so does $|y|$, and hence $x^2 + y^2$ is increasing without bound on the constraint as $|x| \to \infty$. We conclude that the value $468/169$ is the minimum value of f under the constraint, rather than the maximum value.

6. $f(x, y) = 4x^2 + 9y^2, \quad xy = 4$

SOLUTION We find the extreme values of $f(x, y) = 4x^2 + 9y^2$ under the constraint $g(x, y) = xy - 4 = 0$.
Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 8x, 18y\rangle$ and $\nabla g = \langle y, x\rangle$, hence the Lagrange condition is

$$\nabla f = \lambda \nabla g$$
$$\langle 8x, 18y\rangle = \lambda \langle y, x\rangle$$

or

$$8x = \lambda y$$
$$18y = \lambda x$$

Step 2. Solve for λ in terms of x and y. We may assume that $x \neq 0$ and $y \neq 0$, since the points with $x = 0$ or $y = 0$ do not satisfy the constraint. The two equations give

$$\lambda = \frac{8x}{y} \quad \text{and} \quad \lambda = \frac{18y}{x}$$

Step 3. Solve for x and y using the constraint. We equate the two expressions for λ to obtain

$$\frac{8x}{y} = \frac{18y}{x} \quad \Rightarrow \quad 8x^2 = 18y^2 \quad \Rightarrow \quad y = \pm\frac{2}{3}x$$

The constraint $xy = 4$ implies that x and y have the same sign, hence $y = \frac{2}{3}x$. We substitute $y = \frac{2}{3}x$ in the constraint and solve for x and y:

$$x \cdot \frac{2}{3}x = 4 \quad \Rightarrow \quad x^2 = 6 \quad \Rightarrow \quad x_1 = \sqrt{6}, \quad x_2 = -\sqrt{6}$$

The corresponding values of y are obtained by $y = \frac{2}{3}x$:

$$y_1 = \frac{2}{3}\sqrt{6} = 2\sqrt{\frac{2}{3}}, \quad y_2 = \frac{2}{3} \cdot \left(-\sqrt{6}\right) = -2\sqrt{\frac{2}{3}}$$

The critical points are thus

$$\left(\sqrt{6}, 2\sqrt{\frac{2}{3}}\right), \quad \left(-\sqrt{6}, -2\sqrt{\frac{2}{3}}\right)$$

Extreme values can also occur at the point where $\nabla g = \langle y, x\rangle = \langle 0, 0, \rangle$. However, the point $(0, 0)$ is not on the constraint.
Step 4. Calculate f at the critical points. We evaluate $f(x, y) = 4x^2 + 9y^2$ at the critical points:

$$f\left(\sqrt{6}, 2\sqrt{\frac{2}{3}}\right) = 4 \cdot 6 + 9 \cdot 4 \cdot \frac{2}{3} = 48$$

$$f\left(-\sqrt{6}, -2\sqrt{\frac{2}{3}}\right) = 4 \cdot 6 + 9 \cdot 4 \cdot \frac{2}{3} = 48$$

On the constraint, $y = \frac{4}{x}$, thus $f(x, y) = f\left(x, \frac{4}{x}\right) = h(x) = 4x^2 + \frac{144}{x^2}$. Since $\lim\limits_{x\to\infty} h(x) = \lim_{x\to-\infty} h(x) = \infty$, h has a global minimum of 48 (but no maximum!) on $(-\infty, \infty)$.

7. $f(x, y) = xy, \quad 4x^2 + 9y^2 = 32$

SOLUTION We find the extreme values of $f(x, y) = xy$ under the constraint $g(x, y) = 4x^2 + 9y^2 - 32 = 0$.
Step 1. Write out the Lagrange Equation. The gradient vectors are $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 8x, 18y\rangle$, hence the Lagrange Condition is

$$\nabla f = \lambda \nabla g$$
$$\langle y, x\rangle = \lambda \langle 8x, 18y\rangle$$

We obtain the following equations:

$$y = \lambda(8x)$$
$$x = \lambda(18y)$$

Step 2. Solve for λ in terms of x and y. If $x = 0$, then the Lagrange equations also imply that $y = 0$ and vice versa. Since the point $(0, 0)$ does not satisfy the equation of the constraint, we may assume that $x \neq 0$ and $y \neq 0$. The two equations give

$$\lambda = \frac{y}{8x} \quad \text{and} \quad \lambda = \frac{x}{18y}$$

Step 3. Solve for x and y using the constraint. We equate the two expressions for λ to obtain

$$\frac{y}{8x} = \frac{x}{18y} \quad \Rightarrow \quad 18y^2 = 8x^2 \quad \Rightarrow \quad y = \pm\frac{2}{3}x$$

We now substitute $y = \pm\frac{2}{3}x$ in the equation of the constraint and solve for x and y:

$$4x^2 + 9 \cdot \left(\pm\frac{2}{3}x\right)^2 = 32$$

$$4x^2 + 9 \cdot \frac{4x^2}{9} = 32$$
$$8x^2 = 32 \quad \Rightarrow \quad x = -2, \quad x = 2$$

We find y by the relation $y = \pm\frac{2}{3}x$:

$$y = \frac{2}{3} \cdot (-2) = -\frac{4}{3}, \quad y = -\frac{2}{3} \cdot (-2) = \frac{4}{3}, \quad y = \frac{2}{3} \cdot 2 = \frac{4}{3}, \quad y = -\frac{2}{3} \cdot 2 = -\frac{4}{3}$$

We obtain the following critical points:

$$\left(-2, -\frac{4}{3}\right), \quad \left(-2, \frac{4}{3}\right), \quad \left(2, \frac{4}{3}\right), \quad \left(2, -\frac{4}{3}\right)$$

Extreme values can also occur at the point where $\nabla g = \langle 8x, 18y \rangle = \langle 0, 0 \rangle$, that is, at the point $(0, 0)$. However, the point does not lie on the constraint.

Step 4. Calculate f at the critical points. We evaluate $f(x, y) = xy$ at the critical points:

$$f\left(-2, -\frac{4}{3}\right) = f\left(2, \frac{4}{3}\right) = \frac{8}{3}$$
$$f\left(-2, \frac{4}{3}\right) = f\left(2, -\frac{4}{3}\right) = -\frac{8}{3}$$

Since f is continuous and the constraint is a closed and bounded set in R^2 (an ellipse), f attains global extrema on the constraint. We conclude that $\frac{8}{3}$ is the maximum value and $-\frac{8}{3}$ is the minimum value.

8. $f(x, y) = x^2y + x + y, \quad xy = 4$

SOLUTION Under the constraint $xy = 4$, then $f(x, y) = x(xy) + x + y = 4x + x + \frac{4}{x}$. Therefore, as $x \to 0+$, $f(x, y) \to +\infty$ on the constraint, and as $x \to 0-$, $f(x, y) \to -\infty$. Therefore there are no minimum and maximum values of $f(x, y)$ under the constraint.

9. $f(x, y) = x^2 + y^2, \qquad x^4 + y^4 = 1$

SOLUTION We find the extreme values of $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = x^4 + y^4 - 1 = 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \left\langle 4x^3, 4y^3 \right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\langle 2x, 2y \rangle = \lambda \left\langle 4x^3, 4y^3 \right\rangle$$

or

$$\begin{aligned} 2x &= \lambda\left(4x^3\right) \\ 2y &= \lambda\left(4y^3\right) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 2\lambda x^3 \\ y &= 2\lambda y^3 \end{aligned} \tag{1}$$

Step 2. Solve for λ in terms of x and y. We first assume that $x \neq 0$ and $y \neq 0$. Then the Lagrange equations give

$$\lambda = \frac{1}{2x^2} \quad \text{and} \quad \lambda = \frac{1}{2y^2}$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ gives

$$\frac{1}{2x^2} = \frac{1}{2y^2} \quad \Rightarrow \quad y^2 = x^2 \quad \Rightarrow \quad y = \pm x$$

We now substitute $y = \pm x$ in the equation of the constraint $x^4 + y^4 = 1$ and solve for x and y:

$$x^4 + (\pm x)^4 = 1$$
$$2x^4 = 1$$
$$x^4 = \frac{1}{2} \quad \Rightarrow \quad x = \frac{1}{2^{1/4}}, \quad x = -\frac{1}{2^{1/4}}$$

The corresponding values of y are obtained by the relation $y = \pm x$. The critical points are thus

$$\left(\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right), \quad \left(\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}}\right), \quad \left(-\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right), \quad \left(-\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}}\right) \tag{2}$$

We examine the case $x = 0$ or $y = 0$. Notice that the point $(0, 0)$ does not satisfy the equation of the constraint, hence either $x = 0$ or $y = 0$ can hold, but not both at the same time.

Case 1: $x = 0$. Substituting $x = 0$ in the constraint $x^4 + y^4 = 1$ gives $y = \pm 1$. We thus obtain the critical points

$$(0, -1), \quad (0, 1) \tag{3}$$

Case 2: $y = 0$. We may interchange x and y in the discussion in case 1, and obtain the critical points:

$$(-1, 0), \quad (1, 0) \tag{4}$$

Combining (2), (3), and (4) we conclude that the critical points are

$$A_1 = \left(\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right), \quad A_2 = \left(\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}}\right), \quad A_3 = \left(-\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right),$$
$$A_4 = \left(-\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}}\right), \quad A_5 = (0, -1), \quad A_6 = (0, 1), \quad A_7 = (-1, 0), \quad A_8 = (1, 0)$$

The point where $\nabla g = \left\langle 4x^3, 4y^3 \right\rangle = \langle 0, 0 \rangle$, that is, $(0, 0)$, does not lie on the constraint.

Step 4. Compute f at the critical points. We evaluate $f(x, y) = x^2 + y^2$ at the critical points:

$$f(A_1) = f(A_2) = f(A_3) = f(A_4) = \left(\frac{1}{2^{1/4}}\right)^2 + \left(\frac{1}{2^{1/4}}\right)^2 = \frac{2}{2^{1/2}} = \sqrt{2}$$
$$f(A_5) = f(A_6) = f(A_7) = f(A_8) = 1$$

The constraint $x^4 + y^4 = 1$ is a closed and bounded set in R^2 and f is continuous on this set, hence f has global extrema on the constraint. We conclude that $\sqrt{2}$ is the maximum value and 1 is the minimum value.

10. $f(x, y) = x^2 y^4, \quad x^2 + 2y^2 = 6$

SOLUTION We find the extreme values of $f(x, y) = x^2 y^4$ on the constraint $g(x, y) = x^2 + 2y^2 - 6 = 0$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \left\langle 2xy^4, 4y^3x^2 \right\rangle$ and $\nabla g = \langle 2x, 4y \rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\left\langle 2xy^4, 4y^3x^2 \right\rangle = \lambda \langle 2x, 4y \rangle$$

or

$$\begin{aligned} 2xy^4 &= \lambda(2x) \\ 4y^3x^2 &= \lambda(4y) \end{aligned} \quad \Rightarrow \quad \begin{aligned} xy^4 &= \lambda x \\ x^2y^3 &= \lambda y \end{aligned} \tag{1}$$

Step 2. Solve for λ in terms of x and y. Notice that if $x = 0$ or $y = 0$, then $f(x, y) = x^2 y^4$ has the value 0, which is the minimum value (since $f(x, y) \geq 0$). We thus assume that $x \neq 0$ and $y \neq 0$. The Lagrange equations (1) give

$$\lambda = \frac{xy^4}{x} = y^4, \quad \lambda = \frac{x^2y^3}{y} = x^2y^2$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ gives

$$y^4 = x^2y^2 \quad \Rightarrow \quad y^2 = x^2 \quad \Rightarrow \quad y = \pm x$$

Substituting $y = \pm x$ in the equation of the constraint $x^2 + 2y^2 = 6$ and solving for x and y gives

$$\begin{aligned} x^2 + 2x^2 &= 6 \\ 3x^2 &= 6 \\ x^2 &= 2 \quad \Rightarrow \quad x = \sqrt{2}, \quad x = -\sqrt{2} \end{aligned}$$

The corresponding value of y is obtained by the relation $y = \pm x$. We obtain the following points:

$$\left(\sqrt{2}, -\sqrt{2}\right), \quad \left(\sqrt{2}, \sqrt{2}\right), \quad \left(-\sqrt{2}, -\sqrt{2}\right), \quad \left(-\sqrt{2}, \sqrt{2}\right)$$

Extreme values can occur also at the point where $\nabla g = \langle 2x, 4y \rangle = \langle 0, 0 \rangle$, that is, $(0, 0)$. However, this point does not lie on the constraint.

Step 4. Computing f at the critical points. We evaluate $f(x, y) = x^2y^4$ at the critical points:

$$f\left(\sqrt{2}, -\sqrt{2}\right) = f\left(\sqrt{2}, \sqrt{2}\right) = f\left(-\sqrt{2}, -\sqrt{2}\right) = f\left(-\sqrt{2}, \sqrt{2}\right) = \left(\sqrt{2}\right)^2\left(\sqrt{2}\right)^4 = \left(\sqrt{2}\right)^6 = 8$$

Recall that there are critical points with $x = 0$ or $y = 0$ at which the value of f is zero. Since f has global extrema on the ellipse $x^2 + 2y^2 = 6$, we conclude that the minimum value of f on the constraint is 0 and the maximum value is 8.

11. $f(x, y, z) = 3x + 2y + 4z, \quad x^2 + 2y^2 + 6z^2 = 1$

SOLUTION We find the extreme values of $f(x, y, z) = 3x + 2y + 4z$ under the constraint $g(x, y, z) = x^2 + 2y^2 + 6z^2 - 1 = 0$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 3, 2, 4\rangle$ and $\nabla g = \langle 2x, 4y, 12z\rangle$, therefore the Lagrange Condition $\nabla f = \lambda\nabla g$ is:

$$\langle 3, 2, 4\rangle = \lambda \langle 2x, 4y, 12z\rangle$$

The Lagrange equations are, thus:

$$\begin{aligned} 3 &= \lambda(2x) & & \frac{3}{2} = \lambda x \\ 2 &= \lambda(4y) & \Rightarrow \quad & \frac{1}{2} = \lambda y \\ 4 &= \lambda(12z) & & \frac{1}{3} = \lambda z \end{aligned}$$

Step 2. Solve for λ in terms of x, y, and z. The Lagrange equations imply that $x \neq 0$, $y \neq 0$, and $z \neq 0$. Solving for λ we get

$$\lambda = \frac{3}{2x}, \quad \lambda = \frac{1}{2y}, \quad \lambda = \frac{1}{3z}$$

Step 3. Solve for x, y, and z using the constraint. Equating the expressions for λ gives

$$\frac{3}{2x} = \frac{1}{2y} = \frac{1}{3z} \quad \Rightarrow \quad x = \frac{9}{2}z, \quad y = \frac{3}{2}z$$

Substituting $x = \frac{9}{2}z$ and $y = \frac{3}{2}z$ in the equation of the constraint $x^2 + 2y^2 + 6z^2 = 1$ and solving for z we get

$$\begin{aligned} \left(\frac{9}{2}z\right)^2 + 2\left(\frac{3}{2}z\right)^2 + 6z^2 &= 1 \\ \frac{123}{4}z^2 &= 1 \quad \Rightarrow \quad z_1 = \frac{2}{\sqrt{123}}, z_2 = -\frac{2}{\sqrt{123}} \end{aligned}$$

Using the relations $x = \frac{9}{2}z$, $y = \frac{3}{2}z$ we get

$$\begin{aligned} x_1 &= \frac{9}{2}\cdot\frac{2}{\sqrt{123}} = \frac{9}{\sqrt{123}}, \quad y_1 = \frac{3}{2}\cdot\frac{2}{\sqrt{123}} = \frac{3}{\sqrt{123}}, \quad z_1 = \frac{2}{\sqrt{123}} \\ x_2 &= \frac{9}{2}\cdot\frac{-2}{\sqrt{123}} = -\frac{9}{\sqrt{123}}, \quad y_2 = \frac{3}{2}\cdot\frac{-2}{\sqrt{123}} = -\frac{3}{\sqrt{123}}, \quad z_2 = -\frac{2}{\sqrt{123}} \end{aligned}$$

We obtain the following critical points:

$$p_1 = \left(\frac{9}{\sqrt{123}}, \frac{3}{\sqrt{123}}, \frac{2}{\sqrt{123}}\right) \quad \text{and} \quad p_2 = \left(-\frac{9}{\sqrt{123}}, -\frac{3}{\sqrt{123}}, -\frac{2}{\sqrt{123}}\right)$$

Critical points are also the points on the constraint where $\nabla g = 0$. However, $\nabla g = \langle 2x, 4y, 12z\rangle = \langle 0, 0, 0\rangle$ only at the origin, and this point does not lie on the constraint.

Step 4. Computing f at the critical points. We evaluate $f(x, y, z) = 3x + 2y + 4z$ at the critical points:

$$\begin{aligned} f(p_1) &= \frac{27}{\sqrt{123}} + \frac{6}{\sqrt{123}} + \frac{8}{\sqrt{123}} = \frac{41}{\sqrt{123}} = \sqrt{\frac{41}{3}} \approx 3.7 \\ f(p_2) &= -\frac{27}{\sqrt{123}} - \frac{6}{\sqrt{123}} - \frac{8}{\sqrt{123}} = -\frac{41}{\sqrt{123}} = -\sqrt{\frac{41}{3}} \approx -3.7 \end{aligned}$$

Since f is continuous and the constraint is closed and bounded in R^3, f has global extrema under the constraint. We conclude that the minimum value of f under the constraint is about -3.7 and the maximum value is about 3.7.

12. $f(x, y, z) = x^2 - y - z, \quad x^2 - y^2 + z = 0$

SOLUTION We show that the function $f(x, y, z) = x^2 - y - z$ does not have minimum and maximum values subject to the constraint $x^2 - y^2 + z = 0$. Notice that the curve $(x, x, 0)$ lies on the constraint, since it satisfies the equation of the constraint. On this curve we have

$$f(x, y, z) = f(x, x, 0) = x^2 - x - 0 = x^2 - x$$

Since $\lim_{x\to\pm\infty}(x^2 - x) = \infty$, f does not have a maximum value subject to the constraint. Observe that the curve $\left(0, \sqrt{z}, z\right)$ also lies on the constraint, and we have

$$f(x, y, z) = f\left(0, \sqrt{z}, z\right) = 0^2 - \sqrt{z} - z = -\left(z + \sqrt{z}\right)$$

Since $\lim_{z\to\infty} -\left(z + \sqrt{z}\right) = -\infty$, f does not attain a minimum value on the constraint either.

13. $f(x, y, z) = xy + 3xz + 2yz, \quad 5x + 9y + z = 10$

SOLUTION We show that $f(x, y, z) = xy + 3xz + 2yz$ does not have minimum and maximum values subject to the constraint $g(x, y, z) = 5x + 9y + z - 10 = 0$. First notice that the curve $c_1 : (x, x, 10 - 14x)$ lies on the surface of the constraint since it satisfies the equation of the constraint. On c_1 we have,

$$f(x, y, z) = f(x, x, 10 - 14x) = x^2 + 3x(10 - 14x) + 2x(10 - 14x) = -69x^2 + 50x$$

Since $\lim_{x\to\infty}\left(-69x^2 + 50x\right) = -\infty$, f does not have minimum value on the constraint. Notice that the curve c_2 : $(x, -x, 10 + 4x)$ also lies on the surface of the constraint. The values of f on c_2 are

$$f(x, y, z) = f(x, -x, 10 + 4x) = -x^2 + 3x(10 + 4x) - 2x(10 + 4x) = 3x^2 + 10x$$

The limit $\lim_{x\to\infty}(3x^2 + 10x) = \infty$ implies that f does not have a maximum value subject to the constraint.

14. Let

$$f(x, y) = x^3 + xy + y^3, \qquad g(x, y) = x^3 - xy + y^3$$

(a) Show that there is a unique point $P = (a, b)$ on $g(x, y) = 1$ where $\nabla f_P = \lambda \nabla g_P$ for some scalar λ.

(b) Refer to Figure 12 to determine whether $f(P)$ is a local minimum or a local maximum of f subject to the constraint.

(c) Does Figure 12 suggest that $f(P)$ is a global extremum subject to the constraint?

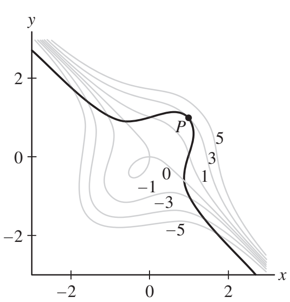


FIGURE 12 Contour map of $f(x, y) = x^3 + xy + y^3$ and graph of the constraint $g(x, y) = x^3 - xy + y^3 = 1$.

SOLUTION

(a) The gradients of f and g are $\nabla f = \left\langle 3x^2 + y, x + 3y^2\right\rangle$ and $\nabla g = \left\langle 3x^2 - y, -x + 3y^2\right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$\left\langle 3x^2 + y, x + 3y^2\right\rangle = \lambda\left\langle 3x^2 - y, -x + 3y^2\right\rangle$$

or

$$\begin{aligned} 3x^2 + y &= \lambda(3x^2 - y) \\ x + 3y^2 &= \lambda(-x + 3y^2) \end{aligned} \tag{1}$$

Notice that if $3x^2 - y = 0$, the first equation implies that also $3x^2 + y = 0$, hence $y = 0$ and $x = 0$. Since the point $(0, 0)$ does not satisfy the equation of the constraint, we may assume that $3x^2 - y \neq 0$. Similarly, if $-x + 3y^2 = 0$, the second equation implies that also $x + 3y^2 = 0$, therefore $x = y = 0$. We thus may also assume that $-x + 3y^2 \neq 0$. Using these assumptions, we have by (1):

$$\lambda = \frac{3x^2 + y}{3x^2 - y}, \quad \lambda = \frac{x + 3y^2}{-x + 3y^2}$$

Equating the two expressions for λ we get

$$\frac{3x^2+y}{3x^2-y}=\frac{x+3y^2}{-x+3y^2}$$
$$\left(3x^2+y\right)\left(-x+3y^2\right)=\left(x+3y^2\right)\left(3x^2-y\right)$$
$$-3x^3+9x^2y^2-yx+3y^3=3x^3-xy+9x^2y^2-3y^3$$
$$x^3=y^3 \quad\Rightarrow\quad x=y$$

We now substitute $x=y$ in the constraint $x^3-xy+y^3=1$ and solve for y:

$$y^3-y^2+y^3=1$$
$$2y^3-y^2-1=0$$

We notice that $y=1$ is a root of $2y^3-y^2-1$, hence this polynomial is divisible by $y-1$. Long division yields

$$(y-1)(2y^2+y+1)=0$$

Since $2y^2+y+1>0$ for all y (the discriminant is negative), the only solution is $y=1$. Then, $x=y=1$ and the only critical point is $(1,1)$.

(b) Figure 12 suggests that the values of $f(x,y)$ are increasing as (x,y) approaches the critical point $(1,1)$ along the constraint. Therefore, f has a local maximum at P, subject to the constraint.

(c) Figure 12 shows the behavior of f and g only in the range $-3\le x\le 3$, so we cannot know whether P is a global maximum, but it is reasonable to guess that it is.

15. Find the point (a,b) on the graph of $y=e^x$ where the value ab is as small as possible.

SOLUTION We must find the point where $f(x,y)=xy$ has a minimum value subject to the constraint $g(x,y)=e^x-y=0$.

Step 1. Write out the Lagrange Equations. Since $\nabla f=\langle y,x\rangle$ and $\nabla g=\left\langle e^x,-1\right\rangle$, the Lagrange Condition $\nabla f=\lambda\nabla g$ is

$$\langle y,x\rangle=\lambda\left\langle e^x,-1\right\rangle$$

The Lagrange equations are thus

$$y=\lambda e^x$$
$$x=-\lambda$$

Step 2. Solve for λ in terms of x and y. The Lagrange equations imply that

$$\lambda=ye^{-x}\quad\text{and}\quad\lambda=-x$$

Step 3. Solve for x and y using the constraint. We equate the two expressions for λ to obtain

$$ye^{-x}=-x\quad\Rightarrow\quad y=-xe^x$$

We now substitute $y=-xe^x$ in the equation of the constraint and solve for x:

$$e^x-(-xe^x)=0$$
$$e^x(1+x)=0$$

Since $e^x\ne 0$ for all x, we have $x=-1$. The corresponding value of y is determined by the relation $y=-xe^x$. That is,

$$y=-(-1)e^{-1}=e^{-1}$$

We obtain the critical point

$$(-1,e^{-1})$$

Step 4. Calculate f at the critical point. We evaluate $f(x,y)=xy$ at the critical point.

$$f(-1,e^{-1})=(-1)\cdot e^{-1}=-e^{-1}$$

We conclude (see Remark) that the minimum value of xy on the graph of $y=e^x$ is $-e^{-1}$, and it is obtained for $x=-1$ and $y=e^{-1}$.

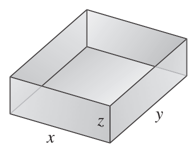
Remark: Since the constraint is not bounded, we need to justify the existence of a minimum value. The values $f(x, y) = xy$ on the constraint $y = e^x$ are $f(x, e^x) = h(x) = xe^x$. Since $h(x) > 0$ for $x > 0$, the minimum value (if it exists) occurs at a point $x < 0$. Since

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0,$$

then for $x <$ some negative number $-R$, we have $|f(x) - 0| < 0.1$, say. Thus, on the bounded region $-R \le x \le 0$, f has a minimum value of $-e^{-1} \approx -0.37$, and this is thus a global minimum (for all x).

16. Find the rectangular box of maximum volume if the sum of the lengths of the edges is 300 cm.

SOLUTION We denote by x, y, and z the dimensions of the rectangular box.



Then the volume of the box is xyz. We must find the values of x, y and z that maximize the volume $f(x, y, z) = xyz$, subject to the constraint $g(x, y, z) = x + y + z = 300$, $x \ge 0$, $y \ge 0$, $z \ge 0$. (One could also argue that the sums of the lengths of the edges is $4x + 4y + 4z = 300$, but that would give a different answer, of course. Instead, we will choose to interpret the problem with the constraint $x + y + z = 300$).

Step 1. Write out the Lagrange Equations. The Lagrange Condition is

$$\nabla f = \lambda \nabla g$$
$$\langle yz, xz, xy\rangle = \lambda \langle 1, 1, 1\rangle$$

We obtain the following equations:

$$yz = \lambda$$
$$xz = \lambda$$
$$xy = \lambda$$

Step 2. Solve for λ in terms of x, y, and z. The Lagrange equations already give λ in terms of x, y, and z. Equating the expressions for λ we get $yz = xz = xy$.

Step 3. Solve for x, y, and z using the constraint. We have

$$\begin{matrix} yz = xz \\ xy = xz \end{matrix} \quad\Rightarrow\quad \begin{matrix} z(x-y) = 0 \\ x(z-y) = 0 \end{matrix}$$

If $x = 0$, $y = 0$, or $z = 0$, the volume has the minimum value 0. We thus may assume that $x \ne 0$, $y \ne 0$, and $z \ne 0$. The first equation implies that $x = y$ and the second equation gives $z = y$. We now substitute $x = y$ and $z = y$ in the constraint $x + y + z = 300$ and solve for y:

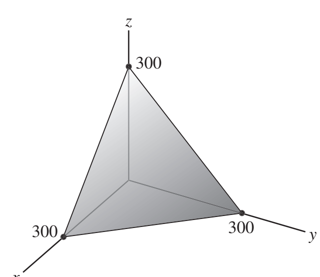
$$y + y + y = 300$$
$$3y = 300 \quad\Rightarrow\quad y = 100$$

Therefore, $x = 100$ and $z = 100$. The critical point is $(100, 100, 100)$.

Step 4. Conclusions. The value of $f(x, y, z) = xyz$ at the critical point is

$$f(100, 100, 100) = 100^3 = 10^6 \text{ cm}^3$$

The constraint $x + y + z = 300$, $x \ge 0$, $y \ge 0$, $z \ge 0$ is the part of the plane $x + y + z = 300$ that lies in the first octant. This is a bounded and closed set in R^3. Since f is continuous on this set, f has global extreme values on this set. The minimum value is zero (obtained if one of the variables is zero), hence the value 10^6 is the maximum value. We conclude that the box with maximum value is a cube of edge 100 cm.



17. The surface area of a right-circular cone of radius r and height h is $S = \pi r\sqrt{r^2+h^2}$, and its volume is $V = \frac{1}{3}\pi r^2 h$.
(a) Determine the ratio h/r for the cone with given surface area S and maximum volume V.
(b) What is the ratio h/r for a cone with given volume V and minimum surface area S?
(c) Does a cone with given volume V and maximum surface area exist?

SOLUTION

(a) Let S_0 denote a given surface area. We must find the ratio $\frac{h}{r}$ for which the function $V(r,h) = \frac{1}{3}\pi r^2 h$ has maximum value under the constraint $S(r,h) = \pi r\sqrt{r^2+h^2} = \pi\sqrt{r^4+h^2r^2} = S_0$.
Step 1. Write out the Lagrange Equation. We have

$$\nabla V = \pi\left\langle \frac{2rh}{3}, \frac{r^2}{3}\right\rangle \quad \text{and} \quad \nabla S = \pi\left\langle \frac{2r^3+h^2r}{\sqrt{r^4+h^2r^2}}, \frac{hr^2}{\sqrt{r^4+h^2r^2}}\right\rangle$$

The Lagrange Condition $\nabla V = \lambda\nabla S$ gives the following equations:

$$\frac{2rh}{3} = \frac{2r^3+h^2r}{\sqrt{r^4+h^2r^2}}\lambda \quad \Rightarrow \quad \frac{2h}{3} = \frac{2r^2+h^2}{\sqrt{r^4+h^2r^2}}\lambda$$
$$\frac{r^2}{3} = \frac{hr^2}{\sqrt{r^4+h^2r^2}}\lambda \quad \Rightarrow \quad \frac{1}{3} = \frac{h}{\sqrt{r^4+h^2r^2}}\lambda$$

Step 2. Solve for λ in terms of r and h. These equations yield two expressions for λ that must be equal:

$$\lambda = \frac{2h}{3}\frac{\sqrt{r^4+h^2r^2}}{2r^2+h^2} = \frac{1}{3h}\sqrt{r^4+h^2r^2}$$

Step 3. Solve for r and h using the constraint. We have

$$\frac{2h}{3}\frac{\sqrt{r^4+h^2r^2}}{2r^2+h^2} = \frac{1}{3h}\sqrt{r^4+h^2r^2}$$
$$2h\frac{1}{2r^2+h^2} = \frac{1}{h}$$
$$2h^2 = 2r^2+h^2 \quad \Rightarrow \quad h^2 = 2r^2 \quad \Rightarrow \quad \frac{h}{r} = \sqrt{2}$$

We substitute $h^2 = 2r^2$ in the constraint $\pi r\sqrt{r^2+h^2} = S_0$ and solve for r. This gives

$$\pi r\sqrt{r^2+2r^2} = S_0$$
$$\pi r\sqrt{3r^2} = S_0$$
$$\sqrt{3}\pi r^2 = S_0 \quad \Rightarrow \quad r^2 = \frac{S_0}{\sqrt{3}\pi}, \quad h^2 = 2r^2 = \frac{2S_0}{\sqrt{3}\pi}$$

Extreme values can occur also at points on the constraint where $\nabla S = \left\langle \frac{2r^2+h^2r}{\sqrt{r^4+h^2r^2}}, \frac{hr^2}{\sqrt{r^4+h^2r^2}}\right\rangle = \langle 0,0\rangle$, that is, at $(r,h) = (0,h)$, $h \neq 0$. However, since the radius of the cone is positive ($r > 0$), these points are irrelevant. We conclude that for the cone with surface area S_0 and maximum volume, the following holds:

$$\frac{h}{r} = \sqrt{2}, \quad h = \sqrt{\frac{2S_0}{\sqrt{3}\pi}}, \quad r = \sqrt{\frac{S_0}{\sqrt{3}\pi}}$$

For the surface area $S_0 = 1$ we get

$$h = \sqrt{\frac{2}{\sqrt{3}\pi}} \approx 0.6, \quad r = \sqrt{\frac{1}{\sqrt{3}\pi}} = 0.43$$

(b) We now must find the ratio $\frac{h}{r}$ that minimizes the function $S(r,h) = \pi r\sqrt{r^2+h^2}$ under the constraint

$$V(r,h) = \frac{1}{3}\pi r^2 h = V_0$$

Using the gradients computed in part (a), the Lagrange Condition $\nabla S = \lambda \nabla V$ gives the following equations:

$$\begin{aligned} \frac{2r^3 + h^2 r}{\sqrt{r^4 + h^2 r^2}} &= \lambda \frac{2rh}{3} \\ \frac{hr^2}{\sqrt{r^4 + h^2 r^2}} &= \lambda \frac{r^2}{3} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{2r^2 + h^2}{\sqrt{r^4 + h^2 r^2}} &= \lambda \frac{2h}{3} \\ \frac{h}{\sqrt{r^4 + h^2 r^2}} &= \frac{\lambda}{3} \end{aligned}$$

These equations give

$$\frac{\lambda}{3} = \frac{1}{2h} \frac{2r^2 + h^2}{\sqrt{r^4 + h^2 r^2}} = \frac{h}{\sqrt{r^4 + h^2 r^2}}$$

We simplify and solve for $\frac{h}{r}$:

$$\begin{aligned} \frac{2r^2 + h^2}{2h} &= h \\ 2r^2 + h^2 &= 2h^2 \\ 2r^2 &= h^2 \quad \Rightarrow \quad \frac{h}{r} = \sqrt{2} \end{aligned}$$

We conclude that the ratio $\frac{h}{r}$ for a cone with a given volume and minimal surface area is

$$\frac{h}{r} = \sqrt{2}$$

(c) The constant $V = 1$ gives $\frac{1}{3}\pi r^2 h = 1$ or $h = \frac{3}{\pi r^2}$. As $r \to \infty$, we have $h \to 0$, therefore

$$\lim_{\substack{r \to \infty \\ h \to 0}} S(r, h) = \lim_{\substack{r \to \infty \\ h \to 0}} \pi r \sqrt{r^2 + h^2} = \infty$$

That is, S does not have maximum value on the constraint, hence there is no cone of volume 1 and maximal surface area.

18. In Example 1, we found the maximum of $f(x, y) = 2x + 5y$ on the ellipse $(x/4)^2 + (y/3)^2 = 1$. Solve this problem again without using Lagrange multipliers. First, show that the ellipse is parametrized by $x = 4\cos t$, $y = 3\sin t$. Then find the maximum value of $f(4\cos t, 3\sin t)$ using single-variable calculus. Is one method easier than the other?

SOLUTION We want to find the maximum of $f(x, y) = 2x + 5y$ on the ellipse $(x/4)^2 + (y/3)^2 = 1$ without using Lagrange multipliers. We rewrite the equation of the ellipse in the form

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

We now identify the following parametrization for the ellipse:

$$x = 4\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi$$

Substituting in the function $f(x, y) = 2x + 5y$ we obtain the following function of t:

$$g(t) = 8\cos t + 15\sin t$$

We now find the maximum value of the single variable function $g(t) = 8\cos t + 15\sin t$ in the interval $0 \le t \le 2\pi$. We first compute the critical points in the interval $0 < t < 2\pi$ by solving $g'(t) = 0$ in this interval. We obtain

$$\begin{aligned} g'(t) &= -8\sin t + 15\cos t = 0 \\ 15\cos t &= 8\sin t \\ \tan t &= \frac{15}{8} \quad \Rightarrow \quad t = \tan^{-1}(15/8) \approx 1.08 \end{aligned}$$

We evaluate $g(t) = 8\cos t + 15\sin t$ at the critical points and at the endpoints $t = 0$, $t = 2\pi$ of the interval:

$$\begin{aligned} g(\tan^{-1}(15/8)) &= 8\cos(\tan^{-1}(15/8)) + 15\sin(\tan^{-1}(15/8)) = 8 \cdot \frac{8}{17} + 15 \cdot \frac{15}{17} = \frac{289}{17} = 17 \\ g(0) &= 8\cos 0 + 15\sin 0 = 8 \\ g(2\pi) &= 8\cos 2\pi + 15\sin 2\pi = 8 \end{aligned}$$

The greatest value is $g(\tan^{-1}(15/8)) = 17$. We conclude that the maximum value of g in the interval $0 \le t \le 2\pi$ is $g(\tan^{-1}(15/8)) = 17$. Therefore, the maximum value of $f(x, y) = 2x + 5y$ on the ellipse $x^2/16 + y^2/9 = 1$ is 17, and it occurs at the point $\left(4\cos(\tan^{-1}(15/8)), 3\sin(\tan^{-1}(15/8))\right) = (4 \cdot 8/17, 3 \cdot 15/17) = (32/17, 45/17)$.

In this example the two methods do not demand much work, hence neither of them is much easier than the other.

19. Find the point on the ellipse

$$x^2 + 6y^2 + 3xy = 40$$

with largest x-coordinate (Figure 13).

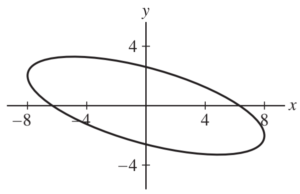


FIGURE 13 Graph of $x^2 + 6y^2 + 3xy = 40$

SOLUTION We need to maximize $f(x, y) = x$ subject to the constraint

$$g(x, y) = x^2 + 6y^2 + 3xy = 40$$

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 1, 0\rangle$ and $\nabla g = \langle 2x + 3y, 12y + 3x\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives:

$$\langle 1, 0\rangle = \lambda \langle 2x + 3y, 12y + 3x\rangle$$

or

$$1 = \lambda(2x + 3y), \qquad 0 = \lambda(12y + 3x)$$

this yields

$$x = -4y$$

Step 2. Solve for x and y using the constraint.

$$x^2 + 6y^2 + 3xy = (-4y)^2 + 6y^2 + 3(-4y)y = (16 + 6 - 12)y^2 = 10y^2 = 40$$

so $y = \pm 2$. If $y = 2$ then $x = -8$ and if $y = -2$ then $x = 8$. The extreme points are $(-8, 2)$ and $(8, -2)$. We conclude that the point with largest x-coordinate is $P = (8, -2)$.

20. Find the maximum area of a rectangle inscribed in the ellipse (Figure 14):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

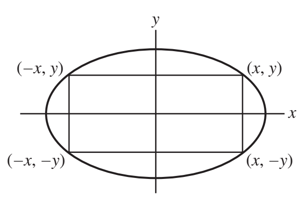


FIGURE 14 Rectangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION Since (x, y) is in the first quadrant, $x > 0$ and $y > 0$. The area of the rectangle is $2x \cdot 2y = 4xy$. The vertices lie on the ellipse, hence their coordinates $(\pm x, \pm y)$ must satisfy the equation of the ellipse. Therefore, we must find the maximum value of the function $f(x, y) = 4xy$ under the constraint

$$g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x > 0, \quad y > 0.$$

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 4y, 4x\rangle$ and $\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}\right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\langle 4y, 4x\rangle = \lambda \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}\right\rangle$$

or

$$\begin{aligned} 4y &= \lambda\left(\frac{2x}{a^2}\right) \\ 4x &= \lambda\left(\frac{2y}{b^2}\right) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2y &= \lambda\frac{x}{a^2} \\ 2x &= \lambda\frac{y}{b^2} \end{aligned}$$

Step 2. Solve for λ in terms of x and y. The Lagrange equations give the following two expressions for λ:

$$\lambda = \frac{2ya^2}{x}, \quad \lambda = \frac{2xb^2}{y}$$

Equating the two equations we get

$$\frac{2ya^2}{x} = \frac{2xb^2}{y}$$

Step 3. Solve for x and y using the constraint. We solve the equation in step 2 for y in terms of x:

$$\begin{aligned} \frac{2ya^2}{x} &= \frac{2xb^2}{y} \\ 2y^2a^2 &= 2x^2b^2 \\ y^2 &= \frac{x^2b^2}{a^2} \quad \Rightarrow \quad y = \frac{b}{a}x \end{aligned}$$

We now substitute $y = \frac{b}{a}x$ in the equation of the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and solve for x:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{\left(\frac{b}{a}x\right)^2}{b^2} &= 1 \\ \frac{x^2}{a^2} + \frac{x^2}{a^2} &= 1 \\ \frac{2x^2}{a^2} &= 1 \\ x^2 = \frac{a^2}{2} \quad &\Rightarrow \quad x = \frac{a}{\sqrt{2}} \end{aligned}$$

The corresponding value of y is obtained by the relation $y = \frac{b}{a}x$:

$$y = \frac{b}{a} \cdot \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}}$$

We obtain the critical point $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. Extreme values can also occur at points on the constraint where $\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2} \right\rangle = \langle 0, 0 \rangle$. However, the point $(0, 0)$ is not on the constraint. We conclude that if $f(x, y) = 4xy$ has a maximum value on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $x > 0$, $y > 0$, then it occurs at the point $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and the maximum value is

$$f\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = 2ab$$

We now justify why the maximum value exists. We consider the problem of finding the extreme values of $f(x, y) = 4xy$ on the quarter ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant. Since the constraint curve is bounded and $f(x, y)$ is continuous, f has a minimum and maximum values on the ellipse. The minimum volume occurs at the end points:

$$x = 0, \quad y = b \quad \Rightarrow \quad 4xy = 0 \qquad \text{or} \qquad x = a, \quad y = 0 \quad \Rightarrow \quad 4xy = 0$$

So the critical point $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ must be a maximum.

21. Find the point (x_0, y_0) on the line $4x + 9y = 12$ that is closest to the origin.

SOLUTION Since we are minimizing distance, we can minimize the square of the distance function without loss of generality:

$$f(x, y) = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$$

subject to the constraint $g(x, y) = 4x + 9y - 12$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle 4, 9\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\langle 2x, 2y\rangle = \lambda \langle 4, 9\rangle$$

or

$$2x = 4\lambda \quad \Rightarrow \quad x = 2\lambda, \quad 2y = 9\lambda$$

Step 2. Solve for λ in terms of x and y. The Lagrange equations give the following two expressions for λ:

$$\lambda = \frac{x}{2}, \quad \lambda = \frac{9}{2}y$$

Equating these two

$$\frac{x}{2} = \frac{9}{2}y \quad \Rightarrow \quad x = 9y$$

Step 3. Solve for x and y using the constraint. We are given $4x + 9y = 12$, therefore we can write:

$$4(9y) + 9y = 12 \quad \Rightarrow \quad 45y = 12 \quad \Rightarrow \quad y = \frac{12}{45} = \frac{4}{15}$$

Since $x = 9y$, then we conclude:

$$y = \frac{4}{15} \quad x = 9 \cdot \frac{4}{15} = \frac{12}{5}$$

Step 4. Conclusions. Therefore the point closest to the origin lying on the plane $4x + 9y = 12$ is the point $(12/5, 4/15)$.

22. Show that the point (x_0, y_0) closest to the origin on the line $ax + by = c$ has coordinates

$$x_0 = \frac{ac}{a^2 + b^2}, \qquad y_0 = \frac{bc}{a^2 + b^2}$$

SOLUTION We need to minimize the distance $d(x, y) = \sqrt{x^2 + y^2}$ subject to the constraint $g(x, y) = ax + by = c$. Notice that the distance $d(x, y)$ is at a minimum at the same points where the square of the distance $d^2(x, y)$ is at a minimum (since the function u^2 is increasing for $u \geq 0$). Therefore, we may find the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $ax + by = c$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle a, b\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$\langle 2x, 2y\rangle = \lambda \langle a, b\rangle$$

or

$$2x = \lambda a$$
$$2y = \lambda b$$

Step 2. Solve for λ in terms of x and y. The Lagrange equations give

$$\lambda = \frac{2x}{a} \quad \text{and} \quad \lambda = \frac{2y}{b}$$

Step 3. Solve for x and y using the constraint. We equate the two expressions for λ and solve for y in terms of x:

$$\frac{2x}{a} = \frac{2y}{b} \quad \Rightarrow \quad y = \frac{b}{a}x$$

We now substitute $y = \frac{bx}{a}$ in the equation of the constraint $ax + by = c$ and solve for x:

$$ax + b \cdot \frac{b}{a}x = c$$
$$\left(a + \frac{b^2}{a}\right)x = c$$
$$\frac{a^2 + b^2}{a}x = c \quad \Rightarrow \quad x = \frac{ac}{a^2 + b^2}$$

We find y using the relation $y = \frac{bx}{a}$:

$$y = \frac{b}{a} \cdot \frac{ac}{a^2 + b^2} = \frac{bc}{a^2 + b^2}$$

The critical point is thus

$$x_0 = \frac{ac}{a^2 + b^2}, \quad y_0 = \frac{bc}{a^2 + b^2} \tag{1}$$

Step 4. Conclusions. It is clear geometrically that the problem has a minimum value and it does not have a maximum value. Therefore the minimum occurs at the critical point. We conclude that the point closest to the origin on the line $ax + by = c$ is given by (1). To show that the vector $\langle x_0, y_0 \rangle$ is perpendicular to the line, we write the line in vector form as $\langle x - x_0, y - y_0 \rangle \cdot \langle a, b \rangle = 0$. Thus, $\langle a, b \rangle$ is perpendicular to the line. Since $\langle x_0, y_0 \rangle = \frac{c}{a^2+b^2} \langle a, b \rangle$, then $\langle x_0, y_0 \rangle$ is parallel to $\langle a, b \rangle$, and thus also perpendicular to the line.

23. Find the maximum value of $f(x, y) = x^a y^b$ for $x \geq 0$, $y \geq 0$ on the line $x + y = 1$, where $a, b > 0$ are constants.

SOLUTION

Step 1. Write the Lagrange Equations. We must find the maximum value of $f(x, y) = x^a y^b$ under the constraints $g(x, y) = x + y - 1$, $x > 0$, $y > 0$. The gradient vectors are $\nabla f = \left\langle ax^{a-1} y^b, bx^a y^{b-1} \right\rangle$ and $\nabla g = \lambda \langle 1, 1 \rangle$, hence the Lagrange Condition$\nabla f = \lambda \nabla g$ is

$$\left\langle ax^{a-1} y^b, bx^a y^{b-1} \right\rangle = \lambda \langle 1, 1 \rangle$$

We obtain the following equations:

$$\begin{aligned} ax^{a-1} y^b &= \lambda \\ bx^a y^{b-1} &= \lambda \end{aligned} \quad \Rightarrow \quad ax^{a-1} y^b = bx^a y^{b-1}$$

Step 2. Solve for x and y using the constraint. We solve the equation in step 1 for y in terms of x. This gives

$$\begin{aligned} ax^{a-1} y^b &= bx^a y^{b-1} \\ ay &= bx \quad \Rightarrow \quad y = \frac{b}{a} x \end{aligned}$$

We now substitute $y = \frac{b}{a} x$ in the constraint $x + y = 1$ and solve for x:

$$\begin{aligned} x + \frac{b}{a} x &= 1 \\ (a + b)x &= a \quad \Rightarrow \quad x = \frac{a}{a + b} \end{aligned}$$

We find y using the relation $y = \frac{b}{a} x$:

$$y = \frac{b}{a} \cdot \frac{a}{a + b} = \frac{b}{a + b}$$

The critical point is thus

$$\left(\frac{a}{a + b}, \frac{b}{a + b} \right) \tag{1}$$

Step 3. Conclusions. We compute $f(x, y) = x^a y^b$ at the critical point:

$$f\left(\frac{a}{a + b}, \frac{b}{a + b} \right) = \left(\frac{a}{a + b} \right)^a \left(\frac{b}{a + b} \right)^b = \frac{a^a b^b}{(a + b)^{a+b}}$$

Now, since f is continuous on the segment $x + y = 1$, $x \geq 0$, $y \geq 0$, which is a closed and bounded set in R^2, then f has minimum and maximum values on this segment. The minimum value is 0 (obtained at $(0, 1)$ and $(1, 0)$), therefore the critical point (1) corresponds to the maximum value. We conclude that the maximum value of $x^a y^b$ on $x + y = 1$, $x > 0$, $y > 0$ is

$$\frac{a^a b^b}{(a + b)^{a+b}}$$

24. Show that the maximum value of $f(x, y) = x^2y^3$ on the unit circle is $\frac{6}{25}\sqrt{\frac{3}{5}}$.

SOLUTION We must maximize $f(x, y) = x^2y^3$ subject to the constraint $x^2 + y^2 = 1$ (the equation of the unit circle). We will write the constraint equation as $g(x, y) = x^2 + y^2 - 1$.

Step 1. Write the Lagrange equations. The gradient vectors are $\nabla f = \left\langle 2xy^3, 3x^2y^2 \right\rangle$ and $\nabla g = \langle 2x, 2y \rangle$, hence the Lagrange condition, $\nabla f = \lambda \nabla g$ gives the following equations:

$$\left\langle 2xy^3, 3x^2y^2 \right\rangle = \lambda \langle 2x, 2y \rangle$$

or

$$2xy^3 = 2\lambda x \quad \Rightarrow \quad xy^3 = \lambda x, \quad 3x^2y^2 = 2\lambda y$$

Step 2. Solve for λ in terms of x and y. Using the first equation above, we can conclude:

$$xy^3 - \lambda x = 0 \quad \Rightarrow \quad x(y^3 - \lambda) = 0 \quad \Rightarrow \quad x = 0 \text{ or } \lambda = y^3$$

If $x = 0$, then using the constraint, $x^2 + y^2 = 1$ we get $y = \pm 1$.

If $\lambda = y^3$, using the second equation we have

$$3x^2y^2 - 2y^4 = 0 \quad \Rightarrow \quad y^2(3x^2 - 2y^2) = 0 \quad \Rightarrow \quad y = 0 \text{ or } x = \pm\sqrt{\frac{2}{3}}y$$

If $y = 0$, then using the constraint we get $x = \pm 1$.

Using the constraint, $x^2 + y^2 = 1$, for $x = \pm\sqrt{\frac{2}{3}}y$, then

$$\frac{2}{3}y^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = \frac{3}{5} \quad \Rightarrow \quad y = \pm\sqrt{\frac{3}{5}}$$

Since $y = \pm\sqrt{\frac{3}{5}}$, then $x = \pm\sqrt{\frac{2}{5}}$.

Step 3. Now to examine the maximum value of the function $f(x, y) = x^2y^3$:

$$f(0, 1) = 0, \quad f(0, -1) = 0, \quad f(1, 0) = 1, \quad f(-1, 0) = 0$$

$$f\left(\sqrt{\frac{2}{5}}, \sqrt{\frac{3}{5}}\right) = \frac{6}{25}\sqrt{\frac{3}{5}}, \quad f\left(-\sqrt{\frac{2}{5}}, \sqrt{\frac{3}{5}}\right) = \frac{6}{25}\sqrt{\frac{3}{5}}$$

$$f\left(\sqrt{\frac{2}{5}}, -\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}}, \quad f\left(-\sqrt{\frac{2}{5}}, -\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}}$$

Step 4. Conclusions. From the analyzing above in Step 3, we see that the maximum value for $f(x, y) = x^2y^3$ on the unit circle is $\frac{6}{25}\sqrt{\frac{3}{5}}$.

25. Find the maximum value of $f(x, y) = x^a y^b$ for $x \geq 0$, $y \geq 0$ on the unit circle, where $a, b > 0$ are constants.

SOLUTION We must find the maximum value of $f(x, y) = x^a y^b$ $(a, b > 0)$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \left\langle ax^{a-1}y^b, bx^a y^{b-1} \right\rangle$ and $\nabla g = \langle 2x, 2y \rangle$. Therefore the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$\left\langle ax^{a-1}y^b, bx^a y^{b-1} \right\rangle = \lambda \langle 2x, 2y \rangle$$

or

$$\begin{aligned} ax^{a-1}y^b &= 2\lambda x \\ bx^a y^{b-1} &= 2\lambda y \end{aligned} \tag{1}$$

Step 2. Solve for λ in terms of x and y. If $x = 0$ or $y = 0$, f has the minimum value 0. We thus may assume that $x > 0$ and $y > 0$. The equations (1) imply that

$$\lambda = \frac{ax^{a-2}y^b}{2}, \quad \lambda = \frac{bx^a y^{b-2}}{2}$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ and solving for y in terms of x gives

$$\begin{aligned}\frac{ax^{a-2}y^b}{2} &= \frac{bx^ay^{b-2}}{2}\\ ax^{a-2}y^b &= bx^ay^{b-2}\\ ay^2 &= bx^2\\ y^2 &= \frac{b}{a}x^2 \quad\Rightarrow\quad y=\sqrt{\frac{b}{a}}x\end{aligned}$$

We now substitute $y=\sqrt{\frac{b}{a}}x$ in the constraint $x^2+y^2=1$ and solve for $x>0$. We obtain

$$\begin{aligned}x^2+\frac{b}{a}x^2 &= 1\\ (a+b)x^2 &= a\\ x^2 &= \frac{a}{a+b} \quad\Rightarrow\quad x=\sqrt{\frac{a}{a+b}}\end{aligned}$$

We find y using the relation $y=\sqrt{\frac{b}{a}}x$:

$$y=\sqrt{\frac{b}{a}}\sqrt{\frac{a}{a+b}}=\sqrt{\frac{ab}{a(a+b)}}=\sqrt{\frac{b}{a+b}}$$

We obtain the critical point:

$$\left(\sqrt{\frac{a}{a+b}},\sqrt{\frac{b}{a+b}}\right)$$

Extreme points can also occur where $\nabla g=\mathbf{0}$, that is, $\langle 2x,2y\rangle=\langle 0,0\rangle$ or $(x,y)=(0,0)$. However, this point is not on the constraint.

Step 4. Conclusions. We compute $f(x,y)=x^ay^b$ at the critical point:

$$f\left(\sqrt{\frac{a}{a+b}},\sqrt{\frac{b}{a+b}}\right)=\left(\frac{a}{a+b}\right)^{a/2}\left(\frac{b}{a+b}\right)^{b/2}=\frac{a^{a/2}b^{b/2}}{(a+b)^{(a+b)/2}}=\sqrt{\frac{a^ab^b}{(a+b)^{a+b}}}$$

The function $f(x,y)=x^ay^b$ is continuous on the set $x^2+y^2=1$, $x\ge 0$, $y\ge 0$, which is a closed and bounded set in R^2, hence f has minimum and maximum values on the set. The minimum value is 0 (obtained at $(0,1)$ and $(1,0)$), hence the critical point that we found corresponds to the maximum value. We conclude that the maximum value of x^ay^b on $x^2+y^2=1$, $x>0$, $y>0$ is

$$\sqrt{\frac{a^ab^b}{(a+b)^{a+b}}}.$$

26. Find the maximum value of $f(x,y,z)=x^ay^bz^c$ for $x,y,z\ge 0$ on the unit sphere, where $a,b,c>0$ are constants.

SOLUTION We must find the maximum value of $f(x,y,z)=x^ay^bz^c$ subject to the constraint $g(x,y,z)=x^2+y^2+z^2-1=0$, $x\ge 0$, $y\ge 0$, $z\ge 0$.

Step 1. Write the Lagrange Equations. The gradient vectors are $\nabla f=\left\langle ax^{a-1}y^bz^c, by^{b-1}x^az^c, cz^{c-1}x^ay^b\right\rangle$ and $\nabla g=\langle 2x,2y,2z\rangle$, hence the Lagrange Condition $\nabla f=\lambda\nabla g$ gives the following equations:

$$\begin{aligned}ax^{a-1}y^bz^c &= \lambda(2x)\\ by^{b-1}x^az^c &= \lambda(2y)\\ cz^{c-1}x^ay^b &= \lambda(2z)\end{aligned}\tag{1}$$

Step 2. Solve for λ in terms of x, y, and z. If $x=0$, $y=0$, or $z=0$, f attains the minimum value 0, therefore we may assume that $x\ne 0$, $y\ne 0$, and $z\ne 0$. The Lagrange equations (1) give

$$\lambda=\frac{ax^{a-2}y^bz^c}{2},\quad \lambda=\frac{by^{b-2}x^az^c}{2},\quad \lambda=\frac{cz^{c-2}x^ay^b}{2}$$

Step 3. Solve for x, y, and z using the constraint. Equating the expressions for λ, we obtain the following equations:

$$\begin{aligned} ax^{a-2}y^bz^c &= by^{b-2}x^az^c \\ ax^{a-2}y^bz^c &= cz^{c-2}x^ay^b \end{aligned} \tag{2}$$

We solve for x and y in terms of z. We first divide the first equation by the second equation to obtain

$$\begin{aligned} 1 &= \frac{by^{b-2}x^az^c}{cz^{c-2}x^ay^b} = \frac{b}{c}\frac{z^2}{y^2} \\ y^2 &= \frac{b}{c}z^2 \quad \Rightarrow \quad y = \sqrt{\frac{b}{c}}z \end{aligned} \tag{3}$$

Both equations (2) imply that

$$\begin{aligned} by^{b-2}x^az^c &= ax^{a-2}y^bz^c \\ by^{b-2}x^az^c &= cz^{c-2}x^ay^b \end{aligned}$$

Dividing the first equation by the second equation gives

$$\begin{aligned} 1 &= \frac{ax^{a-2}y^bz^c}{cz^{c-2}x^ay^b} = \frac{a}{c}\frac{z^2}{x^2} \\ x^2 &= \frac{a}{c}z^2 \quad \Rightarrow \quad x = \sqrt{\frac{a}{c}}z \end{aligned} \tag{4}$$

We now substitute x and y from (3) and (4) in the constraint $x^2 + y^2 + z^2 = 1$ and solve for z. This gives

$$\begin{aligned} \left(\sqrt{\frac{a}{c}}z\right)^2 + \left(\sqrt{\frac{b}{c}}z\right)^2 + z^2 &= 1 \\ \left(\frac{a}{c} + \frac{b}{c} + 1\right)z^2 &= 1 \\ \frac{a+b+c}{c}z^2 &= 1 \quad \Rightarrow \quad z = \sqrt{\frac{c}{a+b+c}} \end{aligned}$$

We find x and y using (4) and (3):

$$\begin{aligned} x &= \sqrt{\frac{a}{c}}\sqrt{\frac{c}{a+b+c}} = \sqrt{\frac{ac}{c(a+b+c)}} = \sqrt{\frac{a}{a+b+c}} \\ y &= \sqrt{\frac{b}{c}}\sqrt{\frac{c}{a+b+c}} = \sqrt{\frac{bc}{c(a+b+c)}} = \sqrt{\frac{b}{a+b+c}} \end{aligned}$$

We obtain the critical point:

$$P = \left(\sqrt{\frac{a}{a+b+c}}, \sqrt{\frac{b}{a+b+c}}, \sqrt{\frac{c}{a+b+c}}\right)$$

We examine the point where $\nabla g = \langle 2x, 2y, 2z\rangle = \langle 0, 0, 0\rangle$, that is, $(0, 0, 0)$: This point does not lie on the constraint, hence it is not a critical point.

Step 4. Conclusions. We compute $f(x, y, z) = x^ay^bz^c$ at the critical point:

$$f(P) = \left(\sqrt{\frac{a}{a+b+c}}\right)^a \left(\sqrt{\frac{b}{a+b+c}}\right)^b \left(\sqrt{\frac{c}{a+b+c}}\right)^c = \sqrt{\frac{a^ab^bc^c}{(a+b+c)^{a+b+c}}}$$

Now, $f(x, y, z) = x^ay^bz^c$ is continuous on the set $x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, which is closed and bounded in R^3. The minimum value is 0 (obtained at the points with at least one zero coordinate), therefore the critical point that we found corresponds to the maximum value. We conclude that the maximum value of $x^ay^bz^c$ subject to the constraint $x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$ is

$$\sqrt{\frac{a^ab^bc^c}{(a+b+c)^{a+b+c}}}$$

27. Show that the minimum distance from the origin to a point on the plane $ax + by + cz = d$ is

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

SOLUTION We want to minimize the distance $P = \sqrt{x^2 + y^2 + z^2}$ subject to $ax + by + cz = d$. Since the square function u^2 is increasing for $u \geq 0$, the square P^2 attains its minimum at the same point where the distance P attains its minimum. Thus, we may minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = ax + by + cz = d$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle 2x, 2y, 2z \rangle$ and $\nabla g = \langle a, b, c \rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives the following equations:

$$2x = \lambda a$$
$$2y = \lambda b$$
$$2z = \lambda c$$

Assume for now that $a \neq 0$, $b \neq 0$, $c \neq 0$.

Step 2. Solve for λ in terms of x, y, and z. The Lagrange Equations imply that

$$\lambda = \frac{2x}{a}, \quad \lambda = \frac{2y}{b}, \quad \lambda = \frac{2z}{c}$$

Step 3. Solve for x, y, and z using the constraint. Equating the expressions for λ give the following equations:

$$\begin{array}{c} \dfrac{2x}{a} = \dfrac{2z}{c} \\ \dfrac{2y}{b} = \dfrac{2z}{c} \end{array} \quad \Rightarrow \quad \begin{array}{c} x = \dfrac{a}{c} z \\ y = \dfrac{b}{c} z \end{array} \tag{1}$$

We now substitute $x = \frac{a}{c} z$ and $y = \frac{b}{c} z$ in the equation of the constraint $ax + by + cz = d$ and solve for z. This gives

$$a \left(\frac{a}{c} z \right) + b \left(\frac{b}{c} z \right) + cz = d$$
$$\frac{a^2}{c} z + \frac{b^2}{c} z + cz = d$$
$$\left(a^2 + b^2 + c^2 \right) z = dc$$

Since $a^2 + b^2 + c^2 \neq 0$, we get $z = \frac{dc}{a^2+b^2+c^2}$. We now use (1) to compute y and x:

$$x = \frac{a}{c} \cdot \frac{dc}{a^2 + b^2 + c^2} = \frac{ad}{a^2 + b^2 + c^2}, \quad y = \frac{b}{c} \cdot \frac{dc}{a^2 + b^2 + c^2} = \frac{bd}{a^2 + b^2 + c^2}$$

We obtain the critical point:

$$P = \left(\frac{ad}{a^2 + b^2 + c^2}, \frac{bd}{a^2 + b^2 + c^2}, \frac{dc}{a^2 + b^2 + c^2} \right) \tag{2}$$

Step 4. Conclusions. It is clear geometrically that f has a minimum value subject to the constraint, hence the minimum value occurs at the point P. We conclude that the point P is the point on the plane closest to the origin. We now consider the case where $a = 0$. We consider the planes $ax + by + cz = d$, where $a \neq 0$ and $a \to 0$. A continuous change in a causes a continuous change in the closest point P. Therefore, the point P closest to the origin in case of $a = 0$ can be obtained by computing the limit of P in (2) as $a \to 0$, that is, by substituting $a = 0$. Similar considerations hold for $b = 0$ or $c = 0$. We conclude that the closest point P in (2) holds also for the planes with $a = 0$, $b = 0$, or $c = 0$ (but not all of them 0). The distance P of that point to the origin is

$$P = \sqrt{\frac{(ad)^2 + (bd)^2 + (dc)^2}{\left(a^2 + b^2 + c^2\right)^2}} = |d| \sqrt{\frac{a^2 + b^2 + c^2}{\left(a^2 + b^2 + c^2\right)^2}} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

28. Antonio has \$5.00 to spend on a lunch consisting of hamburgers (\$1.50 each) and French fries (\$1.00 per order). Antonio's satisfaction from eating x_1 hamburgers and x_2 orders of French fries is measured by a function $U(x_1, x_2) = \sqrt{x_1 x_2}$. How much of each type of food should he purchase to maximize his satisfaction? (Assume that fractional amounts of each food can be purchased.)

SOLUTION Antonio has \$5.00 to spend on the lunch, hence the total cost $1.5x_1 + x_2$ must satisfy

$$1.5x_1 + x_2 = 5$$

We thus want to maximize the function $U(x_1, x_2) = \sqrt{x_1 x_2}$ subject to the constraint $g(x, y) = 1.5x_1 + x_2 = 5$ with $x_1 > 0, x_2 > 0$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla U = \frac{1}{2}\left\langle \sqrt{\frac{x_2}{x_1}}, \sqrt{\frac{x_1}{x_2}} \right\rangle$ and $\nabla g = \langle 1.5, 1 \rangle$, hence the Lagrange Condition $\nabla U = \lambda \nabla g$ gives the following equations:

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{x_2}{x_1}} &= 1.5\lambda \\ \frac{1}{2}\sqrt{\frac{x_1}{x_2}} &= \lambda \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{x_2}{x_1} &= 9\lambda^2 \\ \frac{x_1}{x_2} &= 4\lambda^2 \end{aligned}$$

Step 2. Solve for x_1 and x_2 using the constraint. The two equations in step 1 give

$$\lambda^2 = \frac{x_2}{9x_1} = \frac{x_1}{4x_2}$$

Therefore,

$$\begin{aligned} 4x_2^2 &= 9x_1^2 \\ x_2^2 &= \frac{9}{4}x_1^2 \quad \Rightarrow \quad x_2 = \frac{3}{2}x_1 \end{aligned}$$

We now substitute $x_2 = \frac{3}{2}x_1$ in the constraint $1.5x_1 + x_2 = 5$ and solve for x_1. We get

$$\begin{aligned} 1.5x_1 + \frac{3}{2}x_1 &= 5 \\ 3x_1 &= 5 \quad \Rightarrow \quad x_1 = \frac{5}{3} \end{aligned}$$

We find x_2 by the relation $x_2 = \frac{3}{2}x_1$:

$$x_2 = \frac{3}{2} \cdot \frac{5}{3} = \frac{5}{2}$$

We obtain the critical point:

$$\left(\frac{5}{3}, \frac{5}{2} \right)$$

Step 3. Conclusions. We conclude that Antonio should have $\frac{5}{3}$ hamburgers and $\frac{5}{2}$ orders of fries, to maximize his satisfaction. Notice that $U(x_1, x_2) = \sqrt{x_1 x_2}$ is continuous on the set $1.5x_1 + x_2 = 5$, $x_1 \geq 0$, $x_2 \geq 0$, which is closed and bounded in R^2 (it is a triangle in the first quadrant). f has minimum and maximum values on this set. The minimum value 0 is obtained for $x_1 = 0$ or $x_2 = 0$, hence the critical point that we found corresponds to the maximum value.

29. Let Q be the point on an ellipse closest to a given point P outside the ellipse. It was known to the Greek mathematician Apollonius (third century BCE) that $\overline{PQ}$ is perpendicular to the tangent to the ellipse at Q (Figure 15). Explain in words why this conclusion is a consequence of the method of Lagrange multipliers. *Hint:* The circles centered at P are level curves of the function to be minimized.

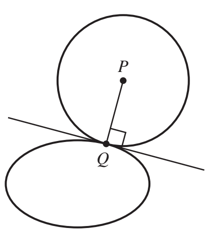


FIGURE 15

SOLUTION Let $P = (x_0, y_0)$. The distance d between the point P and a point $Q = (x, y)$ on the ellipse is minimum where the square d^2 is minimum (since the square function u^2 is increasing for $u \geq 0$). Therefore, we want to minimize the function

$$f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

subject to the constraint

$$g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The method of Lagrange indicates that the solution Q is the point on the ellipse where $\nabla f = \lambda \nabla g$, that is, the point on the ellipse where the gradients ∇f and ∇g are parallel. Since the gradient is orthogonal to the level curves of the function, ∇g is orthogonal to the ellipse $g(x, y) = 1$, and ∇f is orthogonal to the level curve of f passing through Q. But this level curve is a circle through Q centered at P, hence the parallel vectors ∇g and ∇f are orthogonal to the ellipse and to the circle centered at P respectively. We conclude that the point Q is the point at which the tangent to the ellipse is also the tangent to the circle through Q centered at P. That is, the tangent to the ellipse at Q is perpendicular to the radius $\overline{PQ}$ of the circle.

30. In a contest, a runner starting at A must touch a point P along a river and then run to B in the shortest time possible (Figure 16). The runner should choose the point P that minimizes the total length of the path.

(a) Define a function

$$f(x, y) = AP + PB, \quad \text{where } P = (x, y)$$

Rephrase the runner's problem as a constrained optimization problem, assuming that the river is given by an equation $g(x, y) = 0$.

(b) Explain why the level curves of $f(x, y)$ are ellipses.

(c) Use Lagrange multipliers to justify the following statement: The ellipse through the point P minimizing the length of the path is tangent to the river.

(d) Identify the point on the river in Figure 16 for which the length is minimal.

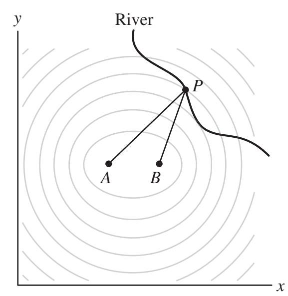
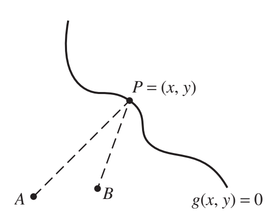


FIGURE 16

SOLUTION

(a) Let A and B be the points $A = (a, b)$ and $B = (c, d)$.



By the Length Formula we have

$$\overline{AP} = \sqrt{(x-a)^2 + (y-b)^2}$$

$$\overline{PB} = \sqrt{(x-c)^2 + (y-d)^2}$$

The distance traveled by the runner is

$$f(x, y) = \sqrt{(x-a)^2 + (y-b)^2} + \sqrt{(x-c)^2 + (y-d)^2}$$

We must minimize the function f subject to the constraint $g(x, y) = 0$ (since the point $P = (x, y)$ must satisfy the equation of the river).

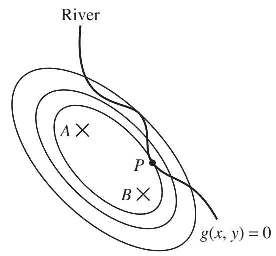
(b) The level curves of $f(x, y)$ are $f(x, y) = k$ for positive constants k. That is,

$$\sqrt{(x-a)^2 + (y-b)^2} + \sqrt{(x-c)^2 + (y-d)^2} = k$$

The level curve consists of all the points $P = (x, y)$ such that the sum of the distances to the two fixed points $A = (a, b)$ and $B = (c, d)$ is constant $k > 0$. Therefore the level curves are ellipses with foci at A and B.

(c) The point P that minimizes the length of the path must satisfy the Lagrange Condition $\nabla f_P = \lambda \nabla g_P$. That is, the gradients of f and g are parallel vectors. Since the gradient at P is orthogonal to the level curve of the function passing through P, the level curve of f through P (which is the ellipse through P) is tangent to the level curve of g through P, that is, it is tangent to the river.

(d) The path-minimizing point is the point closest to the line through A and B such that the ellipse through P is tangent to the river.



In Exercises 31 and 32, let V be the volume of a can of radius r and height h, and let S be its surface area (including the top and bottom).

31. Find r and h that minimize S subject to the constraint $V = 54\pi$.

SOLUTION We see that the surface area of the can is $S = 2\pi rh + 2\pi r^2$ subject to $V = 54\pi = \pi r^2 h$. Let us write the constraint as $V(r, h) = \pi r^2 h - 54\pi$ and use Lagrange Multipliers to solve.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla S = \langle 2\pi h + 4\pi r, 2\pi r\rangle$ and $\nabla V = \left\langle 2\pi rh, \pi r^2\right\rangle$. Then using $\nabla S = \lambda \nabla V$, we see

$$\langle 2\pi h + 4\pi r, 2\pi r\rangle = \lambda \left\langle 2\pi rh, \pi r^2\right\rangle$$

or

$$2\pi h + 4\pi r = 2\pi\lambda rh, \quad 2\pi r = \lambda\pi r^2$$

Consider the second equation, rewriting we have:

$$2\pi r - \lambda\pi r^2 = 0 \quad \Rightarrow \quad \pi r(2 - \lambda r) = 0 \quad \Rightarrow \quad r = 0, \lambda = \frac{2}{r}$$

We can ignore when $r = 0$ since it does not correspond to any point on the constraint curve $54\pi = \pi r^2 h$.

Using the first equation, rewriting we have:

$$2\pi h + 4\pi r = 2\pi\lambda rh \quad \Rightarrow \quad \lambda = \frac{2\pi h + 4\pi r}{2\pi rh} = \frac{h + 2r}{rh}$$

Step 2. Solve for r, h using the constraint to determine the critical point.

Using the two derived equations for λ we have:

$$\frac{2}{r} = \frac{h + 2r}{rh} \quad \Rightarrow \quad 2rh = hr + 2r^2 \quad r(2h - h - 2r) = 0 \quad \Rightarrow \quad h = 2r$$

Then using the constraint, $54\pi = \pi r^2 h$ we see:

$$54\pi = \pi r^2(2r) \quad \Rightarrow \quad 54 = 2r^3 \quad \Rightarrow \quad r^3 = 27 \quad \Rightarrow \quad r = 3$$

Thus $r = 3$ and $h = 2(3) = 6$.

Step 3. Conclusions. The minimum surface area, given that the volume must be 54π is determined by a can having radius $r = 3$ and height $h = 6$. We know this is the minimum surface area because surface area is an increasing function of r and h.

32. Show that for both of the following two problems, $P = (r, h)$ is a Lagrange critical point if $h = 2r$:

- Minimize surface area S for fixed volume V.
- Maximize volume V for fixed surface area S.

Then use the contour plots in Figure 17 to explain why S has a minimum for fixed V but no maximum and, similarly, V has a maximum for fixed S but no minimum.

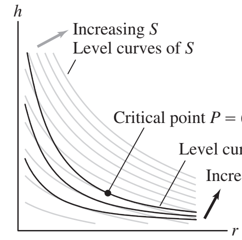


FIGURE 17

SOLUTION

- To minimize surface area $S = 2\pi rh + 2\pi r^2$ for a fixed volume (subject to the constraint $c(r, h) = \pi r^2 h - V$) we use the Lagrange equations. Then using $\nabla S = \lambda \nabla c$, we see

$$\langle 2\pi h + 4\pi r, 2\pi r\rangle = \lambda \left\langle 2\pi rh, \pi r^2 \right\rangle$$

or

$$2\pi h + 4\pi r = 2\pi \lambda rh, \quad 2\pi r = \lambda \pi r^2$$

Consider the second equation, rewriting we have:

$$2\pi r - \lambda \pi r^2 = 0 \quad \Rightarrow \quad \pi r(2 - \lambda r) = 0 \quad \Rightarrow \quad r = 0, \lambda = \frac{2}{r}$$

Since this is a question about surface area, we are not interested in the point when $r = 0$.

Using the first equation, rewriting we have:

$$2\pi h + 4\pi r = 2\pi \lambda rh \quad \Rightarrow \quad \lambda = \frac{2\pi h + 4\pi r}{2\pi rh} = \frac{h + 2r}{rh}$$

Now to solve for r, h using the constraint to determine the critical point. Using the two derived equations for λ we have:

$$\frac{2}{r} = \frac{h + 2r}{rh} \quad \Rightarrow \quad 2rh = hr + 2r^2 \quad r(2h - h - 2r) = 0 \quad \Rightarrow \quad h = 2r$$

Therefore, we see that the critical point is (r, h) where $h = 2r$.

- To maximize the volume $V = \pi r^2 h$ for a fixed surface area (subject to the constraint $c(r, h) = 2\pi rh + 2\pi r^2 - S$) we use the Lagrange equations. Then using $\nabla V = \lambda \nabla c$ we see

$$\left\langle 2\pi rh, \pi r^2 \right\rangle = \lambda \langle 2\pi h + 4\pi r, 2\pi r\rangle$$

or

$$\lambda(2\pi h + 4\pi r) = 2\pi rh, \quad \lambda(2\pi r) = \pi r^2$$

$$\lambda = \frac{rh}{h + 2r}, \quad \lambda = \frac{r}{2}$$

Using these two derived equations for λ, we have:

$$\frac{r}{2} = \frac{rh}{h + 2r} \quad \Rightarrow \quad h = 2r$$

Therefore, we see that the critical point is (r, h) where $h = 2r$.

Using the contour plots in the figure, we can see that S has a minimum for a fixed value of V, but no maximum because it increases without an upper bound, whereas has V has a maximum for a fixed value of S, but no minimum because it decreases without a lower bound.

33. A plane with equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(a, b, c > 0)$ together with the positive coordinate planes forms a tetrahedron of volume $V = \frac{1}{6}abc$ (Figure 18). Find the minimum value of V among all planes passing through the point $P = (1, 1, 1)$.

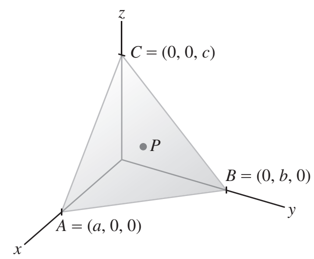


FIGURE 18

SOLUTION The plane is constrained to pass through the point $P = (1, 1, 1)$, hence this point must satisfy the equation of the plane. That is,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

We thus must minimize the function $V(a, b, c) = \frac{1}{6}abc$ subject to the constraint $g(a, b, c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, $a > 0$, $b > 0$, $c > 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla V = \left\langle \frac{1}{6}bc, \frac{1}{6}ac, \frac{1}{6}ab \right\rangle$ and $\nabla g = \left\langle -\frac{1}{a^2}, -\frac{1}{b^2}, -\frac{1}{c^2} \right\rangle$, hence the Lagrange Condition $\nabla V = \lambda \nabla g$ yields the following equations:

$$\frac{1}{6}bc = -\frac{1}{a^2}\lambda$$
$$\frac{1}{6}ac = -\frac{1}{b^2}\lambda$$
$$\frac{1}{6}ab = -\frac{1}{c^2}\lambda$$

Step 2. Solve for λ in terms of a, b, and c. The Lagrange equations imply that

$$\lambda = -\frac{bca^2}{6}, \quad \lambda = -\frac{acb^2}{6}, \quad \lambda = -\frac{abc^2}{6}$$

Step 3. Solve for a, b, and c using the constraint. Equating the expressions for λ, we obtain the following equations:

$$\begin{aligned} bca^2 &= acb^2 \\ abc^2 &= acb^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} abc(a-b) &= 0 \\ abc(c-b) &= 0 \end{aligned}$$

Since a, b, c are positive numbers, we conclude that $a = b$ and $c = b$. We now substitute $a = b$ and $c = b$ in the equation of the constraint $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ and solve for b. This gives

$$\frac{1}{b} + \frac{1}{b} + \frac{1}{b} = 1$$
$$\frac{3}{b} = 1 \quad \Rightarrow \quad b = 3$$

Therefore also $a = b = 3$ and $c = b = 3$. We obtain the critical point $(3, 3, 3)$.

Step 4. Conclusions. If V has a minimum value subject to the constraint then it occurs at the point $(3, 3, 3)$. That is, the plane that minimizes V is

$$\frac{x}{3} + \frac{y}{3} + \frac{z}{3} = 1 \quad \text{or} \quad x + y + z = 3$$

Remark: Since the constraint is not bounded, we need to justify the existence of a minimum value of $V = \frac{1}{6}abc$ under the constraint $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. First notice that since a, b, c are nonnegative and the sum of their reciprocals is 1, none of them can tend to zero. In fact, none of a, b, c can be less than 1. Therefore, if $a \to \infty$, $b \to \infty$, or $c \to \infty$, then $V \to \infty$. This means that we can find a cube that includes the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ such that, on the part of the constraint that is outside the cube, it holds that $V > V\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{162}$. On the part of the constraint inside the cube, V has a minimum value m, since it is a closed and bounded set. Clearly m is the minimum of V on the whole constraint.

34. With the same set-up as in the previous problem, find the plane that minimizes V if the plane is constrained to pass through a point $P = (\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma > 0$.

SOLUTION The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ must pass through the point $P(\alpha, \beta, \gamma)$, hence

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$$

We thus must minimize the function $V(a, b, c) = \frac{1}{6}abc$ subject to the constant $g(a, b, c) = \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$, $a > 0$, $b > 0$, $c > 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla V = \left\langle \frac{1}{6}bc, \frac{1}{6}ac, \frac{1}{6}ab \right\rangle$ and $\nabla g = \left\langle -\frac{\alpha}{a^2}, -\frac{\beta}{b^2}, -\frac{\gamma}{c^2} \right\rangle$, hence the Lagrange Condition $\nabla V = \lambda \nabla g$ yields the following equations:

$$\begin{aligned} \frac{1}{6}bc &= -\frac{\alpha}{a^2}\lambda & & \lambda = -\frac{a^2bc}{6\alpha} \\ \frac{1}{6}ac &= -\frac{\beta}{b^2}\lambda & \Rightarrow \quad & \lambda = -\frac{b^2ac}{6\beta} \\ \frac{1}{6}ab &= -\frac{\gamma}{c^2}\lambda & & \lambda = -\frac{c^2ab}{6\gamma} \end{aligned}$$

Step 2. Solve for a, b, c using the constraint. The Lagrange equations imply the following equations:

$$\frac{a^2bc}{\alpha} = \frac{c^2ab}{\gamma}$$
$$\frac{b^2ac}{\beta} = \frac{c^2ab}{\gamma}$$

We simplify the two equations to obtain

$$abc(\gamma a - \alpha c) = 0$$
$$abc(\gamma b - \beta c) = 0$$

Since $abc \neq 0$, these equations imply that

$$\begin{aligned} \gamma a - \alpha c = 0 \quad &\Rightarrow \quad a = \frac{\alpha}{\gamma}c \\ \gamma b - \beta c = 0 \quad &\Rightarrow \quad b = \frac{\beta}{\gamma}c \end{aligned} \tag{1}$$

We now substitute in the constraint $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$ and solve for c. This gives

$$\begin{aligned} \frac{\alpha}{\frac{\alpha}{\gamma}c} + \frac{\beta}{\frac{\beta}{\gamma}c} + \frac{\gamma}{c} &= 1 \\ \frac{\gamma}{c} + \frac{\gamma}{c} + \frac{\gamma}{c} &= 1 \\ \frac{3\gamma}{c} &= 1 \quad \Rightarrow \quad c = 3\gamma \end{aligned}$$

We find a and b using (1):

$$a = \frac{\alpha}{\gamma} \cdot 3\gamma = 3\alpha, \quad b = \frac{\beta}{\gamma} \cdot 3\gamma = 3\beta$$

We obtain the solution

$$P = (3\alpha, 3\beta, 3\gamma)$$

Step 3. Conclusions. Since V has a minimum value subject to the constraint, it occurs at the critical point. We substitute $a = 3\alpha$, $b = 3\beta$, and $c = 3\gamma$ in the equation of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ to obtain the following plane, which minimizes V:

$$\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1 \quad \text{or} \quad \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

35. Show that the Lagrange equations for $f(x, y) = x + y$ subject to the constraint $g(x, y) = x + 2y = 0$ have no solution. What can you conclude about the minimum and maximum values of f subject to $g = 0$? Show this directly.

SOLUTION Using the methods of Lagrange we can write $\nabla f = \lambda \nabla g$ and see

$$\langle 1, 1 \rangle = \lambda \langle 1, 2 \rangle$$

Which gives us the equations:

$$1 = \lambda, \quad 1 = 2\lambda$$

hence, $\lambda = 1$ or $\lambda = 1/2$. This is an inconsistent set of equations, thus the Lagrange method has no solution. What we can conclude from this is that the maximum and minimum values of f subject to $g = 0$ does not exist. This means that $f(x, y)$ increases without an upper bound and decreases without a lower bound.

To show this directly, we can write $y = -1/2x$ from the constraint equation and substitute it into $f(x, y) = f(x, -1/2x) = x - 1/2x = 1/2x$. We know that $y = 1/2x$ is a straight line having slope $1/2$, increasing, with no maximum nor minimum values.

36. Show that the Lagrange equations for $f(x, y) = 2x + y$ subject to the constraint $g(x, y) = x^2 - y^2 = 1$ have a solution but that f has no min or max on the constraint curve. Does this contradict Theorem 1?

SOLUTION Using the methods of Lagrange we can write $\nabla f = \lambda \nabla g$ and see

$$\langle 2, 1 \rangle = \lambda \langle 2x, -2y \rangle$$

or

$$2 = 2\lambda x, \quad 1 = -2\lambda y$$

and

$$\lambda x = 1, \quad 1 = -2\lambda y$$

hence

$$\lambda x = -2\lambda y \quad \Rightarrow \quad \lambda x + 2\lambda y = 0 \quad \Rightarrow \quad \lambda(x + 2y) = 0$$

Hence $\lambda = 0$ or $x = -2y$. But we see if $\lambda = 0$ above, we get an inconsistent equation, therefore $x = -2y$. Using the constraint equation we see

$$(-2y)^2 - y^2 = 1 \quad \Rightarrow \quad 4y^2 - y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{3}}, x = \mp\frac{2}{\sqrt{3}}$$

Evaluating at these points we see

$$f\left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \sqrt{3}, \quad f\left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\sqrt{3}$$

Now, to show that $f(x, y)$ has no min or max on the constraint curve.

The point $(x, y) = (x, \sqrt{x^2 - 1})$ lies on the constraint for all $x \ge 1$. Consider the following:

$$\lim_{x\to\infty} f(x, y) = \lim_{x\to\infty} f(x, \sqrt{x^2 - 1}) = \lim_{x\to\infty} 2x + \sqrt{x^2 + 1} \to \infty$$

However, the point $(-x, y) = (-x, -\sqrt{x^2 - 1})$ also lies on the constraint curve, and

$$\lim_{x\to\infty} f(x, y) = \lim_{x\to\infty} f(-x, -\sqrt{x^2 - 1}) = \lim_{x\to\infty} -2x - \sqrt{x^2 - 1} \to -\infty$$

Therefore, $f(x, y)$ has no min nor max on the constraint curve.

These calculations do not contradict the Lagrange theorem in the text because the theorem says only that the extrema (if they exist) must satisfy the Lagrange equations.

37. Let L be the minimum length of a ladder that can reach over a fence of height h to a wall located a distance b behind the wall.

(a) Use Lagrange multipliers to show that $L = (h^{2/3} + b^{2/3})^{3/2}$ (Figure 19). *Hint:* Show that the problem amounts to minimizing $f(x, y) = (x + b)^2 + (y + h)^2$ subject to $y/b = h/x$ or $xy = bh$.

(b) Show that the value of L is also equal to the radius of the circle with center $(-b, -h)$ that is tangent to the graph of $xy = bh$.

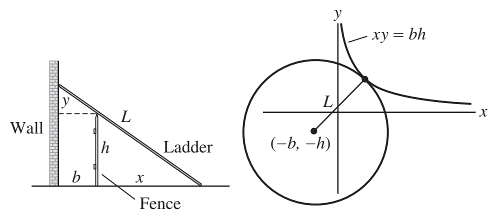
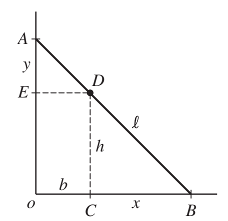


FIGURE 19

SOLUTION

(a) We denote by x and y the lengths shown in the figure, and express the length l of the ladder in terms of x and y.



Using the Pythagorean Theorem, we have

$$l = \sqrt{\overline{OA}^2 + \overline{OB}^2} = \sqrt{(y+h)^2 + (x+b)^2} \tag{1}$$

Since the function u^2 is increasing for $u \geq 0$, l and l^2 have their minimum values at the same point. Therefore, we may minimize the function $f(x, y) = l^2(x, y)$, which is

$$f(x, y) = (x+b)^2 + (y+h)^2$$

We now identify the constraint on the variables x and y. (Notice that h, b are constants while x and y are free to change). Using proportional lengths in the similar triangles ΔAED and ΔDCB, we have

$$\frac{\overline{AE}}{\overline{DC}} = \frac{\overline{ED}}{\overline{CB}}$$

That is,

$$\frac{y}{h} = \frac{b}{x} \quad \Rightarrow \quad xy = bh$$

We thus must minimize $f(x, y) = (x+b)^2 + (y+h)^2$ subject to the constraint $g(x, y) = xy = bh$, $x > 0$, $y > 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle 2(x+b), 2(y+h) \rangle$ and $\nabla g = \langle y, x \rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives the following equations:

$$2(x+b) = \lambda y$$
$$2(y+h) = \lambda x$$

Step 2. Solve for λ in terms of x and y. The equation of the constraint implies that $y \neq 0$ and $x \neq 0$. Therefore, the Lagrange equations yield

$$\lambda = \frac{2(x+b)}{y}, \quad \lambda = \frac{2(y+h)}{x}$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ gives

$$\frac{2(x+b)}{y} = \frac{2(y+h)}{x}$$

We simplify:

$$x(x+b) = y(y+h)$$
$$x^2 + xb = y^2 + yh$$

The equation of the constraint implies that $y = \frac{bh}{x}$. We substitute and solve for $x > 0$. This gives

$$\begin{aligned} x^2 + xb &= \left(\frac{bh}{x}\right)^2 + \frac{bh}{x} \cdot h \\ x^2 + xb &= \frac{b^2h^2}{x^2} + \frac{bh^2}{x} \\ x^4 + x^3 b &= b^2h^2 + bh^2 x \\ x^4 + bx^3 - bh^2 x - b^2h^2 &= 0 \\ x^3(x+b) - bh^2(x+b) &= 0 \\ \left(x^3 - bh^2\right)(x+b) &= 0 \end{aligned}$$

Since $x > 0$ and $b > 0$, also $x + b > 0$ and the solution is

$$x^3 - bh^2 = 0 \quad \Rightarrow \quad x = (bh^2)^{1/3}$$

We compute y. Using the relation $y = \frac{bh}{x}$,

$$y = \frac{bh}{(bh^2)^{1/3}} = \frac{bh}{b^{1/3}h^{2/3}} = b^{2/3}h^{1/3} = (b^2h)^{1/3}$$

We obtain the solution

$$x = \left(bh^2\right)^{1/3}, \quad y = \left(b^2h\right)^{1/3} \tag{2}$$

Extreme values may also occur at the point on the constraint where $\nabla g = \mathbf{0}$. However, $\nabla g = \langle y, x\rangle = \langle 0, 0\rangle$ only at the point $(0, 0)$, which is not on the constraint.

Step 4. Conclusions. Notice that on the constraint $y = \frac{bh}{x}$ or $x = \frac{bh}{y}$, as $x \to 0+$ then $y \to \infty$, and as $x \to \infty$, then $y \to 0+$. Also, as $y \to 0+$, $x \to \infty$ and as $y \to \infty$, $x \to 0+$. In either case, $f(x, y)$ is increasing without bound. Using this property and the theorem on the existence of extreme values for a continuous function on a closed and bounded set (for a certain part of the constraint), one can show that f has a minimum value on the constraint. This minimum value occurs at the point (2). We substitute this point in (1) to obtain the following minimum length L:

$$\begin{aligned} L &= \sqrt{\left((b^2h)^{1/3} + h\right)^2 + \left((bh^2)^{1/3} + b\right)^2} \\ &= \sqrt{(b^2h)^{2/3} + 2h(b^2h)^{1/3} + h^2 + (bh^2)^{2/3} + 2b(bh^2)^{1/3} + b^2} \\ &= \sqrt{b^{\frac{4}{3}}h^{2/3} + 2h^{\frac{4}{3}}b^{2/3} + h^2 + b^{2/3}h^{\frac{4}{3}} + 2b^{\frac{4}{3}}h^{2/3} + b^2} \\ &= \sqrt{3b^{\frac{4}{3}}h^{2/3} + 3h^{\frac{4}{3}}b^{2/3} + h^2 + b^2} \\ &= \sqrt{\left(h^{2/3}\right)^3 + 3\left(h^{2/3}\right)^2 b^{2/3} + 3h^{2/3}\left(b^{2/3}\right)^2 + \left(b^{2/3}\right)^3} \end{aligned}$$

Using the identity $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$, we conclude that

$$L = \sqrt{\left(h^{2/3} + b^{2/3}\right)^3} = \left(h^{2/3} + b^{2/3}\right)^{3/2}.$$

(b) The Lagrange Condition states that the gradient vectors ∇f_P and ∇g_P are parallel (where P is the minimizing point). The gradient ∇f_P is orthogonal to the level curve of f passing through P, which is a circle through P centered at $(-b, -h)$. ∇g_P is orthogonal to the level curve of g passing through P, which is the curve of the constraint $xy = bh$. We conclude that the circle and the curve $xy = bh$, both being perpendicular to parallel vectors, are tangent at P. The radius of the circle is the minimum value L, of $f(x, y)$.

38. Find the maximum value of $f(x, y, z) = xy + xz + yz - xyz$ subject to the constraint $x + y + z = 1$, for $x \geq 0$, $y \geq 0$, $z \geq 0$.

SOLUTION

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle y + z - 4yz, x + z - 4xz, x + y - 4xy\rangle$ and $\nabla g = \langle 1, 1, 1\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ yields the following equations:

$$\begin{aligned} y + z - 4yz &= \lambda \\ x + z - 4xz &= \lambda \\ x + y - 4xy &= \lambda \end{aligned}$$

Step 2. Solve for x, y, and z using the constraint. The Lagrange equations imply that

$$\begin{aligned} x+z-4xz &= y+z-4yz \\ x+y-4xy &= y+z-4yz \end{aligned} \quad \Rightarrow \quad \begin{aligned} x-4xz &= y-4yz \\ x-4xy &= z-4yz \end{aligned} \tag{1}$$

We solve for x and y in terms of z. The first equation gives

$$\begin{aligned} x-y+4yz-4xz &= 0 \\ x-y-4z(x-y) &= 0 \\ (x-y)(1-4z) &= 0 \quad \Rightarrow \quad x=y \quad \text{or} \quad z=\frac{1}{4} \end{aligned} \tag{2}$$

The second equation in (1) gives:

$$\begin{aligned} x-z+4yz-4xy &= 0 \\ x-z-4y(x-z) &= 0 \\ (x-z)(1-4y) &= 0 \quad \Rightarrow \quad x=z \quad \text{or} \quad y=\frac{1}{4} \end{aligned} \tag{3}$$

We examine the possible solutions.

(1) $x=y$, $x=z$. Substituting $x=y=z$ in the equation of the constraint $x+y+z=1$ gives $3z=1$ or $z=\frac{1}{3}$. We obtain the solution

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

(2) $x=y$, $y=\frac{1}{4}$. Substituting $x=y=\frac{1}{4}$ in the constraint $x+y+z=1$ gives

$$\frac{1}{4}+\frac{1}{4}+z=1 \quad \Rightarrow \quad z=\frac{1}{2}$$

We obtain the solution

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)$$

(3) $z=\frac{1}{4}$, $x=z$. Substituting $z=\frac{1}{4}$, $x=\frac{1}{4}$ in the constraint gives

$$\frac{1}{4}+y+\frac{1}{4}=1 \quad \Rightarrow \quad y=\frac{1}{2}$$

We get the point

$$\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$$

(4) $z=\frac{1}{4}$, $y=\frac{1}{4}$. Substituting in the constraint gives $x+\frac{1}{4}+\frac{1}{4}=1$ or $x=\frac{1}{2}$. We obtain the point

$$\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$$

We conclude that the critical points are

$$\begin{aligned} P_1 &= \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \quad P_2=\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right) \\ P_3 &= \left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right), \quad P_4=\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) \end{aligned} \tag{4}$$

Step 3. Conclusions. The constraint $x+y+z=1$, $x\ge 0$, $y\ge 0$, $z\ge 0$ is the part of the plane $x+y+z=1$ in the first octant. This is a closed and bounded set in R^3, hence f (which is a continuous function) has minimum and maximum value subject to the constraint. The extreme values occur at points from (4). We evaluate $f(x,y,z)=xy+xz+yz-4xyz$ at these points:

$$\begin{aligned} f(P_1) &= \frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}-4\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{3}{9}-\frac{4}{27}=\frac{5}{27} \\ f(P_2)=f(P_3)=f(P_4) &= \frac{1}{4}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}-4\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2}=\frac{3}{16} \end{aligned}$$

We conclude that the maximum value of f subject to the constraint is

$$f(P_2) = f(P_3) = f(P_4) = \frac{3}{16}.$$

39. Find the point lying on the intersection of the plane $x + \frac{1}{2}y + \frac{1}{4}z = 0$ and the sphere $x^2 + y^2 + z^2 = 9$ with the largest z-coordinate.

SOLUTION We will use the method of Lagrange Multipliers with two constraints here. We want to maximize $f(x, y, z) = z$ subject to the two surfaces. Set the first constraint as $g(x, y, z) = x + \frac{1}{2}y + \frac{1}{4}z = 0$ and the second as $h(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$.

Write out the Lagrange equations. We have $\nabla f = \langle 0, 0, 1\rangle$, $\nabla g = \left\langle 1, \frac{1}{2}, \frac{1}{4}\right\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$, hence the Lagrange condition, $\nabla f = \lambda\nabla g + \mu\nabla h$ yields the following equations:

$$\langle 0, 0, 1\rangle = \lambda\left\langle 1, \frac{1}{2}, \frac{1}{4}\right\rangle + \mu\,\langle 2x, 2y, 2z\rangle$$

and

$$0 = \lambda + 2\mu x, \quad 0 = \frac{1}{2}\lambda + 2\mu y, \quad 1 = \frac{1}{4}\lambda + 2\mu z$$

Hence, from the first two equations we see

$$\lambda = -2\mu x, \quad \lambda = -4\mu y$$

Therefore

$$-2\mu x = -4\mu y \quad \Rightarrow \quad x = 2y$$

since $\mu \neq 0$. Using the first constraint equation $x + \frac{1}{2}y + \frac{1}{4}z = 0$ we have

$$2y + \frac{1}{2}y + \frac{1}{4}z = 0 \quad \Rightarrow \quad \frac{5}{2}y + \frac{1}{4}z = 0 \quad \Rightarrow \quad y = -\frac{1}{10}z$$

Finally, we can substitute $y = -1/10z$ and $x = 2y = -1/5z$ into the second constraint equation $x^2 + y^2 + z^2 = 9$ to see

$$\left(-\frac{1}{5}z\right)^2 + \left(-\frac{1}{10}z\right)^2 + z^2 = 9 \quad \Rightarrow \quad \frac{1}{25}z^2 + \frac{1}{100}z^2 + z^2 = 9 \quad \Rightarrow \quad 4z^2 + z^2 + 100z^2 = 900$$

Hence

$$105z^2 = 900 \quad \Rightarrow \quad z^2 = \frac{900}{105} = \frac{60}{7}$$

Therefore $z = \pm\sqrt{\frac{60}{7}} = \pm 2\sqrt{\frac{15}{7}}$. The two critical points are:

$$P\left(-\frac{2}{5}\sqrt{\frac{15}{7}}, -\frac{1}{5}\sqrt{\frac{15}{7}}, 2\sqrt{\frac{15}{7}}\right), \quad Q\left(\frac{2}{5}\sqrt{\frac{15}{7}}, \frac{1}{5}\sqrt{\frac{15}{7}}, -2\sqrt{\frac{15}{7}}\right)$$

The critical point with the largest z-coordinate (the maximum of $f(x, y, z)$) is P with z-coordinate $2\sqrt{\frac{15}{7}} \approx 2.928$.

40. Find the maximum of $f(x, y, z) = x + y + z$ subject to the two constraints $x^2 + y^2 + z^2 = 9$ and $\frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 = 9$.

SOLUTION We will use the method of Lagrange Multipliers with two constraints here. We want to maximize $f(x, y, z) = x + y + z$ subject to the two constraints. The first constraint is $g(x, y, z) = x^2 + y^2 + z^2 - 9$ and the second, $h(x, y, z) = \frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 - 9$.

Write out the Lagrange equations. We have $\nabla f = \langle 1, 1, 1\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$, and $\nabla h = \left\langle \frac{1}{2}x, \frac{1}{2}y, 8z\right\rangle$. Therefore the Lagrange condition $\nabla f = \lambda\nabla g + \mu\nabla h$ yields the following equation:

$$\langle 1, 1, 1\rangle = \lambda\,\langle 2x, 2y, 2z\rangle + \mu\left\langle \frac{1}{2}x, \frac{1}{2}y, 8z\right\rangle$$

and

$$1 = 2\lambda x + \frac{1}{2}\mu x, \quad 1 = 2\lambda y + \frac{1}{2}\mu y, \quad 1 = 2\lambda z + 8\mu z$$

Using the first two equations and solving for λ we see:

$$\lambda = \frac{1 - \frac{1}{2}\mu x}{2x}, \quad \lambda = \frac{1 - \frac{1}{2}\mu y}{2y}$$

Setting these equal and solving for x and y we see

$$x = y$$

Now using the first constraint equation we have

$$x^2 + y^2 + z^2 = 9 \quad \Rightarrow \quad 2y^2 + z^2 = 9 \quad \Rightarrow \quad z^2 = 9 - 2y^2$$

Next, using the second constraint equation we have

$$\frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 = 9 \quad \Rightarrow \quad \frac{1}{4}y^2 + \frac{1}{4}y^2 + 4(9 - 2y^2) = 9 \quad \Rightarrow \quad \frac{15}{2}y^2 = 27 \quad \Rightarrow \quad y^2 = \frac{18}{5}$$

Therefore we can conclude $y = \pm 3\sqrt{\frac{2}{5}}$ and, since $x = y$, then $x = \pm 3\sqrt{\frac{2}{5}}$. Then also,

$$x^2 + y^2 + z^2 = 9 \quad \Rightarrow \quad \frac{18}{5} + \frac{18}{5} + z^2 = 9 \quad \Rightarrow \quad z^2 = \frac{11}{5}$$

Hence $z = \pm\frac{3}{\sqrt{5}}$. Our critical points are

$$\left(3\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}, \frac{3}{\sqrt{5}}\right), \quad \left(3\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}, -\frac{3}{\sqrt{5}}\right)$$

$$\left(-3\sqrt{\frac{2}{5}}, -3\sqrt{\frac{2}{5}}, \frac{3}{\sqrt{5}}\right), \quad \left(-3\sqrt{\frac{2}{5}}, -3\sqrt{\frac{2}{5}}, -\frac{3}{\sqrt{5}}\right)$$

We must evaluate $f(x, y, z) = x + y + z$ at the four critical points to determine the maximum value. But note since we are interested in the sum of the coordinates, the maximum value is obtained when they are all positive:

$$f\left(3\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}, \frac{3}{\sqrt{5}}\right) \approx 5.136$$

41. The cylinder $x^2 + y^2 = 1$ intersects the plane $x + z = 1$ in an ellipse. Find the point on that ellipse that is farthest from the origin.

SOLUTION We need to use Lagrange Multipliers with two constraints here. We want to maximize the square of the distance from the origin $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x^2 + y^2 - 1$ and $h(x, y, z) = x + z - 1$. Taking the gradients we have $\nabla f = \langle 2x, 2y, 2z\rangle$, $\nabla g = \langle 2x, 2y, 0\rangle$, and $\nabla h = \langle 1, 0, 1\rangle$. Writing the Lagrange condition $\nabla f = \lambda \nabla g + \mu \nabla h$ we have

$$\langle 2x, 2y, 2z\rangle = \lambda \langle 2x, 2y, 0\rangle + \mu \langle 1, 0, 1\rangle$$

and

$$2x = 2\lambda x + \mu, \quad 2y = 2\lambda y, \quad 2z = \mu$$

Using the second equation we see:

$$2y - 2\lambda y = 0 \quad \Rightarrow \quad 2y(\lambda - 1) = 0$$

Therefore, either $\lambda = 1$ or $y = 0$.

If $\lambda = 1$ then this implies $\mu = 0$ and $z = 0$. Using the constraint $x + z = 1$ then $x = 1$, and using the constraint $x^2 + y^2 = 1$, then $y = 0$. This gives the critical point

$$(1, 0, 0)$$

If $y = 0$, using the constraint $x^2 + y^2 = 1$, then $x = \pm 1$. If $x = 1$, then $z = 0$, if $x = -1$ then $z = 2$. This gives the critical points

$$(1, 0, 0), \quad (-1, 0, 2)$$

Now we examine $f(x, y, z) = x^2 + y^2 + z^2$ at the two critical points for the maximum value:

$$f(1, 0, 0) = 1, \quad f(-1, 0, 2) = 5$$

Thus, the point farthest from the origin on this ellipse is the point $(-1, 0, 2)$ (at $\sqrt{5}$ units away).

42. Find the minimum and maximum of $f(x, y, z) = y + 2z$ subject to two constraints, $2x + z = 4$ and $x^2 + y^2 = 1$.

SOLUTION The constraint equations are:

$$g(x, y) = 2x + z - 4 = 0, \quad h(x, y) = x^2 + y^2 - 1 = 0$$

We now write out the Lagrange Equations. We have, $\nabla f = \langle 0, 1, 2\rangle$, $\nabla g = \langle 2, 0, 1\rangle$, and $\nabla h = \langle 2x, 2y, 0\rangle$, so the Lagrange Condition is

$$\nabla f = \lambda \nabla g + \mu \nabla h$$
$$\langle 0, 1, 2\rangle = \lambda \langle 2, 0, 1\rangle + \mu \langle 2x, 2y, 0\rangle = \langle 2\lambda + 2\mu x, 2\mu y, \lambda\rangle$$

From the third coordinate we get that $\lambda = 2$, which then gives us the following from the first two coordinates:

$$0 = 4 + 2\mu x$$
$$1 = 2\mu y$$

From the second equation, we see that neither μ nor y can be zero, so we can write $\mu = 1/2y$ and substitute it into the first equation, resulting in $0 = 4 + 2(1/2y)x = 4 + x/y$, or in other words, $x = -4y$. Plugging this into the second constraint, we find that $16y^2 + y^2 = 1$, so $y = \pm 1/\sqrt{17}$. Thus, our two points of interest are

$$\left(\frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}}, 4 + \frac{8}{\sqrt{17}}\right) \quad \text{and} \quad \left(\frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}, 4 - \frac{8}{\sqrt{17}}\right)$$

The function f at the first point is $17/\sqrt{17}$, and at the second point is $-17/\sqrt{17}$, so these must be our maximum and minimum values, respectively.

43. Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to two constraints, $x + 2y + z = 3$ and $x - y = 4$.

SOLUTION The constraint equations are

$$g(x, y, z) = x + 2y + z - 3 = 0, \quad h(x, y) = x - y - 4 = 0$$

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle 2x, 2y, 2z\rangle$, $\nabla g = \langle 1, 2, 1\rangle$, and $\nabla h = \langle 1, -1, 0\rangle$, hence the Lagrange Condition is

$$\nabla f = \lambda \nabla g + \mu \nabla h$$
$$\langle 2x, 2y, 2z\rangle = \lambda \langle 1, 2, 1\rangle + \mu \langle 1, -1, 0\rangle$$
$$= \langle \lambda + \mu, 2\lambda - \mu, \lambda\rangle$$

We obtain the following equations:

$$2x = \lambda + \mu$$
$$2y = 2\lambda - \mu$$
$$2z = \lambda$$

Step 2. Solve for λ and μ. The first equation gives $\lambda = 2x - \mu$. Combining with the third equation we get

$$2z = 2x - \mu \tag{1}$$

The second equation gives $\mu = 2\lambda - 2y$, combining with the third equation we get $\mu = 4z - 2y$. Substituting in (1) we obtain

$$2z = 2x - (4z - 2y) = 2x - 4z + 2y$$
$$6z = 2x + 2y \quad \Rightarrow \quad z = \frac{x + y}{3} \tag{2}$$

Step 3. Solve for x, y, and z using the constraints. The constraints give x and y as functions of z:

$$x - y = 4 \quad \Rightarrow \quad y = x - 4$$
$$x + 2y + z = 3 \quad \Rightarrow \quad y = \frac{3 - x - z}{2}$$

Combining the two equations we get

$$x - 4 = \frac{3 - x - z}{2}$$

$$2x - 8 = 3 - x - z$$
$$3x = 11 - z \quad \Rightarrow \quad x = \frac{11 - z}{3}$$

We find y using $y = x - 4$:

$$y = \frac{11 - z}{3} - 4 = \frac{-1 - z}{3}$$

We substitute x and y in (2) and solve for z:

$$z = \frac{\frac{11-z}{3} + \frac{-1-z}{3}}{3} = \frac{11 - z - 1 - z}{9} = \frac{10 - 2z}{9}$$
$$9z = 10 - 2z$$
$$11z = 10 \quad \Rightarrow \quad z = \frac{10}{11}$$

We find x and y:

$$y = \frac{-1 - z}{3} = \frac{-1 - \frac{10}{11}}{3} = -\frac{21}{33} = -\frac{7}{11}$$
$$x = \frac{11 - z}{3} = \frac{11 - \frac{10}{11}}{3} = \frac{111}{33} = \frac{37}{11}$$

We obtain the solution

$$P = \left(\frac{37}{11}, -\frac{7}{11}, \frac{10}{11}\right)$$

Step 4. Calculate the critical values. We compute $f(x, y, z) = z^2 + y^2 + z^2$ at the critical point:

$$f(P) = \left(\frac{37}{11}\right)^2 + \left(-\frac{7}{11}\right)^2 + \left(\frac{10}{11}\right)^2 = \frac{1518}{121} = \frac{138}{11} \approx 12.545$$

As x tends to infinity, so also does $f(x, y, z)$ tend to ∞. Therefore f has no maximum value and the given critical point P must produce a minimum. We conclude that the minimum value of f subject to the two constraints is $f(P) = \frac{138}{11} \approx 12.545$.

Further Insights and Challenges

44. Suppose that both $f(x, y)$ and the constraint function $g(x, y)$ are linear. Use contour maps to explain why $f(x, y)$ does not have a maximum subject to $g(x, y) = 0$ unless $g = af + b$ for some constants a, b.

SOLUTION We denote the linear functions by

$$f(x, y) = Ax + By + C, \quad g(x, y) = Dx + Ey + F$$

If f has a maximum value at a point P subject to g, then at this point $\nabla f_P \parallel \nabla g_P$. Since the gradient is normal to the level curve of the function passing through P, the tangents to the level curves of f and g at P coincide. In our case, the level curves of f (and of g) consist of parallel lines, hence since their tangents coincide, then these parallel contour lines coincide. That is, the contour line $f(x, y) = K$ is also the contour line $g(x, y) = L$ for some K, L, or in other words,

$$Ax + By + C = K, \quad Dx + Ey + F = L$$

Therefore,

$$D = aA, \quad E = aB, \quad F - L = a(C - K)$$

The function g is thus

$$g(x, y) = Dx + Ey + F = aAx + aBy + aC - aK + L$$
$$= a(Ax + By + C) + L - aK = af(x, y) + L - aK$$

Therefore, for $b = L - aK$ we have

$$g(x, y) = af(x, y) + b \tag{1}$$

By Theorem 1 we conclude that if g is not in the form (1), f does not have a maximum subject to $g(x, y) = 0$.

45. Assumptions Matter Consider the problem of minimizing $f(x, y) = x$ subject to $g(x, y) = (x-1)^3 - y^2 = 0$.
(a) Show, without using calculus, that the minimum occurs at $P = (1, 0)$.
(b) Show that the Lagrange condition $\nabla f_P = \lambda \nabla g_P$ is not satisfied for any value of λ.
(c) Does this contradict Theorem 1?

SOLUTION
(a) The equation of the constraint can be rewritten as

$$(x-1)^3 = y^2 \quad \text{or} \quad x = y^{2/3} + 1$$

Therefore, at the points under the constraint, $x \geq 1$, hence $f(x, y) \geq 1$. Also at the point $P = (1, 0)$ we have $f(1, 0) = 1$, hence $f(1, 0) = 1$ is the minimum value of f under the constraint.
(b) We have $\nabla f = \langle 1, 0 \rangle$ and $\nabla g = \left\langle 3(x-1)^2, -2y \right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ yields the following equations:

$$1 = \lambda \cdot 3(x-1)^2$$
$$0 = -2\lambda y$$

The first equation implies that $\lambda \neq 0$ and $x - 1 = \pm \frac{1}{\sqrt{3\lambda}}$. The second equation gives $y = 0$. Substituting in the equation of the constraint gives

$$(x-1)^3 - y^2 = \left(\frac{\pm 1}{\sqrt{3\lambda}} \right)^3 - 0^2 = \frac{\pm 1}{(3\lambda)^{3/2}} \neq 0$$

We conclude that the Lagrange Condition is not satisfied by any point under the constraint.
(c) Theorem 1 is not violated since at the point $P = (1, 0)$, $\nabla g = \mathbf{0}$, whereas the Theorem is valid for points where $\nabla g_P \neq \mathbf{0}$.

46. Marginal Utility Goods 1 and 2 are available at dollar prices of p_1 per unit of good 1 and p_2 per unit of good 2. A utility function $U(x_1, x_2)$ is a function representing the **utility** or benefit of consuming x_j units of good j. The **marginal utility** of the jth good is $\partial U / \partial x_j$, the rate of increase in utility per unit increase in the jth good. Prove the following law of economics: Given a budget of L dollars, utility is maximized at the consumption level (a, b) where the ratio of marginal utility is equal to the ratio of prices:

$$\frac{\text{Marginal utility of good 1}}{\text{Marginal utility of good 2}} = \frac{U_{x_1}(a, b)}{U_{x_2}(a, b)} = \frac{p_1}{p_2}$$

SOLUTION We must maximize the utility $U(x_1, x_2)$ subject to the constraint $p_1 x_1 + p_2 x_2 = L$ or $g(x_1, x_2) = p_1 x_1 + p_2 x_2 - L = 0$, $x_1 \geq 0$, $x_2 \geq 0$. We have $\nabla U = \left\langle U_{x_1}, U_{x_2} \right\rangle$ and $\nabla g = \langle p_1, p_2 \rangle$, hence the Lagrange Condition $\nabla U = \lambda \nabla g$ gives the following equations:

$$\begin{aligned} U_{x_1} &= \lambda p_1 \\ U_{x_2} &= \lambda p_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{U_{x_1}}{p_1} &= \lambda \\ \frac{U_{x_2}}{p_2} &= \lambda \end{aligned}$$

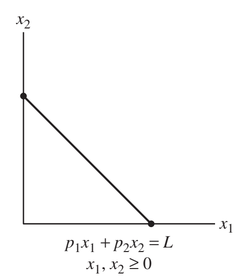
(we assume $p_1, p_2 > 0$). Equating the two expressions for λ we get

$$\frac{U_{x_1}}{p_1} = \frac{U_{x_2}}{p_2} \quad \Rightarrow \quad \frac{U_{x_1}}{U_{x_2}} = \frac{p_1}{p_2}$$

That is, $U(x_1, x_2)$ is maximized at the consumption level (a, b), where the following holds:

$$\frac{\text{marginal utility of good 1}}{\text{marginal utility of good 2}} = \frac{U_{x_1}(a, b)}{U_{x_2}(a, b)} = \frac{p_1}{p_2}$$

Notice that the constraint is a segment in the x_1x_2-plane (if $p_1 > 0$ and $p_2 > 0$), which is a closed and bounded set in this plane. Hence, if U is continuous, it assumes extreme values on this segment.



47. Consider the utility function $U(x_1, x_2) = x_1 x_2$ with budget constraint $p_1 x_1 + p_2 x_2 = c$.

(a) Show that the maximum of $U(x_1, x_2)$ subject to the budget constraint is equal to $c^2/(4p_1 p_2)$.

(b) Calculate the value of the Lagrange multiplier λ occurring in (a).

(c) Prove the following interpretation: λ is the rate of increase in utility per unit increase in total budget c.

SOLUTION

(a) By the earlier exercise, the utility is maximized at a point where the following equality holds:

$$\frac{U_{x_1}}{U_{x_2}} = \frac{p_1}{p_2}$$

Since $U_{x_1} = x_2$ and $U_{x_2} = x_1$, we get

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{p_1}{p_2} x_1$$

We now substitute x_2 in terms of x_1 in the constraint $p_1 x_1 + p_2 x_2 = c$ and solve for x_1. This gives

$$p_1 x_1 + p_2 \cdot \frac{p_1}{p_2} x_1 = c$$

$$2p_1 x_1 = c \quad \Rightarrow \quad x_1 = \frac{c}{2p_1}$$

The corresponding value of x_2 is computed by $x_2 = \frac{p_1}{p_2} x_1$:

$$x_2 = \frac{p_1}{p_2} \cdot \frac{c}{2p_1} = \frac{c}{2p_2}$$

That is, $U(x_1, x_2)$ is maximized at the consumption level $x_1 = \frac{c}{2p_1}$, $x_2 = \frac{c}{2p_2}$. The maximum value is

$$U\left(\frac{c}{2p_1}, \frac{c}{2p_2}\right) = \frac{c}{2p_1} \cdot \frac{c}{2p_2} = \frac{c^2}{4p_1 p_2}$$

(b) The Lagrange condition $\nabla U = \lambda \nabla g$ for $U(x_1, x_2) = x_1 x_2$ and $g(x_1, x_2) = p_1 x_1 + p_2 x_2 - c = 0$ is

$$\langle x_2, x_1 \rangle = \lambda \langle p_1, p_2 \rangle \tag{1}$$

or

$$\begin{aligned} x_2 &= \lambda p_1 \\ x_1 &= \lambda p_2 \end{aligned} \quad \Rightarrow \quad \lambda = \frac{x_2}{p_1} = \frac{x_1}{p_2}$$

In part (a) we showed that at the maximizing point $x_1 = \frac{c}{2p_1}$, therefore the value of λ is

$$\lambda = \frac{x_1}{p_2} = \frac{c}{2p_1 p_2}$$

(c) We compute $\frac{dU}{dc}$ using the Chain Rule:

$$\frac{dU}{dc} = \frac{\partial U}{\partial x_1} x_1'(c) + \frac{\partial U}{\partial x_2} x_2'(c) = x_2 x_1'(c) + x_1 x_2'(c) = \langle x_2, x_1 \rangle \cdot \langle x_1'(c), x_2'(c) \rangle$$

Substituting in (1) we get

$$\frac{dU}{dc} = \lambda \langle p_1, p_2 \rangle \cdot \langle x_1'(c), x_2'(c) \rangle = \lambda \left(p_1 x_1'(c) + p_2 x_2'(c) \right) \tag{2}$$

We now use the Chain Rule to differentiate the equation of the constraint $p_1 x_1 + p_2 x_2 = c$ with respect to c:

$$p_1 x_1'(c) + p_2 x_2'(c) = 1$$

Substituting in (2), we get

$$\frac{dU}{dc} = \lambda \cdot 1 = \lambda$$

Using the approximation $\Delta U \approx \frac{dU}{dc} \Delta c$, we conclude that λ is the rate of increase in utility per unit increase of total budget L.

48. This exercise shows that the multiplier λ may be interpreted as a rate of change in general. Assume that the maximum of $f(x, y)$ subject to $g(x, y) = c$ occurs at a point P. Then P depends on the value of c, so we may write $P = (x(c), y(c))$ and we have $g(x(c), y(c)) = c$.

(a) Show that

$$\nabla g(x(c), y(c)) \cdot \langle x'(c), y'(c) \rangle = 1$$

Hint: Differentiate the equation $g(x(c), y(c)) = c$ with respect to c using the Chain Rule.

(b) Use the Chain Rule and the Lagrange condition $\nabla f_P = \lambda \nabla g_P$ to show that

$$\frac{d}{dc} f(x(c), y(c)) = \lambda$$

(c) Conclude that λ is the rate of increase in f per unit increase in the "budget level" c.

SOLUTION

(a) We differentiate the equation $g(x(c), y(c)) = c$ with respect to c, using the Chain Rule. This gives

$$\frac{\partial g}{\partial x} x'(c) + \frac{\partial g}{\partial y} y'(c) = 1$$

We rewrite this equality using the dot product and the definition of the gradient:

$$\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle \cdot \langle x'(c), y'(c) \rangle = 1$$
$$\nabla g(x(c), y(c)) \cdot \langle x'(c), y'(c) \rangle = 1$$

(b) We now differentiate $f(x(c), y(c))$ with respect to c, using the Chain Rule. We obtain

$$\frac{d}{dc} f(x(c), y(c)) = \frac{\partial f}{\partial x} x'(c) + \frac{\partial f}{\partial y} y'(c) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(c), y'(c) \rangle = \nabla f \cdot \langle x'(c), y'(c) \rangle$$

We use the Lagrange Condition $\nabla f = \lambda \nabla g$ and the result in part (a) to write

$$\frac{d}{dc} f(x(c), y(c)) = \lambda \cdot \nabla g \cdot \langle x'(c), y'(c) \rangle = \lambda \cdot 1 = \lambda$$

(c) The equality obtained in part (b) implies that λ is the rate of change in the maximum value of $f(x, y)$, subject to the constraint $g(x, y) = c$, with respect to c.

49. Let $B > 0$. Show that the maximum of

$$f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$$

subject to the constraints $x_1 + \cdots + x_n = B$ and $x_j \geq 0$ for $j = 1, \dots, n$ occurs for $x_1 = \cdots = x_n = B/n$. Use this to conclude that

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}$$

for all positive numbers $a_1, \dots, a_n$.

SOLUTION We first notice that the constraints $x_1 + \cdots + x_n = B$ and $x_j \geq 0$ for $j = 1, \dots, n$ define a closed and bounded set in the nth dimensional space, hence f (continuous, as a polynomial) has extreme values on this set. The minimum value zero occurs where one of the coordinates is zero (for example, for $n = 2$ the constraint $x_1 + x_2 = B$, $x_1 \geq 0, x_2 \geq 0$ is a triangle in the first quadrant). We need to maximize the function $f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$ subject to the constraints $g(x_1, \dots, x_n) = x_1 + \cdots + x_n - B = 0$, $x_j \geq 0$, $j = 1, \dots, n$.

Step 1. Write out the Lagrange Equations. The gradient vectors are

$$\nabla f = \langle x_2 x_3 \cdots x_n, x_1 x_3 \cdots x_n, \dots, x_1 x_2 \cdots x_{n-1} \rangle$$
$$\nabla g = \langle 1, 1, \dots, 1 \rangle$$

The Lagrange Condition $\nabla f = \lambda \nabla g$ yields the following equations:

$$x_2 x_3 \cdots x_n = \lambda$$
$$x_1 x_3 \cdots x_n = \lambda$$
$$x_1 x_2 \cdots x_{n-1} = \lambda$$

Step 2. Solving for $x_1, x_2, \ldots, x_n$ using the constraint. The Lagrange equations imply the following equations:

$$\begin{aligned} x_2x_3\cdots x_n &= x_1x_2\cdots x_{n-1} \\ x_1x_3\cdots x_n &= x_1x_2\cdots x_{n-1} \\ x_1x_2x_4\cdots x_n &= x_1x_2\cdots x_{n-1} \\ &\vdots \\ x_1x_2\cdots x_{n-2}x_n &= x_1x_2\cdots x_{n-1} \end{aligned}$$

We may assume that $x_j \neq 0$ for $j = 1, \ldots, n$, since if one of the coordinates is zero, f has the minimum value zero. We divide each equation by its right-hand side to obtain

$$\begin{aligned} \frac{x_n}{x_1} &= 1 \\ \frac{x_n}{x_2} &= 1 \\ \frac{x_n}{x_3} &= 1 \\ &\vdots \\ \frac{x_n}{x_{n-1}} &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= x_n \\ x_2 &= x_n \\ x_3 &= x_n \\ &\vdots \\ x_{n-1} &= x_n \end{aligned}$$

Substituting in the constraint $x_1 + \cdots + x_n = B$ and solving for x_n gives

$$\underbrace{x_n + x_n + \cdots + x_n}_{n} = B$$

$$nx_n = B \quad \Rightarrow \quad x_n = \frac{B}{n}$$

Hence $x_1 = \cdots = x_n = \frac{B}{n}$.

Step 3. Conclusions. The maximum value of $f(x_1, \ldots, x_n) = x_1x_2\cdots x_n$ on the constraint $x_1 + \cdots + x_n = B$, $x_j \geq 0$, $j = 1, \ldots, n$ occurs at the point at which all coordinates are equal to $\frac{B}{n}$. The value of f at this point is

$$f\left(\frac{B}{n}, \frac{B}{n}, \ldots, \frac{B}{n}\right) = \left(\frac{B}{n}\right)^n$$

It follows that for any point $(x_1, \ldots, x_n)$ on the constraint, that is, for any point satisfying $x_1 + \cdots + x_n = B$ with x_j positive, the following holds:

$$f(x_1, \ldots, x_n) \leq \left(\frac{B}{n}\right)^n$$

That is,

$$x_1\cdots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n}\right)^n$$

or

$$(x_1\cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

50. Let $B > 0$. Show that the maximum of $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ subject to $x_1^2 + \cdots + x_n^2 = B^2$ is $\sqrt{n}B$. Conclude that

$$|a_1| + \cdots + |a_n| \leq \sqrt{n}(a_1^2 + \cdots + a_n^2)^{1/2}$$

for all numbers $a_1, \ldots, a_n$.

SOLUTION First notice that the function is continuous and the constraint is a sphere centered at the origin in the nth-dimensional space, hence f has extreme values on this set. (For $n = 2$, the constraint defines the circle $x^2 + y^2 = B^2$). We must maximize $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ subject to the constraint $g(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 - B^2 = 0$.

Step 1. Write out the Lagrange Equations. The gradient vectors are

$$\nabla f = \langle 1, 1, \ldots, 1\rangle \quad \text{and} \quad \nabla g = \langle 2x_1, 2x_2, \ldots, 2x_n\rangle$$

Hence, the Lagrange Condition $\nabla f = \lambda \nabla g$ gives the following equations:

$$\begin{aligned} 1 &= \lambda(2x_1) \\ 1 &= \lambda(2x_2) \\ &\vdots \\ 1 &= \lambda(2x_n) \end{aligned}$$

Step 2. Solve for λ in terms of $x_1, \ldots, x_n$. The Lagrange equations imply that $x_j \neq 0$ for $j = 1, \ldots, n$. Therefore we may divide by x_j to obtain

$$\begin{aligned} \lambda &= \frac{1}{2x_1} \\ \lambda &= \frac{1}{2x_2} \\ &\vdots \\ \lambda &= \frac{1}{2x_n} \end{aligned}$$

Step 3. Solving for $x_1, \ldots, x_n$ using the constraint. Equating the expressions for λ gives the following equations:

$$\begin{aligned} \frac{1}{2x_1} &= \frac{1}{2x_n} \\ \frac{1}{2x_2} &= \frac{1}{2x_n} \\ &\vdots \\ \frac{1}{2x_{n-1}} &= \frac{1}{2x_n} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= x_n \\ x_2 &= x_n \\ &\vdots \\ x_{n-1} &= x_n \end{aligned}$$

Substituting $x_1, \ldots, x_{n-1}$ in terms of x_n in the equation of the constraint $x_1^2 + \cdots + x_n^2 = B^2$ and solving for x_n, gives

$$\begin{aligned} \underbrace{x_n^2 + x_n^2 + \cdots + x_n^2}_{n} &= B^2 \\ nx_n^2 &= B^2 \\ x_n^2 &= \frac{B^2}{n} \quad \Rightarrow \quad |x_n| = \frac{B}{\sqrt{n}} \end{aligned}$$

We conclude that $|x_1| = |x_2| = \cdots = |x_n| = \frac{B}{\sqrt{n}}$. Since $x_j = x_n$ for all j, the maximum value occurs when x_n is positive, and the minimum value corresponds to the negative value of x_n. We conclude that the maximizing point is

$$x_1 = x_2 = \cdots = x_n = \frac{B}{\sqrt{n}}$$

Notice that the point where $\nabla g = \langle 2x_1, 2x_2, \ldots, 2x_n \rangle = \mathbf{0}$ is the point at the origin, and this point does not lie on the constraint.

Step 4. Conclusions. The maximum value of $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ under the constraint is

$$f\left(\frac{B}{\sqrt{n}}, \ldots, \frac{B}{\sqrt{n}}\right) = n\frac{B}{\sqrt{n}} = \sqrt{n}B$$

This means that for any point under the constraint, that is, for any $(x_1, \ldots, x_n)$ such that $x_1^2 + \cdots + x_n^2 = B^2$, we have

$$f(x_1, \ldots, x_n) \leq \sqrt{n}B$$

That is,

$$x_1 + \cdots + x_n \leq \sqrt{n}\sqrt{x_1^2 + \cdots + x_n^2} \tag{1}$$

Notice that if $(x_1, \ldots, x_n)$ is under the constraint, then $(|x|_1, \ldots, |x|_n)$ is also under the constraint, and the right-hand side in (1) has the same value at these two points. Therefore, we also have

$$|x_1| + \cdots + |x_n| \leq \sqrt{n}\left(x_1^2 + \cdots + x_n^2\right)^{1/2}.$$

51. Given constants E, E_1, E_2, E_3, consider the maximum of

$$S(x_1, x_2, x_3) = x_1 \ln x_1 + x_2 \ln x_2 + x_3 \ln x_3$$

subject to two constraints:

$$x_1 + x_2 + x_3 = N, \qquad E_1 x_1 + E_2 x_2 + E_3 x_3 = E$$

Show that there is a constant μ such that $x_i = A^{-1} e^{\mu E_i}$ for $i = 1, 2, 3$, where $A = N^{-1}(e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3})$.

SOLUTION The constraints equations are

$$g(x_1, x_2, x_3) = x_1 + x_2 + x_3 - N = 0$$
$$h(x_1, x_2, x_3) = E_1 x_1 + E_2 x_2 + E_3 x_3 - E = 0$$

We first find the Lagrange equations. The gradient vectors are

$$\nabla S = \left\langle \ln x_1 + x_1 \cdot \frac{1}{x_1}, \ln x_2 + x_2 \cdot \frac{1}{x_2}, \ln x_3 + x_3 \cdot \frac{1}{x_3} \right\rangle = \langle 1 + \ln x_1, 1 + \ln x_2, 1 + \ln x_3 \rangle$$
$$\nabla g = \langle 1, 1, 1 \rangle, \quad \nabla h = \langle E_1, E_2, E_3 \rangle$$

The Lagrange Condition $\nabla f = \lambda \nabla g + \mu \nabla h$ gives the following equation:

$$\langle 1 + \ln x_1, 1 + \ln x_2, 1 + \ln x_3 \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle E_1, E_2, E_3 \rangle = \langle \lambda + \mu E_1, \lambda + \mu E_2, \lambda + \mu E_3 \rangle$$

We obtain the Lagrange equations:

$$1 + \ln x_1 = \lambda + \mu E_1$$
$$1 + \ln x_2 = \lambda + \mu E_2$$
$$1 + \ln x_3 = \lambda + \mu E_3$$

We subtract the third equation from the other equations to obtain

$$\ln x_1 - \ln x_3 = \mu (E_1 - E_3)$$
$$\ln x_2 - \ln x_3 = \mu (E_2 - E_3)$$

or

$$\begin{aligned} \ln \frac{x_1}{x_3} &= \mu (E_1 - E_3) \\ \ln \frac{x_2}{x_3} &= \mu (E_2 - E_3) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= x_3 e^{\mu(E_1 - E_3)} \\ x_2 &= x_3 e^{\mu(E_2 - E_3)} \end{aligned} \tag{1}$$

Substituting x_1 and x_2 in the equation of the constraint $g(x_1, x_2, x_3) = 0$ and solving for x_3 gives

$$x_3 e^{\mu(E_1 - E_3)} + x_3 e^{\mu(E_2 - E_3)} + x_3 = N$$

We multiply by $e^{\mu E_3}$:

$$x_3 \left(e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}\right) = N e^{\mu E_3}$$
$$x_3 = \frac{N e^{\mu E_3}}{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}}$$

Substituting in (1) we get

$$x_1 = \frac{N e^{\mu E_3}}{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}} \cdot e^{\mu(E_1 - E_3)} = \frac{N e^{\mu E_1}}{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}}$$
$$x_2 = \frac{N e^{\mu E_3}}{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}} \cdot e^{\mu(E_2 - E_3)} = \frac{N e^{\mu E_2}}{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}}$$

Letting $A = \frac{e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3}}{N}$, we obtain

$$x_1 = A^{-1} e^{\mu E_1}, \quad x_2 = A^{-1} e^{\mu E_2}, \quad x_3 = A^{-1} e^{\mu E_3}$$

The value of μ is determined by the second constraint $h(x_1, x_2, x_3) = 0$.

52. Boltzmann Distribution Generalize Exercise 51 to n variables: Show that there is a constant μ such that the maximum of

$$S = x_1 \ln x_1 + \cdots + x_n \ln x_n$$

subject to the constraints

$$x_1 + \cdots + x_n = N, \qquad E_1 x_1 + \cdots + E_n x_n = E$$

occurs for $x_i = A^{-1} e^{\mu E_i}$, where

$$A = N^{-1}(e^{\mu E_1} + \cdots + e^{\mu E_n})$$

This result lies at the heart of statistical mechanics. It is used to determine the distribution of velocities of gas molecules at temperature T; x_i is the number of molecules with kinetic energy E_i; $\mu = -(kT)^{-1}$, where k is Boltzmann's constant. The quantity S is called the **entropy**.

SOLUTION The constraints equations are

$$g(x_1, \dots, x_n) = x_1 + \cdots + x_n - N$$
$$h(x_1, \dots, x_n) = E_1 x_1 + \cdots + E_n x_n - E$$

We find the Lagrange Equations. The gradient vectors are

$$\nabla S = \left\langle \ln x_1 + x_1 \cdot \frac{1}{x_1}, \dots, \ln x_n + x_n \cdot \frac{1}{x_n} \right\rangle = \langle 1 + \ln x_1, \dots, 1 + \ln x_n \rangle$$
$$\nabla g = \langle 1, \dots, 1 \rangle, \nabla h = \langle E_1, \dots, E_n \rangle$$

We write the Lagrange Condition $\nabla S = \lambda \nabla g + \mu \nabla h$:

$$\langle 1 + \ln x_1, \dots, 1 + \ln x_n \rangle = \lambda \langle 1, \dots, 1 \rangle + \mu \langle E_1, \dots, E_n \rangle = \langle \lambda + \mu E_1, \dots, \lambda + \mu E_n \rangle$$

yielding the following Lagrange equations:

$$\begin{aligned} 1 + \ln x_1 &= \lambda + \mu E_1 \\ 1 + \ln x_2 &= \lambda + \mu E_2 \\ &\vdots \\ 1 + \ln x_n &= \lambda + \mu E_n \end{aligned}$$

Subtracting the ith equation from the jth equation, we

$$\ln x_i - \ln x_j = \ln \frac{x_i}{x_j} = \mu(E_i - E_j)$$

or

$$\ln \frac{x_i}{x_j} = \mu \left(E_i - E_j\right) \quad \Rightarrow \quad x_i e^{-\mu E_i} = x_j e^{-\mu E_j} \tag{1}$$

Let A be the common value of $x_i^{-1} e^{\mu E_i}$. Then

$$x_i = A^{-1} e^{\mu E_i}$$

The constraint $x_1 + \cdots + x_n = N$ gives

$$A^{-1}\left(e^{\mu E_1} + e^{\mu E_2} + \cdots + e^{\mu E_n}\right) = N$$

Therefore

$$A = \frac{e^{\mu E_1} + e^{\mu E_2} + \cdots + e^{\mu E_n}}{N}$$

The value of μ is determined by the second constraint $h(x_1, \dots, x_n) = 0$, although it would be very difficult to calculate.

CHAPTER REVIEW EXERCISES

1. Given $f(x, y) = \dfrac{\sqrt{x^2 - y^2}}{x + 3}$:

(a) Sketch the domain of f.

(b) Calculate $f(3, 1)$ and $f(-5, -3)$.

(c) Find a point satisfying $f(x, y) = 1$.

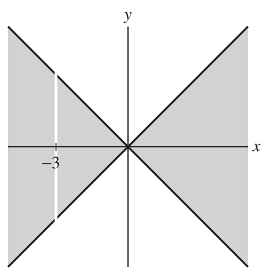
SOLUTION

(a) f is defined where $x^2 - y^2 \geq 0$ and $x + 3 \neq 0$. We solve these two inequalities:

$$x^2 - y^2 \geq 0 \quad \Rightarrow \quad x^2 \geq y^2 \quad \Rightarrow \quad |x| \geq |y|$$
$$x + 3 \neq 0 \quad \Rightarrow \quad x \neq -3$$

Therefore, the domain of f is the following set:

$$D = \{(x, y) : |x| \geq |y|, x \neq -3\}$$



(b) To find $f(3, 1)$ we substitute $x = 3$, $y = 1$ in $f(x, y)$. We get

$$f(3, 1) = \frac{\sqrt{3^2 - 1^2}}{3 + 3} = \frac{\sqrt{8}}{6} = \frac{\sqrt{2}}{3}$$

Similarly, setting $x = -5$, $y = -3$, we get

$$f(-5, -3) = \frac{\sqrt{(-5)^2 - (-3)^2}}{-5 + 3} = \frac{\sqrt{16}}{-2} = -2.$$

(c) We must find a point (x, y) such that

$$f(x, y) = \frac{\sqrt{x^2 - y^2}}{x + 3} = 1$$

We choose, for instance, $y = 1$, substitute and solve for x. This gives

$$\frac{\sqrt{x^2 - 1^2}}{x + 3} = 1$$
$$\sqrt{x^2 - 1} = x + 3$$
$$x^2 - 1 = (x + 3)^2 = x^2 + 6x + 9$$
$$6x = -10 \quad \Rightarrow \quad x = -\frac{5}{3}$$

Thus, the point $\left(-\frac{5}{3}, 1\right)$ satisfies $f\left(-\frac{5}{3}, 1\right) = 1$.

2. Find the domain and range of:

(a) $f(x, y, z) = \sqrt{x - y} + \sqrt{y - z}$

(b) $f(x, y) = \ln(4x^2 - y)$

SOLUTION

(a) $f(x, y, z)$ is defined where the differences under the root signs are nonnegative. That is, $x - y \geq 0$ and $y - z \geq 0$. We solve the inequalities

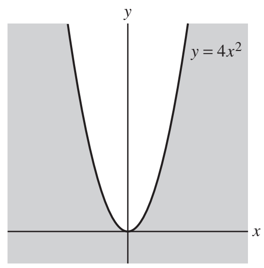
$$\left.\begin{array}{l} x - y \geq 0 \quad \Rightarrow \quad y \leq x \\ y - z \geq 0 \quad \Rightarrow \quad y \geq z \end{array}\right. \quad \Rightarrow \quad z \leq y \leq x$$

The domain of f is the following set:

$$D = \{(x, y, z) | z \le y \le x\}$$

The range is the set of all nonnegative numbers.

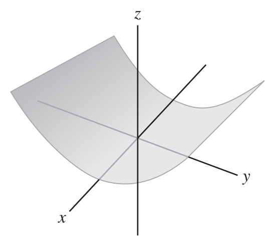
(b) f is defined when $4x^2 - y > 0$ or $y < 4x^2$. The domain $D = \left\{(x, y) : y < 4x^2\right\}$ is shown in the figure.



Since the logarithm function takes on all real values, the range of f is all real values.

3. Sketch the graph $f(x, y) = x^2 - y + 1$ and describe its vertical and horizontal traces.

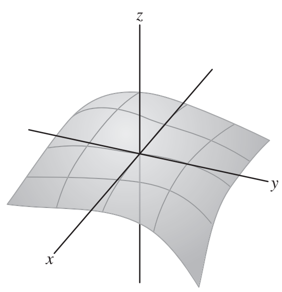
SOLUTION The graph is shown in the following figure.

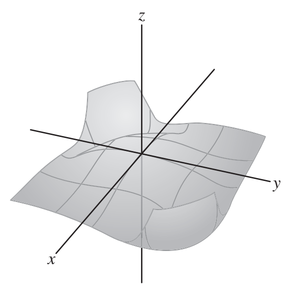


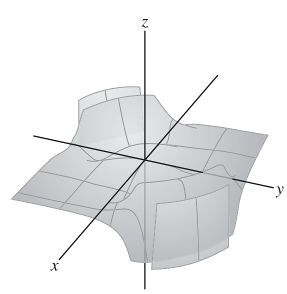
The trace obtained by setting $x = c$ is the line $z = c^2 - y + 1$ or $z = (c^2 + 1) - y$ in the plane $x = c$. The trace obtained by setting $y = c$ is the parabola $z = x^2 - c + 1$ in the plane $y = c$. The trace obtained by setting $z = c$ is the parabola $y = x^2 + 1 - c$ in the plane $z = c$.

4. CAS Use a graphing utility to draw the graph of the function $\cos(x^2 + y^2)e^{1-xy}$ in the domains $[-1, 1] \times [-1, 1]$, $[-2, 2] \times [-2, 2]$, and $[-3, 3] \times [-3, 3]$, and explain its behavior.

SOLUTION The graphs of the function $f(x, y) = \cos(x^2 + y^2)e^{1-xy}$ in the given domains are shown in the following figures.







The graph in the domain $[-1, 1] \times [-1, 1]$ shows a saddle point and two local maxima. In the domain $[-2, 2] \times [-2, 2]$ we see two additional local minima and two maxima and in the last graph two additional maxima and two additional minima appear. We can see that when $|xy| \to 0$, $\cos(x^2 + y^2)$ is the dominant part of the function, and as xy grows, e^{1-xy} gains more effect. When $xy \to -\infty$, the function oscillates between ∞ and $-\infty$, while for $xy \to +\infty$, $f(x, y) \to 0$.

5. Match the functions (a)–(d) with their graphs in Figure 1.

(a) $f(x, y) = x^2 + y$

(b) $f(x, y) = x^2 + 4y^2$

(c) $f(x, y) = \sin(4xy)e^{-x^2-y^2}$

(d) $f(x, y) = \sin(4x)e^{-x^2-y^2}$

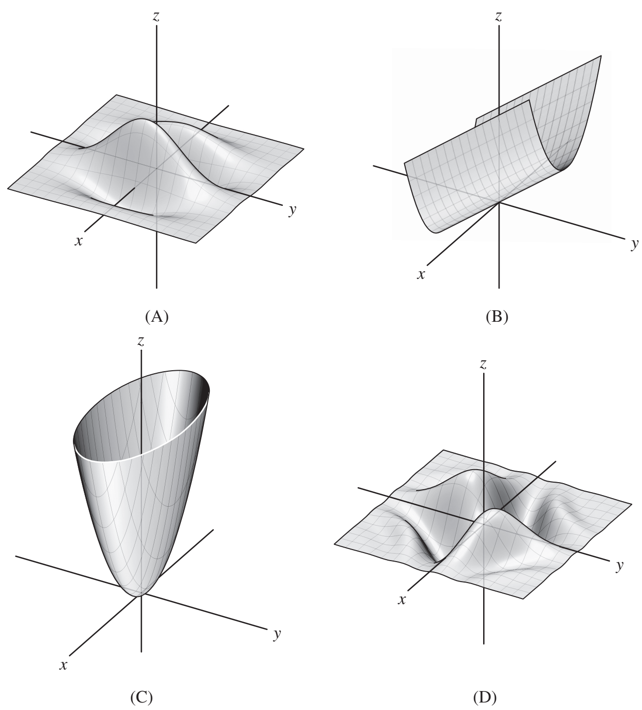


FIGURE 1

SOLUTION The function $f = x^2 + y$ matches picture (b), as can be seen by taking the $x = 0$ slice. The function $f = x^2 + 4y^2$ matches picture (c), as can be seen by taking $z = c$ slices (giving ellipses). Since $\sin(4xy)e^{-x^2-y^2}$ is symmetric with respect to x and y, and so also is picture (d), we match $\sin(4xy)e^{-x^2-y^2}$ with (d). That leaves the third function, $\sin(4x)e^{-x^2-y^2}$, to match with picture (a).

6. Referring to the contour map in Figure 2:

(a) Estimate the average rate of change of elevation from A to B and from A to D.

(b) Estimate the directional derivative at A in the direction of $\mathbf{v}$.

(c) What are the signs of f_x and f_y at D?

(d) At which of the labeled points are both f_x and f_y negative?

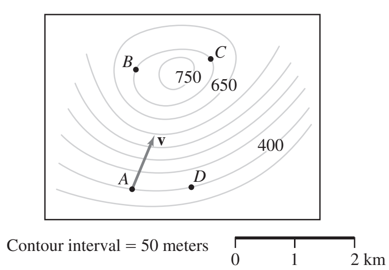


FIGURE 2

SOLUTION

(a) From A to B: The segment $\overline{AB}$ spans 6 level curves and the contour interval is $m = 50$ m, so the change of altitude is $6 \cdot 50 = 300$ m. From the horizontal scale of contour map we see that the horizontal distance from A to B is 2 km or 2000 m. Therefore,

$$\text{Average ROC from } A \text{ to } B = \frac{\Delta\text{altitude}}{\Delta\text{horizontal distance}} = \frac{300}{2000} = 0.15$$

From A to D: A and D lie on the same level curve, hence there is no change in altitude from A to D. Therefore,

$$\text{Average ROC from} A \text{ to } D = \frac{0}{\Delta\text{horizontal distance}} = 0.$$

(b) We first estimate the gradient at A. We get

$$\left.\frac{\partial f}{\partial x}\right|_A \approx \frac{\Delta f}{\Delta x} = \frac{0}{\Delta x} = 0 \qquad \left.\frac{\partial f}{\partial y}\right|_A \approx \frac{\Delta f}{\Delta y} \approx \frac{50}{200} \approx 0.25 \quad \Rightarrow \quad \left.\nabla f\right|_A \approx \langle 0, 0.22\rangle$$

We estimate $\mathbf{v}$, by $\mathbf{v} \approx \left\langle \frac{4}{9}, 1\right\rangle \approx \langle 0.44, 1\rangle$, hence the cosine of the angle between $\mathbf{v}$ and the gradient at A is

$$\cos\theta = \frac{\langle 0, 0.25\rangle \cdot \langle 0.44, 1\rangle}{0.25 \cdot \sqrt{0.44^2 + 1}} = \frac{0.25}{0.25 \cdot 1.093} = 0.915$$

Hence,

$$D_{\mathbf{v}} f(A) = \|\nabla f_A\| \cos\theta = 0.25 \cdot 0.915 \approx 0.229.$$

(Another method is to note that in the direction of $\mathbf{v}$, we cross four contour lines in about 1000 meters; thus, the change of f in that direction is about $4 \cdot 50/10000 = 0.2$.)

(c) At the point D we see that $f_x < 0$ since the elevation is decreasing in the x direction, while $f_y > 0$ since the elevation is increasing in the y direction.

(d) At the point C we see that $f_x < 0$ and $f_y < 0$, the elevation is decreasing in both the x and y direction at the point C.

7. Describe the level curves of:

(a) $f(x, y) = e^{4x-y}$ **(b)** $f(x, y) = \ln(4x - y)$

(c) $f(x, y) = 3x^2 - 4y^2$ **(d)** $f(x, y) = x + y^2$

SOLUTION

(a) The level curves of $f(x, y) = e^{4x-y}$ are the curves $e^{4x-y} = c$ in the xy-plane, where $c > 0$. Taking ln from both sides we get $4x - y = \ln c$. Therefore, the level curves are the parallel lines of slope 4, $4x - y = \ln c$, $c > 0$, in the xy-plane.

(b) The level curves of $f(x, y) = \ln(4x - y)$ are the curves $\ln(4x - y) = c$ in the xy-plane. We rewrite it as $4x - y = e^c$ to obtain the parallel lines of slope 4, with negative y-intercepts.

(c) The level curves of $f(x, y) = 3x^2 - 4y^2$ are the hyperbolas $3x^2 - 4y^2 = c$ in the xy plane.

(d) The level curves of $f(x, y) = x + y^2$ are the curves $x + y^2 = c$ or $x = c - y^2$ in the xy-plane. These are parabolas whose axis is the x-axis.

8. Match each function (a)–(c) with its contour graph (i)–(iii) in Figure 3:

(a) $f(x, y) = xy$

(b) $f(x, y) = e^{xy}$

(c) $f(x, y) = \sin(xy)$

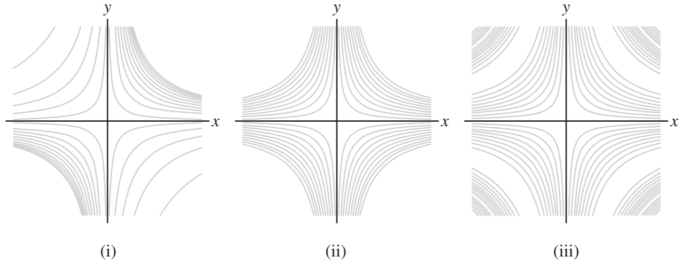


FIGURE 3

SOLUTION We find the level curves of the three functions:

(a) The level curves of $f(x, y) = xy$ are the curves $xy = c$ in the xy-plane, where c is any real value.

(b) The level curves of $f(x, y) = e^{xy}$ are $e^{xy} = c$ or $xy = \ln c$ where $c > 0$.

(c) The level curves of $f(x, y) = \sin xy$ are $\sin xy = c$ for $|c| \le 1$, or $xy = \sin^{-1} c + 2\pi k$.

The contour graphs corresponding to these functions are thus

(a) → (ii)
(b) → (i)
(c) → (iii)

Notice that the curves $xy = \ln c$ become closer and closer when c increases, while the curves $xy = c$ are equidistant for a certain contour interval. The contour map of (b) is in the first and third quadrants for $c > 1$, since then $\ln c > 0$.

In Exercises 9–14, evaluate the limit or state that it does not exist.

9. $\displaystyle\lim_{(x,y)\to(1,-3)} (xy + y^2)$

SOLUTION The function $f(x, y) = xy + y^2$ is continuous everywhere because it is a polynomial, therefore we evaluate the limit using substitution:

$$\lim_{(x,y)\to(1,-3)} \left(xy + y^2\right) = 1 \cdot (-3) + (-3)^2 = 6$$

10. $\displaystyle\lim_{(x,y)\to(1,-3)} \ln(3x + y)$

SOLUTION Approaching $(1, -3)$ along the ray $y = -3$, $x > 1$ gives

$$\lim_{x\to 1+} \ln(3x - 3) = -\infty$$

Therefore f takes on arbitrary small values at the intersection of every disk around the point $(1, -3)$ with the domain of the function. This shows that $\displaystyle\lim_{(x,y)\to(1,-3)} \ln(3x + y)$ does not exist.

11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy + xy^2}{x^2 + y^2}$

SOLUTION We evaluate the limits as (x, y) approaches the origin along the lines $y = x$ and $y = 2x$:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{xy + xy^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x \cdot x + x \cdot x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2 + x^3}{2x^2} = \lim_{x\to 0} \frac{1 + x}{2} = \frac{1}{2}$$

$$\lim_{\substack{x\to(0,0)\\ \text{along } y=2x}} \frac{xy + xy^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x \cdot 2x + x \cdot (2x)^2}{x^2 + (2x)^2} = \lim_{x\to 0} \frac{2x^2 + 4x^3}{5x^2} = \lim_{x\to 0} \frac{2 + 4x}{5} = \frac{2}{5}$$

Since the two limits are different, $f(x, y)$ does not approach one limit as $(x, y) \to (0, 0)$, therefore the limit does not exist.

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2 + x^2y^3}{x^4 + y^4}$

SOLUTION We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $(x, y) \to (0, 0)$ if and only if $r = \sqrt{x^2 + y^2} \to 0+$. Therefore,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^3y^2 + x^2y^3}{x^4 + y^4} &= \lim_{r\to 0+} \frac{r^3\cos^3\theta \cdot r^2\sin^2\theta + r^2\cos^2\theta \cdot r^3\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta} \\
&= \lim_{r\to 0+} \frac{r^5\left(\cos^3\theta\sin^2\theta + \cos^2\theta\sin^3\theta\right)}{r^4\left(\cos^4\theta + \sin^4\theta\right)} \\
&= \lim_{r\to 0+} \frac{r\left(\cos^3\theta\sin^2\theta + \cos^2\theta\sin^3\theta\right)}{\cos^4\theta + \sin^4\theta} \\
&= \lim_{r\to 0+} r \cdot \frac{\cos^3\theta\sin^2\theta + \cos^2\theta\sin^3\theta}{\cos^4\theta + \left(1 - \cos^2\theta\right)^2} \\
&= \lim_{r\to 0+} r \cdot \frac{\cos^2\theta\sin^2\theta(\cos\theta + \sin\theta)}{2\cos^4\theta - 2\cos^2\theta + 1}
\end{aligned}$$

The minimum value of the function $s = 2t^4 - 2t^2 + 1$ is $\frac{1}{2}$. Therefore, since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$, we find that

$$\left|\frac{\cos^2\theta\sin^2\theta(\cos\theta + \sin\theta)}{2\cos^4\theta - 2\cos^2\theta + 1}\right| \le \left|\frac{\cos^2\theta\sin^2\theta(\cos\theta + \sin\theta)}{\frac{1}{2}}\right| \le 2|\cos\theta + \sin\theta| \le 4$$

Hence,

$$0 \le \left| r \frac{\cos^2\theta \sin^2\theta(\cos\theta + \sin\theta)}{2\cos^4\theta - 2\cos^2\theta + 1} \right| \le 4r$$

We now use the Squeeze Theorem to conclude that the limit as $r \to 0+$ is zero, hence also the given limit is zero.

13. $\lim\limits_{(x,y)\to(1,-3)} (2x+y)e^{-x+y}$

SOLUTION The function $f(x,y) = (2x+y)e^{-x+y}$ is continuous, hence we evaluate the limit using substitution:

$$\lim_{(x,y)\to(1,-3)} (2x+y)e^{-x+y} = (2\cdot 1 - 3)e^{-1-3} = -e^{-4}$$

14. $\lim\limits_{(x,y)\to(0,2)} \dfrac{(e^x-1)(e^y-1)}{x}$

SOLUTION We have

$$\lim_{(x,y)\to(0,2)} \frac{\left(e^x-1\right)\left(e^y-1\right)}{x} = \lim_{x\to 0}\frac{e^x-1}{x}\lim_{y\to 2}\left(e^y-1\right) = \left(e^2-1\right)\lim_{x\to 0}\frac{e^x-1}{x} \tag{1}$$

By L'Hôpital's Rules,

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{e^x}{1} = 1 \tag{2}$$

Combining (1) and (2) we conclude that

$$\lim_{(x,y)\to(0,2)} \frac{(e^x-1)(e^y-1)}{x} = (e^2-1)\cdot 1 = e^2-1.$$

15. Let

$$f(x,y) = \begin{cases} \dfrac{(xy)^p}{x^4+y^4} & (x,y)\ne(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Use polar coordinates to show that $f(x,y)$ is continuous at all (x,y) if $p>2$ but is discontinuous at $(0,0)$ if $p\le 2$.

SOLUTION We show using the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, that the limit of $f(x,y)$ as $(x,y)\to(0,0)$ is zero for $p>2$. This will prove that f is continuous at the origin. Since f is a rational function with nonzero denominator for $(x,y)\ne(0,0)$, f is continuous there. We have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0+}\frac{(r\cos\theta)^p(r\sin\theta)^p}{(r\cos\theta)^4+(r\sin\theta)^4} = \lim_{r\to 0+}\frac{r^{2p}(\cos\theta\sin\theta)^p}{r^4\left(\cos^4\theta+\sin^4\theta\right)} \tag{1}$$

$$= \lim_{r\to 0+}\frac{r^{2(p-2)}(\cos\theta\sin\theta)^p}{\cos^4\theta+\sin^4\theta}$$

We use the following inequalities:

$$\left|\cos^4\theta\sin^4\theta\right| \le 1$$

$$\begin{aligned}\cos^4\theta+\sin^4\theta &= \left(\cos^2\theta+\sin^2\theta\right)^2 - 2\cos^2\theta\sin^2\theta = 1 - \frac{1}{2}\cdot(2\cos\theta\sin\theta)^2 \\ &= 1 - \frac{1}{2}\sin^2 2\theta \ge 1 - \frac{1}{2} = \frac{1}{2}\end{aligned}$$

Therefore,

$$0 \le \left|\frac{r^{2(p-2)}(\cos\theta\sin\theta)^p}{\cos^4\theta+\sin^4\theta}\right| \le \frac{r^{2(p-2)}\cdot 1}{\frac{1}{2}} = 2r^{2(p-2)}$$

Since $p-2>0$, $\lim\limits_{r\to 0+} 2r^{2(p-2)} = 0$, hence by the Squeeze Theorem the limit in (1) is also zero. We conclude that f is continuous for $p>2$.

We now show that for $p < 2$ the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist. We compute the limit as (x, y) approaches the origin along the line $y = x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} f(x, y) = \lim_{x\to 0} \frac{(x^2)^p}{x^4 + x^4} = \lim_{x\to 0} \frac{x^{2p}}{2x^4} = \lim_{x\to 0} \frac{x^{2(p-2)}}{2} = \infty$$

Therefore the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist for $p < 2$. We now show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ does not exist for $p = 2$ as well. We compute the limits along the line $y = 0$ and $y = x$:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^2y^2}{x^4 + y^4} = \lim_{x\to 0} \frac{x^2 \cdot 0^2}{x^4 + 0^4} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{x^2y^2}{x^4 + y^4} = \lim_{x\to 0} \frac{x^2 \cdot x^2}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limits along two paths are different, $f(x, y)$ does not approach one limit as $(x, y) \to (0, 0)$. We thus showed that if $p \le 2$, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and f is not continuous at the origin for $p \le 2$.

16. Calculate $f_x(1, 3)$ and $f_y(1, 3)$ for $f(x, y) = \sqrt{7x + y^2}$.

SOLUTION To calculate $f_x(x, y)$ we treat y as a constant and use the Chain Rule. This gives

$$f_x(x, y) = \frac{\partial}{\partial x}\sqrt{7x + y^2} = \frac{1}{2\sqrt{7x + y^2}} \frac{\partial}{\partial x}\left(7x + y^2\right) = \frac{7}{2\sqrt{7x + y^2}}$$

We compute $f_y(x, y)$ similarly, treating x as a constant:

$$f_y(x, y) = \frac{\partial}{\partial y}\sqrt{7x + y^2} = \frac{1}{2\sqrt{7x + y^2}} \frac{\partial}{\partial y}\left(7x + y^2\right) = \frac{2y}{2\sqrt{7x + y^2}} = \frac{y}{\sqrt{7x + y^2}}$$

At the point $(1, 3)$ we have

$$f_x(1, 3) = \frac{7}{2\sqrt{7 \cdot 1 + 3^2}} = \frac{7}{2 \cdot 4} = \frac{7}{8}$$

$$f_y(1, 3) = \frac{3}{\sqrt{7 \cdot 1 + 3^2}} = \frac{3}{4}$$

In Exercises 17–20, compute f_x and f_y.

17. $f(x, y) = 2x + y^2$

SOLUTION To find f_x we treat y as a constant, and to find f_y we treat x as a constant. We get

$$f_x = \frac{\partial}{\partial x}\left(2x + y^2\right) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}\left(y^2\right) = 2 + 0 = 2$$

$$f_y = \frac{\partial}{\partial y}\left(2x + y^2\right) = \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial y}\left(y^2\right) = 0 + 2y = 2y$$

18. $f(x, y) = 4xy^3$

SOLUTION We compute f_x, treating y as a constant:

$$f_x = \frac{\partial}{\partial x}(4xy^3) = 4y^3 \frac{\partial}{\partial x}(x) = 4y^3 \cdot 1 = 4y^3$$

We compute f_y treating x as a constant:

$$f_y = \frac{\partial}{\partial y}(4xy^3) = 4x \frac{\partial}{\partial y}(y^3) = 4x \cdot 3y^2 = 12xy^2.$$

19. $f(x, y) = \sin(xy)e^{-x-y}$

SOLUTION We compute f_x, treating y as a constant and using the Product Rule and the Chain Rule. We get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(\sin(xy)e^{-x-y}\right) = \frac{\partial}{\partial x}\left(\sin(xy)\right)e^{-x-y} + \sin(xy)\frac{\partial}{\partial x}e^{-x-y} \\ &= \cos(xy)\cdot ye^{-x-y} + \sin(xy)\cdot(-1)e^{-x-y} = e^{-x-y}\left(y\cos(xy) - \sin(xy)\right) \end{aligned}$$

We compute f_y similarly, treating x as a constant. Notice that since $f(y, x) = f(x, y)$, the partial derivative f_y can be obtained from f_x by interchanging x and y. That is,

$$f_y = e^{-x-y}\left(x\cos(yx) - \sin(yx)\right).$$

20. $f(x, y) = \ln(x^2 + xy^2)$

SOLUTION Using the Chain Rule we obtain

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\ln\left(x^2 + xy^2\right) = \frac{1}{x^2 + xy^2}\frac{\partial}{\partial x}\left(x^2 + xy^2\right) = \frac{1}{x^2 + xy^2}\cdot\left(2x + y^2\right) = \frac{2x + y^2}{x^2 + xy^2} \\ f_y &= \frac{\partial}{\partial y}\ln\left(x^2 + xy^2\right) = \frac{1}{x^2 + xy^2}\frac{\partial}{\partial y}\left(x^2 + xy^2\right) = \frac{1}{x^2 + xy^2}\cdot(2xy) = \frac{2xy}{x^2 + xy^2} \end{aligned}$$

21. Calculate f_{xxyz} for $f(x, y, z) = y\sin(x + z)$.

SOLUTION We differentiate f twice with respect to x, once with respect to y, and finally with respect to z. This gives

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(y\sin(x + z)\right) = y\cos(x + z) \\ f_{xx} &= \frac{\partial}{\partial x}\left(y\cos(x + z)\right) = -y\sin(x + z) \\ f_{xxy} &= \frac{\partial}{\partial y}\left(-y\sin(x + z)\right) = -\sin(x + z) \\ f_{xxyz} &= \frac{\partial}{\partial z}\left(-\sin(x + z)\right) = -\cos(x + z) \end{aligned}$$

22. Fix $c > 0$. Show that for any constants α, β, the function $u(t, x) = \sin(\alpha ct + \beta)\sin(\alpha x)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

SOLUTION We compute the partial derivatives u_t and u_x using the Chain Rule:

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\left(\sin(\alpha ct + \beta)\sin(\alpha x)\right) = \sin(\alpha x)\frac{\partial}{\partial t}\sin(\alpha ct + \beta) = \sin(\alpha x)\cos(\alpha ct + \beta)\cdot\alpha c \\ u_x &= \frac{\partial}{\partial x}\left(\sin(\alpha ct + \beta)\sin(\alpha x)\right) = \sin(\alpha ct + \beta)\frac{\partial}{\partial x}\sin(\alpha x) = \sin(\alpha ct + \beta)\cos(\alpha x)\cdot\alpha \end{aligned}$$

We find u_{tt} and u_{xx}, differentiating u_t and u_x with respect to t and x respectively, we get

$$\begin{aligned} u_{tt} &= \alpha c\sin(\alpha x)\frac{\partial}{\partial t}\cos(\alpha ct + \beta) = -\alpha^2 c^2\sin(\alpha x)\sin(\alpha ct + \beta) \\ u_{xx} &= \alpha\sin(\alpha ct + \beta)\frac{\partial}{\partial x}\cos(\alpha x) = -\alpha^2\sin(\alpha ct + \beta)\sin(\alpha x) \end{aligned}$$

We see that $u_{tt} = c^2 u_{xx}$.

23. Find an equation of the tangent plane to the graph of $f(x, y) = xy^2 - xy + 3x^3 y$ at $P = (1, 3)$.

SOLUTION The tangent plane has the equation

$$z = f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3) \tag{1}$$

We compute the partial derivatives of $f(x, y) = xy^2 - xy + 3x^3 y$:

$$\left.\begin{aligned} f_x(x, y) &= y^2 - y + 9x^2 y \\ f_y(x, y) &= 2xy - x + 3x^3 \end{aligned}\right. \quad\Rightarrow\quad \begin{aligned} f_x(1, 3) &= 3^2 - 3 + 9\cdot 1^2\cdot 3 = 33 \\ f_y(1, 3) &= 2\cdot 1\cdot 3 - 1 + 3\cdot 1^3 = 8 \end{aligned}$$

Also, $f(1,3) = 1 \cdot 3^2 - 1 \cdot 3 + 3 \cdot 1^3 \cdot 3 = 15$. Substituting these values in (1), we obtain the following equation:

$$z = 15 + 33(x-1) + 8(y-3)$$

or

$$z = 33x + 8y - 42$$

24. Suppose that $f(4,4) = 3$ and $f_x(4,4) = f_y(4,4) = -1$. Use the linear approximation to estimate $f(4.1, 4)$ and $f(3.88, 4.03)$.

SOLUTION The linear approximation is

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k$$

We use the linear approximation at the point $(4, 4)$. Therefore, estimating $f(3.88, 4.03)$,

$$\begin{aligned} h &= 3.88 - 4 = -0.12 \\ k &= 4.03 - 4 = 0.03 \\ f(3.88, 4.03) &\approx f(4,4) + f_x(4,4)\cdot(-0.12) + f_y(4,4)\cdot 0.03 \\ f(3.88, 4.03) &\approx 3 - 1\cdot(-0.12) - 1\cdot 0.03 = 3.09 \end{aligned}$$

Estimating $f(4.1, 4)$,

$$\begin{aligned} h &= 4.1 - 4 = 0.1 \\ k &= 4 - 4 = 0 \\ f(4.1, 4) &\approx f(4,4) + f_x(4,4)(0.1) + f_y(4,4)\cdot 0 \\ f(4.1, 4) &\approx 3 - 1\cdot(0.1) - 1\cdot 0 = 2.9 \end{aligned}$$

We obtain the estimations $f(3.88, 4.03) \approx 3.09$ and $f(4.1, 4) \approx 2.9$.

25. Use a linear approximation of $f(x,y,z) = \sqrt{x^2 + y^2 + z}$ to estimate $\sqrt{7.1^2 + 4.9^2 + 69.5}$. Compare with a calculator value.

SOLUTION The function whose value we want to approximate is

$$f(x,y,z) = \sqrt{x^2 + y^2 + z}$$

We will use the linear approximation at the point $(7, 5, 70)$. Recall that the linear approximation to a surface will be:

$$L(x,y,z) = f(7,5,70) + f_x(7,5,70)(x-7) + f_y(7,5,70)(y-5) + f_z(7,5,70)(z-70)$$

We compute the partial derivatives of f:

$$\begin{aligned} f_x(x,y,z) &= \frac{2x}{2\sqrt{x^2+y^2+z}} = \frac{x}{\sqrt{x^2+y^2+z}} &\Rightarrow\quad f_x(7,5,70) &= \frac{7}{\sqrt{7^2+5^2+70}} = \frac{7}{12} \\ f_y(x,y,z) &= \frac{2y}{2\sqrt{x^2+y^2+z}} = \frac{y}{\sqrt{x^2+y^2+z}} &\Rightarrow\quad f_y(7,5,70) &= \frac{5}{\sqrt{7^2+5^2+70}} = \frac{5}{12} \\ f_z(x,y,z) &= \frac{1}{2\sqrt{x^2+y^2+z}} &\Rightarrow\quad f_z(7,5,70) &= \frac{1}{2\sqrt{7^2+5^2+70}} = \frac{1}{24} \end{aligned}$$

Also, $f(7,5,70) = \sqrt{7^2+5^2+70} = 12$. Substituting the values in the linear approximation equation we obtain the following approximation:

$$L(x,y,z) = 12 + \frac{7}{12}(x-7) + \frac{5}{12}(y-5) + \frac{1}{24}(z-70)$$

Now we are ready to approximate $\sqrt{7.1^2 + 4.9^2 + 69.5}$. That is, using the linear approximation,

$$\begin{aligned} L(7.1, 4.9, 69.5) &= 12 + \frac{7}{12}(7.1-7) + \frac{5}{12}(4.9-5) + \frac{1}{24}(69.5-70) \\ &= 12 + \frac{7}{12}\cdot\frac{1}{10} + \frac{5}{12}\cdot -\frac{1}{10} + \frac{1}{24}\cdot -\frac{1}{2} \\ &= 12 + \frac{7}{120} - \frac{5}{120} - \frac{1}{48} \\ &= \frac{2879}{240} = 11.9958333 \end{aligned}$$

The value obtained using a calculator is 11.996667.

26. The plane $z = 2x - y - 1$ is tangent to the graph of $z = f(x, y)$ at $P = (5, 3)$.
(a) Determine $f(5, 3)$, $f_x(5, 3)$, and $f_y(5, 3)$.
(b) Approximate $f(5.2, 2.9)$.

SOLUTION
(a)

$$f_x(x, y) = 2 \quad \Rightarrow \quad f_x(5, 3) = 2 \tag{1}$$
$$f_y(x, y) = -1 \quad \Rightarrow \quad f_y(5, 3) = -1 \tag{2}$$

and

$$f(5, 3) = 2 \cdot 5 - 3 - 1 = 6$$

(b) Now using the linear approximation:

$$L(x, y) = f(5, 3) + f_x(5, 3)(x - 5) + f_y(5, 3)(y - 3)$$

and therefore

$$L(5.2, 2.9) = 6 + 2(5.2 - 5) - (2.9 - 3) = 6 + 2 \cdot \frac{2}{10} + \frac{1}{10} = 6.5$$

27. Figure 4 shows the contour map of a function $f(x, y)$ together with a path $c(t)$ in the counterclockwise direction. The points $c(1)$, $c(2)$, and $c(3)$ are indicated on the path. Let $g(t) = f(c(t))$. Which of statements (i)–(iv) are true? Explain.
(i) $g'(1) > 0$.
(ii) $g(t)$ has a local minimum for some $1 \le t \le 2$.
(iii) $g'(2) = 0$.
(iv) $g'(3) = 0$.

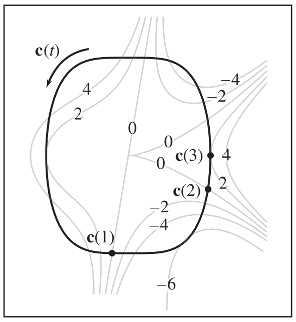


FIGURE 4

SOLUTION (ii) and (iv) are true

28. Jason earns $S(h, c) = 20h\left(1 + \frac{c}{100}\right)^{1.5}$ dollars per month at a used car lot, where h is the number of hours worked and c is the number of cars sold. He has already worked 160 hours and sold 69 cars. Right now Jason wants to go home but wonders how much more he might earn if he stays another 10 minutes with a customer who is considering buying a car. Use the linear approximation to estimate how much extra money Jason will earn if he sells his 70th car during these 10 minutes.

SOLUTION We estimate the money earned in staying for $\frac{1}{6}$ hour more and selling one more car, using the linear approximation

$$\Delta S \approx S_h(a, b)\Delta h + S_c(a, b)\Delta c \tag{1}$$

By the given information, $a = 160$, $b = 69$, $\Delta h = \frac{1}{6}$, and $\Delta c = 1$. We compute the partial derivative of the function:

$$S(h, c) = 20h\left(1 + \frac{c}{100}\right)^{1.5}$$
$$S_h(h, c) = 20\left(1 + \frac{c}{100}\right)^{1.5} \quad \Rightarrow \quad S_h(160, 69) = 43.94$$
$$S_c(h, c) = 20h \cdot 1.5\left(1 + \frac{c}{100}\right)^{0.5} \cdot \frac{1}{100} = 0.3h\left(1 + \frac{c}{100}\right)^{0.5} \quad \Rightarrow \quad S_c(160, 69) = 62.4$$

Substituting the values in (1), we get the following approximation:

$$\Delta S = S_h(160, 69) \cdot \frac{1}{6} + S_c(160, 69) \cdot 1 = 43.94 \cdot \frac{1}{6} + 62.4 \approx \$69.72$$

We see that John will make approximately \$69.72 more if he sells his 70th car during 10 min.

In Exercises 29–32, compute $\frac{d}{dt} f(\mathbf{c}(t))$ *at the given value of* t.

29. $f(x, y) = x + e^y$, $\quad \mathbf{c}(t) = (3t - 1, t^2)$ at $t = 2$

SOLUTION By the Chain Rule for Paths we have

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f \cdot \mathbf{c}'(t) \tag{1}$$

We evaluate the gradient ∇f and $\mathbf{c}'(t)$:

$$\mathbf{c}'(t) = \langle 3, 2t \rangle$$
$$\nabla f = \left\langle f_x, f_y \right\rangle = \left\langle 1, e^y \right\rangle \quad \Rightarrow \quad \nabla f_{\mathbf{c}(t)} = \left\langle 1, e^{t^2} \right\rangle$$

Substituting in (1) we get

$$\frac{d}{dt} f(\mathbf{c}(t)) = \left\langle 1, e^{t^2} \right\rangle \cdot \langle 3, 2t \rangle = 3 + 2te^{t^2}$$

At $t = 2$ we have

$$\frac{d}{dt} f(\mathbf{c}(t)) \bigg|_{t=2} = 3 + 2 \cdot 2 \cdot e^{2^2} = 3 + 4e^4 \approx 221.4.$$

30. $f(x, y, z) = xz - y^2$, $\quad \mathbf{c}(t) = (t, t^3, 1 - t)$ at $t = -2$

SOLUTION We use the Chain Rule for Paths:

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We compute the gradient of f:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle z, -2y, x \rangle$$

On the path, $x = t$, $y = t^3$, and $z = 1 - t$. Therefore,

$$\nabla f_{\mathbf{c}(t)} = \left\langle 1 - t, -2t^3, t \right\rangle$$

Also, $\mathbf{c}'(t) = \left\langle 1, 3t^2, -1 \right\rangle$, hence by (1) we obtain

$$\frac{d}{dt} f(\mathbf{c}(t)) = \left\langle 1 - t, -2t^3, t \right\rangle \cdot \left\langle 1, 3t^2, -1 \right\rangle = 1 - t + 3t^2 \left(-2t^3 \right) - t = -6t^5 - 2t + 1$$

Hence,

$$\frac{d}{dt} f(\mathbf{c}(2) = -6(2)^5 - 2(2) + 1 = -6(32) - 4 + 1 = -195$$

31. $f(x, y) = xe^{3y} - ye^{3x}$, $\quad \mathbf{c}(t) = (e^t, \ln t)$ at $t = 1$

SOLUTION We use the Chain Rule for Paths:

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t) \tag{1}$$

We find the ∇f at the point $\mathbf{c}(1)$ and compute $\mathbf{c}'(1)$. We get

$$\nabla f = \left\langle f_x, f_y \right\rangle = \left\langle e^{3y} - 3ye^{3x}, 3xe^{3y} - e^{3x} \right\rangle$$
$$\mathbf{c}(1) = \left\langle e^1, \ln 1 \right\rangle = \langle e, 0 \rangle$$
$$\nabla f_{\mathbf{c}(1)} = \left\langle e^{3 \cdot 0} - 3 \cdot 0 e^{3e}, 3ee^{3 \cdot 0} - e^{3e} \right\rangle = \left\langle 1, 3e - e^{3e} \right\rangle \tag{2}$$
$$\mathbf{c}'(t) = \frac{d}{dt} \left\langle e^t, \ln t \right\rangle = \left\langle e^t, t^{-1} \right\rangle \quad \Rightarrow \quad \mathbf{c}'(1) = \langle e, 1 \rangle \tag{3}$$

Substituting (2) and (3) in (1) gives

$$\frac{d}{dt} f(\mathbf{c}(t)) \bigg|_{t=1} = \nabla f_{c(1)} \cdot \mathbf{c}'(1) = \left\langle 1, 3e - e^{3e} \right\rangle \cdot \langle e, 1 \rangle = e + 3e - e^{3e} = 4e - e^{3e}$$

32. $f(x,y)=\tan^{-1}\frac{y}{x}$, $\mathbf{c}(t)=(\cos t,\sin t)$, $t=\frac{\pi}{3}$

SOLUTION We use the Chain Rule for Paths. We have

$$\nabla f=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle=\left\langle\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2},\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right\rangle=\left\langle\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right\rangle$$

On the path, $x=\cos t$ and $y=\sin t$. Therefore,

$$\nabla f_{\mathbf{c}(t)}=\left\langle-\frac{\sin t}{\cos^2 t+\sin^2 t},\frac{\cos t}{\cos^2 t+\sin^2 t}\right\rangle=\langle-\sin t,\cos t\rangle$$
$$\mathbf{c}'(t)=\langle-\sin t,\cos t\rangle$$

At the point $t=\frac{\pi}{3}$ we have

$$\nabla f_{\mathbf{c}\left(\frac{\pi}{3}\right)}=\left\langle-\sin\frac{\pi}{3},\cos\frac{\pi}{3}\right\rangle=\left\langle-\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle\quad\text{and}\quad\mathbf{c}'\left(\frac{\pi}{3}\right)=\left(-\sin\frac{\pi}{3},\cos\frac{\pi}{3}\right)=\left\langle-\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle$$

Therefore,

$$\left.\frac{d}{dt}f(\mathbf{c}(t))\right|_{t=\frac{\pi}{3}}=\nabla f_{\mathbf{c}\left(\frac{\pi}{3}\right)}\cdot\mathbf{c}'\left(\frac{\pi}{3}\right)=\left\langle-\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle\cdot\left\langle-\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle=\frac{3}{4}+\frac{1}{4}=1$$

In Exercises 33–36, compute the directional derivative at P in the direction of $\mathbf{v}$.

33. $f(x,y)=x^3y^4$, $P=(3,-1)$, $\mathbf{v}=2\mathbf{i}+\mathbf{j}$

SOLUTION We first normalize $\mathbf{v}$ to find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u}=\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{2\mathbf{i}+\mathbf{j}}{\sqrt{2^2+1^2}}=\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}$$

We compute the directional derivative using the following equality:

$$D_{\mathbf{u}}f(3,-1)=\nabla f_{(3,-1)}\cdot\mathbf{u}$$

The gradient vector at the given point is the following vector:

$$\nabla f=\langle f_x,f_y\rangle=\left\langle 3x^2y^4,4x^3y^3\right\rangle\quad\Rightarrow\quad\nabla f_{(3,-1)}=\langle 27,-108\rangle$$

Hence,

$$D_{\mathbf{u}}f(3,-1)=\langle 27,-108\rangle\cdot\left\langle\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right\rangle=\frac{54}{\sqrt{5}}-\frac{108}{\sqrt{5}}=-\frac{54}{\sqrt{5}}$$

34. $f(x,y,z)=zx-xy^2$, $P=(1,1,1)$, $\mathbf{v}=\langle 2,-1,2\rangle$

SOLUTION We first normalize $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u}=\frac{\langle 2,-1,2\rangle}{\sqrt{2^2+(-1)^2+2^2}}=\left\langle\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right\rangle$$

We compute the directional derivative using the following equality:

$$D_{\mathbf{u}}f(1,1,1)=\nabla f_{(1,1,1)}\cdot\mathbf{u}$$

The gradient vector at the point $(1,1,1)$ is the following vector:

$$\nabla f=\langle f_x,f_y,f_z\rangle=\left\langle z-y^2,-2xy,x\right\rangle\quad\Rightarrow\quad\nabla f_{(1,1,1)}=\langle 0,-2,1\rangle$$

Hence,

$$D_{\mathbf{u}}f(1,1,1)=\langle 0,-2,1\rangle\cdot\left\langle\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right\rangle=0+\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$$

35. $f(x,y)=e^{x^2+y^2}$, $P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $\mathbf{v}=\langle 3,-4\rangle$

SOLUTION We normalize $\mathbf{v}$ to obtain a vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u}=\frac{\langle 3,-4\rangle}{\sqrt{3^2+(-4)^2}}=\left\langle\frac{3}{5},-\frac{4}{5}\right\rangle$$

We use the following theorem:

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u} \tag{1}$$

We find the gradient of f at the given point:

$$\nabla f = \langle f_x, f_y \rangle = \left\langle 2xe^{x^2+y^2}, 2ye^{x^2+y^2} \right\rangle = 2e^{x^2+y^2} \langle x, y \rangle$$

Hence,

$$\nabla f_P = 2e^{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2} \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = e\sqrt{2} \langle 1, 1 \rangle$$

Substituting in (1) we get

$$D_{\mathbf{u}} f(P) = \sqrt{2}e \langle 1, 1 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \sqrt{2}e \left(\frac{3}{5} - \frac{4}{5} \right) = -\frac{\sqrt{2}e}{5}$$

36. $f(x, y, z) = \sin(xy + z), \quad P = (0, 0, 0), \quad \mathbf{v} = \mathbf{j} + \mathbf{k}$

SOLUTION We normalize $\mathbf{v}$ to obtain a vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{1}{\sqrt{0^2 + 1^2 + 1^2}} \cdot \langle 0, 1, 1 \rangle = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$$

By the Theorem on Evaluating Directional Derivatives,

$$D_{\mathbf{v}} f(P) = \nabla f_P \cdot \mathbf{u} \tag{1}$$

We compute the gradient vector:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle y\cos(xy + z), x\cos(xy + z), \cos(xy + z) \rangle$$

Hence,

$$\nabla f_P = \langle 0, 0, 1 \rangle .$$

By (1) we conclude that

$$D_{\mathbf{v}} f(P) = \nabla f_P \cdot \mathbf{u} = \langle 0, 0, 1 \rangle \cdot \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle = \frac{1}{\sqrt{2}}.$$

37. Find the unit vector $\mathbf{e}$ at $P = (0, 0, 1)$ pointing in the direction along which $f(x, y, z) = xz + e^{-x^2+y}$ increases most rapidly.

SOLUTION The gradient vector ∇f_P points in the direction of maximum rate of increase of f. Therefore we need to find a unit vector in the direction of ∇f_P. We first find the gradient of $f(x, y, z) = xz + e^{-x^2+y}$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle z - 2xe^{-x^2+y}, e^{-x^2+y}, x \right\rangle$$

At the point $P = (0, 0, 1)$ we have

$$\nabla f_P = \langle 1, 1, 0 \rangle .$$

We normalize ∇f_P to obtain the unit vector $\mathbf{e}$ at P pointing in the direction of maximum increase of f:

$$\mathbf{e} = \frac{\nabla f_P}{\|\nabla f_P\|} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

38. Find an equation of the tangent plane at $P = (0, 3, -1)$ to the surface with equation

$$ze^x + e^{z+1} = xy + y - 3$$

SOLUTION The surface is defined implicitly by the equation

$$F(x, y, z) = ze^x + e^{z+1} - xy - y + 3$$

The tangent plane to the surface at the point $(0, 3, -1)$ has the following equation:

$$0 = F_x(0, 3, -1)x + F_y(0, 3, -1)(y - 3) + F_z(0, 3, -1)(z + 1) \tag{1}$$

We compute the partial derivatives at the given point:

$$\begin{aligned} F_x(x, y, z) &= ze^x - y && \Rightarrow && F_x(0, 3, -1) = -1e^0 - 3 = -4 \\ F_y(x, y, z) &= -x - 1 && \Rightarrow && F_y(0, 3, -1) = -0 - 1 = -1 \\ F_z(x, y, z) &= e^x + e^{z+1} && \Rightarrow && F_z(0, 3, -1) = e^0 + e^{-1+1} = 2 \end{aligned}$$

Substituting in (1) we obtain the following equation:

$$\begin{aligned} -4x - (y - 3) + 2(z + 1) &= 0 \\ -4x - y + 2z + 5 &= 0 \\ 2z = 4x + y - 5 \quad &\Rightarrow \quad z = 2x + 0.5y - 2.5 \end{aligned}$$

39. Let $n \neq 0$ be an integer and r an arbitrary constant. Show that the tangent plane to the surface $x^n + y^n + z^n = r$ at $P = (a, b, c)$ has equation

$$a^{n-1}x + b^{n-1}y + c^{n-1}z = r$$

SOLUTION The tangent plane to the surface, defined implicitly by $F(x, y, z) = r$ at a point (a, b, c) on the surface, has the following equation:

$$0 = F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) \tag{1}$$

The given surface is defined by the function $F(x, y, z) = x^n + y^n + z^n$. We find the partial derivative of F at a point $P = (a, b, c)$ on the surface:

$$\begin{aligned} F_x(x, y, z) &= nx^{n-1} & & F_x(a, b, c) = na^{n-1} \\ F_y(x, y, z) &= ny^{n-1} & \Rightarrow \quad & F_y(a, b, c) = nb^{n-1} \\ F_z(x, y, z) &= nz^{n-1} & & F_z(a, b, c) = nc^{n-1} \end{aligned}$$

Substituting in (1) we get

$$na^{n-1}(x - a) + nb^{n-1}(y - b) + nc^{n-1}(z - c) = 0$$

We divide by n and simplify:

$$\begin{aligned} a^{n-1}x - a^n + b^{n-1}y - b^n + c^{n-1}z - c^n &= 0 \\ a^{n-1}x + b^{n-1}y + c^{n-1}z &= a^n + b^n + c^n \end{aligned} \tag{2}$$

The point $P = (a, b, c)$ lies on the surface, hence it satisfies the equation of the surface. That is,

$$a^n + b^n + c^n = r$$

Substituting in (2) we obtain the following equation of the tangent plane:

$$a^{n-1}x + b^{n-1}y + c^{n-1}z = r$$

40. Let $f(x, y) = (x - y)e^x$. Use the Chain Rule to calculate $\partial f/\partial u$ and $\partial f/\partial v$ (in terms of u and v), where $x = u - v$ and $y = u + v$.

SOLUTION First we calculate the Primary Derivatives:

$$\frac{\partial f}{\partial x} = e^x(x - y) + e^x = e^x(x - y + 1), \quad \frac{\partial f}{\partial y} = -e^x$$

Since $\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = 1$, $\frac{\partial x}{\partial v} = -1$, and $\frac{\partial y}{\partial v} = 1$, the Chain Rule gives

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = e^x(x - y + 1) \cdot 1 - e^x \cdot 1 = e^x(x - y + 1 - 1) = e^x(x - y) \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = e^x(x - y + 1) \cdot (-1) - e^x \cdot 1 = e^x(y - x - 2) \end{aligned}$$

We now substitute $x = u - v$ and $y = u + v$ to express the partial derivatives in terms of u and v. We get

$$\frac{\partial f}{\partial u} = e^{u-v}(u - v - u - v) = -2ve^{u-v}$$

$$\frac{\partial f}{\partial v} = e^{u-v}(u + v - u + v - 2) = 2e^{u-v}(v - 1)$$

41. Let $f(x, y, z) = x^2y + y^2z$. Use the Chain Rule to calculate $\partial f/\partial s$ and $\partial f/\partial t$ (in terms of s and t), where

$$x = s + t, \quad y = st, \quad z = 2s - t$$

SOLUTION We compute the Primary Derivatives:

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 2yz, \quad \frac{\partial f}{\partial z} = y^2$$

Since $\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial s} = t$, $\frac{\partial z}{\partial s} = 2$, $\frac{\partial x}{\partial t} = 1$, $\frac{\partial y}{\partial t} = s$, and $\frac{\partial z}{\partial t} = -1$, the Chain Rule gives

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 2xy \cdot 1 + \left(x^2 + 2yz\right)t + y^2 \cdot 2 \\
&= 2xy + \left(x^2 + 2yz\right)t + 2y^2 \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = 2xy \cdot 1 + \left(x^2 + 2yz\right)s + y^2 \cdot (-1) \\
&= 2xy + \left(x^2 + 2yz\right)s - y^2
\end{aligned}$$

We now substitute $x = s + t$, $y = st$, and $z = 2s - t$ to express the answer in terms of the independent variables s, t. We get

$$\begin{aligned}
\frac{\partial f}{\partial s} &= 2(s+t)st + \left((s+t)^2 + 2st(2s-t)\right)t + 2s^2t^2 \\
&= 2s^2t + 2st^2 + \left(s^2 + 2st + t^2 + 4s^2t - 2st^2\right)t + 2s^2t^2 \\
&= 3s^2t + 4st^2 + t^3 - 2st^3 + 6s^2t^2 \\
\frac{\partial f}{\partial t} &= 2(s+t)st + \left((s+t)^2 + 2st(2s-t)\right)s - s^2t^2 \\
&= 2s^2t + 2st^2 + \left(s^2 + 2st + t^2 + 4s^2t - 2st^2\right)s - s^2t^2 \\
&= 4s^2t + 3st^2 + s^3 + 4s^3t - 3s^2t^2
\end{aligned}$$

42. Let P have spherical coordinates $(\rho, \theta, \phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$. Calculate $\left.\frac{\partial f}{\partial \phi}\right|_P$ assuming that

$$f_x(P) = 4, \quad f_y(P) = -3, \quad f_z(P) = 8$$

Recall that $x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi$.

SOLUTION Recall the Chain Rule:

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

Taking partial derivatives (with respect to ϕ) and evaluating:

$$\begin{aligned}
\frac{\partial x}{\partial \phi} &= \rho\cos\theta\cos\phi \quad \Rightarrow \quad \left.\frac{\partial x}{\partial \phi}\right|_{(2,\pi/4,\pi/4)} = 1 \\
\frac{\partial y}{\partial \phi} &= \rho\sin\theta\cos\phi \quad \Rightarrow \quad \left.\frac{\partial y}{\partial \phi}\right|_{(2,\pi/4,\pi/4)} = 1 \\
\frac{\partial z}{\partial \phi} &= -\rho\sin\phi \quad \Rightarrow \quad \left.\frac{\partial z}{\partial \phi}\right|_{(2,\pi/4,\pi/4)} = -\sqrt{2}
\end{aligned}$$

Hence,

$$\left.\frac{\partial f}{\partial \phi}\right|_P = 4 \cdot 1 - 3 \cdot 1 - 8\sqrt{2} = 1 - 8\sqrt{2}$$

43. Let $g(u, v) = f(u^3 - v^3, v^3 - u^3)$. Prove that

$$v^2 \frac{\partial g}{\partial u} - u^2 \frac{\partial g}{\partial v} = 0$$

SOLUTION We are given the function $f(x, y)$, where $x = u^3 - v^3$ and $y = v^3 - u^3$. Using the Chain Rule we have the following derivatives:

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \qquad (1)$$

We compute the following partial derivatives:

$$\frac{\partial x}{\partial u} = 3u^2, \quad \frac{\partial y}{\partial u} = -3u^2$$
$$\frac{\partial x}{\partial v} = -3v^2, \quad \frac{\partial y}{\partial v} = 3v^2$$

Substituting in (1) we obtain

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot 3u^2 + \frac{\partial f}{\partial y}\left(-3u^2\right) = 3u^2\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$
$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\left(-3v^2\right) + \frac{\partial f}{\partial y}\left(3v^2\right) = -3v^2\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$

Therefore,

$$v^2 \frac{\partial g}{\partial u} + u^2 \frac{\partial g}{\partial v} = 3u^2v^2\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) - 3u^2v^2\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = 0$$

44. Let $f(x, y) = g(u)$, where $u = x^2 + y^2$ and $g(u)$ is differentiable. Prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 4u\left(\frac{dg}{du}\right)^2$$

SOLUTION We use the Chain Rule and the partial derivatives $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, to differentiate the equation $f(x, y, z) = g(u)$ with respect to x and to y. We get

$$\frac{\partial f}{\partial x} = g'(u) \cdot \frac{\partial u}{\partial x} = g'(u) \cdot 2x$$
$$\frac{\partial f}{\partial y} = g'(u) \cdot \frac{\partial u}{\partial y} = g'(u) \cdot 2y$$

Therefore,

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(g'(u) \cdot 2x\right)^2 + \left(g'(u) \cdot 2y\right)^2 = 4x^2 g'(u)^2 + 4y^2 g'(u)^2$$
$$= 4\left(x^2 + y^2\right) g'(u)^2 = 4ug'(u)^2$$

Since $\frac{\partial f}{\partial u} = g'(u)$, we find that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 4u\left(\frac{\partial f}{\partial u}\right)^2$$

45. Calculate $\partial z/\partial x$, where $xe^z + ze^y = x + y$.

SOLUTION The function $F(x, y, z) = xe^z + ze^y - x - y = 0$ defines z implicitly as a function of x and y. Using implicit differentiation, the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad (1)$$

We compute the partial derivatives F_x and F_z:

$$F_x = e^z - 1$$
$$F_z = xe^z + e^y$$

Substituting in (1) gives

$$\frac{\partial z}{\partial x} = -\frac{e^z - 1}{xe^z + e^y}.$$

46. Let $f(x, y) = x^4 - 2x^2 + y^2 - 6y$.

(a) Find the critical points of f and use the Second Derivative Test to determine whether they are a local minima or a local maxima.

(b) Find the minimum value of f without calculus by completing the square.

SOLUTION

(a) To find the critical points of the function $f(x, y) = x^4 - 2x^2 + y^2 - 6y$ we set the partial derivatives equal to zero and solve. This gives

$$\left.\begin{aligned} f_x(x, y) &= 4x^3 - 4x = 4x\left(x^2 - 1\right) = 0 \\ f_y(x, y) &= 2y - 6 = 2(y - 3) = 0 \end{aligned}\right. \quad \Rightarrow \quad x = 0, \quad x = -1, \quad x = 1, \quad y = 3$$

The critical points are $(0, 3)$, $(-1, 3)$, $(1, 3)$. We now apply the Second Derivative Test to examine the critical points. We compute the second-order partials:

$$f_{xx}(x, y) = 12x^2 - 4, \quad f_{yy} = 2, \quad f_{xy} = 0$$

The discriminant is

$$D = f_{xx} f_{yy} - f_{xy}^2 = 2\left(12x^2 - 4\right) = 8\left(3x^2 - 1\right)$$

Substituting the critical points gives

$$\begin{aligned} D(0, 3) &= -8 < 0 \quad \Rightarrow \quad (0, 3) \text{ is a saddle point} \\ D(-1, 3) &= 16 > 0, \quad f_{xx}(-1, 3) = 8 > 0 \quad \Rightarrow \quad f(-1, 3) \text{ is a local minimum} \\ D(1, 3) &= 16 > 0, \quad f_{xx}(1, 3) = 8 > 0 \quad \Rightarrow \quad f(1, 3) \text{ is a local minimum} \end{aligned}$$

(b) Computing the square in x and y, we obtain

$$\begin{aligned} x^4 - 2x^2 + y^2 - 6y &= \left(x^2 - 1\right)^2 - 1 + (y - 3)^2 - 9 \\ &= \left(x^2 - 1\right)^2 + (y - 3)^2 - 10 \end{aligned}$$

This function has a minimum when $x^2 - 1 = 0$ and $y - 3 = 0$, that is, $x = \pm 1$ and $y = 3$. Therefore, the minimum value is -10 obtained at the points $(1, 3)$ and $(-1, 3)$.

In Exercises 47–50, find the critical points of the function and analyze them using the Second Derivative Test.

47. $f(x, y) = x^4 - 4xy + 2y^2$

SOLUTION To find the critical points, we need the first-order partial derivatives and set them equal to zero to solve for x and y:

$$f_x(x, y) = 4x^3 - 4y = 0, \quad f_y(x, y) = -4x + 4y = 0$$

Looking at the second equation we see $x = y$. Using this in the first equation, then

$$4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x = 0, \pm 1$$

Therefore, our critical points are:

$$(0, 0), (1, 1), (-1, -1)$$

Now to find the discriminant, D, we need the second-order partial derivatives:

$$f_{xx}(x, y) = 12x^2, \quad f_{yy}(x, y) = 4, \quad f_{xy}(x, y) = -4$$

Hence,

$$D(x, y) = f_{xx} f_{yy} - f{xy}^2 = 48x^2 - 16 = 16(3x^2 - 1)$$

Analyzing our three critical points we see:

$$D(0,0) = -16 < 0, \quad D(1,1) = 32 > 0, \quad D(-1,-1) = 32 > 0$$

Since the discriminant for (0, 0) is negative, (0, 0) is a saddle point.

Looking at $f_{xx}(1,1) = 12 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ hence, the points (1, 1) and (−1, −1) are both local minima.

48. $f(x, y) = x^3 + 2y^3 - xy$

SOLUTION We set the partial derivatives of $f(x, y) = x^3 + 2y^3 - xy$ equal to zero and solve to find the critical points. We get

$$f_x(x, y) = 3x^2 - y = 0$$
$$f_y(x, y) = 6y^2 - x = 0$$

The first equation gives $y = 3x^2$. Substituting in the second equation we get

$$6 \cdot \left(3x^2\right)^2 - x = 0$$
$$54x^4 - x = x \cdot \left(54x^3 - 1\right) = 0$$
$$54x^3 - 1 = 0 \quad \Rightarrow \quad x_1 = 0, \quad x_2 = 0.26$$

The corresponding y-coordinates are obtained from $y = 3x^2$. That is,

$$y_1 = 0, \quad y_2 = 3 \cdot 0.26^2 = 0.2$$

There are two critical points, (0, 0) and (0.26, 0.2). We next use the Second Derivative Test to examine the critical points. We compute the second-order partials at these points:

$$\begin{array}{lclll} f_{xx}(x,y) = 6x & & f_{xx}(0,0) = 0 & f_{xx}(0.26, 0.2) = 1.56 \\ f_{yy}(x,y) = 12y & \Rightarrow & f_{yy}(0,0) = 0 & f_{yy}(0.26, 0.2) = 2.4 \\ f_{xy}(x,y) = -1 & & f_{xy}(0,0) = -1 & f_{xy}(0.26, 0.2) = -1 \end{array}$$

We compute the discriminant at the critical points:

$$D(0,0) = f_{xx} \cdot f_{yy} - f_{xy}^2 = -1 < 0$$
$$D(0.26, 0.2) = f_{xx} \cdot f_{yy} - f_{xy}^2 = 1.56 \cdot 2.4 - 1 > 0, \quad f_{xx}(0.26, 0.2) > 0$$

We conclude that (0, 0) is a saddle point, whereas at (0.26, 0.2) the function has a local minimum.

49. $f(x, y) = e^{x+y} - xe^{2y}$

SOLUTION We find the critical point by setting the partial derivatives of $f(x, y) = e^{x+y} - xe^{2y}$ equal to zero and solve. This gives

$$f_x(x, y) = e^{x+y} - e^{2y} = 0$$
$$f_y(x, y) = e^{x+y} - 2xe^{2y} = 0$$

The first equation gives $e^{x+y} = e^{2y}$ and the second equation gives $e^{x+y} = 2xe^{2y}$. Equating the two expressions, dividing by the nonzero function e^{2y}, and solving for x, we obtain

$$e^{2y} = 2xe^{2y} \quad \Rightarrow \quad 1 = 2x \quad \Rightarrow \quad x = \frac{1}{2}$$

We now substitute $x = \frac{1}{2}$ in the first equation and solve for y, to obtain

$$e^{\frac{1}{2}+y} - e^{2y} = 0 \quad \Rightarrow \quad e^{\frac{1}{2}+y} = e^{2y} \quad \Rightarrow \quad \frac{1}{2} + y = 2y \quad \Rightarrow \quad y = \frac{1}{2}$$

There is one critical point, $\left(\frac{1}{2}, \frac{1}{2}\right)$. We examine the critical point using the Second Derivative Test. We compute the second derivatives at this point:

$$f_{xx}(x, y) = e^{x+y} \quad \Rightarrow \quad f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = e^{\frac{1}{2}+\frac{1}{2}} = e$$

$$f_{yy}(x, y) = e^{x+y} - 4xe^{2y} \quad \Rightarrow \quad f_{yy}\left(\frac{1}{2}, \frac{1}{2}\right) = e^{\frac{1}{2}+\frac{1}{2}} - 4 \cdot \frac{1}{2}e^{2\cdot\frac{1}{2}} = -e$$

$$f_{xy}(x, y) = e^{x+y} - 2e^{2y} \quad \Rightarrow \quad f_{xy}\left(\frac{1}{2}, \frac{1}{2}\right) = e^{\frac{1}{2}+\frac{1}{2}} - 2e^{2\cdot\frac{1}{2}} = -e$$

Therefore the discriminant at the critical point is

$$D\left(\frac{1}{2},\frac{1}{2}\right) = f_{xx}f_{yy} - f_{xy}^2 = e\cdot(-e) - (-e)^2 = -2e^2 < 0$$

We conclude that $\left(\frac{1}{2},\frac{1}{2}\right)$ is a saddle point.

50. $f(x,y) = \sin(x+y) - \dfrac{1}{2}(x+y^2)$

SOLUTION We find the critical points by setting the partial derivatives of $f(x,y) = \sin(x+y) - 0.5\left(x+y^2\right)$ equal to zero and solve. We get

$$f_x(x,y) = \cos(x+y) - \frac{1}{2} = 0$$
$$f_y(x,y) = \cos(x+y) - y = 0$$

By the second equation $y = \cos(x+y)$. Substituting in the first equation gives $y - \frac{1}{2} = 0$ or $y = \frac{1}{2}$. We set $y = \frac{1}{2}$ in the first equation and solve for x, to obtain

$$\cos\left(x+\frac{1}{2}\right) - \frac{1}{2} = 0$$
$$\cos\left(x+\frac{1}{2}\right) = \frac{1}{2}$$

The general solution is

$$x + \frac{1}{2} = \pm\frac{\pi}{3} + 2\pi k \quad \Rightarrow \quad x = -\frac{1}{2} \pm \frac{\pi}{3} + 2\pi k$$

The critical points are thus

$$P_k = \left(-\frac{1}{2} + \frac{\pi}{3} + 2\pi k, \frac{1}{2}\right), \quad Q_k = \left(-\frac{1}{2} - \frac{\pi}{3} + 2\pi k, \frac{1}{2}\right)$$

We examine the critical points using the Second Derivative Test. We first compute the second-order partials at the critical points:

$$f_{xx}(x,y) = -\sin(x+y) \quad \Rightarrow \quad f_{xx}(P_k) = -\sin\left(\frac{\pi}{3} + 2\pi k\right) = -\frac{\sqrt{3}}{2}$$
$$f_{xx}(Q_k) = -\sin\left(-\frac{\pi}{3} + 2\pi k\right) = \frac{\sqrt{3}}{2}$$
$$f_{yy}(x,y) = -\sin(x+y) - 1 \quad \Rightarrow \quad f_{yy}(P_k) = -\frac{\sqrt{3}}{2} - 1$$
$$f_{yy}(Q_k) = \frac{\sqrt{3}}{2} - 1$$
$$f_{xy}(x,y) = -\sin(x+y) \quad \Rightarrow \quad f_{xy}(P_k) = -\frac{\sqrt{3}}{2}$$
$$f_{xy}(Q_k) = \frac{\sqrt{3}}{2}$$

We compute the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical points:

$$D(P_k) = \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2} - 1\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{\sqrt{3}}{2} > 0, \quad f_{xx}(P_k) = -\frac{\sqrt{3}}{2} < 0$$
$$D(Q_k) = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2} - 1\right) - \left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{\sqrt{3}}{2} < 0$$

We conclude that $Q_k = \left(-\frac{1}{2} - \frac{\pi}{3} + 2\pi k, \frac{1}{2}\right)$ are saddle points, and at the points $P_k = \left(-\frac{1}{2} + \frac{\pi}{3} + 2\pi k, \frac{1}{2}\right)$ the function has local maxima.

51. Prove that $f(x, y) = (x + 2y)e^{xy}$ has no critical points.

SOLUTION We find the critical points by setting the partial derivatives of $f(x, y) = (x + 2y)e^{xy}$ equal to zero and solving. We get

$$f_x(x, y) = e^{xy} + (x + 2y)ye^{xy} = e^{xy}\left(1 + xy + 2y^2\right) = 0$$

$$f_y(x, y) = 2e^{xy} + (x + 2y)xe^{xy} = e^{xy}\left(2 + x^2 + 2xy\right) = 0$$

We divide the two equations by the nonzero expression e^{xy} to obtain the following equations:

$$1 + xy + 2y^2 = 0$$
$$2 + 2xy + x^2 = 0$$

The first equation implies that $xy = -1 - 2y^2$. Substituting in the second equation gives

$$\begin{aligned} 2 + 2\left(-1 - 2y^2\right) + x^2 &= 0 \\ 2 - 2 - 4y^2 + x^2 &= 0 \\ x^2 = 4y^2 \quad &\Rightarrow \quad x = 2y \quad \text{or} \quad x = -2y \end{aligned}$$

We substitute in the first equation and solve for y:

$$\begin{array}{cc} x = 2y & x = -2y \\ \hline 1 + 2y^2 + 2y^2 = 0 & 1 - 2y^2 + 2y^2 = 0 \\ 1 + 4y^2 = 0 & 1 = 0 \\ y^2 = -\frac{1}{4} & \end{array}$$

In both cases there is no solution. We conclude that there are no solutions for $f_x = 0$ and $f_y = 0$, that is, there are no critical points.

52. Find the global extrema of $f(x, y) = x^3 - xy - y^2 + y$ on the square $[0, 1] \times [0, 1]$.

SOLUTION

Step 1. Examine the critical points. We set the partial derivatives of $f(x, y) = x^3 - xy - y^2 + y$ equal to zero and solve to find the critical points in the interior of the square.

$$f_x(x, y) = 3x^2 - y = 0$$
$$f_y(x, y) = -x - 2y + 1 = 0$$

The first equation gives $y = 3x^2$. We substitute in the second equation and solve for x.

$$\begin{aligned} &-x - 2 \cdot 3x^2 + 1 = 0 \\ &6x^2 + x - 1 = 0 \\ &x_{1,2} = \frac{-1 \pm \sqrt{1 + 24}}{12} = \frac{-1 \pm 5}{12} \quad \Rightarrow \quad x_1 = -\frac{1}{2}, \quad x_2 = \frac{1}{3} \end{aligned}$$

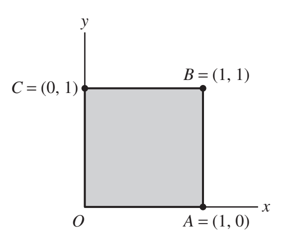
The corresponding y-coordinates are determined by $y = 3x^2$. That is,

$$y_1 = 3 \cdot \left(-\frac{1}{2}\right)^2 = \frac{3}{4}, \quad y_2 = 3 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

Therefore, the critical points are

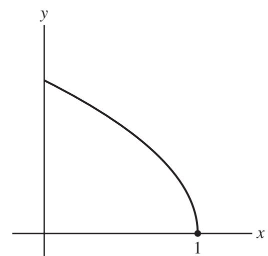
$$\left(-\frac{1}{2}, \frac{3}{4}\right), \quad \left(\frac{1}{3}, \frac{1}{3}\right)$$

Step 2. Find the global extrema on the boundary.



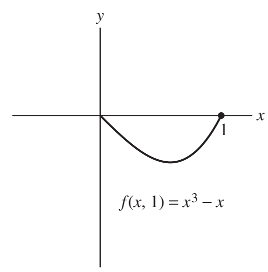
We consider each part of the boundary separately.

The segment $\overline{OA}$: On this segment $y = 0$, $0 \le x \le 1$, hence $f(x, 0) = x^3$. The maximum value occurs at $x = 1$ and the minimum value occurs at $x = 0$. The corresponding points are $(0, 0)$ and $(1, 0)$.
The segment $\overline{AB}$: On this segment $x = 1$, $0 \le y \le 1$, hence $f(1, y) = 1 - y - y^2 + y = 1 - y^2$.



$f(1, y) = 1 - y^2$

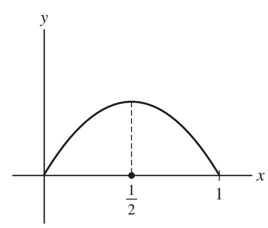
The maximum value in the interval $0 \le y \le 1$ occurs at $y = 0$, and the minimum value occurs at $y = 1$. The corresponding points on the boundary of the square are $(1, 0)$ and $(1, 1)$.
The segment $\overline{BC}$: On this segment $y = 1$, $0 \le x \le 1$, hence $f(x, 1) = x^3 - x - 1 + 1 = x^3 - x$.



Using calculus of one variable and referring to the graph of $f(x, 1)$, we see that the maximum value occurs at $x = 0$ and $x = 1$ and the minimum value occurs at $x = \frac{1}{\sqrt{3}}$. The corresponding points on the segment $\overline{BC}$ are

$$\left(\frac{1}{\sqrt{3}}, 1\right), \quad (0, 1), \quad \text{and} \quad (1, 1)$$

The segment $\overline{OC}$: On this segment $x = 0$, $0 \le y \le 1$, hence $f(0, y) = -y^2 + y$.



$f(0, y) = -y^2 + y$

The maximum value occurs at $y = \frac{1}{2}$ and the minimum value occurs at $y = 0$ and $y = 1$. The corresponding points on the segment $\overline{OC}$ are

$$\left(0, \frac{1}{2}\right), \quad (0, 0), \quad (0, 1)$$

Step 3. Conclusions. Since the global extrema occur either at critical points in the interior of the region or on the boundary of the region, the candidates for global extrema are the following points:

$$\left(-\frac{1}{2}, \frac{3}{4}\right), \quad \left(\frac{1}{3}, \frac{1}{3}\right), \quad (0, 0), \quad (1, 0), \quad (1, 1), \quad (0, 1), \quad \left(0, \frac{1}{2}\right), \quad \left(\frac{1}{\sqrt{3}}, 1\right)$$

We compute $f(x, y) = x^3 - xy - y^2 + y$ at these points:

$$f\left(-\frac{1}{2}, \frac{3}{4}\right) = \left(-\frac{1}{2}\right)^3 + \frac{1}{2} \cdot \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \frac{3}{4} = \frac{7}{16} \approx 0.437$$

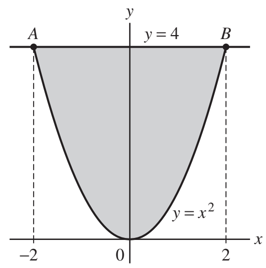
$$f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \frac{1}{3} \cdot \frac{1}{3} - \left(\frac{1}{3}\right)^2 + \frac{1}{3} = \frac{4}{27} \approx 0.148$$

$$f(0,0)=0$$
$$f(1,0)=1$$
$$f(1,1)=1-1-1+1=0$$
$$f(0,1)=-1^2+1=0$$
$$f\left(0,\frac{1}{2}\right)=-\left(\frac{1}{2}\right)^2+\frac{1}{2}=\frac{1}{4}$$
$$f\left(\frac{1}{\sqrt{3}},1\right)=\left(\frac{1}{\sqrt{3}}\right)^3-\frac{1}{\sqrt{3}}-1+1=-\frac{2\sqrt{3}}{9}=-0.38$$

We conclude that the maximum value of f on the square is $f(1,0)=1$ and the minimum value is $f\left(\frac{1}{\sqrt{3}},1\right)=-0.38$.

53. Find the global extrema of $f(x,y)=2xy-x-y$ on the domain $\{y\le 4, y\ge x^2\}$.

SOLUTION The region is shown in the figure.



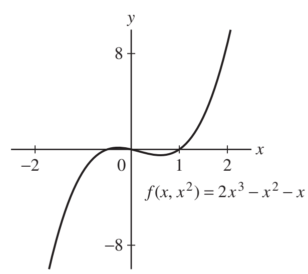
Step 1. Finding the critical points. We find the critical points in the interior of the domain by setting the partial derivatives equal to zero and solving. We get

$$f_x=2y-1=0$$
$$f_y=2x-1=0 \quad\Rightarrow\quad x=\frac{1}{2},\quad y=\frac{1}{2}$$

The critical point is $\left(\frac{1}{2},\frac{1}{2}\right)$. (It lies in the interior of the domain since $\frac{1}{2}<4$ and $\frac{1}{2}>\left(\frac{1}{2}\right)^2$).

Step 2. Finding the global extrema on the boundary. We consider the two parts of the boundary separately.

The parabola $y=x^2$, $-2\le x\le 2$:



On this curve, $f(x,x^2)=2\cdot x\cdot x^2-x-x^2=2x^3-x^2-x$. Using calculus in one variable or the graph of the function, we see that the minimum of $f(x,x^2)$ on the interval occurs at $x=-2$ and the maximum at $x=2$. The corresponding points are $(-2,4)$ and $(2,4)$.

The segment $\overline{AB}$: On this segment $y=4$, $-2\le x\le 2$, hence $f(x,4)=2\cdot x\cdot 4-x-4=7x-4$. The maximum value occurs at $x=2$ and the minimum value at $x=-2$. The corresponding points on the segment $\overline{AB}$ are $(-2,4)$ and $(2,4)$

Step 3. Conclusions. Since the global extrema occur either at critical points in the interior of the domain or on the boundary of the domain, the candidates for global extrema are the following points:

$$\left(\frac{1}{2},\frac{1}{2}\right),\quad (-2,4),\quad (2,4)$$

We compute the values of $f=2xy-x-y$ at these points:

$$f\left(\frac{1}{2},\frac{1}{2}\right)=2\cdot\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}$$

$$f(-2,4) = 2 \cdot (-2) \cdot 4 + 2 - 4 = -18$$
$$f(2,4) = 2 \cdot 2 \cdot 4 - 2 - 4 = 10$$

We conclude that the global maximum is $f(2,4) = 10$ and the global minimum is $f(-2,4) = -18$.

54. Find the maximum of $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 2x + y + 4z = 1$.

SOLUTION

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle yz, xz, xy \rangle$ and $\nabla g = \langle 2, 1, 4 \rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$\langle yz, xz, xy \rangle = \lambda \langle 2, 1, 4 \rangle$$

or

$$yz = 2\lambda, \quad xz = \lambda, \quad xy = 4\lambda$$

Step 2. Solve for λ in terms of x, y, and z. The Lagrange equations imply that

$$\lambda = \frac{yz}{2}, \quad \lambda = xz, \quad \lambda = \frac{xy}{4}$$

Step 3. Solve for x, y, and z using the constraint. Equating the expressions for λ gives the following equations:

$$\begin{aligned} \frac{yz}{2} &= xz \\ \frac{xy}{4} &= xz \end{aligned} \quad \Rightarrow \quad \begin{aligned} z(2x - y) &= 0 \\ x(4z - y) &= 0 \end{aligned}$$

The first equation implies that $z = 0$ or $y = 2x$. The second equation implies that $x = 0$ or $y = 4z$. We examine all possible solutions.

(1) $z = 0$ and $x = 0$: Then substituting in the constraint $2x + y + 4z = 1$ gives $2 \cdot 0 + y + 4 \cdot 0 = 1$ or $y = 1$. We obtain the point $(0, 1, 0)$.

(2) $z = 0$ and $y = 4z$: Then $y = 4 \cdot 0 = 0$. Substituting $z = 0$ and $y = 0$ in the constraint $2x + y + 4z = 1$ gives $2x + 0 + 4 \cdot 0 = 1$ or $x = \frac{1}{2}$. We obtain the point $\left(\frac{1}{2}, 0, 0\right)$.

(3) $y = 2x$ and $x = 0$: Then $y = 2 \cdot 0 = 0$. Substituting $x = y = 0$ in the constraint $2x + y + 4z = 1$ gives $2 \cdot 0 + 0 + 4z = 1$ or $z = \frac{1}{4}$. The corresponding point is $\left(0, 0, \frac{1}{4}\right)$.

(4) $y = 2x$, $y = 4z$: Then $x = \frac{y}{2}$ and $z = \frac{y}{4}$. We substitute in the constraint $2x + y + 4z = 1$ and solve for y:

$$2 \cdot \frac{y}{2} + y + 4 \cdot \frac{y}{4} = 1$$
$$3y = 1 \quad \Rightarrow \quad y = \frac{1}{3}$$

Hence, $x = \frac{y}{2} = \frac{1}{6}$, $z = \frac{y}{4} = \frac{1}{12}$. We obtain the point $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{12}\right)$.

Step 4. Conclusions. We evaluate $f(x,y,z) = xyz$ at the critical points:

$$f(0,1,0) = 0 \cdot 1 \cdot 0 = 0$$
$$f\left(\frac{1}{2}, 0, 0\right) = \frac{1}{2} \cdot 0 \cdot 0 = 0$$
$$f\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{12}\right) = \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{12} = \frac{1}{216}$$
$$f\left(0, 0, \frac{1}{4}\right) = 0 \cdot 0 \cdot \frac{1}{4} = 0$$

We conclude that the local maximum of f subject to the constraint is

$$f\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{12}\right) = \frac{1}{216}.$$

Notice that f does not have a global maximum on the plane $2x + y + 4z = 1$ since, for all t, the point $\left(-t^2, 1 + 6t^2, -t^2\right)$ is on the plane and we have

$$\lim_{t\to\infty} f\left(-t^2, 1 + 6t^2, -t^2\right) = \lim_{t\to\infty} t^4 \left(1 + 6t^2\right) = \infty$$

55. Use Lagrange multipliers to find the minimum and maximum values of $f(x, y) = 3x - 2y$ on the circle $x^2 + y^2 = 4$.

SOLUTION

Step 1. Write out the Lagrange Equations. The constraint curve is $g(x, y) = x^2 + y^2 - 4 = 0$, hence $\nabla g = \langle 2x, 2y\rangle$ and $\nabla f = \langle 3, -2\rangle$. The Lagrange Condition $\nabla f = \lambda \nabla g$ is thus $\langle 3, -2\rangle = \lambda \langle 2x, 2y\rangle$. That is,

$$3 = \lambda \cdot 2x$$
$$-2 = \lambda \cdot 2y$$

Note that $\lambda \neq 0$.

Step 2. Solve for x and y using the constraint. The Lagrange equations gives

$$\begin{aligned} 3 &= \lambda \cdot 2x \\ -2 &= \lambda \cdot 2y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{3}{2\lambda} \\ y &= -\frac{1}{\lambda} \end{aligned} \tag{1}$$

We substitute x and y in the equation of the constraint and solve for λ. We get

$$\left(\frac{3}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 = 4$$
$$\frac{9}{4\lambda^2} + \frac{1}{\lambda^2} = 4$$
$$\frac{1}{\lambda^2} \cdot \frac{13}{4} = 4 \quad \Rightarrow \quad \lambda = \frac{\sqrt{13}}{4} \quad \text{or} \quad \lambda = -\frac{\sqrt{13}}{4}$$

Substituting in (1), we obtain the points

$$x = \frac{6}{\sqrt{13}}, \quad y = -\frac{4}{\sqrt{13}}$$
$$x = -\frac{6}{\sqrt{13}}, \quad y = \frac{4}{\sqrt{13}}$$

The critical points are thus

$$P_1 = \left(\frac{6}{\sqrt{13}}, -\frac{4}{\sqrt{13}}\right)$$
$$P_2 = \left(-\frac{6}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$$

Step 3. Calculate the value at the critical points. We find the value of $f(x, y) = 3x - 2y$ at the critical points:

$$f(P_1) = 3 \cdot \frac{6}{\sqrt{13}} - 2 \cdot \frac{-4}{\sqrt{13}} = \frac{26}{\sqrt{13}}$$
$$f(P_2) = 3 \cdot \frac{-6}{\sqrt{13}} - 2 \cdot \frac{4}{\sqrt{13}} = \frac{-26}{\sqrt{13}}$$

Thus, the maximum value of f on the circle is $\frac{26}{\sqrt{13}}$, and the minimum is $-\frac{26}{\sqrt{13}}$.

56. Find the minimum value of $f(x, y) = xy$ subject to the constraint $5x - y = 4$ in two ways: using Lagrange multipliers and setting $y = 5x - 4$ in $f(x, y)$.

SOLUTION We find the minimum value of $f(x, y) = xy$ subject to the constraint $g(x, y) = 5x - y - 4 = 0$ using the Lagrange multipliers.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 5, -1\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$\langle y, x\rangle = \lambda \langle 5, -1\rangle$$
$$\langle y, x\rangle = \langle 5\lambda, -\lambda\rangle$$

The Lagrange Equations are thus

$$\begin{aligned} y &= 5\lambda \\ x &= -\lambda \end{aligned} \quad \Rightarrow \quad \lambda = \frac{y}{5}, \quad \lambda = -x$$

Step 2. Solve for x and y using the constraint. Equating the two expressions for λ gives

$$\frac{y}{5} = -x \quad \Rightarrow \quad y = -5x$$

We substitute $y = -5x$ in the equation of the constraint $5x - y = 4$ and solve for x. This gives

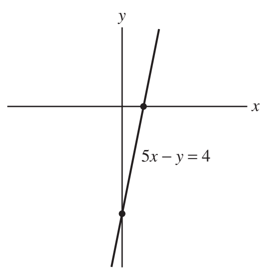
$$\begin{aligned} 5x - (-5x) &= 4 \\ 10x &= 4 \end{aligned} \quad \Rightarrow \quad x = \frac{2}{5}$$

The y-coordinate is $y = -5 \cdot \frac{2}{5} = -2$. We obtain the critical point $\left(\frac{2}{5}, -2\right)$.

Step 3. Calculate the value at the critical point. The value of $f(x, y) = xy$ at the critical point is

$$f\left(\frac{2}{5}, -2\right) = \frac{2}{5} \cdot (-2) = -\frac{4}{5} \tag{1}$$

This value is the minimum value of f subject to the constraint:



Note that since $f(x, y) = xy$ is positive in the first and third quadrant, the minimum value of f subject to the constraint's part in the fourth quadrant is also the minimum value subject to the entire constraint. The part of the constraint in the fourth quadrant is a closed and bounded segment, hence the minimum value of f on this segment exists, and is given in (1).

We now find the minimum value of $f(x, y) = xy$ subject to the constraint $5x - y = 4$ using the second way. On the constraint $5x - y = 4$, we have $y = 5x - 4$. We substitute in the function $f(x, y) = xy$ and then find the minimum of the resulting one-variable function. We get

$$g(x) = f(x, 5x - 4) = x(5x - 4) = 5x^2 - 4x$$

We now find the minimum value of $g(x) = 5x^2 - 4x$ in the interval $-\infty < x < \infty$. We find the critical points:

$$g'(x) = 10x - 4 = 0 \quad \Rightarrow \quad x = \frac{2}{5}$$

The limits

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(5x^2 - 4x\right) = \infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \left(5x^2 - 4x\right) = \infty$$

imply that g has a minimum value for $-\infty < x < \infty$, and it occurs at the critical point. Therefore, the minimum value of g occurs at $x = \frac{2}{5}$. The corresponding y-coordinate is $y = 5 \cdot \frac{2}{5} - 4 = -2$, therefore the minimum value of $f(x, y) = xy$ is

$$f\left(\frac{2}{5}, -2\right) = \frac{2}{5} \cdot (-2) = -\frac{4}{5}$$

57. Find the minimum and maximum values of $f(x, y) = x^2y$ on the ellipse $4x^2 + 9y^2 = 36$.

SOLUTION We must find the minimum and maximum values of $f(x, y) = x^2y$ subject to the constraint $g(x, y) = 4x^2 + 9y^2 - 36 = 0$.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \left\langle 2xy, x^2 \right\rangle$ and $\nabla g = \langle 8x, 18y\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\left\langle 2xy, x^2 \right\rangle = \lambda \langle 8x, 18y\rangle = \langle 8\lambda x, 18\lambda y\rangle$$

We obtain the following Lagrange Equations:

$$\begin{aligned} 2xy &= 8\lambda x \\ x^2 &= 18\lambda y \end{aligned}$$

Step 2. Solve for λ in terms of x and y. If $x = 0$, the equation of the constraint implies that $y = \pm 2$. The points $(0, 2)$ and $(0, -2)$ satisfy the Lagrange Equations for $\lambda = 0$. If $x \neq 0$, the second Lagrange Equation implies that $y \neq 0$. Therefore the Lagrange Equations give

$$2xy = 8\lambda x \quad \Rightarrow \quad \lambda = \frac{y}{4}$$
$$x^2 = 18\lambda y \quad \Rightarrow \quad \lambda = \frac{x^2}{18y}$$

Step 3. Solve for x and y using the constraint. We equate the two expressions for λ to obtain

$$\frac{y}{4} = \frac{x^2}{18y}$$
$$18y^2 = 4x^2$$

We now substitute $4x^2 = 18y^2$ in the equation of the constraint $4x^2 + 9y^2 = 36$ and solve for y. This gives

$$\begin{aligned} 18y^2 + 9y^2 &= 36 \\ 27y^2 &= 36 \end{aligned} \quad \Rightarrow \quad y^2 = \frac{36}{27} \quad \Rightarrow \quad y_1 = \frac{2}{\sqrt{3}}, \quad y_2 = -\frac{2}{\sqrt{3}}$$

We find the x-coordinates using $x^2 = \frac{9y^2}{2}$:

$$x^2 = \frac{9y^2}{2}$$
$$x^2 = \frac{9}{2} \cdot \frac{4}{3} = 6 \quad \Rightarrow \quad x_1 = \sqrt{6}, \quad x_2 = -\sqrt{6}$$

We obtain the following critical points:

$$P_1 = (0, 2), \quad P_2 = (0, -2), \quad P_3 = \left(\sqrt{6}, \frac{2}{\sqrt{3}}\right)$$
$$P_4 = \left(\sqrt{6}, -\frac{2}{\sqrt{3}}\right), \quad P_5 = \left(-\sqrt{6}, \frac{2}{\sqrt{3}}\right), \quad P_6 = \left(-\sqrt{6}, -\frac{2}{\sqrt{3}}\right)$$

Step 4. Conclusions. We evaluate the function $f(x, y) = x^2 y$ at the critical points:

$$f(P_1) = 0^2 \cdot 2 = 0$$
$$f(P_2) = 0^2 \cdot (-2) = 0$$
$$f(P_3) = f(P_5) = 6 \cdot \frac{2}{\sqrt{3}} = \frac{12}{\sqrt{3}}$$
$$f(P_4) = f(P_5) = 6 \cdot \left(-\frac{2}{\sqrt{3}}\right) = -\frac{12}{\sqrt{3}}$$

Since the min and max of f occur on the ellipse, it must occur at critical points. Thus, we conclude that the maximum and minimum of f subject to the constraint are $\frac{12}{\sqrt{3}}$ and $-\frac{12}{\sqrt{3}}$ respectively.

58. Find the point in the first quadrant on the curve $y = x + x^{-1}$ closest to the origin.

SOLUTION We need to minimize the distance $d = \sqrt{x^2 + y^2}$ subject to the constraint $g(x, y) = x + \frac{1}{x} - y = 0$. Since the function u^2 is increasing for $u \geq 0$, the distance d is minimal where the square d^2 is minimal. Therefore, we minimize the function $f(x, y) = d^2 = x^2 + y^2$ subject to the constraint.

Step 1. Write out the Lagrange Equations. The gradient vectors are $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \left\langle 1 - \frac{1}{x^2}, -1 \right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives

$$\langle 2x, 2y \rangle = \lambda \left\langle 1 - \frac{1}{x^2}, -1 \right\rangle = \left\langle \lambda \left(1 - \frac{1}{x^2}\right), -\lambda \right\rangle$$

The Lagrange Equations are

$$2x = \lambda \left(1 - \frac{1}{x^2}\right)$$
$$2y = -\lambda$$

Step 2. Solve for λ in terms of x and y. The second Lagrange equation gives $\lambda = -2y$, and the first equation gives

$$2x = \lambda \frac{x^2 - 1}{x^2} \quad \Rightarrow \quad \lambda = \frac{2x^3}{x^2 - 1}$$

Step 3. Solve for x and y using the constraint. Equating the two expressions for λ, we get

$$-2y = \frac{2x^3}{x^2 - 1} \quad \Rightarrow \quad y = \frac{x^3}{1 - x^2}$$

We now substitute y as a function of x in the equation of the constraint and solve for x. This gives

$$\frac{x^3}{1 - x^2} = x + \frac{1}{x} = \frac{x^2 + 1}{x}$$

$$x^4 = \left(1 - x^2\right)\left(1 + x^2\right) = 1 - x^4$$

$$2x^4 = 1 \quad \Rightarrow \quad x = 2^{-1/4}, \quad x = -2^{-1/4}$$

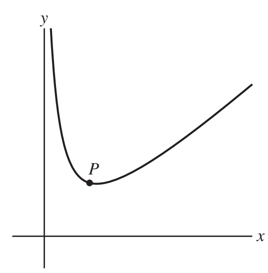
The solution in the first quadrant is $x = 2^{-1/4} = \frac{1}{\sqrt[4]{2}}$. We find the y-coordinate using $y = \frac{x^3}{1-x^2}$:

$$y = \frac{2^{-3/4}}{1 - 2^{-1/2}} = \frac{2^{-1/4}}{2^{1/2} - 1} = 2^{-1/4}\left(2^{1/2} + 1\right) = 2^{1/4} + 2^{-1/4} = \sqrt[4]{2} + \frac{1}{\sqrt[4]{2}}$$

We obtain the critical point:

$$P = \left(\frac{1}{\sqrt[4]{2}}, \sqrt[4]{2} + \frac{1}{\sqrt[4]{2}}\right)$$

Step 4. Conclusion.



Graph of $y = x + \frac{1}{x}$, $x > 0$, $y > 0$

It is clear from the graph of $y = x + \frac{1}{x}$ that the critical point is a minimum. Therefore, the point P is the closest to the origin on the curve $y = x + \frac{1}{x}$ in the first quadrant.

59. Find the extreme values of $f(x, y, z) = x + 2y + 3z$ subject to the two constraints $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$.

SOLUTION We must find the extreme values of $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x + y + z - 1 = 0$ and $h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = < 1, 2, 3 >$, $\nabla g = < 1, 1, 1 >$, $\nabla h = < 2x, 2y, 2z >$, hence the Lagrange condition $\nabla f = \lambda \nabla g + \mu \nabla h$ gives

$$< 1, 2, 3 > = \lambda < 1, 1, 1 > + \mu < 2x, 2y, 2z > = < \lambda + 2\mu x, \lambda + 2\mu y, \lambda + 2\mu z >$$

or

$$1 = \lambda + 2\mu x$$
$$2 = \lambda + 2\mu y$$
$$3 = \lambda + 2\mu z$$

Step 2. Solve for λ and μ. The Lagrange Equations give

$$\begin{aligned} 1 &= \lambda + 2\mu x & & \lambda = 1 - 2\mu x \\ 2 &= \lambda + 2\mu y & \Rightarrow \quad & \lambda = 2 - 2\mu y \\ 3 &= \lambda + 2\mu z & & \lambda = 3 - 2\mu z \end{aligned}$$

Equating the three expressions for λ, we get the following equations:

$$\begin{aligned} 1-2\mu x &= 2-2\mu y \\ 1-2\mu x &= 3-2\mu z \end{aligned} \quad\Rightarrow\quad \begin{aligned} 2\mu(y-x) &= 1 \\ \mu(z-x) &= 2 \end{aligned}$$

The first equation implies that $\mu = \frac{1}{2(y-x)}$, and the second implies that $\mu = \frac{2}{z-x}$. Equating the two expressions for μ, we get

$$\frac{1}{2(y-x)} = \frac{2}{z-x}$$
$$z-x = 4y-4x \quad\Rightarrow\quad z = 4y-3x$$

Step 3. Solve for x, y, and z using the constraints. We substitute $z = 4y-3x$ in the equations of the constraints and solve to find x and y. This gives

$$\begin{aligned} x+y+(4y-3x) &= 1 \\ x^2+y^2+(4y-3x)^2 &= 1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} y &= \frac{1+2x}{5} \\ 10x^2+17y^2-24xy &= 1 \end{aligned}$$

Substituting in the second equation and solving for x, we get

$$y = \frac{1+2x}{5}$$
$$10x^2+17\left(\frac{1+2x}{5}\right)^2 - 24x\cdot\frac{1+2x}{5} = 1$$
$$250x^2+17(1+2x)^2-120x\,(1+2x) = 25$$
$$39x^2-26x-4 = 0$$
$$x_{1,2} = \frac{26\pm\sqrt{1300}}{78}$$
$$\Rightarrow\quad x_1 = \frac{1}{3}+\frac{5\sqrt{13}}{39} \approx 0.8, \quad x_2 = \frac{1}{3}-\frac{5\sqrt{13}}{39} \approx -0.13$$

We find the y-coordinates using $y = \frac{1+2x}{5}$.

$$y_1 = \frac{1+2\cdot 0.8}{5} = 0.52, \quad y_2 = \frac{1-2\cdot 0.13}{5} = 0.15$$

Finally, we find the z-coordinate using $z = 4y-3x$:

$$z_1 = 4\cdot 0.52 - 3\cdot 0.8 = -0.32, \quad z_2 = 4\cdot 0.15 + 3\cdot 0.13 = 0.99$$

We obtain the critical points:

$$P_1 = (0.8, 0.52, -0.32), \quad P_2 = (-0.13, 0.15, 0.99)$$

.

Step 4. Conclusions. We evaluate the function $f(x,y,z) = x+2y+3z$ at the critical points:

$$f(P_1) = 0.8 + 2\cdot 0.52 - 3\cdot 0.32 = 0.88$$
$$f(P_2) = -0.13 + 2\cdot 0.15 + 3\cdot 0.99 = 3.14 \qquad (1)$$

The two constraints determine the common points of the unit sphere $x^2+y^2+z^2 = 1$ and the plane $x+y+z = 1$. This set is a circle that is a closed and bounded set in R^3. Therefore, f has a minimum and maximum values on this set. These extrema are given in (1).

60. Find the minimum and maximum values of $f(x,y,z) = x-z$ on the intersection of the cylinders $x^2+y^2 = 1$ and $x^2+z^2 = 1$ (Figure 5).

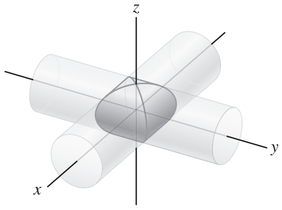


FIGURE 5

SOLUTION Let us use the Lagrange Multipliers method with two constraints for $f(x, y, z) = x - z$ subject to $g(x, y, z) = x^2 + y^2 - 1 = 0$ and $h(x, y, z) = x^2 + z^2 - 1 = 0$. The Lagrange condition would be $\nabla f = \lambda \nabla g + \mu \nabla h$. Noting here that we have $\nabla f = \langle 1, 0, -1 \rangle$, $\nabla g = \langle 2x, 2y, 0 \rangle$, and $\nabla h = \langle 2x, 0, 2z \rangle$. Therefore we have

$$\langle 1, 0, -1 \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 2x, 0, 2z \rangle$$

yielding the equations:

$$1 = 2\lambda x + 2\mu x, \quad 0 = 2\lambda y, \quad -1 = 2\mu z$$

Next, using the second equation, we find either $\lambda = 0$ or $y = 0$.

If $y = 0$, then using the first constraint equation, $x = \pm 1$ and using the second constraint equation we find $z = 0$. The derived critical points are then:

$$(1, 0, 0), \quad (-1, 0, 0)$$

If $\lambda = 0$, then using the first equation above we see $1 = 2\mu x$ which implies

$$\mu = \frac{1}{2x}$$

Using the last equation above we have:

$$-1 = 2 \cdot \frac{1}{2x} z \quad \Rightarrow \quad -x = z$$

Then using the second constraint equation, we have

$$2x^2 = 1 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}, z = \mp \frac{1}{\sqrt{2}}$$

Using the first constraint equation, we have

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = \frac{1}{2} \quad \Rightarrow \quad y = \pm \frac{1}{\sqrt{2}}$$

We have four derived critical points here:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Now to analyze $f(x, y, z) = x - z$ for maximum and minimum values:

$$f(1, 0, 0) = 1, \quad f(-1, 0, 0) = -1$$

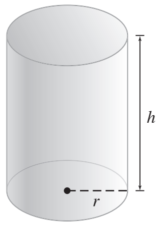
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{2}, \quad f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\sqrt{2}, \quad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

Hence the maximum value of $f(x, y, z) = x - z$ subject to the two constraints is $\sqrt{2}$, while the minimum value is $-\sqrt{2}$.

61. Use Lagrange multipliers to find the dimensions of a cylindrical can with a bottom but no top, of fixed volume V with minimum surface area.

SOLUTION We denote the radius of the cylinder by r and the height by h.



The volume of the cylinder is $g = \pi r^2 h$ and the surface area is

$$f = 2\pi r h + 2\pi r^2$$

We need to minimize $f(r, h) = 2\pi r h + 2\pi r^2$ subject to the constraint $g(r, h) = \pi r^2 h - V = 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \langle 2\pi h + 4\pi r, 2\pi r\rangle = 2\pi \langle h + 2r, r\rangle$ and $\nabla g = \left\langle 2\pi hr, \pi r^2\right\rangle = \pi \left\langle 2hr, r^2\right\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ is

$$2\pi \langle h + 2r, r\rangle = \pi\lambda \left\langle 2hr, r^2\right\rangle$$

or

$$2 \langle h + 2r, r\rangle = \lambda \left\langle 2hr, r^2\right\rangle$$

We obtain the following equations:

$$\begin{aligned} 2(h+2r) &= 2hr\lambda \\ 2r &= \lambda r^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} h+2r &= hr\lambda \\ 2r &= \lambda r^2 \end{aligned}$$

Step 2. Solve for λ in terms of r and h. The equation of the constraint implies that $r \neq 0$ and $h \neq 0$ (we assume that $V > 0$). Therefore, the Lagrange equations give

$$\lambda = \frac{h+2r}{hr} = \frac{1}{r} + \frac{2}{h}, \quad \lambda = \frac{2}{r}$$

Step 3. Solve for r and h using the constraint. Equating the two expressions for λ gives

$$\begin{aligned} \frac{1}{r} + \frac{2}{h} &= \frac{2}{r} \\ \frac{2}{h} &= \frac{1}{r} \quad \Rightarrow \quad h = 2r \end{aligned}$$

We substitute $h = 2r$ in the equation of the constraint $\pi r^2 h = V$ and solve for r. We obtain

$$\begin{aligned} \pi r^2 \cdot 2r &= V \\ 2\pi r^3 &= V \quad \Rightarrow \quad r = \left(\frac{V}{2\pi}\right)^{1/3} \end{aligned}$$

We find h using the relation $h = 2r$:

$$h = 2\left(\frac{V}{2\pi}\right)^{1/3}$$

The critical point is $h = 2\left(\frac{V}{2\pi}\right)^{1/3}$, $r = \left(\frac{V}{2\pi}\right)^{1/3}$.

Step 4. Conclusions. On the constraint $\pi r^2 h = V$ we have $h = \frac{V}{\pi r^2}$ and $r = \sqrt{\frac{V}{\pi h}}$, hence

$$\begin{aligned} f\left(r, \frac{V}{\pi r^2}\right) &= 2\pi r \cdot \frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2 \\ f\left(\sqrt{\frac{V}{\pi h}}, h\right) &= 2\pi \sqrt{\frac{V}{\pi h}} h + 2\pi \cdot \frac{V}{\pi h} = 2\sqrt{\pi V}\sqrt{h} + \frac{2V}{h} \end{aligned}$$

We see that as $h \to 0+$ or $h \to \infty$, we have $f(r, h) \to \infty$, and as $r \to 0+$ or $r \to \infty$, we have $f(r, h) \to \infty$. Therefore, f has a minimum value on the constraint, which occurs at the critical point. We evaluate $f(r, h) = 2\pi rh + 2\pi r^2 = 2\pi(rh + r^2)$ at the critical point P:

$$f(P) = 2\pi\left(\left(\frac{V}{2\pi}\right)^{1/3} \cdot 2\left(\frac{V}{2\pi}\right)^{1/3} + \left(\frac{V}{2\pi}\right)^{2/3}\right) = 2\pi\left(2\left(\frac{V}{2\pi}\right)^{2/3} + \left(\frac{V}{2\pi}\right)^{2/3}\right) = 6\pi\left(\frac{V}{2\pi}\right)^{2/3}$$

We conclude that the minimum surface area is $6\pi\left(\frac{V}{2\pi}\right)^{2/3}$, and the dimensions of the corresponding cylinder are $r = \left(\frac{V}{2\pi}\right)^{1/3}$, $h = 2\left(\frac{V}{2\pi}\right)^{1/3}$.

62. Find the dimensions of the box of maximum volume with its sides parallel to the coordinate planes that can be inscribed in the ellipsoid (Figure 6)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

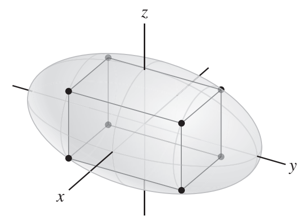


FIGURE 6

SOLUTION We denote the vertices of the box by $(\pm x, \pm y, \pm z)$, where $x \ge 0$, $y \ge 0$, $z \ge 0$. The volume of the box is

$$V(x, y, z) = 8xyz$$

The vertices of the box must satisfy the equation of the ellipsoid, hence,

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

We need to maximize V due to the constraint: $g(x, y, z) = 0$, $x \ge 0$, $y \ge 0$, $z \ge 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla V = 8\langle yz, xz, xy\rangle$ and $\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right\rangle$, hence the Lagrange Condition $\nabla V = \lambda \nabla g$ gives the following equations:

$$yz = \lambda \frac{2x}{a^2}$$
$$xz = \lambda \frac{2y}{b^2}$$
$$xy = \lambda \frac{2z}{c^2}$$

Step 2. Solve for λ in terms of x, y, and z. If $x = 0$, $y = 0$, or $z = 0$, the volume of the box has the minimum value zero. We thus may assume that $x \ne 0$, $y \ne 0$, and $z \ne 0$. The Lagrange equations give

$$\lambda = \frac{a^2 yz}{2x}, \quad \lambda = \frac{b^2 xz}{2y}, \quad \lambda = \frac{c^2 xy}{2z}$$

Step 3. Solve for x, y, and z using the constraint. Equating the three expressions for λ yields the following equations:

$$\begin{array}{c} \frac{a^2}{2}\frac{yz}{x} = \frac{c^2}{2}\frac{xy}{z} \\ \frac{b^2}{2}\frac{xz}{y} = \frac{c^2}{2}\frac{xy}{z} \end{array} \quad \Rightarrow \quad \begin{array}{c} y\left(c^2x^2 - a^2z^2\right) = 0 \\ x\left(c^2y^2 - b^2z^2\right) = 0 \end{array}$$

Since $x > 0$ and $y > 0$, these equations imply that

$$\begin{array}{c} c^2x^2 - a^2z^2 = 0 \\ c^2y^2 - b^2z^2 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} x = \frac{az}{c} \\ y = \frac{bz}{c} \end{array} \tag{1}$$

We now substitute x and y in the equation of the constraint and solve for z. This gives

$$\frac{\left(\frac{az}{c}\right)^2}{a^2} + \frac{\left(\frac{bz}{c}\right)^2}{b^2} + \frac{z^2}{c^2} = 1$$
$$\frac{z^2}{c^2} + \frac{z^2}{c^2} + \frac{z^2}{c^2} = 1$$
$$\frac{3z^2}{c^2} = 1 \quad \Rightarrow \quad z = \frac{c}{\sqrt{3}}$$

We find x and y using (1):

$$x = \frac{a}{c}\frac{c}{\sqrt{3}} = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{c}\frac{c}{\sqrt{3}} = \frac{b}{\sqrt{3}}$$

We obtain the critical point:

$$P = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$$

Step 4. Conclusions. The function $V = 8xyz$ is a polynomial, hence it is continuous. The constraint defines a closed and compact set in R^3, hence f has extreme values on the constraint. The maximum value is obtained at the critical point P. We find it:

$$V(P) = 8\frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = 8\frac{abc}{3\sqrt{3}}$$

We conclude that the dimensions of the box of maximum volume with sides parallel to the coordinate planes, which can be inscribed in the ellipsoid, are

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.$$

63. Given n nonzero numbers $\sigma_1, \dots, \sigma_n$, show that the minimum value of

$$f(x_1, \dots, x_n) = x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2$$

subject to $x_1 + \cdots + x_n = 1$ is c, where $c = \left(\sum_{j=1}^{n} \sigma_j^{-2}\right)^{-1}$.

SOLUTION We must minimize the function $f(x_1, \dots, x_n) = x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2$ subject to the constraint $g(x_1, \dots, x_n) = x_1 + \cdots + x_n - 1 = 0$.

Step 1. Write out the Lagrange Equations. We have $\nabla f = \left\langle 2\sigma_1^2 x_1, \dots, 2\sigma_n^2 x_n \right\rangle$ and $\nabla g = \langle 1, \dots, 1\rangle$, hence the Lagrange Condition $\nabla f = \lambda \nabla g$ gives the following equations:

$$2\sigma_i^2 x_i = \lambda, \quad i = 1, \dots, n$$

Step 2. Solve for $x_1, \dots, x_n$ using the constraint. The Lagrange equations imply the following equations:

$$2\sigma_i^2 x_i = 2\sigma_n^2 x_n, \quad x_i = \frac{\sigma_n^2}{\sigma_i^2} x_n; \quad i = 1, \dots, n-1$$

We substitute these values in the equation of the constraint $x_1 + \cdots + x_n = 1$ and solve for x_n. This gives

$$\frac{\sigma_n^2}{\sigma_1^2} x_n + \frac{\sigma_n^2}{\sigma_2^2} x_n + \cdots + \frac{\sigma_n^2}{\sigma_{n-1}^2} x_n + x_n = 1$$

$$\sigma_n^2 \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots + \frac{1}{\sigma_{n-1}^2} + \frac{1}{\sigma_n^2} \right) x_n = 1$$

$$\sigma_n^2 \left(\sum_{j=1}^{n} \sigma_j^{-2} \right) x_n = 1$$

Denoting $c = \left(\sum_{j=1}^{n} \sigma_j^{-2}\right)^{-1}$, we get $x_n = \frac{c}{\sigma_n^2}$. Using $x_i = \frac{\sigma_n^2}{\sigma_i^2} x_n$ we get

$$x_i = \frac{\sigma_n^2}{\sigma_i^2} \cdot \frac{c}{\sigma_n^2} = \frac{c}{\sigma_i^2}$$

We obtain the following point:

$$P = \left(\frac{c}{\sigma_1^2}, \frac{c}{\sigma_2^2}, \dots, \frac{c}{\sigma_n^2} \right)$$

Step 3. Conclusions. As $x_i \to \infty$ or $x_i \to -\infty$, for one or more i's the function $f(x_1, \dots, x_n)$ tends to ∞. f is continuous since it is a polynomial, hence f has a minimum value on the constraint. This minimum occurs at the critical point. We find it:

$$f(P) = \sum_{j=1}^{n} \sigma_j^2 \left(\frac{c}{\sigma_j^2} \right)^2 = \sum_{j=1}^{n} \frac{\sigma_j^2 c^2}{\sigma_j^4} = c^2 \sum_{j=1}^{n} \sigma_j^{-2} = c^2 \cdot c^{-1} = c$$

15 MULTIPLE INTEGRATION

15.1 Integration in Two Variables (LT Section 16.1)

Preliminary Questions

1. If $S_{8,4}$ is a Riemann sum for a double integral over $\mathcal{R} = [1, 5] \times [2, 10]$ using a regular partition, what is the area of each subrectangle? How many subrectangles are there?

SOLUTION Since the partition is regular, all subrectangles have sides of length

$$\Delta x = \frac{5-1}{8} = \frac{1}{2}, \quad \Delta y = \frac{10-2}{4} = 2$$

Therefore the area of each subrectangle is $\Delta A = \Delta x \Delta y = \frac{1}{2} \cdot 2 = 1$, and the number of subrectangles is $8 \cdot 4 = 32$.

2. Estimate the double integral of a continuous function f over the small rectangle $\mathcal{R} = [0.9, 1.1] \times [1.9, 2.1]$ if $f(1, 2) = 4$.

SOLUTION Since we are given the value of f at one point in $\mathcal{R}$ only, we can only use the approximation S_{11} for the integral of f over $\mathcal{R}$. For S_{11} we have one rectangle wi th sides

$$\Delta x = 1.1 - 0.9 = 0.2, \quad \Delta y = 2.1 - 1.9 = 0.2$$

Hence, the area of the rectangle is $\Delta A = \Delta x \Delta y = 0.2 \cdot 0.2 = 0.04$. We obtain the following approximation:

$$\iint_{\mathcal{R}} f\, dA \approx S_{1,1} = f(1, 2)\Delta A = 4 \cdot 0.04 = 0.16$$

3. What is the integral of the constant function $f(x, y) = 5$ over the rectangle $[-2, 3] \times [2, 4]$?

SOLUTION The integral of f over the unit square $\mathcal{R} = [-2, 3] \times [2, 4]$ is the volume of the box of base $\mathcal{R}$ and height 5. That is,

$$\iint_{\mathcal{R}} 5\, dA = 5 \cdot \text{Area}(\mathcal{R}) = 5 \cdot 5 \cdot 2 = 50$$

4. What is the interpretation of $\iint_{\mathcal{R}} f(x, y)\, dA$ if $f(x, y)$ takes on both positive and negative values on $\mathcal{R}$?

SOLUTION The double integral $\iint_{\mathcal{R}} f(x, y)\, dA$ is the signed volume between the graph $z = f(x, y)$ for $(x, y) \in \mathcal{R}$, and the xy-plane. The region below the xy-plane is treated as negative volume.

5. Which of (a) or (b) is equal to $\int_1^2 \int_4^5 f(x, y)\, dy\, dx$?

(a) $\int_1^2 \int_4^5 f(x, y)\, dx\, dy$

(b) $\int_4^5 \int_1^2 f(x, y)\, dx\, dy$

SOLUTION The integral $\int_1^2 \int_4^5 f(x, y)\, dy\, dx$ is written with dy preceding dx, therefore the integration is first with respect to y over the interval $4 \le y \le 5$, and then with respect to x over the interval $1 \le x \le 2$. By Fubini's Theorem, we may replace the order of integration over the corresponding intervals. Therefore the given integral is equal to (b) rather than to (a).

6. For which of the following functions is the double integral over the rectangle in Figure 15 equal to zero? Explain your reasoning.

(a) $f(x, y) = x^2 y$

(b) $f(x, y) = xy^2$

(c) $f(x, y) = \sin x$

(d) $f(x, y) = e^x$

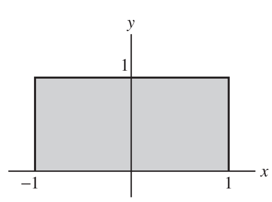


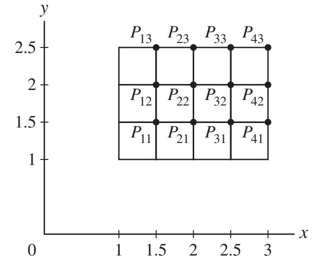
FIGURE 15

SOLUTION The double integral is the signed volume of the region between the graph of $f(x, y)$ and the xy-plane over $\mathcal{R}$. In (b) and (c) the function satisfies $f(-x, y) = -f(x, y)$, hence the region below the xy-plane, where $-1 \le x \le 0$ cancels with the region above the xy-plane, where $0 \le x \le 1$. Therefore, the double integral is zero. In (a) and (d), the function $f(x, y)$ is always positive on the rectangle, so the double integral is greater than zero.

Exercises

1. Compute the Riemann sum $S_{4,3}$ to estimate the double integral of $f(x, y) = xy$ over $\mathcal{R} = [1, 3] \times [1, 2.5]$. Use the regular partition and upper-right vertices of the subrectangles as sample points.

SOLUTION The rectangle $\mathcal{R}$ and the subrectangles are shown in the following figure:



The subrectangles have sides of length

$$\Delta x = \frac{3-1}{4} = 0.5, \quad \Delta y = \frac{2.5-1}{3} = 0.5 \quad \Rightarrow \quad \Delta A = 0.5 \cdot 0.5 = 0.25$$

The upper right vertices are the following points:

$$\begin{array}{llll} P_{11} = (1.5, 1.5) & P_{21} = (2, 1.5) & P_{31} = (2.5, 1.5) & P_{41} = (3, 1.5) \\ P_{12} = (1.5, 2) & P_{22} = (2, 2) & P_{32} = (2.5, 2) & P_{42} = (3, 2) \\ P_{13} = (1.5, 2.5) & P_{23} = (2, 2.5) & P_{33} = (2.5, 2.5) & P_{43} = (3, 2.5) \end{array}$$

We compute $f(x, y) = xy$ at these points:

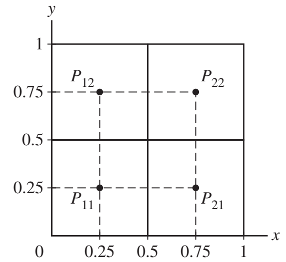
$$\begin{array}{lll} f(P_{11}) = 1.5 \cdot 1.5 = 2.25 & f(P_{12}) = 1.5 \cdot 2 = 3 & f(P_{13}) = 3.75 \\ f(P_{21}) = 2 \cdot 1.5 = 3 & f(P_{22}) = 2 \cdot 2 = 4 & f(P_{23}) = 5 \\ f(P_{31}) = 2.5 \cdot 1.5 = 3.75 & f(P_{32}) = 2.5 \cdot 2 = 5 & f(P_{33}) = 6.25 \\ f(P_{41}) = 3 \cdot 1.5 = 4.5 & f(P_{42}) = 3 \cdot 2 = 6 & f(P_{43}) = 7.5 \end{array}$$

Hence, $S_{4,3}$ is the following sum:

$$S_{4,3} = \sum_{i=1}^{4} \sum_{j=1}^{3} f(P_{ij}) \Delta A = 0.25(2.25 + 3 + 3.75 + 4.5 + 3 + 4 + 5 + 6 + 3.75 + 5 + 6.25 + 7.5) = 13.5$$

2. Compute the Riemann sum with $N = M = 2$ to estimate the integral of $\sqrt{x+y}$ over $\mathcal{R} = [0, 1] \times [0, 1]$. Use the regular partition and midpoints of the subrectangles as sample points.

SOLUTION The rectangle $\mathcal{R}$ and the subintervals are shown in the following figure:



The subrectangles have sides of length $\Delta x = \frac{1}{2} = 0.5$ and $\Delta y = \frac{1}{2} = 0.5$ and area $\Delta A = 0.5 \cdot 0.5 = 0.25$. The midpoints of the subrectangles are:

$$P_{11} = (0.25, 0.25), \quad P_{21} = (0.75, 0.25),$$
$$P_{12} = (0.25, 0.75), \quad P_{22} = (0.75, 0.75)$$

We compute the values of $f(x, y) = \sqrt{x + y}$ at the sample points:

$$f(P_{11}) = \sqrt{0.25 + 0.25} = 0.707$$
$$f(P_{21}) = \sqrt{0.75 + 0.25} = 1$$
$$f(P_{12}) = \sqrt{0.25 + 0.75} = 1$$
$$f(P_{22}) = \sqrt{0.75 + 0.75} = 1.225$$

Hence, S_{22} is the following sum:

$$S_{22} = \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = 0.25(0.707 + 1 + 1 + 1.225) = 0.983$$

In Exercises 3–6, compute the Riemann sums for the double integral $\iint_{\mathcal{R}} f(x, y)\, dA$, *where* $\mathcal{R} = [1, 4] \times [1, 3]$, *for the grid and two choices of sample points shown in Figure 16.*

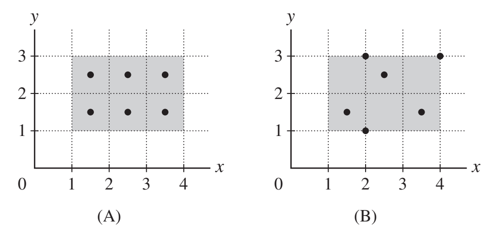


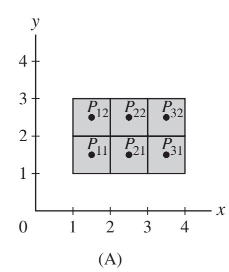
FIGURE 16

3. $f(x, y) = 2x + y$

SOLUTION The subrectangles have sides of length $\Delta x = \frac{4-1}{3} = 1$ and $\Delta y = \frac{3-1}{2} = 1$, and area $\Delta A = \Delta x \Delta y = 1$. We find the sample points in (A) and (B):

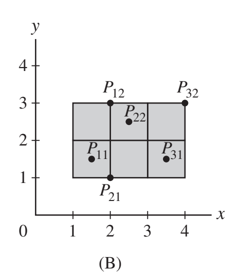
(A)

$$P_{11} = (1.5, 1.5) \quad P_{21} = (2.5, 1.5) \quad P_{31} = (3.5, 1.5)$$
$$P_{12} = (1.5, 2.5) \quad P_{22} = (2.5, 2.5) \quad P_{32} = (3.5, 2.5)$$



(B)

$$P_{11} = (1.5, 1.5) \quad P_{21} = (2, 1) \quad P_{31} = (3.5, 1.5)$$
$$P_{21} = (2, 3) \quad P_{22} = (2.5, 2.5) \quad P_{23} = (4, 3)$$



The Riemann Sum S_{32} is the following estimation of the double integral:

$$\iint_{\mathcal{R}} f(x, y)\, dA \approx S_{32} = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right)$$

We compute S_{32} for the two choices of sample points (A) and (B), and the following function:

$$f(x, y) = 2x + y$$

We compute $f\left(P_{ij}\right)$ for the sample points computed above:

(A)

$$\begin{aligned}
f\left(P_{11}\right) &= f(1.5, 1.5) = 2 \cdot 1.5 + 1.5 = 4.5 \\
f\left(P_{21}\right) &= f(2.5, 1.5) = 2 \cdot 2.5 + 1.5 = 6.5 \\
f\left(P_{31}\right) &= f(3.5, 1.5) = 2 \cdot 3.5 + 1.5 = 8.5 \\
f\left(P_{12}\right) &= f(1.5, 2.5) = 2 \cdot 1.5 + 2.5 = 5.5 \\
f\left(P_{22}\right) &= f(2.5, 2.5) = 2 \cdot 2.5 + 2.5 = 7.5 \\
f\left(P_{32}\right) &= f(3.5, 2.5) = 2 \cdot 3.5 + 2.5 = 9.5
\end{aligned}$$

Hence,

$$S_{32} = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = 4.5 + 6.5 + 8.5 + 5.5 + 7.5 + 9.5 = 42$$

(B)

$$\begin{aligned}
f\left(P_{11}\right) &= f(1.5, 1.5) = 2 \cdot 1.5 + 1.5 = 4.5 \\
f\left(P_{21}\right) &= f(2, 1) = 2 \cdot 2 + 1 = 5 \\
f\left(P_{31}\right) &= f(3.5, 1.5) = 2 \cdot 3.5 + 1.5 = 8.5 \\
f\left(P_{21}\right) &= f(2, 3) = 2 \cdot 2 + 3 = 7 \\
f\left(P_{22}\right) &= f(2.5, 2.5) = 2 \cdot 2.5 + 2.5 = 7.5 \\
f\left(P_{23}\right) &= f(4, 3) = 2 \cdot 4 + 3 = 11
\end{aligned}$$

Hence,

$$S_{32} = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = 4.5 + 5 + 8.5 + 7 + 7.5 + 11 = 43.5$$

4. $f(x, y) = 7$

SOLUTION In this case $f\left(P_{ij}\right) = 7$ for all i and j hence for the sample points in (A) and in (B) we have the same Riemann sum, that is,

$$S_{32} = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = 6 \cdot 7 = 42$$

5. $f(x, y) = 4x$

SOLUTION We compute the values of f at the sample points:

(A)

$$\begin{aligned}
f\left(P_{11}\right) &= f(1.5, 1.5) = 4 \cdot 1.5 = 6 \\
f\left(P_{21}\right) &= f(2.5, 1.5) = 4 \cdot 2.5 = 10 \\
f\left(P_{31}\right) &= f(3.5, 1.5) = 4 \cdot 3.5 = 14 \\
f\left(P_{12}\right) &= f(1.5, 2.5) = 4 \cdot 1.5 = 6 \\
f\left(P_{22}\right) &= f(2.5, 2.5) = 4 \cdot 2.5 = 10 \\
f\left(P_{32}\right) &= f(3.5, 2.5) = 4 \cdot 3.5 = 14 \\
\Delta x = \frac{4-1}{3} &= 1, \quad \Delta y = \frac{3-1}{2} = 1
\end{aligned}$$

Hence $\Delta A = \Delta x \cdot \Delta y = 1$ and we get

$$S_{32} = \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(P_{ij}\right) \Delta A = 6 + 10 + 14 + 6 + 10 + 14 = 60$$

(B)

$$f(P_{11}) = f(1.5, 1.5) = 4 \cdot 1.5 = 6$$
$$f(P_{21}) = f(2, 1) = 4 \cdot 2 = 8$$
$$f(P_{31}) = f(3.5, 1.5) = 4 \cdot 3.5 = 14$$
$$f(P_{12}) = f(2, 3) = 4 \cdot 2 = 8$$
$$f(P_{22}) = f(2.5, 2.5) = 4 \cdot 2.5 = 10$$
$$f(P_{32}) = f(4, 3) = 4 \cdot 4 = 16$$

$\Delta A = 1$. Hence,

$$S_{32} = \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(P_{ij}\right)\Delta A = 6 + 8 + 14 + 8 + 10 + 16 = 62$$

6. $f(x, y) = x - 2y$

SOLUTION We compute the values of f at the sample points:

(A)

$$f(P_{11}) = f(1.5, 1.5) = 1.5 - 2 \cdot 1.5 = -1.5$$
$$f(P_{21}) = f(2.5, 1.5) = 2.5 - 2 \cdot 1.5 = -0.5$$
$$f(P_{31}) = f(3.5, 1.5) = 3.5 - 2 \cdot 1.5 = 0.5$$
$$f(P_{12}) = f(1.5, 2.5) = 1.5 - 2 \cdot 2.5 = -3.5$$
$$f(P_{22}) = f(2.5, 2.5) = 2.5 - 2 \cdot 2.5 = -2.5$$
$$f(P_{32}) = f(3.5, 2.5) = 3.5 - 2 \cdot 2.5 = -1.5$$
$$\Delta x = \frac{4-1}{3} = 1, \quad \Delta y = \frac{3-1}{2} = 1$$

Hence $\Delta A = \Delta x \cdot \Delta y = 1$ and we get

$$S_{32} = \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(P_{ij}\right)\Delta A = -1.5 - 0.5 + 0.5 - 3.5 - 2.5 - 1.5 = -9$$

(B)

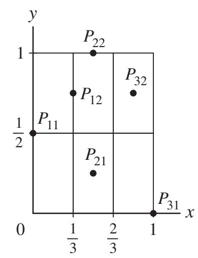
$$f(P_{11}) = f(1.5, 1.5) = 1.5 - 2 \cdot 1.5 = -1.5$$
$$f(P_{21}) = f(2, 1) = 2 - 2 \cdot 1 = 0$$
$$f(P_{31}) = f(3.5, 1.5) = 3.5 - 2 \cdot 1.5 = 0.5$$
$$f(P_{12}) = f(2, 3) = 2 - 2 \cdot 3 = -4$$
$$f(P_{22}) = f(2.5, 2.5) = 2.5 - 2 \cdot 2.5 = -2.5$$
$$f(P_{32}) = f(4, 3) = 4 - 2 \cdot 3 = -2$$

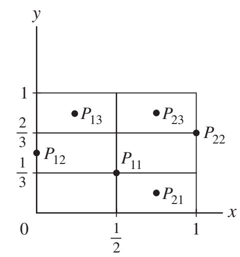
$\Delta A = 1$, hence

$$S_{32} = \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(P_{ij}\right)\Delta A = -1.5 + 0 + 0.5 - 4 - 2.5 - 2 = -9.5$$

7. Let $\mathcal{R} = [0, 1] \times [0, 1]$. Estimate $\iint_{\mathcal{R}} (x + y)\, dA$ by computing two different Riemann sums, each with at least six rectangles.

SOLUTION We define the following subrectangles and sample points:





The sample points defined in the two figures are:

(A)

$$P_{11}=\left(0,\tfrac{1}{2}\right)\quad P_{21}=\left(\tfrac{1}{2},\tfrac{1}{4}\right)\quad P_{31}=(1,0)$$
$$P_{12}=\left(\tfrac{1}{3},\tfrac{3}{4}\right)\quad P_{22}=\left(\tfrac{1}{2},1\right)\quad P_{32}=\left(\tfrac{5}{6},\tfrac{3}{4}\right)$$

(B)

$$P_{11}=\left(\tfrac{1}{2},\tfrac{1}{3}\right)\quad P_{21}=\left(\tfrac{3}{4},\tfrac{1}{6}\right)\quad P_{12}=\left(0,\tfrac{1}{2}\right)$$
$$P_{22}=\left(1,\tfrac{2}{3}\right)\quad P_{13}=\left(\tfrac{1}{4},\tfrac{5}{6}\right)\quad P_{23}=\left(\tfrac{3}{4},\tfrac{5}{6}\right)$$

We compute the values of $f(x,y)=x+y$ at the sample points:

(A)

$$f(P_{11})=f\left(0,\frac{1}{2}\right)=0+\frac{1}{2}=\frac{1}{2}$$
$$f(P_{21})=f\left(\frac{1}{2},\frac{1}{4}\right)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$
$$f(P_{31})=f(1,0)=1+0=1$$
$$f(P_{12})=f\left(\frac{1}{3},\frac{3}{4}\right)=\frac{1}{3}+\frac{3}{4}=\frac{13}{12}$$
$$f(P_{22})=f\left(\frac{1}{2},1\right)=\frac{1}{2}+1=\frac{3}{2}$$
$$f(P_{32})=f\left(\frac{5}{6},\frac{3}{4}\right)=\frac{5}{6}+\frac{3}{4}=\frac{19}{12}$$

Each subrectangle has sides of length $\Delta x=\frac{1}{3}$, $\Delta y=\frac{1}{2}$ and area $\Delta A=\Delta x\Delta y=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$. We obtain the following Riemann sum:

$$S_{32}=\sum_{i=1}^{3}\sum_{j=1}^{2}f\left(P_{ij}\right)\Delta A=\frac{1}{6}\left(\frac{1}{2}+\frac{3}{4}+1+\frac{13}{12}+\frac{3}{2}+\frac{19}{12}\right)=\frac{77}{72}\approx 1.069$$

(B)

$$f(P_{11})=f\left(\frac{1}{2},\frac{1}{3}\right)=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$
$$f(P_{21})=f\left(\frac{3}{4},\frac{1}{6}\right)=\frac{3}{4}+\frac{1}{6}=\frac{11}{12}$$
$$f(P_{12})=f\left(0,\frac{1}{2}\right)=0+\frac{1}{2}=\frac{1}{2}$$
$$f(P_{22})=f\left(1,\frac{2}{3}\right)=1+\frac{2}{3}=\frac{5}{3}$$
$$f(P_{13})=f\left(\frac{1}{4},\frac{5}{6}\right)=\frac{1}{4}+\frac{5}{6}=\frac{13}{12}$$
$$f(P_{23})=f\left(\frac{3}{4},\frac{5}{6}\right)=\frac{3}{4}+\frac{5}{6}=\frac{19}{12}$$

Each subrectangle has sides of length $\Delta x=\frac{1}{2}$, $\Delta y=\frac{1}{3}$ and area $\Delta A=\Delta x\Delta y=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$. We obtain the following Riemann sum:

$$S_{23}=\sum_{i=1}^{3}\sum_{j=1}^{2}f\left(P_{ij}\right)\Delta A=\frac{1}{6}\left(\frac{5}{6}+\frac{11}{12}+\frac{1}{2}+\frac{5}{3}+\frac{13}{12}+\frac{19}{12}\right)=\frac{79}{72}\approx 1.097$$

8. Evaluate $\iint_{\mathcal{R}} 4\,dA$, where $\mathcal{R}=[2,5]\times[4,7]$.

SOLUTION The double integral is the volume of the box of base $\mathcal{R}$ and height 4. That is,

$$\iint_{\mathcal{R}} 4\,dA=4\cdot\text{Area}(R) \tag{1}$$

The rectangle $\mathcal{R}$ has sides of length $5 - 2 = 3$ and $7 - 4 = 3$, therefore its area is $3 \cdot 3 = 9$. Hence, by (1) we get

$$\iint_{\mathcal{R}} 4\,dA = 4 \cdot 9 = 36$$

9. Evaluate $\iint_{\mathcal{R}} (15 - 3x)\,dA$, where $\mathcal{R} = [0, 5] \times [0, 3]$, and sketch the corresponding solid region (see Example 2).

SOLUTION This double integral is the volume V of the solid wedge underneath the graph of $f(x, y) = 15 - 3x$. The triangular face of the wedge has area

$$A = \frac{1}{2} \cdot 5 \cdot 15 = \frac{75}{2}$$

The volume of the wedge is equal to the area A times the length $\ell = 3$; that is

$$V = \ell A = 3\left(\frac{75}{2}\right) = \frac{225}{2}$$

10. Evaluate $\iint_{\mathcal{R}} (-5)\,dA$, where $\mathcal{R} = [2, 5] \times [4, 7]$.

SOLUTION The double integral is the signed volume of the box of base $\mathcal{R}$ and height -5. That is,

$$\iint_{\mathcal{R}} (-5)\,dA = -5 \cdot \text{Area}(\mathcal{R}) = -5 \cdot (5-2) \cdot (7-4) = -5 \cdot 9 = -45$$

11. The following table gives the approximate height at quarter-meter intervals of a mound of gravel. Estimate the volume of the mound by computing the average of the two Riemann sums $S_{4,3}$ with lower-left and upper-right vertices of the subrectangles as sample points.

0.75	0.1	0.2	0.2	0.15	0.1
0.5	0.2	0.3	0.5	0.4	0.2
0.25	0.15	0.2	0.4	0.3	0.2
0	0.1	0.15	0.2	0.15	0.1
$y \backslash x$	0	0.25	0.5	0.75	1

SOLUTION Each subrectangle is a square of side 0.25, hence the area of each subrectangle is $\Delta A = 0.25^2 = 0.0625$. By the given data, the lower-left vertex sample points are:

$$\begin{array}{lll}
f(P_{11}) = f(0,0) & f(P_{12}) = f(0,0.25) & f(P_{13}) = f(0,0.50) \\
f(P_{21}) = f(0.25,0) & f(P_{22}) = f(0.25,0.25) & f(P_{23}) = f(0.25,0.50) \\
f(P_{31}) = f(0.50,0) & f(P_{32}) = f(0.50,0.25) & f(P_{33}) = f(0.50,0.50) \\
f(P_{41}) = f(0.75,0) & f(P_{42}) = f(0.75,0.25) & f(P_{43}) = f(0.75,0.50)
\end{array}$$

The Riemann sum $S_{4,3}$ that corresponds to these lower-left vertex sample points is the following sum:

$$\begin{aligned}
S_{4,3} &= \sum_{i=1}^{4}\sum_{j=1}^{3} f\left(P_{ij}\right)\Delta A \\
&= 0.0625(0.1 + 0.15 + 0.2 + 0.15 + 0.2 + 0.3 + 0.2 + 0.4 + 0.5 + 0.15 + 0.3 + 0.4) \approx 0.190625
\end{aligned}$$

Now by the given data, the upper-right vertex sample points are:

$$\begin{array}{lll}
f(P_{11}) = f(0.25,0.25) & f(P_{12}) = f(0.25,0.50) & f(P_{13}) = f(0.25,0.75) \\
f(P_{21}) = f(0.50,0.25) & f(P_{22}) = f(0.50,0.50) & f(P_{23}) = f(0.50,0.75) \\
f(P_{31}) = f(0.75,0.25) & f(P_{32}) = f(0.75,0.50) & f(P_{33}) = f(0.75,0.75) \\
f(P_{41}) = f(1,0.25) & f(P_{42}) = f(1,0.50) & f(P_{43}) = f(1,0.75)
\end{array}$$

The Riemann sum S'_{43} that corresponds to these upper-right vertex sample points is the following sum:

$$\begin{aligned}
S'_{4,3} &= \sum_{i=1}^{4}\sum_{j=1}^{3} f\left(P_{ij}\right)\Delta A \\
&= 0.0625(0.2 + 0.3 + 0.2 + 0.4 + 0.5 + 0.2 + 0.3 + 0.4 + 0.15 + 0.2 + 0.2 + 0.1) \approx 0.196875
\end{aligned}$$

Taking the average of the two Riemann sums we have:

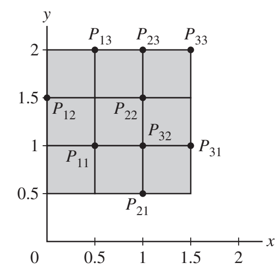
$$\text{volume} \approx \frac{S_{4,3} + S'_{4,3}}{2} = \frac{0.190625 + 0.196875}{2} = 0.19375$$

12. Use the following table to compute a Riemann sum $S_{3,3}$ for $f(x, y)$ on the square $\mathcal{R} = [0, 1.5] \times [0.5, 2]$. Use the regular partition and sample points of your choosing.

Values of $f(x, y)$

y \ x	0	0.5	1	1.5	2
2	2.6	2.17	1.86	1.62	1.44
1.5	2.2	1.83	1.57	1.37	1.22
1	1.8	1.5	1.29	1.12	1
0.5	1.4	1.17	1	0.87	0.78
0	1	0.83	0.71	0.62	0.56

SOLUTION The subrectangles and our choice of sample points are shown in the figure:



Each subrectangle is a square of side 0.5, hence the area of each subrectangle is $\Delta A = 0.5^2 = 0.25$. By the given data, the sample points are:

$$\begin{array}{lll} f(P_{11}) = f(0.5, 1) & f(P_{12}) = f(0, 1.5) & f(P_{13}) = f(0.5, 2) \\ f(P_{21}) = f(1, 0.5) & f(P_{22}) = f(1, 1.5) & f(P_{23}) = f(1, 2) \\ f(P_{31}) = f(1.5, 1) & f(P_{32}) = f(1, 1) & f(P_{33}) = f(1.5, 2) \end{array}$$

The Riemann sum S_{33} that corresponds to these sample points is the following sum:

$$S_{33} = \sum_{i=1}^{3} \sum_{j=1}^{3} f\left(P_{ij}\right) \Delta A = 0.25(1.5 + 1 + 1.12 + 2.2 + 1.57 + 1.29 + 2.17 + 1.86 + 1.62) \approx 3.58$$

13. CAS Let $S_{N,N}$ be the Riemann sum for $\int_0^1 \int_0^1 e^{x^3 - y^3} \, dy \, dx$ using the regular partition and the lower left-hand vertex of each subrectangle as sample points. Use a computer algebra system to calculate $S_{N,N}$ for $N = 25, 50, 100$.

SOLUTION Using a computer algebra system, we compute $S_{N,N}$ to be 1.0731, 1.0783, and 1.0809.

14. CAS Let $S_{N,M}$ be the Riemann sum for

$$\int_0^4 \int_0^2 \ln(1 + x^2 + y^2) \, dy \, dx$$

using the regular partition and the upper right-hand vertex of each subrectangle as sample points. Use a computer algebra system to calculate $S_{2N,N}$ for $N = 25, 50, 100$.

SOLUTION Using a computer algebra system, we compute $S_{2N,N}$ to be 14.632, 14.486, and 14.413.

In Exercises 15–18, use symmetry to evaluate the double integral.

15. $\iint_{\mathcal{R}} x^3 \, dA, \quad \mathcal{R} = [-4, 4] \times [0, 5]$

SOLUTION The double integral is the signed volume of the region between the graph of $f(x, y) = x^3$ and the xy-plane. However, $f(x, y)$ takes opposite values at (x, y) and $(-x, y)$:

$$f(-x, y) = (-x)^3 = -x^3 = -f(x, y)$$

Because of symmetry, the (negative) signed volume of the region below the xy-plane where $-4 \le x \le 0$ cancels with the (positive) signed volume of the region above the xy-plane where $0 \le x \le 4$. The net result is

$$\iint_{\mathcal{R}} x^3 \, dA = 0$$

16. $\displaystyle\iint_{\mathcal{R}} 1 \, dA, \quad \mathcal{R} = [2,4] \times [-7,7]$

SOLUTION This double integral is the signed volume of the region below the graph of $f(x,y) = 1$ (which is a plane at height 1). We can view this integral as the volume of the rectangular box having height 1 over the rectangle $\mathcal{R} = [2,4] \times [-7,7]$. The area of this rectangle is

$$A = (4-2)(7-(-7)) = 2 \cdot 14 = 28$$

Therefore, the volume of the rectangular box is

$$\iint_{\mathcal{R}} 1 \, dA = 1(28) = 28$$

17. $\displaystyle\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi]$

SOLUTION Since $\sin(\pi + x) = -\sin x$, the region between the graph and the xy-plane where $\pi \le x \le 2\pi$, is below the xy-plane, and it cancels with the region above the xy-plane where $0 \le x \le \pi$. Hence,

$$\iint_{\mathcal{R}} \sin x \, dA = 0$$

18. $\displaystyle\iint_{\mathcal{R}} (2 + x^2 y) \, dA, \quad \mathcal{R} = [0,1] \times [-1,1]$

SOLUTION By additivity of the Double Integral, we have

$$\iint_{\mathcal{R}} \left(2 + x^2 y\right) dA = \iint_{\mathcal{R}} 2 \, dA + \iint_{\mathcal{R}} x^2 y \, dA \tag{1}$$

We consider each of the two integrals on the right-hand side:

$$\iint_{\mathcal{R}} 2 \, dA = \iint_{\mathcal{R}} 2 \cdot \text{Area}(\mathcal{R}) = 2 \cdot 1 \cdot 2 = 4 \tag{2}$$

The function $f(x,y) = x^2 y$ satisfies $f(x,-y) = -f(x,y)$. Since the double integral is the signed volume of the region between the graph and the xy-plane, the region below the xy-plane (where $-1 \le y \le 0$) cancels with the region above the xy-plane (where $0 \le y \le 1$). Thus,

$$\iint_{\mathcal{R}} x^2 y \, dA = 0 \tag{3}$$

Combining (1), (2), and (3), we obtain

$$\iint_{\mathcal{R}} \left(2 + x^2 y\right) dA = 4 + 0 = 4$$

In Exercises 19–36, evaluate the iterated integral.

19. $\displaystyle\int_1^3 \int_0^2 x^3 y \, dy \, dx$

SOLUTION We first compute the inner integral, treating x as a constant, then integrate the result with respect to x:

$$\int_1^3 \int_0^2 x^3 y \, dy \, dx = \int_1^3 x^3 \frac{y^2}{2}\bigg|_{y=0}^{2} dx = \int_1^3 x^3 \left(\frac{2^2}{2} - 0\right) dx = \int_1^3 2x^3 \, dx = \frac{x^4}{2}\bigg|_1^3 = 40$$

20. $\int_0^2 \int_1^3 x^3 y \, dx \, dy$

SOLUTION

$$\int_0^2 \int_1^3 x^3 y \, dx dy = \int_0^2 x^3 \left(\int_1^3 y \, dy \right) dx$$

$$= \int_0^2 x^3 \left(\frac{y^2}{2} \bigg|_1^3 \right) dy$$

$$= 4 \int_0^2 x^3 \, dx$$

$$= 4 \cdot \frac{x^4}{4} \bigg|_0^2$$

$$= 16$$

21. $\int_4^9 \int_{-3}^8 1 \, dx \, dy$

SOLUTION

$$\int_4^9 \int_{-3}^8 1 \, dx dy = \int_4^9 1 \left(\int_{-3}^8 1 \, dy \right) dx$$

$$= \int_4^9 1 \left(y \bigg|_{-3}^8 \right) dx$$

$$= \int_4^9 11 \, dx$$

$$= 11x \bigg|_4^9$$

$$= 99 - 44 = 55$$

22. $\int_{-4}^{-1} \int_4^8 (-5) \, dx \, dy$

SOLUTION

$$\int_{-4}^{-1} \int_4^8 (-5) \, dx \, dy = \int_{-4}^{1} \left(\int_4^8 (-5) \, dx \right) dy$$

$$= \int_{-4}^{-1} \left(-5x \bigg|_4^8 \right) dy$$

$$= \int_{-4}^{-1} (-40 + 20) \, dy$$

$$= \int_{-4}^{-1} (-20) \, dy$$

$$= -20y \bigg|_{-4}^{-1}$$

$$= 20 - (80) = -60$$

23. $\int_{-1}^1 \int_0^\pi x^2 \sin y \, dy \, dx$

SOLUTION We first evaluate the inner integral, treating x as a constant, then integrate the result with respect to x. This gives

$$\int_{-1}^1 \int_0^\pi x^2 \sin y \, dy \, dx = \int_{-1}^1 x^2 (-\cos y) \bigg|_{y=0}^{\pi} dx = \int_{-1}^1 x^2 (-\cos \pi + \cos 0) \, dx$$

$$= \int_{-1}^1 x^2 (1+1) \, dx = \int_{-1}^1 2x^2 \, dx = \frac{2}{3} x^3 \bigg|_{-1}^1 = \frac{2}{3} \left(1^3 - (-1)^3 \right) = \frac{4}{3}$$

24. $\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y\,dx\,dy$

SOLUTION

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y\,dx\,dy &= \int_{-1}^{1} \sin y \left(\int_0^{\pi} x^2\,dx\right)\,dy \\ &= \int_{-1}^{1} \sin y \left(\frac{x^3}{3}\bigg|_0^{\pi}\right)\,dy \\ &= \frac{\pi^3}{3}\int_{-1}^{1}\sin y\,dy \\ &= \frac{\pi^3}{3}\left(-\cos y\bigg|_{-1}^{1}\right) \\ &= \frac{\pi^3}{3}(-\cos(1)+\cos(-1)) = 0\end{aligned}$$

25. $\int_{2}^{6}\int_{1}^{4} x^2\,dx\,dy$

SOLUTION We use Iterated Integral of a Product Function to compute the integral as follows:

$$\begin{aligned}\int_2^6\int_1^4 x^2\,dx\,dy = \int_2^6\int_1^4 x^2\cdot 1\,dx\,dy = \left(\int_1^4 x^2\,dx\right)\left(\int_2^6 1\,dy\right) = \left(\frac{x^3}{3}\bigg|_1^4\right)\left(y\bigg|_2^6\right) \\ = \left(\frac{4^3}{3}-\frac{1^3}{3}\right)(6-2) = 21\cdot 4 = 84\end{aligned}$$

26. $\int_{2}^{6}\int_{1}^{4} y^2\,dx\,dy$

SOLUTION Since the integrand is a product $f(x,y)=y^2\cdot 1$ we can compute the double integral as a product of integrals. That is,

$$\int_2^6\int_1^4 y^2\,dx\,dy = \left(\int_2^6 y^2\,dy\right)\left(\int_1^4 1\,dx\right) = \left(\frac{y^3}{3}\bigg|_2^6\right)\left(x\bigg|_1^4\right) = \left(\frac{6^3}{3}-\frac{2^3}{3}\right)(4-1) = \frac{208}{3}\cdot 3 = 208$$

27. $\int_{0}^{1}\int_{0}^{2}(x+4y^3)\,dx\,dy$

SOLUTION We use additivity of the double integral to write

$$\int_0^1\int_0^2 \left(x+4y^3\right)\,dx\,dy = \int_0^1\int_0^2 x\,dx\,dy + \int_0^1\int_0^2 4y^3\,dx\,dy \tag{1}$$

We now compute each of the double integrals using product of iterated integrals:

$$\begin{aligned}\int_0^1\int_0^2 x\,dx\,dy &= \left(\int_0^2 x\,dx\right)\left(\int_0^1 1\,dy\right) = \left(\frac{1}{2}x^2\bigg|_0^2\right)\left(y\bigg|_0^1\right) = 2\cdot 1 = 2 \\ \int_0^1\int_0^2 4y^3\,dx\,dy &= \left(\int_0^1 4y^3\,dy\right)\left(\int_0^2 1\,dx\right) = \left(y^4\bigg|_0^1\right)\left(x\bigg|_0^2\right) = 1\cdot 2 = 2\end{aligned}$$

Substituting in (1) gives

$$\int_0^1\int_0^2 (x+4y^3)\,dx\,dy = 2+2 = 4.$$

28. $\int_0^2 \int_0^2 (x^2 - y^2)\, dy\, dx$

SOLUTION We first use additivity of the double integral to rewrite the integral as the sum of two double integrals. Then we compute each of the resulting integrals as product of iterated integral. We obtain

$$\begin{aligned}\int_0^2 \int_0^2 \left(x^2 - y^2\right) dy\, dx &= \int_0^2 \int_0^2 x^2\, dy\, dx - \int_0^2 \int_0^2 y^2\, dy\, dx \\ &= \left(\int_0^2 x^2\, dx\right)\left(\int_0^2 1\, dy\right) - \left(\int_0^2 y^2\, dy\right)\left(\int_0^2 1\, dx\right) = 0\end{aligned}$$

29. $\int_0^4 \int_0^9 \sqrt{x + 4y}\, dx\, dy$

SOLUTION We compute the inner integral, treating y as a constant. Then we evaluate the resulting integral with respect to y:

$$\begin{aligned}\int_0^4 \int_0^9 \sqrt{x+4y}\, dx\, dy &= \int_0^4 \frac{2}{3}(x+4y)^{3/2}\Big|_{x=0}^{9} dy = \int_0^4 \frac{2}{3}\left((9+4y)^{3/2} - (4y)^{3/2}\right) dy \\ &= \frac{2}{3}\left(\frac{2}{5\cdot 4}(9+4y)^{5/2} - \frac{2}{5\cdot 4}(4y)^{5/2}\right)\Big|_0^4 \\ &= \frac{1}{15}(5^5 - 4^5) - \frac{1}{15}(3^5 - 0) \approx 123.8667\end{aligned}$$

30. $\int_0^{\pi/4} \int_{\pi/4}^{\pi/2} \cos(2x + y)\, dy\, dx$

SOLUTION

$$\begin{aligned}\int_0^{\pi/4} \int_{\pi/4}^{\pi/2} \cos(2x+y)\, dy\, dx &= \int_0^{\pi/4} \left(\sin(2x+y)\Big|_{\pi/4}^{\pi/2}\right) dx \\ &= \int_0^{\pi/4} \sin\left(2x + \frac{\pi}{2}\right) - \sin\left(2x + \frac{\pi}{4}\right) dx \\ &= \frac{1}{2}\left(-\cos\left(2x + \frac{\pi}{2}\right) + \cos\left(2x + \frac{\pi}{4}\right)\right)\Big|_0^{\pi/4} \\ &= \frac{1}{2}\left(-\cos\pi + \cos\frac{3\pi}{4} + \cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{\sqrt{2}}\end{aligned}$$

31. $\int_1^2 \int_0^4 \frac{dy\, dx}{x + y}$

SOLUTION The inner integral is an iterated integral with respect to y. We evaluate it first and then compute the resulting integral with respect to x. This gives

$$\int_1^2 \int_0^4 \frac{dy\, dx}{x+y} = \int_1^2 \left(\int_0^4 \frac{dy}{x+y}\right) dx = \int_1^2 \ln(x+y)\Big|_{y=0}^{4} dx = \int_1^2 (\ln(x+4) - \ln x)\, dx$$

We use the integral formula:

$$\int \ln(x+a)\, dx = (x+a)\left(\ln(x+a) - 1\right) + C$$

We get

$$\begin{aligned}\int_1^2 \int_0^4 \frac{dy\, dx}{x+y} &= (x+4)\left(\ln(x+4) - 1\right) - x(\ln x - 1)\Big|_1^2 \\ &= 6(\ln 6 - 1) - 2(\ln 2 - 1) - (5(\ln 5 - 1) - (\ln 1 - 1)) \\ &= 6\ln 6 - 2\ln 2 - 5\ln 5 \approx 1.31\end{aligned}$$

32. $\int_1^2 \int_2^4 e^{3x-y}\,dy\,dx$

SOLUTION

$$\begin{aligned}
\int_1^2 \int_2^4 e^{3x-y}\,dy\,dx &= \int_1^2 \int_2^4 e^{3x} \cdot e^{-y}\,dy\,dx \\
&= \int_1^2 e^{3x}\,dx \cdot \int_2^4 e^{-y}\,dy \\
&= \left(\frac{1}{3}e^{3x}\bigg|_1^2\right) \cdot \left(-e^{-y}\bigg|_2^4\right) \\
&= \frac{e^6 - e^3}{3} \cdot (e^{-2} - e^{-4}) \approx 14.953
\end{aligned}$$

33. $\int_0^4 \int_0^5 \frac{dy\,dx}{\sqrt{x+y}}$

SOLUTION

$$\begin{aligned}
\int_0^4 \int_0^5 \frac{dy\,dx}{\sqrt{x+y}} &= \int_0^4 \left(\int_0^5 \frac{dy}{\sqrt{x+y}}\right) dx \\
&= \int_0^4 \left(2\sqrt{x+y}\bigg|_{y=0}^5\right) dx \\
&= 2\int_0^4 (\sqrt{x+5} - \sqrt{x})\,dx \\
&= 2\left(\frac{2}{3}(x+5)^{3/2} - \frac{2}{3}x^{3/2}\right)\bigg|_0^4 \\
&= 2\left(\frac{2}{3}\cdot 27 - \frac{2}{3}\cdot 8\right) - 2\left(\frac{2}{3}\cdot 5^{3/2} - 0\right) \\
&= 36 - \frac{32}{3} - \frac{20}{3}\sqrt{5} = \frac{76}{3} - \frac{20}{3}\sqrt{5} \approx 10.426
\end{aligned}$$

34. $\int_0^8 \int_1^2 \frac{x\,dx\,dy}{\sqrt{x^2+y}}$

SOLUTION

$$\begin{aligned}
\int_0^8 \int_1^2 \frac{x\,dx\,dy}{\sqrt{x^2+y}} &= \int_0^8 y \int_1^2 \left(\frac{x\,dx}{\sqrt{x^2+y}}\right) dy \\
&= \int_0^8 \left(\sqrt{x^2+y}\bigg|_1^2\right) dy \\
&= \int_0^8 (\sqrt{4+y} - \sqrt{1+y})\,dy \\
&= \frac{2}{3}(4+y)^{3/2} - \frac{2}{3}(1+y)^{3/2}\bigg|_0^8 \\
&= \frac{2}{3}(12^{3/2} - 27) - \frac{2}{3}(8-1) \\
&= -\frac{68}{3} + 16\sqrt{3} \approx 5.047
\end{aligned}$$

35. $\int_1^2 \int_1^3 \frac{\ln(xy)\,dy\,dx}{y}$

SOLUTION

$$\int_1^2 \int_1^3 \frac{\ln(xy)\,dy\,dx}{y} = \int_1^2 \left(\frac{1}{2}[\ln(xy)]^2\bigg|_1^3\right) dx$$

$$= \frac{1}{2}\int_1^2 [\ln(3x)]^2 - [\ln(x)]^2\,dx$$
$$= \frac{1}{2}\int_1^2 [\ln(3x)]^2\,dx - \frac{1}{2}\int_1^2 [\ln(x)]^2\,dx$$
$$= \frac{1}{2}\left[x(\ln 3x)^2\Big|_1^2 - 2\int_1^2 \ln(3x)\,dx\right] - \frac{1}{2}\left[x(\ln x)^2\Big|_1^2 - 2\int_1^2 \ln x\,dx\right]$$
$$= \frac{1}{2}\left[2(\ln 6)^2 - (\ln 3)^2\right] - \int_1^2 \ln(3x)\,dx - \frac{1}{2}\left[2(\ln 2)^2 - 0\right] + \int_1^2 \ln x\,dx$$
$$= (\ln 6)^2 - \frac{1}{2}(\ln 3)^2 - \left[x\ln(3x) - x\Big|_1^2\right] - (\ln 2)^2 + \left[x\ln x - x\Big|_1^2\right]$$
$$= (\ln 6)^2 - \frac{1}{2}(\ln 3)^2 - (\ln 2)^2 - (2\ln 6 - 2 - \ln 3 + 1) + (2\ln 2 - 2 - 0 + 1)$$
$$= (\ln 6)^2 - \frac{1}{2}(\ln 3)^2 - (\ln 2)^2 - 2\ln 6 + \ln 3 + 1 + 2\ln 2 - 2 + 1$$
$$= (\ln 6)^2 - \frac{1}{2}(\ln 3)^2 - (\ln 2)^2 - 2\ln 6 + \ln 3 + 2\ln 2 \approx 1.028$$

36. $\displaystyle\int_0^1\int_2^3 \frac{1}{(x+4y)^3}\,dx\,dy$

SOLUTION We calculate the inner integral with respect to x, then we compute the resulting integral with respect to y. This gives

$$\int_0^1\int_2^3 \frac{1}{(x+4y)^3}\,dx\,dy = \int_0^1\left(\int_2^3 (x+4y)^{-3}\,dx\right)dy = \int_0^1 -\frac{1}{2}(x+4y)^{-2}\Big|_{x=2}^3\,dy$$
$$= -\frac{1}{2}\int_0^1\left((3+4y)^{-2} - (2+4y)^{-2}\right)dy = -\frac{1}{2\cdot 4}\left(-(3+4y)^{-1} + (2+4y)^{-1}\right)\Big|_0^1$$
$$= -\frac{1}{8}\left(\frac{1}{2+4y} - \frac{1}{3+4y}\right)\Big|_0^1 = -\frac{1}{8}\left(\left(\frac{1}{6}-\frac{1}{7}\right) - \left(\frac{1}{2}-\frac{1}{3}\right)\right)$$
$$= \frac{1}{8}\left(\frac{1}{7} - \frac{1}{6} + \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{56}$$

In Exercises 37–42, use Eq. (1) to evaluate the integral.

37. $\displaystyle\iint_{\mathcal{R}} \frac{x}{y}\,dA, \quad \mathcal{R} = [-2,4]\times[1,3]$

SOLUTION We compute the double integral as the product of two single integrals:

$$\iint_{\mathcal{R}} \frac{x}{y}\,dA = \int_{-2}^4\int_1^3 \frac{x}{y}\,dy\,dx = \int_{-2}^4 x\,dx \cdot \int_1^3 \frac{1}{y}\,dy$$
$$= \left(\frac{1}{2}x^2\Big|_{-2}^4\right)\left(\ln y\Big|_1^3\right) = \frac{1}{2}(16-4)\cdot(\ln 3 - \ln 1)$$
$$= 6\ln 3$$

38. $\displaystyle\iint_{\mathcal{R}} x^2y\,dA, \quad \mathcal{R} = [-1,1]\times[0,2]$

SOLUTION We compute the double integral as the product of two single integrals:

$$\iint_{\mathcal{R}} x^2y\,dA = \int_0^2\int_{-1}^1 x^2y\,dx\,dy = \left(\int_{-1}^1 x^2\,dx\right)\left(\int_0^2 y\,dy\right) = \left(\frac{1}{3}x^3\Big|_{-1}^1\right)\left(\frac{1}{2}y^2\Big|_0^2\right) = \frac{2}{3}\cdot\frac{1}{2}\cdot 4 = \frac{4}{3}$$

39. $\iint_{\mathcal{R}} \cos x \sin 2y\, dA, \quad \mathcal{R} = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

SOLUTION Since the integrand has the form $f(x, y) = g(x)h(y)$, we may compute the double integral as the product of two single integrals. That is,

$$\iint_{\mathcal{R}} \cos x \sin 2y\, dA = \int_0^{\pi/2} \int_0^{\pi/2} \cos x \sin 2y\, dx\, dy = \left(\int_0^{\pi/2} \cos x\, dx\right)\left(\int_0^{\pi/2} \sin 2y\, dy\right)$$

$$= \left(\sin x \Big|_0^{\pi/2}\right)\left(-\frac{1}{2}\cos 2y \Big|_0^{\pi/2}\right) = \left(\sin\frac{\pi}{2} - \sin 0\right)\left(-\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0\right)$$

$$= (1 - 0)\left(\frac{1}{2} + \frac{1}{2}\right) = 1$$

40. $\iint_{\mathcal{R}} \frac{y}{x+1}\, dA, \quad \mathcal{R} = [0, 2] \times [0, 4]$

SOLUTION We evaluate the integral as the product of two single integrals. This can be done since the function has the form $f(x, y) = g(x)h(y)$.

$$\iint_{\mathcal{R}} \frac{y}{x+1}\, dA = \int_0^4 \int_0^2 \frac{y}{x+1}\, dx\, dy = \left(\int_0^2 \frac{dx}{x+1}\right)\left(\int_0^4 y\, dy\right)$$

$$= \left(\ln(x+1)\Big|_0^2\right)\left(\frac{y^2}{2}\Big|_0^4\right) = (\ln 3 - \ln 1)\left(\frac{4^2}{2} - \frac{0^2}{2}\right) = 8\ln 3 \approx 8.79$$

41. $\iint_{\mathcal{R}} e^x \sin y\, dA, \quad \mathcal{R} = [0, 2] \times \left[0, \frac{\pi}{4}\right]$

SOLUTION We compute the double integral as the product of two single integrals. This can be done since the integrand has the form $f(x, y) = g(x)h(y)$. We get

$$\iint_{\mathcal{R}} e^x \sin y\, dA = \int_0^{\pi/4} \int_0^2 e^x \sin y\, dx\, dy = \left(\int_0^2 e^x\, dx\right)\left(\int_0^{\pi/4} \sin y\, dy\right)$$

$$= \left(e^x\Big|_0^2\right)\left(-\cos y\Big|_0^{\pi/4}\right) = \left(e^2 - e^0\right)\left(-\cos\frac{\pi}{4} + \cos 0\right) = \left(e^2 - 1\right)\left(1 - \frac{\sqrt{2}}{2}\right) \approx 1.87$$

42. $\iint_{\mathcal{R}} e^{3x+4y}\, dA, \quad \mathcal{R} = [0, 1] \times [1, 2]$

SOLUTION We can compute this double integral as the product of two single integrals:

$$\iint_{\mathcal{R}} e^{3x+4y}\, dA = \int_0^1 \int_1^2 e^{3x+4y}\, dy\, dx = \int_0^1 \int_1^2 e^{3x} e^{4y}\, dy\, dx$$

$$= \int_0^1 e^{3x}\, dx \cdot \int_1^2 e^{4y}\, dy = \left(\frac{1}{3}e^{3x}\Big|_0^1\right)\left(\frac{1}{4}e^{4y}\Big|_1^2\right)$$

$$= \frac{1}{3}(e^3 - 1) \cdot \frac{1}{4}(e^8 - e^1) = \frac{1}{12}(e^3 - 1)(e^8 - e)$$

43. Let $f(x, y) = mxy^2$, where m is a constant. Find a value of m such that $\iint_{\mathcal{R}} f(x, y)\, dA = 1$, where $\mathcal{R} = [0, 1] \times [0, 2]$.

SOLUTION Since $f(x, y) = mxy^2$ is a product of a function of x and a function of y, we may compute the double integral as the product of two single integrals. That is,

$$\int_0^2 \int_0^1 mxy^2\, dx\, dy = m\left(\int_0^1 x\, dx\right)\left(\int_0^2 y^2\, dy\right) \tag{1}$$

We compute each integral:

$$\int_0^1 x\, dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}\left(1^2 - 0^2\right) = \frac{1}{2}$$

$$\int_0^2 y^2\, dy = \frac{1}{3}y^3\Big|_0^2 = \frac{1}{3}\left(2^3 - 0^3\right) = \frac{8}{3}$$

We substitute in (1), equate to 1 and solve the resulting equation for m. This gives

$$m \cdot \frac{1}{2} \cdot \frac{8}{3} = 1 \quad \Rightarrow \quad m = \frac{3}{4}$$

44. Evaluate $I = \int_1^3 \int_0^1 ye^{xy}\,dy\,dx$. You will need Integration by Parts and the formula

$$\int e^x (x^{-1} - x^{-2})\,dx = x^{-1}e^x + C$$

Then evaluate I again using Fubini's Theorem to change the order of integration (that is, integrate first with respect to x). Which method is easier?

SOLUTION We evaluate the inner integral with respect to y. Using integration by parts with $dv = e^{xy}\,dy$, $u = y$ we obtain

$$\int ye^{xy}\,dy = \frac{y}{x}e^{xy} - \int \frac{1}{x}e^{xy}\,dy = \frac{y}{x}e^{xy} - \frac{1}{x} \cdot \frac{1}{x}e^{xy} + C = \frac{e^{xy}}{x}\left(y - \frac{1}{x}\right) + C$$

Hence,

$$\int_0^1 ye^{xy}\,dy = \frac{e^{xy}}{x}\left(y - \frac{1}{x}\right)\Big|_{y=0}^{1} = \frac{e^x}{x}\left(1 - \frac{1}{x}\right) - \frac{1}{x}\left(0 - \frac{1}{x}\right) = e^x\left(x^{-1} - x^{-2}\right) + x^{-2}$$

We now integrate the result with respect to x, using the given integration formula:

$$\begin{aligned}\int_1^3 \left(e^x\left(x^{-1} - x^{-2}\right) + x^{-2}\right)dx &= \int_1^3 e^x\left(x^{-1} - x^{-2}\right)dx + \int_1^3 x^{-2}\,dx = x^{-1}e^x\Big|_1^3 - x^{-1}\Big|_1^3 \\ &= \frac{e^3}{3} - e - \left(\frac{1}{3} - 1\right) = \frac{e^3}{3} - e + \frac{2}{3} \approx 4.644\end{aligned}$$

The double integral is thus

$$\int_1^3 \int_0^1 ye^{xy}\,dy\,dx = \int_1^3 \left(\int_0^1 ye^{xy}\,dy\right)dx \approx 4.644$$

Using Fubini's Theorem, we now evaluate the double integral first with respect to x and then with respect to y (keeping the corresponding limits of integration). We obtain

$$\begin{aligned}\int_1^3 \int_0^1 ye^{xy}\,dy\,dx &= \int_0^1 \int_1^3 ye^{xy}\,dx\,dy = \int_0^1 \left(\int_1^3 ye^{xy}\,dx\right)dy = \int_0^1 \left(\frac{y}{y}e^{xy}\Big|_{x=1}^{3}\right)dy \\ &= \int_0^1 \left(e^{3y} - e^y\right)dy = \frac{1}{3}e^{3y} - e^y\Big|_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3}e^0 - e^0\right) \\ &= \frac{1}{3}e^3 - e - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644\end{aligned}$$

Obviously, integrating first with respect to x and then with respect to y makes the computation much easier than using the reversed order.

45. Evaluate $\int_0^1 \int_0^1 \frac{y}{1+xy}\,dy\,dx$. *Hint:* Change the order of integration.

SOLUTION Using Fubini's Theorem we change the order of integration, integrating first with respect to x and then with respect to y. This gives

$$\begin{aligned}\int_0^1 \int_0^1 \frac{y}{1+xy}\,dy\,dx &= \int_0^1 \left(\int_0^1 \frac{y}{1+xy}\,dx\right)dy = \int_0^1 \left(y\int_0^1 \frac{dx}{1+xy}\right)dy = \int_0^1 y \cdot \frac{1}{y}\ln(1+xy)\Big|_{x=0}^{1} dy \\ &= \int_0^1 (\ln(1+y) - \ln 1)\,dy = \int_0^1 \ln(1+y)\,dy = (1+y)\left(\ln(1+y) - 1\right)\Big|_{y=0}^{1} \\ &= 2(\ln 2 - 1) - (\ln 1 - 1) = 2\ln 2 - 1 \approx 0.386\end{aligned}$$

46. Calculate a Riemann sum $S_{3,3}$ on the square $\mathcal{R} = [0, 3] \times [0, 3]$ for the function $f(x, y)$ whose contour plot is shown in Figure 17. Choose sample points and use the plot to find the values of $f(x, y)$ at these points.

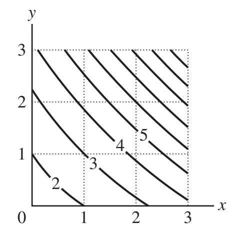


FIGURE 17 Contour plot of $f(x, y)$.

SOLUTION Each subrectangle is a square of side length 1, hence its area is $\Delta A = 1^2 = 1$. We choose the sample points shown in the figure. The contour plot shows the following values of f at the sample points:

$$\begin{array}{lll} f(P_{11}) = 2 & f(P_{21}) = 3 & f(P_{31}) = 4 \\ f(P_{12}) = 3 & f(P_{22}) = 4 & f(P_{32}) = 7 \\ f(P_{13}) = 5 & f(P_{23}) = 6 & f(P_{33}) = 10 \end{array}$$

The Riemann sum S_{33} is thus

$$S_{33} = \sum_{i=1}^{3}\sum_{j=1}^{3} f\left(P_{ij}\right)\Delta A = \sum_{i=1}^{3}\sum_{j=1}^{3} f\left(P_{ij}\right)\cdot 1 = 2+3+4+3+4+7+5+6+10 = 44$$

47. Using Fubini's Theorem, argue that the solid in Figure 18 has volume AL, where A is the area of the front face of the solid.

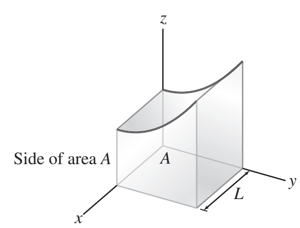
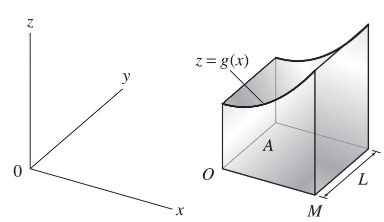


FIGURE 18

SOLUTION We denote by M the length of the other side of the rectangle in the basis of the solid. The volume of the solid is the double integral of the function $f(x, y) = g(x)$ over the rectangle

$$\mathcal{R} = [0, M] \times [0, L]$$

$$V = \iint_{\mathcal{R}} g(x)\, dA.$$



We use Fubini's Theorem to write the double integral as iterated integral, and then compute the resulting integral as the product of two single integrals. This gives

$$V = \iint_{\mathcal{R}} g(x)\, dA = \int_0^L \int_0^M g(x)\, dx\, dy = \left(\int_0^M g(x)\, dx\right)\left(\int_0^L 1\, dy\right) = \left(\int_0^M g(x)\, dx\right)\cdot L \qquad (1)$$

The integral $\int_0^M g(x)dx$ is the area A of the region under the graph of $z = g(x)$ over the interval $0 \le x \le M$. Substituting in (1) gives the following volume of the solid:

$$V = \left(\int_0^M g(x)dx\right)\cdot L = AL$$

Further Insights and Challenges

48. Prove the following extension of the Fundamental Theorem of Calculus to two variables: If $\dfrac{\partial^2 F}{\partial x\,\partial y} = f(x, y)$, then

$$\iint_{\mathcal{R}} f(x, y)\,dA = F(b, d) - F(a, d) - F(b, c) + F(a, c)$$

where $\mathcal{R} = [a, b] \times [c, d]$.

SOLUTION By Fubini's Theorem we get

$$\iint_{\mathcal{R}} f(x, y)\,dx = \int_c^d \int_a^b \frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy = \int_c^d \left(\int_a^b \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)dx\right)dy \tag{1}$$

In the inner integral, y is considered as constant. Therefore, by the Fundamental Theorem of calculus (part I) for the variable x, we have

$$\int_a^b \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)dx = \frac{\partial F}{\partial y}\bigg|_{x=b} - \frac{\partial F}{\partial y}\bigg|_{x=a} = \frac{\partial F}{\partial y}(b, y) - \frac{\partial F}{\partial y}(a, y)$$

We substitute in (1), use additivity of the single integral and use again the Fundamental Theorem, this time for the variable y. This gives

$$\begin{aligned}
\iint_{\mathcal{R}} f(x, y)\,dx &= \int_c^d \left(\frac{\partial F}{\partial y}(b, y) - \frac{\partial F}{\partial y}(a, y)\right)dy = \int_c^d \frac{\partial F}{\partial y}(b, y)\,dy - \int_c^d \frac{\partial F}{\partial y}(a, y)\,dy \\
&= \left(F(b, y)\bigg|_{y=d} - F(b, y)\bigg|_{y=c}\right) - \left(F(a, y)\bigg|_{y=d} - F(a, y)\bigg|_{y=c}\right) \\
&= F(b, d) - F(b, c) - F(a, d) + F(a, c)
\end{aligned}$$

49. Let $F(x, y) = x^{-1}e^{xy}$. Show that $\dfrac{\partial^2 F}{\partial x\,\partial y} = ye^{xy}$ and use the result of Exercise 48 to evaluate $\iint_{\mathcal{R}} ye^{xy}\,dA$ for the rectangle $\mathcal{R} = [1, 3] \times [0, 1]$.

SOLUTION Differentiating $F(x, y) = x^{-1}e^{xy}$ with respect to y gives

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^{-1}e^{xy}\right) = x^{-1}xe^{xy} = e^{xy}$$

We now differentiate $\frac{\partial F}{\partial y}$ with respect to x:

$$\frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(e^{xy}\right) = ye^{xy}$$

In Exercise 48 we showed that

$$\int_c^d \int_a^b \frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy = F(b, d) - F(b, c) - F(a, d) + F(a, c)$$

Therefore, for $F(x, y) = x^{-1}e^{xy} = \frac{e^{xy}}{x}$ we obtain

$$\begin{aligned}
\iint_{\mathcal{R}} ye^{xy}\,dA &= \int_0^1 \int_1^3 ye^{xy}\,dx\,dy = F(3, 1) - F(3, 0) - F(1, 1) + F(1, 0) \\
&= \frac{e^3}{3} - \frac{e^0}{3} - \frac{e^1}{1} + \frac{e^0}{1} = \frac{e^3}{3} - \frac{1}{3} - e + 1 = 4.644
\end{aligned}$$

50. Find a function $F(x, y)$ satisfying $\dfrac{\partial^2 F}{\partial x\,\partial y} = 6x^2y$ and use the result of Exercise 48 to evaluate $\iint_{\mathcal{R}} 6x^2y\,dA$ for the rectangle $\mathcal{R} = [0, 1] \times [0, 4]$.

SOLUTION We integrate $\frac{\partial^2 F}{\partial x \partial y}$ with respect to x, taking zero as the constant of integration. We get $\frac{\partial F}{\partial y} = 6y\frac{x^3}{3} = 2yx^3$. We now integrate with respect to y, obtaining

$$F(x, y) = 2 \cdot \frac{y^2}{2}x^3 = y^2x^3$$

In Exercise 48 we showed that

$$\int_c^d \int_a^b \frac{\partial^2 F}{\partial x\, \partial y}\, dx\, dy = F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

For $F(x, y) = y^2 x^3$ we get

$$\iint_{\mathcal{R}} 6x^2 y\, dA = \int_0^4 \int_0^1 6x^2 y\, dx\, dy = F(1,4) - F(1,0) - F(0,4) + F(0,0) = 4^2 \cdot 1^3 - 0 - 0 + 0 = 16$$

51. In this exercise, we use double integration to evaluate the following improper integral for $a > 0$ a positive constant:

$$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\, dx$$

(a) Use L'Hôpital's Rule to show that $f(x) = \dfrac{e^{-x} - e^{-ax}}{x}$, though not defined at $x = 0$, can be made continuous by assigning the value $f(0) = a - 1$.

(b) Prove that $|f(x)| \le e^{-x} + e^{-ax}$ for $x > 1$ (use the triangle inequality), and apply the Comparison Theorem to show that $I(a)$ converges.

(c) Show that $I(a) = \displaystyle\int_0^\infty \int_1^a e^{-xy}\, dy\, dx$.

(d) Prove, by interchanging the order of integration, that

$$I(a) = \ln a - \lim_{T \to \infty} \int_1^a \frac{e^{-Ty}}{y}\, dy \qquad \boxed{2}$$

(e) Use the Comparison Theorem to show that the limit in Eq. (2) is zero. *Hint:* If $a \ge 1$, show that $e^{-Ty}/y \le e^{-T}$ for $y \ge 1$, and if $a < 1$, show that $e^{-Ty}/y \le e^{-aT}/a$ for $a \le y \le 1$. Conclude that $I(a) = \ln a$ (Figure 19).

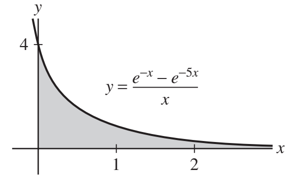


FIGURE 19 The shaded region has area ln 5.

SOLUTION

(a) The function $f(x) = \frac{e^{-x} - e^{-ax}}{x}$, $f(0) = a - 1$ is continuous if $\lim_{x \to 0} f(x) = f(0) = a - 1$. We verify this limit using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{e^{-x} - e^{-ax}}{x} = \lim_{x \to 0} \frac{-e^{-x} + ae^{-ax}}{1} = -1 + a = a - 1$$

Therefore, f is continuous.

(b) We now show that the following integral converges:

$$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\, dx \qquad (a > 0)$$

Since $e^{-x} - e^{-ax} < e^{-x} + e^{-ax}$ then also $\frac{e^{-x} - e^{-ax}}{x} < \frac{e^{-x} + e^{-ax}}{x}$ for $x > 0$. Therefore, if $x > 1$ we have

$$\frac{e^{-x} - e^{-ax}}{x} < \frac{e^{-x} + e^{-ax}}{x} < e^{-x} + e^{-ax}$$

That is, for $x > 1$,

$$f(x) < e^{-x} + e^{-ax} \qquad (1)$$

Also, since $e^{-ax} - e^{-x} < e^{-ax} + e^{-x}$ we have for $x > 1$,

$$\frac{e^{-ax} - e^{-x}}{x} < \frac{e^{-ax} + e^{-x}}{x} < e^{-ax} + e^{-x}$$

Thus, we get

$$-f(x) < e^{-x} + e^{-ax} \qquad (2)$$

Together with (1) we have

$$0 \le |f(x)| < e^{-x} + e^{-ax} \tag{3}$$

We now show that the integral of the right hand-side converges:

$$\begin{aligned}
\int_0^{\infty} \left(e^{-x} + e^{-ax}\right) dx &= \lim_{\mathcal{R}\to\infty} \int_0^{\mathcal{R}} \left(e^{-x} + e^{-ax}\right) dx \\
&= \lim_{\mathcal{R}\to\infty} \left(-e^{-x} - \frac{e^{-ax}}{a} \Big|_{x=0}^{\mathcal{R}} \right) \\
&= \lim_{R\to\infty} \left(-e^{-\mathcal{R}} - \frac{e^{-aR}}{a} + e^0 + \frac{e^0}{a} \right) \\
&= \lim_{\mathcal{R}\to\infty} \left(-e^{-\mathcal{R}} - \frac{e^{-a\mathcal{R}}}{a} + 1 + \frac{1}{a} \right) \\
&= 1 + \frac{1}{a}
\end{aligned}$$

Since the integral converges, we conclude by (3) and the Comparison Test for Improper Integrals that

$$\int_0^{\infty} \frac{e^{-x} - e^{-ax}}{x}\, dx$$

also converges for $a > 0$.

(c) We compute the inner integral with respect to y:

$$\int_1^a e^{-xy}\, dy = -\frac{1}{x} e^{-xy} \Big|_{y=1}^{a} = -\frac{1}{x}\left(e^{-xa} - e^{-x\cdot 1}\right) = \frac{e^{-x} - e^{-xa}}{x}$$

Therefore,

$$\int_0^{\infty} \int_1^a e^{-xy}\, dy\, dx = \int_0^{\infty} \left(\int_1^a e^{-xy}\, dy \right) dx = \int_0^{\infty} \frac{e^{-x} - e^{-xa}}{x}\, dx = I(a)$$

(d) By the definition of the improper integral,

$$I(a) = \lim_{T\to\infty} \int_0^T \int_1^a e^{-xy}\, dy\, dx \tag{4}$$

We compute the double integral. Using Fubini's Theorem we may compute the iterated integral using reversed order of integration. That is,

$$\begin{aligned}
\int_0^T \int_1^a e^{-xy}\, dy\, dx &= \int_1^a \int_0^T e^{-xy}\, dx\, dy = \int_1^a \left(\int_0^T e^{-xy}\, dx \right) dy = \int_1^a \left(-\frac{1}{y} e^{-xy} \Big|_{x=0}^{T} \right) dy \\
&= \int_1^a \left(-\frac{1}{y}\left(e^{-Ty} - e^{-0\cdot y}\right)\right) dy = \int_1^a \frac{1 - e^{-Ty}}{y}\, dy = \int_1^a \frac{dy}{y} - \int_1^a \frac{e^{-Ty}}{y}\, dy \\
&= \ln y \Big|_1^a - \int_1^a \frac{e^{-Ty}}{y}\, dy = \ln a - \ln 1 - \int_1^a \frac{e^{-Ty}}{y}\, dy = \ln a - \int_1^a \frac{e^{-Ty}}{y}\, dy
\end{aligned}$$

Combining with (4) we get

$$I(a) = \ln a - \lim_{T\to\infty} \int_1^a \frac{e^{-Ty}}{y}\, dy \tag{5}$$

(e) We now show, using the Comparison Theorem, that

$$\lim_{T\to\infty} \int_1^a \frac{e^{-Ty}}{y}\, dy = 0$$

We consider the following possible cases:

Case 1: $a \ge 1$. Then in the interval of integration $y \ge 1$. Also since $T \to \infty$, we may assume that $T > 0$. Thus,

$$\frac{e^{-Ty}}{y} \le \frac{e^{-T\cdot 1}}{1} = e^{-T}$$

Hence,

$$0 \le \int_1^a \frac{e^{-Ty}}{y}\,dy \le \int_1^a e^{-T}\,dy = e^{-T}(a-1)$$

By the limit $\lim_{T\to\infty} e^{-T}(a-1) = 0$ and the Squeeze Theorem we conclude that,

$$\lim_{T\to\infty}\int_1^a \frac{e^{-Ty}}{y}\,dy = 0$$

Case 2: $0 < a < 1$. Then,

$$\int_1^a \frac{e^{-Ty}}{y}\,dy = -\int_a^1 \frac{e^{-Ty}}{y}\,dy$$

and in the interval of integration $a \le y \le 1$, therefore

$$\frac{e^{-Ty}}{y} \le \frac{e^{-Ta}}{a}$$

(the function $\frac{e^{-Ty}}{y}$ is decreasing). Hence,

$$0 \le \int_a^1 \frac{e^{-Ty}}{y}\,dy \le \int_a^1 \frac{e^{-Ta}}{a}\,dy = \frac{(1-a)}{a}e^{-Ta}$$

By the limit $\lim_{T\to\infty} \frac{1-a}{a}e^{-Ta} = 0$ and the Squeeze Theorem we conclude also that

$$\lim_{T\to\infty}\int_1^a \frac{e^{-Ty}}{y} = -\lim_{T\to\infty}\int_a^1 \frac{e^{-Ty}}{y} = 0$$

We thus showed that for all $a > 0$, $\lim_{T\to\infty}\int_1^a \frac{e^{-Ty}}{y} = 0$. Combining with Eq. (5) obtained in part (c), we find that $I(a) = \ln a$.

15.2 Double Integrals over More General Regions (LT Section 16.2)

Preliminary Questions

1. .Which of the following expressions do not make sense?

(a) $\int_0^1 \int_1^x f(x,y)\,dy\,dx$

(b) $\int_0^1 \int_1^y f(x,y)\,dy\,dx$

(c) $\int_0^1 \int_x^y f(x,y)\,dy\,dx$

(d) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

SOLUTION

(a) This integral is the following iterated integral:

$$\int_0^1 \int_1^x f(x,y)\,dy\,dx = \int_0^1 \left(\int_1^x f(x,y)\,dy\right)dx$$

The inner integral is a function of x and it is integrated with respect to x over the interval $0 \le x \le 1$. The result is a number. This integral makes sense.

(b) This integral is the same as

$$\int_0^1 \int_1^y f(x,y)\,dy\,dx = \int_0^1 \left(\int_1^y f(x,y)\,dy\right)dx$$

The inner integral is an integral with respect to y, over the interval $[1, y]$. This does not make sense.

(c) This integral is the following iterated integral:

$$\int_0^1 \left(\int_x^y f(x,y)\right)dy\,dx$$

The inner integral is a function of x and y and it is integrated with respect to y over the interval $x \le y \le y$. This does not make sense.

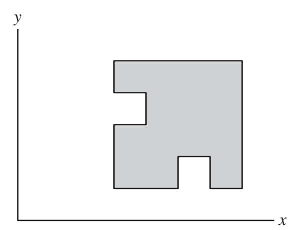
(d) This integral is the following iterated integral:

$$\int_0^1 \left(\int_x^1 f(x, y)\, dy \right) dx$$

The inner integral is a function of x and it is integrated with respect to x. This makes sense.

2. Draw a domain in the plane that is neither vertically nor horizontally simple.

SOLUTION The following region cannot be described in the form $\{a \le x \le b, \alpha(x) \le y \le \beta(x)\}$ nor in the form $\{c \le y \le d, \alpha(y) \le x \le \beta(y)\}$, hence it is neither vertically nor horizontally simple.



3. Which of the four regions in Figure 18 is the domain of integration for $\int_{-\sqrt{2}/2}^{0} \int_{-x}^{\sqrt{1-x^2}} f(x, y)\, dy\, dx$?

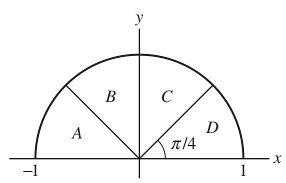
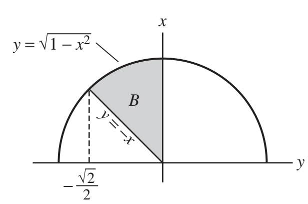


FIGURE 18

SOLUTION The region B is defined by the inequalities

$$-x \le y \le \sqrt{1 - x^2}, \qquad -\frac{\sqrt{2}}{2} \le x \le 0$$

To compute $\int_{-\sqrt{2}/2}^{0} \int_{-x}^{\sqrt{1-x^2}} f(x, y)\, dy\, dx$, we first integrate with respect to y over the interval $-x \le y \le \sqrt{1 - x^2}$, and then with respect to x over $-\frac{\sqrt{2}}{2} \le x \le 0$. That is, the domain of integration is B.



4. Let $\mathcal{D}$ be the unit disk. If the maximum value of $f(x, y)$ on $\mathcal{D}$ is 4, then the largest possible value of $\iint_{\mathcal{D}} f(x, y)\, dA$ is (choose the correct answer):

(a) 4 **(b)** 4π **(c)** $\frac{4}{\pi}$

SOLUTION The area of the unit disk is π and the maximum value of $f(x, y)$ on this region is $M = 4$, therefore we have,

$$\iint_{\mathcal{D}} f(x, y)\, dx\, dy \le 4\pi$$

The correct answer is (b).

Exercises

1. Calculate the Riemann sum for $f(x, y) = x - y$ and the shaded domain $\mathcal{D}$ in Figure 19 with two choices of sample points, $\bullet$ and $\circ$. Which do you think is a better approximation to the integral of f over $\mathcal{D}$? Why?

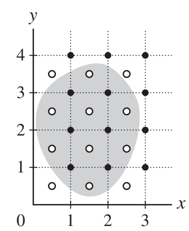


FIGURE 19

SOLUTION The subrectangles in Figure 17 have sides of length $\Delta x = \Delta y = 1$ and area $\Delta A = 1 \cdot 1 = 1$.

(a) Sample points $\bullet$. There are six sample points that lie in the domain $\mathcal{D}$. We compute the values of $f(x, y) = x - y$ at these points:

$$f(1,1) = 0, \qquad f(1,2) = -1, \qquad f(1,3) = -2,$$
$$f(2,1) = 1, \qquad f(2,2) = 0, \qquad f(2,3) = -1$$

The Riemann sum is

$$S_{3,4} = (0 - 1 - 2 + 1 + 0 - 1) \cdot 1 = -3$$

(b) Sample points $\circ$. We compute the values of $f(x, y) = x - y$ at the eight sample points that lie in $\mathcal{D}$:

$$\begin{array}{lll} & f(0.5, 1.5) = -1, & f(0.5, 2.5) = -2, \\ f(1.5, 0.5) = 1, & f(1.5, 1.5) = 0, & f(1.5, 2.5) = -1, \\ f(1.5, 3.5) = -2, & f(2.5, 1.5) = 1, & f(2.5, 2.5) = 0. \end{array}$$

The corresponding Riemann sum is thus

$$S_{34} = (1 - 1 - 2 + 0 - 1 - 2 + 1 + 0) \cdot 1 = -4.$$

2. Approximate values of $f(x, y)$ at sample points on a grid are given in Figure 20. Estimate $\iint_{\mathcal{D}} f(x, y)\,dx\,dy$ for the shaded domain by computing the Riemann sum with the given sample points.

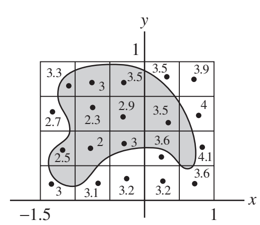


FIGURE 20

SOLUTION The subrectangles have sides of length $\Delta x = 0.5$ and $\Delta y = 0.25$, so the area is $\Delta A = 0.5 \cdot 0.25 = 0.125$. Only nine of the sample points lie in $\mathcal{D}$, hence the corresponding Riemann sum is

$$S_{5,4} = (2.5 + 3.3 + 2 + 2.3 + 3 + 3 + 2.9 + 3.5 + 3.5) \cdot 0.125 = 3.25$$

3. Express the domain $\mathcal{D}$ in Figure 21 as both a vertically simple region and a horizontally simple region, and evaluate the integral of $f(x, y) = xy$ over $\mathcal{D}$ as an iterated integral in two ways.

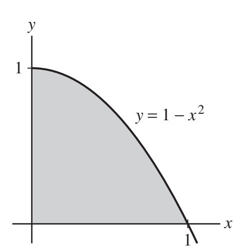
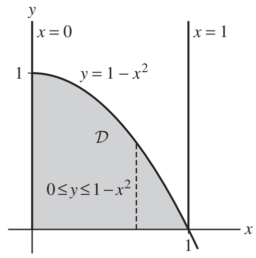


FIGURE 21

SOLUTION The domain $\mathcal{D}$ can be described as a vertically simple region as follows:

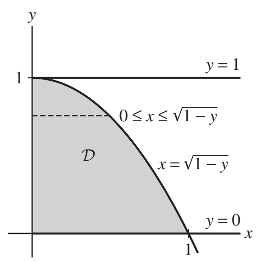
$$0 \le x \le 1, \quad 0 \le y \le 1 - x^2 \tag{1}$$



Vertically simple region

The domain $\mathcal{D}$ can also be described as a horizontally simple region. To do this, we must express x in terms of y, for nonnegative values of x. This gives

$$y = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y \quad \Rightarrow \quad x = \sqrt{1-y}$$



Horizontally simple region

Therefore, we can describe $\mathcal{D}$ by the following inequalities:

$$0 \le y \le 1, \quad 0 \le x \le \sqrt{1-y} \tag{2}$$

We now compute the integral of $f(x, y) = xy$ over $\mathcal{D}$ first using definition (1) and then using definition (2). We obtain

$$\iint_{\mathcal{D}} xy\, dA = \int_0^1 \int_0^{1-x^2} xy\, dy\, dx = \int_0^1 \frac{xy^2}{2}\bigg|_{y=0}^{1-x^2} dx = \int_0^1 \frac{x}{2}\left((1-x^2)^2 - 0^2\right) dx = \int_0^1 \frac{x(1-x^2)^2}{2}\, dx$$

$$= \frac{1}{2}\int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6}\right)\bigg|_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{12}$$

Using definition (2) gives

$$\iint_{\mathcal{D}} xy\, dA = \int_0^1 \int_0^{\sqrt{1-y}} xy\, dx\, dy = \int_0^1 \frac{yx^2}{2}\bigg|_{x=0}^{\sqrt{1-y}} dy = \int_0^1 \frac{y}{2}\left(\left(\sqrt{1-y}\right)^2 - 0^2\right) dy$$

$$= \int_0^1 \frac{y}{2}(1-y)\, dy = \int_0^1 \left(\frac{y}{2} - \frac{y^2}{2}\right) dy = \frac{y^2}{4} - \frac{y^3}{6}\bigg|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

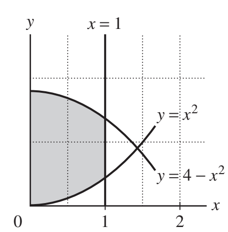
The answers agree as expected.

4. Sketch the domain

$$\mathcal{D} : 0 \le x \le 1, \quad x^2 \le y \le 4 - x^2$$

and evaluate $\iint_{\mathcal{D}} y\, dA$ as an iterated integral.

SOLUTION The domain $\mathcal{D}$ is shown in the following figure:



$\mathcal{D}$ is a vertically simple region and the limits of integration are

$$\underbrace{0 \le x \le 1}_{\text{limits of outer integral}} \qquad \underbrace{x^2 \le y \le 4 - x^2}_{\text{limits of inner integral}}$$

We follow three steps.

Step 1. Set up the iterated integral. The iterated integral is

$$\iint_{\mathcal{D}} y\, dA = \int_0^1 \int_{x^2}^{4-x^2} y\, dy\, dx.$$

Step 2. Evaluate the inner integral. We evaluate the inner integral with respect to y:

$$\int_{x^2}^{4-x^2} y\, dy = \frac{y^2}{2}\bigg|_{y=x^2}^{y=4-x^2} = \frac{1}{2}\left((4-x^2)^2 - (x^2)^2\right) = \frac{1}{2}(16 - 8x^2 + x^4 - x^4) = 8 - 4x^2$$

Step 3. Complete the computation. We integrate the resulting function with respect to x:

$$\iint_{\mathcal{D}} y\, dy = \int_0^1 (8 - 4x^2)\, dx = 8x - \frac{4}{3}x^3\bigg|_0^1 = 8 - \frac{4}{3} = \frac{20}{3} \approx 6.67$$

In Exercises 5–7, compute the double integral of $f(x, y) = x^2 y$ over the given shaded domain in Figure 22.

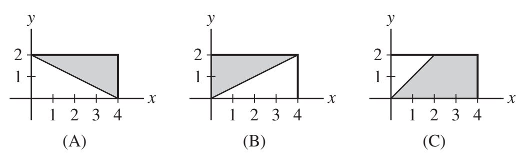
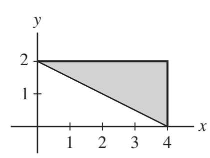


FIGURE 22

5. (A)

SOLUTION



We describe the domain $\mathcal{D}$ as a vertically simple region. We find the equation of the line connecting the points (0, 2) and (4, 0).

$$y - 0 = \frac{2-0}{0-4}(x-4) \quad \Rightarrow \quad y = -\frac{1}{2}x + 2$$

Therefore the domain is described as a vertically simple region by the inequalities

$$0 \le x \le 4, \quad -\frac{1}{2}x + 2 \le y \le 2$$

We use Theorem 2 to evaluate the double integral:

$$\iint_{\mathcal{D}} x^2 y\, dA = \int_0^4 \int_{-\frac{x}{2}+2}^{2} x^2 y\, dy\, dx = \int_0^4 \frac{x^2 y^2}{2}\bigg|_{y=-\frac{x}{2}+2}^{2} dx = \int_0^4 \frac{x^2}{2}\left(2^2 - \left(-\frac{x}{2}+2\right)^2\right) dx$$

$$= \int_0^4 \left(x^3 - \frac{x^4}{8}\right) dx = \frac{x^4}{4} - \frac{x^5}{40}\bigg|_0^4 = \frac{4^4}{4} - \frac{4^5}{40} = \frac{192}{5} = 38.4$$

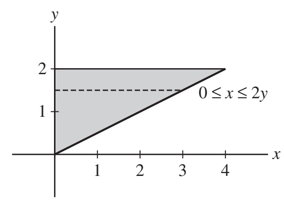
6. (B)

SOLUTION We describe the domain $\mathcal{D}$ as a horizontally simple region. We first find the equation of the line connecting the points (0, 0) and (4, 2).

$$y = \frac{1}{2}x \quad \Rightarrow \quad x = 2y$$

Therefore, the domain of integration is described by the following inequalities:

$$0 \le y \le 2, \quad 0 \le x \le 2y$$



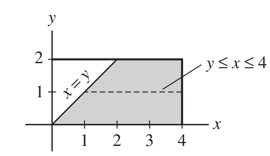
We use Theorem 2 to evaluate the double integral as follows:

$$\iint_{\mathcal{D}} x^2 y\, dA = \int_0^2 \int_0^{2y} x^2 y\, dx\, dy = \int_0^2 \left.\frac{x^3 y}{3}\right|_{x=0}^{2y} dy = \int_0^2 \frac{y}{3}\left((2y)^3 - 0^3\right) dy = \int_0^2 \frac{y}{3}\cdot 8y^3\, dy$$
$$= \int_0^2 \frac{8y^4}{3}\, dy = \left.\frac{8}{15}y^5\right|_0^2 = \frac{256}{15} \approx 17.07$$

7. (C)

SOLUTION The domain in (C) is a horizontally simple region, described by the inequalities

$$0 \le y \le 2, \quad y \le x \le 4$$

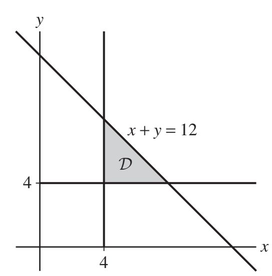


Using Theorem 2 we obtain the following integral:

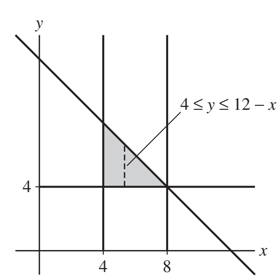
$$\iint_{\mathcal{D}} x^2 y\, dA = \int_0^2 \int_y^4 x^2 y\, dx\, dy = \int_0^2 \left.\frac{x^3 y}{3}\right|_{x=y}^{x=4} dy = \int_0^2 \frac{y}{3}\left(4^3 - y^3\right) dy = \int_0^2 \left(\frac{64y}{3} - \frac{y^4}{3}\right) dy$$
$$= \left.\frac{32}{3}y^2 - \frac{y^5}{15}\right|_0^2 = \frac{32\cdot 2^2}{3} - \frac{2^5}{15} = \frac{608}{15} \approx 40.53$$

8. Sketch the domain $\mathcal{D}$ defined by $x + y \le 12$, $x \ge 4$, $y \ge 4$ and compute $\iint_{\mathcal{D}} e^{x+y}\, dA$.

SOLUTION The domain $\mathcal{D} = \{(x, y) : x + y \le 12, x \ge 4, y \ge 4\}$ is shown in the following figure:



To compute the integral we described $\mathcal{D}$ as a vertically simple region by the following inequalities (see figure):



$$4 \le x \le 8, \quad 4 \le y \le 12 - x$$

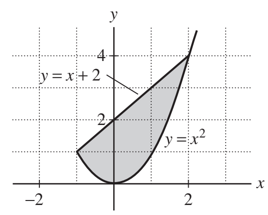
Using Theorem 2, we obtain the following integral:

$$\iint_{\mathcal{D}} e^{x+y}\,dA = \int_4^8 \int_4^{12-x} e^{x+y}\,dy\,dx = \int_4^8 e^{x+y}\Big|_{y=4}^{12-x}\,dx = \int_4^8 \left(e^{x+(12-x)} - e^{x+4}\right)dx$$

$$= \int_4^8 \left(e^{12} - e^{x+4}\right)dx = e^{12}x - e^{x+4}\Big|_4^8 = 8e^{12} - e^{12} - \left(4e^{12} - e^8\right) = 3e^{12} + e^8 \approx 491245.3$$

9. Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$.

SOLUTION The domain of integration is shown in the following figure:

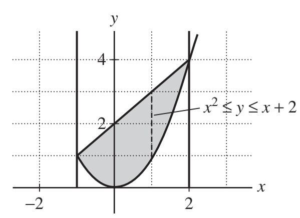


To find the inequalities defining the domain as a vertically simple region we first must find the x-coordinates of the two points where the line $y = x + 2$ and the parabola $y = x^2$ intersect. That is,

$$x + 2 = x^2 \quad\Rightarrow\quad x^2 - x - 2 = (x-2)(x+1) = 0$$
$$\Rightarrow\quad x_1 = -1, \quad x_2 = 2$$

We describe the domain by the following inequalities:

$$-1 \le x \le 2, \quad x^2 \le y \le x + 2$$



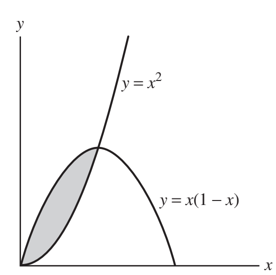
We now evaluate the integral of $f(x, y) = x$ over the vertically simple region $\mathcal{D}$:

$$\iint_{\mathcal{D}} x\,dA = \int_{-1}^2 \int_{x^2}^{x+2} x\,dy\,dx = \int_{-1}^2 xy\Big|_{y=x^2}^{x+2}\,dx = \int_{-1}^2 x\left(x + 2 - x^2\right)dx$$

$$= \int_{-1}^2 \left(x^2 + 2x - x^3\right)dx = \frac{x^3}{3} + x^2 - \frac{x^4}{4}\Big|_{-1}^2 = \left(\frac{8}{3} + 4 - 4\right) - \left(-\frac{1}{3} + 1 - \frac{1}{4}\right) = 2\frac{1}{4}$$

10. Sketch the region $\mathcal{D}$ between $y = x^2$ and $y = x(1 - x)$. Express $\mathcal{D}$ as a simple region and calculate the integral of $f(x, y) = 2y$ over $\mathcal{D}$.

SOLUTION The region $\mathcal{D}$ between $y = x^2$ and $y = x(1 - x)$ is shown in the following figure:

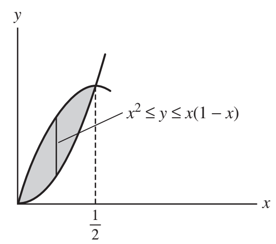


To find the inequalities for the vertically simple region $\mathcal{D}$, we first compute the x-coordinate of the point where the curves $y = x^2$ and $y = x(1-x)$ intersect.

$$x^2 = x(1-x) \quad \Rightarrow \quad x^2 = x - x^2$$
$$\Rightarrow \quad 2x^2 - x = x(2x-1) = 0$$
$$\Rightarrow \quad x_1 = 0, \quad x_2 = \frac{1}{2}$$

The region $\mathcal{D}$ is defined by the following inequalities:

$$0 \le x \le \frac{1}{2}, \quad x^2 \le y \le x(1-x)$$



The double integral of f over $\mathcal{D}$ is computed using Theorem 2. That is,

$$\iint_{\mathcal{D}} 2y\,dA = \int_0^{1/2} \int_{x^2}^{x(1-x)} 2y\,dy\,dx = \int_0^{1/2} y^2 \Big|_{y=x^2}^{y=x(1-x)} dx = \int_0^{1/2} x^2(1-x)^2 - x^4\,dx$$
$$= \int_0^{1/2} x^2 - 2x^3 + x^4 - x^4\,dx = \int_0^{1/2} x^2 - 2x^3\,dx = \frac{1}{96} \approx 0.010$$

11. Evaluate $\iint_{\mathcal{D}} \frac{y}{x}\,dA$, where $\mathcal{D}$ is the shaded part of the semicircle of radius 2 in Figure 23.

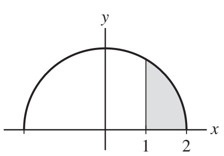


FIGURE 23 $y = \sqrt{4-x^2}$

SOLUTION The region is defined by the following inequalities:

$$1 \le x \le 2, \quad 0 \le y \le \sqrt{4-x^2}$$

Therefore, the double integral of f over $\mathcal{D}$ is:

$$\iint_{\mathcal{D}} \frac{y}{x}\,dA = \int_1^2 \int_0^{\sqrt{4-x^2}} \frac{y}{x}\,dy\,dx$$
$$= \int_1^2 \frac{1}{x}\left(\frac{1}{2}y^2\Big|_0^{\sqrt{4-x^2}}\right) dx$$
$$= \frac{1}{2}\int_1^2 \frac{1}{x}(4-x^2)\,dx$$
$$= \frac{1}{2}\int_1^2 \frac{4}{x} - x\,dx$$
$$= \frac{1}{2}\left(4\ln|x| - \frac{1}{2}x^2\right)\Big|_1^2$$
$$= \frac{1}{2}(4\ln 2 - 2) - \frac{1}{2}\left(0 - \frac{1}{2}\right)$$
$$= 2\ln 2 - 1 + \frac{1}{4} = 2\ln 2 - \frac{3}{4} \approx 0.636$$

12. Calculate the double integral of $f(x, y) = y^2$ over the rhombus $\mathcal{R}$ in Figure 24.

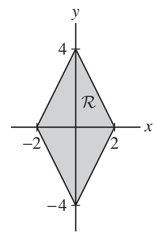


FIGURE 24 $|x| + \frac{1}{2}|y| \le 1$

SOLUTION Since $f(x, -y) = f(x, y)$ and since the rhombus is symmetric with respect to the x-axis, the double integral equals twice the integral over the upper half of the rhombus. Moreover, since $f(-x, y) = f(x, y)$ and $\mathcal{R}$ is symmetric with respect to the y-axis, the double integral over $\mathcal{R}$ equals twice the integral over the right half of the rhombus. Therefore, denoting by $\mathcal{D}$ the part of the rhombus in the first quadrant, we have

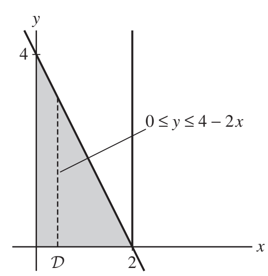
$$\iint_{\mathcal{R}} y^2\,dA = 4\iint_{\mathcal{D}} y^2\,dA \tag{1}$$

We now compute the double integral over $\mathcal{D}$. We express $\mathcal{D}$ as a vertically simple region. The line connecting the point $(0, 4)$ and $(2, 0)$ has the equation

$$y - 4 = \frac{4-0}{0-2}(x-0) \quad \Rightarrow \quad y - 4 = -2x \quad \Rightarrow \quad y = 4 - 2x$$

Thus, $\mathcal{D}$ is defined by the following inequalities:

$$0 \le x \le 2, \quad 0 \le y \le 4 - 2x$$



We now compute the integral over $\mathcal{D}$ using Theorem 2:

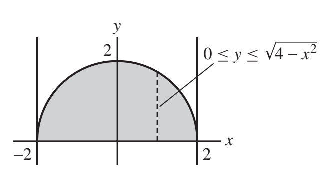
$$\iint_{\mathcal{D}} y^2\,dA = \int_0^2 \int_0^{4-2x} y^2\,dy\,dx = \int_0^2 \frac{y^3}{3}\bigg|_{y=0}^{4-2x} dx = \int_0^2 \frac{(4-2x)^3}{3}\,dx = \frac{(4-2x)^4}{12\cdot(-2)}\bigg|_0^2 = 0 + \frac{4^4}{24} = \frac{32}{3}$$

Combining with (1) we get

$$\iint_{\mathcal{R}} y^2\,dA = 4\cdot\frac{32}{3} \approx 42.67$$

13. Calculate the double integral of $f(x, y) = x + y$ over the domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \le 4,\ y \ge 0\}$.

SOLUTION



The domain $\mathcal{D}$

The semicircle can be described as a vertically simple region, by the following inequalities:

$$-2 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}$$

We evaluate the double integral by the following iterated integral:

$$\iint_{\mathcal{D}} (x+y)\,dA = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \int_{-2}^{2} xy + \frac{1}{2}y^2\Big|_{y=0}^{\sqrt{4-x^2}} dx = \int_{-2}^{2}\left(x\sqrt{4-x^2} + \frac{1}{2}\left(\sqrt{4-x^2}\right)^2\right) dx$$

$$= \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \frac{1}{2}\int_{-2}^{2}(4-x^2)\,dx = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + 2x - \frac{x^3}{6}\Big|_{x=-2}^{2}$$

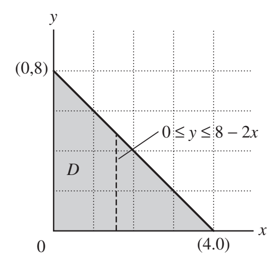
$$= \int_{-2}^{2} x\sqrt{4-x^2}\,dx + 4 - \frac{8}{6} - \left(-4 - \frac{-8}{6}\right) = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \frac{16}{3}$$

The integral of an odd function over an interval that is symmetric with respect to the origin is zero. Hence $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$, so we get

$$\iint_{\mathcal{D}} (x+y)\,dA = 0 + \frac{16}{3} = \frac{16}{3} \approx 5.33$$

14. Integrate $f(x,y) = (x+y+1)^{-2}$ over the triangle with vertices $(0,0)$, $(4,0)$, and $(0,8)$.

SOLUTION



We describe the region $\mathcal{D}$ as a vertically simple region, but first we need to find the equation of the line joining the points $(4,0)$ and $(0,8)$:

$$y - 8 = \frac{8-0}{0-4}(x-0) \quad \Rightarrow \quad y - 8 = -2x$$
$$\Rightarrow \quad y = 8 - 2x$$

We obtain the following inequalities for $\mathcal{D}$:

$$0 \le x \le 4, \quad 0 \le y \le 8 - 2x$$

We now evaluate the double integral of $f(x,y) = (x+y+1)^{-2}$ over the triangle $\mathcal{D}$, by the following iterated integral:

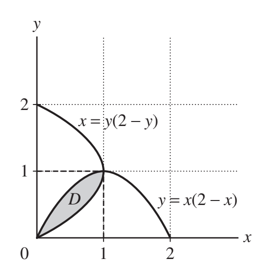
$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_{0}^{4}\int_{0}^{8-2x} (x+y+1)^{-2}\,dy\,dx = \int_{0}^{4} -(x+y+1)^{-1}\Big|_{y=0}^{8-2x} dx$$

$$= \int_{0}^{4}\left(-(x+8-2x+1)^{-1} + (x+0+1)^{-1}\right) dx = \int_{0}^{4}\left(\frac{1}{x+1} - \frac{1}{9-x}\right) dx$$

$$= \ln(x+1) + \ln(9-x)\Big|_{0}^{4} = \ln 5 + \ln 5 - (\ln 1 + \ln 9) = 2\ln 5 - \ln 9 = \ln\frac{25}{9} \approx 1.02$$

15. Calculate the integral of $f(x,y) = x$ over the region $\mathcal{D}$ bounded above by $y = x(2-x)$ and below by $x = y(2-y)$. *Hint:* Apply the quadratic formula to the lower boundary curve to solve for y as a function of x.

SOLUTION The two graphs are symmetric with respect to the line $y = x$, thus their point of intersection is $(1,1)$. The region $\mathcal{D}$ is shown in the following figure:

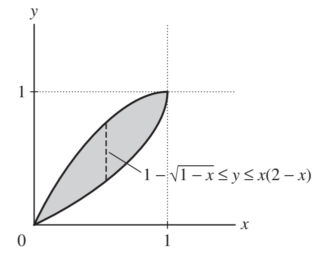


To find the inequalities defining the region $\mathcal{D}$ as a vertically simple region, we first must solve the lower boundary curve for y in terms of x. We get

$$x = y(2-y) = 2y - y^2$$
$$y^2 - 2y + x = 0$$

We solve the quadratic equation in y:

$$y = 1 \pm \sqrt{1-x}$$



The domain $\mathcal{D}$ lies below the line $y = 1$, hence the appropriate solution is $y = 1 - \sqrt{1-x}$. We obtain the following inequalities for $\mathcal{D}$:

$$0 \le x \le 1, \quad 1 - \sqrt{1-x} \le y \le x(2-x)$$

We now evaluate the double integral of $f(x,y) = x$ over $\mathcal{D}$:

$$\iint_{\mathcal{D}} x\,dA = \int_0^1 \int_{1-\sqrt{1-x}}^{x(2-x)} x\,dy\,dx = \int_0^1 xy\Big|_{y=1-\sqrt{1-x}}^{x(2-x)} dx = \int_0^1 \left(x^2(2-x) - \left(x - x\sqrt{1-x}\right)\right) dx$$
$$= \int_0^1 \left(2x^2 - x^3 - x + x\sqrt{1-x}\right) dx = \frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^1 + \int_0^1 x\sqrt{1-x}\,dx$$
$$= -\frac{1}{12} + \int_0^1 x\sqrt{1-x}\,dx$$

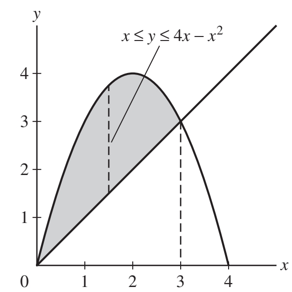
Using the substitution $u = \sqrt{1-x}$ it can be shown that $\int_0^1 x\sqrt{1-x}\,dx = \frac{4}{15}$. Therefore we get

$$\iint_{\mathcal{D}} x\,dA = -\frac{1}{12} + \frac{4}{15} = \frac{11}{60}$$

16. Integrate $f(x,y) = x$ over the region bounded by $y = x$, $y = 4x - x^2$, and $y = 0$ in two ways: as a vertically simple region and as a horizontally simple region.

SOLUTION

(a) The region $\mathcal{D}$ between $y = x$ and $y = 4x - x^2$ is a vertically simple region.



To find the inequalities for this region, we first compute the x-coordinates of the points of intersection of the two curves, by solving the following equation:

$$x = 4x - x^2 \quad \Rightarrow \quad x^2 - 3x = x(x-3) = 0 \quad \Rightarrow \quad x_1 = 0, \quad x_2 = 3$$

The region $\mathcal{D}$ is defined by the following inequalities:

$$0 \le x \le 3, \quad x \le y \le 4x - x^2$$

We now compute the double integral of $f(x, y) = x$ over $\mathcal{D}$ by computing the following iterated integral:

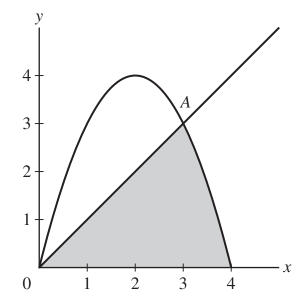
$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_0^3 \int_x^{4x-x^2} (x)\, dy\, dx = \int_0^3 xy\Big|_{y=x}^{4x-x^2} dx$$

$$= \int_0^3 (4x^2 - x^3 - x^2)\, dx$$

$$= \int_0^3 3x^2 - x^3\, dx$$

$$= x^3 - \frac{x^4}{4}\Big|_0^3 = 27 - \frac{81}{4} = \frac{27}{4}$$

(b) The region (shown in the figure) is the union of two horizontally simple regions.



To determine the inequalities defining this region, we first find the y-coordinate of A. In part (a) we found that the x-coordinate of this point is $x = 3$, hence also $y = 3$. We now solve the equation of the right curve $y = 4x - x^2$ for x in terms of y:

$$y = 4x - x^2$$
$$x^2 - 4x + y = 0$$
$$x_{1,2} = 2 \pm \sqrt{4 - y}$$

In this part of the boundary $x \le 3$, hence the negative root must be taken. That is,

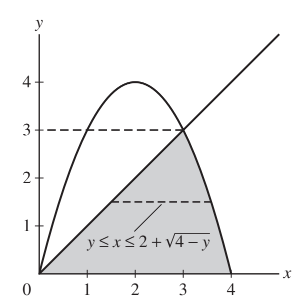
$$x = 2 - \sqrt{4 - y}$$

We obtain the following inequalities for the domain $\mathcal{D}$:

$$0 \le y \le 3, \quad y \le x \le 2 + \sqrt{4 - y}$$

The other part of the domain in question will have to be:

$$3 \le y \le 4, \quad 2 - \sqrt{4 - y} \le x \le 2 + \sqrt{4 - y}$$



The double integral of $f(x, y) = x$ can now be computed by the following iterated integral:

$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_0^3 \int_{2-\sqrt{4-y}}^{y} (x)\, dx\, dy + \int_3^4 \int_{2-\sqrt{4-y}}^{2+\sqrt{4-y}} x\, dx\, dy$$

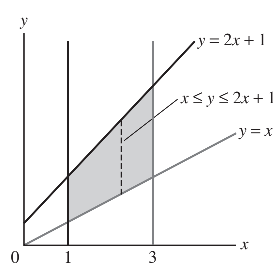
$$= \int_0^3 \frac{1}{2}x^2\Big|_{x=2-\sqrt{4-y}}^{x=y} dy + \int_3^4 \frac{1}{2}x^2\Big|_{x=2-\sqrt{4-y}}^{x=2+\sqrt{4-y}}$$

$$= \frac{1}{2}\int_0^3 \left(y^2 - \left(2-\sqrt{4-y}\right)^2\right) dy + \frac{1}{2}\int_3^4 (2+\sqrt{4-y})^2 - (2-\sqrt{4-y})^2\, dy$$
$$= \frac{1}{2}\int_0^3 \left(y^2 - (4-4\sqrt{4-y}+(4-y))\right) dy + \frac{1}{2}\int_3^4 8\sqrt{4-y}\,dy$$
$$= \frac{1}{2}\int_0^3 \left(y^2 - 8 + 4\sqrt{4-y}+y\right) dy + \int_3^4 4\sqrt{4-y}\,dy$$
$$= \frac{1}{2}\left(\frac{y^3}{3} - 8y - \frac{8}{3}(4-y)^{3/2} + \frac{1}{2}y^2\right)\Bigg|_0^3 - \frac{8}{3}(4-y)^{3/2}\Bigg|_3^4$$
$$= \frac{1}{2}\left(9 - 24 - \frac{8}{3} + \frac{9}{2} + \frac{64}{3}\right) - \left(\frac{8}{3}(0-1)\right)$$
$$= \frac{49}{12} + \frac{8}{3} = \frac{27}{4}$$

In Exercises 17–24, compute the double integral of $f(x, y)$ over the domain $\mathcal{D}$ indicated.

17. $f(x,y) = x^2y$; $1 \le x \le 3$, $x \le y \le 2x+1$

SOLUTION These inequalities describe $\mathcal{D}$ as a vertically simple region.

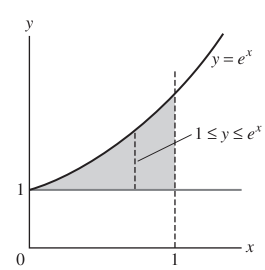


We compute the double integral of $f(x,y) = x^2y$ on $\mathcal{D}$ by the following iterated integral:

$$\iint_{\mathcal{D}} x^2y\,dA = \int_1^3 \int_x^{2x+1} x^2y\,dy\,dx = \int_1^3 \frac{x^2y^2}{2}\Bigg|_{y=x}^{2x+1} dx = \int_1^3 \frac{x^2}{2}\left((2x+1)^2 - x^2\right) dx$$
$$= \int_1^3 \left(\frac{3}{2}x^4 + 2x^3 + \frac{x^2}{2}\right) dx = \frac{3}{10}x^5 + \frac{x^4}{2} + \frac{x^3}{6}\Bigg|_1^3$$
$$= \frac{3\cdot 3^5}{10} + \frac{3^4}{2} + \frac{3^3}{6} - \left(\frac{3}{10} + \frac{1}{2} + \frac{1}{6}\right) = \frac{1754}{15} \approx 116.93$$

18. $f(x,y) = 1$; $0 \le x \le 1$, $1 \le y \le e^x$

SOLUTION The domain $\mathcal{D}$ is a vertically simple region.



We compute the double integral of $f(x,y) = 1$ over $\mathcal{D}$ as the following iterated integral:

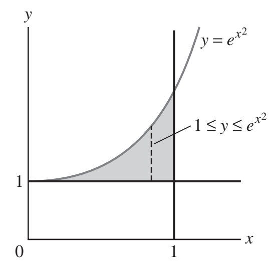
$$\iint_{\mathcal{D}} 1\,dA = \int_0^1 \int_1^{e^x} 1\,dy\,dx = \int_0^1 y\Bigg|_{y=1}^{e^x} dx = \int_0^1 \left(e^x - 1\right) dx = e^x - x\Bigg|_0^1$$
$$= (e^1 - 1) - (e^0 - 0) = e^1 - 1 - 1 = e - 2 \approx 0.718$$

19. $f(x, y) = x; \quad 0 \le x \le 1, \quad 1 \le y \le e^{x^2}$

SOLUTION We compute the double integral of $f(x, y) = x$ over the vertically simple region $\mathcal{D}$, as the following iterated integral:

$$\iint_{\mathcal{D}} x\, dA = \int_0^1 \int_1^{e^{x^2}} x\, dy\, dx = \int_0^1 xy\Big|_{y=1}^{e^{x^2}} dx = \int_0^1 \left(xe^{x^2} - x \cdot 1\right) dx$$

$$= \int_0^1 xe^{x^2}\, dx - \int_0^1 x\, dx = \int_0^1 xe^{x^2} dx - \frac{x^2}{2}\Big|_0^1 = \int_0^1 xe^{x^2} dx - \frac{1}{2} \tag{1}$$



The resulting integral can be computed using the substitution $u = x^2$. The value of this integral is

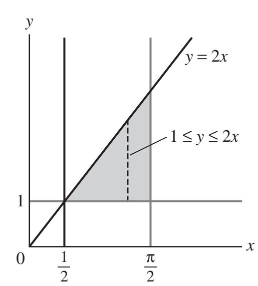
$$\int_0^1 xe^{x^2}\, dx = \frac{e-1}{2}$$

Combining with (1) we get

$$\iint_{\mathcal{D}} x\, dA = \frac{e-1}{2} - \frac{1}{2} = \frac{e-2}{2} \approx 0.359$$

20. $f(x, y) = \cos(2x + y); \quad \frac{1}{2} \le x \le \frac{\pi}{2}, \quad 1 \le y \le 2x$

SOLUTION The vertically simple region $\mathcal{D}$ defined by the given inequalities is shown in the figure:



We compute the double integral of $f(x, y) = \cos(2x + y)$ over $\mathcal{D}$ as an iterated integral, as stated in Theorem 2. This gives

$$\iint_{\mathcal{D}} \cos(2x+y)\, dA = \int_{1/2}^{\pi/2} \int_1^{2x} \cos(2x+y)\, dy\, dx = \int_{1/2}^{\pi/2} \sin(2x+y)\Big|_{y=1}^{2x} dx$$

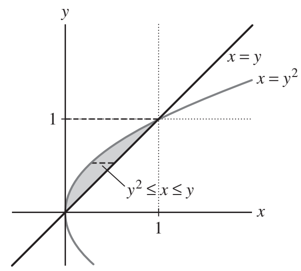
$$= \int_{1/2}^{\pi/2} (\sin(2x+2x) - \sin(2x+1))\, dx = \int_{1/2}^{\pi/2} (\sin(4x) - \sin(2x+1))\, dx$$

$$= -\frac{\cos 4x}{4} + \frac{\cos(2x+1)}{2}\Big|_{x=1/2}^{\pi/2} = -\frac{\cos\frac{4\pi}{2}}{4} + \frac{\cos\left(\frac{2\pi}{2}+1\right)}{2} - \left(-\frac{\cos 2}{4} + \frac{\cos 2}{2}\right)$$

$$= -\frac{1}{4} + \frac{\cos(\pi+1)}{2} - \frac{\cos 2}{4} = -0.416$$

21. $f(x, y) = 2xy$; bounded by $x = y$, $x = y^2$

SOLUTION The intersection points of the graphs $x = y$ and $x = y^2$ are $(0, 0)$ $(1, 1)$. The horizontally simple region $\mathcal{D}$ is shown in the figure:



We compute the double integral of $f(x, y) = 2xy$ over $\mathcal{D}$, using Theorem 2. The limits of integration are determined by the inequalities:

$$0 \le y \le 1, \quad y^2 \le x \le y.$$

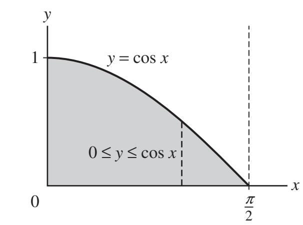
Defining $\mathcal{D}$, we get

$$\iint_{\mathcal{D}} 2xy\, dA = \int_0^1 \int_{y^2}^{y} 2xy\, dx\, dy = \int_0^1 x^2 y \Big|_{x=y^2}^{y} dy = \int_0^1 (y^2 \cdot y - y^4 \cdot y)\, dy$$

$$= \int_0^1 (y^3 - y^5)\, dy = \frac{y^4}{4} - \frac{y^6}{6}\Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

22. $f(x, y) = \sin x$; bounded by $x = 0$, $x = 1$, $y = \cos x$

SOLUTION These curves describe $\mathcal{D}$ as a vertically simple region.



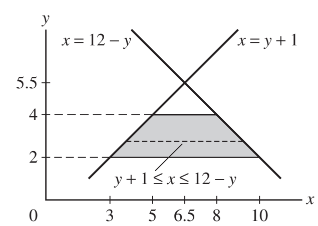
We compute the double integral of $f(x, y) = \sin x$ over $\mathcal{D}$. This gives

$$\iint_{\mathcal{D}} \sin x\, dA = \int_0^1 \int_0^{\cos x} \sin x\, dy\, dx = \int_0^1 y \sin x \Big|_{y=0}^{\cos x} dx = \int_0^1 (\cos x \sin x - 0)\, dx$$

$$= \int_0^1 \frac{\sin 2x}{2}\, dx = -\frac{\cos 2x}{4}\Big|_0^1 = -\left(\frac{\cos(2) - \cos 0}{4}\right) = -\frac{\cos 2}{4} + \frac{1}{4} \approx 0.354$$

23. $f(x, y) = e^{x+y}$; bounded by $y = x - 1$, $y = 12 - x$ for $2 \le y \le 4$

SOLUTION The horizontally simple region $\mathcal{D}$ is shown in the figure:



We compute the double integral of $f(x, y) = e^{x+y}$ over $\mathcal{D}$ by evaluating the following iterated integral:

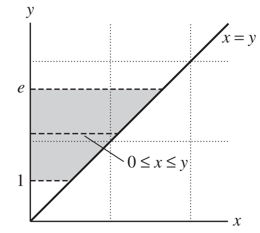
$$\iint_{\mathcal{D}} e^{x+y}\, dA = \int_2^4 \int_{y+1}^{12-y} e^{x+y}\, dx\, dy = \int_2^4 e^{x+y} \Big|_{x=y+1}^{12-y} dy = \int_2^4 \left(e^{12-y+y} - e^{y+1+y}\right) dy$$

$$= \int_2^4 \left(e^{12} - e^{2y+1}\right) dy = e^{12} \cdot y - \frac{1}{2} e^{2y+1}\Big|_2^4 = \left(e^{12} \cdot 4 - \frac{1}{2} e^{2\cdot 4+1}\right) - \left(e^{12} \cdot 2 - \frac{1}{2} e^{2\cdot 2+1}\right)$$

$$= 4e^{12} - \frac{1}{2} e^9 - 2e^{12} + \frac{1}{2} e^5 = 2e^{12} - \frac{1}{2} e^9 + \frac{1}{2} e^5 \approx 321532.2$$

24. $f(x, y) = (x + y)^{-1}$; bounded by $y = x$, $y = 1$, $y = e$, $x = 0$

SOLUTION



The double integral $f(x, y) = (x + y)^{-1}$ over the horizontally simple region $\mathcal{D}$ (shown in the figure) is computed, using Theorem 2, by the following iterated integral:

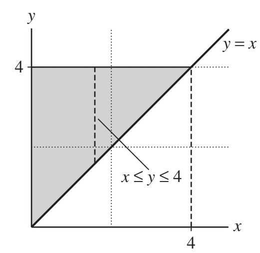
$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_1^e \int_0^y (x + y)^{-1}\, dx\, dy = \int_1^e \ln(x + y)\Big|_{x=0}^{y} dy = \int_1^e (\ln(y + y) - \ln(0 + y))\, dy$$
$$= \int_1^e (\ln(2y) - \ln y)\, dy = \int_1^e \ln \frac{2y}{y}\, dy = \int_1^e \ln 2\, dy = (e - 1) \cdot \ln 2 \approx 1.19$$

In Exercises 25–28, sketch the domain of integration and express as an iterated integral in the opposite order.

25. $\int_0^4 \int_x^4 f(x, y)\, dy\, dx$

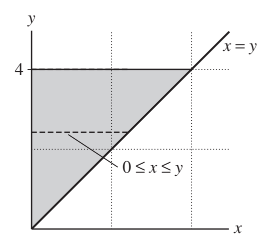
SOLUTION The limits of integration correspond to the inequalities describing the following domain $\mathcal{D}$:

$$0 \le x \le 4, \quad x \le y \le 4$$



From the sketch of $\mathcal{D}$ we see that $\mathcal{D}$ can also be expressed as a horizontally simple region as follows:

$$0 \le y \le 4, \quad 0 \le x \le y$$



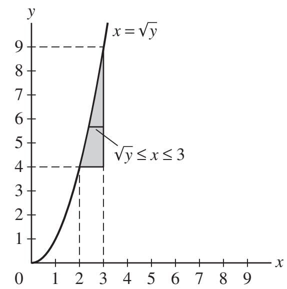
Therefore we can reverse the order of integration as follows:

$$\int_0^4 \int_x^4 f(x, y)\, dy\, dx = \int_0^4 \int_0^y f(x, y)\, dx\, dy.$$

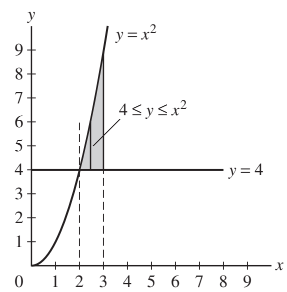
26. $\int_4^9 \int_{\sqrt{y}}^3 f(x, y)\, dx\, dy$

SOLUTION The limits of integration correspond to the inequalities describing the following horizontally simple region $\mathcal{D}$:

$$4 \le y \le 9, \quad \sqrt{y} \le x \le 3$$



The sketch of $\mathcal{D}$ shows that $\mathcal{D}$ can also be expressed as a vertically simple region. We first express the curve $x = \sqrt{y}$ in the form $y = x^2$.



Now we get

$$2 \le x \le 3, \quad 4 \le y \le x^2$$

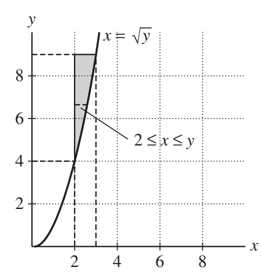
We obtain the following equality:

$$\int_4^9 \int_{\sqrt{y}}^3 f(x, y)\,dx\,dy = \int_2^3 \int_4^{x^2} f(x, y)\,dy\,dx.$$

27. $\displaystyle\int_4^9 \int_2^{\sqrt{y}} f(x, y)\,dx\,dy$

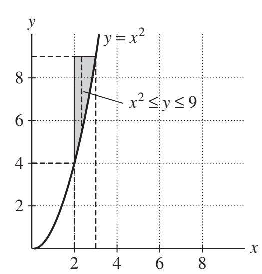
SOLUTION The limits of integration correspond to the following inequalities defining the horizontally simple region $\mathcal{D}$:

$$4 \le y \le 9, \quad 2 \le x \le \sqrt{y}$$



The region $\mathcal{D}$ can also be expressed as a vertically simple region. We first need to write the equation of the curve $x = \sqrt{y}$ in the form $y = x^2$. The corresponding inequalities are

$$2 \le x \le 3, \quad x^2 \le y \le 9$$



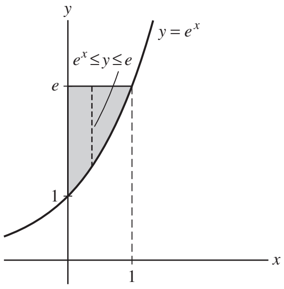
We now can write the iterated integral with reversed order of integration:

$$\int_4^9 \int_2^{\sqrt{y}} f(x, y)\, dx\, dy = \int_2^3 \int_{x^2}^9 f(x, y)\, dy\, dx.$$

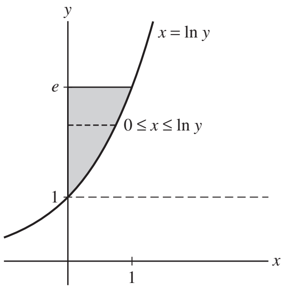
28. $\int_0^1 \int_{e^x}^e f(x, y)\, dy\, dx$

SOLUTION The limits of integration define a vertically simple region $\mathcal{D}$ by the following inequalities:

$$0 \le x \le 1, \quad e^x \le y \le e$$



This region can also be expressed as a horizontally simple region.



The curve $y = e^x$ can be rewritten as $x = \ln y$, and we obtain the following inequalities for $\mathcal{D}$ (see figure):

$$1 \le y \le e, \quad 0 \le x \le \ln y$$

Using this description we obtain the integral with reversed order of integration:

$$\int_0^1 \int_{e^x}^e f(x, y)\, dy\, dx = \int_1^e \int_0^{\ln y} f(x, y)\, dx\, dy.$$

29. Sketch the domain $\mathcal{D}$ corresponding to

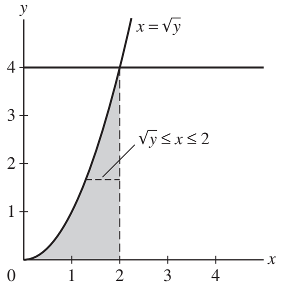
$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y}\, dx\, dy$$

Then change the order of integration and evaluate.

SOLUTION The limits of integration correspond to the following inequalities describing the domain $\mathcal{D}$:

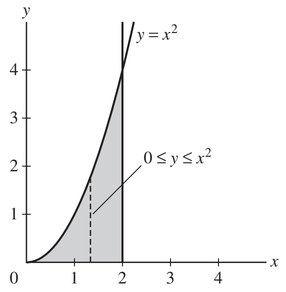
$$0 \le y \le 4, \quad \sqrt{y} \le x \le 2$$

The horizontally simple region $\mathcal{D}$ is shown in the figure:



The domain $\mathcal{D}$ can also be described as a vertically simple region. Rewriting the equation $x = \sqrt{y}$ in the form $y = x^2$, we define $\mathcal{D}$ by the following inequalities (see figure):

$$0 \le x \le 2, \quad 0 \le y \le x^2$$



The corresponding iterated integral is

$$\int_0^2 \int_0^{x^2} \sqrt{4x^2 + 5y}\, dy\, dx$$

We evaluate this integral:

$$\begin{aligned}
\int_0^2 \int_0^{x^2} \sqrt{4x^2 + 5y}\, dy\, dx &= \int_0^2 \left(\int_0^{x^2} \left(4x^2 + 5y\right)^{1/2} dy \right) dx \\
&= \int_0^2 \frac{2}{15}\left(4x^2 + 5y\right)^{3/2} \Big|_{y=0}^{x^2} dx \\
&= \int_0^2 \left(\frac{2}{15}\left(4x^2 + 5x^2\right)^{3/2} - \frac{2}{15}\left(4x^2 + 0\right)^{3/2} \right) dx \\
&= \int_0^2 \left(\frac{2}{15}\left(9x^2\right)^{3/2} - \frac{2}{15}\left(4x^2\right)^{3/2} \right) dx \\
&= \int_0^2 \frac{18}{5}x^3 - \frac{16}{15}x^3\, dx \\
&= \int_0^2 \frac{38}{15}x^3\, dx \\
&= \frac{38}{15} \cdot \frac{x^4}{4}\Big|_0^2 = \frac{38}{15} \cdot \frac{2^4}{4} = \frac{152}{15} \approx 10.133
\end{aligned}$$

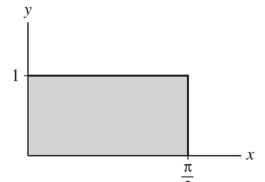
30. Change the order of integration and evaluate

$$\int_0^1 \int_0^{\pi/2} x\cos(xy)\, dx\, dy$$

Explain the simplification achieved by changing the order.

SOLUTION The domain of integration is the rectangle defined by the following inequalities:

$$0 \le y \le 1, \quad 0 \le x \le \frac{\pi}{2}$$



By Fubini's Theorem, the double integral of $f(x, y) = x\cos(xy)$ over the rectangle is equal to the iterated integral in either order. Hence,

$$\int_0^1 \int_0^{\pi/2} x\cos(xy)\, dx\, dy = \int_0^{\pi/2} \int_0^1 x\cos(xy)\, dy\, dx = \int_0^{\pi/2} x \cdot \frac{1}{x}(\sin xy)\Big|_{y=0}^1 dx = \int_0^{\pi/2} \sin(xy)\Big|_{y=0}^1 dx$$

$$= \int_0^{\pi/2} (\sin x - \sin 0)\,dx = \int_0^{\pi/2} \sin x\,dx = -\cos x \Big|_0^{\pi/2}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1) = 1$$

Trying to integrate in the original order of integration, we obtain

$$\int_0^1 \int_0^{\pi/2} x\cos(xy)\,dx\,dy = \int_0^1 \left(\int_0^{\pi/2} x\cos(xy)\,dx\right) dy \tag{1}$$

To compute the inner integral we would have to use integration by parts, whereas the integrals involved in computing the integral first with respect to x and then with respect to y were quite easy to compute.

31. Compute the integral of $f(x, y) = (\ln y)^{-1}$ over the domain $\mathcal{D}$ bounded by $y = e^x$ and $y = e^{\sqrt{x}}$. *Hint:* Choose the order of integration that enables you to evaluate the integral.

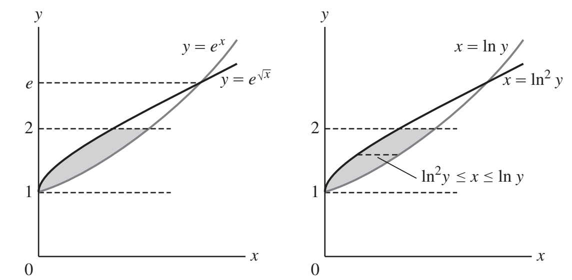
SOLUTION To express $\mathcal{D}$ as a horizontally simple region, we first must rewrite the equations of the curves $y = e^x$ and $y = e^{\sqrt{x}}$ with x as a function of y. That is,

$$y = e^x \quad \Rightarrow \quad x = \ln y$$

$$y = e^{\sqrt{x}} \quad \Rightarrow \quad \sqrt{x} = \ln y \quad \Rightarrow \quad x = \ln^2 y$$

We obtain the following inequalities:

$$1 \le y \le e, \quad \ln^2 y \le x \le \ln y$$



Using Theorem 2, we compute the double integral of $f(x, y) = (\ln y)^{-1}$ over $\mathcal{D}$ as the following iterated integral:

$$\iint_{\mathcal{D}} (\ln y)^{-1}\,dA = \int_1^e \int_{\ln^2 y}^{\ln y} (\ln y)^{-1}\,dx\,dy = \int_1^e (\ln y)^{-1} x \Big|_{x=\ln^2 y}^{\ln y} dy = \int_1^e (\ln y)^{-1}\left(\ln y - \ln^2 y\right) dy$$

$$= \int_1^e (1 - \ln y)\,dy = \int_1^e 1\,dy - \int_1^e \ln y\,dy = y\Big|_1^e - y(\ln y - 1)\Big|_1^e$$

$$= (e-1) - [e(0) - 1(-1)] = e - 2$$

32. Evaluate by changing the order of integration: $\displaystyle\int_0^9 \int_0^{\sqrt{y}} \frac{x\,dx\,dy}{(3x^2 + y)^{1/2}}$

SOLUTION The region of integration is bounded by:

$$0 \le y \le 9, \quad 0 \le x \le \sqrt{y}$$

which also gives the region

$$0 \le x \le 3, \quad x^2 \le y \le 9$$

So changing the order of integration gives the integral:

$$\int_0^3 \int_{x^2}^9 \frac{x\,dy\,dx}{(3x^2 + y)^{1/2}}$$

Now evaluating we get

$$\int_0^3 \int_{x^2}^9 \frac{x\,dy\,dx}{(3x^2 + y)^{1/2}} = \int_0^3 x\left(\int_{x^2}^9 \frac{1}{(3x^2 + y)^{1/2}}\,dy\right) dx$$

$$= \int_0^3 x\left(2(3x^2 + y)^{1/2}\Big|_{x^2}^9\right) dx$$

$$= \int_0^3 2x(3x^2+9)^{1/2} - 2x(3x^2+x^2)^{1/2}\,dx$$
$$= \int_0^3 2x(3x^2+9)^{1/2} - 4x^2\,dx$$
$$= \frac{2}{9}(3x^2+9)^{3/2} - \frac{4}{3}x^3\Big|_0^3$$
$$= \frac{2}{9}(36)^{3/2} - \frac{4}{3}(27) - \frac{2}{9}(9)^{3/2} = 6$$

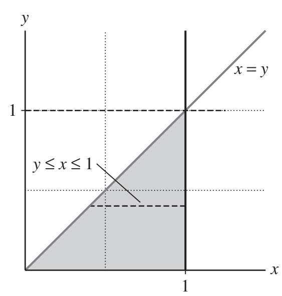
In Exercises 33–36, sketch the domain of integration. Then change the order of integration and evaluate. Explain the simplification achieved by changing the order.

33. $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$

SOLUTION The limits of integration correspond to the following inequalities:

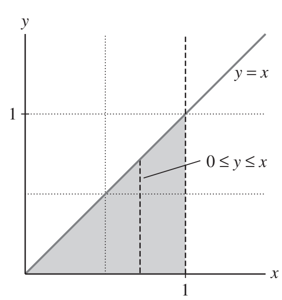
$$0 \le y \le 1, \quad y \le x \le 1$$

The horizontally simple region $\mathcal{D}$ is shown in the figure.



We see that $\mathcal{D}$ can also be described as a vertically simple region, by the following inequalities:

$$0 \le x \le 1, \quad 0 \le y \le x$$



We evaluate the corresponding iterated integral:

$$\int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x} y\Big|_{y=0}^{x} dx = \int_0^1 \frac{\sin x}{x}(x-0)\,dx = \int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = 1 - \cos 1 \approx 0.46$$

Trying to integrate in reversed order we obtain a complicated integral in the inner integral. That is,

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\,dx \right) dy$$

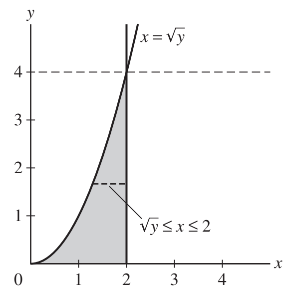
Remark: $f(x, y) = \frac{\sin x}{x}$ is not continuous at the point $(0, 0)$ in $\mathcal{D}$. To make it continuous we need to define $f(0, 0) = 1$.

34. $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$

SOLUTION The limits of integration correspond to the following inequalities describing the domain $\mathcal{D}$:

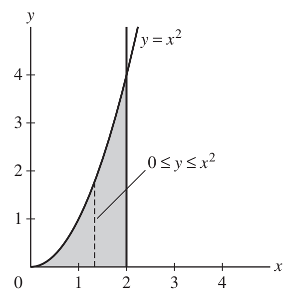
$$0 \le y \le 4, \quad \sqrt{y} \le x \le 2$$

The domain $\mathcal{D}$ is a horizontally simple region, as shown in the figure.



From the sketch of $\mathcal{D}$, we see that $\mathcal{D}$ can be expressed as a vertically simple region. Rewriting the equation of the curve $x = \sqrt{y}$ as $y = x^2$, we obtain the following inequalities for $\mathcal{D}$:

$$0 \le x \le 2, \quad 0 \le y \le x^2$$



The integral in reversed order of integration is thus

$$\int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^2 \sqrt{x^3+1}\,y\Big|_{y=0}^{x^2} dx = \int_0^2 \sqrt{x^3+1}\left(x^2-0\right) dx = \int_0^2 \sqrt{x^3+1}\cdot x^2\,dx$$

This integral is easy to compute using the substitution $u = x^3+1$, $du = 3x^2\,dx$. This gives

$$\int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^2 \sqrt{x^3+1}\cdot x^2\,dx = \int_1^9 \sqrt{u}\cdot\frac{du}{3} = \frac{2}{9}u^{3/2}\Big|_1^9 = \frac{2}{9}\left(9^{3/2}-1\right) = \frac{52}{9} \approx 5.78$$

Trying to compute the double integral in the original order we find that the inner integral is impossible to compute:

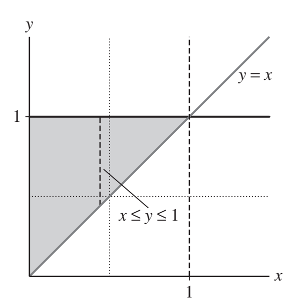
$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy = \int_0^4 \left(\int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\right) dy$$

35. $\displaystyle\int_0^1 \int_{y=x}^1 xe^{y^3}\,dy\,dx$

SOLUTION The limits of integration define a vertically simple region $\mathcal{D}$ by the following inequalities:

$$0 \le x \le 1, \quad x \le y \le 1$$

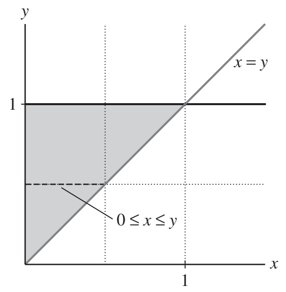
This region can also be described as a horizontally simple region by the following inequalities (see figure):



$$0 \le y \le 1, \quad 0 \le x \le y$$

We thus can rewrite the given integral in reversed order of integration as follows:

$$\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \frac{x^2}{2}e^{y^3}\Big|_{x=0}^{y}\,dy = \int_0^1 e^{y^3}\left(\frac{y^2}{2} - 0\right)dy = \int_0^1 \frac{1}{2}e^{y^3}y^2\,dy$$



We compute this integral using the substitution $u = y^3,\ du = 3y^2\,dy$. This gives

$$\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \frac{1}{2}e^{y^3}y^2\,dy = \int_0^1 e^u \cdot \frac{1}{6}\,du = \frac{e^u}{6}\Big|_0^1 = \frac{e-1}{6} \approx 0.286$$

Trying to evaluate the double integral in the original order of integration, we find that the inner integral is impossible to compute:

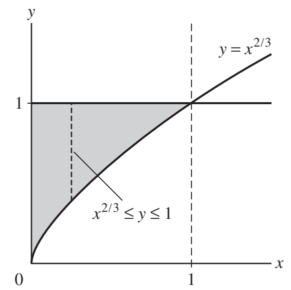
$$\int_0^1 \int_x^1 xe^{y^3}\,dy\,dx = \int_0^1 \left(\int_x^1 xe^{y^3}\,dy\right)dx$$

36. $\displaystyle\int_0^1 \int_{y=x^{2/3}}^1 xe^{y^4}\,dy\,dx$

SOLUTION The limits of integration define a vertically simple region $\mathcal{D}$ by the following inequalities:

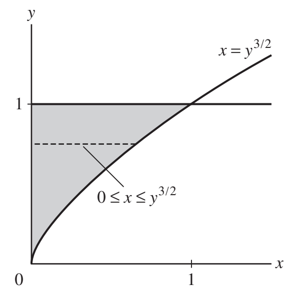
$$0 \le x \le 1, \quad x^{2/3} \le y \le 1$$

The region $\mathcal{D}$ shown in the figure can also be described as a horizontally simple region.



We rewrite the equation of the curve $y = x^{2/3}$ as $x = y^{3/2}$ and express the domain $\mathcal{D}$ as a horizontally simple region by the following inequalities:

$$0 \le y \le 1, \quad 0 \le x \le y^{3/2}$$



The corresponding iterated integral is:

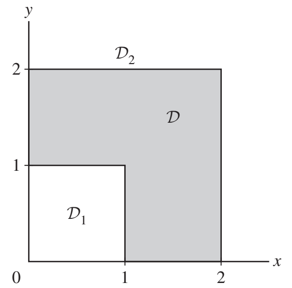
$$\int_0^1 \int_0^{y^{3/2}} xe^{y^4}\,dx\,dy = \int_0^1 \frac{x^2}{2}e^{y^4}\bigg|_{x=0}^{y^{3/2}} dy = \int_0^1 \left(\frac{(y^{3/2})^2}{2}e^{y^4} - 0\right) dy = \int_0^1 \frac{1}{2}y^3 e^{y^4}\,dy$$
$$= \int_0^1 \frac{1}{8}\left(\frac{d}{dy}e^{y^4}\right) dy = \frac{1}{8}e^{y^4}\bigg|_0^1 = \frac{1}{8}\left(e^1 - e^0\right) = \frac{1}{8}(e-1) \approx 0.215$$

Trying to evaluate the original integral we find that the inner integral is impossible to evaluate:

$$\int_0^1 \int_{x^{2/3}}^1 xe^{y^4}\,dy\,dx = \int_0^1 \left(\int_{x^{2/3}}^1 xe^{y^4}\,dy\right) dx$$

37. Sketch the domain $\mathcal{D}$ where $0 \le x \le 2$, $0 \le y \le 2$, and x *or* y is greater than 1. Then compute $\iint_{\mathcal{D}} e^{x+y}\,dA$.

SOLUTION The domain $\mathcal{D}$ within the square $0 \le x,\ y \le 2$ is shown in the figure.



We denote the unit square $0 \le x,\ y \le 1$ and the square $0 \le x,\ y \le 2$ by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Then $\mathcal{D}_2$ is the union of $\mathcal{D}_1$ and $\mathcal{D}$, and these two domains do not overlap except on the boundary of $\mathcal{D}_1$. Therefore, by properties of the double integral, we have

$$\iint_{\mathcal{D}_2} e^{x+y}dA = \iint_{\mathcal{D}_1} e^{x+y}dA + \iint_{\mathcal{D}} e^{x+y}dA$$

Hence,

$$\iint_{\mathcal{D}} e^{x+y}dA = \iint_{\mathcal{D}_2} e^{x+y}\,dA - \iint_{\mathcal{D}_1} e^{x+y}\,dA = \int_0^2 \int_0^2 e^{x+y}\,dx\,dy - \int_0^1 \int_0^1 e^{x+y}\,dx\,dy$$
$$= \int_0^2 e^{x+y}\bigg|_{x=0}^2 dy - \int_0^1 e^{x+y}\bigg|_{x=0}^1 dy = \int_0^2 (e^{2+y} - e^y)\,dy - \int_0^1 (e^{1+y} - e^y)\,dy$$
$$= e^{2+y} - e^y\bigg|_{y=0}^2 - (e^{1+y} - e^y)\bigg|_{y=0}^1 = e^4 - e^2 - (e^2 - e^0) - \left(e^2 - e - (e - e^0)\right)$$
$$= e^4 - 3e^2 + 2e \approx 37.87$$

38. Calculate $\iint_{\mathcal{D}} e^x\,dA$, where $\mathcal{D}$ is bounded by the lines $y = x+1$, $y = x$, $x = 0$, and $x = 1$.

SOLUTION This region is a vertically simple region

$$\mathcal{D} : 0 \le x \le 1, \quad x \le y \le x+1$$

So that

$$\int_{\mathcal{D}} e^x\,dA = \int_0^1 \int_x^{x+1} e^x\,dy\,dx = \int_0^1 ye^x\bigg|_x^{x+1} dx$$
$$= \int_0^1 e^x(x+1-x)\,dx = \int_0^1 e^x\,dx = e^x\bigg|_0^1 = e - 1$$

In Exercises 39–42, calculate the double integral of $f(x, y)$ over the triangle indicated in Figure 25.

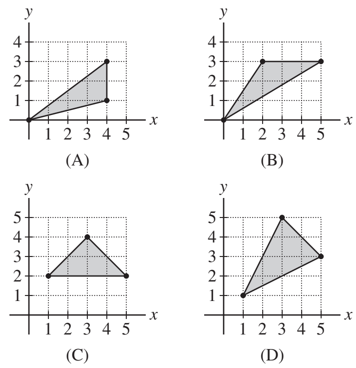
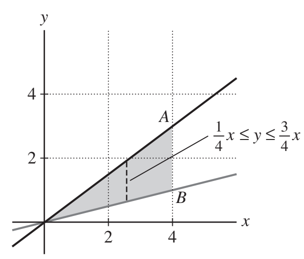


FIGURE 25

39. $f(x, y) = e^{x^2}$, (A)

SOLUTION The equations of the lines OA and OB are $y = \frac{3}{4}x$ and $y = \frac{1}{4}x$, respectively. Therefore, the triangle may be expressed as a vertically simple region by the following inequalities:

$$0 \le x \le 4, \quad \frac{x}{4} \le y \le \frac{3x}{4}$$



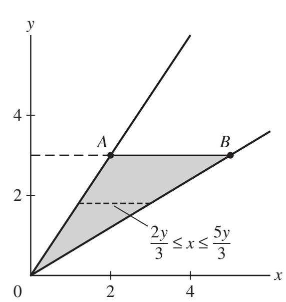
The double integral of $f(x, y) = e^{x^2}$ over the triangle is the following iterated integral:

$$\begin{aligned}
\int_0^4 \int_{x/4}^{3x/4} e^{x^2}\, dy\, dx &= \int_0^4 y e^{x^2} \Big|_{y=x/4}^{3x/4} dx \\
&= \int_0^4 e^{x^2} \left(\frac{3x}{4} - \frac{x}{4} \right) dx \\
&= \frac{1}{2} \int_0^4 x e^{x^2}\, dx \\
&= \frac{1}{4} e^{x^2} \Big|_0^4 = \frac{1}{4}(e^{16} - 1)
\end{aligned}$$

40. $f(x, y) = 1 - 2x$, (B)

SOLUTION The equations of the lines OA and OB are $y = \frac{3x}{2}$ and $y = \frac{3x}{5}$, respectively. We describe the triangle as a horizontally simple region. The equations $y = \frac{3x}{2}$ and $y = \frac{3x}{5}$ can be written as $x = \frac{2y}{3}$ and $x = \frac{5y}{3}$, therefore the inequalities defining the triangle are

$$0 \le y \le 3, \quad \frac{2}{3}y \le x \le \frac{5}{3}y$$



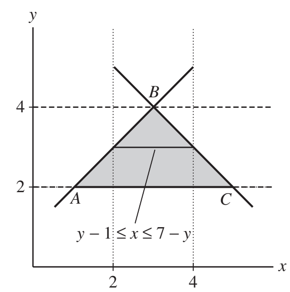
We now compute the double integral of $f(x, y) = 1 - 2x$ over the triangle by evaluating the following iterated integral:

$$\int_0^3 \int_{2y/3}^{5y/3} (1-2x)\,dx\,dy = \int_0^3 x - x^2\Big|_{y=2y/3}^{5y/3} dy = \int_0^3 \left(\left(\frac{5y}{3} - \frac{25y^2}{9}\right) - \left(\frac{2y}{3} - \frac{4y^2}{9}\right)\right) dy$$

$$= \int_0^3 \left(y - \frac{7}{3}y^2\right) dy = \frac{y^2}{2} - \frac{7y^3}{9}\Big|_0^3 = \frac{9}{2} - \frac{7 \cdot 3^3}{9} = -16.5$$

41. $f(x, y) = \dfrac{x}{y^2}$, (C)

SOLUTION To find the inequalities defining the triangle as a horizontally simple region, we first find the inequalities of the lines AB and BC:



$$AB:\quad y - 2 = \frac{4-2}{3-1}(x-1) \quad \Rightarrow \quad y - 2 = x - 1 \quad \Rightarrow \quad x = y - 1$$

$$BC:\quad y - 2 = \frac{4-2}{3-5}(x-5) \quad \Rightarrow \quad y - 2 = 5 - x \quad \Rightarrow \quad x = 7 - y$$

We obtain the following inequalities for the triangle:

$$2 \le y \le 4, \quad y - 1 \le x \le 7 - y$$

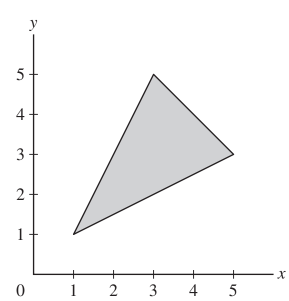
The double integral of $f(x, y) = \frac{x}{y^2}$ over the triangle is the following iterated integral:

$$\int_2^4 \int_{y-1}^{7-y} \frac{x}{y^2}\,dx\,dy = \int_2^4 \frac{x^2}{2y^2}\Big|_{x=y-1}^{7-y} dy = \int_2^4 \frac{(7-y)^2 - (y-1)^2}{2y^2}\,dy = \int_2^4 \left(\frac{24}{y^2} - \frac{6}{y}\right) dy$$

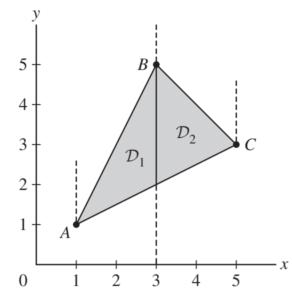
$$= -\frac{24}{y} - 6\ln y\Big|_2^4 = -\frac{24}{4} - 6\ln 4 - \left(-\frac{24}{2} - 6\ln 2\right) = 6 - 6\ln 2 = 1.84$$

42. $f(x, y) = x + 1$, (D)

SOLUTION



This triangle is not a simple region, therefore we decompose it into two vertically simple regions $\mathcal{D}_1$ and $\mathcal{D}_2$, shown in the figure.



By properties of the double integral, the double integral of $f(x, y) = x + 1$ over the given triangle $\mathcal{D}$ is the following sum:

$$\iint_{\mathcal{D}} x + 1\, dA = \iint_{\mathcal{D}_1} x + 1\, dA + \iint_{\mathcal{D}_2} x + 1\, dA \tag{1}$$

To describe $\mathcal{D}_1$ and $\mathcal{D}_2$ as vertically simple regions we first must find the equations of the lines AB, AC, BC:

$$AB:\quad y - 1 = \frac{5-1}{3-1}(x-1) \quad\Rightarrow\quad y - 1 = 2(x-1) \quad\Rightarrow\quad y = 2x - 1$$

$$AC:\quad y - 1 = \frac{3-1}{5-1}(x-1) \quad\Rightarrow\quad y - 1 = \frac{1}{2}(x-1) \quad\Rightarrow\quad y = \frac{1}{2}x + \frac{1}{2}$$

$$BC:\quad y - 3 = \frac{5-3}{3-5}(x-5) \quad\Rightarrow\quad y - 3 = -(x-5) \quad\Rightarrow\quad y = -x + 8$$

We obtain the following inequalities for $\mathcal{D}_1$ and $\mathcal{D}_2$:

$\mathcal{D}_1$	$\mathcal{D}_2$
$1 \le x \le 3$	$3 \le x \le 5$
$\frac{1}{2}x + \frac{1}{2} \le y \le 2x - 1$	$\frac{1}{2}x + \frac{1}{2} \le y \le -x + 8$

We compute the double integral of f over $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\begin{aligned}
\iint_{\mathcal{D}_1} (x+1)\, dA &= \int_1^3 \int_{\frac{x+1}{2}}^{2x-1} (x+1)\, dy\, dx \\
&= \int_1^3 (x+1) \left(y \Big|_{\frac{x+1}{2}}^{2x-1} \right) dx \\
&= \int_1^3 (x+1) \left(2x - 1 - \frac{x+1}{2} \right) dx \\
&= \int_1^3 \left(\frac{3}{2}x^2 - \frac{3}{2} \right) dx \\
&= \frac{1}{2}x^3 - \frac{3}{2}x \Big|_1^3 = 10
\end{aligned}$$

The double integral of f over $\mathcal{D}_2$ is

$$\begin{aligned}
\iint_{\mathcal{D}_2} (x+1)\, dA &= \int_3^5 \int_{\frac{x+1}{2}}^{-x+8} (x+1)\, dy\, dx \\
&= \int_3^5 (x+1) \left(y \Big|_{\frac{x+1}{2}}^{-x+8} \right) dx \\
&= \int_3^5 (x+1) \left(-x + 8 - \frac{x+1}{2} \right) dx \\
&= \int_3^5 -\frac{3}{2}x^2 + 6x + \frac{15}{2}x\, dx \\
&= -\frac{1}{2}x^3 + 3x^2 + \frac{15}{2}x \Big|_3^5 \\
&= \left(-\frac{125}{2} + 75 + \frac{75}{2} \right) - \left(-\frac{27}{2} + 27 + \frac{45}{2} \right) = 14
\end{aligned}$$

Finally we substitute the results above to obtain the following solution:

$$\iint_{\mathcal{D}} (x+1)\,dA = \iint_{\mathcal{D}_1} (x+1)\,dA + \iint_{\mathcal{D}_2} (x+1)\,dA = 10 + 14 = 24$$

43. Calculate the double integral of $f(x, y) = \dfrac{\sin y}{y}$ over the region $\mathcal{D}$ in Figure 26.

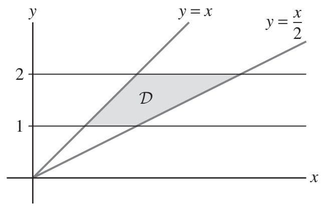
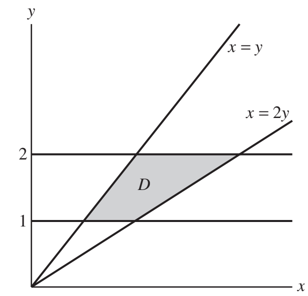


FIGURE 26

SOLUTION To describe $\mathcal{D}$ as a horizontally simple region, we rewrite the equations of the lines with x as a function of y, that is, $x = y$ and $x = 2y$. The inequalities for $\mathcal{D}$ are

$$1 \le y \le 2, \quad y \le x \le 2y$$



We now compute the double integral of $f(x, y) = \frac{\sin y}{y}$ over $\mathcal{D}$ by the following iterated integral:

$$\begin{aligned}\iint_{\mathcal{D}} \frac{\sin y}{y}\,dA &= \int_1^2 \int_y^{2y} \frac{\sin y}{y}\,dx\,dy = \int_1^2 \frac{\sin y}{y} x\Big|_{x=y}^{2y}\,dy = \int_1^2 \frac{\sin y}{y}(2y - y)\,dy \\ &= \int_1^2 \frac{\sin y}{y} \cdot y\,dy = \int_1^2 \sin y\,dy = -\cos y\Big|_1^2 = \cos 1 - \cos 2 \approx 0.956\end{aligned}$$

44. Evaluate $\iint_{\mathcal{D}} x\,dA$ for $\mathcal{D}$ in Figure 27.

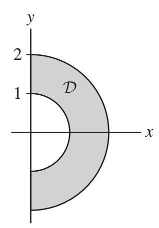
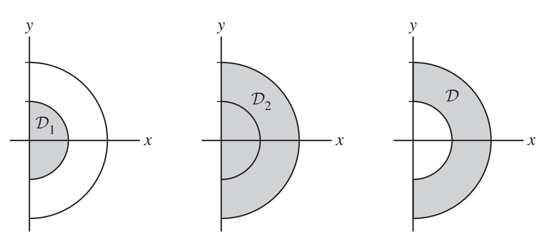


FIGURE 27

SOLUTION We compute the integral using decomposition of a domain into smaller domains. We denote by $\mathcal{D}_2$ and $\mathcal{D}_1$ the right semicircles of radius 2 and 1, respectively.



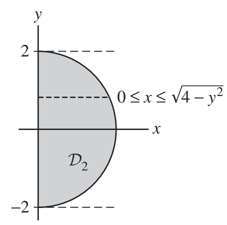
Then $\mathcal{D}_2$ is the union of $\mathcal{D}_1$ and $\mathcal{D}$, which do not overlap except on a boundary curve. Therefore,

$$\iint_{\mathcal{D}_2} x\,dA = \iint_{\mathcal{D}_1} x\,dA + \iint_{\mathcal{D}} x\,dA$$

or

$$\iint_{\mathcal{D}} x\,dA = \iint_{\mathcal{D}_2} x\,dA - \iint_{\mathcal{D}_1} x\,dA \tag{1}$$

To express $\mathcal{D}_1$ and $\mathcal{D}_2$ as horizontally simple regions, we write the equations of the circles in the form $x = \sqrt{1-y^2}$ and $x = \sqrt{4-y^2}$, respectively. The equations describing the regions $\mathcal{D}_1$ and $\mathcal{D}_2$ are

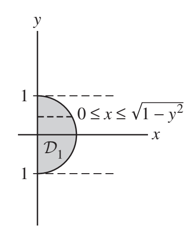


$$\begin{array}{cc} -2 \le y \le 2, & -1 \le y \le 1, \\ 0 \le x \le \sqrt{4-y^2} & 0 \le x \le \sqrt{1-y^2} \\ \mathcal{D}_2 & \mathcal{D}_1 \end{array}$$

We compute the double integrals over $\mathcal{D}_2$:

$$\iint_{\mathcal{D}_2} x\,dA = \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} x\,dx\,dy = \int_{-2}^{2} \frac{x^2}{2}\bigg|_{x=0}^{\sqrt{4-y^2}} dy = \int_{-2}^{2} \frac{\left(\sqrt{4-y^2}\right)^2 - 0^2}{2}\,dy = \int_{-2}^{2}\left(2 - \frac{y^2}{2}\right) dy$$

$$= \int_{0}^{2} (4-y^2)\,dy = 4y - \frac{y^3}{3}\bigg|_{0}^{2} = 8 - \frac{8}{3} = \frac{16}{3} \tag{2}$$



The integral over $\mathcal{D}_1$ is

$$\iint_{\mathcal{D}_1} x\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} x\,dx\,dy = \int_{-1}^{1} \frac{x^2}{2}\bigg|_{x=0}^{\sqrt{1-y^2}} dy = \int_{-1}^{1} \frac{\left(\sqrt{1-y^2}\right)^2 - 0^2}{2}\,dy = \int_{-1}^{1}\left(\frac{1}{2} - \frac{y^2}{2}\right) dy$$

$$= \int_{0}^{1} (1-y^2)\,dy = y - \frac{y^3}{3}\bigg|_{0}^{1} = 1 - \frac{1}{3} = \frac{2}{3} \tag{3}$$

We now combine (1), (2), and (3) to obtain the following solution:

$$\iint_{\mathcal{D}} x\,dA = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \approx 4.67$$

45. Find the volume of the region bounded by $z = 40 - 10y$, $z = 0$, $y = 0$, $y = 4 - x^2$.

SOLUTION The volume of the region is the double integral of $f(x, y) = 40 - 10y$ over the domain $\mathcal{D}$ in the xy-plane between the curves $y = 0$ and $y = 4 - x^2$. This is a vertically simple region described by the inequalities:

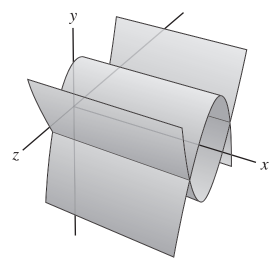
$$-2 \le x \le 2, \quad 0 \le y \le 4 - x^2$$

We compute the double integral as the following iterated integral:

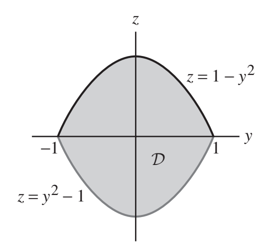
$$\begin{aligned}
V &= \iint_{\mathcal{D}} 40 - 10y\, dA = \int_{-2}^{2} \int_{0}^{4-x^2} (40 - 10y)\, dy\, dx \\
&= \int_{-2}^{2} \left(40y - 5y^2 \Big|_{0}^{4-x^2} \right) dx \\
&= \int_{-2}^{2} 40(4 - x^2) - 5(4 - x^2)^2\, dx = \int_{-2}^{2} 160 - 40x^2 - 5(16 - 8x^2 + x^4)\, dx \\
&= \int_{-2}^{2} 80 - 5x^4\, dx = 80x - x^5 \Big|_{-2}^{2} \\
&= (160 - 32) - (-160 + 32) = 256
\end{aligned}$$

46. Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \le x \le 2$.

SOLUTION



The volume of the region is the double integral of $f(y, z) = 2$ over the domain $\mathcal{D}$ in the yz-plane between the curves $z = 1 - y^2$ and $z = y^2 - 1$.



This domain is the vertically simple region described by the inequalities

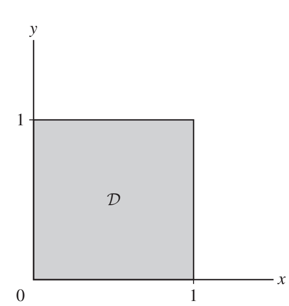
$$-1 \le y \le 1, \quad y^2 - 1 \le z \le 1 - y^2$$

We compute the double integral as the following iterated integral:

$$\begin{aligned}
V &= \iint_{\mathcal{D}} 2\, dA = \int_{-1}^{1} \int_{y^2-1}^{1-y^2} 2\, dz\, dy = \int_{-1}^{1} 2z \Big|_{z=y^2-1}^{1-y^2} dy = \int_{-1}^{1} 2\left((1 - y^2) - (y^2 - 1) \right) dy \\
&= \int_{-1}^{1} (4 - 4y^2)\, dy = \int_{0}^{1} (8 - 8y^2)\, dy = 8y - \frac{8}{3}y^3 \Big|_{0}^{1} = 8 - \frac{8}{3} = \frac{16}{3} \approx 5.33
\end{aligned}$$

47. Calculate the average value of $f(x, y) = e^{x+y}$ on the square $[0, 1] \times [0, 1]$.

SOLUTION



Since the area of the square $\mathcal{D}$ is 1, the average value of $f(x, y) = e^{x+y}$ on $\mathcal{D}$ is the following value:

$$\overline{f} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x, y)\, dA = \frac{1}{1} \int_0^1 \int_0^1 e^{x+y}\, dx\, dy = \int_0^1 e^{x+y} \Big|_{x=0}^{1} dy = \int_0^1 \left(e^{1+y} - e^{0+y} \right) dy$$
$$= \int_0^1 \left(e^{1+y} - e^{y} \right) dy = e^{1+y} - e^{y} \Big|_0^1 = \left(e^2 - e \right) - \left(e^1 - e^0 \right) = e^2 - 2e + 1 \approx 2.95$$

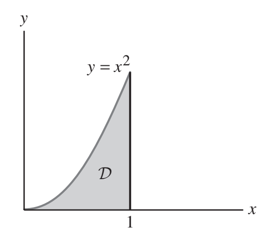
48. Calculate the average height above the x-axis of a point in the region $0 \le x \le 1$, $0 \le y \le x^2$.

SOLUTION The height of the point (x, y) in the region $\mathcal{D}$ above the x-axis is $f(x, y) = y$. Therefore, the average height is the following value:

$$\overline{H} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} y\, dA \tag{1}$$

We first compute the integral. The region $\mathcal{D}$ is a vertically simple region defined by the inequalities

$$0 \le x \le 1, \quad 0 \le y \le x^2$$



Therefore,

$$\iint_{\mathcal{D}} y\, dA = \int_0^1 \int_0^{x^2} y\, dy\, dx = \int_0^1 \frac{1}{2} y^2 \Big|_{y=0}^{x^2} dx = \int_0^1 \frac{1}{2} \left(x^4 - 0 \right) dx = \int_0^1 \frac{1}{2} x^4\, dx = \frac{x^5}{10} \Big|_0^1 = \frac{1}{10} \tag{2}$$

We compute the area of $\mathcal{D}$ using the formula for the area as a double integral:

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\, dA = \int_0^1 \int_0^{x^2} dy\, dx = \int_0^1 y \Big|_{y=0}^{x^2} dx = \int_0^1 x^2\, dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3} \tag{3}$$

Substituting (2) and (3) in (1) we obtain

$$\overline{H} = \frac{1}{\frac{1}{3}} \cdot \frac{1}{10} = \frac{3}{10}$$

49. Find the average height of the "ceiling" in Figure 28 defined by $z = y^2 \sin x$ for $0 \le x \le \pi$, $0 \le y \le 1$.

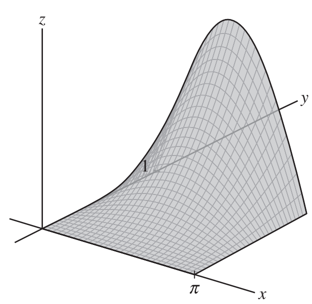
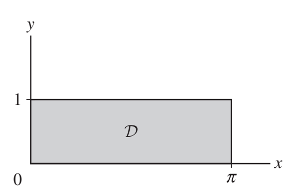


FIGURE 28

SOLUTION



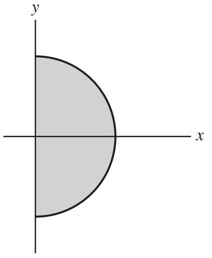
The average height is

$$\overline{H} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} y^2 \sin x \, dA = \frac{1}{\pi \cdot 1} \int_0^1 \int_0^\pi y^2 \sin x \, dx \, dy = \frac{1}{\pi} \int_0^1 y^2(-\cos x) \Big|_{x=0}^{\pi} dy$$
$$= \frac{1}{\pi} \int_0^1 y^2(-\cos \pi + \cos 0) \, dy = \frac{1}{\pi} \int_0^1 2y^2 \, dy = \frac{1}{\pi} \cdot \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3\pi}$$

50. Calculate the average value of the x-coordinate of a point on the semicircle $x^2 + y^2 \le R^2$, $x \ge 0$. What is the average value of the y-coordinate?

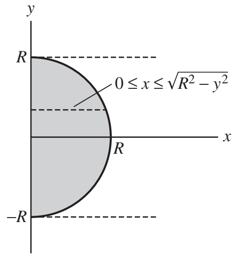
SOLUTION The average value of the x-coordinates of a point on the semicircle $\mathcal{D}$ is

$$\overline{x} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} x \, dA$$



The area of the semicircle is $\frac{\pi R^2}{2}$. To compute the double integral, we identify the inequalities defining $\mathcal{D}$ as a horizontally simple region:

$$-R \le y \le R, \quad 0 \le x \le \sqrt{R^2 - y^2}$$



Therefore,

$$\overline{x} = \frac{1}{\frac{\pi R^2}{2}} \int_{-R}^{R} \int_0^{\sqrt{R^2-y^2}} x \, dx \, dy = \frac{2}{\pi R^2} \int_{-R}^{R} \frac{x^2}{2} \Big|_{x=0}^{\sqrt{R^2-y^2}} dy = \frac{2}{\pi R^2} \int_{-R}^{R} \frac{\left(\sqrt{R^2 - y^2}\right)^2 - 0^2}{2} \, dy$$
$$= \frac{1}{\pi R^2} \int_{-R}^{R} \left(R^2 - y^2\right) dy = \frac{2}{\pi R^2} \int_0^R \left(R^2 - y^2\right) dy = \frac{2}{\pi R^2} \left(R^2 y - \frac{y^3}{3}\right) \Big|_{y=0}^{R}$$
$$= \frac{2}{\pi R^2} \left(R^3 - \frac{R^3}{3}\right) = \frac{2}{\pi R^2} \cdot \frac{2R^3}{3} = \frac{4R}{3\pi}$$

The average value of the x-coordinate is $\overline{x} = \frac{4R}{3\pi}$. The average value of the y-coordinate is

$$\overline{y} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} y \, dA = \frac{1}{\frac{\pi R^2}{2}} \int_{-R}^{R} \int_0^{\sqrt{R^2-y^2}} y \, dx \, dy = \frac{2}{\pi R^2} \int_{-R}^{R} yx \Big|_{x=0}^{\sqrt{R^2-y^2}} dy$$
$$= \frac{2}{\pi R^2} \int_{-R}^{R} y \left(\sqrt{R^2 - y^2} - 0\right) dy = \frac{2}{\pi R^2} \int_{-R}^{R} y \sqrt{R^2 - y^2} \, dy = 0$$

(The integral of an odd function over a symmetric interval with respect to the x-axis is zero). The average value of the y-coordinate is $\overline{y} = 0$.

Remark: Since the region is symmetric with respect to the x-axis, we expect the average value of y to be zero.

51. What is the average value of the linear function

$$f(x,y) = mx + ny + p$$

on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$? Argue by symmetry rather than calculation.

SOLUTION The average value of the linear function $f(x,y) = mx + ny + p$ over the ellipse $\mathcal{D}$ is

$$\begin{aligned}\overline{f} &= \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} f(x,y)\,dA = \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} (mx+ny+p)\,dA \\ &= m\cdot\underbrace{\frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} x\,dA}_{I_1} + n\cdot\underbrace{\frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} y\,dA}_{I_2} + \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} p\,dA \qquad (1)\end{aligned}$$

I_1 and I_2 are the average values of the x and y coordinates of a point in the region enclosed by the ellipse. This region is symmetric with respect to the y-axis, hence $I_1 = 0$. It is also symmetric with respect to the x-axis, hence $I_2 = 0$. We use the formula

$$\iint_{\mathcal{D}} p\,dA = p\cdot\text{Area}(\mathcal{D})$$

to conclude by (1) that

$$\overline{f} = m\cdot 0 + n\cdot 0 + \frac{1}{\text{Area}(\mathcal{D})}\cdot p\cdot\text{Area}(\mathcal{D}) = p$$

52. Find the average square distance from the origin to a point in the domain $\mathcal{D}$ in Figure 29.

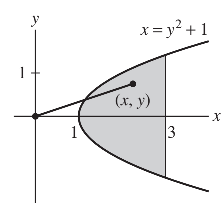
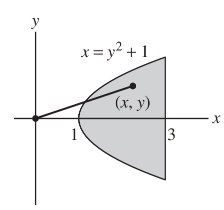


FIGURE 29

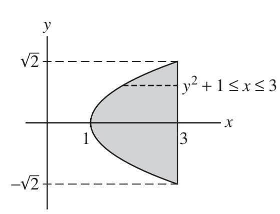
SOLUTION The square distance from the origin to a point (x,y) is given by the following function:

$$f(x,y) = (x-0)^2 + (y-0)^2 = x^2 + y^2$$



The domain $\mathcal{D}$ is a horizontally simple region described by the inequalities (see figure)

$$-\sqrt{2} \le y \le \sqrt{2}, \quad y^2 + 1 \le x \le 3$$



The average value of f in $\mathcal{D}$ is

$$\overline{f} = \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} \left(x^2 + y^2\right)\,dA \qquad (1)$$

We first compute the integral:

$$\iint_{\mathcal{D}} \left(x^2+y^2\right) dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2+1}^{3} \left(x^2+y^2\right) dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^3}{3} + y^2 x \bigg|_{x=y^2+1}^{3} dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(9 + 3y^2 - \left(\frac{(y^2+1)^3}{3} + y^2(y^2+1)\right)\right) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(-\frac{y^6}{3} - 2y^4 + y^2 + \frac{26}{3}\right) dy = \int_{0}^{\sqrt{2}} \left(-\frac{2y^6}{3} - 4y^4 + 2y^2 + \frac{52}{3}\right) dy$$

$$= -\frac{2y^7}{21} - \frac{4}{5}y^5 + \frac{2}{3}y^3 + \frac{52}{3}y \bigg|_0^{\sqrt{2}} = -\frac{2\left(\sqrt{2}\right)^7}{21} - \frac{4\left(\sqrt{2}\right)^5}{5} + \frac{2\left(\sqrt{2}\right)^3}{3} + \frac{52\sqrt{2}}{3}$$

$$= -\frac{16\sqrt{2}}{21} - \frac{16\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} + \frac{52\sqrt{2}}{3} = \frac{1544}{105}\sqrt{2} \tag{2}$$

We compute the area of $\mathcal{D}$:

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2+1}^{3} 1\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} x \bigg|_{x=y^2+1}^{3} dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left(3-(y^2+1)\right) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)\,dy = \int_0^{\sqrt{2}} (4-2y^2)\,dy = 4y - \frac{2}{3}y^3 \bigg|_0^{\sqrt{2}} = 4\sqrt{2} - \frac{2}{3}\left(\sqrt{2}\right)^3 = \frac{8}{3}\sqrt{2} \tag{3}$$

Substituting (2) and (3) into (1), we obtain the following solution:

$$\overline{f} = \frac{1}{\frac{8\sqrt{2}}{3}} \cdot \frac{1544}{105}\sqrt{2} = \frac{3 \cdot 1544\sqrt{2}}{8\sqrt{2} \cdot 105} = \frac{193}{35} \approx 5.51$$

53. Let $\mathcal{D}$ be the rectangle $0 \le x \le 2$, $-\frac{1}{8} \le y \le \frac{1}{8}$, and let $f(x,y) = \sqrt{x^3+1}$. Prove that

$$\iint_{\mathcal{D}} f(x,y)\,dA \le \frac{3}{2}$$

SOLUTION Recall that we can write

$$\iint_{\mathcal{D}} f(x,y)\,dA \le M \cdot \text{Area}(\mathcal{D})$$

where M is a constant such that $f(x,y) \le M$. We can see that $\text{Area}(\mathcal{D}) = 2(1/4) = 1/2$. So it remains to show that there is some constant M so that $f(x,y) \le M$. Consider the following:

$$x \le 2 \quad \Rightarrow \quad x^3 + 1 \le 0 \quad \Rightarrow \quad \sqrt{x^3+1} \le 3$$

Thus we can let $M = 3$. So then we have

$$\iint_{\mathcal{D}} f(x,y)\,dA \le M \cdot \text{Area}(\mathcal{D}) \quad \Rightarrow \quad \iint_{\mathcal{D}} \sqrt{x^3+1}\,dA \le 3 \cdot \frac{1}{2} = \frac{3}{2}$$

54. (a) Use the inequality $0 \le \sin x \le x$ for $x \ge 0$ to show that

$$\int_0^1 \int_0^1 \sin(xy)\,dx\,dy \le \frac{1}{4}$$

(b) Use a computer algebra system to evaluate the double integral to three decimal places.

SOLUTION Since $\sin(xy) \le xy$, we get that

$$\int_0^1 \int_0^1 \sin(xy)\,dx\,dy \le \int_0^1 \int_0^1 xy\,dx\,dy = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}$$

Using a CAS, we find that the double integral is approximately 0.240.

55. Prove the inequality $\iint_{\mathcal{D}} \frac{dA}{4+x^2+y^2} \le \pi$, where $\mathcal{D}$ is the disk $x^2+y^2 \le 4$.

SOLUTION The function $f(x,y) = \frac{1}{4+x^2+y^2}$ satisfies

$$f(x,y) = \frac{1}{4+x^2+y^2} \le \frac{1}{4}$$

Also, the area of the disk is

$$\text{Area}(\mathcal{D}) = \pi \cdot 2^2 = 4\pi$$

Therefore, by Theorem 3, we have

$$\iint_{\mathcal{D}} \frac{dA}{4+x^2+y^2} \le \frac{1}{4} \cdot 4\pi = \pi.$$

56. Let $\mathcal{D}$ be the domain bounded by $y = x^2+1$ and $y = 2$. Prove the inequality

$$\frac{4}{3} \le \iint_{\mathcal{D}} (x^2+y^2)dA \le \frac{20}{3}$$

SOLUTION Recall that we can write:

$$m \cdot \text{Area}(\mathcal{D}) \le \iint_{\mathcal{D}} f(x,y)\,dA \le M \cdot \text{Area}(\mathcal{D})$$

where m and M are constants such that $m \le f(x,y) \le M$.

First let us compute the area of the domain, $\mathcal{D}$:

$$\text{Area} = \int_{-1}^{1}\int_{x^2+1}^{2} 1\,dy\,dx = \int_{-1}^{1} 2-(x^2+1)\,dy\,dx$$

$$= \int_{-1}^{1} 1-x^2\,dy\,dx = x - \frac{1}{3}x^3\Big|_{-1}^{1}$$

$$= \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3}$$

Then we must bound $f(x,y) = x^2+y^2$ by constants (above and below). We know that $-1 \le x \le 1$ and the largest x^2 can be is 1 and the smallest x^2 can be is 0. The largest y can be is 2 and the smallest y can be is 0. Therefore,

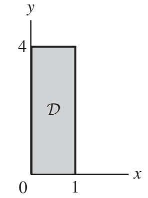
$$(-1)^2+0^2 \le x^2+y^2 \le 1^2+2^2 \quad\Rightarrow\quad 1 \le x^2+y^2 \le 5$$

Putting this all together for the inequality we see

$$m \cdot \text{Area}(\mathcal{D}) \le \iint_{\mathcal{D}} f(x,y)\,dA \le M \cdot \text{Area}(\mathcal{D}) \quad\Rightarrow\quad \frac{4}{3} \le \iint f(x,y)\,dA \le 5 \cdot \frac{4}{3} = \frac{20}{3}$$

57. Let $\overline{f}$ be the average of $f(x,y) = xy^2$ on $\mathcal{D} = [0,1]\times[0,4]$. Find a point $P \in \mathcal{D}$ such that $f(P) = \overline{f}$ (the existence of such a point is guaranteed by the Mean Value Theorem for Double Integrals).

SOLUTION We first compute the average $\overline{f}$ of $f(x,y) = xy^2$ on $\mathcal{D}$.



$\overline{f}$ is

$$\overline{f} = \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} xy^2\,dA = \frac{1}{4\cdot 1}\int_0^1\int_0^4 xy^2\,dy\,dx = \frac{1}{4}\int_0^1 \frac{xy^3}{3}\Big|_{y=0}^{4}\,dx$$

$$= \frac{1}{4}\int_0^1 \left(\frac{x\cdot 4^3}{3} - \frac{x\cdot 0^3}{3}\right)dx = \int_0^1 \frac{16x}{3}\,dx = \frac{8x^2}{3}\Big|_0^1 = \frac{8}{3}$$

We now must find a point $P = (a, b)$ in $\mathcal{D}$ such that

$$f(P) = ab^2 = \frac{8}{3}$$

We choose $b = 2$, obtaining

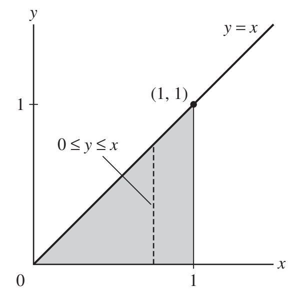
$$a \cdot 2^2 = \frac{8}{3} \quad \Rightarrow \quad a = \frac{2}{3}$$

The point $P = \left(\frac{2}{3}, 2\right)$ in the rectangle $\mathcal{D}$ satisfies

$$f(P) = \overline{f} = \frac{8}{3}$$

58. Verify the Mean Value Theorem for Double Integrals for $f(x, y) = e^{x-y}$ on the triangle bounded by $y = 0$, $x = 1$, and $y = x$.

SOLUTION



We must find a point $P = (a, b)$ in the triangle $\mathcal{D}$ such that

$$\iint_{\mathcal{D}} e^{x-y}\, dA = e^{a-b} \cdot \frac{1}{2} \tag{1}$$

(The area of the triangle is $\frac{1\cdot 1}{2} = \frac{1}{2}$.) We first compute the double integral. The inequalities defining $\mathcal{D}$ as a vertically simple region are

$$0 \le x \le 1, \quad 0 \le y \le x \tag{2}$$

Therefore,

$$\iint_{\mathcal{D}} e^{x-y}\, dA = \int_0^1 \int_0^x e^{x-y}\, dy\, dx = \int_0^1 -e^{x-y}\Big|_{y=0}^{x} dx = \int_0^1 -(e^{x-x} - e^{x-0})\, dx = \int_0^1 (e^x - 1)\, dx$$
$$= e^x - x\Big|_0^1 = (e^1 - 1) - (e^0 - 0) = e - 2$$

Substituting in (1) we get

$$e - 2 = e^{a-b} \cdot \frac{1}{2} \qquad \text{or} \qquad e^{a-b} = 2e - 4$$

or

$$a - b = \ln(2e - 4) \quad \Rightarrow \quad b = a - \ln(2e - 4)$$

We choose $a = 0.5$ and find b:

$$b = 0.5 - \ln(2e - 4) \approx 0.138$$

The point $(a, b) = (0.5, 0.138)$ satisfies the inequalities (2), hence it lies in $\mathcal{D}$. This point satisfies

$$\iint_{\mathcal{D}} e^{x-y}\, dA = e^{a-b} \text{Area}(\mathcal{D})$$

In Exercises 59 and 60, use (11) to estimate the double integral.

59. The following table lists the areas of the subdomains $\mathcal{D}_j$ of the domain $\mathcal{D}$ in Figure 30 and the values of a function $f(x, y)$ at sample points $P_j \in \mathcal{D}_j$. Estimate $\iint_{\mathcal{D}} f(x, y)\, dA$.

j	1	2	3	4	5	6
Area($\mathcal{D}_j$)	1.2	1.1	1.4	0.6	1.2	0.8
$f(P_j)$	9	9.1	9.3	9.1	8.9	8.8

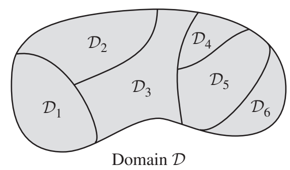


FIGURE 30

SOLUTION By Eq. (11) we have

$$\iint_{\mathcal{D}} f(x, y)\, dA \approx \sum_{j=1}^{6} f\left(P_j\right) \text{Area}\left(\mathcal{D}_j\right)$$

Substituting the data given in the table, we obtain

$$\iint_{\mathcal{D}} f(x, y)\, dA \approx 9 \cdot 1.2 + 9.1 \cdot 1.1 + 9.3 \cdot 1.4 + 9.1 \cdot 0.6 + 8.9 \cdot 1.2 + 8.8 \cdot 0.8 = 57.01$$

Thus,

$$\iint_{\mathcal{D}} f(x, y)\, dA \approx 57.01$$

60. The domain $\mathcal{D}$ between the circles of radii 5 and 5.2 in the first quadrant in Figure 31 is divided into six subdomains of angular width $\Delta\theta = \frac{\pi}{12}$, and the values of a function $f(x, y)$ at sample points are given. Compute the area of the subdomains and estimate $\iint_{\mathcal{D}} f(x, y)\, dA$.

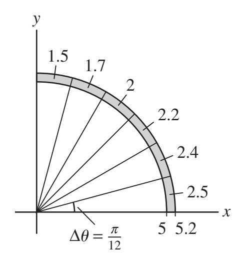


FIGURE 31

SOLUTION The area of a sector of angular width θ in a circle of radius $\mathcal{R}$ is $\frac{\mathcal{R}^2\theta}{2}$. Hence, the area of each subdomain is the following difference:

$$\text{Area}\left(\mathcal{D}_j\right) = \frac{(5.2)^2 \cdot \frac{\pi}{12}}{2} - \frac{5^2 \cdot \frac{\pi}{12}}{2} = \frac{2.04 \cdot \frac{\pi}{12}}{2} = 0.085\pi$$

We now use Eq. (11) and the given values at sample points to estimate the double integral:

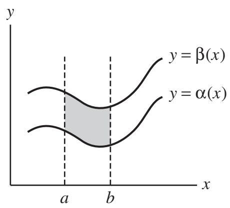
$$\iint_{\mathcal{D}} f(x, y)\, dA \approx \sum_{j=1}^{6} f\left(P_j\right) \text{Area}\left(\mathcal{D}_j\right) = 0.085\pi(2.5 + 2.4 + 2.2 + 2 + 1.7 + 1.5) = 1.0455\pi \approx 3.285$$

61. According to Eq. (3), the area of a domain $\mathcal{D}$ is equal to $\iint_{\mathcal{D}} 1\, dA$. Prove that if $\mathcal{D}$ is the region between two curves $y = g_1(x)$ and $y = g_2(x)$ with $g_2(x) \le g_1(x)$ for $a \le x \le b$, then

$$\iint_{\mathcal{D}} 1\, dA = \int_a^b (g_1(x) - g_2(x))\, dx$$

SOLUTION The region $\mathcal{D}$ is defined by the inequalities

$$a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$



We compute the double integral of $f(x, y) = 1$ on $\mathcal{D}$, using Theorem 2, by evaluating the following iterated integral:

$$\int_{\mathcal{D}} 1\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} 1\, dy \right) dx = \int_a^b y \Big|_{y=g_1(x)}^{g_2(x)} dx = \int_a^b (g_2(x) - g_1(x))\, dx$$

Further Insights and Challenges

62. Let $\mathcal{D}$ be a closed connected domain and let $P, Q \in \mathcal{D}$. The Intermediate Value Theorem (IVT) states that if f is continuous on $\mathcal{D}$, then $f(x, y)$ takes on every value between $f(P)$ and $f(Q)$ at some point in $\mathcal{D}$.

(a) Show, by constructing a counterexample, that the IVT is false if $\mathcal{D}$ is not connected.

(b) Prove the IVT as follows: Let $\mathbf{c}(t)$ be a path such that $\mathbf{c}(0) = P$ and $\mathbf{c}(1) = Q$ (such a path exists because $\mathcal{D}$ is connected). Apply the IVT in one variable to the composite function $f(\mathbf{c}(t))$.

SOLUTION

(a) Let $\mathcal{D}$ be the union of the disc $\mathcal{D}_1$ of radius $\frac{1}{2}$ centered at the origin, and the disc $\mathcal{D}_2$ of radius $\frac{1}{2}$ centered at $(1, 1)$.

Obviously $\mathcal{D}$ is not connected. We define a function $f(x, y)$ on $\mathcal{D}$ as follows:

$$f(x, y) = \begin{cases} 1 & (x, y) \in \mathcal{D}_1 \\ 2 & (x, y) \in \mathcal{D}_2 \end{cases}$$

f is continuous on $\mathcal{D}$, but it does not take on any value between 1 and 2.

(b) Let $\mathcal{D}$ be a closed connected domain and $f(x, y)$ a continuous function on $\mathcal{D}$. Suppose that $f(P) = a$ and $f(Q) = b$ where $P, Q \in \mathcal{D}$, and $a < c < b$. We show that f takes on the value c at a point in $\mathcal{D}$. Since $\mathcal{D}$ is connected, there is a curve $\gamma(t) = (x(t), y(t))$ lying entirely in $\mathcal{D}$, such that $\gamma(0) = P$ and $\gamma(1) = Q$. We consider the function

$$g(t) = f(x(t), y(t)), \quad 0 \le t \le 1$$

The composition $g(t)$ is continuous on the segment $0 \le t \le 1$, and c is an intermediate value of g on this segment (since $g(0) = a < c < b = g(1)$). Therefore, by the IVT, there exists $t_0 \in (0, 1)$ such that $g(t_0) = c$. The curve $\gamma(t)$ lies in $\mathcal{D}$, hence the point $\mathcal{R} = \gamma(t_0) = (x(t_0), y(t_0))$ is in $\mathcal{D}$ and the following holds:

$$f(\mathcal{R}) = f(x(t_0), y(t_0)) = g(t_0) = c.$$

63. Use the fact that a continuous function on a closed domain $\mathcal{D}$ attains both a minimum value m and a maximum value M, together with Theorem 3, to prove that the average value $\overline{f}$ lies between m and M. Then use the IVT in Exercise 62 to prove the Mean Value Theorem for Double Integrals.

SOLUTION Suppose that $f(x, y)$ is continuous and $\mathcal{D}$ is closed, bounded, and connected. By Theorem 3 in Chapter 15.7 ("Existence of Global Extrema"), $f(x, y)$ takes on a minimum value (call it m) at some point (x_m, y_m) and a maximum value (call it M) at some point (x_M, y_M) in the domain $\mathcal{D}$. Now, by Theorem 3,

$$m \,\text{Area}(\mathcal{D}) \le \iint_{\mathcal{D}} f(x, y)\, dA \le M \,\text{Area}(\mathcal{D})$$

which can be restated as

$$m \le \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x, y)\, dA \le M$$

By the IVT in two variables (stated and proved in the previous problem), $f(x, y)$ takes on every value between m and M at some point in $\mathcal{D}$. In particular, f must take on the value $\frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x, y)\, dA$ at some point P. So, $f(P) = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x, y)\, dA$, which is rewritten as

$$f(P)\,\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} f(x, y)\, dA$$

64. Let $f(y)$ be a function of y alone and set $G(t) = \int_0^t \int_0^x f(y)\,dy\,dx$.

(a) Use the Fundamental Theorem of Calculus to prove that $G''(t) = f(t)$.

(b) Show, by changing the order in the double integral, that $G(t) = \int_0^t (t-y) f(y)\,dy$. This shows that the "second antiderivative" of $f(y)$ can be expressed as a single integral.

SOLUTION

(a) Let $H(x) = \int_0^x f(y)\,dy$. Then $G(t) = \int_0^t H(x)\,dx$, and by the FTC we have

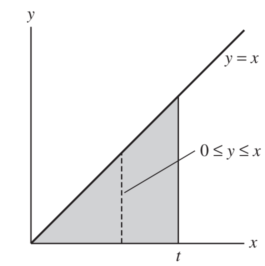
$$G'(t) = \frac{d}{dt}\int_0^t H(x)\,dx = H(t) = \int_0^t f(y)\,dy$$

We again use the FTC to differentiate $G'(t)$. This gives

$$G''(t) = \frac{d}{dt}\int_0^t f(y)\,dy = f(t)$$

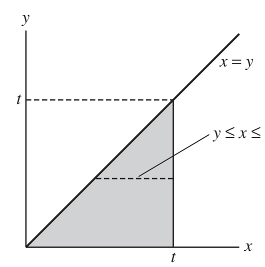
(b) For a fixed t, the domain of integration is described by the following inequalities:

$$0 \le x \le t, \quad 0 \le y \le x$$



We describe the domain as a horizontally simple region by the inequalities

$$0 \le y \le t, \quad y \le x \le t$$



Then, the iterated integral for $G(t)$ can be computed in reverse order of integration as follows:

$$G(t) = \int_0^t \int_y^t f(y)\,dx\,dy = \int_0^t f(y)x\Big|_{x=y}^{t}\,dy = \int_0^t f(y)(t-y)\,dy = \int_0^t (t-y)f(y)\,dy$$

We, thus showed that

$$G(t) = \int_0^t (t-y)f(y)\,dy$$

15.3 Triple Integrals (LT Section 16.3)

Preliminary Questions

1. Which of (a)–(c) is not equal to $\int_0^1 \int_3^4 \int_6^7 f(x, y, z)\,dz\,dy\,dx$?

(a) $\int_6^7 \int_0^1 \int_3^4 f(x, y, z)\,dy\,dx\,dz$

(b) $\int_3^4 \int_0^1 \int_6^7 f(x, y, z)\,dz\,dx\,dy$

(c) $\int_0^1 \int_3^4 \int_6^7 f(x, y, z)\,dx\,dz\,dy$

SOLUTION The given integral, I, is a triple integral of f over the box $B = [0, 1] \times [3, 4] \times [6, 7]$. In (a) the limits of integration are $0 \le x \le 1$, $3 \le y \le 4$, $6 \le z \le 7$, hence this integral is equal to I. In (b) the limits of integration are $0 \le x \le 1$, $3 \le y \le 4$, $6 \le z \le 7$, hence it is also equal to I. In (c) the limits of integration are $6 \le x \le 7$, $0 \le y \le 1$, $3 \le z \le 4$. This is the triple integral of f over the box $[6, 7] \times [0, 1] \times [3, 4]$, which is different from $\mathcal{B}$. Therefore, the triple integral is usually unequal to I.

2. Which of the following is not a meaningful triple integral?

(a) $\displaystyle\int_0^1 \int_0^x \int_{x+y}^{2x+y} e^{x+y+z}\,dz\,dy\,dx$

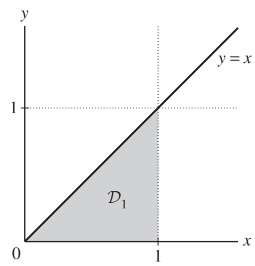
(b) $\displaystyle\int_0^1 \int_0^z \int_{x+y}^{2x+y} e^{x+y+z}\,dz\,dy\,dx$

SOLUTION

(a) The limits of integration determine the following inequalities:

$$0 \le x \le 1, \quad 0 \le y \le x, \quad x + y \le z \le 2x + y$$

The integration is over the simple region $\mathcal{W}$, which lies between the planes $z = x + y$ and $z = 2x + y$ over the domain $\mathcal{D}_1 = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$ in the xy-plane.



Thus, the integral represents a meaningful triple integral.

(b) Note that the inner integral is with respect to z, but then the middle integral has limits from 0 to z! This makes no sense.

3. Describe the projection of the region of integration $\mathcal{W}$ onto the xy-plane:

(a) $\displaystyle\int_0^1 \int_0^x \int_0^{x^2+y^2} f(x, y, z)\,dz\,dy\,dx$

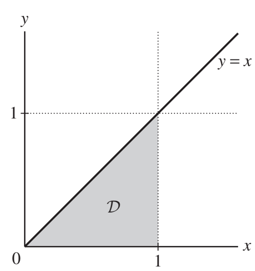
(b) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_2^4 f(x, y, z)\,dz\,dy\,dx$

SOLUTION

(a) The region of integration is defined by the limits of integration, yielding the following inequalities:

$$0 \le x \le 1, \quad 0 \le y \le x, \quad 0 \le z \le x^2 + y^2$$

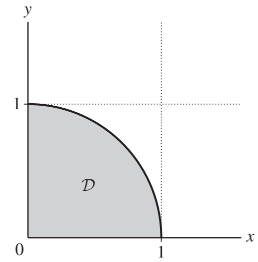
$\mathcal{W}$ is the region between the paraboloid $z = x^2 + y^2$ and the xy-plane which is above the triangle $\mathcal{D} = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$ in the xy-plane. This triangle is the projection of $\mathcal{W}$ onto the xy-plane.



(b) The inequalities determined by the limits of integration are

$$0 \le x \le 1, \quad 0 \le y \le \sqrt{1 - x^2}, \quad 2 \le z \le 4$$

This is the region between the planes $z = 2$ and $z = 4$, which is above the region $\mathcal{D} = \left\{(x, y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\right\}$ in the xy-plane. The projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is the part of the unit disk in the first quadrant.



Exercises

In Exercises 1–8, evaluate $\iiint_{\mathcal{B}} f(x, y, z)\, dV$ *for the specified function* f *and box* $\mathcal{B}$.

1. $f(x, y, z) = z^4; \quad 2 \le x \le 8, \quad 0 \le y \le 5, \quad 0 \le z \le 1$

SOLUTION We write the triple integral as an iterated integral and compute it to obtain

$$\iiint_{\mathcal{B}} z^4\, dV = \int_2^8 \int_0^5 \int_0^1 z^4\, dz\, dy\, dx = \int_2^8 \int_0^5 \left(\int_0^1 z^4\, dz \right) dy\, dx = \int_2^8 \int_0^5 \frac{1}{5} z^5 \Big|_{z=0}^{1} dy\, dx$$
$$= \int_2^8 \int_0^5 \frac{1}{5}\, dy\, dx = \frac{1}{5} \int_2^8 \int_0^5 dy\, dx = \frac{1}{5} \cdot 6 \cdot 5 = 6$$

2. $f(x, y, z) = xz^2; \quad [-2, 3] \times [1, 3] \times [1, 4]$

SOLUTION The box $[-2, 3] \times [1, 3] \times [1, 4]$ corresponds to the inequalities $-2 \le x \le 3$, $1 \le y \le 3$, $1 \le z \le 4$. We write the integral as an iterated integral in any order we choose, and evaluate the inner, middle, and outer integral one after the other. This gives

$$\iint_{\mathcal{B}} xz^2\, dV = \int_{-2}^{3} \int_1^3 \int_1^4 xz^2\, dz\, dy\, dx = \int_{-2}^{3} \int_1^3 \left(\int_1^4 xz^2\, dz \right) dy\, dx = \int_{-2}^{3} \int_1^3 \frac{xz^3}{3} \Big|_{z=1}^{4} dy\, dx$$
$$= \int_{-2}^{3} \int_1^3 \frac{x(4^3 - 1^3)}{3}\, dy\, dx = \int_{-2}^{3} \left(\int_1^3 21x\, dy \right) dx = \int_{-2}^{3} 21xy \Big|_{y=1}^{3} dx$$
$$= \int_{-2}^{3} 42x\, dx = 21x^2 \Big|_{-2}^{3} = 21(9 - 4) = 105$$

Alternatively, we can use the form $f(x, y, z) = xz^2 = h(x)g(y)l(z)$ to compute the triple integral as the product:

$$\iint_{\mathcal{B}} xz^2\, dV = \int_{-2}^{3} \int_1^3 \int_1^4 xz^2\, dz\, dy\, dx = \left(\int_{-2}^{3} x\, dx \right) \left(\int_1^3 1\, dy \right) \left(\int_1^4 z^2\, dz \right)$$
$$= \left(\frac{x^2}{2} \Big|_{-2}^{3} \right) \left(y \Big|_1^3 \right) \left(\frac{z^3}{3} \Big|_1^4 \right) = \frac{5}{2} \cdot 2 \cdot \frac{\left(4^3 - 1^3\right)}{3} = 5 \cdot 21 = 105$$

3. $f(x, y, z) = xe^{y-2z}; \quad 0 \le x \le 2, \quad 0 \le y \le 1, \quad 0 \le z \le 1$

SOLUTION We write the triple integral as an iterated integral. Since $f(x, y, z) = xe^y \cdot e^{-2z}$, we may evaluate the iterated integral as the product of three single integrals. We get

$$\iiint_{\mathcal{B}} xe^{y-2z}\, dV = \int_0^2 \int_0^1 \int_0^1 xe^{y-2z}\, dz\, dy\, dx = \left(\int_0^2 x\, dx \right) \left(\int_0^1 e^y\, dy \right) \left(\int_0^1 e^{-2z}\, dz \right)$$
$$= \left(\frac{1}{2} x^2 \Big|_0^2 \right) \left(e^y \Big|_0^1 \right) \left(-\frac{1}{2} e^{-2z} \Big|_0^1 \right) = 2(e - 1) \cdot -\frac{1}{2}(e^{-2} - 1) = (e - 1)(1 - e^{-2})$$

4. $f(x, y, z) = \dfrac{x}{(y+z)^2}$; $[0, 2] \times [2, 4] \times [-1, 1]$

SOLUTION We write the triple integral as an iterated integral in any order we choose, and then evaluate the resulting integrals successively. We get:

$$\begin{aligned}\iiint_{\mathcal{B}} f(x, y, z)\, dx &= \int_{-1}^{1}\int_{2}^{4}\int_{0}^{2} \frac{x}{(y+z)^2}\, dx\, dy\, dz = \int_{-1}^{1}\int_{2}^{4}\left(\int_{0}^{2} \frac{x}{(y+z)^2}\, dx\right) dy\, dz \\ &= \int_{-1}^{1}\int_{2}^{4}\left(\frac{1}{(y+z)^2}\int_{0}^{2} x\, dx\right) dy\, dz = \int_{-1}^{1}\int_{2}^{4} \frac{1}{(y+z)^2}\, \frac{x^2}{2}\bigg|_{x=0}^{2} dy\, dz \\ &= \int_{-1}^{1}\int_{2}^{4} \frac{2}{(y+z)^2}\, dy\, dz = \int_{-1}^{1}\left(\int_{2}^{4} \frac{2}{(y+z)^2}\, dy\right) dz \\ &= \int_{-1}^{1} \frac{-2}{y+z}\bigg|_{y=2}^{4} dz = \int_{-1}^{1}\left(\frac{-2}{4+z} + \frac{2}{2+z}\right) dz = -2\ln(4+z) + 2\ln(2+z)\bigg|_{z=-1}^{1} \\ &= -2\ln 5 + 2\ln 3 - (-2\ln 3 + 2\ln 1) = -2\ln 5 + 4\ln 3 = \ln\frac{3^4}{5^2} \approx 1.176\end{aligned}$$

5. $f(x, y, z) = (x - y)(y - z)$; $[0, 1] \times [0, 3] \times [0, 3]$

SOLUTION We write the triple integral as an iterated integral and evaluate the inner, middle, and outer integrals successively. This gives

$$\begin{aligned}\iiint_{\mathcal{B}} (x-y)(y-z)\, dV &= \int_{0}^{1}\int_{0}^{3}\int_{0}^{3} (x-y)(y-z)\, dz\, dy\, dx = \int_{0}^{1}\int_{0}^{3}\left(\int_{0}^{3} (x-y)(y-z)\, dz\right) dy\, dx \\ &= \int_{0}^{1}\int_{0}^{3} (x-y)\left(yz - \frac{1}{2}z^2\right)\bigg|_{z=0}^{3} dy\, dx = \int_{0}^{1}\int_{0}^{3} (x-y)\left(3y - \frac{9}{2}\right) dy\, dx \\ &= \int_{0}^{1}\int_{0}^{3}\left(\left(3x + \frac{9}{2}\right)y - \frac{9}{2}x - 3y^2\right) dy\, dx = \int_{0}^{1}\left(\frac{3}{2}x + \frac{9}{4}\right)y^2 - \frac{9}{2}xy - y^3\bigg|_{y=0}^{3} dx \\ &= \int_{0}^{1}\left(\left(\frac{3}{2}x + \frac{9}{4}\right)\cdot 9 - \frac{9}{2}x \cdot 3 - 27\right) dx = \int_{0}^{1} -\frac{27}{4}\, dx = -\frac{27}{4} = -6.75\end{aligned}$$

6. $f(x, y, z) = \dfrac{z}{x}$; $1 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 4$

SOLUTION We write the triple integral as an iterated integral and evaluate it using iterated integral of a product function. We get

$$\begin{aligned}\iiint_{\mathcal{B}} f(x, y, z)\, dV &= \int_{1}^{3}\int_{0}^{2}\int_{0}^{4} \frac{z}{x}\, dz\, dy\, dx = \left(\int_{0}^{4} z\, dz\right)\left(\int_{0}^{2} 1\, dy\right)\left(\int_{1}^{3} \frac{1}{x}\, dx\right) \\ &= \left(\frac{1}{2}z^2\bigg|_{0}^{4}\right)\left(y\bigg|_{0}^{2}\right)\left(\ln x\bigg|_{1}^{3}\right) = 8 \cdot 2 \cdot (\ln 3 - \ln 1) = 16\ln 3\end{aligned}$$

7. $f(x, y, z) = (x + z)^3$; $[0, a] \times [0, b] \times [0, c]$

SOLUTION We write the triple integral as an iterated integral and evaluate it to obtain

$$\begin{aligned}\iiint_{\mathcal{B}} f(x, y, z)\, dV &= \int_{0}^{a}\int_{0}^{b}\int_{0}^{c} (x+z)^3\, dz\, dy\, dx = \int_{0}^{a}\int_{0}^{b} \frac{(x+z)^4}{4}\bigg|_{z=0}^{c} dy\, dx \\ &= \int_{0}^{a}\int_{0}^{b}\left(\frac{(x+c)^4}{4} - \frac{x^4}{4}\right) dy\, dx = \int_{0}^{a} \frac{(x+c)^4 - x^4}{4}\, y\bigg|_{y=0}^{b} dx \\ &= \int_{0}^{a} \frac{b}{4}\left[(x+c)^4 - x^4\right] dx = \frac{b}{4}\left[\frac{(x+c)^5}{5} - \frac{x^5}{5}\right]\bigg|_{x=0}^{a} \\ &= \frac{b}{4}\,\frac{(a+c)^5 - a^5 - c^5}{5} = \frac{b}{20}\left[(a+c)^5 - a^5 - c^5\right]\end{aligned}$$

8. $f(x, y, z) = (x + y - z)^2$; $[0, a] \times [0, b] \times [0, c]$

SOLUTION We evaluate the triple integral using Theorem 1. This gives

$$\iiint_{\mathcal{B}} f(x, y, z)\, dV = \int_0^a \int_0^b \int_0^c (x + y - z)^2\, dz\, dy\, dx = \int_0^a \int_0^b \left(-\frac{(x + y - z)^3}{3} \bigg|_{z=0}^{c} \right) dy\, dx$$

$$= \int_0^a \int_0^b -\frac{(x + y - c)^3}{3} + \frac{(x + y)^3}{3}\, dy\, dx$$

$$= \int_0^a \left(-\frac{(x + y - c)^4}{12} + \frac{(x + y)^4}{12} \bigg|_{y=0}^{b} \right) dx$$

$$= \int_0^a -\frac{(x + b - c)^4}{12} + \frac{(x + b)^4}{12} + \frac{(x - c)^4}{12} - \frac{x^4}{12}\, dx$$

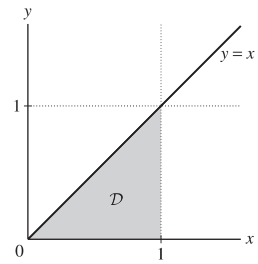
$$= -\frac{(x + b - c)^5}{60} + \frac{(x + b)^5}{60} + \frac{(x - c)^5}{60} - \frac{x^5}{60} \bigg|_{x=0}^{a}$$

$$= -\frac{(a + b - c)^5}{60} + \frac{(a + b)^5}{60} + \frac{(a - c)^5}{60} - \frac{a^5}{60} + \frac{(b - c)^5}{60} - \frac{b^5}{60} + \frac{(-c)^5}{60}$$

In Exercises 9–14, evaluate $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ *for the function* f *and region* $\mathcal{W}$ *specified.*

9. $f(x, y, z) = x + y$; $\mathcal{W} : y \le z \le x,\ \ 0 \le y \le x,\ \ 0 \le x \le 1$

SOLUTION $\mathcal{W}$ is the region between the planes $z = y$ and $z = x$ lying over the triangle $\mathcal{D}$ in the xy-plane defined by the inequalities $0 \le y \le x$, $0 \le x \le 1$.



We compute the integral, using Theorem 2, by evaluating the following iterated integral:

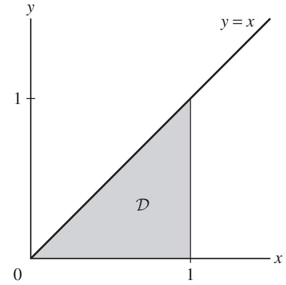
$$\iiint_{\mathcal{W}} (x + y)\, dV = \iint_{\mathcal{D}} \left(\int_y^x (x + y)\, dz \right) dA = \iint_{\mathcal{D}} (x + y)z \bigg|_{z=y}^{x} dA = \iint_{\mathcal{D}} (x + y)(x - y)\, dA$$

$$= \iint_{\mathcal{D}} \left(x^2 - y^2 \right) dA = \int_0^1 \int_0^x \left(x^2 - y^2 \right) dy\, dx = \int_0^1 \left(\int_0^x \left(x^2 - y^2 \right) dy \right) dx$$

$$= \int_0^1 x^2 y - \frac{y^3}{3} \bigg|_{y=0}^{x} dx = \int_0^1 \frac{2x^3}{3}\, dx = \frac{2}{12} x^4 \bigg|_0^1 = \frac{1}{6}$$

10. $f(x, y, z) = e^{x+y+z}$; $\mathcal{W} : 0 \le z \le 1,\ \ 0 \le y \le x,\ \ 0 \le x \le 1$

SOLUTION $\mathcal{W}$ is the region between the planes $z = 0$ and $z = 1$ lying over the triangle $\mathcal{D}$ in the xy-plane described in Exercise 9.

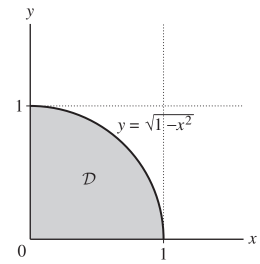


We compute the triple integral as the following iterated integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} e^{x+y+z}\,dV &= \iint_{\mathcal{D}} \left(\int_0^1 e^{x+y+z}\,dz\right) dA = \iint_{\mathcal{D}} e^{x+y+z}\Big|_{z=0}^{1} dA \\
&= \iint_{\mathcal{D}} (e^{x+y+1} - e^{x+y})\,dA = \int_0^1 \int_0^x (e^{x+y+1} - e^{x+y})\,dy\,dx \\
&= \int_0^1 \left(\int_0^x (e^{x+y+1} - e^{x+y})\,dy\right) dx = \int_0^1 e^{x+y+1} - e^{x+y}\Big|_{y=0}^{x} dx \\
&= \int_0^1 \left(e^{2x+1} - e^{2x} - e^{x+1} + e^x\right) dx = \frac{1}{2}e^{2x+1} - \frac{1}{2}e^{2x} - e^{x+1} + e^x\Big|_0^1 \\
&= \frac{1}{2}e^3 - \frac{1}{2}e^2 - e^2 + e - \left(\frac{1}{2}e - \frac{1}{2}e^0 - e + e^0\right) \\
&= \frac{1}{2}e^3 - \frac{3}{2}e^2 + \frac{3}{2}e - \frac{1}{2} = \frac{1}{2}(e^3 - 3e^2 + 3e - 1)
\end{aligned}$$

11. $f(x, y, z) = xyz;\ \mathcal{W}: 0 \le z \le 1,\quad 0 \le y \le \sqrt{1-x^2},\quad 0 \le x \le 1$

SOLUTION $\mathcal{W}$ is the region between the planes $z = 0$ and $z = 1$, lying over the part $\mathcal{D}$ of the disk in the first quadrant.



Using Theorem 2, we compute the triple integral as the following iterated integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} xyz\,dV &= \iint_{\mathcal{D}} \left(\int_0^1 xyz\,dz\right) dA = \iint_{\mathcal{D}} \frac{xyz^2}{2}\Big|_{z=0}^{1} dA = \iint_{\mathcal{D}} \frac{xy}{2}\,dA \\
&= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \frac{xy}{2}\,dy\right) dx = \int_0^1 \frac{xy^2}{4}\Big|_{y=0}^{\sqrt{1-x^2}} dx = \int_0^1 \frac{x(1-x^2)}{4}\,dx \\
&= \int_0^1 \frac{x - x^3}{4}\,dx = \frac{x^2}{8} - \frac{x^4}{16}\Big|_0^1 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}
\end{aligned}$$

12. $f(x, y, z) = x;\quad \mathcal{W}: x^2 + y^2 \le z \le 4$

SOLUTION Here, $\mathcal{W}$ is the upper half of the solid cone underneath the plane $z = 4$. First we must determine the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane. First considering the intersection of the cone and the plane $z = 4$ we have:

$$x^2 + y^2 = 4$$

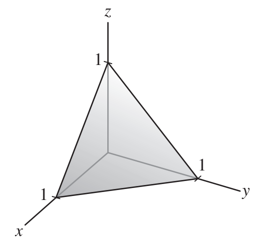
which is a circle centered at the origin with radius 2. The projection into the xy-plane is still $x^2 + y^2 = 4$.

The triple integral can be written as the following iterated integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} x\,dV &= \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{4} x\,dz\right) dA = \iint_{\mathcal{D}} xz\Big|_{z=x^2+y^2}^{4} dA \\
&= \iint_{\mathcal{D}} x(4 - x^2 - y^2)\,dA = \int_0^{2\pi} \int_0^2 r\cos\theta(4 - r^2\cos^2\theta - r^2\sin^2\theta)r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 4r^2\cos\theta - r^4\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{4}{3}r^3\cos\theta - \frac{1}{5}r^5\cos\theta\Big|_0^2 d\theta \\
&= \int_0^{2\pi} \frac{64}{15}\cos\theta\,d\theta = \frac{64}{15}\sin\theta\Big|_0^{2\pi} = 0
\end{aligned}$$

13. $f(x, y, z) = e^z$; $\mathcal{W} : x + y + z \le 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$

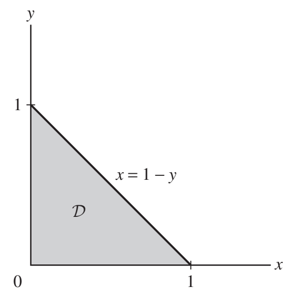
SOLUTION Notice that $\mathcal{W}$ is the tetrahedron under the plane $x + y + z = 1$ above the first quadrant.



First, we must determine the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane. The intersection of the plane $x + y + z = 1$ with the xy-plane is obtained by solving

$$\begin{aligned} x + y + z &= 1 \\ z &= 0 \end{aligned} \quad \Rightarrow \quad x + y = 1$$

Therefore, the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is the triangle enclosed by the line $x + y = 1$ and the positive axes.



The region $\mathcal{W}$ is the region between the planes $z = 1 - x - y$ and $z = 0$, lying above the triangle $\mathcal{D}$ in the xy-plane. The triple integral can be written as the following iterated integral:

$$\begin{aligned} \iiint_{\mathcal{W}} e^z \, dV &= \iint_{\mathcal{D}} \left(\int_0^{1-x-y} e^z \, dz \right) dA = \iint_{\mathcal{D}} e^z \Big|_{z=0}^{1-x-y} dA \\ &= \iint_{\mathcal{D}} \left(e^{1-x-y} - 1 \right) dA \\ &= \int_0^1 \left(\int_0^{1-y} \left(e^{1-x-y} - 1 \right) dx \right) dy = \int_0^1 -e^{1-x-y} - x \Big|_{x=0}^{1-y} dy \\ &= \int_0^1 -e^{1-1+y-y} - (1-y) + e^{1-y} \, dy \\ &= \int_0^1 e^{1-y} + y - 2 \, dy = -e^{1-y} + \frac{1}{2} y^2 - 2y \Big|_{y=0}^{1} \\ &= -1 + \frac{1}{2} - 2 - \left(-e^1 \right) = e - \frac{5}{2} \end{aligned}$$

14. $f(x, y, z) = z$; $\mathcal{W} : x^2 \le y \le 2$, $0 \le x \le 1$, $x - y \le z \le x + y$

SOLUTION The triple integral can be written as the following iterated integral:

$$\begin{aligned} \iiint_{\mathcal{W}} z \, dV &= \int_0^1 \int_{x^2}^2 \int_{x-y}^{x+y} z \, dz \, dy \, dx = \int_0^1 \int_{x^2}^2 \frac{1}{2} z^2 \Big|_{z=x-y}^{x+y} dy \, dx \\ &= \frac{1}{2} \int_0^1 \int_{x^2}^2 (x+y)^2 - (x-y)^2 \, dy \, dx = \frac{1}{2} \int_0^1 \int_{x^2}^2 4xy \, dy \, dx \\ &= \int_0^1 xy^2 \Big|_{y=x^2}^2 dx = \int_0^1 4x - x^5 \, dx \\ &= 2x^2 - \frac{1}{6} x^6 \Big|_0^1 = 2 - \frac{1}{6} = \frac{11}{6} \end{aligned}$$

15. Calculate the integral of $f(x, y, z) = z$ over the region $\mathcal{W}$ in Figure 10 below the hemisphere of radius 3 and lying over the triangle $\mathcal{D}$ in the xy-plane bounded by $x = 1$, $y = 0$, and $x = y$.

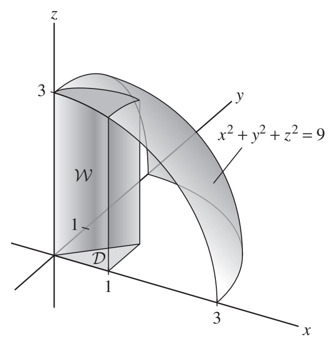
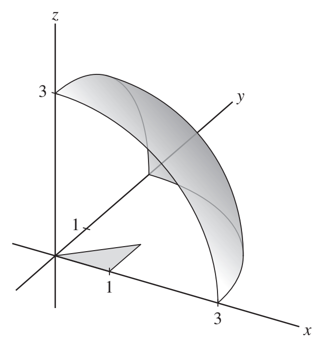
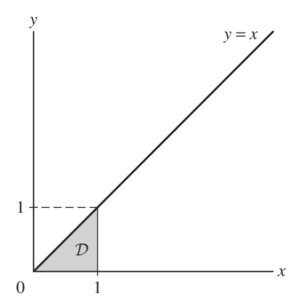


FIGURE 10

SOLUTION



The upper surface is the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the lower surface is the xy-plane $z = 0$. The projection of $\mathcal{V}$ onto the xy-plane is the triangle $\mathcal{D}$ shown in the figure.



We compute the triple integral as the following iterated integral:

$$\iiint_{\mathcal{V}} z\,dV = \iint_{\mathcal{D}} \left(\int_0^{\sqrt{9-x^2-y^2}} z\,dz \right) dA = \iint_{\mathcal{D}} \left. \frac{z^2}{2} \right|_0^{\sqrt{9-x^2-y^2}} dA = \iint_{\mathcal{D}} \frac{9 - x^2 - y^2}{2}\,dA$$

$$= \int_0^1 \left(\int_0^x \frac{9 - x^2 - y^2}{2}\,dy \right) dx = \int_0^1 \left. \frac{9y - x^2 y - \frac{y^3}{3}}{2} \right|_{y=0}^{x} dx = \int_0^1 \left(\frac{9x}{2} - \frac{2x^3}{3} \right) dx$$

$$= \left. \frac{9x^2}{4} - \frac{x^4}{6} \right|_0^1 = 2\frac{1}{12}$$

16. Calculate the integral of $f(x, y, z) = e^{2z}$ over the tetrahedron $\mathcal{W}$ in Figure 11.

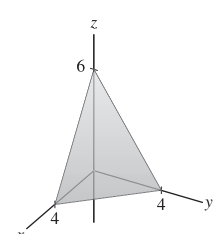
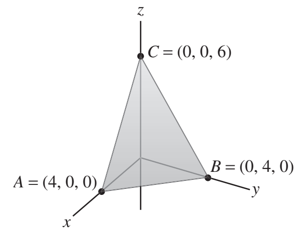


FIGURE 11

SOLUTION We first must find the equation of the upper surface, which is the plane through the points $A = (4, 0, 0)$, $B = (0, 4, 0)$, and $C = (0, 0, 6)$.



A vector normal to the plane is the following cross product:

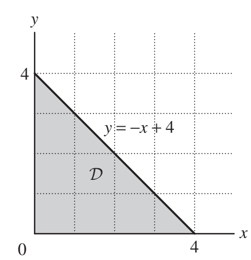
$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -4, 4, 0\rangle \times \langle -4, 0, 6\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 4 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 24\mathbf{i} + 24\mathbf{j} + 16\mathbf{k} = 8\,(3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

The equation of the plane with normal $3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ passing through $A = (4, 0, 0)$ is

$$3(x-4) + 3y + 2z = 0$$
$$3x + 3y + 2z = 12 \quad \Rightarrow \quad z = 6 - \frac{3}{2}x - \frac{3}{2}y$$

Thus, the tetrahedron $\mathcal{V}$ lies between the planes $z = 6 - \frac{3x}{2} - \frac{3y}{2}$ and $z = 0$. The projection $\mathcal{D}$ of $\mathcal{V}$ onto the xy-plane is the triangle enclosed by the line AB and the positive axes. We compute the equation of the line AB:

$$y - 0 = \frac{4-0}{0-4}(x-4) \quad \Rightarrow \quad y = -x + 4$$



We now compute the triple integral of f over $\mathcal{V}$ as the following iterated integral:

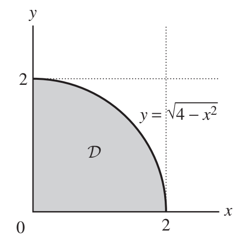
$$\iiint_{\mathcal{V}} e^{2z}\,dV = \iint_{\mathcal{D}} \left(\int_0^{6-\frac{3x}{2}-\frac{3y}{2}} e^{2z}\,dz\right) dA = \frac{1}{2}\iint_{\mathcal{D}} e^{2z}\Big|_{z=0}^{6-\frac{3x}{2}-\frac{3y}{2}} dA$$
$$= \frac{1}{2}\iint_{\mathcal{D}} e^{12-3x-3y} - 1\,dA = \frac{1}{2}\int_0^4 \int_0^{4-x} e^{12-3x-3y} - 1\,dy\,dx$$
$$= \frac{1}{2}\int_0^4 -\frac{1}{3}e^{12-3x-3y} - y\Big|_{y=0}^{4-x} dx = \frac{1}{2}\int_0^4 -\frac{1}{3}e^{12-3x-3(4-x)} - (4-x) + \frac{1}{3}e^{12-3x}\,dx$$
$$= \frac{1}{2}\int_0^4 x - \frac{13}{3} + \frac{1}{3}e^{12-3x}\,dx = \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{13}{3}x - \frac{1}{9}e^{12-3x}\right)\Big|_0^4$$
$$= \frac{1}{2}\left(8 - \frac{52}{3} - \frac{1}{9}\right) - \frac{1}{2}\left(-\frac{1}{9}e^{12}\right) = \frac{1}{18}e^{12} - \frac{85}{18} \approx 9037$$

17. Integrate $f(x, y, z) = x$ over the region in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ above $z = y^2$ and below $z = 8 - 2x^2 - y^2$.

SOLUTION We first find the projection of the region $\mathcal{W}$ onto the xy-plane. We find the curve of intersection between the upper and lower surfaces, by solving the following equation for $x, y \ge 0$:

$$8 - 2x^2 - y^2 = y^2 \quad \Rightarrow \quad y^2 = 4 - x^2 \quad \Rightarrow \quad y = \sqrt{4 - x^2},\, x \ge 0$$

The projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is the region bounded by the circle $x^2 + y^2 = 4$ and the positive axes.



We now compute the triple integral over $\mathcal{W}$ by evaluating the following iterated integral:

$$\begin{aligned}\iiint_{\mathcal{W}} x\,dV &= \iint_{\mathcal{D}} \left(\int_{y^2}^{8-2x^2-y^2} x\,dz\right) dA = \iint_{\mathcal{D}} xz\Big|_{z=y^2}^{8-2x^2-y^2} dA \\ &= \iint_{\mathcal{D}} x(8-2x^2-y^2-y^2)\,dA = \iint_{\mathcal{D}} 8x-2x^3-2xy^2\,dA \\ &= \int_0^2\int_0^{\sqrt{4-x^2}} 8x-2x^3-2xy^2\,dy\,dx = \int_0^2 8xy-2x^3y-x^2y^2\Big|_{y=0}^{\sqrt{4-x^2}} dx \\ &= \int_0^2 8x\sqrt{4-x^2}\,dx - \int_0^2 2x^3\sqrt{4-x^2}\,dx - \int_0^2 x^2(4-x^2)\,dx \\ &= 8\int_0^2 x\sqrt{4-x^2}\,dx - 2\int_0^2 x^3\sqrt{4-x^2}\,dx - \int_0^2 4x^2-x^4\,dx\end{aligned}$$

The first and third integrals are easily computing using u-substitution and term by term integration, respectively. The second integral requires a clever u-substitution, let $u = 4-x^2$, then $du = -2x\,dx$ and $x^2 = 4-u$. Using this information we see

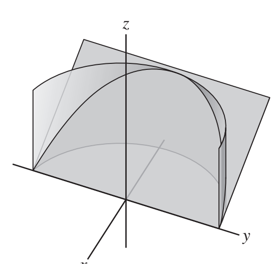
$$\begin{aligned}-2\int_0^2 x^3\sqrt{4-x^2}\,dx &= -2\int_0^2 x\cdot x^2\sqrt{4-x^2}\,dx \\ &= \int_{u=4}^0 (4-u)\sqrt{u}\,du \\ &= \int_4^0 4\sqrt{u}-u^{3/2}\,du \\ &= \frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\Big|_{u=4}^0 \\ &= -\left(\frac{64}{3}-\frac{64}{5}\right) = -\frac{128}{15}\end{aligned}$$

Hence,

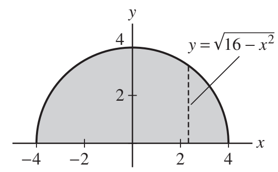
$$\begin{aligned}\iiint_{\mathcal{W}} x\,dV &= 8\int_0^2 x\sqrt{4-x^2}\,dx - 2\int_0^2 x^3\sqrt{4-x^2}\,dx - \int_0^2 4x^2-x^4\,dx \\ &= 8\int_0^2 x\sqrt{4-x^2}\,dx - \frac{128}{15} - \int_0^2 4x^2-x^4\,dx \\ &= 8\cdot -\frac{1}{3}(4-x^2)^{3/2}\Big|_0^2 - \frac{128}{15} - \left(\frac{4}{3}x^3-\frac{1}{5}x^5\Big|_0^2\right) \\ &= -\frac{8}{3}(0-8) - \frac{128}{15} - \left(\frac{32}{3}-\frac{32}{5}\right) = \frac{128}{15}\end{aligned}$$

18. Compute the integral of $f(x,y,z) = y^2$ over the region within the cylinder $x^2+y^2=4$ where $0 \le z \le y$.

SOLUTION



The upper surface is the plane $z = y$ and the lower surface is the plane $z = 0$. The region of integration, $\mathcal{W}$, projects onto the domain $\mathcal{D}$ bounded by the semicircle and the x-axis.



The triple integral of f over $\mathcal{W}$ is equal to the following iterated integral:

$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,dx &= \iint_{\mathcal{D}}\left(\int_0^y y^2\,dz\right)dA = \iint_{\mathcal{D}} y^2 z\Big|_{z=0}^{y}\,dA\\ &= \iint_{\mathcal{D}} y^3\,dA = \int_{-4}^{4}\int_0^{\sqrt{16-x^2}} y^3\,dy\,dx\\ &= \int_{-4}^{4}\frac{y^4}{4}\Big|_{y=0}^{\sqrt{16-x^2}}dx = \frac{1}{4}\int_{-4}^{4}(16-x^2)^2\,dx\\ &= \frac{1}{4}\int_{-4}^{4}256-32x^2+x^4\,dx = 256x-\frac{32}{3}x^3+\frac{1}{5}x^5\Big|_{-4}^{4}\\ &= \frac{1}{4}\left(256\cdot 4-\frac{32}{3}\cdot 64-\frac{1024}{5}\right)-\frac{1}{4}\left(256\cdot -4+\frac{32}{3}\cdot 64-\frac{1024}{5}\right)\\ &= \frac{4096}{15}\approx 273.067\end{aligned}$$

19. Find the triple integral of the function z over the ramp in Figure 12. Here, z is the height above the ground.

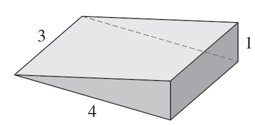
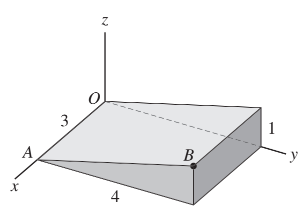


FIGURE 12

SOLUTION We place the coordinate axes as shown in the figure:



The upper surface is the plane passing through the points $O = (0,0,0)$, $A = (3,0,0)$, and $B = (3,4,1)$. We find a normal to this plane and then determine the equation of the plane. We get

$$\overrightarrow{OA}\times\overrightarrow{AB} = \langle 3,0,0\rangle\times\langle 0,4,1\rangle = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 0 & 0\\ 0 & 4 & 1\end{vmatrix} = -3\mathbf{j}+12\mathbf{k} = 3(-\mathbf{j}+4\mathbf{k})$$

The plane is orthogonal to the vector $\langle 0,-1,4\rangle$ and passes through the origin, hence the equation of the plane is

$$0\cdot x - y + 4z = 0 \quad\Rightarrow\quad z = \frac{y}{4}$$

The projection of the region of integration $\mathcal{W}$ onto the xy-plane is the rectangle $\mathcal{D}$ defined by

$$0\le x\le 3,\quad 0\le y\le 4.$$

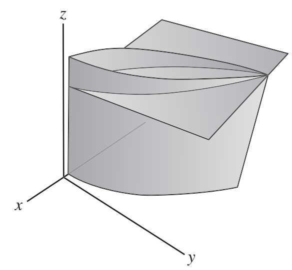
We now compute the triple integral of f over $\mathcal{W}$, as the following iterated integral:

$$\iiint_{\mathcal{W}} z\,dV = \iint_{\mathcal{D}} \left(\int_0^{y/4} z\,dz \right) dA = \iint_{\mathcal{D}} \frac{z^2}{2} \Big|_{z=0}^{y/4} dA = \iint_{\mathcal{D}} \frac{y^2}{32}\,dA = \int_0^4 \left(\int_0^3 \frac{y^2}{32}\,dx \right) dy$$

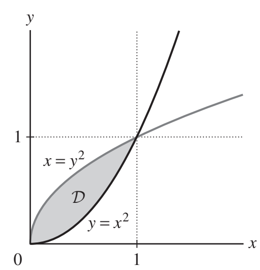
$$= \int_0^4 \frac{y^2 x}{32} \Big|_{x=0}^{3} dy = \int_0^4 \frac{3y^2}{32}\,dy = \frac{y^3}{32} \Big|_0^4 = \frac{4^3}{32} = 2$$

20. Find the volume of the solid in $\mathbf{R}^3$ bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

SOLUTION The solid $\mathcal{W}$ is shown in the following figure:



The upper surface is the plane $z = x + y + 5$ and the lower surface is the plane $z = 0$. The projection of $\mathcal{W}$ onto the xy-plane is the region in the first quadrant enclosed by the curves $y = x^2$ and $x = y^2$.



We use the formula for the volume as a triple integral to write

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\,dV$$

The triple integral is equal to the following iterated integral:

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\,dV = \iint_{\mathcal{D}} \left(\int_0^{x+y+5} 1\,dz \right) dA = \iint_{\mathcal{D}} z \Big|_{z=0}^{x+y+5} dA$$

$$= \iint_{\mathcal{D}} (x + y + 5)\,dA = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (x + y + 5)\,dy \right) dx = \int_0^1 xy + \frac{y^2}{2} + 5y \Big|_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x\sqrt{x} + \frac{x}{2} + 5\sqrt{x} - \left(x^3 + \frac{x^4}{2} + 5x^2 \right) \right) dx$$

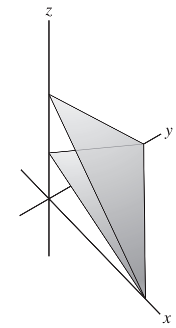
$$= \int_0^1 \left(-\frac{x^4}{2} - x^3 - 5x^2 + x^{3/2} + \frac{x}{2} + 5x^{1/2} \right) dx$$

$$= -\frac{x^5}{10} - \frac{x^4}{4} - \frac{5x^3}{3} + \frac{2}{5}x^{5/2} + \frac{x^2}{4} + \frac{10}{3}x^{3/2} \Big|_0^1$$

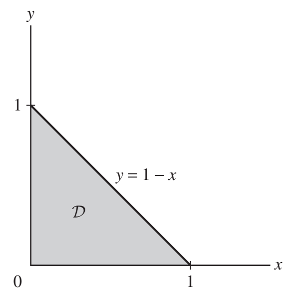
$$= -\frac{1}{10} - \frac{1}{4} - \frac{5}{3} + \frac{2}{5} + \frac{1}{4} + \frac{10}{3} = \frac{59}{30} = 1\frac{29}{30}$$

21. Find the volume of the solid in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

SOLUTION The solid $\mathcal{W}$ is shown in the figure:



The upper and lower surfaces are the planes $x + y + z = 1$ (or $z = 1 - x - y$) and $x + y + 2z = 1$ (or $z = \frac{1-x-y}{2}$), respectively. The projection of $\mathcal{W}$ onto the xy-plane is the triangle enclosed by the line $AB : y = 1 - x$ and the positive x and y-axes.

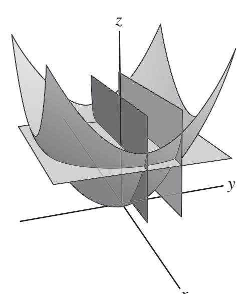


Using the volume of a solid as a triple integral, we have

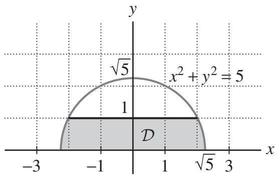
$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\, dV = \iint_{\mathcal{D}} \left(\int_{(1-x-y)/2}^{1-x-y} 1\, dz \right) dA = \iint_{\mathcal{D}} z \Big|_{z=(1-x-y)/2}^{1-x-y} dA \\
&= \iint_{\mathcal{D}} \left((1-x-y) - \frac{1-x-y}{2} \right) dA = \iint_{\mathcal{D}} \frac{1-x-y}{2}\, dA \\
&= \int_0^1 \left(\int_0^{1-x} \frac{1-x-y}{2}\, dy \right) dx = \int_0^1 \frac{y - xy - \frac{y^2}{2}}{2} \Big|_{y=0}^{1-x} dx \\
&= \int_0^1 \frac{1 - x - x(1-x) - \frac{(1-x)^2}{2}}{2}\, dx = \frac{1}{2} \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx \\
&= \frac{1}{2} \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{1}{2}x \right) \Big|_0^1 = \frac{1}{2} \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{12}
\end{aligned}$$

22. Calculate $\iiint_{\mathcal{W}} y\, dV$, where $\mathcal{W}$ is the region above $z = x^2 + y^2$ and below $z = 5$, and bounded by $y = 0$ and $y = 1$.

SOLUTION The region $\mathcal{W}$ is shown in the figure:



The upper surface is the plane $z = 5$ and the lower surface is the paraboloid $z = x^2 + y^2$. The projection of $\mathcal{W}$ onto the xy-plane is the part of the disk $x^2 + y^2 \le 5$ between the lines $y = 0$ and $y = 1$.



The triple integral of $f(x, y, z) = y$ over $\mathcal{W}$ is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} y\,dV = \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{5} y\,dz\right) dA = \iint_{\mathcal{D}} yz\Big|_{z=x^2+y^2}^{5} dA = \iint_{\mathcal{D}} y\left(5 - x^2 - y^2\right) dA$$

$$= \int_0^1 \left(\int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y\left(5 - x^2 - y^2\right) dx\right) dy = 2\int_0^1 y\left(5x - \frac{x^3}{3} - y^2 x\right)\Big|_{x=0}^{\sqrt{5-y^2}} dy$$

$$= 2\int_0^1 y\left(\left(5 - y^2\right)x - \frac{x^3}{3}\right)\Big|_{x=0}^{\sqrt{5-y^2}} dy = 2\int_0^1 y\left(\left(5 - y^2\right)^{3/2} - \frac{1}{3}\left(5 - y^2\right)^{3/2}\right) dy$$

$$= \int_0^1 \frac{4}{3}\left(5 - y^2\right)^{3/2} y\,dy \qquad (1)$$

We compute the integral using the substitution $u = 5 - y^2$, $du = -2y\,dy$:

$$\iiint_{\mathcal{W}} y\,dV = \int_0^1 \frac{4}{3}\left(5 - y^2\right)^{3/2} y\,dy = \int_5^4 -\frac{2}{3}u^{3/2}\,du = \int_4^5 \frac{2}{3}u^{3/2}\,du = \frac{4}{15}u^{5/2}\Big|_4^5$$

$$= \frac{4}{15}\left(5^{5/2} - 4^{5/2}\right) \approx 6.37$$

23. Evaluate $\iiint_{\mathcal{W}} xz\,dV$, where $\mathcal{W}$ is the domain bounded by the elliptic cylinder $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and the sphere $x^2 + y^2 + z^2 = 16$ in the first octant $x \ge 0$, $y \ge 0$, $z \ge 0$ (Figure 13).

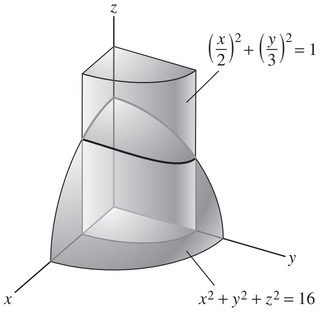
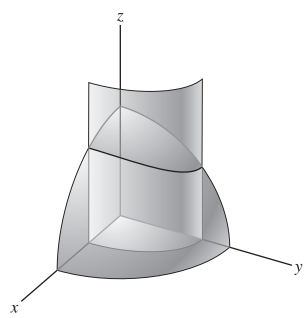
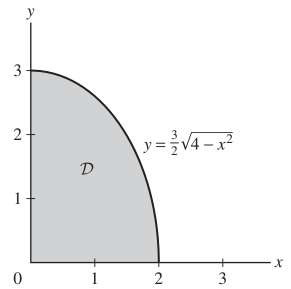


FIGURE 13

SOLUTION



The upper surface is the sphere $x^2 + y^2 + z^2 = 16$, or $z = \sqrt{16 - x^2 - y^2}$, and the lower surface is the xy-plane, $z = 0$. The projection of $\mathcal{W}$ onto the xy-plane is the region in the first quadrant bounded by the ellipse $(x/2)^2 + (y/3)^2 = 1$, or $y = \frac{3}{2}\sqrt{4 - x^2}$.



Therefore, the triple integral over $\mathcal{W}$ is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} xz\,dV = \iint_{\mathcal{D}} \left(\int_0^{\sqrt{16-x^2-y^2}} xz\,dz\right) dA = \iint_{\mathcal{D}} \frac{1}{2}xz^2\Big|_{z=0}^{\sqrt{16-x^2-y^2}} dA$$
$$= \frac{1}{2}\iint_{\mathcal{D}} x(16-x^2-y^2)\,dA = \frac{1}{2}\int_0^2 \int_0^{\frac{3}{2}\sqrt{4-x^2}} 16x - x^3 - xy^2\,dy\,dx$$
$$= \frac{1}{2}\int_0^2 16xy - x^3y - \frac{1}{3}xy^3\Big|_{y=0}^{\frac{3}{2}\sqrt{4-x^2}}$$
$$= \frac{1}{2}\int_0^2 24x\sqrt{4-x^2} - \frac{3}{2}x^3\sqrt{4-x^2} - \frac{9}{8}x(4-x^2)^{3/2}\,dx$$
$$= 12\int_0^2 x\sqrt{4-x^2}\,dx - \frac{3}{4}\int_0^2 x^3\sqrt{4-x^2}\,dx - \frac{9}{16}\int_0^2 x(4-x^2)^{3/2}\,dx$$

The first and third integrals are simple u-substitution problems. For the second integral, let us use $u = 4 - x^2$ and thus $du = -2x\,dx$ and $x^2 = 4 - u$. Thus, we can write

$$-\frac{3}{4}\int_0^2 x^3\sqrt{4-x^2}\,dx = -\frac{3}{4}\int_0^2 x \cdot x^2\sqrt{4-x^2}\,dx$$
$$= \frac{3}{8}\int_{u=4}^0 (4-u)\sqrt{u}\,du$$
$$= \frac{3}{8}\int_4^0 4u^{1/2} - u^{3/2}\,du$$
$$= \frac{3}{8}\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big|_{u=4}^0\right)$$
$$= -\frac{3}{8}\left(\frac{64}{3} - \frac{64}{5}\right) = -\frac{16}{5}$$

Therefore we have:

$$\iiint_{\mathcal{W}} xz\,dV = 12\int_0^2 x\sqrt{4-x^2}\,dx - \frac{3}{4}\int_0^2 x^3\sqrt{4-x^2}\,dx - \frac{9}{16}\int_0^2 x(4-x^2)^{3/2}$$
$$= 12\int_0^2 x\sqrt{4-x^2}\,dx - \frac{16}{5} - \frac{9}{16}\int_0^2 x(4-x^2)^{3/2}\,dx$$
$$= -6\cdot\frac{2}{3}\left((4-x^2)^{3/2}\Big|_0^2\right) - \frac{16}{5} + \frac{9}{32}\cdot\frac{2}{5}\left((4-x^2)^{5/2}\Big|_0^2\right)$$
$$= -4\,(0-8) - \frac{16}{5} + \frac{9}{80}\,(0-32) = \frac{126}{5}$$

24. Describe the domain of integration and evaluate:

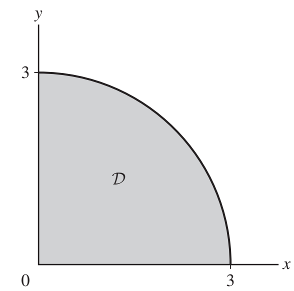
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy\,dz\,dy\,dx$$

SOLUTION The domain of integration $\mathcal{W}$ is defined by the following inequalities:

$$0 \le x \le 3, \quad 0 \le y \le \sqrt{9-x^2}, \quad 0 \le z \le \sqrt{9-x^2-y^2}$$

This domain is the simple region in which the upper surface is the hemisphere of radius 3, $z = \sqrt{9 - x^2 - y^2}$, and the lower surface is the plane $z = 0$. The projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is defined by the inequalities

$$0 \le x \le 3, \quad 0 \le y \le \sqrt{9 - x^2}$$



$\mathcal{D}$ is the part of the disk of radius 3 that lies in the first quadrant. We conclude that $\mathcal{W}$ is the part of the ball of radius 3 in the first octant. We evaluate the triple integral as the following iterated integral:

$$\iiint_{\mathcal{W}} xy\, dV = \iint_{\mathcal{D}} \left(\int_0^{\sqrt{9-x^2-y^2}} xy\, dz \right) = \iint_{\mathcal{D}} xyz \Big|_{z=0}^{\sqrt{9-x^2-y^2}} dA = \iint_{\mathcal{D}} xy\sqrt{9 - x^2 - y^2}\, dA$$

$$= \int_0^3 \left(x \int_0^{\sqrt{9-x^2}} \sqrt{9 - x^2 - y^2} \cdot y\, dy \right) dx \tag{1}$$

We evaluate the inner integral using the substitution $u = 9 - x^2 - y^2$, $du = -2y\, dy$. The upper limit of integration is $u = 9 - x^2 - (9 - x^2) = 0$ and the lower limit is $u = 9 - x^2 - 0^2 = 9 - x^2$. Therefore,

$$\int_0^{\sqrt{9-x^2}} \sqrt{9 - x^2 - y^2} \cdot y\, dy = \int_{9-x^2}^0 u^{1/2} \left(-\frac{du}{2} \right) = \frac{1}{2} \int_0^{9-x^2} u^{1/2}\, du$$

$$= \frac{1}{3} u^{3/2} \Big|_{u=0}^{9-x^2} = \frac{(9 - x^2)^{3/2}}{3}$$

We substitute in (1) and use the substitution $u = 9 - x^2$, $du = -2x\, dx$ to compute the resulting integral:

$$\iiint_{\mathcal{W}} xy\, dV = \int_0^3 \frac{1}{3} x (9 - x^2)^{3/2}\, dx = \int_9^0 \frac{1}{3} u^{3/2} \left(\frac{du}{-2} \right) = -\int_9^0 \frac{1}{6} u^{3/2}\, du$$

$$= \frac{1}{6} \cdot \frac{2}{5} u^{5/2} \Big|_0^9 = \frac{1}{15} \cdot 9^{5/2} = \frac{3^5}{15} = \frac{243}{15} = 16.2$$

25. Describe the domain of integration of the following integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_1^{\sqrt{5-x^2-z^2}} f(x, y, z)\, dy\, dx\, dz$$

SOLUTION The domain of integration of $\mathcal{W}$ is defined by the following inequalities:

$$-2 \le z \le 2, \quad -\sqrt{4 - z^2} \le x \le \sqrt{4 - z^2}, \quad 1 \le y \le \sqrt{5 - x^2 - z^2}$$

This region is bounded by the plane $y = 1$ and the sphere $y^2 = 5 - x^2 - z^2$ or $x^2 + y^2 + z^2 = 5$, lying over the disk $x^2 + z^2 \le 4$ in the xz-plane. This is the central cylinder oriented along the y-axis of radius 2 inside a sphere of radius $\sqrt{5}$.

26. Let $\mathcal{W}$ be the region below the paraboloid

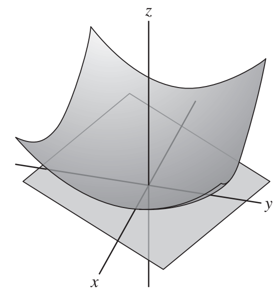
$$x^2 + y^2 = z - 2$$

that lies above the part of the plane $x + y + z = 1$ in the first octant ($x \ge 0$, $y \ge 0$, $z \ge 0$). Express

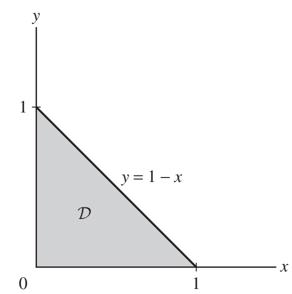
$$\iiint_{\mathcal{W}} f(x, y, z)\, dV$$

as an iterated integral (for an arbitrary function f).

SOLUTION The region of integration is shown in the figure:



The upper surface is $z = 2 + x^2 + y^2$ and the lower surface is the plane $z = 1 - x - y$. The projection $\mathcal{D}$ of W into the xy-plane is the triangle enclosed by the line $y = 1 - x$ (found by setting $z = 0$ in the equation $x + y + z = 1$) and the positive x and y axes.



Therefore, the triple integral over $\mathcal{W}$ is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \iint_{\mathcal{D}} \left(\int_{1-x-y}^{2+x^2+y^2} f(x, y, z)\, dz \right) dA = \int_0^1 \int_0^{1-x} \int_{1-x-y}^{2+x^2+y^2} f(x, y, z)\, dz\, dy\, dx$$

27. In Example 5, we expressed a triple integral as an iterated integral in the three orders

$$dz\, dy\, dx, \quad dx\, dz\, dy, \quad \text{and} \quad dy\, dz\, dx$$

Write this integral in the three other orders:

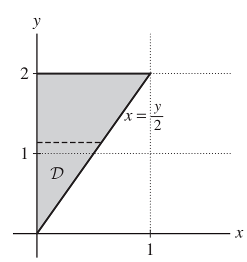
$$dz\, dx\, dy, \quad dx\, dy\, dz, \quad \text{and} \quad dy\, dx\, dz$$

SOLUTION In Example 5 we considered the triple integral $\iiint_{\mathcal{W}} xyz^2\, dV$, where $\mathcal{W}$ is the region bounded by

$$z = 4 - y^2, \quad z = 0, \quad y = 2x, \quad x = 0.$$

We now write the triple integral as an iterated integral in the orders $dz\, dx\, dy$, $dx\, dy\, dz$, and $dy\, dx\, dz$.

- $dz\, dx\, dy$: The upper surface $z = 4 - y^2$ projects onto the xy-plane on the triangle defined by the lines $y = 2$, $y = 2x$, and $x = 0$.

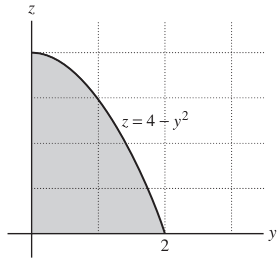


We express the line $y = 2x$ as $x = \frac{y}{2}$ and write the triple integral as

$$\iiint_{\mathcal{W}} xyz^2\, dV = \iint_{\mathcal{D}} \left(\int_0^{4-y^2} xyz^2\, dz \right) dA = \int_0^2 \int_0^{y/2} \int_0^{4-y^2} xyz^2\, dz\, dx\, dy$$

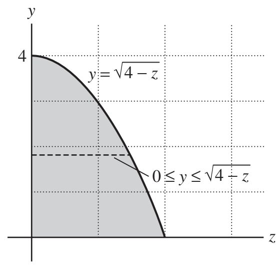
- $dx\,dy\,dz$: The projection of $\mathcal{W}$ onto the yz-plane is the domain T (see Example 5) defined by the inequalities

$$T:\ 0 \le y \le 2,\ 0 \le z \le 4 - y^2$$



This region can also be expressed as

$$0 \le z \le 4, \quad 0 \le y \le \sqrt{4-z}$$



As explained in Example 5, the region $\mathcal{W}$ consists of all points lying between T and the "left-face" $y = 2x$, or $x = \frac{y}{2}$. Therefore, we obtain the following inequalities for $\mathcal{W}$:

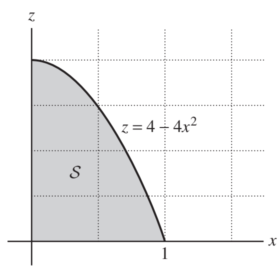
$$0 \le z \le 4, \quad 0 \le y \le \sqrt{4-z}, \quad 0 \le x \le \frac{1}{2}y$$

This yields the following iterated integral:

$$\iiint_{\mathcal{W}} xyz^2\,dV = \iint_T \left(\int_0^{y/2} xyz^2\,dx\right) dA = \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{y/2} xyz^2\,dx\,dy\,dz$$

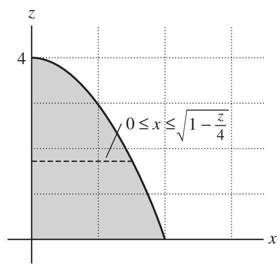
- $dy\,dx\,dz$: As explained in Example 5, the projection of $\mathcal{W}$ onto the xz-plane is determined by the inequalities

$$S:\ 0 \le x \le 1,\ 0 \le z \le 4 - 4x^2$$



This region can also be described if we write x as a function of z:

$$z = 4 - 4x^2 \quad \Rightarrow \quad 4x^2 = 4 - z \quad \Rightarrow \quad x = \sqrt{1 - \frac{z}{4}}$$



This gives the following inequalities of S:

$$S: \ 0 \le z \le 4, \ 0 \le x \le \sqrt{1 - \frac{z}{4}}$$

The upper surface $z = 4 - y^2$ can be described by $y = \sqrt{4 - z}$, hence the limits of y are $2x \le y \le \sqrt{4 - z}$. We obtain the following iterated integral:

$$\iiint_{\mathcal{W}} xyz^2 \, dV = \iint_S \left(\int_{2x}^{\sqrt{4-z}} xyz^2 \, dy \right) dA = \int_0^4 \int_0^{\sqrt{1-\frac{z}{4}}} \int_{2x}^{\sqrt{4-z}} xyz^2 \, dy \, dx \, dz$$

28. Let $\mathcal{W}$ be the region bounded by

$$y + z = 2, \quad 2x = y, \quad x = 0, \quad \text{and } z = 0$$

(Figure 14). Express and evaluate the triple integral of $f(x, y, z) = z$ by projecting $\mathcal{W}$ onto the:

(a) xy-plane **(b)** yz-plane **(c)** xz-plane

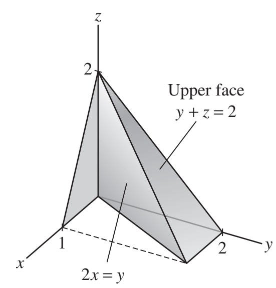
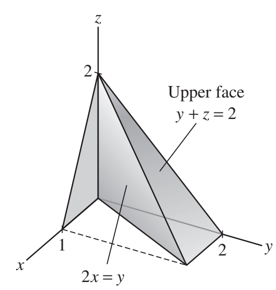


FIGURE 14

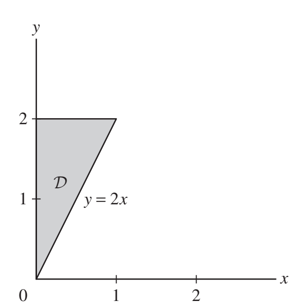
SOLUTION

(a)



The upper face $z = 2 - y$ intersects the first quadrant of the xy-plane ($z = 0$) in the line $y = 2$. Therefore the projection of $\mathcal{W}$ onto the xy-plane is the triangle $\mathcal{D}$ defined by

$$\mathcal{D}: \ 0 \le x \le 1, \ 2x \le y \le 2$$



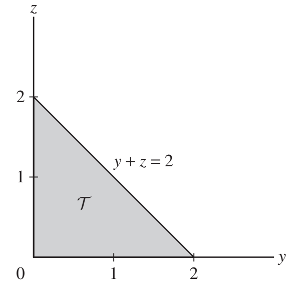
Therefore, $\mathcal{W}$ is the following region:

$$0 \le x \le 1, \quad 2x \le y \le 2, \quad 0 \le z \le 2 - y$$

We obtain the following iterated integral:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^1 \int_{2x}^2 \int_0^{2-y} z\,dz\,dy\,dx = \int_0^1 \int_{2x}^2 \frac{1}{2}z^2\bigg|_{z=0}^{2-y} dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_{2x}^2 (2-y)^2\,dy\,dx = \frac{1}{2}\int_0^1 -\frac{1}{3}(2-y)^3\bigg|_{2x}^2 dx$$
$$= -\frac{1}{6}\int_0^1 -(2-2x)^3\,dx = \frac{1}{6}\cdot -\frac{1}{8}(2-2x)^4\bigg|_0^1$$
$$= -\frac{1}{48}(0) + \frac{1}{48}\cdot 16 = \frac{1}{3}$$

(b) The projection of $\mathcal{W}$ onto the yz-plane is the triangle defined by the line $y+z=2$ and the positive y and z axes.



That is,

$$T:\ 0 \le y \le 2,\ 0 \le z \le 2-y$$

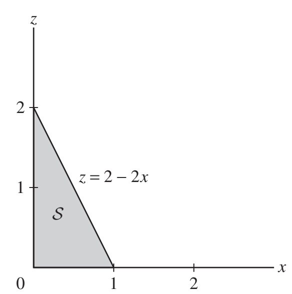
The region $\mathcal{W}$ consists of all points lying between T and the "left-face" $y=2x$. Therefore, $\mathcal{W}$ is defined by the following inequalities:

$$\mathcal{W}:\ 0 \le y \le 2,\ 0 \le z \le 2-y,\ 0 \le x \le \frac{y}{2}$$

We obtain the following iterated integral:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^2 \int_0^{2-y} \int_0^{y/2} z\,dx\,dz\,dy = \int_0^2 \int_0^{2-y} xz\bigg|_{x=0}^{y/2} dz\,dy$$
$$= \int_0^2 \int_0^{2-y} \frac{1}{2}yz\,dz\,dy = \frac{1}{2}\int_0^2 \frac{1}{2}yz^2\bigg|_{z=0}^{2-y} dy$$
$$= \frac{1}{4}\int_0^2 y(2-y)^2\,dy = \frac{1}{4}\int_0^2 4y - 4y^2 + y^3\,dy$$
$$= \frac{1}{4}\left(2y^2 - \frac{4}{3}y^3 - \frac{1}{4}y^4\bigg|_0^2\right) = \frac{1}{4}\left(8 - \frac{4}{3}\cdot 8 + 4\right) = \frac{1}{3}$$

(c) We first find the points on the intersection of the faces $2x-y=0$ and $y+z=2$. These are the points $(x, 2x, 2-2x)$. Therefore, the projection of this intersection onto the xz-plane consists of the points $(x, 0, 2-2x)$. That is, the projection of $\mathcal{W}$ onto the xz-plane is the triangles defined by the line $z=2-2x$ and the positive x and z axes.



The corresponding inequalities are

$$S:\ 0 \le x \le 1,\ 0 \le z \le 2-2x$$

The limits for y are $2x \le y \le 2 - z$, where $y = 2 - z$ is obtained from the equation $y + z = 2$ of the upper face. We obtain the following iterated integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} z\,dV &= \int_0^1 \int_0^{2-2x} \int_{2x}^{2-z} z\,dy\,dz\,dx = \int_0^1 \int_0^{2-2x} yz\Big|_{y=2x}^{2-z} dz\,dx \\
&= \int_0^1 \int_0^{2-2x} (2-z)z - 2xz\,dz\,dx = \int_0^1 \int_0^{2-2x} 2z - z^2 - 2xz\,dz\,dx \\
&= \int_0^1 z^2 - \frac{1}{3}z^3 - xz^2\Big|_{z=0}^{2-2x} dx = \int_0^1 (2-2x)^2 - \frac{1}{3}(2-2x)^3 - x(2-2x)^2\,dx \\
&= \int_0^1 \frac{4}{3} - 4x - \frac{4}{3}x^3 + 4x^2\,dx = \frac{4}{3}x - 2x^2 - \frac{1}{3}x^3 + \frac{4}{3}x^3\Big|_0^1 \\
&= \frac{4}{3} - 2 - \frac{1}{3} + \frac{4}{3} = \frac{1}{3}
\end{aligned}$$

The three answers agree as expected.

29. Let

$$\mathcal{W} = \left\{(x, y, z) : \sqrt{x^2 + y^2} \le z \le 1\right\}$$

(see Figure 15). Express $\iiint_{\mathcal{W}} f(x, y, z)\,dV$ as an iterated integral in the order $dz\,dy\,dx$ (for an arbitrary function f).

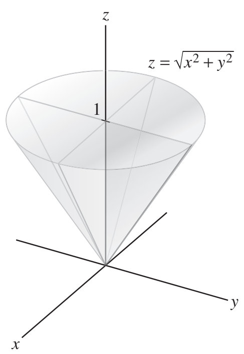
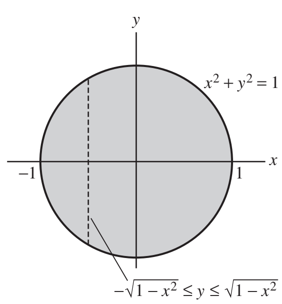


FIGURE 15

SOLUTION To express the triple integral as an iterated integral in order $dz\,dy\,dx$, we must find the projection of $\mathcal{W}$ onto the xy-plane. The upper circle is $\sqrt{x^2 + y^2} = 1$ or $x^2 + y^2 = 1$, hence the projection of $\mathcal{W}$ onto the xy plane is the disk

$$\mathcal{D} : x^2 + y^2 \le 1$$

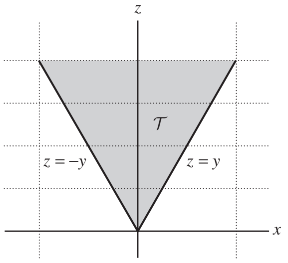


The upper surface is the plane $z = 1$ and the lower surface is $z = \sqrt{x^2 + y^2}$, therefore the triple integral over $\mathcal{W}$ is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} f(x, y, z)\,dV = \iint_{\mathcal{D}} \left(\int_{\sqrt{x^2+y^2}}^1 f(x, y, z)\,dz\right) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f(x, y, z)\,dz\,dy\,dx$$

30. Repeat Exercise 29 for the order $dx\,dy\,dz$.

SOLUTION To express the triple integral as an iterated integral in order $dx\,dy\,dz$, we must find the projection of $\mathcal{W}$ onto the yz-plane. The points on the surface $z = \sqrt{x^2 + y^2}$ are $\left(x, y, \sqrt{x^2 + y^2}\right)$, hence the projection on the yz-plane consists of the points $(0, y, |y|)$, that is, $z = |y|$.



Therefore, the projection of $\mathcal{W}$ onto the yz-plane is the triangle defined by the lines $z = |y|$ and $z = 1$. The region $\mathcal{W}$ consists of all points lying between T and the surface $z = \sqrt{x^2 + y^2}$, or $x = \pm\sqrt{z^2 - y^2}$. $\mathcal{W}$ can be described by the inequalities

$$0 \le z \le 1$$
$$-z \le y \le z$$
$$-\sqrt{z^2 - y^2} \le x \le \sqrt{z^2 - y^2}$$

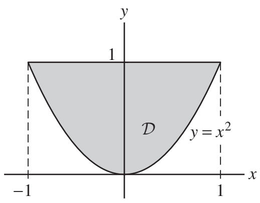
The triple integral is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x, y, z)\, dx\, dy\, dz.$$

31. Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2$, $y = x^2$, and the planes $z = 0$, $y = 1$. Calculate the volume of $\mathcal{W}$ as a triple integral in the order $dz\, dy\, dx$.

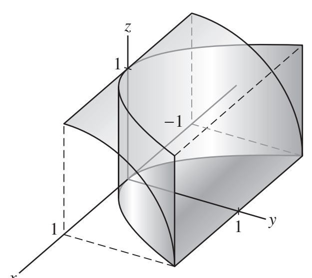
SOLUTION $dz\, dy\, dx$:

The projection of $\mathcal{W}$ onto the xy-plane is the region $\mathcal{D}$ bounded by the curve $y = x^2$ and the line $y = 1$. The region $\mathcal{W}$ consists of all points lying between $\mathcal{D}$ and the cylinder $z = 1 - y^2$.



Therefore, $\mathcal{W}$ can be described by the following inequalities:

$$-1 \le x \le 1, \quad x^2 \le y \le 1, \quad 0 \le z \le 1 - y^2$$



We use the formula for the volume as a triple integral, write the triple integral as an iterated integral, and compute it. We obtain

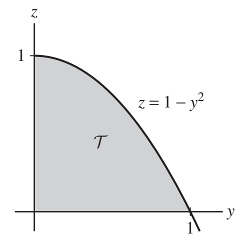
$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\, dV = \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y^2} 1\, dz\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} z \Big|_{z=0}^{1-y^2} dy\, dx \\
&= \int_{-1}^{1} \int_{x^2}^{1} \left(1 - y^2\right) dy\, dx = \int_{-1}^{1} y - \frac{y^3}{3} \Big|_{y=x^2}^{1} dx = \int_{-1}^{1} \left(1 - \frac{1}{3} - \left(x^2 - \frac{x^6}{3}\right)\right) dx \\
&= 2 \int_0^1 \left(\frac{x^6}{3} - x^2 + \frac{2}{3}\right) dx = 2\left(\frac{x^7}{21} - \frac{x^3}{3} + \frac{2x}{3}\right)\Big|_0^1 = 2\left(\frac{1}{21} - \frac{1}{3} + \frac{2}{3}\right) = \frac{16}{21}
\end{aligned}$$

32. Calculate the volume of the region $\mathcal{W}$ in Exercise 31 as a triple integral in the following orders:

(a) $dx\,dz\,dy$ **(b)** $dy\,dz\,dx$

SOLUTION $dx\,dz\,dy$:

The projection of $\mathcal{W}$ onto the yz-plane is the region $\mathcal{T}$ in the first quadrant bounded by the curve $z = 1 - y^2$.



The region $\mathcal{W}$ consists of all points lying between $\mathcal{T}$ and the faces $y = x^2$ or $x = \pm\sqrt{y}$. Therefore, $\mathcal{W}$ is described by the following inequalities:

$$0 \le y \le 1, \quad 0 \le z \le 1 - y^2, \quad -\sqrt{y} \le x \le \sqrt{y}$$

We obtain the following iterated integral:

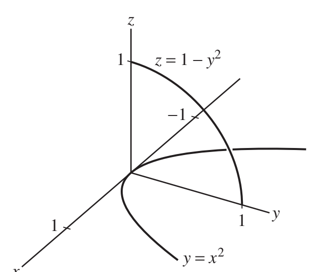
$$\text{Volume}(\mathcal{W}) = \int_0^1 \int_0^{1-y^2} \int_{-\sqrt{y}}^{\sqrt{y}} 1 \cdot dx\,dz\,dy = \int_0^1 \int_0^{1-y^2} x\Big|_{x=-\sqrt{y}}^{\sqrt{y}} dz\,dy = \int_0^1 \int_0^{1-y^2} 2\sqrt{y}dz\,dy$$

$$= \int_0^1 2\sqrt{y}z\Big|_{z=0}^{1-y^2} dy = \int_0^1 2\sqrt{y}\left(1-y^2\right) dy = \int_0^1 \left(2y^{1/2} - 2y^{5/2}\right) dy$$

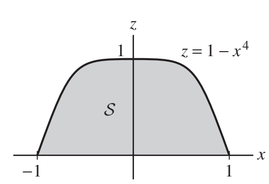
$$= \frac{4}{3}y^{3/2} - \frac{4}{7}y^{7/2}\Big|_0^1 = \frac{4}{3} - \frac{4}{7} = \frac{16}{21}$$

$dy\,dz\,dx$:

To find the projection of $\mathcal{W}$ onto the xz-plane we first must find the points Q that lie on both cylinders $y = x^2$ and $z = 1 - y^2$.



At these points $y = x^2$ and $z = 1 - y^2$, hence $z = 1 - \left(x^2\right)^2 = 1 - x^4$. Therefore, $Q = (x, x^2, 1 - x^4)$. The projection onto the xz-plane are the points $(x, 0, 1 - x^4)$, that is, the curve $z = 1 - x^4$.



The region $\mathcal{W}$ consists of all points between the cylinders $y = x^2$ and $z = 1 - y^2$, or $y = \sqrt{1-z}$. Therefore, $\mathcal{W}$ can be described by the following inequalities:

$$-1 \le x \le 1, \quad 0 \le z \le 1 - x^4, \quad x^2 \le y \le \sqrt{1-z}$$

We obtain the following integral:

$$\text{Volume}(\mathcal{W}) = \int_{-1}^1 \int_0^{1-x^4} \int_{x^2}^{\sqrt{1-z}} dy\,dz\,dx = \int_{-1}^1 \int_0^{1-x^4} y\Big|_{x^2}^{\sqrt{1-z}} dz\,dx$$

$$= \int_{-1}^1 \int_0^{1-x^4} \left(\sqrt{1-z} - x^2\right) dz\,dx = \int_{-1}^1 \left(-\frac{2}{3}(1-z)^{3/2} - x^2 z\right)\Big|_{z=0}^{1-x^4} dx$$

$$= \int_{-1}^{1} \left(-\frac{2}{3}\left(1 - \left(1 - x^4\right)\right)^{3/2} - x^2\left(1 - x^4\right) + \frac{2}{3} \right) dx$$

$$= \int_{-1}^{1} \left(-\frac{2x^6}{3} - x^2 + x^6 + \frac{2}{3} \right) dx = 2\int_{0}^{1} \left(\frac{x^6}{3} - x^2 + \frac{2}{3} \right) dx$$

$$= 2\left(\frac{x^7}{21} - \frac{x^3}{3} + \frac{2x}{3} \right)\Big|_0^1 = 2\left(\frac{1}{21} - \frac{1}{3} + \frac{2}{3} \right) = \frac{16}{21}$$

(The three answers agree as expected.)

In Exercises 33–36, compute the average value of $f(x, y, z)$ over the region $\mathcal{W}$.

33. $f(x, y, z) = xy\sin(\pi z)$; $\quad \mathcal{W} = [0, 1] \times [0, 1] \times [0, 1]$

SOLUTION The volume of the cube is $V = 1$, hence the average of f over the cube is the following value:

$$\overline{f} = \iiint_{\mathcal{W}} xy\sin(\pi z)\, dV = \int_0^1 \int_0^1 \int_0^1 xy\sin(\pi z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \frac{1}{2}x^2 y\sin(\pi z)\Big|_{x=0}^1 dy\, dz = \int_0^1 \int_0^1 \frac{1}{2} y\sin(\pi z)\, dy\, dz$$

$$= \int_0^1 \frac{y^2}{4}\sin(\pi z)\Big|_{y=0}^1 dz = \int_0^1 \frac{1}{4}\sin(\pi z)\, dz = -\frac{1}{4\pi}\cos(\pi z)\Big|_0^1$$

$$= -\frac{1}{4\pi}(\cos\pi - \cos 0) = -\frac{1}{4\pi}(-1 - 1) = \frac{1}{2\pi}$$

34. $f(x, y, z) = xyz$; $\quad \mathcal{W} = [0, 1] \times [0, 1] \times [0, 1]$

SOLUTION This solid is a tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ and volume $1/6$. The equation of the plane that forms this tetrahedron is $y + z = 0$. Therefore, the average of f over the tetrahedron is the following value:

$$\overline{f} = \frac{1}{V}\iiint_{\mathcal{W}} xyz\, dV = 6\int_0^1 \int_0^x \int_0^{-y} xyz\, dz\, dy\, dx$$

$$= 6\int_0^1 \int_0^x \frac{1}{2}xyz^2\Big|_{z=0}^{-y} dy\, dx = 3\int_0^1 \int_0^x xy(-y)^2 dy\, dx$$

$$= 3\int_0^1 \int_0^x xy^3 dy\, dx = \frac{3}{4}\int_0^1 xy^4\Big|_{y=0}^x dx$$

$$= \frac{3}{4}\int_0^1 x^5 dx = \frac{1}{8}\left(x^6\Big|_0^1 \right) = \frac{1}{8}$$

35. $f(x, y, z) = e^y$; $\quad \mathcal{W} : 0 \le y \le 1 - x^2, \quad 0 \le z \le x$

SOLUTION First we must calculate the volume of $\mathcal{W}$. We will use the symmetry of $y = 1 - x^2$ to write:

$$V = \iiint_{\mathcal{W}} 1\, dV = 2\int_0^1 \int_0^{1-x^2} \int_0^x 1\, dz\, dy\, dx$$

$$= 2\int_0^1 \int_0^{1-x^2} z\Big|_{z=0}^x dy\, dx = 2\int_0^1 \int_0^{1-x^2} x\, dy\, dx$$

$$= 2\int_0^1 xy\Big|_{y=0}^{1-x^2} dx = 2\int_0^1 x(1 - x^2)\, dx$$

$$= 2\int_0^1 x - x^3\, dx = x^2 - \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}$$

Now we can compute the average value for $f(x, y, z) = e^y$:

$$\overline{f} = \frac{1}{V}\iiint_{\mathcal{W}} e^y\, dV = 2 \cdot \frac{1}{1/2}\int_0^1 \int_0^{1-x^2} \int_0^x e^y\, dz\, dy\, dx$$

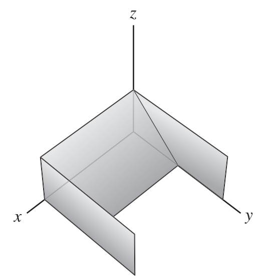
$$= 4\int_0^1 \int_0^{1-x^2} ze^y \Big|_{z=0}^{x} dy\,dx = 4\int_0^1 \int_0^{1-x^2} xe^y\,dy\,dx$$

$$= 4\int_0^1 xe^y \Big|_{y=0}^{1-x^2} dx = 4\int_0^1 xe^{1-x^2} - x\,dx$$

$$= 4\left(-\frac{1}{2}e^{1-x^2} - \frac{1}{2}x^2\right)\Big|_0^1 = \left(-2e^0 - 2\right) - \left(-2e^1 - 0\right) = 2e - 4$$

36. $f(x, y, z) = x^2 + y^2 + z^2$; $\mathcal{W}$ bounded by the planes $2y + z = 1$, $x = 0$, $x = 1$, $z = 0$, and $y = 0$.

SOLUTION The prism $\mathcal{W}$ bounded by the planes $2y + z = 1$, $x = 0$, $x = 1$, $z = 0$, and $y = 0$ is shown in the figure:

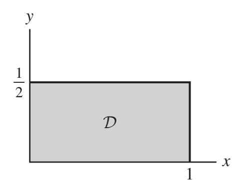


The average of f over $\mathcal{W}$ is

$$\overline{f} = \frac{1}{V}\iiint_{\mathcal{W}} f(x, y, z)\,dV \tag{1}$$

where $V = \text{Volume}(\mathcal{W})$. The projection of $\mathcal{W}$ onto the xy-plane is the rectangle

$$\mathcal{D}:\ 0 \le x \le 1,\ 0 \le y \le \frac{1}{2}$$



The region $\mathcal{W}$ consists of all points lying between the upper plane $z = 1 - 2y$ and the lower plane $z = 0$. The volume of the prism is

$$V = \frac{\frac{1}{2}\cdot 1}{2}\cdot 1 = \frac{1}{4}.$$

We compute the triple integral (1) using an iterated integral. That is,

$$\overline{f} = \frac{1}{\frac{1}{4}}\int_0^1 \int_0^{1/2} \int_0^{1-2y} \left(x^2 + y^2 + z^2\right) dz\,dy\,dx = 4\int_0^1 \int_0^{1/2} \left(x^2 + y^2\right) z + \frac{z^3}{3}\Big|_{z=0}^{1-2y} dy\,dx$$

$$= 4\int_0^1 \int_0^{1/2} \left(\left(x^2 + y^2\right)(1 - 2y) + \frac{(1-2y)^3}{3}\right) dy\,dx$$

$$= 4\int_0^1 \int_0^{1/2} \left(x^2 - 2x^2y + y^2 - 2y^3 + \frac{(1-2y)^3}{3}\right) dy\,dx$$

$$= 4\int_0^1 x^2y - x^2y^2 + \frac{y^3}{3} - \frac{y^4}{2} - \frac{(1-2y)^4}{24}\Big|_{y=0}^{1/2} dx$$

$$= 4\int_0^1 \left(\frac{x^2}{4} + \frac{5}{96}\right) dx = 4\left(\frac{x^3}{12} + \frac{5x}{96}\Big|_0^1\right) = \frac{13}{24}$$

The average value is $\overline{f} = \frac{13}{24}$.

In Exercises 37 and 38, let $I = \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dV$ *and let* $S_{N,N,N}$ *be the Riemann sum approximation*

$$S_{N,N,N} = \frac{1}{N^3} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} f\left(\frac{i}{N}, \frac{j}{N}, \frac{k}{N}\right)$$

37. CAS Calculate $S_{N,N,N}$ for $f(x, y, z) = e^{x^2-y-z}$ for $N = 10, 20, 30$. Then evaluate I and find an N such that $S_{N,N,N}$ approximates I to two decimal places.

SOLUTION Using a CAS, we get $S_{N,N,N} \approx 0.561, 0.572$, and 0.576 for $N = 10, 20$, and 30, respectively. We get $I \approx 0.584$, and using $N = 100$ we get $S_{N,N,N} \approx 0.582$, accurate to two decimal places.

38. CAS Calculate $S_{N,N,N}$ for $f(x, y, z) = \sin(xyz)$ for $N = 10, 20, 30$. Then use a computer algebra system to calculate I numerically and estimate the error $|I - S_{N,N,N}|$.

SOLUTION Using a CAS, we get $S_{N,N,N} \approx 0.162, 0.141$, and 0.135 for $N = 10, 20$, and 30, respectively. We get $I \approx 0.122$, giving an error of 0.040, 0.019, and 0.013, respectively.

Further Insights and Challenges

39. Use Integration by Parts to verify Eq. (7).

SOLUTION If $C_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta\, d\theta$, we use integration by parts to show that

$$C_n = \left(\frac{n-1}{n}\right) C_{n-2}.$$

We use integration by parts with $u = \cos^{n-1}\theta$ and $V' = \cos\theta$. Hence, $u' = (n-1)\cos^{n-2}\theta(-\sin\theta)$ and $v = \sin\theta$. Thus,

$$\begin{aligned}
C_n &= \int_{-\pi/2}^{\pi/2} \cos^n \theta\, d\theta = \int_{-\pi/2}^{\pi/2} \cos^{n-1}\theta \cos\theta\, d\theta = \cos^{n-1}\theta \sin\theta \Big|_{\theta=-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} (n-1)\cos^{n-2}\theta \sin^2\theta\, d\theta \\
&= \cos^{n-1}\frac{\pi}{2}\sin\frac{\pi}{2} - \cos^{n+1}\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + (n-1)\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta \sin^2\theta\, d\theta \\
&= 0 + (n-1)\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta\left(1-\cos^2\theta\right) d\theta = (n-1)\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta\, d\theta - (n-1)\int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta \\
&= (n-1)C_{n-2} - (n-1)C_n
\end{aligned}$$

We obtain the following equality:

$$C_n = (n-1)C_{n-2} - (n-1)C_n$$

or

$$\begin{aligned}
C_n + (n-1)C_n &= (n-1)C_{n-2} \\
nC_n &= (n-1)C_{n-2} \\
C_n &= \frac{n-1}{n}C_{n-2}
\end{aligned}$$

40. Compute the volume A_n of the unit ball in $\mathbf{R}^n$ for $n = 8, 9, 10$. Show that $C_n \le 1$ for $n \ge 6$ and use this to prove that of all unit balls, the five-dimensional ball has the largest volume. Can you explain why A_n tends to 0 as $n \to \infty$?

SOLUTION We use the following formulas:

$$A_{2m} = \frac{\pi^m}{m!}, \quad A_{2m+1} = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$$

We get

$$\begin{aligned}
A_8 &= A_{2\cdot 4} = \frac{\pi^4}{4!} = \frac{\pi^4}{24} \approx 4.06 \\
A_9 &= A_{2\cdot 4+1} = \frac{2^{4+1}\pi^4}{1\cdot 3\cdot 5\cdot 7\cdot 9} = \frac{32\pi^4}{945} \approx 3.3 \\
A_{10} &= A_{2\cdot 5} = \frac{\pi^5}{5!} = \frac{\pi^5}{120} \approx 2.55
\end{aligned}$$

We now show that $C_n < 1$ for $n \geq 6$. We first compute C_6, using the formula $C_n = \frac{n-1}{n} C_{n-2}$ successively and the value $C_0 = \pi$. This gives

$$C_6 = \frac{6-1}{6} C_4 = \frac{5}{6} \cdot \frac{4-1}{4} C_2 = \frac{5}{8} \cdot \frac{2-1}{2} C_0 = \frac{5\pi}{16} \approx 0.98 < 1$$

$$C_7 = \frac{7-1}{7} C_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot C_3 = \frac{24}{35} \cdot \frac{2}{3} C_1 = \frac{16}{35} C_1$$

$$C_1 = \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_{-\pi/2}^{\pi/2} = 2$$

Hence,

$$C_7 = \frac{16}{35} \cdot 2 = \frac{32}{35} < 1$$

Thus, $C_n < 1$ for $n = 6$ and 7. We now assume that $C_k < 1$ for all k, $6 \leq k \leq n$, for some $n \geq 7$. We show that $C_{n+1} < 1$.

$$C_{n+1} = \frac{(n+1)-1}{n+1} C_{(n+1)-2} = \frac{n}{n+1} C_{n-1}$$

By the assumption, $n - 1 \geq 6$, and so $C_{n-1} < 1$. Therefore,

$$C_{n+1} = \frac{n}{n+1} C_{n-1} < C_{n-1} < 1 \quad \Rightarrow \quad C_{n+1} < 1$$

We thus proved by mathematical induction that $C_n < 1$ for all $n \geq 6$. We now show that of all unit balls, the five-dimensional ball has the largest volume. The volume of the n-dimensional unit ball is

$$A_n = A_{n-1} C_n$$

Since $A_{n-1} > 0$, and for $n \geq 6$, $C_n < 1$, we have

$$A_n < A_{n-1} \quad \text{for } n \geq 6 \tag{1}$$

Therefore $\{A_n\}_{n=5}^{\infty}$ is a decreasing sequence, and we have

$$A_n < A_5 \text{ for } n \geq 6 \tag{2}$$

Combining (1) and (2) gives

$$A_n < A_5 = \frac{8\pi^2}{15} \approx 5.26 \quad \text{for } n \geq 6 \tag{3}$$

We compute $A_1, \ldots, A_5$:

$$A_1 = A_0 C_1 = 1 \cdot 2 = 2$$

$$A_2 = A_1 C_2 = 2 \cdot \frac{\pi}{2} = \pi \approx 3.14$$

$$A_3 = A_2 C_3 = \pi \cdot \frac{4}{3} = \frac{4\pi}{3} \approx 4.19$$

$$A_4 = A_3 C_4 = \frac{4\pi}{3} \cdot \frac{3\pi}{8} = \frac{\pi^2}{2} \approx 4.93$$

$$A_5 = A_4 C_5 = \frac{\pi^2}{2} \cdot \frac{16}{15} = \frac{8\pi^2}{15} \approx 5.26$$

The maximum value is A_5. Combining with (3) we conclude that the five-dimensional ball has the largest volume. By the closed formulas for A_{2m} and A_{2m+1}, it follows that these sequences tend to zero as $n \to \infty$ (factorials grow faster than exponentials). Therefore, the sequence $\{A_n\}$ also tends to zero as $n \to \infty$.

15.4 Integration in Polar, Cylindrical, and Spherical Coordinates (LT Section 16.4)

Preliminary Questions

1. Which of the following represent the integral of $f(x, y) = x^2 + y^2$ over the unit circle?

(a) $\int_0^1 \int_0^{2\pi} r^2 \, dr \, d\theta$

(b) $\int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta$

(c) $\int_0^1 \int_0^{2\pi} r^3 \, dr \, d\theta$

(d) $\int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta$

SOLUTION The unit circle is described in polar coordinates by the inequalities

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1$$

Using double integral in polar coordinates, we have

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_0^{2\pi}\int_0^1 \left((r\cos\theta)^2 + (r\sin\theta)^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\left(\cos^2\theta + \sin^2\theta\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

Therefore (d) is the correct answer.

2. What are the limits of integration in $\iiint f(r,\theta,z) r\,dr\,d\theta\,dz$ if the integration extends over the following regions?

(a) $x^2 + y^2 \le 4, \quad -1 \le z \le 2$

(b) Lower hemisphere of the sphere of radius 2, center at origin

SOLUTION

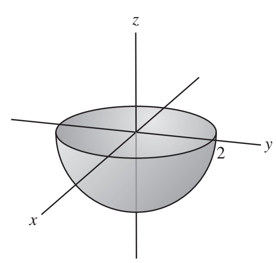
(a) This is a cylinder of radius 2. In the given region the z coordinate is changing between the values -1 and 2, and the angle θ is changing between the values $\theta = 0$ and 2π. Therefore the region is described by the inequalities

$$-1 \le z \le 2, \quad 0 \le \theta < 2\pi, \quad 0 \le r \le 2$$

Using triple integral in cylindrical coordinates gives

$$\int_{-1}^{2}\int_0^{2\pi}\int_0^2 f(P)\,r\,dr\,d\theta\,dz$$

(b) The sphere of radius 2 is $x^2 + y^2 + z^2 = r^2 + z^2 = 4$, or $r = \sqrt{4 - z^2}$.



In the lower hemisphere we have $-2 \le z \le 0$ and $0 \le \theta < 2\pi$. Therefore, it has the description

$$-2 \le z \le 0, \quad 0 \le \theta < 2\pi, \quad 0 \le r \le \sqrt{4 - z^2}$$

We obtain the following integral in cylindrical coordinates:

$$\int_{-2}^{0}\int_0^{2\pi}\int_0^{\sqrt{4-z^2}} r\,dr\,d\theta\,dz$$

3. What are the limits of integration in

$$\iiint f(\rho,\phi,\theta)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

if the integration extends over the following spherical regions centered at the origin?

(a) Sphere of radius 4

(b) Region between the spheres of radii 4 and 5

(c) Lower hemisphere of the sphere of radius 2

SOLUTION

(a) In the sphere of radius 4, θ varies from 0 to 2π, ϕ varies from 0 to π, and ρ varies from 0 to 4. Using triple integral in spherical coordinates, we obtain the following integral:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^4 f(P)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

(b) In the region between the spheres of radii 4 and 5, ρ varies from 4 to 5, ϕ varies from 0 to π, and θ varies from 0 to 2π. We obtain the following integral:

$$\int_0^{2\pi} \int_0^{\pi} \int_4^5 f(P)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(c) The inequalities in spherical coordinates for the lower hemisphere of radius 2 are

$$0 \le \theta \le 2\pi, \quad \frac{\pi}{2} \le \phi \le \pi, \quad 0 \le \rho \le 2$$

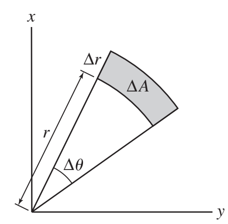
Therefore we obtain the following integral:

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^2 f(P)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

4. An ordinary rectangle of sides Δx and Δy has area $\Delta x \, \Delta y$, no matter where it is located in the plane. However, the area of a polar rectangle of sides Δr and $\Delta\theta$ depends on its distance from the origin. How is this difference reflected in the Change of Variables Formula for polar coordinates?

SOLUTION The area ΔA of a small polar rectangle is

$$\Delta A = \frac{1}{2}(r+\Delta r)^2 \Delta\theta - \frac{1}{2}r^2\Delta\theta = r\,(\Delta r \Delta\theta) + \frac{1}{2}(\Delta r)^2 \Delta\theta \approx r\,(\Delta r \Delta\theta)$$



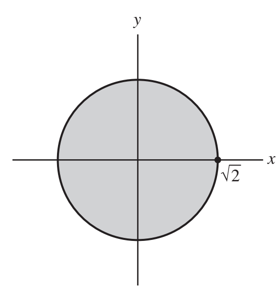
The factor r, due to the distance of the polar rectangle from the origin, appears in $dA = r\,dr\,d\theta$, in the Change of Variables formula.

Exercises

In Exercises 1–6, sketch the region $\mathcal{D}$ indicated and integrate $f(x,y)$ over $\mathcal{D}$ using polar coordinates.

1. $f(x,y) = \sqrt{x^2+y^2}, \quad x^2+y^2 \le 2$

SOLUTION The domain $\mathcal{D}$ is the disk of radius $\sqrt{2}$ shown in the figure:



The inequalities defining $\mathcal{D}$ in polar coordinates are

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{2}$$

We describe $f(x,y) = \sqrt{x^2+y^2}$ in polar coordinates:

$$f(x,y) = \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

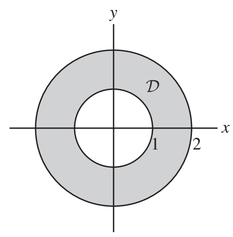
Using change of variables in polar coordinates, we get

$$\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} r \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r^2 \, dr \, d\theta = \int_0^{2\pi} \left. \frac{r^3}{3} \right|_{r=0}^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \frac{\left(\sqrt{2}\right)^3}{3} \, d\theta = \left. \frac{2\sqrt{2}}{3}\theta \right|_0^{2\pi} = \frac{4\sqrt{2}\pi}{3}$$

2. $f(x, y) = x^2 + y^2; \quad 1 \le x^2 + y^2 \le 4$

SOLUTION The domain $\mathcal{D}$ is shown in the figure:



The inequalities defining $\mathcal{D}$ in polar coordinates are

$$0 \le \theta \le 2\pi, \quad 1 \le r \le 2$$

We describe f in polar coordinates:

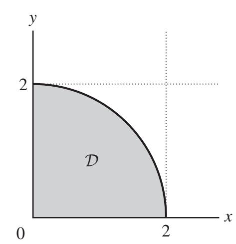
$$f(x, y) = x^2 + y^2 = r^2$$

Using change of variables in polar coordinates gives

$$\iint_{\mathcal{D}} \left(x^2 + y^2\right) dA = \int_0^{2\pi} \int_1^2 r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\bigg|_{r=1}^2 d\theta$$
$$= \int_0^{2\pi} \left(\frac{2^4}{4} - \frac{1}{4}\right) d\theta = 2\pi \cdot 3\frac{3}{4} = \frac{15\pi}{2}$$

3. $f(x, y) = xy; \quad x \ge 0, \quad y \ge 0, \quad x^2 + y^2 \le 4$

SOLUTION The domain $\mathcal{D}$ is the quarter circle of radius 2 in the first quadrant.



It is described by the inequalities

$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2$$

We write f in polar coordinates:

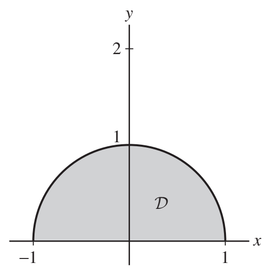
$$f(x, y) = xy = (r\cos\theta)(r\sin\theta) = r^2 \cos\theta \sin\theta = \frac{1}{2} r^2 \sin 2\theta$$

Using change of variables in polar coordinates gives

$$\iint_{\mathcal{D}} xy\, dA = \int_0^{\pi/2} \int_0^2 \left(\frac{1}{2} r^2 \sin 2\theta\right) r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 \frac{1}{2} r^3 \sin 2\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2} \cdot \frac{r^4}{4} \sin 2\theta \bigg|_{r=0}^2 d\theta$$
$$= \int_0^{\pi/2} 2\sin 2\theta\, d\theta = -\cos 2\theta \bigg|_0^{\pi/2} = -(\cos\pi - \cos 0) = 2$$

4. $f(x, y) = y(x^2 + y^2)^3; \quad y \geq 0, \quad x^2 + y^2 \leq 1$

SOLUTION The region $\mathcal{D}$ is the upper half of the unit circle, shown in the figure:



In this region, θ varies from 0 to π, and r varies from 0 to 1. Therefore, $\mathcal{D}$ is described by the inequalities

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 1$$

We write f in polar coordinates:

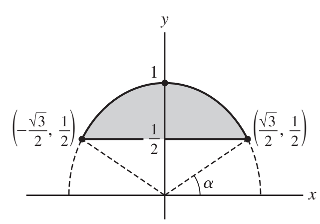
$$f(x, y) = y(x^2 + y^2)^3 = (r \sin\theta)(r^2)^3 = r^7 \sin\theta$$

Using change of variables in polar coordinates, we get

$$\iint_{\mathcal{D}} y\left(x^2 + y^2\right)^3 dA = \int_0^{\pi} \int_0^1 \left(r^7 \sin\theta\right) r\, dr\, d\theta = \int_0^{\pi} \int_0^1 r^8 \sin\theta\, dr\, d\theta = \int_0^{\pi} \frac{r^9}{9} \sin\theta \Big|_{r=0}^{1} d\theta$$
$$= \int_0^{\pi} \frac{1}{9} \sin\theta\, d\theta = -\frac{1}{9}\cos\theta \Big|_0^{\pi} = -\frac{1}{9}(\cos\pi - \cos 0) = \frac{2}{9}$$

5. $f(x, y) = y(x^2 + y^2)^{-1}; \quad y \geq \frac{1}{2}, \quad x^2 + y^2 \leq 1$

SOLUTION The region $\mathcal{D}$ is the part of the unit circle lying above the line $y = \frac{1}{2}$.

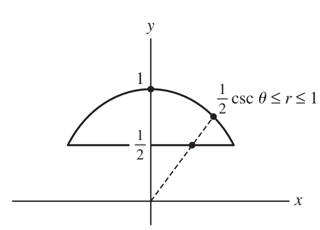


The angle α in the figure is

$$\alpha = \tan^{-1} \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

Therefore, θ varies between $\frac{\pi}{6}$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The horizontal line $y = \frac{1}{2}$ has polar equation $r \sin\theta = \frac{1}{2}$ or $r = \frac{1}{2}\csc\theta$. The circle of radius 1 centered at the origin has polar equation $r = 1$. Therefore, r varies between $\frac{1}{2}\csc\theta$ and 1. The inequalities describing $\mathcal{D}$ in polar coordinates are thus

$$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}, \quad \frac{1}{2}\csc\theta \leq r \leq 1$$



We write f in polar coordinates:

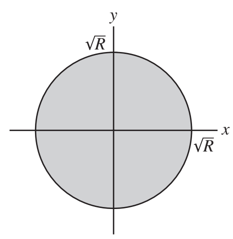
$$f(x, y) = y(x^2 + y^2)^{-1} = (r \sin\theta)(r^2)^{-1} = r^{-1} \sin\theta$$

Using change of variables in polar coordinates, we obtain

$$\iint_{\mathcal{D}} y(x^2+y^2)^{-1}\,dA = \int_{\pi/6}^{5\pi/6}\int_{\frac{1}{2}\csc\theta}^{1} r^{-1}\sin\theta\, r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\int_{\frac{1}{2}\csc\theta}^{1}\sin\theta\,dr\,d\theta$$
$$= \int_{\pi/6}^{5\pi/6} r\sin\theta\Big|_{r=\frac{1}{2}\csc\theta}^{1}\,d\theta = \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\sin\theta\csc\theta\right)d\theta$$
$$= \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\right)d\theta = -\cos\theta - \frac{\theta}{2}\Big|_{\pi/6}^{5\pi/6} = -\cos\frac{5\pi}{6} - \frac{5\pi}{12} - \left(-\cos\frac{\pi}{6} - \frac{\pi}{12}\right)$$
$$= \frac{\sqrt{3}}{2} - \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685$$

6. $f(x, y) = e^{x^2+y^2}$; $\quad x^2 + y^2 \le R$

SOLUTION The region $\mathcal{D}$ is the circle of radius $\sqrt{R}$.



The inequalities describing $\mathcal{D}$ in polar coordinates are

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{R}$$

We write f in polar coordinates as $f(x, y) = e^{x^2+y^2} = e^{r^2}$, and use change of variables in polar coordinates to obtain

$$\iint_{\mathcal{D}} e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{R}} e^{r^2} r\,dr\,d\theta = \int_0^{2\pi}\left(\int_0^{\sqrt{R}} e^{r^2} r\,dr\right)d\theta \tag{1}$$

We compute the inner integral using the substitution $u = e^{r^2}$, $du = 2re^{r^2}\,dr$. We get

$$\int_0^{\sqrt{R}} e^{r^2} r\,dr = \int_1^{e^R}\frac{1}{2}\,du = \frac{1}{2}u\Big|_1^{e^R} = \frac{1}{2}\left(e^R - 1\right)$$

Substituting in (1) gives

$$\iint_{\mathcal{D}} e^{x^2+y^2}\,dA = \int_0^{2\pi}\frac{1}{2}(e^R-1)\,d\theta = \frac{1}{2}(e^R-1)\theta\Big|_0^{2\pi} = \pi(e^R-1)$$

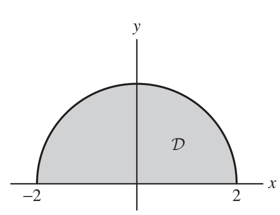
In Exercises 7–14, sketch the region of integration and evaluate by changing to polar coordinates.

7. $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}(x^2+y^2)\,dy\,dx$

SOLUTION The domain $\mathcal{D}$ is described by the inequalities

$$\mathcal{D}:\ -2 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}$$

That is, $\mathcal{D}$ is the semicircle $x^2 + y^2 \le 4$, $0 \le y$.



We describe $\mathcal{D}$ in polar coordinates:

$$\mathcal{D}: 0 \le \theta \le \pi,\ 0 \le r \le 2$$

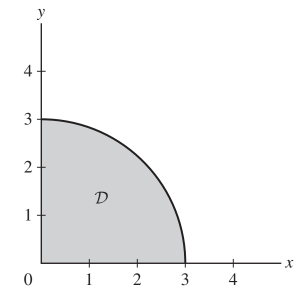
The function f in polar coordinates is $f(x, y) = x^2 + y^2 = r^2$. We use the Change of Variables Formula to write

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \left(x^2 + y^2\right) dy\, dx = \int_{0}^{\pi} \int_{0}^{2} r^2 \cdot r\, dr\, d\theta = \int_{0}^{\pi} \int_{0}^{2} r^3\, dr\, d\theta = \int_{0}^{\pi} \frac{r^4}{4}\bigg|_{r=0}^{2} d\theta = \int_{0}^{\pi} \frac{2^4}{4}\, d\theta = 4\pi$$

8. $\displaystyle\int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \sqrt{x^2 + y^2}\, dx\, dy$

SOLUTION The region $\mathcal{D}$ is defined by the following inequalities:

$$0 \le y \le 3, \quad 0 \le x \le \sqrt{9 - y^2}$$



We see that $\mathcal{D}$ is the quarter of the circle $x^2 + y^2 = 9$, $x \ge 0$, $y \ge 0$. We describe $\mathcal{D}$ in polar coordinates by the following inequalities:

$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 3$$

The function in polar coordinates is $f(x, y) = \sqrt{x^2 + y^2} = r$. Using change of variables we get

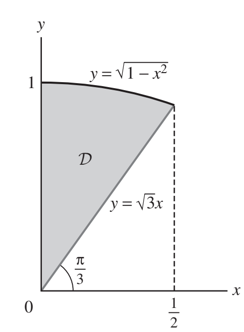
$$\int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \sqrt{x^2 + y^2}\, dx\, dy = \int_{0}^{\pi/2} \int_{0}^{3} r \cdot r dr\, d\theta = \int_{0}^{\pi/2} \int_{0}^{3} r^2\, dr\, d\theta$$
$$= \int_{0}^{\pi/2} \frac{r^3}{3}\bigg|_{0}^{3} d\theta = \int_{0}^{\pi/2} 9\, d\theta = 9 \cdot \frac{\pi}{2} = 4.5\pi$$

9. $\displaystyle\int_{0}^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\, dy\, dx$

SOLUTION The region of integration is described by the inequalities

$$0 \le x \le \frac{1}{2}, \quad \sqrt{3}x \le y \le \sqrt{1 - x^2}$$

$\mathcal{D}$ is the circular sector shown in the figure.



The ray $y = \sqrt{3}x$ in the first quadrant has the polar equation

$$r \sin\theta = \sqrt{3} r \cos\theta \quad \Rightarrow \quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

Therefore, $\mathcal{D}$ lies in the angular sector $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$. Also, the circle $y = \sqrt{1-x^2}$ has the polar equation $r = 1$, hence $\mathcal{D}$ can be described by the inequalities

$$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1$$

We use change of variables to obtain

$$\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\,dy\,dx = \int_{\pi/3}^{\pi/2} \int_0^1 r(\cos\theta) r\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \int_0^1 r^2 \cos\theta\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \frac{r^3 \cos\theta}{3}\bigg|_{r=0}^{1} d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{\cos\theta}{3}\,d\theta = \frac{\sin\theta}{3}\bigg|_{\pi/3}^{\pi/2} = \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) = \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right) \approx 0.045$$

10. $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \tan^{-1}\frac{y}{x}\,dy\,dx$

SOLUTION We note that this is an integral over the quarter circle of radius 4 in the first quadrant. Using the standard polar coordinates, we get:

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \tan^{-1}\frac{y}{x}\,dy\,dx = \int_0^{\pi/2} \int_0^4 \tan^{-1}\frac{r\sin\theta}{r\cos\theta}\,r\,dr\,d\theta = \int_0^{\pi/2} \int_0^4 \tan^{-1}\tan\theta\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^4 \theta\,r\,dr\,d\theta = \frac{1}{2}r^2\bigg|_0^4 \cdot \frac{1}{2}\theta^2\bigg|_0^{\pi/2} = \pi^2$$

11. $\displaystyle\int_0^5 \int_0^y x\,dx\,dy$

SOLUTION

$$\int_0^5 \int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2} \int_{r=0}^{5/\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{3}r^3\cos\theta\bigg|_{r=0}^{5/\sin\theta} d\theta$$

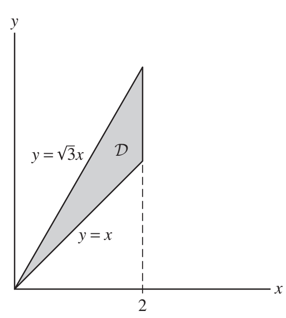
$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \frac{125}{\sin^3\theta}\cos\theta\,d\theta = \frac{125}{3}\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\,d\theta$$

$$= -\frac{125}{6}\frac{1}{\sin^2\theta}\bigg|_{\pi/4}^{\pi/2} = -\frac{125}{6}(1-2) = \frac{125}{6}$$

12. $\displaystyle\int_0^2 \int_x^{\sqrt{3}x} y\,dy\,dx$

SOLUTION The region is determined by the inequalities

$$\mathcal{D}: \ 0 \le x \le 2, \ x \le y \le \sqrt{3}x$$



The rays $y = x$ and $y = \sqrt{3}x$ in the first quadrant have polar equations $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$, respectively, hence the region lies in the angular sector $\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$. The line $x = 2$ has polar equation $r\cos\theta = 2$ or $r = 2\sec\theta$. Therefore, the inequalities describing $\mathcal{D}$ in polar equations are

$$\mathcal{D}: \ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}, \ 0 \le r \le 2\sec\theta$$

Using change of variables we have

$$\int_0^2 \int_x^{\sqrt{3}x} y\,dy\,dx = \int_{\pi/4}^{\pi/3} \int_0^{2\sec\theta} (r\sin\theta) r\,dr\,d\theta = \int_{\pi/4}^{\pi/3} \int_0^{2\sec\theta} r^2 \sin\theta\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/3} \frac{r^3 \sin\theta}{3}\bigg|_{r=0}^{2\sec\theta} d\theta = \int_{\pi/4}^{\pi/3} \frac{8\sec^3\theta\sin\theta}{3}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{8}{3}\frac{\sin\theta}{\cos^3\theta}\,d\theta$$

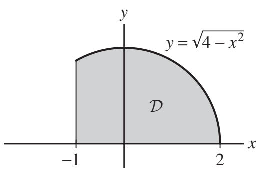
We compute the integral using the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$. We get

$$\int_0^2 \int_x^{\sqrt{3}x} y\,dy\,dx = \int_{1/\sqrt{2}}^{1/2} \frac{8}{3}u^{-3}(-du) = \int_{1/2}^{1/\sqrt{2}} \frac{8}{3}u^{-3}\,du = -\frac{4}{3}u^{-2}\bigg|_{1/2}^{1/\sqrt{2}} = -\frac{4}{3}(2-4) = \frac{8}{3}$$

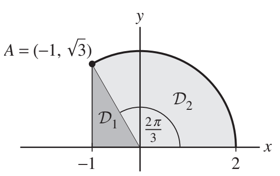
13. $\displaystyle\int_{-1}^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

SOLUTION The domain $\mathcal{D}$, shown in the figure, is described by the inequalities

$$-1 \le x \le 2, \quad 0 \le y \le \sqrt{4-x^2}$$



We denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the triangle and the circular sections, respectively, shown in the figure.



By properties of integrals we have

$$\iint_{\mathcal{D}} \left(x^2+y^2\right) dA = \iint_{\mathcal{D}_1} \left(x^2+y^2\right) dA + \iint_{\mathcal{D}_2} \left(x^2+y^2\right) dA \tag{1}$$

We compute each integral separately, starting with $\mathcal{D}_1$. The vertical line $x = -1$ has polar equation $r\cos\theta = -1$ or $r = -\sec\theta$. The ray OA has polar equation $\theta = \frac{2\pi}{3}$. Therefore $\mathcal{D}_1$ is described by

$$\frac{2\pi}{3} \le \theta \le \pi, \quad 0 \le r \le -\sec\theta$$

Using change of variables gives

$$\iint_{\mathcal{D}_1} \left(x^2+y^2\right) dA = \int_{2\pi/3}^{\pi} \int_0^{-\sec\theta} r^2 \cdot r\,dr\,d\theta = \int_{2\pi/3}^{\pi} \int_0^{-\sec\theta} r^3\,dr\,d\theta$$
$$= \int_{2\pi/3}^{\pi} \frac{r^4}{4}\bigg|_{r=0}^{-\sec\theta} d\theta = \int_{2\pi/3}^{\pi} \frac{\sec^4\theta}{4}\,d\theta = \frac{1}{4}\int_{2\pi/3}^{\pi} \sec^4\theta\,d\theta \tag{2}$$

We compute the integral (we use substitution $u = \tan\theta$ for the second integral):

$$\int_{2\pi/3}^{\pi} \frac{1}{\cos^4\theta}\,d\theta = \int_{2\pi/3}^{\pi} \frac{\sin^2\theta + \cos^2\theta}{\cos^4\theta}\,d\theta = \int_{2\pi/3}^{\pi} \frac{d\theta}{\cos^2\theta} + \int_{2\pi/3}^{\pi} \tan^2\theta \cdot \frac{1}{\cos^2\theta}\,d\theta$$
$$= \tan\theta\bigg|_{\theta=2\pi/3}^{\pi} + \int_{-\sqrt{3}}^{0} u^2\,du = \tan\pi - \tan\frac{2\pi}{3} + \frac{u^3}{3}\bigg|_{-\sqrt{3}}^{0} = \sqrt{3} + \frac{3\sqrt{3}}{3} = 2\sqrt{3}$$

Hence, by (2) we get

$$\iint_{\mathcal{D}_1} \left(x^2+y^2\right) dA = \frac{\sqrt{3}}{2} \tag{3}$$

$\mathcal{D}_2$ is described by the inequalities

$$0 \le \theta \le \frac{2\pi}{3}, \quad 0 \le r \le 2$$

Hence,

$$\iint_{\mathcal{D}_2} \left(x^2 + y^2\right) dA = \int_0^{2\pi/3} \int_0^2 r^2 \cdot r \, dr \, d\theta = \int_0^{2\pi/3} \int_0^2 r^3 \, dr \, d\theta$$
$$= \int_0^{2\pi/3} \left.\frac{r^4}{4}\right|_{r=0}^{2} d\theta = \int_0^{2\pi/3} 4 \, d\theta = 4 \cdot \theta \Big|_0^{2\pi/3} = \frac{8\pi}{3} \qquad (4)$$

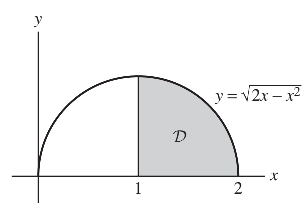
Combining (1), (3) and (4) we obtain the following solution:

$$\iint_{\mathcal{D}} \left(x^2 + y^2\right) dA = \frac{\sqrt{3}}{2} + \frac{8\pi}{3} \approx 9.24$$

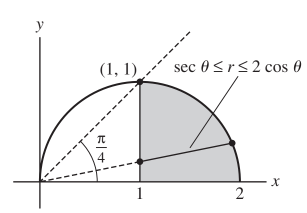
14. $\displaystyle\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}} \, dy \, dx$

SOLUTION The region is described by the inequalities

$$1 \le x \le 2, \quad 0 \le y \le \sqrt{2x - x^2}$$



We first describe $\mathcal{D}$ in polar coordinates. The region lies in the angular sector $0 \le \theta \le \frac{\pi}{4}$. The circle $y = \sqrt{2x - x^2}$ or $(x-1)^2 + y^2 = 1$, $y \ge 0$ (obtained by completing the square) is the circle of radius 1 and center $(1, 0)$. Its polar equation is $r = 2\cos\theta$. The polar equation of the line $x = 1$ is $r\cos\theta = 1$ or $r = \sec\theta$.



Therefore, $\mathcal{D}$ has the following description:

$$0 \le \theta \le \frac{\pi}{4}, \quad \sec\theta \le r \le 2\cos\theta$$

Using change of variables we get

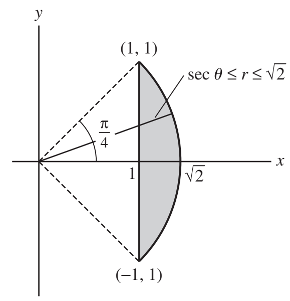
$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{dy\,dx}{\sqrt{x^2+y^2}} = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r} \cdot r \, dr \, d\theta = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} dr \, d\theta = \int_0^{\pi/4} r \Big|_{r=\sec\theta}^{2\cos\theta} d\theta$$
$$= \int_0^{\pi/4} (2\cos\theta - \sec\theta) \, d\theta = 2\sin\theta \Big|_0^{\pi/4} - \ln(\sec\theta + \tan\theta) \Big|_0^{\pi/4}$$
$$= 2\sin\frac{\pi}{4} - \left(\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln 1\right) = 2 \cdot \frac{\sqrt{2}}{2} - \ln\left(\sqrt{2} + 1\right)$$
$$= \sqrt{2} - \ln\left(1 + \sqrt{2}\right) \approx 0.533$$

In Exercises 15–20, calculate the integral over the given region by changing to polar coordinates.

15. $f(x, y) = (x^2 + y^2)^{-2}$; $x^2 + y^2 \le 2$, $x \ge 1$

SOLUTION The region $\mathcal{D}$ lies in the angular sector

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$



The vertical line $x = 1$ has polar equation $r\cos\theta = 1$ or $r = \sec\theta$. The circle $x^2 + y^2 = 2$ has polar equation $r = \sqrt{2}$. Therefore, $\mathcal{D}$ has the following description:

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad \sec\theta \le r \le \sqrt{2}$$

The function in polar coordinates is

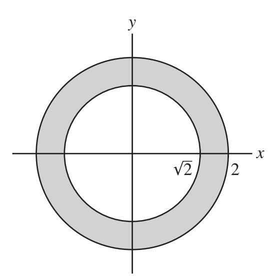
$$f(x, y) = \left(x^2 + y^2\right)^{-2} = \left(r^2\right)^{-2} = r^{-4}.$$

Using change of variables we obtain

$$\iint_{\mathcal{D}} \left(x^2 + y^2\right)^{-2} dA = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r^{-4} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r^{-3} \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \left. \frac{r^{-2}}{-2} \right|_{\sec\theta}^{\sqrt{2}} d\theta$$
$$= \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2\sec^2\theta} - \frac{1}{4} \right) d\theta = 2\int_0^{\pi/4} \left(\frac{1}{2}\cos^2\theta - \frac{1}{4} \right) d\theta$$
$$= \left. \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \right|_0^{\pi/4} - \left. \frac{\theta}{2} \right|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4} - \frac{\pi}{8} = \frac{1}{4}$$

16. $f(x, y) = x$; $2 \le x^2 + y^2 \le 4$

SOLUTION $\mathcal{D}$ is the annulus shown in the figure.



The inequalities describing $\mathcal{D}$ in polar coordinates are

$$0 \le \theta \le 2\pi, \quad \sqrt{2} \le r \le 2$$

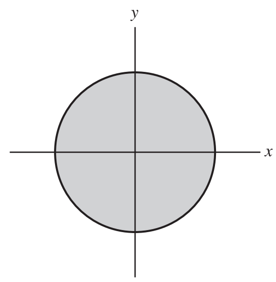
The function is $f(x, y) = x = r\cos\theta$, therefore using change of variables gives

$$\iint_{\mathcal{D}} x \, dA = \int_0^{2\pi} \int_{\sqrt{2}}^{2} (r\cos\theta) r \, dr \, d\theta = \int_0^{2\pi} \int_{\sqrt{2}}^{2} r^2 \cos\theta \, dr \, d\theta = \int_0^{2\pi} \left. \frac{r^3 \cos\theta}{3} \right|_{r=\sqrt{2}}^{2} d\theta$$
$$= \int_0^{2\pi} \frac{8 - 2\sqrt{2}}{3} \cos\theta \, d\theta = \left. \frac{8 - 2\sqrt{2}}{3} \sin\theta \right|_0^{2\pi} = 0$$

17. $f(x, y) = |xy|; \quad x^2 + y^2 \le 1$

SOLUTION The unit disk is described in polar coordinates by

$$\mathcal{D}: 0 \le \theta \le 2\pi, \ 0 \le r \le 1$$



$x^2 + y^2 \le 1$

The function is $f(x, y) = |xy| = |r\cos\theta \cdot r\sin\theta| = \frac{1}{2}r^2|\sin 2\theta|$. Using change of variables we obtain

$$\begin{aligned}\iint_{\mathcal{D}} |xy|\, dA &= \int_0^{2\pi}\int_0^1 \frac{1}{2}r^2|\sin 2\theta| \cdot r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^3|\sin 2\theta|\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{r^4}{8}|\sin 2\theta|\Big|_{r=0}^{1} d\theta = \int_0^{2\pi} \frac{1}{8}|\sin 2\theta|\, d\theta \qquad (1)\end{aligned}$$

The signs of $\sin 2\theta$ in the interval of integration are

For $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$, $\sin 2\theta \ge 0$, hence $|\sin 2\theta| = \sin 2\theta$.
For $\frac{\pi}{2} \le \theta \le \pi$ or $\frac{3\pi}{2} \le \theta \le 2\pi$, $\sin 2\theta \le 0$, hence $|\sin 2\theta| = -\sin 2\theta$.

Therefore, by (1) we get

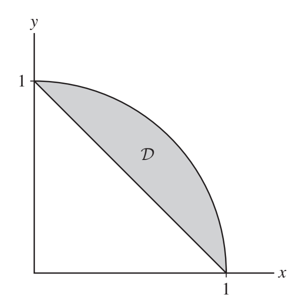
$$\begin{aligned}\iint_{\mathcal{D}} |xy|\, dA &= \int_0^{\pi/2} \frac{1}{8}\sin 2\theta\, d\theta - \int_{\pi/2}^{\pi} \frac{1}{8}\sin 2\theta\, d\theta + \int_{\pi}^{3\pi/2} \frac{1}{8}\sin 2\theta\, d\theta - \int_{3\pi/2}^{2\pi} \frac{1}{8}\sin 2\theta\, d\theta \\ &= -\frac{1}{16}\cos 2\theta\Big|_0^{\pi/2} + \frac{1}{16}\cos 2\theta\Big|_{\pi/2}^{\pi} - \frac{1}{16}\cos 2\theta\Big|_{\pi}^{3\pi/2} + \frac{1}{16}\cos 2\theta\Big|_{3\pi/2}^{2\pi} \\ &= -\frac{1}{16}(\cos\pi - 1) + \frac{1}{16}(\cos 2\pi - \cos\pi) - \frac{1}{16}(\cos 3\pi - \cos 2\pi) + \frac{1}{16}(\cos 4\pi - \cos 3\pi) \\ &= \frac{2}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{1}{2}\end{aligned}$$

That is,

$$\iint_{\mathcal{D}} |xy|\, dA = \frac{1}{2}$$

18. $f(x, y) = (x^2 + y^2)^{-3/2}; \quad x^2 + y^2 \le 1, \quad x + y \ge 1$

SOLUTION The domain $\mathcal{D}$ is shown in the figure.

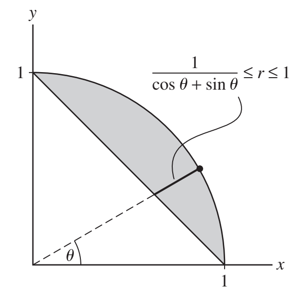


Step 1. Describe $\mathcal{D}$ and f in polar coordinates. In $\mathcal{D}$, the angle θ is changing from 0 to $\frac{\pi}{2}$. The circle $x^2 + y^2 = 1$ has polar equation $r = 1$. The line $x + y = 1$ has the following polar equation:

$$r\cos\theta + r\sin\theta = 1 \quad \Rightarrow \quad r = \frac{1}{\cos\theta + \sin\theta}$$

Therefore, the polar inequalities describing the region $\mathcal{D}$ are

$$\mathcal{D}: \ 0 \le \theta \le \frac{\pi}{2}, \ \frac{1}{\cos\theta + \sin\theta} \le r \le 1$$



The function f in polar coordinates is

$$f(x, y) = (x^2 + y^2)^{-3/2} = (r^2)^{-3/2} = r^{-3}$$

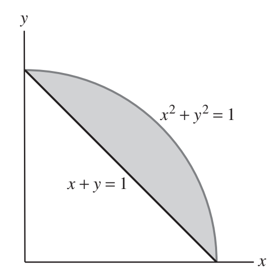
Step 2. Change variables in the Integral and Evaluate. Using the Change of Variables Formula we get

$$\begin{aligned}
\iint_{\mathcal{D}} \left(x^2 + y^2\right)^{-3/2} dA &= \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^{-3} \cdot r\,dr\,d\theta = \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^{-2}\,dr\,d\theta \\
&= \int_0^{\pi/2} -\frac{1}{r}\bigg|_{r=\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta = \int_0^{\pi/2} (-1 + \cos\theta + \sin\theta)\,d\theta \\
&= -\theta + \sin\theta - \cos\theta\bigg|_0^{\pi/2} = \left(-\frac{\pi}{2} + \sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (0 + 0 - 1) \\
&= 2 - \frac{\pi}{2} \approx 0.43
\end{aligned}$$

19. $f(x, y) = x - y; \quad x^2 + y^2 \le 1, \quad x + y \ge 1$

SOLUTION As shown in Exercise 24, the region $\mathcal{D}$ is described by the following inequalities:

$$\mathcal{D}: \ 0 \le \theta \le \frac{\pi}{2}, \ \frac{1}{\cos\theta + \sin\theta} \le r \le 1$$



The function in polar coordinates is

$$f(x, y) = x - y = r\cos\theta - r\sin\theta = r(\cos\theta - \sin\theta)$$

Using the Change of Variables Formula we get

$$\begin{aligned}
\iint_{\mathcal{D}} (x - y)\,dA &= \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r(\cos\theta - \sin\theta) r\,dr\,d\theta = \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^2(\cos\theta - \sin\theta)\,dr\,d\theta \\
&= \int_0^{\pi/2} \frac{r^3(\cos\theta - \sin\theta)}{3}\bigg|_{r=\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta = \int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{3}\left(1 - \frac{1}{(\cos\theta + \sin\theta)^3}\right) d\theta \\
&= \int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{3}\,d\theta - \frac{1}{3}\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(\cos\theta + \sin\theta)^3}\,d\theta \qquad (1)
\end{aligned}$$

We compute the two integrals:

$$\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{3}\,d\theta = \frac{\sin\theta + \cos\theta}{3}\bigg|_0^{\pi/2} = \frac{(1 + 0) - (0 + 1)}{3} = 0 \qquad (2)$$

To compute the second integral we will use u-substitution and let $u = \sin\theta + \cos\theta$:

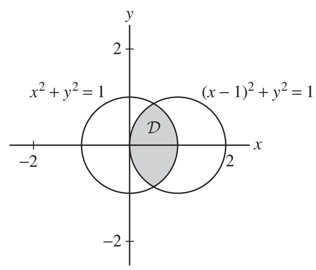
$$\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(\cos\theta + \sin\theta)^3}\,d\theta = \int_{u=1}^{1} u^{-3}\,du = 0 \tag{3}$$

Combining (1), (2), and (3) we conclude that

$$\iint_{\mathcal{D}} (x - y)\,dA = 0$$

20. $f(x, y) = y;\quad x^2 + y^2 \le 1,\quad (x - 1)^2 + y^2 \le 1$

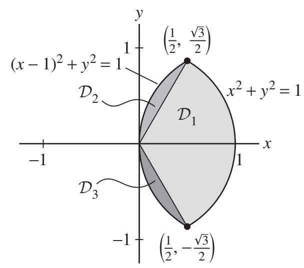
SOLUTION $\mathcal{D}$ is the common region of the two circles shown in the figure.



To evaluate the integral we decompose $\mathcal{D}$ into three regions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ shown in the figure. Thus,

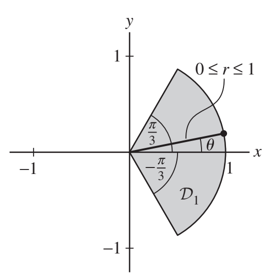
$$\iint_{\mathcal{D}} y\,dA = \iint_{\mathcal{D}_1} y\,dA + \iint_{\mathcal{D}_2} y\,dA + \iint_{\mathcal{D}_3} y\,dA \tag{1}$$

We compute each integral separately.



$\mathcal{D}_1$: The domain $\mathcal{D}_1$ lies in the angular sector $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ (where $\frac{\pi}{3} = \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \tan^{-1}\sqrt{3}$). The circle $x^2 + y^2 = 1$ has polar equation $r = 1$, therefore $\mathcal{D}_1$ has the following definition:

$$\mathcal{D}_1 : -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ 0 \le r \le 1$$



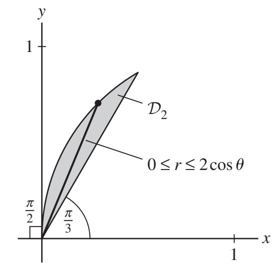
We use change of variables in the integral to write

$$\iint_{\mathcal{D}_1} y\,dA = \int_{-\pi/3}^{\pi/3}\int_0^1 (r\sin\theta)r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\int_0^1 r^2\sin\theta\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \left.\frac{r^3\sin\theta}{3}\right|_{r=0}^{1} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\sin\theta}{3}\,d\theta = -\left.\frac{\cos\theta}{3}\right|_{-\pi/3}^{\pi/3} = 0 \tag{2}$$

$\mathcal{D}_2$: The angle θ is changing in $\mathcal{D}_2$ from $\frac{\pi}{3}$ to $\frac{\pi}{2}$. The circle $(x - 1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$. Therefore, $\mathcal{D}_2$ has the following description:

$$\mathcal{D}_2 : \frac{\pi}{3} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$$



Using the Change of Variables Formula gives

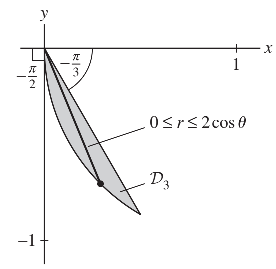
$$\iint_{\mathcal{D}_2} y\,dA = \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta}(r\sin\theta)r\,dr\,d\theta = \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta} r^2\sin\theta\,dr\,d\theta$$
$$= \int_{\pi/3}^{\pi/2} \frac{r^3\sin\theta}{3}\bigg|_{r=0}^{2\cos\theta} d\theta = \int_{\pi/3}^{\pi/2}\frac{8}{3}\cos^3\theta\sin\theta\,d\theta$$

We compute the integral using the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$:

$$\iint_{\mathcal{D}_2} y\,dA = \int_{1/2}^{0}\frac{8}{3}u^3(-du) = \int_0^{1/2}\frac{8}{3}u^3\,du = \frac{2}{3}u^4\bigg|_0^{1/2} = \frac{1}{24} \tag{3}$$

$\mathcal{D}_3$: The domain $\mathcal{D}_3$ has the following description:

$$\mathcal{D}_3 : -\frac{\pi}{2} \le \theta \le -\frac{\pi}{3},\ 0 \le r \le 2\cos\theta$$



We obtain the integral (the inner integral was computed previously)

$$\iint_{\mathcal{D}_3} y\,dA = \int_{-\pi/2}^{-\pi/3}\int_0^{2\cos\theta}(r\sin\theta)r\,dr\,d\theta = \int_{-\pi/2}^{-\pi/3}\left(\int_0^{2\cos\theta} r^2\sin\theta\,dr\right)d\theta = \int_{-\pi/2}^{-\pi/3}\frac{8}{3}\cos^3\theta\sin\theta\,d\theta$$

We use the substitution $\omega = -\theta$ and the integral computed previously to obtain

$$\iint_{\mathcal{D}_3} y\,dA = \int_{\pi/2}^{\pi/3}\frac{8}{3}\cos^3\omega(\sin-\omega)(-d\omega) = -\int_{\pi/3}^{\pi/2}\frac{8}{3}\cos^3\omega\sin\omega\,d\omega = -\frac{1}{24} \tag{4}$$

Finally, we combine (1), (2), (3), and (4) to obtain the following solution:

$$\iint_{\mathcal{D}} y\,dA = 0 + \frac{1}{24} - \frac{1}{24} = 0$$

Remark: The integral is zero since the average value of the y-coordinates of the points in $\mathcal{D}$ is zero ($\mathcal{D}$ is symmetric with respect to the x-axis).

21. Find the volume of the wedge-shaped region (Figure 17) contained in the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$ and below by the xy-plane.

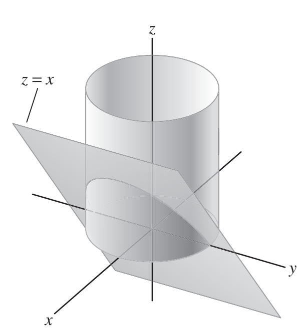
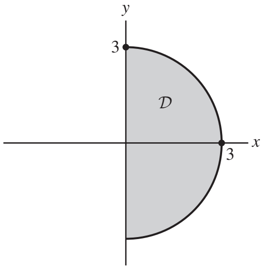


FIGURE 17

SOLUTION

Step 1. Express $\mathcal{W}$ in cylindrical coordinates. $\mathcal{W}$ is bounded above by the plane $z = x$ and below by $z = 0$, therefore $0 \le z \le x$, in particular $x \ge 0$. Hence, $\mathcal{W}$ projects onto the semicircle $\mathcal{D}$ in the xy-plane of radius 3, where $x \ge 0$.



In polar coordinates,

$$\mathcal{D}: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 3$$

The upper surface is $z = x = r\cos\theta$ and the lower surface is $z = 0$. Therefore,

$$\mathcal{W}: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 3,\ 0 \le z \le r\cos\theta$$

Step 2. Set up an integral in cylindrical coordinates and evaluate. The volume of $\mathcal{W}$ is the triple integral $\iiint_{\mathcal{W}} 1\,dV$. Using change of variables in cylindrical coordinates gives

$$\iiint_{\mathcal{W}} 1\,dV = \int_{-\pi/2}^{\pi/2}\int_0^3\int_0^{r\cos\theta} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^3 rz\Big|_{z=0}^{r\cos\theta} dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^3 r^2\cos\theta\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^3}{3}\cos\theta\Big|_{r=0}^{3} d\theta = \int_{-\pi/2}^{\pi/2} 9\cos\theta\,d\theta = 9\sin\theta\Big|_{-\pi/2}^{\pi/2} = 9\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = 18$$

22. Let $\mathcal{W}$ be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$.
(a) Show that the projection of $\mathcal{W}$ on the xy-plane is the disk $x^2 + y^2 \le 2$ (Figure 18).
(b) Compute the volume of $\mathcal{W}$ using polar coordinates.

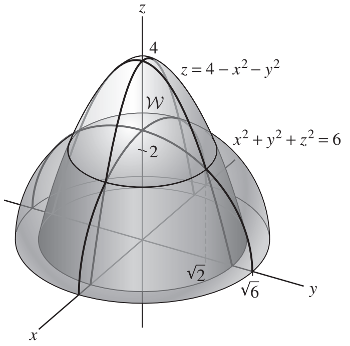


FIGURE 18

SOLUTION

(a) We find the intersection of the sphere $x^2 + y^2 + z^2 = 6$ and the paraboloid $z = 4 - x^2 - y^2$. The equation $z = 4 - x^2 - y^2$ implies $x^2 + y^2 = 4 - z$. Substituting in $x^2 + y^2 + z^2 = 6$ and solving for z gives

$$4 - z + z^2 = 6 \quad\Rightarrow\quad z^2 - z - 2 = (z-2)(z+1) = 0$$
$$\Rightarrow\quad z = 2,\quad z = -1$$

We omit the negative solution $z = -1$, since in $\mathcal{W}$, $z \ge 0$. Substituting $z = 2$ in $z = 4 - x^2 - y^2$, we get

$$2 = 4 - x^2 - y^2 \quad\Rightarrow\quad x^2 + y^2 = 2$$

That is, the projection of $\mathcal{W}$ on the xy-plane is the disk $x^2 + y^2 \le 2$.

(b) Using part (a) we obtain the following inequalities for $\mathcal{D}$:

$$\mathcal{D}: \ 0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{2}$$

The upper boundary is the surface $z = 4 - x^2 - y^2$ or $z = 4 - r^2$. The lower boundary is the sphere $x^2 + y^2 + z^2 = 6$, $r^2 + z^2 = 6$, or $z = \sqrt{6 - r^2}$. Therefore, the inequalities describing $\mathcal{W}$ in cylindrical coordinates are

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{2}, \quad \sqrt{6 - r^2} \le z \le 4 - r^2$$

We use the volume as a triple integral and the change of variables in cylindrical coordinates to obtain

$$V = \iiint_{\mathcal{W}} 1\, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{6-r^2}}^{4-r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r \cdot z \Big|_{z=\sqrt{6-r^2}}^{4-r^2} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \left(4 - r^2 - \sqrt{6 - r^2}\right) r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \left(4r - r^3 - r\sqrt{6 - r^2}\right) dr\, d\theta \qquad (1)$$

We compute the inner integral. Using the substitution $u = \sqrt{6 - r^2}$, $du = -\frac{r}{u}\, dr$ for the second integral, we get

$$\int_0^{\sqrt{2}} (4r - r^3)\, dr = 2r^2 - \frac{r^4}{4}\Big|_0^{\sqrt{2}} = 4 - 1 = 3$$

$$\int_0^{\sqrt{2}} r\sqrt{6 - r^2}\, dr = \int_{\sqrt{6}}^{2} -u^2\, du = \int_2^{\sqrt{6}} u^2\, du = \frac{u^3}{3}\Big|_2^{\sqrt{6}} = 2\sqrt{6} - \frac{8}{3}$$

Hence,

$$\int_0^{\sqrt{2}} \left(4r - r^3 - r\sqrt{6 - r^2}\right) dr = 3 - \left(2\sqrt{6} - \frac{8}{3}\right) = \frac{17}{3} - 2\sqrt{6} \approx 0.77$$

Substituting in (1) we obtain

$$V = \int_0^{2\pi} 0.77\, d\theta = 1.54\pi \approx 4.84$$

23. Evaluate $\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dA$, where $\mathcal{D}$ is the domain in Figure 19. *Hint:* Find the equation of the inner circle in polar coordinates and treat the right and left parts of the region separately.

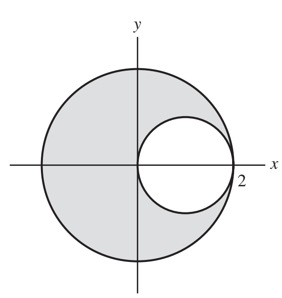


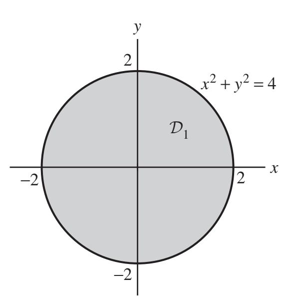
FIGURE 19

SOLUTION We denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the regions enclosed by the circles $x^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 1$. Therefore,

$$\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dx\, dy = \iint_{\mathcal{D}_1} \sqrt{x^2 + y^2}\, dx\, dy - \iint_{\mathcal{D}_2} \sqrt{x^2 + y^2}\, dx\, dy \qquad (1)$$

We compute the integrals on the right hand-side.

$\mathcal{D}_1$:



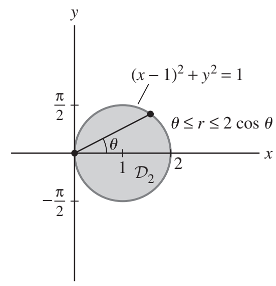
The circle $x^2 + y^2 = 4$ has polar equation $r = 2$, therefore $\mathcal{D}_1$ is determined by the following inequalities:

$$\mathcal{D}_1 : 0 \le \theta \le 2\pi, \ 0 \le r \le 2$$

The function in polar coordinates is $f(x, y) = \sqrt{x^2 + y^2} = r$. Using change of variables in the integral gives

$$\iint_{\mathcal{D}_1} \sqrt{x^2 + y^2}\, dx\, dy = \int_0^{2\pi} \int_0^2 r \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^2\, dr\, d\theta = \int_0^{2\pi} \frac{r^3}{3}\bigg|_{r=0}^{2} d\theta = \int_0^{2\pi} \frac{8}{3}\, d\theta = \frac{16\pi}{3} \tag{2}$$

$\mathcal{D}_2$:



$\mathcal{D}_2$ lies in the angular sector $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We find the polar equation of the circle $(x-1)^2 + y^2 = 1$:

$$\begin{aligned}(x-1)^2 + y^2 = x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2x + 1 = 1 \quad &\Rightarrow \quad x^2 + y^2 = 2x \\ &\Rightarrow \quad r^2 = 2r\cos\theta \\ &\Rightarrow \quad r = 2\cos\theta\end{aligned}$$

Thus, the domain $\mathcal{D}_2$ is defined by the following inequalities:

$$\mathcal{D}_2 : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2\cos\theta$$

We use the change of variables in the integral and integration table to obtain

$$\begin{aligned}\iint_{\mathcal{D}_2} \sqrt{x^2 + y^2}\, dx\, dy &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{r^3}{3}\bigg|_{r=0}^{2\cos\theta} d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta}{3}\, d\theta = 2\int_0^{\pi/2} \frac{8\cos^3\theta}{3}\, d\theta = \frac{16}{3}\left(\frac{\cos^2\theta \sin\theta}{3} + \frac{2}{3}\sin\theta\right)\bigg|_{\theta=0}^{\pi/2} \\ &= \frac{16}{3} \cdot \frac{2}{3}\sin\frac{\pi}{2} = \frac{32}{9}\end{aligned} \tag{3}$$

Substituting (2) and (3) in (1), we obtain the following solution:

$$\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dx\, dy = \frac{16\pi}{3} - \frac{32}{9} = \frac{48\pi - 32}{9} \approx 13.2.$$

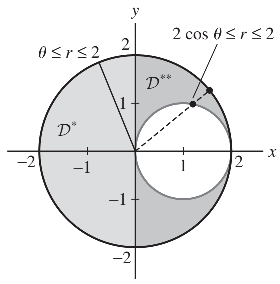
Remark: The integral can also be evaluated using the hint as the sum of

$$\int_{\mathcal{D}^*} \iint \sqrt{x^2 + y^2}\, dA \quad \text{and} \quad \int_{\mathcal{D}^{**}} \iint \sqrt{x^2 + y^2}\, dA$$

where $\mathcal{D}^*$ is the left semicircle $x^2 + y^2 = 4$ and $\mathcal{D}^*$ is the right part of $\mathcal{D}$. Since

$$\mathcal{D}^*: \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \ 0 \le r \le 2$$

$$\mathcal{D}^{**}: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 2\cos\theta \le r \le 2$$



we get

$$\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA = \int_{\pi/2}^{3\pi/2} \int_0^2 r^2\, dr\, d\theta + \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{2} r^2\, dr\, d\theta$$

Obviously, computing the integrals leads to the same result.

24. Evaluate $\iint_{\mathcal{D}} x\sqrt{x^2+y^2}\, dA$, where $\mathcal{D}$ is the shaded region enclosed by the lemniscate curve $r^2 = \sin 2\theta$ in Figure 20.

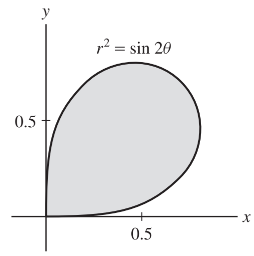
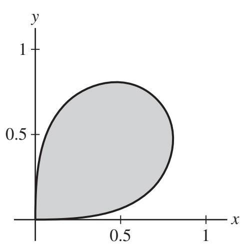


FIGURE 20

SOLUTION In $\mathcal{D}$, the angle θ varies between 0 and $\frac{\pi}{2}$, and r varies between 0 and $\sqrt{\sin 2\theta}$. Therefore $\mathcal{D}$ has the following description:

$$\mathcal{D}:\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{\sin 2\theta}$$



The function in polar coordinates is $f(x,y) = x\sqrt{x^2+y^2} = r\cos\theta \cdot r = r^2\cos\theta$. Therefore, using the Change of Variables Formula we have

$$\begin{aligned}
\iint_{\mathcal{D}} x\sqrt{x^2+y^2}\, dA &= \int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}} r^2\cos\theta \cdot r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}} r^3\cos\theta\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{r^4\cos\theta}{4}\Bigg|_{r=0}^{\sqrt{\sin 2\theta}} d\theta = \int_0^{\pi/2} \frac{\sin^2 2\theta\cos\theta}{4}\, d\theta = \int_0^{\pi/2} \sin^2\theta\cos^3\theta\, d\theta \\
&= \int_0^{\pi/2} \sin^2\theta(1-\sin^2\theta)\cos\theta\, d\theta = \int_0^{\pi/2} (\sin^2\theta - \sin^4\theta)\cos\theta\, d\theta \\
&= \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta\Bigg|_0^{\pi/2} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}
\end{aligned}$$

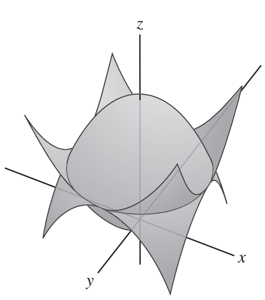
25. Let $\mathcal{W}$ be the region between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

(a) Describe $\mathcal{W}$ in cylindrical coordinates.

(b) Use cylindrical coordinates to compute the volume of $\mathcal{W}$.

SOLUTION

(a)



The paraboloids $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$ have the polar equations $z = r^2$ and $z = 8 - r^2$, respectively. We find the curve of intersection by solving

$$8 - r^2 = r^2 \quad \Rightarrow \quad 2r^2 = 8 \quad \Rightarrow \quad r = 2$$

Therefore, the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is the region enclosed by the circle $r = 2$, and $\mathcal{D}$ has the following description:

$$\mathcal{D}: 0 \le \theta \le 2\pi,\ 0 \le r \le 2$$

The upper and lower boundaries of $\mathcal{W}$ are $z = 8 - r^2$ and $z = r^2$, respectively. Hence,

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r^2 \le z \le 8 - r^2$$

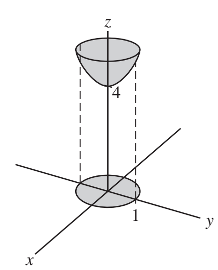
(b) Using change of variables in cylindrical coordinates, we get

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 rz\Big|_{z=r^2}^{8-r^2} dr\,d\theta = \int_0^{2\pi} \int_0^2 r\left(8 - 2r^2\right) dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(8r - 2r^3\right) dr\,d\theta = \int_0^{2\pi} 4r^2 - \frac{r^4}{2}\Big|_{r=0}^2 d\theta = \int_0^{2\pi} 8\,d\theta = 16\pi$$

26. Use cylindrical coordinates to calculate the integral of the function $f(x, y, z) = z$ over the region above the disk $x^2 + y^2 = 1$ in the xy-plane and below the surface $z = 4 + x^2 + y^2$.

SOLUTION



The upper boundary is the surface $z = 4 + r^2$, and the lower boundary is the xy-plane $z = 0$. The projection $\mathcal{D}$ onto the xy-plane is the region $x^2 + y^2 \le 1$, having the polar description

$$\mathcal{D}: 0 \le \theta \le 2\pi,\ 0 \le r \le 1$$

Therefore, the inequalities for $\mathcal{W}$ in cylindrical coordinates are

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 4 + r^2$$

Using cylindrical coordinates we obtain the following integral:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^{2\pi} \int_0^1 \int_0^{4+r^2} zr\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \frac{z^2 r}{2}\Big|_{z=0}^{4+r^2} dr\,d\theta = \int_0^{2\pi} \int_0^1 \frac{\left(4 + r^2\right)^2 r}{2}\,dr\,d\theta \qquad (1)$$

We compute the inner integral using the substitution $u = 4 + r^2$, $du = 2r\,dr$:

$$\int_0^1 \frac{\left(4 + r^2\right)^2 r}{2}\,dr = \int_4^5 \frac{u^2}{2} \cdot \frac{du}{2} = \int_4^5 \frac{u^2}{4}\,du = \frac{u^3}{12}\Big|_4^5 = \frac{61}{12}.$$

Substituting in (1) gives

$$\iiint_{\mathcal{W}} z\,dV = \int_0^{2\pi} \frac{61}{12}\,d\theta = \frac{61}{12}\theta\Big|_0^{2\pi} = \frac{61}{12} \cdot 2\pi = \frac{61\pi}{6} \approx 31.94$$

In Exercises 27–32, use cylindrical coordinates to calculate $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ *for the given function and region.*

27. $f(x, y, z) = x^2 + y^2;\quad x^2 + y^2 \le 9,\quad 0 \le z \le 5$

SOLUTION The projection of $\mathcal{W}$ onto the xy-plane is the region inside the circle $x^2 + y^2 = 9$. In polar coordinates,

$$\mathcal{D}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 3$$

The upper and lower boundaries are the planes $z = 5$ and $z = 0$, respectively. Therefore, $\mathcal{W}$ has the following description in cylindrical coordinates:

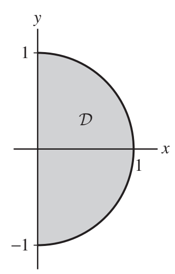
$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ 0 \le z \le 5$$

The integral in cylindrical coordinates is thus

$$\iiint_{\mathcal{W}} (x^2 + y^2)\, dV = \int_0^{2\pi}\int_0^3\int_0^5 r^2 \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^3\int_0^5 r^3\, dz\, dr\, d\theta$$
$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^3 r^3\, dr\right)\left(\int_0^5 1\, dz\right) = 2\pi \cdot 5 \cdot \frac{r^4}{4}\bigg|_0^3 = \frac{5 \cdot 3^4 \pi}{2} \approx 636.17$$

28. $f(x, y, z) = xz;\quad x^2 + y^2 \le 1,\quad x \ge 0,\quad 0 \le z \le 2$

SOLUTION



$\mathcal{W}$ projects onto the semicircle $\mathcal{D}$ in the xy-plane of radius 1, where $x \ge 0$. In polar coordinates we have

$$\mathcal{D}:\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1$$

The upper and lower boundaries of $\mathcal{W}$ are the planes $z = 2$ and $z = 0$, respectively; therefore, $\mathcal{W}$ has the following description in cylindrical coordinates:

$$\mathcal{W}:\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ 0 \le z \le 2$$

The function in cylindrical coordinates is

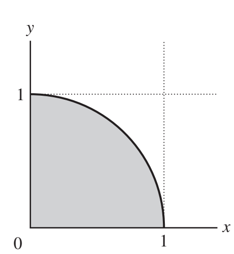
$$f(x, y, z) = xz = (r\cos\theta)z = zr\cos\theta$$

We obtain the following integral:

$$\iiint_{\mathcal{W}} xz\, dV = \int_{-\pi/2}^{\pi/2}\int_0^1\int_0^2 zr\cos\theta \cdot r\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_0^2 zr^2\cos\theta\, dz\, dr\, d\theta$$
$$= \left(\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta\right)\left(\int_0^1 r^2\, dr\right)\left(\int_0^2 z\, dz\right) = \left(\sin\theta\bigg|_{-\pi/2}^{\pi/2}\right)\left(\frac{r^3}{3}\bigg|_0^1\right)\left(\frac{z^2}{2}\bigg|_0^2\right) = 2 \cdot \frac{1}{3} \cdot 2 = \frac{4}{3}$$

29. $f(x, y, z) = y;\quad x^2 + y^2 \le 1,\quad x \ge 0,\quad y \ge 0,\quad 0 \le z \le 2$

SOLUTION



The projection of $\mathcal{W}$ onto the xy-plane is the quarter of the unit circle in the first quadrant. It is defined by the following polar equations:

$$\mathcal{D}: 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1$$

The upper and lower boundaries of $\mathcal{W}$ are the planes $z = 2$ and $z = 0$, respectively; hence, $\mathcal{W}$ has the following definition:

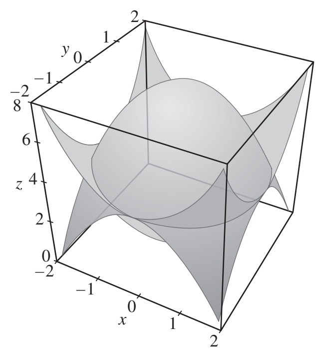
$$\mathcal{W}: 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ 0 \le z \le 2$$

The function is $f(x, y, z) = y = r\sin\theta$. The integral in cylindrical coordinates is thus

$$\iiint_{\mathcal{W}} y\,dV = \int_0^{\pi/2}\int_0^1\int_0^2 (r\sin\theta)r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^2 r^2\sin\theta\,dz\,dr\,d\theta$$
$$= \left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^1 r^2\,dr\right)\left(\int_0^2 1\,dz\right) = \left(-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^1\right)\left(z\Big|_0^2\right) = 1\cdot\frac{1}{3}\cdot 2 = \frac{2}{3}$$

30. $f(x, y, z) = z\sqrt{x^2 + y^2}$; $\quad x^2 + y^2 \le z \le 8 - (x^2 + y^2)$

SOLUTION

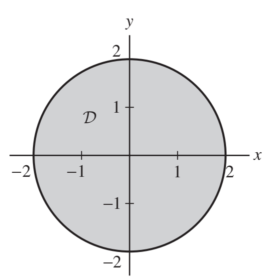


$\mathcal{W}$ is the region enclosed by the upper surface $z = 8 - (x^2 + y^2) = 8 - r^2$ and the lower surface $z = x^2 + y^2 = r^2$, that is, $r^2 \le z \le 8 - r^2$. To find the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane, we first find the curve of intersection of $z = 8 - r^2$ and $z = r^2$ by solving the equation

$$8 - r^2 = r^2 \quad\Rightarrow\quad 8 = 2r^2 \quad\Rightarrow\quad r^2 = 4 \quad\Rightarrow\quad r = 2$$

We conclude that $\mathcal{D}$ is the region in the xy-plane enclosed by the circle of radius 2:

$$\mathcal{D}: 0 \le \theta \le 2\pi,\ 0 \le r \le 2$$



The inequalities describing $\mathcal{W}$ in cylindrical coordinates are thus

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r^2 \le z \le 8 - r^2$$

The function in cylindrical coordinates is

$$f(x, y, z) = z\sqrt{x^2 + y^2} = zr$$

We obtain the following integral:

$$\iiint_{\mathcal{W}} z\sqrt{x^2 + y^2}\,dA = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} zr\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} zr^2\,dz\,dr\,d\theta$$

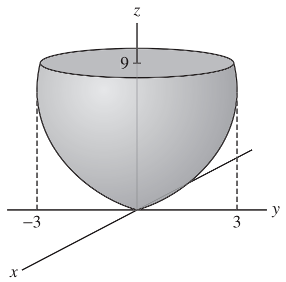
$$= \int_0^{2\pi} \int_0^2 \frac{z^2 r^2}{2}\bigg|_{z=r^2}^{8-r^2} dr\, d\theta = \int_0^{2\pi} \int_0^2 \frac{r^2\left((8-r^2)^2 - r^4\right)}{2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 8r^2(4-r^2)\, dr\, d\theta = \left(\int_0^{2\pi} 8\, d\theta\right)\left(\int_0^2 \left(4r^2 - r^4\right) dr\right)$$

$$= 16\pi \cdot \left(\frac{4r^3}{3} - \frac{r^5}{5}\bigg|_0^2\right) = \frac{1024}{15}\pi \approx 214.47$$

31. $f(x, y, z) = z;\quad x^2 + y^2 \le z \le 9$

SOLUTION



The upper boundary of $\mathcal{W}$ is the plane $z = 9$, and the lower boundary is $z = x^2 + y^2 = r^2$. Therefore, $r^2 \le z \le 9$. The projection $\mathcal{D}$ onto the xy-plane is the circle $x^2 + y^2 = 9$ or $r = 3$. That is,

$$\mathcal{D}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 3$$

The inequalities defining $\mathcal{W}$ in cylindrical coordinates are thus

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ r^2 \le z \le 9$$

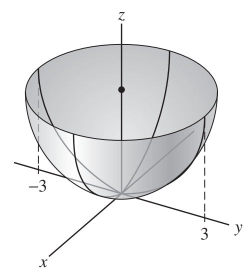
Therefore, we obtain the following integral:

$$\iiint_{\mathcal{W}} z\, dV = \int_0^{2\pi} \int_0^3 \int_{r^2}^9 zr\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \frac{z^2 r}{2}\bigg|_{z=r^2}^9 dr\, d\theta = \int_0^{2\pi} \int_0^3 \frac{r(81-r^4)}{2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \frac{81r - r^5}{2}\, dr\, d\theta = \int_0^{2\pi} \frac{81r^2}{4} - \frac{r^6}{12}\bigg|_0^3 d\theta = \int_0^{2\pi} 121.5\, d\theta = 243\pi$$

32. $f(x, y, z) = z;\quad 0 \le z \le x^2 + y^2 \le 9$

SOLUTION

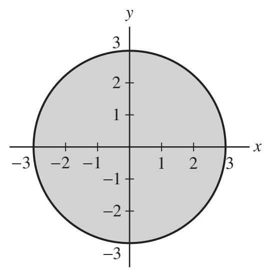


The condition $0 \le z \le x^2 + y^2 = r^2$ tells us that the upper surface is $z = r^2$ and the lower surface is $z = 0$. Also, $x^2 + y^2 \le 9$ or $r \le 3$, hence the projection of $\mathcal{W}$ onto the xy-plane is the region inside the circle $r = 3$. That is,

$$\mathcal{D}: 0 \le \theta \le 2\pi,\ 0 \le r \le 3$$

And $\mathcal{W}$ is determined by

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ 0 \le z \le r^2$$



Therefore, we obtain the following integral in cylindrical coordinates:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^{2\pi}\int_0^3\int_0^{r^2} z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \frac{z^2 r}{2}\bigg|_{z=0}^{r^2} dr\,d\theta = \int_0^{2\pi}\int_0^3 \frac{r^5}{2}\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{r^6}{12}\bigg|_0^3 d\theta = \int_0^{2\pi} \frac{729}{12}\,d\theta = \frac{729}{12}\cdot 2\pi = 121.5\pi$$

In Exercises 33–36, express the triple integral in cylindrical coordinates.

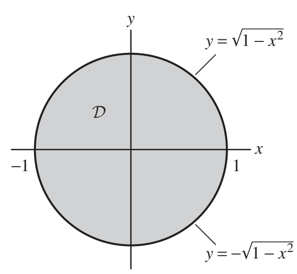
33. $\displaystyle\int_{-1}^{1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=0}^{4} f(x,y,z)\,dz\,dy\,dx$

SOLUTION The region of integration is determined by the limits of integration. That is,

$$\mathcal{W}:\ -1\le x\le 1,\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2},\ 0\le z\le 4$$

Therefore the projection of $\mathcal{W}$ onto the xy-plane is the disk $x^2+y^2\le 1$. This region has the following definition in polar coordinates:

$$\mathcal{D}:\ 0\le\theta\le 2\pi,\ 0\le r\le 1$$



The upper and lower boundaries of $\mathcal{W}$ are the planes $z=4$ and $z=0$, respectively. Hence,

$$\mathcal{W}:\ 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 0\le z\le 4$$

Using change of variables in cylindrical coordinates, we get the integral

$$\int_0^{2\pi}\int_0^1\int_0^4 f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta$$

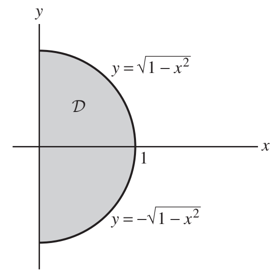
34. $\displaystyle\int_{0}^{1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=0}^{4} f(x,y,z)\,dz\,dy\,dx$

SOLUTION The region of integration is determined by the limits of integration. That is,

$$\mathcal{W}:\ 0\le x\le 1,\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2},\ 0\le z\le 4$$

Thus, the projection $\mathcal{D}$ of $\mathcal{W}$ onto the xy-plane is the semicircle $x^2+y^2=1$, where $0\le x\le 1$. This region is described by the polar inequalities

$$\mathcal{D}:\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le r\le 1$$



The upper and lower boundaries of $\mathcal{W}$ are the planes $z = 4$ and $z = 0$, respectively. Hence,

$$\mathcal{W}: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ 0 \le z \le 4$$

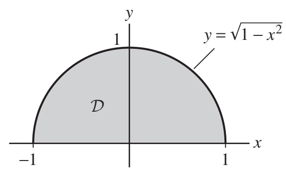
The integral in cylindrical coordinates is thus

$$\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^4 f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

35. $\displaystyle\int_{-1}^{1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{x^2+y^2} f(x,y,z)\, dz\, dy\, dx$

SOLUTION The inequalities defining the region of integration are

$$\mathcal{W}: -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le x^2 + y^2$$



The projection of $\mathcal{W}$ onto the xy-plane is the semicircle $x^2 + y^2 = 1$, where $-1 \le x \le 1$. This domain is defined by the polar inequalities

$$\mathcal{D}: 0 \le \theta \le \pi,\ 0 \le r \le 1$$

The lower surface is $z = 0$ and upper surface is $z = x^2 + y^2 = r^2$, hence $\mathcal{W}$ has the following description in cylindrical coordinates:

$$\mathcal{W}: 0 \le \theta \le \pi,\ 0 \le r \le 1,\ 0 \le z \le r^2$$

We obtain the following integral:

$$\int_0^{\pi} \int_0^1 \int_0^{r^2} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

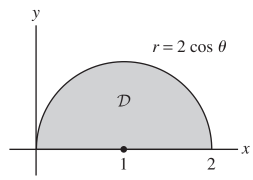
36. $\displaystyle\int_0^2 \int_{y=0}^{y=\sqrt{2x-x^2}} \int_{z=0}^{\sqrt{x^2+y^2}} f(x,y,z)\, dz\, dy\, dx$

SOLUTION The inequalities defining the region of integration are

$$\mathcal{W}: 0 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2},\ 0 \le z \le \sqrt{x^2+y^2}$$

The curve $y = \sqrt{2x - x^2}$ is the semicircle of radius 1 centered at $(1, 0)$, where $y \ge 0$. We find the polar equation of the semicircle:

$$\begin{aligned} r\sin\theta &= \sqrt{2r\cos\theta - r^2\cos^2\theta} \\ r^2\sin^2\theta &= 2r\cos\theta - r^2\cos^2\theta \\ r^2(\sin^2\theta + \cos^2\theta) &= 2r\cos\theta \\ r^2 &= 2r\cos\theta \quad \Rightarrow \quad r = 2\cos\theta \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{2} \end{aligned}$$



Therefore the projection of $\mathcal{W}$ onto the xy-plane is defined by the polar inequalities

$$\mathcal{D} : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$$

The lower boundary of $\mathcal{W}$ is the plane $z = 0$ and the upper boundary is $z = \sqrt{x^2 + y^2} = r$, hence $\mathcal{W}$ has the following cylindrical definition:

$$\mathcal{W} : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta,\ 0 \le z \le r$$

The integral in cylindrical coordinates is

$$\int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^r f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

37. Find the equation of the right-circular cone in Figure 21 in cylindrical coordinates and compute its volume.

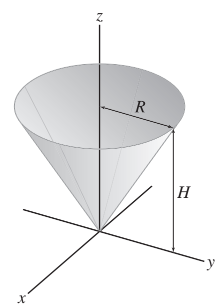
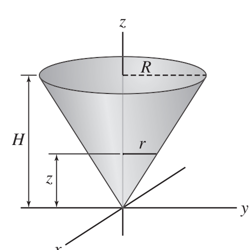


FIGURE 21

SOLUTION To find the equation of the surface we use proportion in similar triangles.



This gives

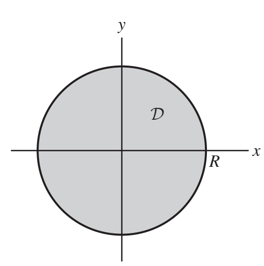
$$\frac{z}{H} = \frac{r}{R} \quad \Rightarrow \quad z = \frac{H}{R} r$$

The volume of the right circular cone is

$$V = \iiint_{\mathcal{W}} 1\, dV$$

The projection of $\mathcal{W}$ onto the xy-plane is the region $x^2 + y^2 \le R^2$, or in polar coordinates,

$$\mathcal{D} : 0 \le \theta \le 2\pi,\ 0 \le r \le R$$



The upper and lower boundaries are the surfaces $z = H$ and $z = \frac{H}{R}r$, respectively. Hence $\mathcal{W}$ is determined by the following cylindrical inequalities:

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le r \le R,\ \frac{H}{R}r \le z \le H$$

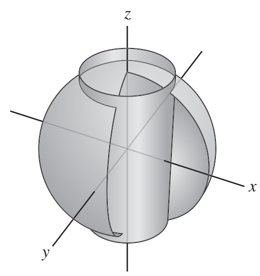
We compute the volume using the following integral:

$$V = \iiint_{\mathcal{W}} 1\,dv = \int_0^{2\pi}\int_0^R\int_{\frac{H}{R}r}^{H} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R rz\Big|_{z=\frac{Hr}{R}}^{H} dr\,d\theta = \int_0^{2\pi}\int_0^R r\left(H - \frac{Hr}{R}\right) dr\,d\theta$$

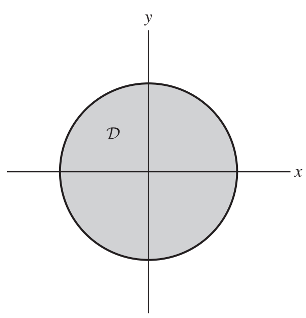
$$= \int_0^{2\pi}\int_0^R \left(rH - \frac{r^2H}{R}\right) dr\,d\theta = \int_0^{2\pi} \frac{r^2H}{2} - \frac{r^3H}{3R}\Big|_{r=0}^{R} d\theta = \int_0^{2\pi} \frac{R^2H}{6}\,d\theta = \frac{R^2H}{6}\cdot 2\pi = \frac{\pi R^2 H}{3}$$

38. Use cylindrical coordinates to integrate $f(x, y, z) = z$ over the intersection of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$, and the cylinder $x^2 + y^2 = 1$.

SOLUTION



The region of integration projects onto the circle $\mathcal{D}$ of radius 1 in the xy-plane.



In polar coordinates,

$$\mathcal{D}: 0 \le \theta \le 2\pi,\ 0 \le r \le 1$$

The upper boundary of $\mathcal{W}$ is the sphere

$$x^2 + y^2 + z^2 = 4 \quad \text{or} \quad z = \sqrt{4 - (x^2 + y^2)} = \sqrt{4 - r^2}$$

and the lower boundary is $z = -\sqrt{4 - r^2}$. Therefore,

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}$$

We obtain the following integral in cylindrical coordinates:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{z^2r}{2}\Big|_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dr\,d\theta = \int_0^{2\pi}\int_0^1 0\,dr\,d\theta = 0$$

This should not surprise us since $\mathcal{W}$ is symmetric with respect to $z = 0$ and f is antisymmetric.

39. Use cylindrical coordinates to calculate the volume of the solid obtained by removing a central cylinder of radius b from a sphere of radius a where $b < a$.

SOLUTION Firstly, the equation of the sphere having radius a is

$$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad r^2 + z^2 = a^2$$

in cylindrical coordinates. Next, the equation of the cylinder with radius b that is being removed from the sphere is

$$x^2 + y^2 = b^2 \quad \Rightarrow \quad r^2 = b^2 \quad \Rightarrow \quad r = b$$

in cylindrical coordinates. Thus the region that is remaining can be described by the following inequalities in cylindrical coordinates:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le b, \quad -\sqrt{a^2 - r^2} \le z \le \sqrt{a^2 - r^2}$$

Thus the volume can be computed:

$$\begin{aligned} V &= \iiint_{\mathcal{W}} 1\, dV = \iiint_{\mathcal{W}} r\, dz\, dr\, d\theta \\ &= 2\int_0^{2\pi} \int_b^a \int_0^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta \\ &= 2\int_0^{2\pi} \int_b^a rz\Big|_{z=0}^{\sqrt{a^2-r^2}} dr\, d\theta \\ &= 2\int_0^{2} \pi \int_b^a r\sqrt{a^2 - r^2}\, dr\, d\theta \\ &= 2\int_0^{2\pi} -\frac{1}{2} \cdot \frac{2}{3}\left(a^2 - r^2\right)^{3/2}\Big|_{r=b}^{a} d\theta \\ &= \frac{2}{3}\int_0^{2\pi} \left(a^2 - b^2\right)^{3/2} d\theta = \frac{2}{3}\left(a^2 - b^2\right)^{3/2} \cdot \theta\Big|_{\theta=0}^{2\pi} \\ &= \frac{4}{3}\pi\left(a^2 - b^2\right)^{3/2} \end{aligned}$$

40. Find the volume of the region in Figure 22.

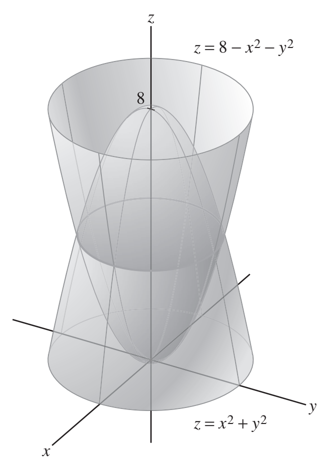
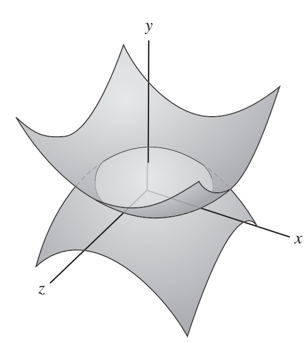


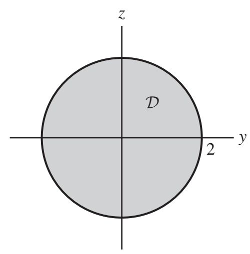
FIGURE 22

SOLUTION



We first must identify the domain of integration in the xy-plane. To do this we equate the equations of the two surfaces. That is,

$$x^2 + y^2 = 8 - x^2 - y^2 \quad \Rightarrow \quad 2(x^2 + y^2) = 8 \quad \Rightarrow \quad x^2 + y^2 = 4$$



Therefore, the domain of integration is

$$\mathcal{D} = \left\{(x, y); x^2 + y^2 \le 4\right\}$$

The double integrals $\iint_{\mathcal{D}} (x^2 + y^2)\, dA$ and $\iint_{\mathcal{D}} (8 - x^2 - y^2)\, dA$ give the volumes of the regions bounded by the graphs of $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$, respectively, and the xy-plane over the region $\mathcal{D}$. Therefore, the volume of the required region is the difference between these two integrals (see figure). That is,

$$V = \iint_{\mathcal{D}} (8 - x^2 - y^2)\, dA - \iint_{\mathcal{D}} (x^2 + y^2)\, dA = \iint_{\mathcal{D}} \Big((8 - x^2 - y^2) - (x^2 + y^2)\Big)\, dA$$

or

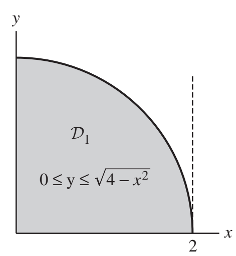
$$V = \iint_{\mathcal{D}} (8 - 2x^2 - 2y^2)\, dA$$

To compute the double integral we first notice that, since the function $f(x, y) = 8 - 2x^2 - 2y^2$ satisfies $f(x, -y) = f(x, y)$ and $f(-x, y) = f(x, y)$, and the circle $\mathcal{D}$ is symmetric with respect to the x and y axes, the double integral over $\mathcal{D}$ is four times the integral over the part $\mathcal{D}_1$ of the circle in the first quadrant. That is,

$$V = 4 \iint_{\mathcal{D}_1} (8 - 2x^2 - 2y^2)\, dA$$

$\mathcal{D}_1$ is the vertically simple region described by the inequalities

$$0 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}$$



Since this is a portion of a circle, using cylindrical coordinates we can write this vertically simple region using the inequalities:

$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2$$

$$V = 4 \iint_{\mathcal{D}_1} (8 - 2x^2 - 2y^2)\, dA = 4 \int_0^{\pi/2} \int_0^2 (8 - 2r^2) r\, dr\, d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^2 8r - 2r^3\, dr\, d\theta = 4 \int_0^{\pi/2} 4r^2 - \frac{1}{2} r^4 \bigg|_0^2 d\theta$$

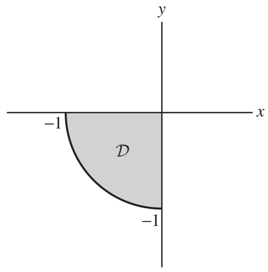
$$= 4 \int_0^{\pi/2} 8\, d\theta = 32 \cdot \frac{\pi}{2} = 16\pi$$

So the volume is 16π.

In Exercises 41–46, use spherical coordinates to calculate the triple integral of $f(x, y, z)$ over the given region.

41. $f(x, y, z) = y; \quad x^2 + y^2 + z^2 \le 1, \quad x, y, z \le 0$

SOLUTION



The region inside the unit sphere in the octant $x, y, z \le 0$ is defined by the inequalities

$$\mathcal{W}: \pi \le \theta \le \frac{3\pi}{2}, \ \frac{\pi}{2} \le \phi \le \pi, \ 0 \le \rho \le 1$$

The function in spherical coordinates is $f(x, y, z) = y = \rho \sin\theta \sin\phi$. Using a triple integral in spherical coordinates, we obtain

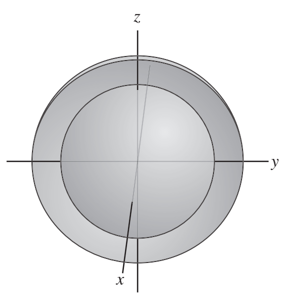
$$\iiint_{\mathcal{W}} y\, dV = \int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi} \int_0^1 (\rho \sin\theta \sin\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi} \int_0^1 \rho^3 \sin\theta \sin^2\phi\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_{\pi}^{3\pi/2} \sin\theta\, d\theta\right)\left(\int_{\pi/2}^{\pi} \sin^2\phi\, d\phi\right)\left(\int_0^1 \rho^3 d\rho\right) = \left(-\cos\theta \Big|_{\pi}^{3\pi/2}\right)\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_{\pi/2}^{\pi}\right)\left(\frac{\rho^4}{4}\Big|_0^1\right)$$

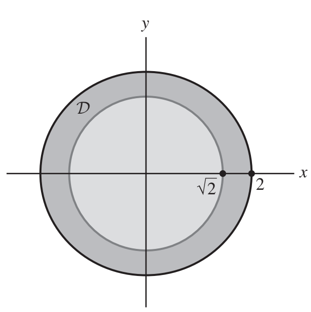
$$= (-1)\cdot\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\cdot\frac{1}{4} = -\frac{\pi}{16}$$

42. $f(x, y, z) = \rho^{-3}; \quad 2 \le x^2 + y^2 + z^2 \le 4$

SOLUTION



The lower and upper boundaries of $\mathcal{W}$ are the spheres $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 4$. Therefore, ρ varies from $\sqrt{2}$ to 2, ϕ varies from 0 to π, and θ varies from 0 to 2π.



That is, $\mathcal{W}$ is defined by the inequalities

$$\mathcal{W}: 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi, \ \sqrt{2} \le \rho \le 2$$

The triple integral in spherical coordinates is thus

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{2}}^{2} \rho^{-3}\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{2}}^{2} \rho^{-1}\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_{\sqrt{2}}^{2}\rho^{-1}\,d\rho\right) = \left(\theta\Big|_0^{2\pi}\right)\left(-\cos\phi\Big|_0^{\pi}\right)\left(\ln\rho\Big|_{\sqrt{2}}^{2}\right)$$
$$= 2\pi\cdot 2\cdot\left(\ln 2 - \frac{1}{2}\ln 2\right) = 2\pi\ln 2$$

43. $f(x,y,z) = x^2+y^2;\quad \rho\le 1$

SOLUTION $\mathcal{W}$ is the region inside the unit sphere, therefore it is described by the following inequalities:

$$\mathcal{W}:\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi,\ 0\le\rho\le 1$$

The function in spherical coordinates is

$$f(x,y,z) = x^2+y^2 = (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2$$
$$= \rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right) = \rho^2\sin^2\phi$$

Using triple integrals in spherical coordinates we get

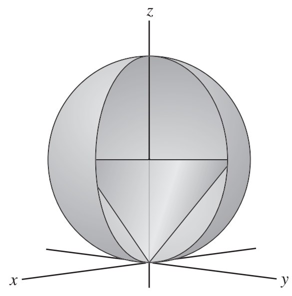
$$\iiint_{\mathcal{W}} (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = \left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^1\rho^4 d\rho\right)$$
$$= \left(\theta\Big|_0^{2\pi}\right)\left(-\frac{\sin^2\theta\cos\theta}{3} - \frac{2}{3}\cos\theta\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^1\right) = 2\pi\cdot\left(\frac{2}{3}+\frac{2}{3}\right)\cdot\frac{1}{5} = \frac{8\pi}{15}$$

44. $f(x,y,z) = 1;\quad x^2+y^2+z^2\le 4z,\quad z\ge\sqrt{x^2+y^2}$

SOLUTION The inequality $x^2+y^2+z^2\le 4z$ can be rewritten as

$$x^2+y^2+z^2-4z\le 0 \quad\Rightarrow\quad x^2+y^2+(z-2)^2\le 4$$

This inequality defines the region inside the sphere of radius 2 centered at $(0,0,2)$. Therefore, $\mathcal{W}$ is the region inside the sphere, above the cone $z=\sqrt{x^2+y^2}$.



We write the equation of the sphere $x^2+y^2+z^2=4z$ in spherical coordinates:

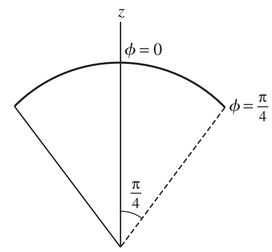
$$(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 + (\rho\cos\phi)^2 = 4\rho\cos\phi$$
$$\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right) + \rho^2\cos^2\phi = 4\rho\cos\phi$$
$$\rho^2\sin^2\phi + \rho^2\cos^2\phi = 4\rho\cos\phi$$
$$\rho^2 = 4\rho\cos\phi$$
$$\rho = 4\cos\phi$$

We write the equation of the cone $z=\sqrt{x^2+y^2}$ in spherical coordinates:

$$\rho\cos\phi = \sqrt{(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2} = \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$$

or

$$\tan\phi = 1 \quad \Rightarrow \quad \phi = \frac{\pi}{4}$$



The spherical inequalities for $\mathcal{W}$ are thus

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le 4\cos\phi$$

We obtain the following integral:

$$\iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4} \left.\frac{\rho^3\sin\phi}{3}\right|_{\rho=0}^{4\cos\phi} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{64\cos^3\phi\sin\phi}{3}\,d\phi\,d\theta = \left(\int_0^{2\pi}\frac{64}{3}\,d\theta\right)\left(\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\right)$$

$$= \frac{128\pi}{3}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi$$

We compute the integral using the substitution $u = \cos\phi$, $du = -\sin\phi\,d\phi$:

$$\iiint_{\mathcal{W}} 1\,dV = \frac{128\pi}{3}\int_1^{1/\sqrt{2}} u^3(-du) = \frac{128\pi}{3}\int_{1/\sqrt{2}}^1 u^3\,du = \frac{128\pi}{3}\cdot\left.\frac{u^4}{4}\right|_{1/\sqrt{2}}^1 = 8\pi$$

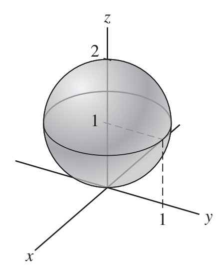
45. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$; $\quad x^2+y^2+z^2 \le 2z$

SOLUTION We rewrite the inequality for the region using spherical coordinates:

$$\rho^2 \le 2\rho\cos\phi \quad \Rightarrow \quad \rho \le 2\cos\phi$$

Completing the square in $x^2+y^2+z^2 = 2z$, we see that this is the equation of the sphere of radius 1 centered at (0, 0, 1). That is,

$$x^2 + y^2 + z - 2z = 0$$
$$x^2 + y^2 + (z-1)^2 = 1$$



$\mathcal{W}$ is the region inside the sphere, hence θ varies from 0 to 2π, and ϕ varies from 0 to $\frac{\pi}{2}$. The inequalities describing $\mathcal{W}$ in spherical coordinates are thus

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 2\cos\phi$$

The function in spherical coordinates is

$$f(x,y,z) = \sqrt{x^2+y^2+z^2} = \rho.$$

We obtain the following integral:

$$I = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\phi} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\phi} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^4 \sin\phi}{4}\Big|_{\rho=0}^{2\cos\phi} d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \frac{16\cos^4\phi \sin\phi}{4} \, d\phi \, d\theta$$

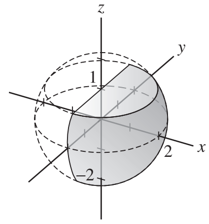
$$= \left(\int_0^{2\pi} 4 \, d\theta\right)\left(\int_0^{\pi/2} \cos^4\phi \sin\phi \, d\phi\right) = 8\pi \int_0^{\pi/2} \cos^4\phi \sin\phi \, d\phi$$

We compute the integral using the substitution $u = \cos\phi$, $du = -\sin\phi \, d\phi$. We obtain

$$I = 8\pi \int_1^0 u^4(-du) = 8\pi \int_0^1 u^4 \, du = 8\pi \frac{u^5}{5}\Big|_0^1 = \frac{8\pi}{5}$$

46. $f(x, y, z) = \rho; \quad x^2 + y^2 + z^2 \le 4, \quad z \le 1, \quad x \ge 0$

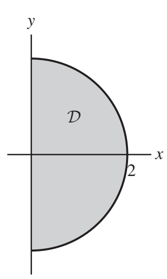
SOLUTION



$\mathcal{W}$ is the region inside the sphere of radius 2, below the plane $z = 1$ and above and below the right xy-plane. The equation of the sphere $x^2 + y^2 + z^2 = 4$ in spherical coordinates is $\rho = 2$, and the equation of the plane $z = 1$ is,

$$\rho\cos\phi = 1 \quad \Rightarrow \quad \rho = \frac{1}{\cos\phi} \tag{1}$$

To evaluate the triple integral we let $\mathcal{W}_1$ be the region inside the sphere above the plane $z = 1$ for $x \ge 0$, and $\mathcal{W}_2$ be the region enclosed by the sphere, for $x \ge 0$.

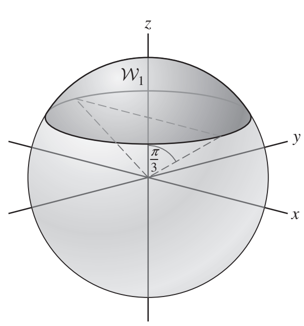


Thus,

$$\iiint_{\mathcal{W}} f(x, y, z) \, dV = \iiint_{\mathcal{W}_2} f(x, y, z) \, dV - \iiint_{\mathcal{W}_1} f(x, y, z) \, dV$$

Setting $\rho = 2$ in (1) gives

$$2 = \frac{1}{\cos\phi} \quad \Rightarrow \quad \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3}$$



Therefore, we obtain the following definition:

$$\mathcal{W}_1 : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{3}, \quad \frac{1}{\cos\phi} \le \rho \le 2$$
$$\mathcal{W}_2 : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \pi, \quad 0 \le \rho \le 2$$

We compute the integral over $\mathcal{W}_1$:

$$\begin{aligned}
\iiint_{\mathcal{W}_1} f(x,y,z)\,dV &= \int_{-\pi/2}^{\pi/2}\int_0^{\pi/3}\int_{1/\cos\phi}^{2} \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/3}\int_{1/\cos\phi}^{2} \rho^3\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_0^{\pi/3} \frac{\rho^4\sin\phi}{4}\bigg|_{\rho=\frac{1}{\cos\phi}}^{2} d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/3}\left(4\sin\phi - \frac{\sin\phi}{4\cos^4\phi}\right)d\phi\,d\theta \\
&= \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^{\pi/3}\left(4\sin\phi - \frac{\sin\phi}{4\cos^4\phi}\right)\right)d\phi \\
&= \pi\int_0^{\pi/3} 4\sin\phi\,d\phi - \pi\int_0^{\pi/3}\frac{\sin\phi}{4\cos^4\phi}\,d\phi \qquad (2)
\end{aligned}$$

We compute the second integral using the substitution $u = \cos\theta$, $du = -\sin\phi\,d\phi$, and the first using an integration formula. We get

$$\int_0^{\pi/3} 4\,\sin\phi\,d\phi = -4\cos\phi\bigg|_0^{\pi/3} = -4\left(\frac{1}{2}-1\right) = 2$$
$$\int_0^{\pi/3}\frac{\sin\phi}{4\cos^4\phi}d\phi = \int_1^{1/2}\frac{1}{4u^4}(-du) = \int_{1/2}^{1}\frac{u^{-4}}{4}\,du = \frac{u^{-3}}{-12}\bigg|_{1/2}^{1} = \frac{7}{12}$$

Substituting the integrals in (2) we get

$$\iiint_{\mathcal{W}_1} f(x,y,z)\,dV = 2\pi - \frac{7\pi}{12} = \frac{17\pi}{12} \qquad (3)$$

We compute the integral over $\mathcal{W}_2$:

$$\begin{aligned}
\iiint_{\mathcal{W}_2} f(x,y,z)\,dV &= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \left(\int_{-\pi/2}^{\pi/2} 1\,d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^2\rho^3 d\rho\right) = \pi\cdot\left(-\cos\phi\bigg|_0^{\pi}\right)\left(\frac{\rho^4}{4}\bigg|_0^2\right) \\
&= \pi\cdot 2\cdot 4 = 8\pi \qquad (4)
\end{aligned}$$

We now substitute (3) and (4) in (1) to obtain the following solution:

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = 8\pi - \frac{17\pi}{12} = \frac{79\pi}{12}$$

47. Use spherical coordinates to evaluate the triple integral of $f(x,y,z) = z$ over the region

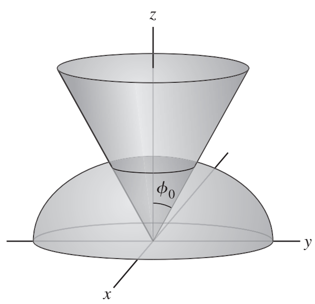
$$0 \le \theta \le \frac{\pi}{3}, \qquad 0 \le \phi \le \frac{\pi}{2}, \qquad 1 \le \rho \le 2$$

SOLUTION The function in spherical coordinates is $f(x,y,z) = z = \rho\cos\phi$. Using triple integral in spherical coordinates gives

$$\begin{aligned}
\iiint_{\mathcal{W}} z\,dV &= \int_0^{\pi/3}\int_0^{\pi/2}\int_1^2(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/3}\int_0^{\pi/2}\int_1^2\rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \left(\int_0^{\pi/3}1\,d\theta\right)\left(\int_0^{\pi/2}\frac{1}{2}\sin 2\phi\,d\phi\right)\left(\int_1^2\rho^3 d\rho\right) \\
&= \frac{\pi}{3}\cdot\left(-\frac{1}{4}\cos 2\phi\right)\bigg|_0^{\pi/2}\cdot\left(\frac{\rho^4}{4}\bigg|_1^2\right) = \frac{\pi}{3}\cdot\frac{1}{2}\cdot\left(4-\frac{1}{4}\right) = \frac{5}{8}\pi
\end{aligned}$$

48. Find the volume of the region lying above the cone $\phi = \phi_0$ and below the sphere $\rho = R$.

SOLUTION



The region is described by the following inequalities in spherical coordinates:

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le \phi \le \phi_0,\ 0 \le \rho \le R$$

We compute the volume V of $\mathcal{W}$ using triple integrals in spherical coordinates:

$$V = \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi} \int_0^{\phi_0} \int_0^R \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\phi_0} \sin\phi\,d\phi\right)\left(\int_0^R \rho^2 d\rho\right)$$

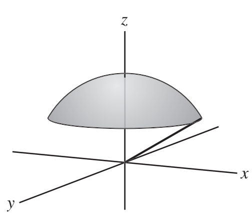
$$= \left(\theta\Big|_0^{2\pi}\right)\left(-\cos\phi\Big|_0^{\phi_0}\right)\left(\frac{\rho^3}{3}\Big|_0^R\right) = 2\pi\,(1-\cos\phi_0)\cdot\frac{R^3}{3} = \frac{2\pi R^3\,(1-\cos\phi_0)}{3}$$

49. Calculate the integral of

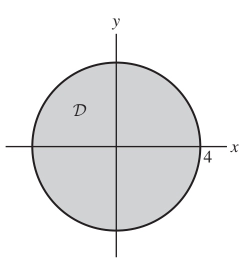
$$f(x, y, z) = z(x^2 + y^2 + z^2)^{-3/2}$$

over the part of the ball $x^2 + y^2 + z^2 \le 16$ defined by $z \ge 2$.

SOLUTION



The equation of the sphere in spherical coordinates is $\rho^2 = 16$ or $\rho = 4$.



We write the equation of the plane $z = 2$ in spherical coordinates:

$$\rho\cos\phi = 2 \quad\Rightarrow\quad \rho = \frac{2}{\cos\phi}$$

To compute the interval of ϕ, we must find the value of ϕ corresponding to $\rho = 4$ on the plane $z = 2$. We get

$$4 = \frac{2}{\cos\phi} \quad\Rightarrow\quad \cos\phi = \frac{1}{2} \quad\Rightarrow\quad \phi = \frac{\pi}{3}$$

Therefore, ϕ is changing from 0 to $\frac{\pi}{3}$, θ is changing from 0 to 2π, and ρ is changing from $\frac{2}{\cos\phi}$ to 4. We obtain the following description for $\mathcal{W}$:

$$\mathcal{W}: 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3},\ \frac{2}{\cos\phi} \le \rho \le 4$$

The function is

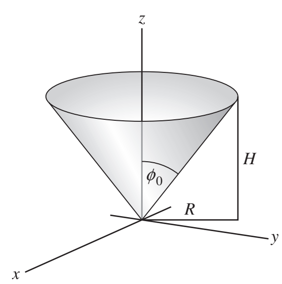
$$f(x, y, z) = z(x^2 + y^2 + z^2)^{-3/2} = \rho \cos\phi \cdot (\rho^2)^{-3/2} = \rho^{-2} \cos\phi$$

We use triple integrals in spherical coordinates to write

$$\begin{aligned}
\iiint_{\mathcal{W}} f(x, y, z)\, dV &= \int_0^{2\pi} \int_0^{\pi/3} \int_{2/\cos\phi}^{4} (\rho^{-2}\cos\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_{2/\cos\phi}^{4} \frac{\sin 2\phi}{2}\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/3} \frac{\sin 2\phi}{2} \rho \Big|_{\rho=\frac{2}{\cos\phi}}^{4} d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left(2\sin 2\phi - \frac{\sin 2\phi}{2} \cdot \frac{2}{\cos\phi}\right) d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/3} (2\sin 2\phi - 2\sin\phi)\, d\phi\, d\theta = 2\pi \cdot \left(-\cos 2\phi + 2\cos\phi \Big|_{\phi=0}^{\pi/3}\right) \\
&= 2\pi \cdot \left(-\cos\frac{2\pi}{3} + 2\cos\frac{\pi}{3} + 1 - 2\right) = \pi
\end{aligned}$$

50. Calculate the volume of the cone in Figure 21 using spherical coordinates.

SOLUTION



First, we write the equation of the upper plane $z = H$ in spherical coordinates:

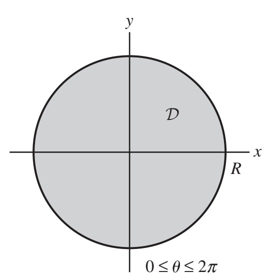
$$\rho\cos\phi = H \quad \Rightarrow \quad \rho = \frac{H}{\cos\phi}$$

To write the equation of the circular cone, we must find the angle ϕ_0 in terms of R and H. That is,

$$\tan\phi_0 = \frac{R}{H} \quad \Rightarrow \quad \phi_0 = \tan^{-1}\frac{R}{H}$$

Therefore, the equation of the lower surface (the circular cone) is $\phi = \tan^{-1}\frac{R}{H}$. We now write the inequalities for the region $\mathcal{W}$:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \tan^{-1}\frac{R}{H},\ 0 \le \rho \le \frac{H}{\cos\phi}$$



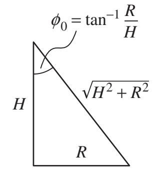
We compute the volume of $\mathcal{W}$ using a triple integral in spherical coordinates. We get

$$\begin{aligned}
V = \iiint_{\mathcal{W}} 1\, dv &= \int_0^{2\pi} \int_0^{\tan^{-1}\frac{R}{H}} \int_0^{H/\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\tan^{-1}\frac{R}{H}} \frac{\rho^3 \sin\phi}{3}\Big|_{\rho=0}^{H/\cos\phi} d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\tan^{-1}\frac{R}{H}} \frac{H^3 \sin\phi}{3\cos^3\phi} d\phi\, d\theta = \left(\int_0^{2\pi} \frac{H^3}{3}\, d\theta\right)\left(\int_0^{\tan^{-1}\frac{R}{H}} \frac{\sin\phi}{\cos^3\phi}\, d\phi\right) = \frac{2\pi H^3}{3} \int_0^{\tan^{-1}\frac{R}{H}} \frac{\sin\phi}{\cos^3\phi}\, d\phi
\end{aligned}$$

We compute the integral using the substitution $u = \cos\phi$, $du = -\sin\phi\, d\phi$. We get

$$V = \frac{2\pi H^3}{3}\int_1^{\cos\left(\tan^{-1}\frac{R}{H}\right)} u^{-3}(-du) = \frac{2\pi H^3}{3}\int_{\cos\left(\tan^{-1}\frac{R}{H}\right)}^1 u^{-3}\,du = \frac{2\pi H^3}{3}\left.\frac{u^{-2}}{-2}\right|_{\cos\left(\tan^{-1}\frac{R}{H}\right)}^1$$

$$= \frac{\pi H^3}{3}\left(\frac{1}{\cos^2\left(\tan^{-1}\frac{R}{H}\right)} - 1\right) \tag{1}$$



Using the triangle shown in the figure, we see that

$$\cos\left(\tan^{-1}\frac{R}{H}\right) = \frac{H}{\sqrt{H^2+R^2}} \quad\Rightarrow\quad \cos^2\left(\tan^{-1}\frac{R}{H}\right) = \frac{H^2}{H^2+R^2} \quad\Rightarrow\quad \frac{1}{\cos^2\left(\tan^{-1}\frac{R}{H}\right)} = 1 + \frac{R^2}{H^2}$$

Substituting in (1) gives

$$V = \frac{\pi H^3}{3}\left(1 + \frac{R^2}{H^2} - 1\right) = \frac{\pi H^3}{3}\cdot\frac{R^2}{H^2} = \frac{\pi R^2 H}{3}$$

51. Calculate the volume of the sphere $x^2 + y^2 + z^2 = a^2$, using both spherical and cylindrical coordinates.

SOLUTION Spherical coordinates: In the entire sphere of radius a, we have

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le \rho \le a$$

Using triple integral in spherical coordinates we get

$$V = \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \left(\int_0^a \rho^2 d\rho\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_0^{2\pi} 1\, d\theta\right)$$

$$= \left(\left.\frac{\rho^3}{3}\right|_0^a\right)\left(\left.-\cos\phi\right|_0^{\pi}\right)\left(\left.\theta\right|_0^{2\pi}\right) = \frac{a^3}{3}\cdot 2\cdot 2\pi = \frac{4\pi a^3}{3}$$

Cylindrical coordinates: The projection of $\mathcal{W}$ onto the xy-plane is the circle of radius a, that is,

$$\mathcal{D}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le a$$

The upper surface is $z = \sqrt{a^2 - (x^2+y^2)} = \sqrt{a^2 - r^2}$ and the lower surface is $z = -\sqrt{a^2 - r^2}$. Therefore, $\mathcal{W}$ has the following description in cylindrical coordinates:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le a,\ -\sqrt{a^2-r^2} \le z \le \sqrt{a^2-r^2}$$

We obtain the following integral:

$$V = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a \left. rz\right|_{z=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dr\, d\theta = \int_0^{2\pi}\int_0^a 2r\sqrt{a^2-r^2}\, dr\, d\theta \tag{1}$$

We compute the inner integral using the substitution $u = \sqrt{a^2 - r^2}$, $du = -\frac{r}{u}\,dr$. We get

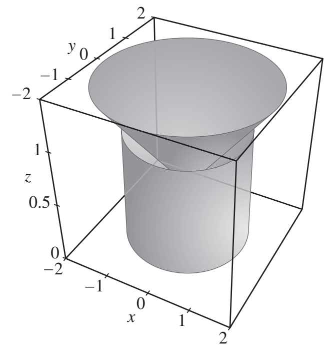
$$\int_0^a 2r\sqrt{a^2-r^2}\, dr = \int_a^0 -2u^2\, du = \int_0^a 2u^2\, du = \left.\frac{2u^3}{3}\right|_0^a = \frac{2a^3}{3}$$

Substituting in (1) gives

$$V = \int_0^{2\pi}\frac{2a^3}{3}\, d\theta = \left.\frac{2a^3}{3}\theta\right|_0^{2\pi} = \frac{2a^3}{3}\cdot 2\pi = \frac{4\pi a^3}{3}.$$

52. Let $\mathcal{W}$ be the region within the cylinder $x^2 + y^2 = 2$ between $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$. Calculate the integral of $f(x, y, z) = x^2 + y^2$ over $\mathcal{W}$, using both spherical and cylindrical coordinates.

SOLUTION Spherical coordinates:



We write the equation of the cylinder $x^2 + y^2 = 2$ in spherical coordinates:

$$2 = (\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 = \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right)$$

$$2 = \rho^2 \sin^2\phi \quad \Rightarrow \quad \rho = \frac{\sqrt{2}}{\sin\phi}$$

We write the equation of the cone $z = \sqrt{x^2 + y^2}$ in spherical coordinates:

$$\rho\cos\phi = \sqrt{(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2} = \sqrt{\rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)} = \rho\sin\phi$$

or

$$\tan\phi = 1 \quad \Rightarrow \quad \phi = \frac{\pi}{4}$$

Therefore the region $\mathcal{W}$ is described by the following inequalities:

$$0 \le \theta \le 2\pi, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{2}, \quad 0 \le \rho \le \frac{\sqrt{2}}{\sin\phi}$$

The function is

$$f(x, y, z) = x^2 + y^2 = (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 = \rho^2\sin^2\phi$$

We obtain the following integral:

$$\begin{aligned}
\iiint_{\mathcal{W}} (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}/\sin\phi} \left(\rho^2 \sin^2\phi\right) \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}/\sin\phi} \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{\rho^5 \sin^3\phi}{5}\bigg|_{\rho=0}^{\sqrt{2}/\sin\phi} d\phi\, d\theta \\
&= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{\frac{4\sqrt{2}}{\sin^5\phi}\cdot\sin^3\phi}{5}\, d\phi\, d\theta = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{4\sqrt{2}}{5}\cdot\frac{1}{\sin^2\phi}\, d\phi\, d\theta \\
&= \left(\int_0^{2\pi} \frac{4\sqrt{2}}{5}\, d\theta\right)\left(\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2\phi}\, d\phi\right) = \left(\frac{4\sqrt{2}}{5}\theta\bigg|_0^{2\pi}\right)\left(-\cot\theta\bigg|_{\pi/4}^{\pi/2}\right) \\
&= \frac{4\sqrt{2}}{5}\cdot 2\pi\cdot 1 = \frac{8\pi\sqrt{2}}{5}
\end{aligned}$$

Cylindrical coordinates: The region of integration has the following definition in cylindrical coordinates:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{2},\ 0 \le z \le \sqrt{x^2 + y^2} = r$$

The function in cylindrical coordinates is $f(x, y, z) = x^2 + y^2 = r^2$. Using triple integrals in cylindrical coordinates, we obtain

$$\iiint_{\mathcal{W}} (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^r r^2 \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^r r^3\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 z\Big|_{z=0}^{r} dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r^4\, dr\, d\theta = \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{\sqrt{2}} r^4\, dr\right)$$

$$= \left(\theta\Big|_0^{2\pi}\right)\left(\frac{r^5}{5}\Big|_0^{\sqrt{2}}\right) = 2\pi \cdot \frac{4\sqrt{2}}{5} = \frac{8\sqrt{2}\pi}{5}$$

53. Bell-Shaped Curve One of the key results in calculus is the computation of the area under the bell-shaped curve (Figure 23):

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

This integral appears throughout engineering, physics, and statistics, and although e^{-x^2} does not have an elementary antiderivative, we can compute I using multiple integration.

(a) Show that $I^2 = J$, where J is the improper double integral

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

Hint: Use Fubini's Theorem and $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$.

(b) Evaluate J in polar coordinates.

(c) Prove that $I = \sqrt{\pi}$.

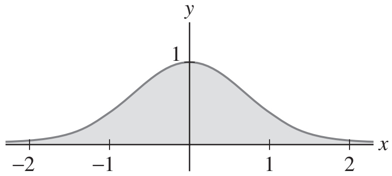


FIGURE 23 The bell-shaped curve $y = e^{-x^2}$.

SOLUTION

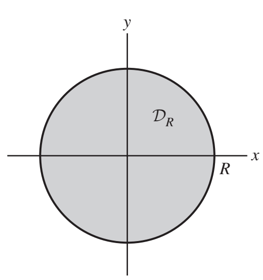
(a) We must show that $I^2 = J$. Firstly, consider the following:

$$I^2 = I \cdot I = \int_{-\infty}^{\infty} e^{-x^2}\, dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2}\, dx\, dy = \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

This works because each integral after the first equals sign is independent of the other.

(b) The improper integral over the xy-plane can be computed as the limit as $\mathcal{R} \to \infty$ of the double integrals over the disk. $\mathcal{D}_{\mathcal{R}}$ is defined by

$$\mathcal{D}_{\mathcal{R}} : 0 \le \theta \le 2\pi,\ 0 \le r \le \mathcal{R}$$



That is,

$$J = \lim_{\mathcal{R}\to\infty} \iint_{\mathcal{D}_{\mathcal{R}}} e^{-(x^2+y^2)}\, dx\, dy \tag{1}$$

We compute the double integral using polar coordinates. The function is $f(x, y) = e^{-(x^2+y^2)} = e^{-r^2}$, hence

$$\iint_{\mathcal{D}_{\mathcal{R}}} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi} \int_0^{\mathcal{R}} e^{-r^2} r\, dr\, d\theta = \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{\mathcal{R}} e^{-r^2} r\, dr\right) = 2\pi \int_0^{\mathcal{R}} e^{-r^2} r\, dr$$

We compute the integral using the substitution $u = r^2$, $du = 2r\,dr$. We get

$$\iint_{\mathcal{D}_{\mathcal{R}}} e^{-(x^2+y^2)}\,dx\,dy = 2\pi \int_0^{\mathcal{R}^2} e^{-u}\frac{du}{2} = \pi \int_0^{\mathcal{R}^2} e^{-u}\,du = \pi(-e^{-u})\Big|_0^{R^2} = \pi(1-e^{-R^2}) \qquad (2)$$

Combining (1) and (2), we get

$$J = \lim_{\mathcal{R}\to\infty}\left(\pi\left(1-e^{-R^2}\right)\right) = \pi$$

On the other hand, using the Iterated Integral of a Product Function, we get

$$\begin{aligned}\pi = J &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\,dx\,dy\\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = I^2\end{aligned}$$

(c) That is,

$$I^2 = \pi \quad\Rightarrow\quad I = \sqrt{\pi}$$

Further Insights and Challenges

54. An Improper Multiple Integral Show that a triple integral of $(x^2+y^2+z^2+1)^{-2}$ over all of $\mathbf{R}^3$ is equal to π^2. This is an improper integral, so integrate first over $\rho \le R$ and let $R \to \infty$.

SOLUTION The triple integral I over $\mathbf{R}^3$ can be computed as the limit as $R \to \infty$ of the triple integral over the balls of radius R. These balls have the following definition in spherical coordinates:

$$\mathcal{D}_R:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le \rho \le R$$

The function in spherical coordinates is $f(x,y,z) = (x^2+y^2+z^2+1)^{-2} = (\rho^2+1)^{-2}$. We obtain the following integral:

$$\begin{aligned}I_R &= \iiint_{\mathcal{W}_R} (x^2+y^2+z^2+1)^{-2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R (1+\rho^2)^{-2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^R (1+\rho^2)^{-2}\rho^2 d\rho\right) = 2\pi\cdot\left(-\cos\phi\Big|_0^{\pi}\right)\int_0^R (1+\rho^2)^{-2}\rho^2 d\rho\\ &= 4\pi\int_0^R (1+\rho^2)^{-2}\rho^2 d\rho\end{aligned}$$

We compute the integral using the trigonometric substitution $\rho = \tan u$, $d\rho = \frac{1}{\cos^2 u}\,du$. Therefore, $1+\rho^2 = 1+\tan^2 u = \frac{1}{\cos^2 u}$, and we obtain

$$\begin{aligned}I_R &= 4\pi\int_0^{\tan^{-1}R}\left(\frac{1}{\cos^2 u}\right)^{-2}\tan^2 u\cdot\frac{1}{\cos^2 u}\,du = 4\pi\int_0^{\tan^{-1}R}\tan^2 u\,\cos^2 u\,du\\ &= 4\pi\int_0^{\tan^{-1}R}\sin^2 u\,du = 4\pi\left(\frac{u}{2}-\frac{\sin 2u}{4}\Big|_0^{\tan^{-1}R}\right) = 4\pi\left(\frac{\tan^{-1}R}{2}-\frac{\sin 2(\tan^{-1}R)}{4}\right)\end{aligned}$$

We now let $R \to \infty$. Using the limit $\lim_{R\to\infty}\tan^{-1}R = \frac{\pi}{2}$, we obtain

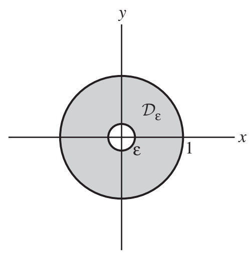
$$I_R = 4\pi\lim_{R\to\infty}\left(\frac{\tan^{-1}R}{2}-\frac{\sin 2(\tan^{-1}R)}{4}\right) = 4\pi\left(\frac{\pi}{4}-\frac{\sin\left(2\cdot\frac{\pi}{2}\right)}{4}\right) = 4\pi\left(\frac{\pi}{4}-0\right) = \pi^2$$

55. Prove the formula

$$\iint_{\mathcal{D}} \ln r\,dA = -\frac{\pi}{2}$$

swhere $r = \sqrt{x^2+y^2}$ and $\mathcal{D}$ is the unit disk $x^2+y^2 \le 1$. This is an improper integral since $\ln r$ is not defined at $(0,0)$, so integrate first over the annulus $a \le r \le 1$ where $0 < a < 1$, and let $a \to 0$.

SOLUTION



The improper integral I is computed by the limit as $a \to 0^+$ of the integrals over the annulus $\mathcal{D}_a$ defined by

$$\mathcal{D}_a : 0 \le \theta \le 2\pi,\ a \le r \le 1$$

Using double integrals in polar coordinates and integration by parts, we get

$$I_a = \int_0^{2\pi} \int_a^1 (\ln r) \cdot r\,dr\,d\theta = 2\pi \int_a^1 r \ln r\,dr = 2\pi \left(\frac{r^2 \ln r}{2} - \frac{r^2}{4} \bigg|_a^1 \right)$$
$$= 2\pi \left(\frac{\ln 1}{2} - \frac{1}{4} - \frac{a^2 \ln a}{2} + \frac{a^2}{4} \right) = \frac{\pi}{2} \left(a^2 - 2a^2 \ln a - 1 \right)$$

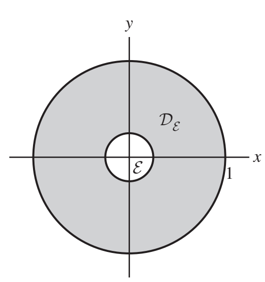
We now compute the limit of I_a as $a \to 0^+$. We use L'Hôpital's rule to obtain

$$I = \lim_{a\to 0^+} \frac{\pi}{2}(a^2 - 2a^2 \ln a - 1) = -\frac{\pi}{2} - \pi \lim_{a\to 0^+} a^2 \ln a = -\frac{\pi}{2} - \pi \lim_{a\to 0^+} \frac{\ln a}{a^{-2}}$$
$$= -\frac{\pi}{2} - \pi \lim_{a\to 0^+} \frac{a^{-1}}{-2a^{-3}} = -\frac{\pi}{2} + \frac{\pi}{2} \lim_{a\to 0^+} a^2 = -\frac{\pi}{2}$$

56. Recall that the improper integral $\int_0^1 x^{-a}\,dx$ converges if and only if $a < 1$. For which values of a does $\iint_{\mathcal{D}} r^{-a}\,dA$ converge, where $r = \sqrt{x^2 + y^2}$ and $\mathcal{D}$ is the unit disk $x^2 + y^2 \le 1$?

SOLUTION The improper integral $I = \iint_{\mathcal{D}} r^{-a}\,dA$ is computed as the limit $\epsilon \to 0^+$ of the double integrals over the annulus $\mathcal{D}_\epsilon$ defined by

$$\mathcal{D}_\epsilon : 0 \le \theta \le 2\pi,\ \epsilon \le r \le 1$$



We compute the integral over $\mathcal{D}_a$ using double integral in polar coordinates. We obtain

$$\iint_{\mathcal{D}_\epsilon} \left(\sqrt{x^2 + y^2} \right)^{-a} dA = \int_0^{2\pi} \int_\epsilon^1 r^{-a} \cdot r\,dr\,d\theta = 2\pi \int_\epsilon^1 r^{1-a}\,dr$$

Therefore,

$$I = \lim_{\epsilon\to 0^+} 2\pi \int_\epsilon^1 r^{1-a}\,dr = 2\pi \int_0^1 r^{1-a}\,dr = 2\pi \int_0^1 r^{-(a-1)}\,dr$$

This integral converges only if $a - 1 < 1$, or $a < 2$.

15.5 Applications of Multiple Integrals (LT Section 16.5)

Preliminary Questions

1. What is the mass density $\rho(x, y, z)$ of a solid of volume 5 m^3 with uniform mass density and total mass 25 kg?

SOLUTION Here, recall that

$$\text{total mass} = \iiint_{\mathcal{W}} \rho(x, y, z)\, dV$$

Since we are told that the solid has volume 5, and $\rho(x, y, z)$ is uniform (i.e. constant, let $\rho(x, y, z) = \rho$), we can write:

$$25 = \iiint_{\mathcal{W}} \rho(x, y, z)\, dV = \rho \cdot V(\mathcal{W}) = 5\rho, \quad \Rightarrow \quad \rho = 5 \text{ kg/m}^3$$

2. A domain $\mathcal{D}$ in $\mathbf{R}^2$ with uniform mass density is symmetric with respect to the y-axis. Which of the following are true?

(a) $x_{\text{CM}} = 0$ **(b)** $y_{\text{CM}} = 0$ **(c)** $I_x = 0$ **(d)** $I_y = 0$

SOLUTION Here, the x-coordinate of the center of mass, $x_{\text{CM}} = 0$ (a) since $x_{\text{CM}} = \frac{M_y}{M}$ and $M_y = \iint_{\mathcal{D}} x\rho(x, y)\, dA$. Since $\rho(x, y) = \rho(-x, y)$, then we see that $(-x)\rho(-x, y) = -x\rho(x, y)$ and M_y is an integral of an odd function over a symmetric region, hence $M_y = 0$.

3. If $p(x, y)$ is the joint probability density function of random variables X and Y, what does the double integral of $p(x, y)$ over $[0, 1] \times [0, 1]$ represent? What does the integral of $p(x, y)$ over the triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ represent?

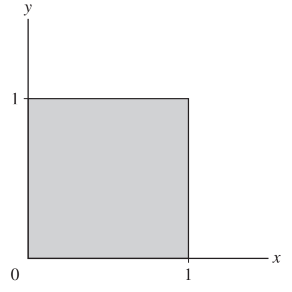
SOLUTION The double integral of $p(x, y)$ over $[0, 1] \times [0, 1]$ represents the probability that both X and Y are between 0 and 1. The integral of $p(x, y)$ over the triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ represents the probability that both X and Y are nonnegative and $X + Y \le 1$.

Exercises

1. Find the total mass of the square $0 \le x \le 1$, $0 \le y \le 1$ assuming a mass density of

$$\rho(x, y) = x^2 + y^2$$

SOLUTION



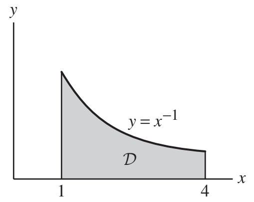
The total mass M is obtained by integrating the mass density $\rho(x, y) = x^2 + y^2$ over the square $\mathcal{D}$ in the xy-plane. This gives

$$M = \iint_{\mathcal{D}} \rho(x, y)\, dA = \int_0^1 \int_0^1 \left(x^2 + y^2\right) dx\, dy = \int_0^1 \frac{x^3}{3} + y^2 x \Big|_{x=0}^{1} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2 - 0\right) dy = \int_0^1 \left(\frac{1}{3} + y^2\right) dy = \frac{y}{3} + \frac{y^3}{3}\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

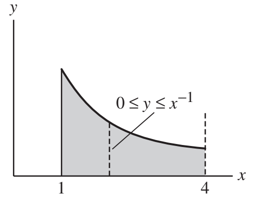
2. Calculate the total mass of a plate bounded by $y = 0$ and $y = x^{-1}$ for $1 \le x \le 4$ (in meters) assuming a mass density of $\rho(x, y) = y/x$ kg/m^2.

SOLUTION The total mass M of the plate is obtained by computing the double integral of mass density $\rho(x, y) = \frac{y}{x}$ over the region $\mathcal{D}$ shown in the figure.



$\mathcal{D}$ is a vertically simple region defined by the following inequalities:

$$1 \le x \le 4, \quad 0 \le y \le x^{-1}$$



We compute the double integral using Theorem 2. That is,

$$M = \iint_{\mathcal{D}} \rho(x, y)\, dA = \int_1^4 \int_0^{x^{-1}} \frac{y}{x}\, dy\, dx = \int_1^4 \frac{y^2}{2x}\bigg|_{y=0}^{x^{-1}} dx = \int_1^4 \frac{\left(x^{-1}\right)^2 - 0^2}{2x}\, dx = \int_1^4 \frac{1}{2} x^{-3}\, dx$$
$$= -\frac{1}{4} x^{-2}\bigg|_{x=1}^{4} = -\frac{1}{4}(4^{-2} - 1^{-2}) = \frac{15}{64}$$

3. Find the total charge in the region under the graph of $y = 4e^{-x^2/2}$ for $0 \le x \le 10$ (in centimeters) assuming a charge density of $\rho(x, y) = 10^{-6} xy$ coulombs per square centimeter.

SOLUTION The total charge C of the region is obtained by computing the double integral of charge density $\rho(x, y) = 10^{-6} xy$ over the region defined by the inequalities

$$0 \le x \le 10, \quad 0 \le y \le 4e^{-x^2/2}$$

Therefore, we compute the double integral

$$C = \iint_{\mathcal{D}} \rho(x, y)\, dA = \int_0^{10} \int_0^{4e^{-x^2/2}} 10^{-6} xy\, dy\, dx = 10^{-6} \int_0^{10} \left(\frac{1}{2} xy^2 \bigg|_{y=0}^{4e^{-x^2/2}} \right) dx$$
$$= \frac{1}{2} \cdot 10^{-6} \int_0^{10} 16xe^{-x^2}\, dx = -4 \cdot 10^{-6} \int_0^{10} e^{-x^2} (-2x\, dx) = -4 \cdot 10^{-6} \left(e^{-x^2} \bigg|_0^{10} \right)$$
$$= -4 \cdot 10^{-6} \left(e^{-10^2} - 1 \right) = \frac{1}{250{,}000} \left(1 - e^{-100} \right)$$

4. Find the total population within a 4-kilometer radius of the city center (located at the origin) assuming a population density of $\rho(x, y) = 2000(x^2 + y^2)^{-0.2}$ people per square kilometer.

SOLUTION The total population P of the region is obtained by computing the double integral of population density $\rho(x, y) = 2000(x^2 + y^2)^{-0.2}$ over the region defined by the inequalities:

$$0 \le x^2 + y^2 \le 16$$

We can think of this region with polar coordinates and write:

$$x = 4\cos\theta, \quad y = 4\sin\theta, \quad 0 \le r \le 4, 0 \le \theta \le 2\pi$$

Transforming with this information we compute the double integral:

$$\iint_{\mathcal{D}} \rho(x, y)\, dA = \iint_{\mathcal{D}} 2000(x^2 + y^2)^{-0.2}\, dA = \int_0^{2\pi} \int_0^4 2000(r^2)^{-0.2} \cdot r\, dr\, d\theta$$
$$= 2000 \int_0^{2\pi} \int_0^4 r^{0.6}\, dr\, d\theta = 2000 \int_0^{2\pi} \frac{1}{1.6} r^{1.6} \bigg|_0^4 d\theta$$

$$= \frac{2000}{1.6} \cdot 4^{1.6} \int_0^{2\pi} 1\, d\theta = 2500\pi \cdot 4^{1.6} \approx 72{,}175$$

5. Find the total population within the sector $2|x| \le y \le 8$ assuming a population density of $\rho(x, y) = 100e^{-0.1y}$ people per square kilometer.

SOLUTION The total population P of the region is obtained by computing the double integral of population density $\rho(x, y) = 100e^{-0.1y}$ over the region defined by the inequality $2|x| \le y \le 8$. This means the region can be split into two vertically simple regions described by the inequalities:

$$0 \le x \le 4, \quad 2x \le y \le 8$$

and

$$-4 \le x \le 4, \quad -2x \le y \le 8$$

Now to compute the double integral:

$$\iint_{\mathcal{D}} \rho(x, y)\, dA = \iint_{\mathcal{D}_1} \rho(x, y)\, dA + \iint_{\mathcal{D}_2} \rho(x, y)\, dA$$

$$\begin{aligned}
\iint_{\mathcal{D}_1} \rho(x, y)\, dA + \iint_{\mathcal{D}_2} \rho(x, y)\, dA &= \int_0^4 \int_{2x}^8 100e^{-0.1y}\, dy\, dx + \int_{-4}^0 \int_{-2x}^8 100e^{-0.1y}\, dy\, dx \\
&= \int_0^4 \frac{100}{-0.1} e^{-0.1y}\Big|_{y=2x}^{8} dx + \int_{-4}^0 \frac{100}{-0.1} e^{-0.1y}\Big|_{y=-2x}^{8} dx \\
&= -1000 \int_0^4 e^{-0.8} - e^{-0.2x}\, dx - 1000 \int_{-4}^0 e^{-0.8} - e^{0.2x}\, dx \\
&= -1000 \left(e^{-0.8}x + 5e^{-0.2x}\Big|_0^4 \right) - 1000 \left(e^{-0.8}x - 5e^{0.2x}\Big|_{-4}^0 \right) \\
&= -1000 \left(4e^{-0.8} + 5e^{-0.8} - 5 \right) - 1000 \left(-5 + 4e^{-0.8} + 5e^{-0.8} \right) \\
&= -1000 \left(18e^{-0.8} - 10 \right) \approx 1912
\end{aligned}$$

6. Find the total mass of the solid region $\mathcal{W}$ defined by $x \ge 0$, $y \ge 0$, $x^2 + y^2 \le 4$, and $x \le z \le 32 - x$ (in centimeters) assuming a mass density of $\rho(x, y, z) = 6y$ g/cm^3.

SOLUTION To find the total mass of this solid region, we will think of it using cylindrical coordinates:

$$0 \le r \le 2, \quad r\cos\theta \le z \le 32 - r\cos\theta$$

and the mass density is:

$$\rho(x, y, z) = 6y \quad \Rightarrow \quad \rho(r\cos\theta, r\sin\theta, z) = 6r\sin\theta$$

Since we are given $x \ge 0$ and $y \ge 0$, then we know

$$0 \le \theta \le \frac{\pi}{2}$$

Therefore, the total mass can be computed:

$$\begin{aligned}
\iiint_{\mathcal{W}} \rho(r\cos\theta, r\sin\theta, z)\, dV &= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \int_{z=r\cos\theta}^{z=32-r\cos\theta} (6r\sin\theta) r\, dz\, dr\, d\theta \\
&= 6 \int_0^{\pi/2} \int_0^2 \int_{r\cos\theta}^{32-r\cos\theta} r^2 \sin\theta\, dz\, dr\, d\theta \\
&= 6 \int_0^{\pi/2} \int_0^2 r^2 \sin\theta \left(z\Big|_{r\cos\theta}^{32-r\cos\theta} \right) dr\, d\theta \\
&= 6 \int_0^{\pi/2} \int_0^2 r^2 \sin\theta\, (32 - 2r\cos\theta)\; dr\, d\theta \\
&= 6 \int_0^{\pi/2} \int_0^2 32r^2 \sin\theta - 2r^3 \sin\theta\cos\theta\, dr\, d\theta
\end{aligned}$$

$$= 6 \int_0^{\pi/2} \frac{32}{3} r^3 \sin\theta - \frac{1}{2} r^4 \sin\theta\cos\theta \Big|_0^2 \, d\theta$$

$$= 6 \int_0^{\pi/2} \frac{256}{3} \sin\theta - 8\sin\theta\cos\theta \, d\theta$$

$$= 6 \left(-\frac{256}{3}\cos\theta - 4\sin^2\theta \right) \Big|_0^{\pi/2}$$

$$= 6 \left(-4 + \frac{256}{3} \right) = 488$$

7. Calculate the total charge of the solid ball $x^2 + y^2 + z^2 \le 5$ (in centimeters) assuming a charge density (in coulombs per cubic centimeter) of

$$\rho(x, y, z) = (3 \cdot 10^{-8})(x^2 + y^2 + z^2)^{1/2}$$

SOLUTION To calculate total charge, first we consider the solid ball in spherical coordinates:

$$x^2 + y^2 + z^2 \le 5 \quad \Rightarrow \quad 0 \le \rho \le \sqrt{5}, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

And the charge density function too, let us rename it $R(x, y, z)$:

$$R(x, y, z) = (3 \cdot 10^{-8})(x^2 + y^2 + z^2)^{1/2} \quad \Rightarrow \quad R(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = (3 \cdot 10^{-8})\rho$$

Then integrating to compute the total charge we have:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{5}} (3 \cdot 10^{-8})\rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 3 \cdot 10^{-8} \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{5}} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \cdot 10^{-8} \int_0^{2\pi}\int_0^{\pi} \sin\phi \left(\frac{1}{4}\rho^4 \Big|_0^{\sqrt{5}} \right) d\phi \, d\theta = 3 \cdot 10^{-8} \cdot \frac{25}{4} \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= 3 \cdot 10^{-8} \cdot \frac{25}{4} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta = 3 \cdot 10^{-8} \cdot \frac{25}{4} \int_0^{2\pi} 2 \, d\theta = 3 \cdot 10^{-8} \cdot 25\pi$$

$$\approx 2.356 \cdot 10^{-6}$$

8. Compute the total mass of the plate in Figure 10 assuming a mass density of $f(x, y) = x^2/(x^2 + y^2)$ g/cm^2.

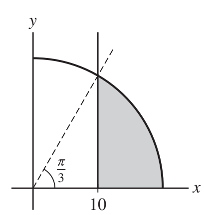


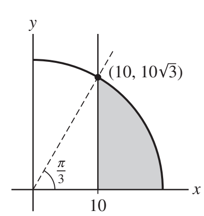
FIGURE 10

SOLUTION The total mass of the plate is

$$M = \iint_{\mathcal{D}} \frac{x^2}{x^2 + y^2} \, dA$$

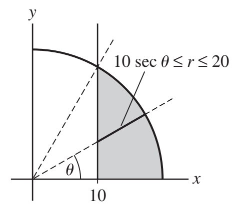
We compute the integral using polar coordinates.

Step 1. Describe $\mathcal{D}$ and f in polar coordinates. The region $\mathcal{D}$ lies in the angular sector $0 \le \theta \le \frac{\pi}{3}$.



The vertical line $x = 10$ has polar equation $r\cos\theta = 10$ or $r = 10\sec\theta$. The circle is of radius $r = \sqrt{10^2 + (10\sqrt{3})^2} = 20$, hence its polar equation is $r = 20$. A ray of angle θ intersects $\mathcal{D}$ for r between $10\sec\theta$ and 20. Therefore, $\mathcal{D}$ has the following description in polar coordinates:

$$0 \le \theta \le \frac{\pi}{3}, \quad 10\sec\theta \le r \le 20$$



The function f can be rewritten as

$$f(x, y) = \frac{x^2}{x^2 + y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Step 2. Change variables in the integral and evaluate. The Change of Variables Formula gives

$$\begin{aligned} M &= \iint_{\mathcal{D}} \frac{x^2}{x^2+y^2}\,dA = \int_0^{\pi/3}\int_{10\sec\theta}^{20} \cos^2\theta \cdot r\,dr\,d\theta = \int_0^{\pi/3} \frac{r^2\cos^2\theta}{2}\bigg|_{r=10\sec\theta}^{20} d\theta \\ &= \int_0^{\pi/3} \frac{\cos^2\theta}{2}\left(400 - 100\sec^2\theta\right) d\theta = \int_0^{\pi/3} 200\cos^2\theta - 50\,d\theta \\ &= 100\int_0^{\pi/3} 1 + \cos 2\theta\,d\theta - 50\int_0^{\pi/3} d\theta = 100\left(\theta + \frac{1}{2}\sin 2\theta\bigg|_0^{\pi/3}\right) - \frac{50\pi}{3} \\ &= 100\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right) - \frac{50\pi}{3} = \frac{50\pi}{3} + 25\sqrt{3} \approx 95.661 \end{aligned}$$

9. Assume that the density of the atmosphere as a function of altitude h (in km) above sea level is $\rho(h) = ae^{-bh}$ kg/km^3, where $a = 1.225 \times 10^9$ and $b = 0.13$. Calculate the total mass of the atmosphere contained in the cone-shaped region $\sqrt{x^2 + y^2} \le h \le 3$.

SOLUTION First we must consider the given cone in cylindrical coordinates:

$$\sqrt{x^2 + y^2} \le z \le 3 \quad \Rightarrow \quad r \le z \le 3$$

while

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

And the density function as well:

$$\rho(x, y, z) = ae^{-bz} \quad \Rightarrow \quad \rho(r\cos\theta, r\sin\theta, z) = ae^{-bz}$$

Now to compute the total mass of the atmosphere in question:

$$\begin{aligned} \int_0^{\theta=2\pi}\int_{r=0}^3\int_{z=r}^3 ae^{-bz}\cdot r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^3\int_r^3 r(ae^{-bz})\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 r\left(-\frac{1}{b}ae^{-bz}\bigg|_{z=r}^3\right) dr\,d\theta \\ &= -\frac{a}{b}\int_0^{2\pi}\int_0^3 re^{-3b} - re^{-br}\,dr\,d\theta \\ &= -\frac{a}{b}\int_0^{2\pi} \frac{1}{2}r^2e^{-3b}\bigg|_0^3 - \left(-\frac{1}{b}re^{-br} - \frac{1}{b^2}e^{-br}\bigg|_0^3\right) d\theta \\ &= -\frac{a}{b}\cdot 2\pi\left(\frac{9}{2}e^{-3b} + \frac{3}{b}e^{-3b} + \frac{1}{b^2}e^{-3b} - \frac{1}{b^2}\right) \end{aligned}$$

Now, since $a = 1.225 \times 10^9$ and $b = 0.13$ we have that the total mass is

$$-\frac{a}{b} \cdot 2\pi \left(\frac{9}{2}e^{-3b} + \frac{3}{b}e^{-3b} + \frac{1}{b^2}e^{-3b} - \frac{1}{b^2}\right) \approx 2.593 \times 10^{10}$$

10. Calculate the total charge on a plate $\mathcal{D}$ in the shape of the ellipse with the polar equation

$$r^2 = \left(\frac{1}{6}\sin^2\theta + \frac{1}{9}\cos^2\theta\right)^{-1}$$

with the disk $x^2 + y^2 \le 1$ removed (Figure 11) assuming a charge density of $\rho(r,\theta) = 3r^{-4}$ C/cm^2.

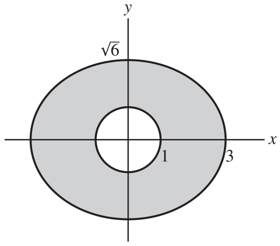
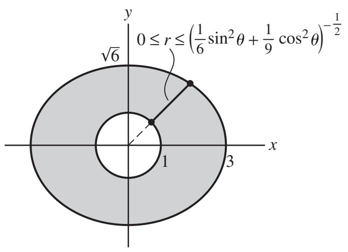


FIGURE 11

SOLUTION We first describe the region in polar coordinates. The circle $x^2 + y^2 = 1$ has the polar equation $r = 1$, and the ellipse has the polar equation $r = \left(\frac{1}{6}\sin^2\theta + \frac{1}{9}\cos^2\theta\right)^{-1/2}$.



The angle θ is changing between 0 and 2π, therefore the polar inequalities for the region are

$$\mathcal{D}: \ 0 \le \theta \le 2\pi, \ 1 \le r \le \left(\frac{1}{6}\sin^2\theta + \frac{1}{9}\cos^2\theta\right)^{-1/2}$$

Using the Double Integral in Polar Coordinates, we obtain the following iterated integral for the total charge on the plate:

$$\begin{aligned}
Q &= \int_0^{2\pi}\int_1^{\left(\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{9}\right)^{-1/2}} 3r^{-4}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\left(\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{9}\right)^{-1/2}} 3r^{-3}\,dr\,d\theta \\
&= \int_0^{2\pi} \left.\frac{-3r^{-2}}{2}\right|_{r=1}^{\left(\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{9}\right)^{-1/2}} d\theta = \int_0^{2\pi} -\frac{3}{2}\left(\frac{\sin^2\theta}{6} + \frac{\cos^2\theta}{9} - 1\right) d\theta \\
&= -\frac{3}{2}\int_0^{2\pi} \frac{2+\sin^2\theta}{18} - 1\,d\theta = -\frac{3}{2}\int_0^{2\pi} \frac{1}{18}\sin^2\theta - \frac{8}{9}\,d\theta \\
&= -\frac{3}{2}\left(\frac{1}{36}\theta - \frac{1}{36}\sin\theta\cos\theta - \frac{8}{9}\theta\right)\Bigg|_0^{2\pi} \\
&= -\frac{3}{2}\left(\frac{\pi}{18} - \frac{16\pi}{9}\right) = \frac{3}{2}\cdot\frac{31}{18}\pi = \frac{31}{12}\pi
\end{aligned}$$

In Exercises 11–14, find the centroid of the given region.

11. Region bounded by $y = 1 - x^2$ and $y = 0$

SOLUTION First we will compute the area of the region:

$$\text{Area}(\mathcal{D}) = \int_{-1}^{1}\int_0^{1-x^2} dy\,dx = \int_{-1}^{1} 1 - x^2\,dx = x - \frac{1}{3}x^3\Big|_{-1}^{1} = 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

It is clear from symmetry that $\overline{x} = 0$, and

$$\begin{aligned}
\overline{y} &= \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} y\, dA = \frac{3}{4} \int_{-1}^{1} \int_{0}^{1-x^2} y\, dy\, dx \\
&= \frac{3}{4} \int_{-1}^{1} \frac{1}{2} y^2 \Big|_0^{1-x^2} dx = \frac{3}{8} \int_{-1}^{1} (1-x^2)^2\, dx \\
&= \frac{3}{8} \int_{-1}^{1} 1 - 2x^2 + x^4\, dx = \frac{3}{8} \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_{-1}^{1} \\
&= \frac{3}{8} \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \frac{3}{8} \left(-1 + \frac{2}{3} - \frac{1}{5} \right) = \frac{2}{5}
\end{aligned}$$

The centroid has coordinates $(\overline{x}, \overline{y}) = \left(0, \frac{2}{5}\right)$.

12. Region bounded by $y^2 = x + 4$ and $x = 4$

SOLUTION First we compute the area of the region. Note that when $x = 4$, $y = \pm\sqrt{8} = \pm 2\sqrt{2}$, so that

$$\text{Area}(\mathcal{D}) = \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{y^2-4}^{4} 1\, dx\, dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} 8 - y^2\, dy = \left(8y - \frac{1}{3}y^3 \right) \Big|_{-2\sqrt{2}}^{2\sqrt{2}} = \frac{64}{3}\sqrt{2}$$

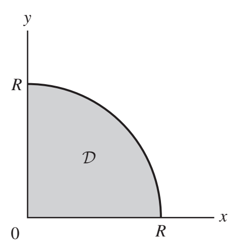
Since the region is symmetric around the x-axis, it is clear that $\overline{y} = 0$, and

$$\begin{aligned}
\overline{x} &= \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} x\, dA = \frac{3}{64\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{y^2-4}^{4} x\, dx\, dy \\
&= \frac{3}{64\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{1}{2} x^2 \Big|_{y^2-4}^{4} dy = \frac{3}{64\sqrt{2}} \cdot \frac{1}{2} \int_{-2\sqrt{2}}^{2\sqrt{2}} 16 - (y^2 - 4)^2\, dy \\
&= \frac{3}{128\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} 8y^2 - y^4\, dy = \frac{3}{128\sqrt{2}} \left(\frac{8}{3}y^3 - \frac{1}{5}y^5 \Big|_{-2\sqrt{2}}^{2\sqrt{2}} \right) \\
&= \frac{3}{128\sqrt{2}} \left(\frac{128\sqrt{2}}{3} - \frac{128\sqrt{2}}{5} + \frac{128\sqrt{2}}{3} - \frac{128\sqrt{2}}{5} \right) \\
&= \frac{3}{128\sqrt{2}} \cdot \frac{512\sqrt{2}}{15} = \frac{4}{5}
\end{aligned}$$

The centroid has coordinates $(\overline{x}, \overline{y}) = \left(\frac{4}{5}, 0\right)$.

13. Quarter circle $x^2 + y^2 \le R^2$, $x \ge 0$, $y \ge 0$

SOLUTION



The centroid $P = (\overline{x}, \overline{y})$ has the following coordinates:

$$\begin{aligned}
\overline{x} &= \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} x\, dA = \frac{4}{\pi R^2} \iint_{\mathcal{D}} x\, dA \\
\overline{y} &= \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} y\, dA = \frac{4}{\pi R^2} \iint_{\mathcal{D}} y\, dA
\end{aligned}$$

We compute the integrals using polar coordinates. The domain $\mathcal{D}$ is described in polar coordinates by the inequalities

$$\mathcal{D}: 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le R$$

The functions are $x = r\cos\theta$ and $y = r\sin\theta$, respectively. Using the Change of Variables Formula gives

$$\begin{aligned}\overline{x} &= \frac{4}{\pi R^2}\int_0^{\pi/2}\int_0^R r\cos\theta\cdot r\,dr\,d\theta = \frac{4}{\pi R^2}\int_0^{\pi/2}\int_0^R r^2\cos\theta\,dr\,d\theta = \frac{4}{\pi R^2}\int_0^{\pi/2}\frac{r^3\cos\theta}{3}\bigg|_{r=0}^R d\theta \\ &= \frac{4}{\pi R^2}\int_0^{\pi/2}\frac{R^3\cos\theta}{3}\,d\theta = \frac{4R}{3\pi}\sin\theta\bigg|_0^{\pi/2} = \frac{4R}{3\pi}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{4R}{3\pi}\end{aligned}$$

And,

$$\begin{aligned}\overline{y} &= \frac{4}{\pi R^2}\int_0^{\pi/2}\int_0^R r\sin\theta\cdot r\,dr\,d\theta = \frac{4}{\pi R^2}\int_0^{\pi/2}\int_0^R r^2\sin\theta\,dr\,d\theta = \frac{4}{\pi R^2}\int_0^{\pi/2}\frac{r^3\sin\theta}{3}\bigg|_{r=0}^R d\theta \\ &= \frac{4}{\pi R^2}\int_0^{\pi/2}\frac{R^3\sin\theta}{3}\,d\theta = \frac{4R}{3\pi}(-\cos\theta)\bigg|_0^{\pi/2} = \frac{4R}{3\pi}\left(-\cos\frac{\pi}{2} + \cos 0\right) = \frac{4R}{3\pi}\end{aligned}$$

Notice that we can use the symmetry of $\mathcal{D}$ with respect to x and y to conclude that $\overline{y} = \overline{x}$, and save the computation of $\overline{y}$. We obtain the centroid $P = \left(\frac{4R}{3\pi}, \frac{4R}{3\pi}\right)$.

14. Infinite lamina bounded by the x- and y-axes and the graph of $y = e^{-x}$

SOLUTION The area of this region is

$$A = \int_0^\infty e^{-x}\,dx = \left(-e^{-x}\right)\bigg|_0^\infty = 1$$

Using integration by parts, we have

$$\begin{aligned}\overline{x} &= \frac{1}{A}\iint_{\mathcal{D}} x\,dA = \int_0^\infty\int_0^{e^{-x}} x\,dy\,dx = \int_0^\infty xe^{-x}\,dx = \left(-xe^{-x}\right)\bigg|_0^\infty + \int_0^\infty e^{-x}\,dx \\ &= 0 - \left(-e^{-x}\bigg|_0^\infty\right) = 1\end{aligned}$$

and

$$\begin{aligned}\overline{y} &= \frac{1}{A}\iint_{\mathcal{D}} y\,dA = \int_0^\infty\int_0^{e^{-x}} y\,dy\,dx = \int_0^\infty\left(\frac{1}{2}y^2\right)\bigg|_0^{e^{-x}} dx \\ &= \int_0^\infty \frac{1}{2}e^{-2x}\,dx = \left(-\frac{1}{4}e^{-2x}\right)\bigg|_0^\infty = \frac{1}{4}\end{aligned}$$

and therefore the centroid is $(\overline{x}, \overline{y}) = \left(1, \dfrac{1}{4}\right)$.

15. CAS Use a computer algebra system to compute numerically the centroid of the shaded region in Figure 12 bounded by $r^2 = \cos 2\theta$ for $x \geq 0$.

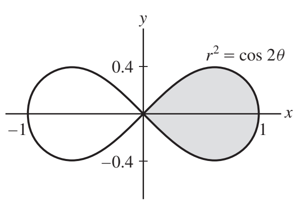


FIGURE 12

SOLUTION Using symmetry, it is easy to see $\overline{y} = 0$. Also, computing the area of the region,

$$\text{Area} = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\,d\theta = \int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{2}\sin 2\theta\bigg|_{-\pi/4}^{\pi/4} = 1$$

and we will compute $\overline{x}$ as

$$\overline{x} = \frac{1}{A}\iint_{\mathcal{D}} x\,dA = \int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} r\cos\theta\cdot r\,dr\,d\theta = \frac{\sqrt{2}}{16}\pi \approx 0.278$$

Therefore, we have that the centroid is $(\overline{x}, \overline{y}) = (\sqrt{2}\pi/16, 0)$.

16. Show that the centroid of the sector in Figure 13 has y-coordinate

$$\overline{y} = \left(\frac{2R}{3}\right)\left(\frac{\sin\theta}{\theta}\right)$$

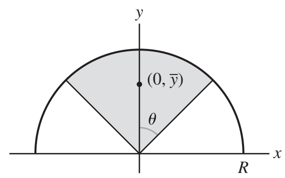
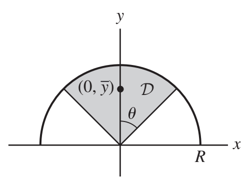


FIGURE 13

SOLUTION



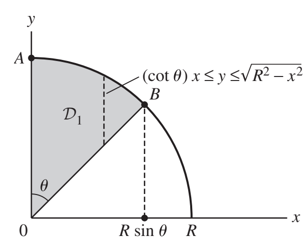
The y-coordinate of the centroid of the sector $\mathcal{D}$ is the average value of the y-coordinate of a point in the sector. That is,

$$\overline{y} = \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} y\,dA$$

The sector is symmetric with respect to the y-axis, and the integrand $f(x, y) = y$ satisfies $f(-x, y) = f(x, y)$, hence the double integral over $\mathcal{D}$ is twice the integral over the right half $\mathcal{D}_1$ of the sector. Also $\text{Area}(\mathcal{D}) = 2\text{Area}\,(\mathcal{D}_1)$, therefore

$$\overline{y} = \frac{1}{\text{Area}\,(\mathcal{D}_1)}\iint_{\mathcal{D}_1} y\,dA \tag{1}$$

We now find the inequalities describing $\mathcal{D}_1$ as a vertically simple region.



The circle bounding the region has the equation $y = \sqrt{R^2 - x^2}$ and the line OB has the equation $y = (\cot\theta)x$. We obtain the following inequalities for $\mathcal{D}_1$:

$$0 \le x \le R\sin\theta, \quad (\cot\theta)x \le y \le \sqrt{R^2 - x^2}$$

Hence,

$$\iint_{\mathcal{D}_1} y\,dA = \int_0^{R\sin\theta}\int_{(\cot\theta)x}^{\sqrt{R^2-x^2}} y\,dy\,dx = \int_0^{R\sin\theta} \frac{y^2}{2}\bigg|_{y=(\cot\theta)x}^{\sqrt{R^2-x^2}} dx = \int_0^{R\sin\theta} \frac{R^2 - x^2 - (\cot\theta)^2x^2}{2}\,dx$$

$$= \int_0^{R\sin\theta}\left(\frac{R^2}{2} - \frac{x^2}{2\sin^2\theta}\right)dx = \frac{R^2}{2}x - \frac{x^3}{6\sin^2\theta}\bigg|_{x=0}^{R\sin\theta} = \frac{R^3\sin\theta}{2} - \frac{R^3\sin^3\theta}{6\sin^2\theta} = \frac{R^3\sin\theta}{3} \tag{2}$$

The area of the sector $\mathcal{D}_1$ is

$$\text{Area}\,(\mathcal{D}_1) = \frac{R^2\theta}{2} \tag{3}$$

Substituting (2) and (3) in (1), we obtain the following solution:

$$\overline{y} = \frac{1}{\frac{R^2\theta}{2}} \cdot \frac{R^3\sin\theta}{3} = \frac{2R^3\sin\theta}{3R^2\theta} = \left(\frac{2R}{3}\right)\left(\frac{\sin\theta}{\theta}\right)$$

In Exercises 17–19, find the centroid of the given solid region.

17. Hemisphere $x^2 + y^2 + z^2 \le R^2, z \ge 0$

SOLUTION First we need to find the volume of the solid in question. It is a hemisphere, so using geometry, we have

$$\text{Volume} = \frac{1}{2} \cdot \frac{4}{3}\pi R^3 = \frac{2\pi R^3}{3}$$

The centroid is the point P with the following coordinates:

$$\overline{x} = \frac{1}{V}\iiint_{\mathcal{W}} x\,dV, \quad \overline{y} = \frac{1}{V}\iiint_{\mathcal{W}} y\,dV, \quad \overline{z} = \frac{1}{V}\iiint_{\mathcal{W}} z\,dV$$

By symmetry, it is clear that $\overline{x} = \overline{y} = 0$, and using spherical coordinates,

$$\begin{aligned}
\overline{z} &= \frac{1}{V}\iiint_{region} z\,dV = \frac{3}{2\pi R^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R \rho\cos\phi \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{3}{2\pi R^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R \rho^3 \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{3}{2\pi R^3}\int_0^{2\pi} 1\,d\theta \cdot \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi \cdot \int_0^R \rho^3\,d\rho \\
&= \frac{3}{2\pi R^3}\cdot 2\pi\left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\right)\left(\frac{1}{4}\rho^4\Big|_0^R\right) = \frac{3}{2\pi R^3}\cdot 2\pi \cdot \frac{1}{2}\cdot\frac{1}{4}R^4 = \frac{3R}{8}
\end{aligned}$$

Therefore, the coordinates of the centroid of a hemisphere having radius R, are $(0, 0, 3R/8)$.

18. Region bounded by the xy-plane, the cylinder $x^2 + y^2 = R^2$, and the plane $x/R + z/H = 1$, where $R > 0$ and $H > 0$

SOLUTION First to find the volume of this solid. The first equation lends itself well to cylindrical coordinates:

$$x^2 + y^2 = R^2 \quad \Rightarrow \quad r = R, 0 \le \theta \le 2\pi$$

and

$$\frac{x}{R} + \frac{z}{H} = 1 \quad \Rightarrow \quad z = H\left(1 - \frac{x}{R}\right) = H\left(1 - \frac{r\cos\theta}{R}\right)$$

The volume is:

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} 1\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R H\left(1 - \frac{r\cos\theta}{R}\right)dr\,d\theta \\
&= H\int_0^{2\pi} r - \frac{1}{2}\cdot\frac{r^2\cos\theta}{R}\Big|_{r=0}^R d\theta = H\int_0^{2\pi} R - \frac{1}{2}R\cos\theta\,d\theta \\
&= H\left(R\theta - \frac{1}{2}R\sin\theta\Big|_0^{2\pi}\right) = 2\pi HR
\end{aligned}$$

Now to compute the coordinates of the centroid:

$$\begin{aligned}
\overline{x} &= \frac{1}{V}\iiint_{\mathcal{W}} x\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} r\cos\theta\,dz\,dr\,d\theta \\
&= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R r\cos\theta\cdot z\Big|_0^{H(1-r\cos\theta/R)} dr\,d\theta = \frac{H}{2\pi HR}\int_0^{2\pi}\int_0^R r\cos\theta\left(1 - \frac{r\cos\theta}{R}\right)dr\,d\theta \\
&= \frac{1}{2\pi R}\int_0^{2\pi}\int_0^R r\cos\theta - \frac{1}{R}r^2\cos^2\theta\,dr\,d\theta = \frac{1}{2\pi R}\int_0^{2\pi}\frac{1}{2}r^2\cos\theta - \frac{1}{3R}r^3\cos^2\theta\Big|_0^R d\theta \\
&= \frac{1}{2\pi R}\int_0^{2\pi}\frac{1}{2}R^2\cos\theta - \frac{R^2}{6}(1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2\pi R}\left(\frac{1}{2}R^2\sin\theta - \frac{R^2}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right) = \frac{1}{2\pi R}\cdot -\frac{R^2}{6}(2\pi) = -\frac{R}{6}
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{1}{V}\iiint_{\mathcal{W}} y\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} r\sin\theta\,dz\,dr\,d\theta\\
&= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R r\sin\theta\cdot z\Big|_0^{H(1-r\cos\theta/R)} dr\,d\theta = \frac{H}{2\pi HR}\int_0^{2\pi}\int_0^R r\sin\theta\left(1-\frac{r\cos\theta}{R}\right)dr\,d\theta\\
&= \frac{1}{2\pi R}\int_0^{2\pi}\int_0^R r\sin\theta - \frac{1}{R}r^2\sin\theta\cos\theta\,dr\,d\theta = \frac{1}{2\pi R}\int_0^{2\pi}\frac{1}{2}r^2\sin\theta - \frac{1}{3R}r^3\sin\theta\cos\theta\Big|_0^R d\theta\\
&= \frac{1}{2\pi R}\int_0^{2\pi}\frac{1}{2}R^2\sin\theta - \frac{R^2}{3}\sin\theta\cos\theta\,d\theta = \frac{1}{2\pi R}\left(-\frac{1}{2}R^2\cos\theta - \frac{R^2}{6}\sin^2\theta\Big|_0^{2\pi}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\overline{z} &= \frac{1}{V}\iiint_{\mathcal{W}} z\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} z\,dz\,dr\,d\theta\\
&= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R \frac{1}{2}z^2\Big|_0^{H(1-r\cos\theta/R)} dr\,d\theta = \frac{H^2}{4\pi HR}\int_0^{2\pi}\int_0^R\left(1-\frac{r\cos\theta}{R}\right)^2 dr\,d\theta\\
&= \frac{H}{4\pi R}\int_0^{2\pi}\int_0^R 1 - \frac{2r\cos\theta}{R} + \frac{r^2\cos^2\theta}{R^2}\,dr\,d\theta = \frac{H}{4\pi R}\int_0^{2\pi} r - \frac{r^2\cos\theta}{R} + \frac{r^3\cos^2\theta}{3R^2}\Big|_0^R d\theta\\
&= \frac{H}{4\pi R}\int_0^{2\pi} R - R\cos\theta + \frac{1}{6}R(1+\cos 2\theta)\,d\theta = \frac{H}{4\pi R}\left(R\theta - R\sin\theta + \frac{1}{6}R\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right)\\
&= \frac{H}{4\pi R}\left(2\pi R + \frac{1}{3}\pi R\right)\\
&= \frac{H}{4\pi R}\cdot\frac{7\pi R}{3} = \frac{7}{12}H
\end{aligned}$$

The coordinates of the centroid are $(-R/6, 0, 7H/12)$.

19. The "ice cream cone" region $\mathcal{W}$ bounded, in spherical coordinates, by the cone $\phi = \pi/3$ and the sphere $\rho = 2$

SOLUTION First we must find the volume of this solid:

$$\begin{aligned}
V &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=0}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\
&= 2\pi\left(\int_0^{\pi/3}\sin\phi\,d\phi\right)\left(\int_0^2\rho^2\,d\rho\right) = 2\pi\cdot\frac{8}{3}\left(-\cos\phi\Big|_0^{\pi/3}\right)\\
&= \frac{16\pi}{3}\cdot\frac{1}{2} = \frac{8\pi}{3}
\end{aligned}$$

And now compute the coordinates of the centroid. By symmetry, it is clear that $\overline{x} = \overline{y} = 0$.

$$\begin{aligned}
\overline{z} &= \frac{1}{V}\iiint_{\mathcal{W}} z\,dV = \frac{3}{8\pi}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=0}^{2}\rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\
&= \frac{3}{8\pi}\int_0^{2\pi}d\theta\cdot\int_0^{\pi/3}\cos\phi\sin\phi\,d\phi\cdot\int_0^2\rho^3\,d\rho\\
&= \frac{3}{8\pi}\cdot 2\pi\cdot\left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/3}\right)\left(\frac{1}{4}\rho^4\Big|_0^2\right) = \frac{3}{4}\left(\frac{1}{2}\cdot\frac{3}{4}\right)(4) = \frac{9}{8}
\end{aligned}$$

The coordinates of the centroid are $(0, 0, 9/8)$.

20. Show that the z-coordinate of the centroid of the tetrahedron bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

in Figure 14 is $\overline{z} = c/4$. Conclude by symmetry that the centroid is $(a/4, b/4, c/4)$.

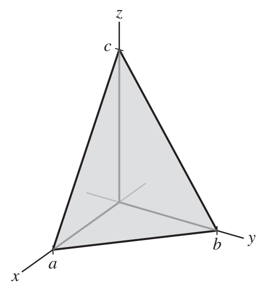


FIGURE 14

SOLUTION First we must find the volume of the tetrahedron. Solving the equation of the plane for z we get:

$$z = c - \frac{c}{a}x - \frac{c}{b}y$$

and the projection into the xy-plane is:

$$y = b - \frac{b}{a}x$$

Hence, volume can be computed:

$$\begin{aligned}
Volume &= \iiint_{\mathcal{W}} 1\, dV = \int_{x=0}^{a}\int_{y=0}^{b-\frac{b}{a}x}\int_{z=0}^{c-\frac{c}{a}x-\frac{c}{b}y} 1\, dz\, dy\, dx \\
&= \int_0^a \int_0^{b-\frac{b}{a}x} z\Big|_0^{c-\frac{c}{a}x-\frac{c}{b}y} dy\, dx = \int_0^a\int_0^{b-\frac{b}{a}x} c - \frac{c}{a}x - \frac{c}{b}y\, dy\, dx \\
&= \int_0^a cy - \frac{c}{a}xy - \frac{c}{2b}y^2\Big|_0^{b-\frac{b}{a}x} dx \\
&= \int_0^a c\left(b - \frac{b}{a}x\right) - \frac{c}{a}x\left(b - \frac{b}{a}x\right) - \frac{c}{2b}\left(b - \frac{b}{a}x\right)^2 dx \\
&= \int_0^a \frac{bc}{2} - \frac{bc}{a}x + \frac{bc}{2a^2}x^2\, dx \\
&= \frac{bc}{2}x - \frac{bc}{2a}x^2 + \frac{bc}{6a^2}x^3\Big|_0^a = \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{6} = \frac{abc}{6}
\end{aligned}$$

Now to find $\overline{z}$, the z-coordinate of the centroid:

$$\begin{aligned}
\overline{z} &= \frac{1}{V}\iiint_{\mathcal{W}} 1\, dV = \frac{6}{abc}\int_{x=0}^{a}\int_{y=0}^{b-\frac{b}{a}x}\int_{z=0}^{c-\frac{c}{a}x-\frac{c}{b}y} z\, dz\, dy\, dx \\
&= \frac{3}{abc}\int_{x=0}^{a}\int_{y=0}^{b-\frac{b}{a}x} z^2\Big|_{z=0}^{c-\frac{c}{a}x-\frac{c}{b}y} dy\, dx \\
&= \frac{3}{abc}\int_0^a\int_0^{b-\frac{b}{a}x}\left(c - \frac{c}{a}x - \frac{c}{b}y\right)^2 dy\, dx \\
&= \frac{3}{abc}\int_0^a\int_0^{b-\frac{b}{a}x} c^2 - \frac{2c^2}{a}x - \frac{2c^2}{b}y + \frac{2c^2}{ab}xy + \frac{c^2}{a^2}x^2 + \frac{c^2}{b^2}y^2\, dy\, dx \\
&= \frac{3}{abc}\int_0^a c^2y - \frac{2c^2}{a}xy - \frac{c^2}{b}y^2 + \frac{c^2}{ab}xy^2 + \frac{c^2}{a^2}x^2y + \frac{c^2}{3b^2}y^3\Big|_0^{b-\frac{b}{a}x} dx \\
&= \frac{3}{abc}\int_0^a c^2\left(b - \frac{b}{a}x\right) - \frac{2c^2}{a}x\left(b - \frac{b}{a}x\right) - \frac{c^2}{b}\left(b - \frac{b}{a}x\right)^2 \\
&\qquad + \frac{c^2}{ab}x\left(b - \frac{b}{a}x\right)^2 + \frac{c^2}{a^2}x^2\left(b - \frac{b}{a}x\right) + \frac{c^2}{3b^2}\left(b - \frac{b}{a}x\right)^3 dx
\end{aligned}$$

$$= \frac{3}{abc}\int_0^a \frac{bc^2}{3} - \frac{bc^2}{a}x + \frac{bc^2}{a^2}x^2 - \frac{bc^2}{3a^3}x^3\,dx$$
$$= \frac{3}{abc}\left(\frac{bc^2}{3}x - \frac{bc^2}{2a}x^2 + \frac{bc^2}{3a^2}x^3 - \frac{bc^2}{12a^3}x^4\Big|_0^a\right)$$
$$= \frac{3}{abc}\left(\frac{abc^2}{3} - \frac{abc^2}{2} + \frac{abc^2}{3} - \frac{abc^2}{12}\right)$$
$$= \frac{3}{abc}\cdot\frac{abc^2}{12} = \frac{c}{4}$$

Then, using symmetry, we can conclude $\overline{x} = a/4$ and $\overline{y} = b/4$, therefore, the coordinates of the centroid are $(a/4, b/4, c/4)$.

21. Find the centroid of the region $\mathcal{W}$ lying above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$ (Figure 15).

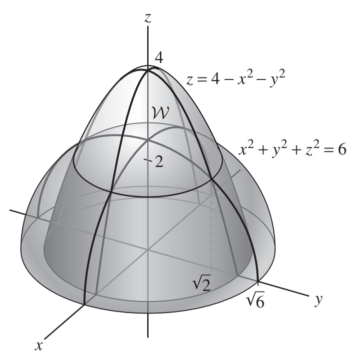


FIGURE 15

SOLUTION The centroid is the point P with the following coordinates:

$$\overline{x} = \frac{1}{V}\iiint_{\mathcal{W}} x\,dV,\quad \overline{y} = \frac{1}{V}\iiint_{\mathcal{W}} y\,dV,\quad \overline{z} = \frac{1}{V}\iiint_{\mathcal{W}} z\,dV$$

In a previous section we showed that the volume of the region is $V = 1.54\pi$. We also showed that $\mathcal{D}$ has the following definition in cylindrical coordinates:

$$0 \le \theta \le 2\pi,\quad 0 \le r \le \sqrt{2},\quad \sqrt{6 - r^2} \le z \le 4 - r^2$$

Using this information we compute the coordinates of the centroid by the following integrals:

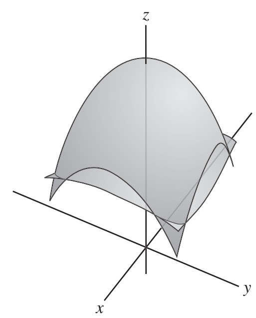
$$\overline{x} = \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\sqrt{6-r^2}}^{4-r^2}(r\cos\theta)r\,dz\,dr\,d\theta = \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} r^2\cos\theta z\Big|_{z=\sqrt{6-r^2}}^{4-r^2}dr\,d\theta$$
$$= \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} r^2\cos\theta\left(4 - r^2 - \sqrt{6-r^2}\right)dr\,d\theta$$
$$= \frac{1}{1.54\pi}\int_0^{2\pi}\cos\theta\int_0^{\sqrt{2}}\left(4r^2 - r^4 - r^2\sqrt{6-r^2}\right)dr\,d\theta \tag{1}$$

We denote the inner integral by a and compute the second integral to obtain

$$\overline{x} = \frac{1}{1.54\pi}\int_0^{2\pi}\cos\theta\cdot a\,d\theta = \frac{1}{1.54\pi}a\sin\theta\Big|_0^{2\pi} = 0$$

The value $\overline{x} = 0$ is the result of the symmetry of $\mathcal{W}$ with respect to the yz-plane. Similarly, since $\mathcal{W}$ is symmetric with respect to the xz-plane, the average value of the y-coordinate is zero.

$\overline{y} = 0$:



We compute the z-coordinate of the centroid:

$$\overline{z} = \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\sqrt{6-r^2}}^{4-r^2} zr\,dz\,dr\,d\theta = \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} \frac{z^2 r}{2}\bigg|_{z=\sqrt{6-r^2}}^{4-r^2} dr\,d\theta$$

$$= \frac{1}{1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} \frac{r}{2}\left((4-r^2)^2 - \left(\sqrt{6-r^2}\right)^2\right) dr\,d\theta$$

$$= \frac{1}{2\cdot 1.54\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} (r^5 - 7r^3 + 10r)\,dr\,d\theta$$

$$= \frac{1}{3.08\pi}\int_0^{2\pi} \frac{r^6}{6} - \frac{7r^4}{4} + 5r^2\bigg|_{r=0}^{\sqrt{2}} d\theta = \frac{1}{3.08\pi}\cdot\frac{13}{3}\cdot 2\pi \approx 2.81$$

Therefore the centroid of $\mathcal{W}$ is

$$P = (0, 0, 2.81).$$

22. Let $R > 0$ and $H > 0$, and let $\mathcal{W}$ be the upper half of the ellipsoid $x^2 + y^2 + (Rz/H)^2 = R^2$ where $z \geq 0$ (Figure 16). Find the centroid of $\mathcal{W}$ and show that it depends on the height H but not on the radius R.

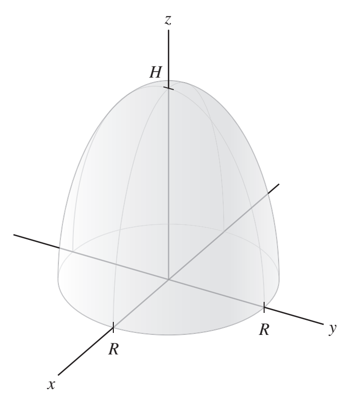


FIGURE 16 Upper half of ellipsoid $x^2 + y^2 + (Rz/H)^2 = R^2$, $z \geq 0$.

SOLUTION By symmetry, it is clear that the x and y-coordinates of the centroid are both zero. To find the z-coordinate, we first compute the volume. Using cylindrical coordinates, the equation of the ellipsoid is

$$z = \sqrt{\frac{H^2}{R^2}(R^2 - x^2 - y^2)} = \frac{H}{R}\sqrt{R^2 - r^2}$$

so that

$$M = \int_0^{2\pi}\int_0^R\int_0^{\frac{H}{R}\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta = 2\pi\frac{H}{R}\int_0^R r\sqrt{R^2-r^2}\,dr$$

$$= 2\pi\frac{H}{R}\cdot\left(-\frac{1}{3}(R^2-r^2)^{3/2}\right)\bigg|_0^R = 2\pi\frac{H}{R}\cdot\left(\frac{1}{3}R^3\right) = \frac{2}{3}\pi H R^2$$

and the z-moment is

$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi} \int_0^R \int_0^{\frac{H}{R}\sqrt{R^2-r^2}} zr\,dz\,dr\,d\theta = 2\pi \int_0^R \frac{1}{2} r z^2 \Big|_0^{\frac{H}{R}\sqrt{R^2-r^2}} dr$$
$$= \pi \frac{H^2}{R^2} \int_0^R R^2 r - r^3\,dr = \pi \frac{H^2}{R^2} \left(\frac{1}{2} R^2 r^2 - \frac{1}{4} r^4 \right) \Big|_0^R = \pi \frac{H^2}{R^2} \left(\frac{1}{2} R^4 - \frac{1}{4} R^4 \right)$$
$$= \frac{1}{4} \pi H^2 R^2$$

Thus the z-coordinate of the centroid is

$$\frac{M_{xy}}{M} = \frac{1}{4} \pi H^2 R^2 \cdot \frac{3}{2\pi H R^2} = \frac{3}{8} H$$

which depends on H but not on R.

In Exercises 23–26, find the center of mass of the region with the given mass density ρ.

23. Region bounded by $y = 4 - x$, $x = 0$, $y = 0$; $\rho(x, y) = x$

SOLUTION The mass of the region is

$$M = \int_0^4 \int_0^{4-x} x\,dy\,dx = \int_0^4 xy \Big|_0^{4-x} dx = \int_0^4 4x - x^2\,dx = 2x^2 - \frac{1}{3} x^3 \Big|_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$$

and we have

$$M_x = \int_0^4 \int_0^{4-x} yx\,dy\,dx = \int_0^4 \frac{1}{2} x y^2 \Big|_0^{4-x} dx = \frac{1}{2} \int_0^4 16x - 8x^2 + x^3\,dx$$
$$= \frac{1}{2} \left(8x^2 - \frac{8}{3} x^3 + \frac{1}{4} x^4 \right) \Big|_0^4 = \frac{1}{2} \left(128 - \frac{512}{3} + 64 \right) = \frac{32}{3}$$

and

$$M_y = \int_0^4 \int_0^{4-x} x^2\,dy\,dx = \int_0^4 x^2 y \Big|_0^{4-x} dx = \int_0^4 4x^2 - x^3\,dx$$
$$= \left(\frac{4}{3} x^3 - \frac{1}{4} x^4 \right) \Big|_0^4 = \frac{256}{3} - 64 = \frac{64}{3}$$

and thus the center of mass is

$$\left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{64}{3} \cdot \frac{3}{32}, \frac{32}{3} \cdot \frac{3}{32} \right) = (2, 1)$$

24. Region bounded by $y^2 = x + 4$ and $x = 0$; $\rho(x, y) = |y|$

SOLUTION Solving for x gives $x = y^2 - 4$. Now, the mass of the region is

$$M = \int_{-2}^2 \int_{y^2-4}^0 |y|\,dx\,dy = 2 \int_0^2 \int_{y^2-4}^0 y\,dx\,dy = 2 \int_0^2 yx \Big|_{y^2-4}^0 dy$$
$$= 2 \int_0^2 4y - y^3\,dy = 2 \left(2y^2 - \frac{1}{4} y^4 \right) \Big|_0^2 = 2(8 - 4) = 8$$

and

$$M_y = \iint_{\mathcal{D}} x\rho(x, y)\,dA = \int_{-2}^2 \int_{y^2-4}^0 x|y|\,dx\,dy = 2 \int_0^2 \int_{y^2-4}^0 xy\,dx\,dy$$
$$= 2 \int_0^2 y \cdot \frac{1}{2} x^2 \Big|_{y^2-4}^0 dy = - \int_0^2 y(y^2 - 4)^2\,dy$$
$$= -\frac{1}{2} \int_{-4}^0 u^2\,du = -\frac{1}{6} u^3 \Big|_{-4}^0 = \frac{-32}{3}$$

Clearly $M_x = 0$ by symmetry, since both the region and the density are symmetric around the x-axis. Thus the center of mass is

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(-\frac{32}{3} \cdot \frac{1}{8}, 0\right) = \left(-\frac{4}{3}, 0\right)$$

25. Region $|x| + |y| \leq 1$; $\quad \rho(x, y) = (x+1)(y+1)$

SOLUTION For $x \leq 0$, the region is defined by $-1 \leq x \leq 0$ and $-1 - x \leq y \leq 1 + x$; for $x \geq 0$, it is parameterized by $0 \leq x \leq 1$ and $-1 + x \leq y \leq 1 - x$. The mass of the region is thus

$$\begin{aligned}
M &= \int_{-1}^{0} \int_{-1-x}^{1+x} (x+1)(y+1)\, dy\, dx + \int_{0}^{1} \int_{x-1}^{1-x} (x+1)(y+1)\, dy\, dx \\
&= \frac{1}{2}\left(\int_{-1}^{0} (x+1)(y+1)^2 \Big|_{y=-1-x}^{1+x} dx + \int_{0}^{1} (x+1)(y+1)^2 \Big|_{y=x-1}^{1-x} dx\right) \\
&= \frac{1}{2}\left(\int_{-1}^{0} (x+1)((x+2)^2 - (-x)^2)\, dx + \int_{0}^{1} (x+1)((2-x)^2 - x^2)\, dx\right) \\
&= \frac{1}{2}\left(\int_{-1}^{0} 4(x+1)^2\, dx + \int_{0}^{1} 4(1-x^2)\, dx\right) \\
&= 2\left(\left(\frac{1}{3}(x+1)^3\right)\Big|_{-1}^{0} + \left(x - \frac{1}{3}x^3\right)\Big|_{0}^{1}\right) = 2\left(\frac{1}{3} + \left(1 - \frac{1}{3}\right)\right) = 2
\end{aligned}$$

We have

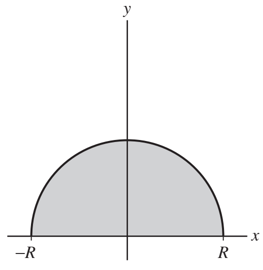
$$\begin{aligned}
M_x &= \int_{-1}^{0} \int_{-1-x}^{1+x} y(x+1)(y+1)\, dy\, dx + \int_{0}^{1} \int_{x-1}^{1-x} y(x+1)(y+1)\, dy\, dx \\
&= \int_{-1}^{0} \int_{-1-x}^{1+x} (x+1)(y^2+y)\, dy\, dx + \int_{0}^{1} \int_{x-1}^{1-x} (x+1)(y^2+y)\, dy\, dx \\
&= \int_{-1}^{0} (x+1)\left(\frac{1}{3}y^3 + \frac{1}{2}y^2\right)\Big|_{y=-1-x}^{1+x} dx + \int_{0}^{1} (x+1)\left(\frac{1}{3}y^3 + \frac{1}{2}y^2\right)\Big|_{y=x-1}^{1-x} dx \\
&= \int_{-1}^{0} (x+1)\left(\frac{2}{3} + 2x + 2x^2 + \frac{2}{3}x^3\right) dx + \int_{0}^{1} (x+1)\left(\frac{2}{3} - 2x + 2x^2 - \frac{2}{3}x^3\right) dx \\
&= \int_{-1}^{0} \frac{2}{3}x^4 + \frac{8}{3}x^3 + 4x^2 + \frac{8}{3}x + \frac{2}{3}\, dx + \int_{0}^{1} -\frac{2}{3}x^4 + \frac{4}{3}x^3 - \frac{4}{3}x + \frac{2}{3}\, dx \\
&= \left(\frac{2}{15}x^5 + \frac{2}{3}x^4 + \frac{4}{3}x^3 + \frac{4}{3}x^2 + \frac{2}{3}x\right)\Big|_{-1}^{0} + \left(-\frac{2}{15}x^5 + \frac{1}{3}x^4 - \frac{2}{3}x^3 + \frac{2}{3}x\right)\Big|_{0}^{1} \\
&= \frac{2}{15} + \frac{1}{5} = \frac{1}{3}
\end{aligned}$$

Since the region and the density function are symmetric in x and y, we must have also $M_y = M_x = \frac{1}{3}$. Then the center of mass is

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{1}{3} \cdot \frac{1}{2}, \frac{1}{3} \cdot \frac{1}{2}\right) = \left(\frac{1}{6}, \frac{1}{6}\right)$$

26. Semicircle $x^2 + y^2 \leq R^2$, $y \geq 0$; $\quad \rho(x, y) = y$

SOLUTION



The center of mass has the following coordinates:

$$x_{\text{CM}} = \frac{1}{M}\iint_{\mathcal{D}} x\rho\, dA = \frac{1}{M}\iint_{\mathcal{D}} xy\, dA$$
$$y_{\text{CM}} = \frac{1}{M}\iint_{\mathcal{D}} y\rho\, dA = \frac{1}{M}\iint_{\mathcal{D}} y^2\, dA$$

We compute the integrals using polar coordinates. The semicircle $\mathcal{D}$ has the following description:

$$0 \le \theta \le \pi, \quad 0 \le r \le R$$

We first compute the total mass M. Using the Change of Variables Formula we get

$$M = \iint_{\mathcal{D}} \rho(x, y, z)\, dA = \iint_{\mathcal{D}} y\, dA = \int_0^{\pi}\int_0^R r\sin\theta \cdot r\, dr\, d\theta = \int_0^{\pi}\int_0^R r^2 \sin\theta\, dr\, d\theta$$
$$= \int_0^{\pi} \left.\frac{r^3\sin\theta}{3}\right|_{r=0}^{R} d\theta = \int_0^{\pi} \frac{R^3\sin\theta}{3}\, d\theta = -\left.\frac{R^3\cos\theta}{3}\right|_0^{\pi} = -\frac{R^3}{3}(\cos\pi - \cos 0) = \frac{2R^3}{3}$$

We compute x_{CM}:

$$x_{\text{CM}} = \frac{1}{M}\iint_{\mathcal{D}} xy\, dA = \frac{3}{2R^3}\int_0^{\pi}\int_0^R (r\cos\theta)(r\sin\theta)\cdot r\, dr\, d\theta = \frac{3}{2R^3}\int_0^{\pi}\int_0^R \frac{r^3\sin 2\theta}{2}\, dr\, d\theta$$
$$= \frac{3}{2R^3}\int_0^{\pi} \left.\frac{r^4\sin 2\theta}{8}\right|_{r=0}^{R} d\theta = \frac{3}{2R^3}\cdot\frac{R^4}{8}\int_0^{\pi}\sin 2\theta\, d\theta = \frac{3R}{16}\cdot \left.\frac{-\cos 2\theta}{2}\right|_{\theta=0}^{\pi}$$
$$= -\frac{3R}{32}(\cos 2\pi - \cos 0) = 0$$

We compute y_{CM}:

$$y_{\text{CM}} = \frac{1}{M}\iint_{\mathcal{D}} y^2\, dA = \frac{3}{2R^3}\int_0^{\pi}\int_0^R r^2\sin^2\theta\cdot r\, dr\, d\theta = \frac{3}{2R^3}\int_0^{\pi}\int_0^R r^3\sin^2\theta\, dr\, d\theta$$
$$= \frac{3}{2R^3}\int_0^{\pi}\left.\frac{r^4\sin^2\theta}{4}\right|_{r=0}^{R} d\theta = \frac{3}{2R^3}\cdot\frac{R^4}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3R}{8}\left.\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right|_0^{\pi} = \frac{3R}{8}\cdot\frac{\pi}{2} = \frac{3\pi R}{16}$$

We obtain the following center of mass:

$$\left(0, \frac{3\pi R}{16}\right)$$

27. Find the z-coordinate of the center of mass of the first octant of the unit sphere with mass density $\rho(x, y, z) = y$ (Figure 17).

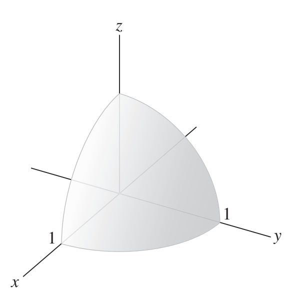


FIGURE 17

SOLUTION We use spherical coordinates:

$$x = \rho\cos\theta\sin\phi, \; y = \rho\sin\theta\sin\phi, \; z = \rho\cos\phi$$

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

The octant $\mathcal{W}$ is defined by $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 1$, so we have

$$M_{xy} = \iiint_{\mathcal{W}} z\,\rho(x, y, z)\, dV = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} (\rho\cos\phi)(\rho\sin\theta\sin\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \right) \left(\int_{\phi=0}^{\pi/2} \cos\phi \sin^2\phi \, d\phi \right) \left(\int_{\rho=0}^{1} \rho^4 \, d\rho \right)$$

$$= (1) \left(\frac{1}{3} \sin^3\phi \Big|_0^{\pi/2} \right) \left(\frac{1}{5} \right) = \frac{1}{15}$$

The total mass M of $\mathcal{W}$ is equal to the integral of the mass density $\rho(x, y, z)$:

$$M = \iiint_{\mathcal{W}} \rho(x, y, z)\, dV = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} (\rho \sin\theta \sin\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

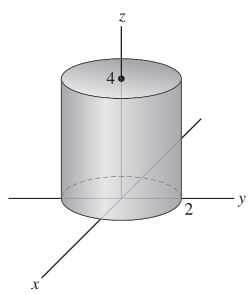
$$= \left(\int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \right) \left(\int_{\phi=0}^{\pi/2} \sin^2\phi \, d\phi \right) \left(\int_{\rho=0}^{1} \rho^3 \, d\rho \right) = (1) \left(\frac{\pi}{4} \right) \left(\frac{1}{4} \right) = \frac{\pi}{16}$$

We conclude that

$$z_{\rm CM} = \frac{1}{M} \iiint_{\mathcal{W}} z\, \rho(x, y, z)\, dV = \frac{1/15}{\pi/16} = \frac{16}{15\pi} \approx 0.34$$

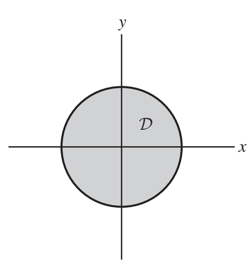
28. Find the center of mass of a cylinder of radius 2 and height 4 and mass density e^{-z}, where z is the height above the base.

SOLUTION



The center of mass is the point with the following coordinates:

$$x_{\rm CM} = \frac{1}{M} \iiint_{\mathcal{W}} x e^{-z}\, dV, \quad y_{\rm CM} = \frac{1}{M} \iiint_{\mathcal{W}} y e^{-z}\, dV, \quad z_{\rm CM} = \frac{1}{M} \iiint_{\mathcal{W}} z e^{-z}\, dV$$



Since $\mathcal{W}$ is symmetric with respect to the z axis, and the functions xe^{-z} and ye^{-z} are odd with respect to the variables x and y, respectively, we have

$$x_{\rm CM} = y_{\rm CM} = 0$$

We need only to find $z_{\rm CM}$. We first compute the mass M by the triple integral

$$M = \iiint_{\mathcal{W}} e^{-z}\, dV$$

To evaluate the integral we use cylindrical coordinates. The region $\mathcal{W}$ is described by the inequalities

$$\mathcal{W}: \; 0 \le \theta \le 2\pi, \; 0 \le r \le 2, \; 0 \le z \le 4$$

Using triple integrals in cylindrical coordinates, we obtain

$$M = \int_0^{2\pi} \int_0^2 \int_0^4 e^{-z} r \, dz \, dr \, d\theta = \left(\int_0^{2\pi} 1 \, d\theta \right) \left(\int_0^2 r \, dr \right) \left(\int_0^4 e^{-z} \, dz \right)$$

$$= 2\pi \cdot \left(\frac{r^2}{2} \Big|_0^2 \right) \left(-e^{-z} \Big|_0^4 \right) = 2\pi \cdot 2 \cdot (1 - e^{-4}) = 4\pi (1 - e^{-4}) \approx 12.34$$

We compute z_{CM}:

$$z_{CM} = \frac{1}{12.34}\int_0^{2\pi}\int_0^2\int_0^4 ze^{-z} r\,dz\,dr\,d\theta = 0.08\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r\,dr\right)\left(\int_0^4 ze^{-z}\,dz\right)$$
$$= 0.08\cdot 2\pi\cdot 2\int_0^4 ze^{-z}\,dz = 0.32\pi\int_0^4 ze^{-z}\,dz$$

We compute the integral using integration by parts:

$$z_{CM} = 0.32\pi\left(-ze^{-z}\Big|_0^4 - e^{-z}\Big|_0^4\right) = -0.32\pi e^{-z}(1+z)\Big|_0^4 = 0.32\pi(1-5e^{-4}) = 0.91$$

The center of mass is the point $(0, 0, 0.91)$.

29. Let $\mathcal{R}$ be the rectangle $[-a, a]\times[b, -b]$ with uniform density and total mass M. Calculate:

(a) The mass density ρ of $\mathcal{R}$

(b) I_x and I_0

(c) The radius of gyration about the x-axis

SOLUTION

(a) The mass density is simply the mass per unit area since the density is uniform; this is

$$\frac{M}{4ab}$$

(b) We have

$$I_x = \iint_{\mathcal{R}} y^2\rho(x,y)\,dA = \frac{M}{4ab}\int_{-a}^{a}\int_{-b}^{b} y^2\,dy\,dx = \frac{2aM}{4ab}\int_{-b}^{b} y^2\,dy$$
$$= \frac{M}{2b}\cdot\frac{1}{3}y^3\Big|_{-b}^{b} = \frac{1}{3}Mb^2$$

and

$$I_0 = \iint_{\mathcal{R}}(x^2+y^2)\rho(x,y)\,dA = \frac{M}{4ab}\int_{-a}^{a}\int_{-b}^{b} x^2+y^2\,dy\,dx = \frac{M}{4ab}\int_{-a}^{a} x^2y+\frac{1}{3}y^3\Big|_{-b}^{b}\,dx$$
$$= \frac{2M}{4ab}\int_{-a}^{a} x^2b+\frac{1}{3}b^3\,dx = \frac{M}{2ab}\left(\frac{b}{3}x^3+\frac{b^3}{3}x\right)\Big|_{-a}^{a}$$
$$= \frac{M}{2ab}\left(\frac{2}{3}ba^3+\frac{2}{3}b^3a\right) = \frac{1}{3}M(a^2+b^2)$$

(c) The radius of gyration about the x-axis is defined to be

$$\sqrt{\frac{I_x}{M}} = \sqrt{\frac{Mb^2}{3}\cdot\frac{1}{M}} = \frac{b}{\sqrt{3}}$$

30. Calculate I_x and I_0 for the rectangle in Exercise 29 assuming a mass density of $\rho(x,y)=x$.

SOLUTION

$$I_x = \iint_{\mathcal{R}} y^2\rho(x,y)\,dA = \int_{-b}^{b}\int_{-a}^{a} xy^2\,dx\,dy = \left(\int_{-b}^{b} y^2\,dy\right)\cdot\left(\int_{-a}^{a} x\,dx\right) = 0$$

since x is an odd function. Also

$$I_0 = \iint_{\mathcal{R}}(x^2+y^2)\rho(x,y)\,dA = \int_{-b}^{b}\int_{-a}^{a} x^3+y^2x\,dx\,dy = 0$$

since the integrand is an odd function of x.

31. Calculate I_0 and I_x for the disk $\mathcal{D}$ defined by $x^2 + y^2 \le 16$ (in meters), with total mass 1000 kg and uniform mass density. *Hint:* Calculate I_0 first and observe that $I_0 = 2I_x$. Express your answer in the correct units.

SOLUTION Note that the area of the disk is $\pi r^2 = 16\pi$ so that the mass density is

$$\rho(x, y) = \frac{1000}{16\pi} = \frac{125}{2\pi}$$

Then using polar coordinates we have

$$I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\frac{125}{2\pi}\, dA = \frac{125}{2\pi}\int_0^4 \int_0^{2\pi} r^2 \cdot r\, d\theta\, dr = 125 \cdot \frac{1}{4}r^4\Big|_0^4 = 125 \cdot 64 = 8000 \text{ kg-m}^2$$

Since both the region and the mass density are symmetric in x and y, we have $I_x = I_y$. But then $I_0 = I_x + I_y = 2I_x$ so that

$$I_x = 4000 \text{ kg-m}^2$$

32. Calculate I_x and I_y for the half-disk $x^2 + y^2 \le R^2$, $x \ge 0$ (in meters), of total mass M kg and uniform mass density.

SOLUTION This is quite similar to the preceding exercise. The area of the disk is πR^2, so that its mass density is

$$\rho(x, y) = \frac{M}{\pi R^2}$$

Using polar coordinates,

$$I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\rho(x, y)\, dA = \frac{M}{\pi R^2}\int_0^R \int_0^{2\pi} r^2 \cdot r\, d\theta\, dr = \frac{M}{\pi R^2} \cdot 2\pi \int_0^R r^3\, dr = \frac{2M}{R^2} \cdot \frac{1}{4}r^4\Big|_0^R$$
$$= \frac{1}{2}MR^2$$

Since both the region and the mass density are symmetric in x and y, we have $I_x = I_y$. But then $I_0 = I_x + I_y = 2I_x$, so that

$$I_x = \frac{1}{4}MR^2$$

In Exercises 33–36, let $\mathcal{D}$ be the triangular domain bounded by the coordinate axes and the line $y = 3 - x$, with mass density $\rho(x, y) = y$. Compute the given quantities.

33. Total mass

SOLUTION The total mass is simply

$$\iint_{\mathcal{D}} \rho(x, y)\, dA = \int_0^3 \int_0^{3-x} y\, dy\, dx = \frac{1}{2}\int_0^3 y^2\Big|_0^{3-x} dx = \frac{1}{2}\int_0^3 (3 - x)^2\, dx = -\frac{1}{6}(3 - x)^3\Big|_0^3 = \frac{27}{6} = \frac{9}{2}$$

34. Center of Mass

SOLUTION We have

$$M_x = \iint_{\mathcal{D}} y\rho(x, y)\, dA = \int_0^3 \int_0^{3-x} y^2\, dy\, dx = \frac{1}{3}\int_0^3 (3 - x)^3\, dx = -\frac{1}{12}(3 - x)^4\Big|_0^3$$
$$= \frac{1}{12} \cdot 81 = \frac{27}{4}$$

and

$$M_y = \iint_{\mathcal{D}} x\rho(x, y)\, dA = \int_0^3 \int_0^{3-x} xy\, dy\, dx = \frac{1}{2}\int_0^3 xy^2\Big|_0^{3-x} dx$$
$$= \frac{1}{2}\int_0^3 9x - 6x^2 + x^3\, dx = \frac{1}{2}\left(\frac{9}{2}x^2 - 2x^3 + \frac{1}{4}x^4\right)\Big|_0^3 = \frac{1}{2}\left(\frac{81}{2} - 54 + \frac{81}{4}\right) = \frac{27}{8}$$

so that the center of mass is

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{27}{8} \cdot \frac{2}{9}, \frac{27}{4} \cdot \frac{2}{9}\right) = \left(\frac{3}{4}, \frac{3}{2}\right)$$

35. I_x

SOLUTION

$$I_x = \iint_{\mathcal{D}} y^2 \rho(x, y)\, dA = \int_0^3 \int_0^{3-x} y^3\, dy\, dx = \frac{1}{4} \int_0^3 (3-x)^4\, dx$$
$$= -\frac{1}{20}(3-x)^5 \Big|_0^3 = \frac{1}{20} 3^5 = \frac{243}{20}$$

36. I_0

SOLUTION

$$I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\rho(x, y)\, dA = \int_0^3 \int_0^{3-x} (x^2 + y^2) y\, dy\, dx$$
$$= \int_0^3 \int_0^{3-x} x^2 y + y^3\, dy\, dx = \int_0^3 \frac{1}{2} x^2 y^2 + \frac{1}{4} y^4 \Big|_0^{3-x} dx$$
$$= \int_0^3 \frac{1}{2} x^2 (3-x)^2 + \frac{1}{4}(3-x)^4\, dx$$
$$= \int_0^3 \frac{9}{2} x^2 - 3x^3 + \frac{1}{2} x^4 + \frac{1}{4}(3-x)^4\, dx$$
$$= \left(\frac{3}{2} x^3 - \frac{3}{4} x^4 + \frac{1}{10} x^5 - \frac{1}{20}(3-x)^5 \right) \Big|_0^3$$
$$= \frac{81}{5}$$

In Exercises 37–40, let $\mathcal{D}$ be the domain between the line $y = bx/a$ and the parabola $y = bx^2/a^2$ where $a, b > 0$. Assume the mass density is $\rho(x, y) = xy$. Compute the given quantities.

37. Centroid

SOLUTION The curves intersect at $x = 0$ and at $x = a$. The area is

$$A = \iint_{\mathcal{D}} 1\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} 1\, dy\, dx = \int_0^a \frac{bx}{a} - \frac{bx^2}{a^2}\, dx$$
$$= \left(\frac{bx^2}{2a} - \frac{bx^3}{3a^2} \right) \Big|_0^a = \frac{ab}{2} - \frac{ab}{3} = \frac{ab}{6}$$

Then

$$\overline{x} = \frac{1}{A} \iint_{\mathcal{D}} x\, dA = \frac{6}{ab} \int_0^a \int_{bx^2/a^2}^{bx/a} x\, dy\, dx = \frac{6}{ab} \int_0^a x \left(\frac{bx}{a} - \frac{bx^2}{a^2} \right) dx$$
$$= \frac{6}{ab} \int_0^a \frac{b}{a} x^2 - \frac{b}{a^2} x^3\, dx = \frac{6}{ab} \left(\frac{b}{3a} a^3 - \frac{b}{4a^2} a^4 \right)$$
$$= \frac{6}{ab} \left(\frac{a^2 b}{3} - \frac{a^2 b}{4} \right) = \frac{a}{2}$$

and

$$\overline{y} = \frac{1}{A} \iint_{\mathcal{D}} y\, dA = \frac{6}{ab} \int_0^a \int_{bx^2/a^2}^{bx/a} y\, dy\, dx = \frac{6}{2ab} \int_0^a y^2 \Big|_{bx^2/a^2}^{bx/a} dx$$
$$= \frac{3}{ab} \int_0^a \frac{b^2}{a^2} x^2 - \frac{b^2}{a^4} x^4\, dx = \frac{3}{ab} \left(\frac{b^2}{3a^2} x^3 - \frac{b^2}{5a^4} x^5 \right) \Big|_0^a$$
$$= \frac{3}{ab} \left(\frac{ab^2}{3} - \frac{ab^2}{5} \right) = \frac{2b}{5}$$

38. Center of Mass

SOLUTION The curves intersect at $x = 0$ and at $x = a$, so the mass is

$$M = \iint_{\mathcal{D}} \rho(x, y)\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} xy\, dy\, dx = \frac{1}{2}\int_0^a xy^2\Big|_{bx^2/a^2}^{bx/a} dx$$

$$= \frac{1}{2}\int_0^a x\left(\frac{b^2x^2}{a^2} - \frac{b^2x^4}{a^4}\right) dx = \frac{b^2}{2a^4}\int_0^a a^2x^3 - x^5\, dx$$

$$= \frac{b^2}{2a^4}\left(\frac{a^2}{4}x^4 - \frac{1}{6}x^6\right)\Big|_0^a = \frac{b^2}{2a^4}\left(\frac{a^6}{3} - \frac{a^6}{4}\right) = \frac{a^2b^2}{24}$$

We also have

$$M_x = \iint_{\mathcal{D}} y\rho(x, y)\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} xy^2\, dy\, dx = \frac{1}{3}\int_0^a xy^3\Big|_{bx^2/a^2}^{bx/a} dx$$

$$= \frac{1}{3}\int_0^a x\left(\frac{b^3x^3}{a^3} - \frac{b^3x^6}{a^6}\right) dx = \frac{b^3}{3a^6}\int_0^a a^3x^4 - x^7\, dx$$

$$= \frac{b^3}{3a^6}\left(\frac{a^3}{5}x^5 - \frac{1}{8}x^8\right)\Big|_0^a = \frac{b^3}{3a^6}\left(\frac{a^8}{5} - \frac{a^8}{8}\right) = \frac{a^2b^3}{40}$$

and

$$M_y = \iint_{\mathcal{D}} x\rho(x, y)\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} x^2y\, dy\, dx = \frac{1}{2}\int_0^a x^2y^2\Big|_{bx^2/a^2}^{bx/a} dx$$

$$= \frac{1}{2}\int_0^a x^2\left(\frac{b^2x^2}{a^2} - \frac{b^2x^4}{a^4}\right) dx = \frac{b^2}{2a^4}\int_0^a a^2x^4 - x^6\, dx$$

$$= \frac{b^2}{2a^4}\left(\frac{a^2}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^a = \frac{b^2}{2a^4}\left(\frac{a^7}{5} - \frac{a^7}{7}\right) = \frac{a^3b^2}{35}$$

so that the center of mass is at

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{a^3b^2}{35}\cdot\frac{24}{a^2b^2}, \frac{a^2b^3}{40}\cdot\frac{24}{a^2b^2}\right) = \left(\frac{24}{35}a, \frac{3}{5}b\right)$$

39. I_x

SOLUTION The curves intersect at $x = 0$ and at $x = a$, so

$$I_x = \iint_{\mathcal{D}} y^2\rho(x, y)\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} xy^3\, dy\, dx = \frac{1}{4}\int_0^a xy^4\Big|_{bx^2/a^2}^{bx/a} dx$$

$$= \frac{1}{4}\int_0^a x\left(\frac{b^4}{a^4}x^4 - \frac{b^4}{a^8}x^8\right) dx = \frac{b^4}{4a^8}\int_0^a a^4x^5 - x^9\, dx$$

$$= \frac{b^4}{4a^8}\left(\frac{a^4}{6}x^6 - \frac{1}{10}x^10\right)\Big|_0^a = \frac{b^4}{4a^8}\left(\frac{a^{10}}{6} - \frac{a^{10}}{10}\right) = \frac{a^2b^4}{60}$$

40. I_0

SOLUTION The curves intersect at $x = 0$ and at $x = a$. We computed I_x in the previous exercise, so we will compute I_y and add the two to get I_0:

$$I_y = \iint_{\mathcal{D}} x^2\rho(x, y)\, dA = \int_0^a \int_{bx^2/a^2}^{bx/a} x^3y\, dy\, dx = \frac{1}{2}\int_0^a x^3y^2\Big|_{bx^2/a^2}^{bx/a} dx$$

$$= \frac{1}{2}\int_0^a x^3\left(\frac{b^2}{a^2}x^2 - \frac{b^2}{a^4}x^4\right) dx = \frac{b^2}{2a^4}\int_0^a a^2x^5 - x^7\, dx$$

$$= \frac{b^2}{2a^4}\left(\frac{a^2}{6}x^6 - \frac{1}{8}x^8\right)\Big|_0^a$$

$$= \frac{b^2}{2a^4}\left(\frac{a^8}{6} - \frac{a^8}{8}\right) = \frac{a^4b^2}{48}$$

Thus

$$I_0 = I_x + I_y = \frac{a^2 b^4}{60} + \frac{a^4 b^2}{48} = a^2 b^2 \frac{5a^2 + 4b^2}{240}$$

41. Calculate the moment of inertia I_x of the disk $\mathcal{D}$ defined by $x^2 + y^2 \le R^2$ (in meters) with total mass M kg. How much kinetic energy (in joules) is required to rotate the disk about the x-axis with angular velocity 10 rad/s?

SOLUTION The area of the disk is πR^2, so its mass density is

$$\rho(x, y) = \frac{M}{\pi R^2}$$

We compute I_x using polar coordinates:

$$\begin{aligned} I_x &= \iint_{\mathcal{D}} y^2 \rho(x, y)\, dA = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R (r \sin\theta)^2 r\, dr\, d\theta = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R r^3 \sin^2\theta\, dr\, d\theta \\ &= \frac{M}{\pi R^2} \left(\int_0^{2\pi} \sin^2\theta\, d\theta \right) \left(\int_0^R r^3\, dr \right) \\ &= \frac{M}{\pi R^2} \cdot \pi \cdot \frac{R^4}{4} = \frac{1}{4} M R^2 \end{aligned}$$

It follows that the kinetic energy required to rotate the disk about the x-axis with angular velocity 10 rad/s is

$$\frac{1}{2} I_x \omega^2 = \frac{1}{8} M R^2 \cdot 100 = \frac{25}{2} M R^2 \text{ joules}$$

42. Calculate the moment of inertia I_z of the box $\mathcal{W} = [-a, a] \times [-a, a] \times [0, H]$ assuming that $\mathcal{W}$ has total mass M.

SOLUTION The volume of the region is $2a \cdot 2a \cdot H = 4a^2 H$, so that the mass density is

$$\rho = \rho(x, y, z) = \frac{M}{4a^2 H}$$

Then

$$\begin{aligned} I_z &= \iint_{\mathcal{W}} (x^2 + y^2) \rho(x, y, z)\, dA = \frac{M}{4a^2 H} \int_{-a}^{a} \int_{-a}^{a} \int_0^H x^2 + y^2\, dz\, dy\, dx \\ &= \frac{M}{4a^2 H} H \int_{-a}^{a} \int_{-a}^{a} x^2 + y^2\, dy\, dx = \frac{M}{4a^2} \int_{-a}^{a} \left(x^2 y + \frac{1}{3} y^3 \right) \Big|_{-a}^{a} dx \\ &= \frac{2M}{4a^2} \int_{-a}^{a} a x^2 + \frac{a^3}{3}\, dx = \frac{M}{2a^2} \left(\frac{a}{3} x^3 + \frac{a^3}{3} x \right) \Big|_{-a}^{a} \\ &= \frac{M}{2a^2} \left(\frac{2a^4}{3} + \frac{2a^4}{3} \right) = \frac{2}{3} M a^2 \end{aligned}$$

43. Show that the moment of inertia of a sphere of radius R of total mass M with uniform mass density about any axis passing through the center of the sphere is $\frac{2}{5} M R^2$. Note that the mass density of the sphere is $\rho = M / \left(\frac{4}{3}\pi R^3\right)$.

SOLUTION Since the sphere is symmetric under an arbitrary rotation, and since the mass density is uniform, it follows that the moments of inertia of the sphere about all axes passing through its center are equal. Thus it suffices to prove the result for an arbitrary axis; we choose the z-axis. Then, using spherical coordinates, we have

$$\begin{aligned} I_z &= \iiint_{\mathcal{S}} (x^2 + y^2) \rho(x, y, z)\, dA \\ &= \frac{3M}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R ((r \cos\theta \sin\phi)^2 + (r \sin\theta \sin\phi)^2) r^2 \sin\phi\, dr\, d\theta\, d\phi \\ &= \frac{3M}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R r^4 \sin^3\phi\, dr\, d\theta\, d\phi = \frac{3M}{4\pi R^3} \cdot 2\pi \left(\int_0^{\pi} \sin^3\phi\, d\phi \right) \left(\int_0^R r^4\, dr \right) \\ &= \frac{3M}{2R^3} \cdot \frac{4}{3} \cdot \frac{1}{5} R^5 = \frac{2}{5} M R^2 \end{aligned}$$

44. Use the result of Exercise 43 to calculate the radius of gyration of a uniform sphere of radius R about any axis through the center of the sphere.

SOLUTION From the referenced exercise, we know that the moment of inertia of the sphere around the given axis a is

$$I_a = \frac{2}{5}MR^2$$

so that the radius of gyration about a is

$$r_g = \sqrt{\frac{I_a}{M}} = \sqrt{\frac{2}{5}MR^2 \cdot \frac{1}{M}} = \frac{\sqrt{10}}{5}R$$

In Exercises 45 and 46, prove the formula for the right circular cylinder in Figure 18.

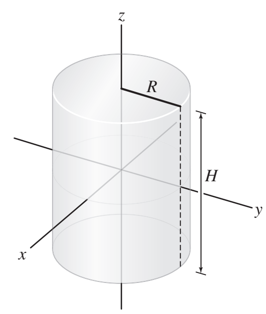


FIGURE 18

45. $I_z = \frac{1}{2}MR^2$

SOLUTION Assuming the cylinder has uniform mass density 1, and using cylindrical coordinates, we have

$$I_z = \iiint_C (x^2+y^2)\rho(x,y,z)\,dA = \int_0^R \int_0^{2\pi} \int_{-H/2}^{H/2} r^2 \cdot r\,dz\,d\theta\,dr$$
$$= 2\pi H \int_0^R r^3\,dr = \frac{1}{2}\pi R^4 H$$

But the volume of the cylinder, which is equal to its mass, is $\pi R^2 H$, so that

$$I_z = \frac{1}{2}\pi R^4 H = \frac{1}{2}MR^2$$

46. $I_x = \frac{1}{4}MR^2 + \frac{1}{12}MH^2$

SOLUTION Assuming the cylinder has uniform mass density 1, and using cylindrical coordinates, we have

$$I_x = \iiint_C (y^2+z^2)\rho(x,y,z)\,dA = \int_0^R \int_0^{2\pi} \int_{-H/2}^{H/2} (r^2\sin^2\theta + z^2)r\,dz\,d\theta\,dr$$
$$= \int_0^R \int_0^{2\pi} \left(r^3 z\sin^2\theta + \frac{1}{3}rz^3\right)\Big|_{-H/2}^{H/2} d\theta\,dr = \int_0^R \int_0^{2\pi} Hr^3\sin^2\theta + \frac{1}{12}H^3 r\,d\theta\,dr$$
$$= \int_0^R \pi Hr^3 + \frac{2\pi}{12}H^3 r\,dr = \frac{1}{4}\pi HR^4 + \frac{1}{12}\pi H^3 R^2$$

The volume of the cylinder, which is equal to its mass, is $\pi R^2 H$; substituting, we get

$$I_x = \frac{1}{4}\pi HR^4 + \frac{1}{12}\pi H^3 R^2 = \frac{1}{4}MR^2 + \frac{1}{12}MH^2$$

47. The yo-yo in Figure 19 is made up of two disks of radius $r = 3$ cm and an axle of radius $b = 1$ cm. Each disk has mass $M_1 = 20$ g, and the axle has mass $M_2 = 5$ g.

(a) Use the result of Exercise 45 to calculate the moment of inertia I of the yo-yo with respect to the axis of symmetry. Note that I is the sum of the moments of the three components of the yo-yo.

(b) The yo-yo is released and falls to the end of a 100-cm string, where it spins with angular velocity ω. The total mass of the yo-yo is $m = 45$ g, so the potential energy lost is $\mathrm{PE} = mgh = (45)(980)100$ g-cm^2/s^2. Find ω under the assumption that one-third of this potential energy is converted into rotational kinetic energy.

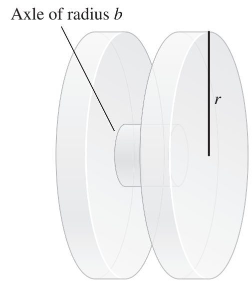


FIGURE 19

SOLUTION

(a) If the figure is rotated by 90°, it looks like three right circular cylinders oriented as in Exercise 45. The moment of inertia of each of the disks around the axis of rotation is thus

$$\frac{1}{2}MR^2 = \frac{1}{2} \cdot 20 \cdot 3^2 = 90 \text{ g-cm}^2$$

and the moment of inertia of the axle around the axis of rotation is

$$\frac{1}{2}MR^2 = \frac{1}{2} \cdot 5 \cdot 1^2 = \frac{5}{2} \text{ g-cm}^2$$

Thus the total moment of inertia of the yo-yo around its axis of rotation is

$$2 \cdot 90 + \frac{5}{2} = 182.5 \text{ g-cm}^2$$

(b) If one third of the potential energy is converted to kinetic energy, then

$$\frac{1}{3} \cdot 45 \cdot 980 \cdot 100 = \frac{1}{2} \cdot 182.5 \cdot \omega^2 \text{ g-cm}^2$$

so that

$$\omega = \sqrt{\frac{2}{3} \cdot \frac{45 \cdot 980 \cdot 100}{182.5}} \approx 127 \text{ radians/sec} = \frac{127}{2\pi} \approx 20.2 \text{ rotations/sec}$$

48. Calculate I_z for the solid region $\mathcal{W}$ inside the hyperboloid $x^2 + y^2 = z^2 + 1$ between $z = 0$ and $z = 1$.

SOLUTION We have

$$\begin{aligned} I_z &= \iiint_{\mathcal{W}} (x^2 + y^2)\rho(x, y, z)\, dA = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z^2+1}} r^3\, dr\, d\theta\, dz = \frac{1}{4} \int_0^1 \int_0^{2\pi} r^4 \Big|_0^{\sqrt{z^2+1}} d\theta\, dz \\ &= \frac{1}{4} \int_0^1 \int_0^{2\pi} (z^2 + 1)^2\, d\theta\, dz = \frac{\pi}{2} \int_0^1 z^4 + 2z^2 + 1\, dz \\ &= \frac{\pi}{2} \left(\frac{1}{5} z^5 + \frac{2}{3} z^2 + z \right) \Big|_0^1 = \frac{14}{15}\pi \end{aligned}$$

49. Calculate $P(0 \le X \le 2; 1 \le Y \le 2)$, where X and Y have joint probability density function

$$p(x, y) = \begin{cases} \frac{1}{72}(2xy + 2x + y) & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

SOLUTION The region $0 \le X \le 2; 1 \le Y \le 2$ falls into the region where $p(x, y)$ is defined by the first line of the given formula, so that

$$\begin{aligned} P(0 \le X \le 2; 1 \le Y \le 2) &= \int_0^2 \int_1^2 \frac{1}{72}(2xy + 2x + y)\, dy\, dx = \frac{1}{72} \int_0^2 \left(xy^2 + 2xy + \frac{1}{2}y^2 \right) \Big|_1^2 dx \\ &= \frac{1}{72} \int_0^2 5x + \frac{3}{2}\, dx = \frac{1}{72} \left(\frac{5}{2}x^2 + \frac{3}{2}x \right) \Big|_0^2 = \frac{1}{72} \cdot 13 = \frac{13}{72} \approx 0.18 \end{aligned}$$

50. Calculate the probability that $X + Y \le 2$ for random variables with joint probability density function as in Exercise 49.

SOLUTION Note that $p(x, y) = 0$ if either x or y is negative, so that $X + Y \le 2$ corresponds to the region of integration $0 \le x \le 2$ and $0 \le y \le 2 - x$. Thus

$$\begin{aligned}
P(X + Y \le 2) &= \int_0^2 \int_0^{2-x} \frac{1}{72}(2xy + 2x + y)\,dy\,dx = \frac{1}{72}\int_0^2 \left(xy^2 + 2xy + \frac{1}{2}y^2\right)\Big|_0^{2-x} dx \\
&= \frac{1}{72}\int_0^2 x(2-x)^2 + 2x(2-x) + \frac{1}{2}(2-x)^2\,dx \\
&= \frac{1}{72}\int_0^2 x^3 - 4x^2 + 4x + 4x - 2x^2 + 2 - 2x + \frac{1}{2}x^2\,dx \\
&= \frac{1}{72}\int_0^2 x^3 - \frac{11}{2}x^2 + 6x + 2\,dx = \frac{1}{72}\left(\frac{1}{4}x^4 - \frac{11}{6}x^3 + 3x^2 + 2x\right)\Big|_0^2 \\
&= \frac{1}{72}\left(4 - \frac{11}{6}\cdot 8 + 12 + 4\right) = \frac{2}{27} \approx 0.074
\end{aligned}$$

51. The lifetime (in months) of two components in a certain device are random variables X and Y that have joint probability distribution function

$$p(x, y) = \begin{cases} \frac{1}{9216}(48 - 2x - y) & \text{if } x \ge 0,\ y \ge 0,\ 2x + y \le 48 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the probability that both components function for at least 12 months without failing. Note that $p(x, y)$ is nonzero only within the triangle bounded by the coordinate axes and the line $2x + y = 48$ shown in Figure 20.

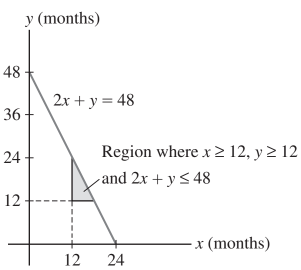


FIGURE 20

SOLUTION Both components function for at least 12 months without failing if $X + Y \ge 12$; however, we must also have $2X + Y \le 48$. Then the region of integration is the shaded triangle in the figure; the lower left corner of that triangle is (12, 12). One of the remaining vertices is the intersection of $x = 12$ and $2x + y = 48$; solving for y we have $y = 24$, so the point is (12, 24). The other vertex is the intersection of $y = 12$ and $2x + y = 48$; solving for x gives $x = 18$, so the point is (18, 12). The region of integration is then $12 \le x \le 18$ and $12 \le y \le 48 - 2x$. Thus the probability is

$$\begin{aligned}
P(X \ge 12, Y \ge 12) &= \int_{12}^{18} \int_{12}^{48-2x} \frac{1}{9216}(48 - 2x - y)\,dy\,dx \\
&= \frac{1}{9216}\int_{12}^{18} \left(48y - 2xy - \frac{1}{2}y^2\right)\Big|_{12}^{48-2x} dx \\
&= \frac{1}{9216}\int_{12}^{18} 1800 - 96x - 2x(36 - 2x) - \frac{1}{2}(48 - 2x)^2\,dx \\
&= \frac{1}{9216}\left(1800x - 48x^2 - 36x^2 + \frac{4}{3}x^3 + \frac{1}{12}(48 - 2x)^3\right)\Big|_{12}^{18} \\
&= \frac{144}{9216} = \frac{1}{64} = .015625
\end{aligned}$$

52. Find a constant C such that

$$p(x, y) = \begin{cases} Cxy & \text{if } 0 \le x \text{ and } 0 \le y \le 1 - x \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function. Then calculate

(a) $P\left(X \le \frac{1}{2}; Y \le \frac{1}{4}\right)$

(b) $P(X \ge Y)$

SOLUTION In order for $p(x, y)$ to be a joint probability density function, we must have

$$1 = \int_0^1 \int_0^{1-x} p(x, y) = \int_0^1 \int_0^{1-x} Cxy\,dy\,dx = \frac{C}{2} \int_0^1 xy^2 \Big|_0^{1-x} dx$$
$$= \frac{C}{2} \int_0^1 x(1-x)^2\,dx = \frac{C}{2} \int_0^1 x - 2x^2 + x^3\,dx = \frac{C}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{C}{24}$$

so that $C = 24$.

(a)

$$P\left(X \le \frac{1}{2}; Y \le \frac{1}{4}\right) = \int_0^{1/2} \int_0^{1/4} 24xy\,dy\,dx = 24\left(\int_0^{1/2} x\,dx\right)\left(\int_0^{1/4} y\,dy\right)$$
$$= 24 \cdot \frac{1}{8} \cdot \frac{1}{32} = \frac{3}{32} = 0.09375$$

(b) The probability density function is nonzero only for $0 \le y \le 1 - x$; in addition, $X \ge Y$ means that $y \le x$. The region of integration is then the region below both the lines $y = x$ and $y = 1 - x$ for $y \ge 0$; this is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1/2, 1/2)$. Integrating first in the x direction gives:

$$P(X \ge Y) = \int_0^{1/2} \int_y^{1-y} 24xy\,dx\,dy = \int_0^{1/2} 12x^2 y \Big|_{x=y}^{1-y} dy$$
$$= \int_0^{1/2} 12y((1-y)^2 - y^2)\,dy = \int_0^{1/2} 12y - 24y^2\,dy = (6y^2 - 8y^3)\Big|_0^{1/2}$$
$$= \frac{3}{2} - 1 = \frac{1}{2}$$

53. Find a constant C such that

$$p(x, y) = \begin{cases} Cy & \text{if } 0 \le x \le 1 \text{ and } x^2 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function. Then calculate the probability that $Y \ge X^{3/2}$.

SOLUTION $p(x, y)$ is a joint probability density function if

$$1 = \int_0^1 \int_{x^2}^x p(x, y)\,dy\,dx = \int_0^1 \int_{x^2}^x Cy\,dy\,dx$$
$$= \frac{C}{2} \int_0^1 y^2 \Big|_{x^2}^x dx = \frac{C}{2} \int_0^1 x^2 - x^4\,dx = \frac{C}{15}$$

so that we must have $C = 15$. Now, for $0 \le x \le 1$ we have $x^2 \le x^{3/2} \le x$, so that

$$P(Y \ge X^{3/2}) = \int_0^1 \int_{x^{3/2}}^x 15y\,dy\,dx = \frac{15}{2} \int_0^1 y^2 \Big|_{x^{3/2}}^x dx$$
$$= \frac{15}{2} \int_0^1 x^2 - x^3\,dx = \frac{15}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{15}{24} = \frac{5}{8} = 0.375$$

54. Numbers X and Y between 0 and 1 are chosen randomly. The joint probability density is $p(x, y) = 1$ if $0 \le x \le 1$ and $0 \le y \le 1$, and $p(x, y) = 0$ otherwise. Calculate the probability P that the product XY is at least $\frac{1}{2}$.

SOLUTION Since the probability density function is 1, the probability P is the integral of 1 over the region $\mathcal{W} = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1, xy \ge \frac{1}{2}\}$, which is just the area of $\mathcal{W}$. Now, $\mathcal{W}$ is the area bounded the curves $y = \frac{1}{2x}$ and $y = 1$ for $0 \le x \le 1$. Since these curves cross at $x = \frac{1}{2}$, the area is simply

$$P = \int_{1/2}^1 1 - \frac{1}{2x}\,dx = \left(x - \frac{1}{2}\ln x\right)\Big|_{1/2}^1 = 1 - \frac{1}{2} + \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}(1 - \ln 2)$$

55. According to quantum mechanics, the x- and y-coordinates of a particle confined to the region $\mathcal{R} = [0, 1] \times [0, 1]$ are random variables with joint probability density function

$$p(x, y) = \begin{cases} C\sin^2(2\pi\ell x)\sin^2(2\pi ny) & \text{if } (x, y) \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

The integers ℓ and n determine the energy of the particle, and C is a constant.

(a) Find the constant C.
(b) Calculate the probability that a particle with $\ell = 2$, $n = 3$ lies in the region $\left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{8}\right]$.

SOLUTION
(a) We have

$$\int_0^1 \int_0^1 C \sin^2(2\pi \ell x) \sin^2(2\pi n y)\, dx\, dy = C \left(\int_0^1 \sin^2(2\pi \ell x)^2\, dx\right)\left(\int_0^1 \sin^2(2\pi n y)^2\, dy\right)$$

Now, since ℓ is an integer, using the substitution $u = 2\pi\ell x$, $du = 2\pi\ell\, dx$, we have

$$\int_0^1 \sin^2(2\pi\ell x)\, dx = \frac{1}{2\pi\ell}\int_0^{2\pi\ell} \sin^2 u\, du = \frac{1}{2\pi\ell}\left(\frac{1}{2}u - \frac{1}{2}\sin u \cos u\right)\Big|_0^{2\pi\ell}$$
$$= \frac{1}{2\pi\ell}\left(\pi\ell - \frac{1}{2}\sin(2\pi\ell)\cos(2\pi\ell) + \frac{1}{2}\sin 0 \cos 0\right) = \frac{1}{2}$$

and the same is true of $\int_0^1 \sin^2(2\pi n y)\, dy$. Thus the value of the entire integral is $C\frac{1}{2} \cdot \frac{1}{2} = \frac{C}{4}$. In order for this to be a joint probability density function, then, we must have $C = 4$.
(b) We compute

$$\int_0^{1/4}\int_0^{1/8} 4\sin^2(2\pi \cdot 2x)\sin^2(2\pi \cdot 3y)\, dy\, dx = 4\left(\int_0^{1/4}\sin^2(4\pi x)\, dx\right)\left(\int_0^{1/8}\sin^2(6\pi y)\, dy\right)$$
$$= 4\left(\frac{1}{4\pi}\int_0^{\pi}\sin^2 u\, du\right)\left(\frac{1}{6\pi}\int_0^{3\pi/4}\sin^2 v\, dv\right)$$
$$= 4\left(\frac{1}{4\pi}\cdot\frac{\pi}{2}\right)\left(\frac{1}{6\pi}\cdot\left(\frac{3\pi}{8} + \frac{1}{4}\right)\right)$$
$$= 4\left(\frac{1}{8}\right)\left(\frac{1}{16} + \frac{1}{24\pi}\right) = \frac{1}{32} + \frac{1}{48\pi} \approx 0.03788$$

56. The wave function for the 1s state of an electron in the hydrogen atom is

$$\psi_{1s}(\rho) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\rho/a_0}$$

where a_0 is the Bohr radius. The probability of finding the electron in a region $\mathcal{W}$ of $\mathbf{R}^3$ is equal to

$$\iiint_{\mathcal{W}} p(x, y, z)\, dV$$

where, in spherical coordinates,

$$p(\rho) = |\psi_{1s}(\rho)|^2$$

Use integration in spherical coordinates to show that the probability of finding the electron at a distance greater than the Bohr radius is equal to $5/e^2 \approx 0.677$. The Bohr radius is $a_0 = 5.3 \times 10^{-11}$ m, but this value is not needed.

SOLUTION According to the problem statement, the probability of finding the electron at a distance greater than the Bohr radius is the probability of finding it in the region $\mathcal{W}$ of $\mathbf{R}^3$ with $\rho \geq a_0$ in spherical coordinates. This probability is

$$\iiint_{\mathcal{W}} p(x, y, z)\, dV = \int_{a_0}^{\infty}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-\rho/a_0}\right)^2 \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$
$$= \int_{a_0}^{\infty}\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-\rho/a_0}\right)^2 \cdot \rho^2 \cdot (-\cos\phi)\Big|_0^{\pi} d\theta\, d\rho$$
$$= 2\int_{a_0}^{\infty}\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-\rho/a_0}\right)^2 \cdot \rho^2\, d\theta\, d\rho$$
$$= 4\pi \cdot \frac{1}{\pi a_0^3}\int_{a_0}^{\infty}\rho^2 e^{-2\rho/a_0}\, d\rho$$

Using integration by parts twice, we continue

$$\iiint_{\mathcal{W}} p(x, y, z)\,dV = \frac{4}{a_0^3}\left(-\frac{a_0}{2}\rho^2 e^{-2\rho/a_0}\Big|_{a_0}^{\infty} + a_0\int_{a_0}^{\infty} \rho e^{-2\rho/a_0}\,d\rho\right)$$

$$= \frac{4}{a_0^3}\left(\frac{a_0^3}{2}e^{-2} + a_0\left(-\frac{a_0}{2}\rho e^{-2\rho/a_0}\Big|_{a_0}^{\infty} + a_0\int_{a_0}^{\infty} e^{-2\rho/a_0}\,d\rho\right)\right)$$

$$= \frac{4}{a_0^3}\left(\frac{a_0^3}{2}e^{-2} + \frac{a_0^3}{2}e^{-2} - a_0\left(\frac{a_0^2}{4}e^{-2\rho/a_0}\right)\Big|_{a_0}^{\infty}\right)$$

$$= \frac{4}{a_0^3}\left(a_0^3 e^{-2} + \frac{a_0^3}{4}e^{-2}\right) = \frac{4}{a_0^3}\left(\frac{5}{4}a_0^3 e^{-2}\right) = 5e^{-2} \approx 0.68$$

57. According to Coulomb's Law, the force between two electric charges of magnitude q_1 and q_2 separated by a distance r is kq_1q_2/r^2 (k is a negative constant). Let F be the net force on a charged particle P of charge Q coulombs located d centimeters above the center of a circular disk of radius R with a uniform charge distribution of density ρ C/m^2 (Figure 21). By symmetry, F acts in the vertical direction.

(a) Let $\mathcal{R}$ be a small polar rectangle of size $\Delta r \times \Delta\theta$ located at distance r. Show that $\mathcal{R}$ exerts a force on P whose vertical component is

$$\left(\frac{k\rho Q d}{(r^2 + d^2)^{3/2}}\right) r\,\Delta r\,\Delta\theta$$

(b) Explain why F is equal to the following double integral, and evaluate:

$$F = k\rho Q d\int_0^{2\pi}\int_0^R \frac{r\,dr\,d\theta}{(r^2 + d^2)^{3/2}}$$

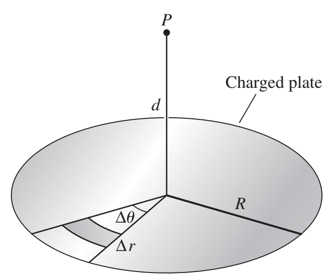


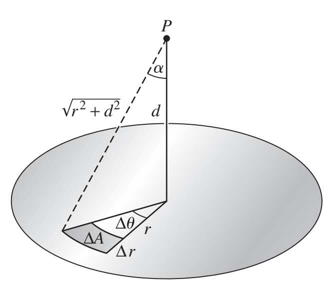
FIGURE 21

SOLUTION

(a) The area of the small polar rectangle $\mathcal{R}$ is

$$\Delta A = \frac{1}{2}(r + \Delta r)^2\Delta\theta - \frac{1}{2}r^2\Delta\theta = r(\Delta r\,\Delta\theta) + \frac{1}{2}\Delta r^2\Delta\theta \approx r(\Delta r\,\Delta\theta)$$

Therefore, the charge on $\mathcal{R}$ is $q_1 = \rho r(\Delta r\,\Delta\theta)$. The distance between P and $\mathcal{R}$ is, by the Pythagorean Law, $\sqrt{r^2 + d^2}$.



Therefore, the magnitude of the force between P and $\mathcal{R}$ is

$$\frac{kq_1q_2}{\left(\sqrt{r^2 + d^2}\right)^2} = \frac{k\,(\rho r\,\Delta r\,\Delta\theta)\,Q}{r^2 + d^2} = \frac{k\rho Q}{r^2 + d^2} r\,\Delta r\,\Delta\theta$$

The vertical component of this force is obtained by multiplying the force by $\cos\alpha = \frac{d}{\sqrt{r^2+d^2}}$. That is,

$$F_{\text{vert}} = \frac{k\rho Q}{r^2+d^2} \cdot \frac{d}{\sqrt{r^2+d^2}} r\,\Delta r\,\Delta\theta = \frac{k\rho Q d}{(r^2+d^2)^{3/2}} r\,\Delta r\,\Delta\theta$$

(b) Since F acts in the vertical direction, it is approximated by the Riemann sum of the forces F_{vert} in part (a), over the polar rectangles. This Riemann sum approximates F in higher precision if we let $\Delta\theta \to 0$ and $\Delta r \to 0$. The result is the double integral of $\frac{k\rho Q d}{(r^2+d^2)^{3/2}}$ over the disk. The disk is determined by $0 \le \theta \le 2\pi$ and $0 \le r \le R$. Therefore we get

$$F = \int_0^R \int_0^{2\pi} \frac{k\rho Q d}{(r^2+d^2)^{3/2}} r\,dr\,d\theta = 2\pi k\rho Q d \int_0^R \int_0^{2\pi} \frac{r\,dr\,d\theta}{(r^2+d^2)^{3/2}}$$

Using the u-substitution of $u = r^2 + d^2$, $du = 2r\,dr$, we continue

$$F = 2\pi k\rho Q d \int_0^R \frac{r\,dr}{(r^2+d^2)^{3/2}} = 2\pi k\rho Q d \cdot \frac{1}{2} \int_{d^2}^{R^2+d^2} u^{-3/2}\,du = -\pi k\rho Q d u^{-1/2}\Big|_{d^2}^{R^2+d^2}$$

$$= \pi k\rho Q d \left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

58. Let $\mathcal{D}$ be the annular region

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad a \le r \le b$$

where $b > a > 0$. Assume that $\mathcal{D}$ has a uniform charge distribution of ρ C/m^2. Let F be the net force on a charged particle of charge Q coulombs located at the origin (by symmetry, F acts along the x-axis).

(a) Argue as in Exercise 57 to show that

$$F = k\rho Q \int_{\theta=-\pi/2}^{\pi/2} \int_{r=a}^{b} \left(\frac{\cos\theta}{r^2}\right) r\,dr\,d\theta$$

(b) Compute F.

SOLUTION As in Exercise 57, let $\mathcal{R}$ be a small polar rectangle of size $\Delta r \times \Delta\theta$ located at distance r ($a \le r \le b$) from the origin and at an angle θ to the x-axis. As before, its area is approximately $r\,\Delta r\,\Delta\theta$, so the charge on $\mathcal{R}$ is $q_1 = \rho r\,\Delta r\,\Delta\theta$. The distance between P, at the origin, and $\mathcal{R}$ is r, so the magnitude of the force between P and $\mathcal{R}$ is

$$F = \frac{kq_1q_2}{r^2} = \frac{k\rho Q}{r^2} r\,\Delta r\,\Delta\theta = \frac{k\rho Q}{r^2} r\,\Delta r\,\Delta\theta$$

Now, since the force acts along the x-axis, we want the horizontal component of F; this is just

$$F_{\text{horiz}} = F\cos\theta = k\rho Q \frac{\cos\theta}{r^2} r\,\Delta r\,\Delta\theta$$

Then F is approximated by the Riemann sum of the forces F_{horiz} over the polar rectangles. As $\Delta r \to 0$, $\Delta\theta \to 0$, this becomes the double integral of $k\rho Q \frac{\cos\theta}{r^2} r$ over the half-annulus. Since the region is determined by $a \le r \le b$ and $-\pi/2 \le \theta \le \pi/2$, we hve

$$F = k\rho Q \int_a^b \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{r^2} r\,d\theta\,dr = 2k\rho Q \int_a^b \frac{1}{r}\,dr = 2k\rho Q \ln\left(\frac{b}{a}\right)$$

Further Insights and Challenges

59. Let $\mathcal{D}$ be the domain in Figure 22. Assume that $\mathcal{D}$ is symmetric with respect to the y-axis; that is, both $g_1(x)$ and $g_2(x)$ are even functions.

(a) Prove that the centroid lies on the y-axis—that is, that $\bar{x} = 0$.

(b) Show that if the mass density satisfies $\rho(-x, y) = \rho(x, y)$, then $M_y = 0$ and $x_{\text{CM}} = 0$.

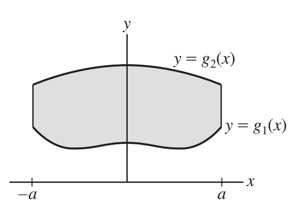


FIGURE 22

SOLUTION

(a) Assume $\mathcal{D}$ has area A. Then the x coordinate of the centroid is

$$\overline{x} = \frac{1}{A}\iint_{\mathcal{D}} x\,dA = \frac{1}{A}\int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} x\,dy\,dx = \frac{1}{A}\int_{-a}^{a} x(g_2(x) - g_1(x))\,dx$$

Since g_1 and g_2 are both even functions, $x(g_2(x) - g_1(x))$ is an odd function, so its integral over a region symmetric about the x-axis is zero. Thus $\overline{x} = 0$.

(b) Let $R(x, y)$ be an antiderivative of $\rho(x, y)$ with respect to y, i.e. $R(x, y) = \int \rho(x, y)\,dy$. Note that

$$R(-x, y) = \int \rho(-x, y)\,dy = \int \rho(x, y)\,dy = R(x, y)$$

so that R is an even function with respect to x. Now, we have

$$M_y \iint_{\mathcal{D}} x\rho(x, y)\,dA = \int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} x\rho(x, y)\,dy\,dx = \int_{-a}^{a} x(R(x, g_2(x)) - R(x, g_1(x))\,dx$$

Since g_1, g_2, and R are all even functions of x, we have

$$R(-x, g_2(-x)) - R(-x, g_1(-x)) = R(-x, g_2(x)) - R(-x, g_1(x)) = R(x, g_2(x)) - R(x, g_1(x))$$

so that the second factor in the integrand is an even function of x. But x is an odd function of x, so their product is odd. It follows that the integral over the range $-a \le x \le a$ is zero. Thus $M_y = x_{\mathrm{CM}} = 0$.

60. Pappus's Theorem Let A be the area of the region $\mathcal{D}$ between two graphs $y = g_1(x)$ and $y = g_2(x)$ over the interval $[a, b]$, where $g_2(x) \ge g_1(x) \ge 0$. Prove Pappus's Theorem: The volume of the solid obtained by revolving $\mathcal{D}$ about the x-axis is $V = 2\pi A\overline{y}$, where $\overline{y}$ is the y-coordinate of the centroid of $\mathcal{D}$ (the average of the y-coordinate). *Hint:* Show that

$$A\overline{y} = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)} y\,dy\,dx$$

SOLUTION Using the washer method, the volume of the solid of revolution about the x axis is

$$V = \int_{a}^{b} \pi(g_2(x)^2 - g_1(x)^2)\,dx$$

Now, we have by definition

$$A\overline{y} = \iint_{\mathcal{D}} y\,dA = \int_{a}^{b}\int_{g_1(x)}^{g_2(x)} y\,dy\,dx = \frac{1}{2}\int_{a}^{b} y^2\Big|_{g_1(x)}^{g_2(x)}\,dx = \frac{1}{2}\int_{a}^{b} g_2(x)^2 - g_1(x)^2\,dx$$

so that

$$2\pi A\overline{y} = 2\pi\cdot\frac{1}{2}\int_{a}^{b} g_2(x)^2 - g_1(x)^2\,dx = \int_{a}^{b}\pi(g_2(x)^2 - g_1(x)^2)\,dx = V$$

61. Use Pappus's Theorem in Exercise 60 to show that the torus obtained by revolving a circle of radius b centered at $(0, a)$ about the x-axis (where $b < a$) has volume $V = 2\pi^2 ab^2$.

SOLUTION The centroid of the circle is obviously at the center, $(0, a)$, and the area of the circle is πb^2, so that by Pappus' theorem,

$$V = 2\pi\pi b^2 a = 2\pi ab^2$$

62. Use Pappus's Theorem to compute $\overline{y}$ for the upper half of the disk $x^2 + y^2 \le a^2$, $y \ge 0$. *Hint:* The disk revolved about the x-axis is a sphere.

SOLUTION Since the disk revolved about the x-axis is a sphere of radius a, it has volume $\frac{4}{3}\pi a^3$. The area of the upper half of the disk is $\frac{1}{2}\pi a^2$. Thus by Pappus' theorem,

$$\frac{4}{3}\pi a^3 = 2\pi\cdot\frac{1}{2}\pi a^2\cdot\overline{y}$$

Solving for $\overline{y}$ gives

$$\overline{y} = \frac{4a}{3\pi}$$

63. Parallel-Axis Theorem Let $\mathcal{W}$ be a region in $\mathbf{R}^3$ with center of mass at the origin. Let I_z be the moment of inertia of $\mathcal{W}$ about the z-axis, and let I_h be the moment of inertia about the vertical axis through a point $P = (a, b, 0)$, where $h = \sqrt{a^2 + b^2}$. By definition,

$$I_h = \iiint_{\mathcal{W}} ((x-a)^2 + (y-b)^2)\rho(x, y, z)\,dV$$

Prove the Parallel-Axis Theorem: $I_h = I_z + Mh^2$.

SOLUTION We have

$$\begin{aligned}
I_h - I_z &= \iiint_{\mathcal{W}} ((x-a)^2 + (y-b)^2)\rho(x, y, z)\,dV - \iiint_{\mathcal{W}} (x^2 + y^2)\rho(x, y, z)\,dV \\
&= \iiint_{\mathcal{W}} (-2ax + a^2 - 2by + b^2)\rho(x, y, z)\,dV \\
&= (a^2 + b^2)\iiint_{\mathcal{W}} \rho(x, y, z)\,dV - 2a\iiint_{\mathcal{W}} x\rho(x, y, z)\,dV - 2b\iiint_{\mathcal{W}} y\rho(x, y, z)\,dV \\
&= (a^2 + b^2)M - 2aM_{yz} - 2bM_{xz} = Mh^2
\end{aligned}$$

since the last two terms are zero because the center of mass of $\mathcal{W}$ is at the origin.

64. Let $\mathcal{W}$ be a cylinder of radius 10 cm and height 20 cm, with total mass $M = 500$ g. Use the Parallel-Axis Theorem (Exercise 63) and the result of Exercise 45 to calculate the moment of inertia of $\mathcal{W}$ about an axis that is parallel to and at a distance of 30 cm from the cylinder's axis of symmetry.

SOLUTION For this cylinder, using Exercise 45, we see that the moment of inertia about the cylinder's axis of symmetry is

$$I_{\text{symm}} = \frac{1}{2}MR^2 = \frac{1}{2}\cdot 500 \cdot 100 = 25{,}000 \text{ g-cm}^2$$

Since the axis we are considering is at a distance of 30 cm from the axis of symmetry, the Parallel-Axis theorem tells us that

$$I = I_{\text{symm}} + Mh^2 = 25000 + 500 \cdot 900 = 475{,}000 \text{ g-cm}^2$$

15.6 Change of Variables (LT Section 16.6)

Preliminary Questions

1. Which of the following maps is linear?

(a) (uv, v) **(b)** $(u + v, u)$ **(c)** $(3, e^u)$

SOLUTION

(a) This map is not linear since it does not satisfy the linearity property:

$$\Phi(2u, 2v) = (2u \cdot 2v, 2v) = (4uv, 2v) = 2(2uv, v)$$
$$2\Phi(u, v) = 2(uv, v) \quad \Rightarrow \quad \Phi(2u, 2v) \neq 2\Phi(u, v)$$

(b) This map is linear since it has the form $\Phi(u, v) = (Au + Cv, Bu + Dv)$ where $A = C = 1$, $B = 1$, $D = 0$.

(c) This map is not linear since it does not satisfy the linearity properties. For example,

$$\left.\begin{aligned}\Phi(2u, 2v) &= (3, e^{2u}) \\ 2\Phi(u, v) &= 2(3, e^u)\end{aligned}\right. \quad \Rightarrow \quad \Phi(2u, 2v) \neq 2\Phi(u, v)$$

2. Suppose that Φ is a linear map such that $\Phi(2, 0) = (4, 0)$ and $\Phi(0, 3) = (-3, 9)$. Find the images of:

(a) $\Phi(1, 0)$ **(b)** $\Phi(1, 1)$ **(c)** $\Phi(2, 1)$

SOLUTION We denote the linear map by $\Phi(u, v) = (Au + Cv, Bu + Dv)$. By the given information we have

$$\Phi(2, 0) = (A \cdot 2 + C \cdot 0, B \cdot 2 + D \cdot 0) = (2A, 2B) = (4, 0)$$
$$\Phi(0, 3) = (A \cdot 0 + C \cdot 3, B \cdot 0 + D \cdot 3) = (3C, 3D) = (-3, 9)$$

Therefore,

$$\begin{aligned} 2A &= 4 \\ 2B &= 0 \\ 3C &= -3 \\ 3D &= 9 \end{aligned} \quad \Rightarrow \quad A = 2, \quad B = 0, \quad C = -1, \quad D = 3$$

The linear map is thus

$$\Phi(u, v) = (2u - v, 3v)$$

We now compute the images:

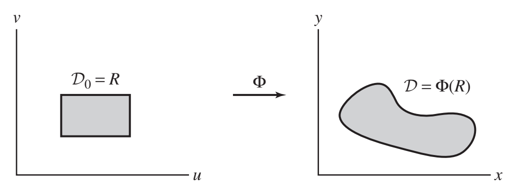
(a) $\Phi(1, 0) = (2 \cdot 1 - 0, 3 \cdot 0) = (2, 0)$

(b) $\Phi(1, 1) = (2 \cdot 1 - 1, 3 \cdot 1) = (1, 3)$

(c) $\Phi(2, 1) = (2 \cdot 2 - 1, 3 \cdot 1) = (3, 3)$

3. What is the area of $\Phi(\mathcal{R})$ if $\mathcal{R}$ is a rectangle of area 9 and Φ is a mapping whose Jacobian has constant value 4?

SOLUTION



The areas of $\mathcal{D}_0 = \Phi(\mathcal{R})$ and $\mathcal{D} = \mathcal{R}$ are the following integrals:

$$\text{Area}(\mathcal{R}) = 9 = \iint_{\mathcal{D}_0} 1\, du\, dv$$

$$\text{Area}(\Phi(\mathcal{R})) = \iint_{\mathcal{D}} 1\, dx\, dy$$

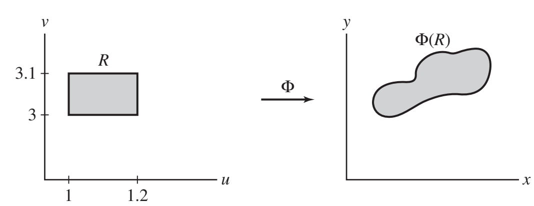
Using the Change of Variables Formula, we have

$$\text{Area}\,(\Phi(\mathcal{R})) = \iint_{\mathcal{D}} 1\, dx\, dy = \iint_{\mathcal{D}_0} 1|\text{Jac}\Phi|\, du\, dv = \iint_{\mathcal{D}_0} 4\, du\, dv = 4 \iint_{\mathcal{D}_0} 1\, du\, dv = 4 \cdot 9 = 36$$

The area of $\Phi(\mathcal{R})$ is 36.

4. Estimate the area of $\Phi(\mathcal{R})$, where $\mathcal{R} = [1, 1.2] \times [3, 3.1]$ and Φ is a mapping such that $\text{Jac}(\Phi)(1, 3) = 3$.

SOLUTION



We use the following estimation:

$$\text{Area}\,(\Phi(\mathcal{R})) \approx |\text{Jac}\,(\Phi)\,(P)|\text{Area}(\mathcal{R})$$

The area of the rectangle $\mathcal{R}$ is

$$\text{Area}(\mathcal{R}) = 0.2 \cdot 0.1 = 0.02$$

We choose the sample point $P = (1, 3)$ in $\mathcal{R}$ to obtain the following estimation:

$$\text{Area}\,(\Phi(\mathcal{R})) \approx |\text{Jac}\,(\Phi)\,(1, 3)|\text{Area}(\mathcal{R}) = 3 \cdot 0.02 = 0.06$$

Exercises

1. Determine the image under $\Phi(u, v) = (2u, u + v)$ of the following sets:

(a) The u- and v-axes

(b) The rectangle $\mathcal{R} = [0, 5] \times [0, 7]$

(c) The line segment joining $(1, 2)$ and $(5, 3)$

(d) The triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

SOLUTION

(a) The image of the u-axis is obtained by substituting $v = 0$ in $\Phi(u, v) = (2u, u + v)$. That is,

$$\Phi(u, 0) = (2u, u + 0) = (2u, u).$$

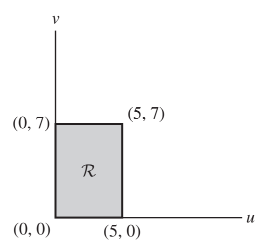
The image of the u-axis is the set of points $(x, y) = (2u, u)$, which is the line $y = \frac{1}{2}x$ in the xy-plane. The image of the v-axis is obtained by substituting $u = 0$ in $\Phi(u, v) = (2u, u + v)$. That is,

$$\Phi(0, v) = (0, 0 + v) = (0, v).$$

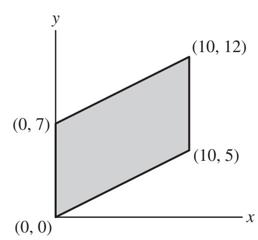
Therefore, the image of the v-axis is the set $(x, y) = (0, v)$, which is the vertical line $x = 0$ (the y-axis).

(b) Since Φ is a linear map, the segment through points P and Q is mapped to the segment through $\Phi(P)$ and $\Phi(Q)$. We thus must find the images of the vertices of $\mathcal{R}$:

$$\Phi(0, 0) = (2 \cdot 0, 0 + 0) = (0, 0)$$
$$\Phi(5, 0) = (2 \cdot 5, 5 + 0) = (10, 5)$$
$$\Phi(5, 7) = (2 \cdot 5, 5 + 7) = (10, 12)$$
$$\Phi(0, 7) = (2 \cdot 0, 0 + 7) = (0, 7)$$

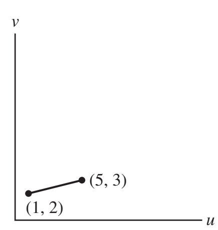


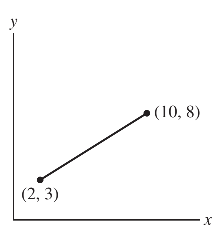
The image of $\mathcal{R}$ is the parallelogram with vertices $(0, 0)$, $(10, 5)$, $(10, 12)$, and $(0, 7)$ in the xy-plane.



(c) We compute the images of the endpoints of the segment:

$$\Phi(1, 2) = (2 \cdot 1, 1 + 2) = (2, 3)$$
$$\Phi(5, 3) = (2 \cdot 5, 5 + 3) = (10, 8)$$





The image is the segment in the xy-plane joining the points $(2, 3)$ and $(10, 8)$.

(d) Since Φ is linear, the image of the triangle is the triangle whose vertices are the images of the vertices of the triangle. We compute these images:

$$\Phi(0, 1) = (2 \cdot 0, 0 + 1) = (0, 1)$$
$$\Phi(1, 0) = (2 \cdot 1, 1 + 0) = (2, 1)$$
$$\Phi(1, 1) = (2 \cdot 1, 1 + 1) = (2, 2)$$

Therefore the image is the triangle in the xy-plane whose vertices are at the points $(0, 1)$, $(2, 1)$, and $(2, 2)$.

2. Describe [in the form $y = f(x)$] the images of the lines $u = c$ and $v = c$ under the mapping $\Phi(u, v) = (u/v, u^2 - v^2)$.

SOLUTION The image of the vertical line $u = c$ is the set of points

$$(x, y) = \Phi(c, v) = \left(\frac{c}{v}, c^2 - v^2\right).$$

That is, $x = \frac{c}{v}$ and $y = c^2 - v^2$. We substitute $v = \frac{c}{x}$ in the equation for y to obtain

$$y = c^2 - \frac{c^2}{x^2} = c^2\left(1 - \frac{1}{x^2}\right)$$

The image of the horizontal line $v = c$ is the set of points

$$(x, y) = \Phi(u, c) = \left(\frac{u}{c}, u^2 - c^2\right).$$

That is, $x = \frac{u}{c}$ and $y = u^2 - c^2$. Substituting $u = cx$ in the equation for y gives the parabola

$$y = c^2x^2 - c^2 = c^2(x^2 - 1).$$

3. Let $\Phi(u, v) = (u^2, v)$. Is Φ one-to-one? If not, determine a domain on which Φ is one-to-one. Find the image under Φ of:

(a) The u- and v-axes
(b) The rectangle $\mathcal{R} = [-1, 1] \times [-1, 1]$
(c) The line segment joining $(0, 0)$ and $(1, 1)$
(d) The triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$

SOLUTION Φ is not one-to-one since for any $u \neq 0$, (u, v) and $(-u, v)$ are two different points with the same image. However, Φ is one-to-one on the domain $\{(u, v) : u \geq 0\}$ and on the domain $\{(u, v) : u \leq 0\}$.

(a) The image of the u-axis is the set of the points

$$(x, y) = \Phi(u, 0) = (u^2, 0) \quad \Rightarrow \quad x = u^2, \quad y = 0$$

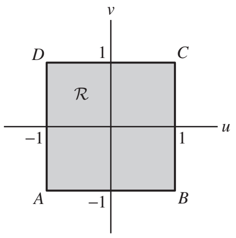
That is, the positive x-axis, including the origin. The image of the v-axis is the set of the following points:

$$(x, y) = \Phi(0, v) = (0^2, v) = (0, v) \quad \Rightarrow \quad x = 0, \quad y = v$$

That is, the line $x = 0$, which is the y-axis.

(b) The rectangle $\mathcal{R}$ is defined by

$$|u| \leq 1, \quad |v| \leq 1$$



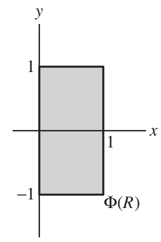
Since $x = u^2$ and $y = v$, we have $u = \pm\sqrt{x}$ and $v = y$ (depending on our choice of domain). Therefore, the inequalities for x and y are

$$|\pm\sqrt{x}| \leq 1, \quad |y| \leq 1$$

or

$$0 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1.$$

We conclude that the image of $\mathcal{R}$ in the xy-plane is the rectangle $[0, 1] \times [-1, 1]$.



(c) The line segment joining the points (0, 0) and (1, 1) in the uv-plane is defined by

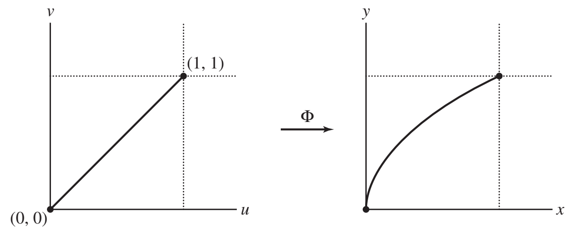
$$0 \le u \le 1, \quad v = u.$$

Substituting $u = \sqrt{x}$ and $v = y$, we get

$$0 \le \sqrt{x} \le 1, \quad y = \sqrt{x}$$

or

$$0 \le x \le 1, \quad y = \sqrt{x}$$



The image is the curve $y = \sqrt{x}$ for $0 \le x \le 1$.

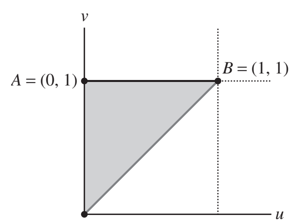
(d) We identify the image of the sides of the triangle OAB.

The image of $\overline{OA}$: This segment is defined by $u = 0$ and $0 \le v \le 1$. That is,

$$\pm\sqrt{x} = 0 \quad \text{and} \quad 0 \le y \le 1$$

or

$$x = 0, \quad 0 \le y \le 1.$$



This is the segment joining the points (0, 0) and (0, 1) in the xy-plane.

The image of $\overline{AB}$: This segment is defined by $0 \le u \le 1$ and $v = 1$. That is,

$$0 \le \sqrt{x} \le 1, \quad y = 1$$

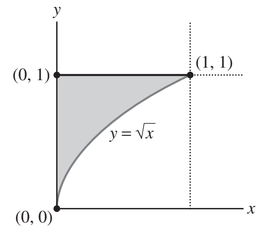
or

$$0 \le x \le 1, \quad y = 1.$$

This is the segment joining the points (0, 1) and (1, 1) in the xy-plane.

The image of $\overline{OB}$: In part (c) we showed that the image of the segment is the curve $y = \sqrt{x}$, $0 \le x \le 1$.

Therefore, the image of the triangle is the region shown in the figure:



4. Let $\Phi(u, v) = (e^u, e^{u+v})$.

(a) Is Φ one-to-one? What is the image of Φ?

(b) Describe the images of the vertical lines $u = c$ and the horizontal lines $v = c$.

SOLUTION

(a) Suppose that $\Phi(u_1, v_1) = \Phi(u_2, v_2)$. We show that $(u_1, v_1) = (u_2, v_2)$, hence Φ is one-to-one. We have

$$\left(e^{u_1}, e^{u_1+v_1}\right) = \left(e^{u_2}, e^{u_2+v_2}\right) \quad \Rightarrow \quad e^{u_1} = e^{u_2} \quad \Rightarrow \quad u_1 = u_2$$

and

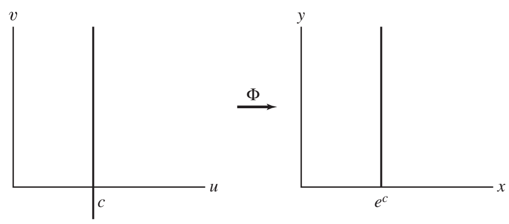
$$e^{u_1+v_1} = e^{u_2+v_2} \quad \Rightarrow \quad u_1 + v_1 = u_2 + v_2 \quad \Rightarrow \quad v_1 = v_2$$

We find the image of Φ. The image of the exponent function is $(0, \infty)$. Since u and v can take any value, the points (e^u, e^{u+v}) are all the points (x, y) where $x > 0$ and $y > 0$; that is, the image of Φ is the first quadrant of the xy-plane, with the axes excluded.

(b) The image of the vertical line $u = c$ is the set of the following points:

$$(x, y) = \Phi(c, v) = (e^c, e^{c+v}) \quad \Rightarrow \quad x = e^c, \quad y = e^v x \quad \Rightarrow \quad x = e^c, \quad y > 0$$

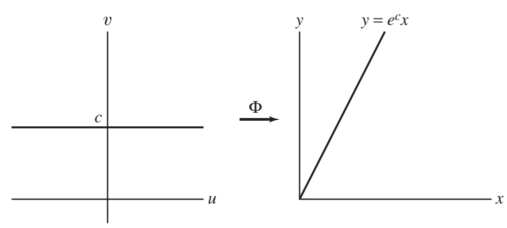
That is, the image is the ray $x = e^c$, $y > 0$ in the xy-plane.



The image of the horizontal line $v = c$ is the set of the points

$$(x, y) = \Phi(u, c) = (e^u, e^{u+c}) \quad \Rightarrow \quad x = e^u, \quad y = e^c x \quad \Rightarrow \quad y = e^c x, \quad x > 0$$

Since u can take any value, x can take any positive value, and hence the image is the ray $y = e^c x$, $x > 0$.



In Exercises 5–12, let $\Phi(u, v) = (2u + v, 5u + 3v)$ be a map from the uv-plane to the xy-plane.

5. Show that the image of the horizontal line $v = c$ is the line $y = \frac{5}{2}x + \frac{1}{2}c$. What is the image (in slope-intercept form) of the vertical line $u = c$?

SOLUTION The image of the horizontal line $v = c$ is the set of the following points:

$$(x, y) = \Phi(u, c) = (2u + c, 5u + 3c) \quad \Rightarrow \quad x = 2u + c, \quad y = 5u + 3c$$

The first equation implies $u = \frac{x-c}{2}$. Substituting in the second equation gives

$$y = 5\frac{(x - c)}{2} + 3c = \frac{5x}{2} + \frac{c}{2}$$

Therefore, the image of the line $v = c$ is the line $y = \frac{5x}{2} + \frac{c}{2}$ in the xy-plane.

The image of the vertical line $u = c$ is the set of the following points:

$$(x, y) = \Phi(c, v) = (2c + v, 5c + 3v) \quad \Rightarrow \quad x = 2c + v, \quad y = 5c + 3v$$

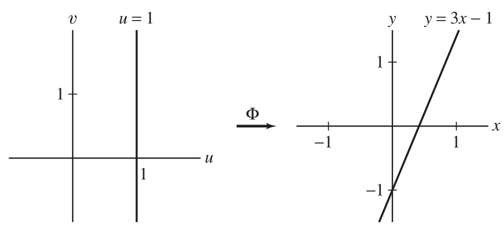
By the first equation, $v = x - 2c$. Substituting in the second equation gives

$$y = 5c + 3(x - 2c) = 5c + 3x - 6c = 3x - c$$

Therefore, the image of the line $u = c$ is the line $y = 3x - c$ in the xy-plane.

6. Describe the image of the line through the points $(u, v) = (1, 1)$ and $(u, v) = (1, -1)$ under Φ in slope-intercept form.

SOLUTION



The line is the vertical line $u = 1$ in the uv-plane. The image of the line under the linear map $\Phi(u, v) = (2u + v, 5u + 3v)$ is the line through the images of the points $(u, v) = (1, 1)$ and $(u, v) = (1, -1)$. We find these images:

$$\Phi(1, 1) = (2 \cdot 1 + 1, 5 \cdot 1 + 3 \cdot 1) = (3, 8)$$
$$\Phi(1, -1) = (2 \cdot 1 - 1, 5 \cdot 1 - 3) = (1, 2)$$

We find the slope-intercept form of the equation of the line in the xy-plane, through the points $(3, 8)$ and $(1, 2)$:

$$y - 2 = \frac{8 - 2}{3 - 1}(x - 1) = 3(x - 1) \quad \Rightarrow \quad y = 3x - 1$$

7. Describe the image of the line $v = 4u$ under Φ in slope-intercept form.

SOLUTION We choose any two points on the line $v = 4u$, for example $(u, v) = (1, 4)$ and $(u, v) = (0, 0)$. By a property of linear maps, the image of the line $v = 4u$ under the linear map $\Phi(u, v) = (2u + v, 5u + 3v)$ is the line in the xy-plane through the points $\Phi(1, 4)$ and $\Phi(0, 0)$. We find these points:

$$\Phi(0, 0) = (2 \cdot 0 + 0, 5 \cdot 0 + 3 \cdot 0) = (0, 0)$$
$$\Phi(1, 4) = (2 \cdot 1 + 4, 5 \cdot 1 + 3 \cdot 4) = (6, 17)$$

We now find the slope-intercept equation of the line in the xy-plane through the points $(0, 0)$ and $(6, 17)$:

$$y - 0 = \frac{17 - 0}{6 - 0}(x - 0) \quad \Rightarrow \quad y = \frac{17}{6}x$$

8. Show that Φ maps the line $v = mu$ to the line of slope $(5 + 3m)/(2 + m)$ through the origin in the xy-plane.

SOLUTION The points $(0, 0)$ and $(1, m)$ lie on the line $v = mu$. The images of these points under the map $\Phi(u, v) = (2u + v, 5u + 3v)$ are

$$\Phi(0, 0) = (2 \cdot 0 + 0, 5 \cdot 0 + 3 \cdot 0) = (0, 0)$$
$$\Phi(1, m) = (2 \cdot 1 + m, 5 \cdot 1 + 3m) = (2 + m, 5 + 3m)$$

By properties of linear maps, the image of the line $v = mu$ under the linear map Φ is a line through the points $(0, 0)$ and $(2 + m, 5 + 3m)$. The slope of this line is

$$\frac{(5 + 3m) - 0}{(2 + m) - 0} = \frac{5 + 3m}{2 + m}.$$

9. Show that the inverse of Φ is

$$\Phi^{-1}(x, y) = (3x - y, -5x + 2y)$$

Hint: Show that $\Phi(\Phi^{-1}(x, y)) = (x, y)$ and $\Phi^{-1}(\Phi(u, v)) = (u, v)$.

SOLUTION By the definition of the inverse map, we must show that the given maps $\Phi^{-1}(x, y) = (3x - y, -5x + 2y)$ and $\Phi(u, v) = (2u + v, 5u + 3v)$ satisfy $\Phi\left(\Phi^{-1}(x, y)\right) = (x, y)$ and $\Phi^{-1}(\Phi(u, v)) = (u, v)$. We have

$$\Phi\left(\Phi^{-1}(x, y)\right) = \Phi(3x - y, -5x + 2y) = (2(3x - y) + (-5x + 2y), 5(3x - y) + 3(-5x + 2y)) = (x, y)$$
$$\Phi^{-1}(\Phi(u, v)) = \Phi^{-1}(2u + v, 5u + 3v) = (3(2u + v) - (5u + 3v), -5(2u + v) + 2(5u + 3v)) = (u, v)$$

We conclude that Φ^{-1} is the inverse of Φ.

10. Use the inverse in Exercise 9 to find:

(a) A point in the uv-plane mapping to $(2, 1)$

(b) A segment in the uv-plane mapping to the segment joining $(-2, 1)$ and $(3, 4)$

SOLUTION

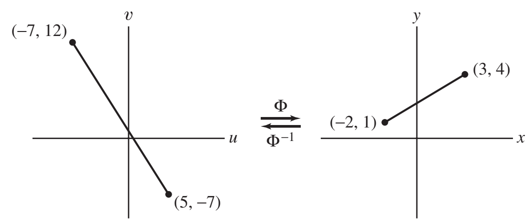
(a) The inverse of $\Phi(u, v)$ is $\Phi^{-1}(x, y) = (3x - y, -5x + 2y)$. Therefore, the point in the uv-plane mapping to $(2, 1)$ in the xy-plane is $\Phi^{-1}(2, 1)$. We find it:

$$(u, v) = \Phi^{-1}(2, 1) = (3 \cdot 2 - 1, -5 \cdot 2 + 2 \cdot 1) = (5, -8)$$

(b) The segment we need to find is the image of the segment joining the points $(-2, 1)$ and $(3, 4)$ in the xy-plane, under the inverse map $\Phi^{-1}(x, y) = (3x - y, -5x + 2y)$. By properties of linear maps, this image is the segment in the uv-plane joining the points $\Phi^{-1}(-2, 1)$ and $\Phi^{-1}(3, 4)$. We find these points:

$$\Phi^{-1}(-2, 1) = (3 \cdot (-2) - 1, -5 \cdot (-2) + 2 \cdot 1) = (-7, 12)$$
$$\Phi^{-1}(3, 4) = (3 \cdot 3 - 4, -5 \cdot 3 + 2 \cdot 4) = (5, -7)$$

Therefore, the segment in the uv-plane mapping to the given segment under Φ is the segment joining the points $(-7, 12)$ and $(5, -7)$ in the uv-plane.



11. Calculate $\text{Jac}(\Phi) = \dfrac{\partial(x, y)}{\partial(u, v)}$.

SOLUTION The Jacobian of the linear mapping $\Phi(u, v) = (2u + v, 5u + 3v)$ is the following determinant:

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 2 \cdot 3 - 5 \cdot 1 = 1$$

12. Calculate $\text{Jac}(\Phi^{-1}) = \dfrac{\partial(u, v)}{\partial(x, y)}$.

SOLUTION We use the formula for the Jacobian of the inverse map, and the result of Exercise 11 to obtain

$$\text{Jac}(\Phi^{-1}) = \left(\text{Jac}(\Phi)^{-1}\right) = 1^{-1} = 1$$

In Exercises 13–18, compute the Jacobian (at the point, if indicated).

13. $\Phi(u, v) = (3u + 4v, u - 2v)$

SOLUTION Using the Jacobian of linear mappings we get

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 3 \cdot (-2) - 1 \cdot 4 = -10$$

14. $\Phi(r, s) = (rs, r + s)$

SOLUTION We have $x = rs$ and $y = r + s$, therefore,

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(r, s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} s & r \\ 1 & 1 \end{vmatrix} = s \cdot 1 - r \cdot 1 = s - r$$

15. $\Phi(r, t) = (r \sin t, r - \cos t), \quad (r, t) = (1, \pi)$

SOLUTION We have $x = r \sin t$ and $y = r - \cos t$. Therefore,

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(r, t)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \sin t & r \cos t \\ 1 & \sin t \end{vmatrix} = \sin^2 t - r \cos t$$

At the point $(r, t) = (1, \pi)$ we get

$$\text{Jac}(\Phi)(1, \pi) = \sin^2 \pi - 1 \cdot \cos \pi = 0 - 1 \cdot (-1) = 1$$

16. $\Phi(u, v) = (v \ln u, u^2 v^{-1})$, $\quad (u, v) = (1, 2)$

SOLUTION We have $x = v \ln u$ and $y = u^2 v^{-1}$. Therefore,

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{u} & \ln u \\ \frac{2u}{v} & \frac{-u^2}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{2u}{v} \ln u = -\frac{u}{v}(1 + 2 \ln u)$$

At the point $(u, v) = (1, 2)$ we get:

$$\text{Jac}(\Phi)(1, 2) = -\frac{1}{2}(1 + 2 \ln 1) = -\frac{1}{2}$$

17. $\Phi(r, \theta) = (r \cos\theta, r \sin\theta)$, $\quad (r, \theta) = \left(4, \frac{\pi}{6}\right)$

SOLUTION Since $x = r \cos\theta$ and $y = r \sin\theta$, the Jacobian of Φ is the following determinant:

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r \cdot 1 = r$$

At the point $(r, \theta) = (4, \pi/6)$ we get:

$$\text{Jac}(\Phi)(4, \pi/6) = 4$$

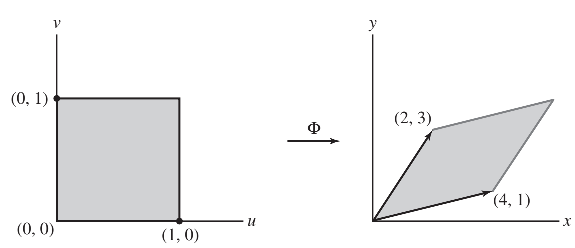
18. $\Phi(u, v) = (ue^v, e^u)$

SOLUTION We have $x = ue^v$ and $y = e^u$, hence the Jacobian of Φ is the following determinant:

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^v & ue^v \\ e^u & 0 \end{vmatrix} = e^v \cdot 0 - ue^v \cdot e^u = -ue^{u+v}$$

19. Find a linear mapping Φ that maps $[0, 1] \times [0, 1]$ to the parallelogram in the xy-plane spanned by the vectors $\langle 2, 3\rangle$ and $\langle 4, 1\rangle$.

SOLUTION



We denote the linear map by

$$\Phi(u, v) = (Au + Cv, Bu + Dv) \tag{1}$$

The image of the unit square $\mathcal{R} = [0, 1] \times [0, 1]$ under the linear map is the parallelogram whose vertices are the images of the vertices of $\mathcal{R}$. Two of vertices of the given parallelogram are $(2, 3)$ and $(4, 1)$. To find A, B, C, and D it suffices to determine four equations. Therefore, we ask that (notice that for linear maps $\Phi(0, 0) = (0, 0)$)

$$\Phi(0, 1) = (2, 3), \quad \Phi(1, 0) = (4, 1)$$

We substitute in (1) and solve for A, B, C, and D:

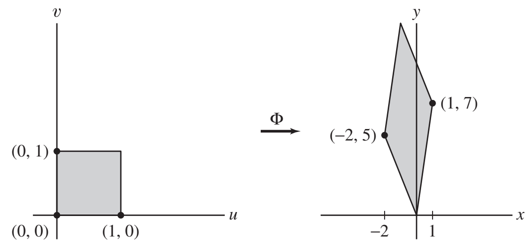
$$\begin{matrix} (A \cdot 0 + C \cdot 1, B \cdot 0 + D \cdot 1) = (C, D) = (2, 3) \\ (A \cdot 1 + C \cdot 0, B \cdot 1 + D \cdot 0) = (A, B) = (4, 1) \end{matrix} \quad \Rightarrow \quad \begin{matrix} C = 2, \quad D = 3 \\ A = 4, \quad B = 1 \end{matrix}$$

Substituting in (1) we obtain the following map:

$$\Phi(u, v) = (4u + 2v, u + 3v).$$

20. Find a linear mapping Φ that maps $[0,1] \times [0,1]$ to the parallelogram in the xy-plane spanned by the vectors $\langle -2, 5\rangle$ and $\langle 1, 7\rangle$.

SOLUTION



We denote the linear map by

$$\Phi(u, v) = (Au + Cv, Bu + Dv) \tag{1}$$

By properties of linear mapping, the images of the vertices of the unit square in the uv-plane are the vertices of the parallelogram in the xy-plane. Since we need four equations to determine A, B, C, and D, we ask that (notice that for any linear map, $\Phi(0,0) = (0,0)$)

$$\Phi(0,1) = (-2, 5), \quad \Phi(1,0) = (1,7)$$

We substitute in (1) and solve for A, B, C, and D:

$$\Phi(0,1) = (A\cdot 0 + C\cdot 1, B\cdot 0 + D\cdot 1) = (C, D) = (-2, 5) \quad \Rightarrow \quad C = -2, \quad D = 5$$
$$\Phi(1,0) = (A\cdot 1 + C\cdot 0, B\cdot 1 + D\cdot 0) = (A, B) = (1, 7) \quad \Rightarrow \quad A = 1, \quad B = 7$$

We substitute in (1) to obtain the following map:

$$\Phi(u, v) = (u - 2v, 7u + 5v)$$

21. Let $\mathcal{D}$ be the parallelogram in Figure 13. Apply the Change of Variables Formula to the map $\Phi(u, v) = (5u + 3v, u + 4v)$ to evaluate $\iint_{\mathcal{D}} xy\,dx\,dy$ as an integral over $\mathcal{D}_0 = [0,1] \times [0,1]$.

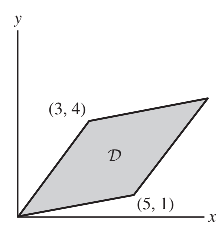
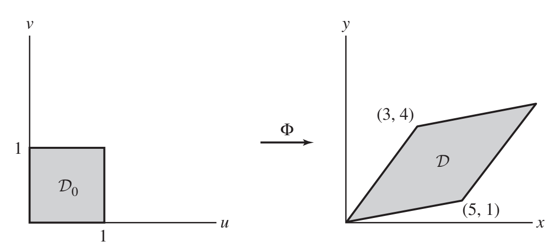


FIGURE 13

SOLUTION



We express $f(x, y) = xy$ in terms of u and v. Since $x = 5u + 3v$ and $y = u + 4v$, we have

$$f(x, y) = xy = (5u + 3v)(u + 4v) = 5u^2 + 12v^2 + 23uv$$

The Jacobian of the linear map $\Phi(u, v) = (5u + 3v, u + 4v)$ is

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix} = 20 - 3 = 17$$

Applying the Change of Variables Formula we get

$$\iint_{\mathcal{D}} xy\,dA = \iint_{\mathcal{D}_0} f(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1\int_0^1 (5u^2+12v^2+23uv)\cdot 17\,du\,dv$$

$$= 17\int_0^1 \frac{5u^3}{3}+12v^2u+\frac{23u^2v}{2}\bigg|_{u=0}^{1} dv = 17\int_0^1\left(\frac{5}{3}+12v^2+\frac{23v}{2}\right)dv$$

$$= 17\left(\frac{5v}{3}+4v^3+\frac{23v^2}{4}\bigg|_0^1\right) = 17\left(\frac{5}{3}+4+\frac{23}{4}\right) = \frac{2329}{12} \approx 194.08$$

22. Let $\Phi(u,v)=(u-uv,uv)$.

(a) Show that the image of the horizontal line $v=c$ is $y=\dfrac{c}{1-c}x$ if $c\neq 1$, and is the y-axis if $c=1$.

(b) Determine the images of vertical lines in the uv-plane.

(c) Compute the Jacobian of Φ.

(d) Observe that by the formula for the area of a triangle, the region $\mathcal{D}$ in Figure 14 has area $\frac{1}{2}(b^2-a^2)$. Compute this area again, using the Change of Variables Formula applied to Φ.

(e) Calculate $\iint_{\mathcal{D}} xy\,dx\,dy$.

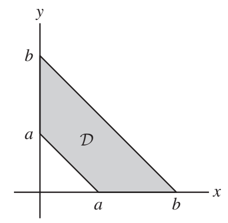


FIGURE 14

SOLUTION

(a) The image of the horizontal line $v=c$ in the (u,v)-plane is the set of the following points:

$$(x,y)=\Phi(u,c)=(u-uc,uc) \quad\Rightarrow\quad x=u-uc,\quad y=uc$$

Substituting $u=\frac{y}{c}$ (for $c\neq 0$) in $x=u-uc$ gives $x=\frac{y}{c}-\frac{y}{c}\cdot c$, $x=\frac{y}{c}-y$, or $y=\frac{cx}{1-c}$ (for $c\neq 1$). The image of $v=0$ (the u-axis) is

$$(x,y)=\Phi(u,0)=(u,0) \quad\Rightarrow\quad y=0$$

The image of $v=1$ is

$$(x,y)=\Phi(u,1)=(0,u) \quad\Rightarrow\quad x=0$$

We conclude that

The image of the line $v=c$ is the line $y=\frac{cx}{1-c}$ for $c\neq 1$, and the image of the line $v=1$ is the y-axis.
The image of the vertical line $u=c$ is the set of the following points:

$$(x,y)=\Phi(c,v)=(c-cv,cv) \quad\Rightarrow\quad x=c-cv,\quad y=cv$$

Therefore, for $c\neq 0$ we have $v=\frac{y}{c}$ and $x=c-c\cdot\frac{y}{c}=c-y$ or $y=c-x$.

The image of the vertical line $u=0$ (the v-axis) is

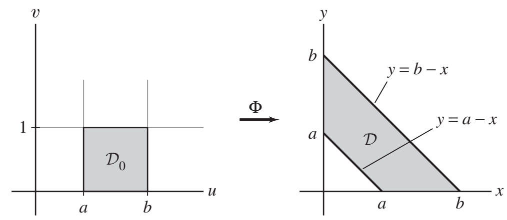
$$(x,y)=\Phi(0,v)=(0,0)$$

We conclude that the image of the line $u=c$ is the line $y=c-x$ if $c\neq 0$, and the image of the line $u=0$ is the point at the origin of the xy-plane.

(b) Since $x=u-uv$ and $y=uv$, the Jacobian of Φ is the following determinant:

$$\operatorname{Jac}(\Phi)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=(1-v)u+uv=u$$

(c)



The area of $\mathcal{D}$ is the following determinant:

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\, dA \tag{1}$$

We compute this integral using the Change of Variables Formula with the mapping $\Phi(u, v) = (u - uv, uv)$. By part (a) the lines $y = b - x$ and $y = a - x$ are the images of the lines $u = b$ and $u = a$, respectively, in the uv-plane. Also the lines $y = 0$ and $x = 0$ are the images of the lines $v = 0$ and $v = 1$, respectively. Therefore, $\mathcal{D}$ is the image of the rectangle $\mathcal{D}_0$ in the uv-plane shown in the figure. We compute the integral (1) using the Change of Variables Formula and the Jacobian computed in part (b). We get

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\, dA = \iint_{\mathcal{D}_0} 1|\text{Jac}\Phi|\, du\, dv = \int_a^b \int_0^1 u\, dv\, du = \left(\int_a^b u\, du\right)\left(\int_0^1 dv\right) = \left(\frac{u^2}{2}\Big|_a^b\right) = \frac{b^2 - a^2}{2}$$

(d) The function expressed in the new variables u and v is

$$f(x, y) = xy = (u - uv)uv = u^2 v - u^2 v^2$$

Using the Change of Variables Formula and the Jacobian obtained in part (b) we get

$$\iint_{\mathcal{D}} xy\, dA = \iint_{\mathcal{D}_0} (u^2 v - u^2 v^2) u\, du\, dv = \int_a^b \int_0^1 (u^3 v - u^3 v^2)\, dv\, du = \int_a^b \frac{u^3 v^2}{2} - \frac{u^3 v^3}{3}\Big|_{v=0}^1 du$$

$$= \int_a^b \left(\frac{u^3}{2} - \frac{u^3}{3}\right) du = \int_a^b \frac{u^3}{6}\, du = \frac{u^4}{24}\Big|_a^b = \frac{b^4 - a^4}{24}$$

23. Let $\Phi(u, v) = (3u + v, u - 2v)$. Use the Jacobian to determine the area of $\Phi(\mathcal{R})$ for:
(a) $\mathcal{R} = [0, 3] \times [0, 5]$ **(b)** $\mathcal{R} = [2, 5] \times [1, 7]$

SOLUTION The Jacobian of the linear map $\Phi(u, v) = (3u + v, u - 2v)$ is the following determinant:

$$\text{Jac}\Phi = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -6 - 1 = -7$$

By properties of linear maps, we have

$$\text{Area}\,(\Phi(\mathcal{R})) = |\text{Jac}\Phi|\text{Area}(\mathcal{R}) = 7 \cdot \text{Area}(\mathcal{R})$$

(a) The area of the rectangle $R = [0, 3] \times [0, 5]$ is $3 \cdot 5 = 15$, therefore the area of $\Phi(\mathcal{R})$ is

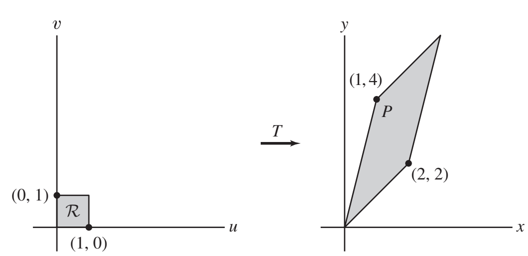
$$\text{Area}\,(\Phi(\mathcal{R})) = 7 \cdot 15 = 105$$

(b) The area of the rectangle $\mathcal{R} = [2, 5] \times [1, 7]$ is $3 \cdot 6 = 18$ hence the area of $\Phi(\mathcal{R})$ is

$$\text{Area}\,(\Phi(\mathcal{R})) = 7 \cdot 18 = 126.$$

24. Find a linear map T that maps $[0, 1] \times [0, 1]$ to the parallelogram $\mathcal{P}$ in the xy-plane with vertices $(0, 0)$, $(2, 2)$, $(1, 4)$, $(3, 6)$. Then calculate the double integral of e^{2x-y} over $\mathcal{P}$ via change of variables.

SOLUTION



By properties of linear maps, the vertices of the square $\mathcal{R} = [0, 1] \times [0, 1]$ are mapped to the vertices of the parallelogram $\mathcal{P}$. Denoting the linear map by $T(u, v) = (Au + Cv, Bu + Dv)$, we ask that

$$T(0, 1) = (1, 4), \quad T(1, 0) = (2, 2)$$

We substitute the points in $T(u, v)$ and solve for A, B, C, D to obtain

$$T(0, 1) = (A \cdot 0 + C \cdot 1, B \cdot 0 + D \cdot 1) = (C, D) = (1, 4) \quad \Rightarrow \quad C = 1, \quad D = 4$$
$$T(1, 0) = (A \cdot 1 + C \cdot 0, B \cdot 1 + D \cdot 0) = (A, B) = (2, 2) \quad \Rightarrow \quad A = 2, \quad B = 2$$

We obtain the following map:

$$T(u, v) = (2u + v, 2u + 4v)$$

We now compute the integral $\iint_{\mathcal{P}} e^{2x-y}\, dA$ using change of variables. We express $f(x, y) = e^{2x-y}$ in terms of the new variables u and v. Since $x = 2u + v$ and $y = 2u + 4v$, we obtain

$$f(x, y) = e^{2(2u+v)-(2u+4v)} = e^{2u-2v}$$

The Jacobian of linear map T is

$$\text{Jac}(T) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 2 \cdot 4 - 2 \cdot 1 = 6$$

Using the Change of Variables Formula, we get

$$\begin{aligned}
\iint_{\mathcal{P}} e^{2x-y}\, dA &= \iint_{\mathcal{R}} e^{2u-2v} |\text{Jac}(T)|\, du\, dv = \int_0^1 \int_0^1 e^{2u-2v} \cdot 6\, du\, dv = 6 \int_0^1 \int_0^1 e^{2u} \cdot e^{-2v}\, du\, dv \\
&= 6 \left(\int_0^1 e^{2u}\, du \right) \left(\int_0^1 e^{-2v}\, dv \right) = 6 \left(\frac{e^{2u}}{2} \bigg|_0^1 \right) \cdot \left(\frac{e^{-2v}}{-2} \bigg|_0^1 \right) \\
&= -\frac{3}{2}(e^2 - 1)(e^{-2} - 1) = \frac{3}{2}(e^2 - 1)(1 - e^{-2}) = \frac{3}{2}(e^2 + e^{-2} - 2)
\end{aligned}$$

25. With Φ as in Example 3, use the Change of Variables Formula to compute the area of the image of $[1, 4] \times [1, 4]$.

SOLUTION Let $\mathcal{R}$ represent the rectangle $[1, 4] \times [1, 4]$. We proceed as follows. $\text{Jac}(\Phi)$ is easily calculated as

$$\text{Jac}(T) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = 2u/v$$

Now, the area is given by the Change of Variables Formula as

$$\begin{aligned}
\iint_{\Phi(\mathcal{R})} 1\, dA &= \iint_{\mathcal{R}} 1 |\text{Jac}(\Phi)|\, du\, dv = \iint_{\mathcal{R}} 1 |2u/v|\, du\, dv = \int_1^4 \int_1^4 2u/v\, du\, dv \\
&= \int_1^4 2u\, du \cdot \int_1^4 \frac{1}{v}\, dv = (16 - 1)(\ln 4 - \ln 1) = 15 \ln 4
\end{aligned}$$

In Exercises 26–28, let $\mathcal{R}_0 = [0, 1] \times [0, 1]$ be the unit square. The translate of a map $\Phi_0(u, v) = (\phi(u, v), \psi(u, v))$ is a map

$$\Phi(u, v) = (a + \phi(u, v), b + \psi(u, v))$$

where a, b are constants. Observe that the map Φ_0 in Figure 15 maps $\mathcal{R}_0$ to the parallelogram $\mathcal{P}_0$ and that the translate

$$\Phi_1(u, v) = (2 + 4u + 2v, 1 + u + 3v)$$

maps $\mathcal{R}_0$ to $\mathcal{P}_1$.

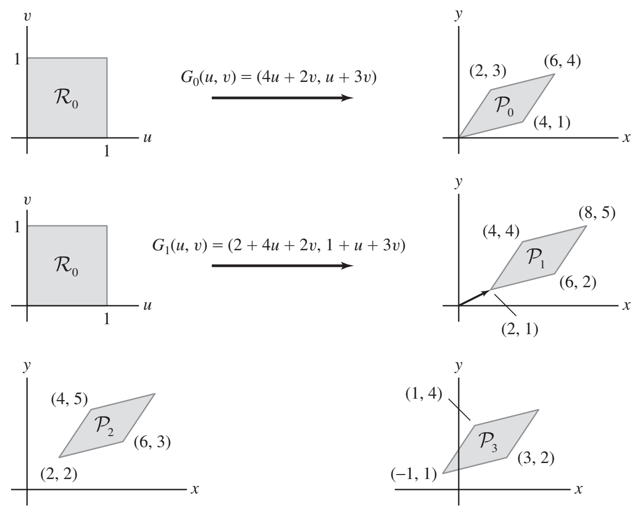


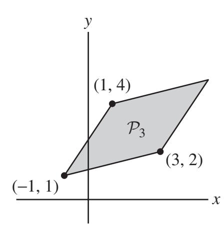
FIGURE 15

26. Find translates Φ_2 and Φ_3 of the mapping Φ_0 in Figure 15 that map the unit square $\mathcal{R}_0$ to the parallelograms $\mathcal{P}_2$ and $\mathcal{P}_3$.

SOLUTION The parallelogram $\mathcal{P}_2$ is obtained by translating $\mathcal{P}_0$ two units upward and two units to the left. Therefore the translate that maps the unit square $\mathcal{R}_0$ to $\mathcal{P}_2$ is

$$\Phi_2(u, v) = (2 + 4u + 2v, 2 + u + 3v)$$

The parallelogram $\mathcal{P}_3$ is obtained by translating $\mathcal{P}_0$ one unit upward and one unit to the left.

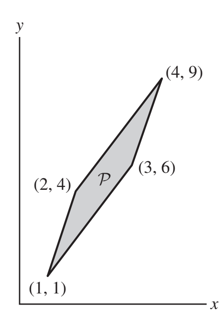


Therefore, the translate that maps $\mathcal{R}_0$ to $\mathcal{P}_2$ is

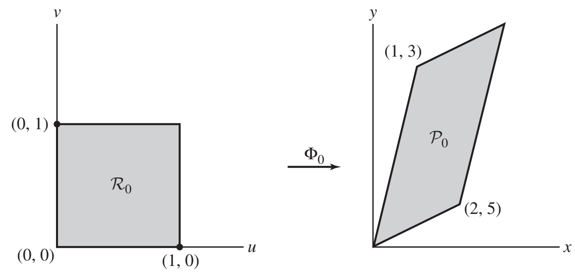
$$\Phi_3(u, v) = (-1 + 4u + 2v, 1 + u + 3v)$$

27. Sketch the parallelogram $\mathcal{P}$ with vertices (1, 1), (2, 4), (3, 6), (4, 9) and find the translate of a linear mapping that maps $\mathcal{R}_0$ to $\mathcal{P}$.

SOLUTION The parallelogram $\mathcal{P}$ is shown in the figure:



We first translate the parallelogram $\mathcal{P}$ one unit to the left and one unit downward to obtain a parallelogram $\mathcal{P}_0$ with a vertex at the origin.



We find a linear map $\Phi_0(u, v) = (Au + Cv, Bu + Dv)$ that maps $\mathcal{R}_0$ to $\mathcal{P}_0$:

$$\Phi_0(0, 1) = (1, 3) \quad \Rightarrow \quad (C, D) = (1, 3) \quad \Rightarrow \quad C = 1, \quad D = 3$$
$$\Phi_0(1, 0) = (2, 5) \quad \Rightarrow \quad (A, B) = (2, 5) \quad \Rightarrow \quad A = 2, \quad B = 5$$

Therefore,

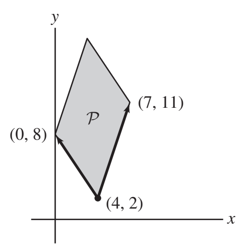
$$\Phi_0(u, v) = (2u + v, 5u + 3v)$$

Now we can determine the translate Φ of Φ_0 that maps $\mathcal{R}_0$ to $\mathcal{P}$. Since $\mathcal{P}$ is obtained by translating $\mathcal{P}_0$ one unit upward and one unit to the right, the map Φ is the following translate of Φ_0:

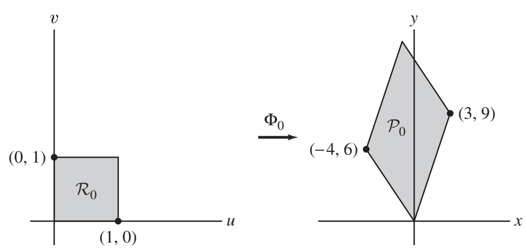
$$\Phi(u, v) = (1 + 2u + v, 1 + 5u + 3v)$$

28. Find the translate of a linear mapping that maps $\mathcal{R}_0$ to the parallelogram spanned by the vectors $\langle 3, 9\rangle$ and $\langle -4, 6\rangle$ based at $(4, 2)$.

SOLUTION The parallelogram $\mathcal{P}$ spanned by $\langle 3, 9\rangle$ and $\langle -4, 6\rangle$ based at $(4, 2)$ is shown in the figure:



We first translate $\mathcal{P}$ four units to the left and two units downward, so that the base $(4, 2)$ is moved to the origin. The translated parallelogram $\mathcal{P}_0$ is shown in the figure:



We find a linear map $\Phi_0(u, v) = (Au + Cv, Bu + Dv)$ that maps $\mathcal{R}_0$ to $\mathcal{P}_0$. We demand that

$$\Phi(1, 0) = (A, B) = (3, 9) \quad \Rightarrow \quad A = 3, \quad B = 9$$
$$\Phi(0, 1) = (C, D) = (-4, 6) \quad \Rightarrow \quad C = -4, \quad D = 6$$

Therefore,

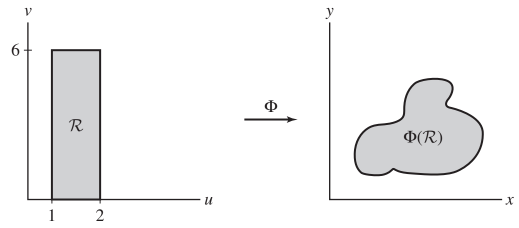
$$\Phi_0(u, v) = (3u - 4v, 9u + 6v)$$

Since $\mathcal{P}$ is obtained by translating $\mathcal{P}_0$ four units to the right and two units upward, the translate of Φ_0 that maps $\mathcal{R}_0$ to $\mathcal{P}$ is

$$\Phi(u, v) = (4 + 3u - 4v, 2 + 9u + 6v)$$

29. Let $\mathcal{D} = \Phi(\mathcal{R})$, where $\Phi(u, v) = (u^2, u + v)$ and $\mathcal{R} = [1, 2] \times [0, 6]$. Calculate $\iint_{\mathcal{D}} y\,dx\,dy$. *Note:* It is not necessary to describe $\mathcal{D}$.

SOLUTION



Changing variables, we have

$$\iint_{\mathcal{D}} y\,dA = \iint_{\mathcal{R}} (u + v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv \tag{1}$$

We compute the Jacobian of Φ. Since $x = u^2$ and $y = u + v$, we have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 1 & 1 \end{vmatrix} = 2u$$

We substitute in (1) and compute the resulting integral:

$$\iint_{\mathcal{D}} y\,dA = \int_0^6 \int_1^2 (u + v) \cdot 2u\,du\,dv = \int_0^6 \int_1^2 (2u^2 + 2uv)\,du\,dv = \int_0^6 \frac{2u^3}{3} + u^2 v \Big|_{u=1}^{2} dv$$
$$= \int_0^6 \left(\left(\frac{16}{3} + 4v \right) - \left(\frac{2}{3} + v \right) \right) dv = \int_0^6 \left(3v + \frac{14}{3} \right) dv = \frac{3}{2} v^2 + \frac{14}{3} v \Big|_0^6 = 82$$

30. Let $\mathcal{D}$ be the image of $\mathcal{R} = [1, 4] \times [1, 4]$ under the map $\Phi(u, v) = (u^2/v, v^2/u)$.

(a) Compute $\mathrm{Jac}(\Phi)$.

(b) Sketch $\mathcal{D}$.

(c) Use the Change of Variables Formula to compute $\mathrm{Area}(\mathcal{D})$ and $\iint_{\mathcal{D}} f(x, y)\,dx\,dy$, where $f(x, y) = x + y$.

SOLUTION

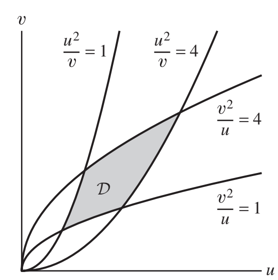
(a) Since $x = \frac{u^2}{v}$ and $y = \frac{v^2}{u}$, the Jacobian of Φ is the following determinant:

$$\mathrm{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = \frac{2u}{v} \cdot \frac{2v}{u} - \frac{u^2}{v^2} \cdot \frac{v^2}{u^2} = 3$$

(b) In the rectangle $\mathcal{R}$, we have $1 \le x \le 4$ and $1 \le y \le 4$. Therefore, the image $\mathcal{D}$ is defined by the following inequalities:

$$\mathcal{D}: 1 \le \frac{u^2}{v} \le 4 \quad \text{and} \quad 1 \le \frac{v^2}{u} \le 4$$

The domain $\mathcal{D}$ is shown in the figure:



(c) Since $x = \frac{u^2}{v}$ and $y = \frac{v^2}{u}$, the function in the new variables is

$$f(x,y) = \left(\frac{u^2}{v}\right)^2 + \left(\frac{v^2}{u}\right)^2 = \frac{u^4}{v^2} + \frac{v^4}{u^2}$$

Using change of variables, we get

$$\iint_{\mathcal{D}} f(x,y)\,dA = \iint_{\mathcal{R}} \left(\frac{u^4}{v^2} + \frac{v^4}{u^2}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^4\int_1^4 \left(\frac{u^4}{v^2} + \frac{v^4}{u^2}\right)\cdot 3\,du\,dv = 3\int_1^4 \left(\frac{u^5}{5v^2} - \frac{v^4}{u}\right)\Bigg|_{u=1}^4 dv$$

$$= 3\int_1^4 \left(\left(\frac{1024}{5v^2} - \frac{v^4}{4}\right) - \left(\frac{1}{5v^2} - v^4\right)\right) dv = 3\int_1^4 \left(\frac{1023}{5v^2} + \frac{3v^4}{4}\right) dv$$

$$= \frac{3069}{5}\left(-\frac{1}{v}\right) + \frac{9v^5}{20}\Bigg|_1^4 = \left(-\frac{3069}{20} + \frac{2304}{5}\right) - \left(-\frac{3069}{5} + \frac{9}{20}\right) = 920.7$$

31. Compute $\iint_{\mathcal{D}} (x+3y)\,dx\,dy$, where $\mathcal{D}$ is the shaded region in Figure 16. *Hint:* Use the map $\Phi(u,v) = (u-2v, v)$.

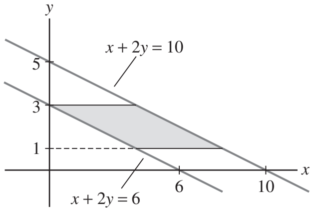
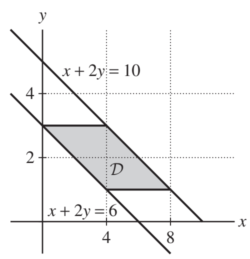


FIGURE 16

SOLUTION The boundary of $\mathcal{D}$ is defined by the lines $x+2y=6$, $x+2y=10$, $y=1$, and $y=3$.



Therefore, $\mathcal{D}$ is mapped to a rectangle $\mathcal{D}_0$ in the uv-plane under the map

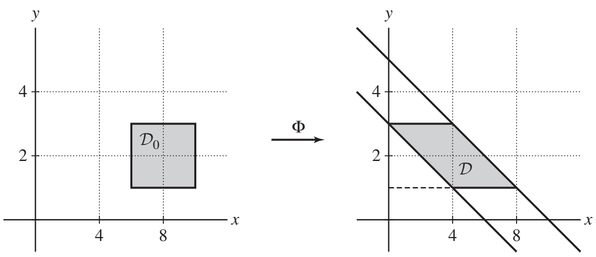
$$u = x + 2y, \quad v = y \tag{1}$$

or

$$(u,v) = \Phi^{-1}(x,y) = (x+2y, y)$$

Since $\mathcal{D}$ is defined by the inequalities $6 \le x+2y \le 10$ and $1 \le y \le 3$, the corresponding domain in the uv-plane is the rectangle

$$\mathcal{D}_0:\ 6 \le u \le 10,\ 1 \le v \le 3$$



To find $\Phi(u, v)$ we must solve the equations (1) for x and y in terms of u and v. We obtain

$$\begin{array}{l} u = x + 2y \\ v = y \end{array} \quad \Rightarrow \quad \begin{array}{l} x = u - 2v \\ y = v \end{array} \quad \Rightarrow \quad \Phi(u, v) = (u - 2v, v)$$

We compute the Jacobian of the linear mapping Φ:

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 + 2 \cdot 0 = 1$$

The function $f(x, y) = x + 3y$ expressed in terms of the new variables u and v is

$$f(x, y) = u - 2v + 3v = u + v$$

We now use the Change of Variables Formula to compute the required integral. We get

$$\begin{aligned} \iint_{\mathcal{D}} f(x, y)\, dx\, dy &= \iint_{\mathcal{D}_0} (u + v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv = \int_1^3 \int_6^{10} (u + v) \cdot 1\, du\, dv \\ &= \int_1^3 \frac{u^2}{2} + vu \Big|_{u=6}^{10} dv = \int_1^3 ((50 + 10v) - (18 + 6v))\ dv \\ &= \int_1^3 (32 + 4v)\, dv = 32v + 2v^2 \Big|_1^3 = (96 + 18) - (32 + 2) = 80 \end{aligned}$$

32. Use the map $\Phi(u, v) = \left(\frac{u}{v + 1}, \frac{uv}{v + 1} \right)$ to compute

$$\iint_{\mathcal{D}} (x + y)\, dx\, dy$$

where $\mathcal{D}$ is the shaded region in Figure 17.

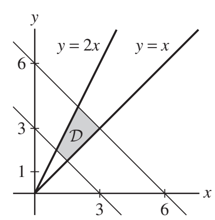
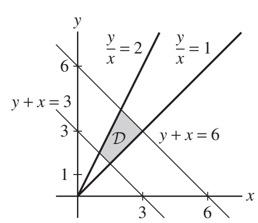


FIGURE 17

SOLUTION



We first identify the region $\mathcal{D}_0$ in the uv-plane, mapped to $\mathcal{D}$ under Φ. The equations of the lines defining the boundary of $\mathcal{D}$ can be rewritten as

$$\frac{y}{x} = 2, \quad \frac{y}{x} = 1, \quad y + x = 3, \quad y + x = 6.$$

Therefore, $\mathcal{D}$ is defined by the inequalities

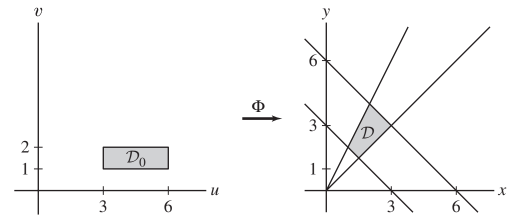
$$1 \le \frac{y}{x} \le 2, \quad 3 \le y + x \le 6$$

Since $x = \frac{u}{v+1}$ and $y = \frac{uv}{v+1}$, we have

$$\frac{y}{x} = \frac{\frac{uv}{v+1}}{\frac{u}{v+1}} = v \quad \text{and} \quad y + x = \frac{uv}{v + 1} + \frac{u}{v + 1} = \frac{u(v + 1)}{v + 1} = u$$

Therefore, the corresponding domain $\mathcal{D}_0$ in the uv-plane is the rectangle

$$\mathcal{D}_0 : 1 \le v \le 2,\ 3 \le u \le 6$$



We express the function $f(x, y) = x + y$ in terms of u and v:

$$f(x, y) = \frac{uv}{v+1} + \frac{u}{v+1} = \frac{uv+u}{v+1} = \frac{u(v+1)}{v+1} = u$$

We compute the Jacobian of Φ. Since $x = \frac{u}{v+1}$ and $y = \frac{uv}{v+1}$, we have

$$\frac{\partial x}{\partial u} = \frac{1}{v+1}, \quad \frac{\partial x}{\partial v} = -\frac{u}{(v+1)^2}, \quad \frac{\partial y}{\partial u} = \frac{v}{v+1},$$
$$\frac{\partial y}{\partial v} = \frac{u(v+1) - uv \cdot 1}{(v+1)^2} = \frac{u}{(v+1)^2}$$

Therefore,

$$\text{Jac}(\Phi) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u}{(v+1)^2} \end{vmatrix} = \frac{1}{v+1} \cdot \frac{u}{(v+1)^2} + \frac{u}{(v+1)^2} \cdot \frac{v}{v+1}$$
$$= \frac{u+uv}{(v+1)^3} = \frac{u(1+v)}{(v+1)^3} = \frac{u}{(1+v)^2}$$

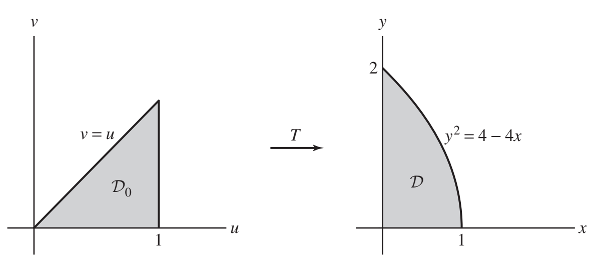
Now we can apply the Change of Variables Formula to compute the integral:

$$\iint_{\mathcal{D}} (x+y)\, dx\, dy = \iint_{\mathcal{D}_0} u \cdot \frac{u}{(1+v)^2}\, du\, dv = \int_1^2 \int_3^6 u^2 \cdot \frac{1}{(1+v)^2}\, du\, dv$$
$$= \left(\int_1^2 \frac{1}{(1+v)^2}\, dv \right) \left(\int_3^6 u^2\, du \right) = \left(-\frac{1}{1+v}\Big|_1^2 \right) \left(\frac{u^3}{3}\Big|_3^6 \right)$$
$$= \left(-\frac{1}{3} + \frac{1}{2} \right) \left(\frac{216}{3} - \frac{27}{3} \right) = \frac{63}{6} = 10.5$$

33. Show that $T(u, v) = (u^2 - v^2, 2uv)$ maps the triangle $\mathcal{D}_0 = \{(u, v) : 0 \le v \le u \le 1\}$ to the domain $\mathcal{D}$ bounded by $x = 0$, $y = 0$, and $y^2 = 4 - 4x$. Use T to evaluate

$$\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dx\, dy$$

SOLUTION We show that the boundary of $\mathcal{D}_0$ is mapped to the boundary of $\mathcal{D}$.



We have

$$x = u^2 - v^2 \quad \text{and} \quad y = 2uv$$

The line $v = u$ is mapped to the following set:

$$(x, y) = (u^2 - u^2, 2u^2) = (0, 2u^2) \quad \Rightarrow \quad x = 0, \quad y \ge 0$$

That is, the image of the line $u = v$ is the positive y-axis. The line $v = 0$ is mapped to the following set:

$$(x, y) = (u^2, 0) \quad \Rightarrow \quad x = u^2, \quad y = 0 \quad \Rightarrow \quad y = 0, \quad x \geq 0$$

Thus, the line $v = 0$ is mapped to the positive x-axis. We now show that the vertical line $u = 1$ is mapped to the curve $y^2 + 4x = 4$. The image of the line $u = 1$ is the following set:

$$(x, y) = (1 - v^2, 2v) \quad \Rightarrow \quad x = 1 - v^2, \quad y = 2v$$

We substitute $v = \frac{y}{2}$ in the equation $x = 1 - v^2$ to obtain

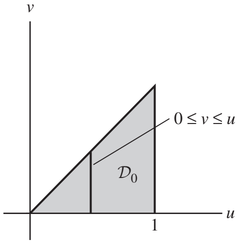
$$x = 1 - \left(\frac{y}{2}\right)^2 = 1 - \frac{y^2}{4} \quad \Rightarrow \quad 4x = 4 - y^2 \quad \Rightarrow \quad y^2 + 4x = 4$$

Since the boundary of $\mathcal{D}_0$ is mapped to the boundary of $\mathcal{D}$, we conclude that the domain $\mathcal{D}_0$ is mapped by T to the domain $\mathcal{D}$ in the xy-plane. We now compute the integral $\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dx\, dy$. We express the function $f(x, y) = \sqrt{x^2 + y^2}$ in terms of the new variables u and v:

$$\begin{aligned} f(x, y) &= \sqrt{(u^2 - v^2)^2 + (2uv)^2} = \sqrt{u^4 - 2u^2v^2 + v^4 + 4u^2v^2} \\ &= \sqrt{u^4 + 2u^2v^2 + v^4} = \sqrt{(u^2 + v^2)^2} = u^2 + v^2 \end{aligned}$$

We compute the Jacobian of T:

$$\operatorname{Jac}(T) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2 + v^2)$$



Using the Change of Variables Formula gives

$$\begin{aligned} \iint_{\mathcal{D}} \sqrt{x^2 + y^2}\, dx\, dy &= \iint_{\mathcal{D}_0} (u^2 + v^2) \cdot 4(u^2 + v^2)\, du\, dv = 4 \int_0^1 \int_0^u (u^4 + 2u^2v^2 + v^4)\, dv\, du \\ &= 4 \int_0^1 u^4 v + \frac{2}{3}u^2v^3 + \frac{v^5}{5} \bigg|_{v=0}^{u} du = 4 \int_0^1 \left(u^5 + \frac{2}{3}u^5 + \frac{u^5}{5}\right) du \\ &= 4 \int_0^1 \frac{28}{15} u^5\, du = \frac{112}{15} \cdot \frac{u^6}{6} \bigg|_0^1 = \frac{56}{45} \end{aligned}$$

34. Find a mapping Φ that maps the disk $u^2 + v^2 \leq 1$ onto the interior of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$. Then use the Change of Variables Formula to prove that the area of the ellipse is πab.

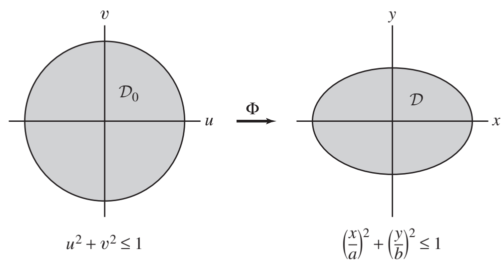
SOLUTION We define the following mapping:

$$x = au, \quad y = bv$$

or

$$(x, y) = \Phi(u, v) = (au, bv)$$

Then, $u^2 + v^2 \leq 1$ if and only if $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$. Hence Φ maps the disk $u^2 + v^2 \leq 1$ onto the interior of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$.



We find the Jacobian of Φ:

$$\text{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

We compute the area of $\mathcal{D}$ using the Change of Variables Formula (notice the area of the disk $\mathcal{D}_0$ is π):

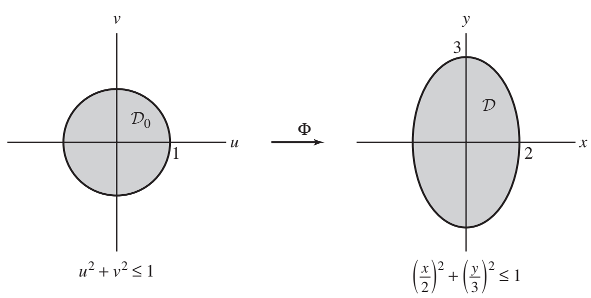
$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\,dx\,dy = \iint_{\mathcal{D}_0} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{\mathcal{D}_0} ab\,du\,dv$$
$$= ab\iint_{\mathcal{D}_0} 1\,du\,dv = ab\text{Area}\,(\mathcal{D}_0) = ab\pi$$

35. Calculate $\iint_{\mathcal{D}} e^{9x^2+4y^2}\,dx\,dy$, where $\mathcal{D}$ is the interior of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$.

SOLUTION We define a map that maps the unit disk $u^2+v^2 \le 1$ onto the interior of the ellipse. That is,

$$x = 2u, \quad y = 3v \quad \Rightarrow \quad \Phi(u,v) = (2u, 3v)$$

Since $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$ if and only if $u^2+v^2 \le 1$, Φ is the map we need.



We express the function $f(x,y) = e^{9x^2+4y^2}$ in terms of u and v:

$$f(x,y) = e^{9(2u)^2+4(3v)^2} = e^{36u^2+36v^2} = e^{36(u^2+v^2)}$$

We compute the Jacobian of Φ:

$$\text{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Using the Change of Variables Formula gives

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\,dA = \iint_{\mathcal{D}_0} e^{36(u^2+v^2)} \cdot 6\,du\,dv$$

We compute the integral using polar coordinates $u = r\cos\theta$, $v = r\sin\theta$:

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\,dA = \int_0^{2\pi}\int_0^1 6e^{36r^2}\cdot r\,dr\,d\theta = \left(6\int_0^{2\pi} d\theta\right)\left(\int_0^1 e^{36r^2} r\,dr\right)$$
$$= 12\pi \left.\frac{e^{36r^2}}{72}\right|_{r=0}^{1} = \frac{12\pi(e^{36}-1)}{72} = \frac{\pi(e^{36}-1)}{6}$$

36. Compute the area of the region enclosed by the ellipse $x^2 + 2xy + 2y^2 - 4y = 8$ as an integral in the variables $u = x + y$, $v = y - 2$.

SOLUTION We complete the square to rewrite the equation of the boundary of the region as follows:

$$x^2 + 2xy + 2y^2 - 4y = (x^2 + 2xy + y^2) + y^2 - 4y = (x + y)^2 + (y - 2)^2 - 4 = 8$$
$$\Rightarrow \quad (x + y)^2 + (y - 2)^2 = 12$$

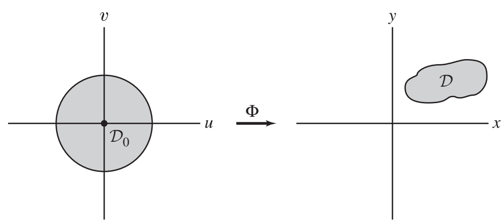
We use the following mapping:

$$u = x + y, \quad v = y - 2 \quad \Rightarrow \quad x = u - v - 2, \quad y = v + 2$$

or

$$\Phi(u, v) = (u - v - 2, v + 2)$$

Then $(x + y)^2 + (y - 2)^2 \le 12$ if and only if $u^2 + v^2 \le 12$, hence Φ maps the disc of radius $\sqrt{12}$ centered at $(0, 0)$ in the uv-plane to the given region $\mathcal{D}$.



The Jacobian of Φ is

$$\text{Jac}(\Phi) = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Using the Change of Variables Formula and the area of the disk, Area $(\mathcal{D}_0) = 12\pi$, we get

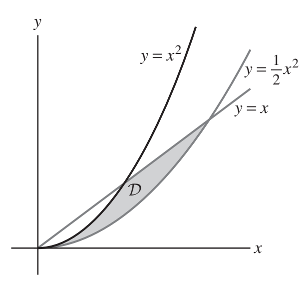
$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\, dx\, dy = \iint_{\mathcal{D}_0} 1 \cdot |\text{Jac}\Phi|\, du\, dv = \iint_{\mathcal{D}_0} du\, dv = \text{Area}\,(\mathcal{D}_0) = 12\pi$$

37. Sketch the domain $\mathcal{D}$ bounded by $y = x^2$, $y = \frac{1}{2}x^2$, and $y = x$. Use a change of variables with the map $x = uv$, $y = u^2$ to calculate

$$\iint_{\mathcal{D}} y^{-1}\, dx\, dy$$

This is an improper integral since $f(x, y) = y^{-1}$ is undefined at $(0, 0)$, but it becomes proper after changing variables.

SOLUTION The domain $\mathcal{D}$ is shown in the figure.



We must identify the domain $\mathcal{D}_0$ in the uv-plane. Notice that Φ is one-to-one, where $u \ge 0$ (or $u \le 0$), since in $\mathcal{D}$, $x \ge 0$, so it also follows by $x = uv$ that $v \ge 0$. Therefore, we search the domain $\mathcal{D}_0$ in the first quadrant of the uv-plane. To do this, we examine the curves that are mapped to the curves defining the boundary of $\mathcal{D}$. We examine each curve separately.

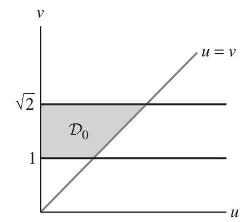
$y = x^2$: Since $x = uv$ and $y = u^2$ we get

$$u^2 = (uv)^2 \quad \Rightarrow \quad 1 = v^2 \quad \Rightarrow \quad v = 1$$

$y = \frac{1}{2}x^2$:

$$u^2 = \frac{1}{2}(uv)^2 \quad \Rightarrow \quad 1 = \frac{1}{2}v^2 \quad \Rightarrow \quad v^2 = 2 \quad \Rightarrow \quad v = \sqrt{2}$$

$y = x$: $u^2 = uv \quad \Rightarrow \quad v = u$. The region $\mathcal{D}_0$ is the region in the first quadrant of the uv-plane enclosed by the curves $v = 1$, $v = \sqrt{2}$, and $v = u$.



We now use change of variables to compute the integral $\iint_{\mathcal{D}} y^{-1}\,dx\,dy$. The function in terms of the new variables is $f(x, y) = u^{-2}$. We compute the Jacobian of $\Phi(u, v) = (x, y) = (uv, u^2)$:

$$\operatorname{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 2u & 0 \end{vmatrix} = -2u^2$$

Using the Change of Variables Formula gives

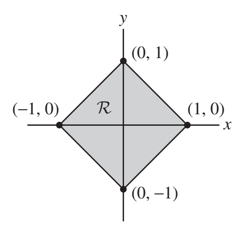
$$\iint_{\mathcal{D}} y^{-1}\,dx\,dy = \iint_{\mathcal{D}_0} u^{-2} \cdot 2u^2\,du\,dv = \int_1^{\sqrt{2}} \int_0^v 2\,du\,dv = \int_1^{\sqrt{2}} 2u\Big|_{u=0}^{v}\,dv = \int_1^{\sqrt{2}} 2v\,dv = v^2\Big|_1^{\sqrt{2}} = 2 - 1 = 1$$

38. Find an appropriate change of variables to evaluate

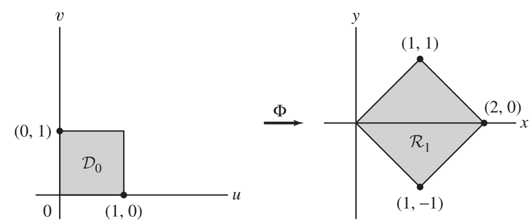
$$\iint_{\mathcal{R}} (x + y)^2 e^{x^2 - y^2}\,dx\,dy$$

where $\mathcal{R}$ is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

SOLUTION



We first find a linear map $\Phi(u, v)$ that maps the unit square $\mathcal{D}_0$ in the uv-plane to the square $\mathcal{R}_1$ in the xy-plane, obtained by translating $\mathcal{R}$ one unit to the right.



Let us use $u = x + y$ and $v = x - y$ then

$$x = \frac{u + v}{2}, \quad y = \frac{u - v}{2}$$

Therefore, $\Phi(u, v) = (x + y, x - y) = \left(\frac{u + v}{2}, \frac{u - v}{2}\right)$. This transformation maps the square $\mathcal{R}_1$ to the square defined by $u = \pm 1$ and $v = \pm 1$.

We use the change of variables $x = \frac{u + v}{2}$, $y = \frac{u - v}{2}$ to compute the integral $\iint_{\mathcal{R}} (x + y)^2 e^{x^2 - y^2}\,dx\,dy$. The function expressed in the new variables u and v is

$$f(u, v) = \left(\frac{u + v}{2} + \frac{u - v}{2}\right)^2 e^{1/4((u+v)^2 - (u-v)^2)} = u^2 e^{uv}$$

We compute the Jacobian of Φ:

$$\mathrm{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

Using the Change of Variables Formula gives

$$\begin{aligned} I &= \iint_{\mathcal{R}} (x+y)^2 e^{x^2-y^2}\,dx\,dy = \iint_{\mathcal{D}_0} u^2 e^{uv}\cdot\frac{1}{2}\,dv\,du \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2 e^{uv}\,dv\,du = \frac{1}{2}\int_{-1}^{1} u^2\cdot\frac{1}{u}e^{uv}\bigg|_{v=-1}^{1}\,du \\ &= \frac{1}{2}\int_{-1}^{1} ue^u - ue^{-u}\,du = \frac{1}{2}\int_{-1}^{1} ue^u\,du - \frac{1}{2}\int_{-1}^{1} ue^{-u}\,du \\ &= \frac{1}{2}\left(ue^u - e^u\bigg|_{-1}^{1}\right) - \frac{1}{2}\left(-ue^{-u} - e^{-u}\bigg|_{-1}^{1}\right) \\ &= \frac{1}{2}(e-e) - \frac{1}{2}(-e^{-1}-e^{-1}) - \left[\frac{1}{2}(-e^{-1}-e^{-1}) - \frac{1}{2}(e-e)\right] \\ &= \frac{1}{e} + \frac{1}{e} = \frac{2}{e} \end{aligned}$$

39. Let Φ be the inverse of the map $F(x, y) = (xy, x^2 y)$ from the xy-plane to the uv-plane. Let $\mathcal{D}$ be the domain in Figure 18. Show, by applying the Change of Variables Formula to the inverse $\Phi = F^{-1}$, that

$$\iint_{\mathcal{D}} e^{xy}\,dx\,dy = \int_{10}^{20}\int_{20}^{40} e^u v^{-1}\,dv\,du$$

and evaluate this result. *Hint:* See Example 8.

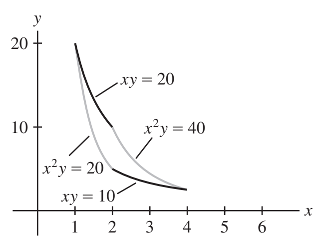
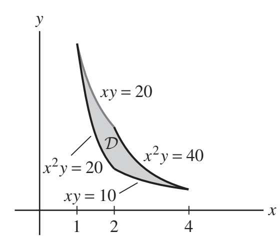


FIGURE 18

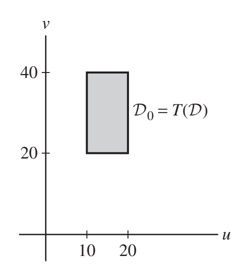
SOLUTION The domain $\mathcal{D}$ is defined by the inequalities

$$\mathcal{D}:\ 10 \le xy \le 20,\ 20 \le x^2 y \le 40$$



Since $u = xy$ and $v = x^2 y$, the image $\mathcal{D}_0$ of $\mathcal{D}$ (in the uv-plane) under F is the rectangle

$$\mathcal{D}_0:\ 10 \le u \le 20,\ 20 \le v \le 40$$



The function expressed in the new variables is

$$f(x, y) = e^{xy} = e^u$$

To find the Jacobian of the inverse Φ of F, we use the formula for the Jacobian of the inverse mapping. That is,

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}$$

We find the Jacobian of F. Since $u = xy$ and $v = x^2 y$, we have

$$\text{Jac}(F) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2xy & x^2 \end{vmatrix} = yx^2 - 2x^2 y = -x^2 y$$

Hence,

$$\text{Jac}(\Phi) = -\frac{1}{x^2 y}$$

We now compute the double integral $\iint_{\mathcal{D}} e^{xy}\, dA$ using the Change of Variables Formula. Since $y > 0$ in $\mathcal{D}$, we have $|\text{Jac}(\Phi)| = |-\frac{1}{x^2 y}| = \frac{1}{x^2 y} = v^{-1}$. Therefore,

$$\iint_{\mathcal{D}} e^{xy}\, dA = \iint_{\mathcal{D}_0} e^u v^{-1}\, dv\, du = \int_{10}^{20} \int_{20}^{40} e^u v^{-1}\, dv\, du = \left(\int_{10}^{20} e^u\, du\right)\left(\int_{20}^{40} v^{-1}\, dv\right)$$

$$= e^u \Big|_{10}^{20} \cdot \ln v \Big|_{20}^{40} = (e^{20} - e^{10})(\ln(40) - \ln(20)) = (e^{20} - e^{10}) \ln 2$$

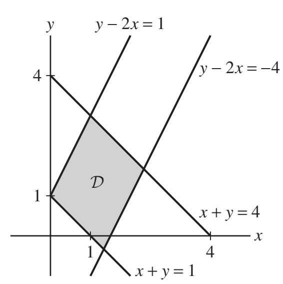
40. Sketch the domain

$$\mathcal{D} = \{(x, y) : 1 \le x + y \le 4,\ -4 \le y - 2x \le 1\}$$

(a) Let F be the map $u = x + y$, $v = y - 2x$ from the xy-plane to the uv-plane, and let Φ be its inverse. Use Eq. (14) to compute $\text{Jac}(\Phi)$.

(b) Compute $\iint_{\mathcal{D}} e^{x+y}\, dx\, dy$ using the Change of Variables Formula with the map Φ. *Hint:* It is not necessary to solve for Φ explicitly.

SOLUTION The domain $\mathcal{D} = \{(x, y) : 1 \le x + y \le 4, -4 \le y - 2x \le 1\}$ is shown in the figure.



(a) By Eq. (**??**), the Jacobian of the inverse map Φ is the reciprocal of the Jacobian of F. We compute the Jacobian of the linear mapping F:

$$\text{Jac}(F) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 1 + 2 = 3$$

Therefore,

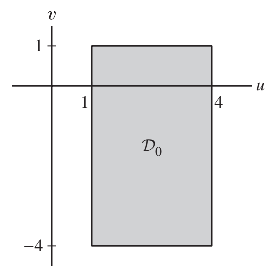
$$\text{Jac}(\Phi) = (\text{Jac}(F))^{-1} = \frac{1}{3}$$

(b) We first must identify the region of integration $\mathcal{D}_0$ in the uv-plane. The region $\mathcal{D}$ is defined by the inequalities

$$\mathcal{D}: \ 1 \le x + y \le 4, \ -4 \le y - 2x \le 1$$

Since $u = x + y$ and $v = y - 2x$, the image of $\mathcal{D}$ under F is the following rectangle:

$$\mathcal{D}_0: \ 1 \le u \le 4, \ -4 \le v \le 1$$



The function expressed in the new variables is $f(x, y) = e^{x+y} = e^u$. We now compute the integral $\iint_{\mathcal{D}} e^{x+y}\, dA$ using change of variables with the map Φ. We obtain

$$\iint_{\mathcal{D}} e^{x+y}\, dA = \iint_{\mathcal{D}_0} e^u |\mathrm{Jac}(\Phi)|\, du\, dv = \int_{-4}^{1}\int_{1}^{4} e^u \cdot \frac{1}{3}\, du\, dv = \left(\int_{-4}^{1} \frac{1}{3}\, dv\right)\left(\int_{1}^{4} e^u\, du\right)$$
$$= \left(\frac{1}{3}v\Big|_{-4}^{1}\right)\left(e^u\Big|_{1}^{4}\right) = \frac{5}{3}(e^4 - e)$$

41. Let $I = \iint_{\mathcal{D}} (x^2 - y^2)\, dx\, dy$, where

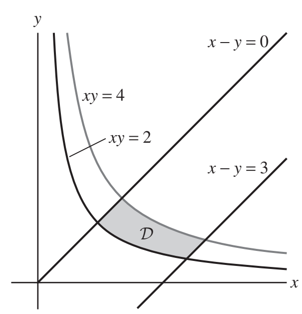
$$\mathcal{D} = \{(x, y) : 2 \le xy \le 4, 0 \le x - y \le 3, x \ge 0, y \ge 0\}$$

(a) Show that the mapping $u = xy$, $v = x - y$ maps $\mathcal{D}$ to the rectangle $\mathcal{R} = [2, 4] \times [0, 3]$.
(b) Compute $\partial(x, y)/\partial(u, v)$ by first computing $\partial(u, v)/\partial(x, y)$.
(c) Use the Change of Variables Formula to show that I is equal to the integral of $f(u, v) = v$ over $\mathcal{R}$ and evaluate.

SOLUTION

(a) The domain $\mathcal{D}$ is defined by the inequalities

$$\mathcal{D}: \ 2 \le xy \le 4, \ 0 \le x - y \le 3, \ x \ge 0, \ y \ge 0$$



Since $u = xy$ and $v = x - y$, the image of $\mathcal{D}$ under this mapping is the rectangle defined by

$$\mathcal{D}_0: \ 2 \le u \le 4, \ 0 \le v \le 3$$

That is, $\mathcal{D}_0 = [2, 4] \times [0, 3]$.

(b) We compute the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ and then use the formula for the Jacobian of the inverse mapping to compute $\frac{\partial(x,y)}{\partial(u,v)}$. Since $u = xy$ and $v = x - y$, we have

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix} = -y - x = -(x + y)$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{x+y}$$

(c) In $\mathcal{D}$, $x \ge 0$ and $y \ge 0$, hence $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{x+y}$. Using the change of variable formula gives:

$$I = \iint_{\mathcal{D}_0} (x^2 - y^2) \cdot \frac{1}{x+y}\, du\, dv = \iint_{\mathcal{D}_0} (x-y)\, du\, dv = \int_0^3 \int_2^4 v\, du\, dv$$
$$= \left(\int_0^3 v\, dv\right)\left(\int_2^4 du\right) = \left(\frac{v^2}{2}\Big|_0^3\right)\left(u\Big|_2^4\right) = \frac{9}{2} \cdot 2 = 9$$

42. Derive formula (5) in Section 15.4 for integration in cylindrical coordinates from the general Change of Variables Formula.

SOLUTION The cylindrical coordinates are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Suppose that a region $\mathcal{W}$ in the (x, y, z)-space is the image of a region $\mathcal{W}_0$ in the (θ, r, z)-space defined by

$$\mathcal{W}_0 : \theta_1 \le \theta \le \theta_2,\ \alpha(\theta) \le r \le \beta(\theta),\ z_1(r,\theta) \le z \le z_2(r,\theta)$$

Then by the change of variables formula, we have

$$\iiint_{\mathcal{W}} f(x,y,z)\, dV = \iiint_{\mathcal{W}_0} f(r\cos\theta, r\sin\theta, z) \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dz\, dr\, d\theta$$

We compute the Jacobian:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta \begin{vmatrix} r\cos\theta & 0 \\ 0 & 1 \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0$$
$$= \cos\theta(r\cos\theta - 0) + r\sin\theta(\sin\theta - 0) = r\cos^2\theta + r\sin^2\theta = r$$

Thus,

$$\iiint_{\mathcal{W}} f(x,y,z)\, dv = \int_{\theta_1}^{\theta_2} \int_{\alpha(\theta)}^{\beta(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) \cdot r\, dz\, dr\, d\theta$$

43. Derive formula (9) in Section 15.4 for integration in spherical coordinates from the general Change of Variables Formula.

SOLUTION The spherical coordinates are

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi$$

Suppose that a region $\mathcal{W}$ in the (x, y, z)-plane is the image of a region $\mathcal{W}_0$ in the (θ, ϕ, ρ)-space defined by:

$$\mathcal{W}_0 : \theta_1 \le \theta \le \theta_2,\ \phi_1 \le \phi \le \phi_2, \quad \rho_1(\theta,\phi) \le \rho \le \rho_2(\theta,\phi) \tag{1}$$

Then, by the Change of Variables Formula, we have

$$\iiint_{\mathcal{W}} f(x,y,z) = \iiint_{\mathcal{W}_0} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) = \left|\frac{\partial(x,y,z)}{\partial(\theta,\phi,\rho)}\right| d\rho\, d\phi\, d\theta \tag{2}$$

We compute the Jacobian:

$$\frac{\partial(x,y,z)}{\partial(\theta,\phi,\rho)} = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\rho} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\rho} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\rho} \end{vmatrix} = \begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi & \cos\theta\sin\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi & \sin\theta\sin\phi \\ 0 & -\rho\sin\phi & \cos\phi \end{vmatrix}$$

$$\begin{aligned}
&= -\rho\sin\theta\sin\phi \begin{vmatrix} \rho\sin\theta\cos\phi & \sin\theta\sin\phi \\ -\rho\sin\phi & \cos\phi \end{vmatrix} - \rho\cos\theta\cos\phi \begin{vmatrix} \rho\cos\theta\sin\phi & \sin\theta\sin\phi \\ 0 & \cos\phi \end{vmatrix} \\
&\quad + \cos\theta\sin\phi \begin{vmatrix} \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ 0 & -\rho\sin\phi \end{vmatrix} \\
&= -\rho\sin\theta\sin\phi(\rho\sin\theta\cos^2\phi + \rho\sin\theta\sin^2\phi) - \rho\cos\theta\cos\phi(\rho\cos\theta\cos\phi\sin\phi - 0) \\
&\quad + \cos\theta\sin\phi(-\rho^2\cos\theta\sin^2\phi - 0) \\
&= -\rho^2\sin^2\theta\sin\phi(\cos^2\phi + \sin^2\phi) - \rho^2\cos^2\theta\cos^2\phi\sin\phi - \rho^2\cos^2\theta\sin^3\phi \\
&= -\rho^2\sin^2\theta\sin\phi - \rho^2\cos^2\theta\cos^2\phi\sin\phi - \rho^2\cos^2\theta\sin^3\phi \\
&= -\rho^2\sin\phi(\sin^2\theta + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi) \\
&= -\rho^2\sin\phi\left(\sin^2\theta + \cos^2\theta(\cos^2\phi + \sin^2\phi)\right) \\
&= -\rho^2\sin\phi(\sin^2\theta + \cos^2\theta) = -\rho^2\sin\phi
\end{aligned}$$

Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$. Therefore,

$$\left|\frac{\partial(x,y,z)}{\partial(\theta,\phi,\rho)}\right| = \rho^2\sin\phi \tag{3}$$

Combining (1), (2), and (3) gives

$$\iiint_{\mathcal{W}} f(x,y,z)\,dv = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

44. Use the Change of Variables Formula in three variables to prove that the volume of the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ is equal to $abc \times$ the volume of the unit sphere.

SOLUTION We define a map $\Phi(u,v,w)$ that maps the interior of the unit ball $\mathcal{W}_0$ in the (u,v,w)-space onto the interior of the ellipsoid $\mathcal{W}$. We define

$$\begin{aligned} x &= au \\ y &= bv \quad \Rightarrow \quad \Phi(u,v,w) = (au, bv, cw) \\ z &= cw \end{aligned}$$

Then, $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1$ if and only if $u^2 + v^2 + w^2 \le 1$. Therefore, Φ is the map we need. Using the Change of Variables Formula gives

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\,dv = \iiint_{\mathcal{W}_0} 1\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \tag{1}$$

We compute the Jacobian:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} + 0 + 0 = abc$$

We substitute in (1) and compute the integral:

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}_0} abc\,du\,dv\,dw = abc\iiint_{\mathcal{W}_0} du\,dv\,dw = abc \cdot \frac{4}{3}\pi.$$

Further Insights and Challenges

45. Use the map

$$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u}$$

to evaluate the integral

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1 - x^2y^2}$$

This is an improper integral since the integrand is infinite if $x = \pm 1$, $y = \pm 1$, but applying the Change of Variables Formula shows that the result is finite.

SOLUTION We express the function $f(x, y) = \frac{1}{1-x^2y^2}$ in terms of the new variables u and v:

$$1 - x^2y^2 = 1 - \frac{\sin^2 u}{\cos^2 v}\frac{\sin^2 v}{\cos^2 u} = 1 - \left(\frac{\sin u}{\cos u}\right)^2 \cdot \left(\frac{\sin v}{\cos v}\right)^2 = 1 - \tan^2 u \tan^2 v$$

Hence,

$$f(x, y) = \frac{1}{1 - \tan^2 u \tan^2 v}$$

We compute the Jacobian of the mapping:

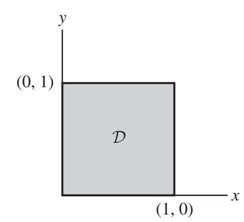
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v} \\ \frac{\sin v \sin u}{\cos^2 u} & \frac{\cos v}{\cos u} \end{vmatrix} = \frac{\cos u}{\cos v} \cdot \frac{\cos v}{\cos u} - \frac{\sin u \sin v}{\cos^2 v} \cdot \frac{\sin v \sin u}{\cos^2 u}$$

$$= 1 - \frac{\sin^2 u}{\cos^2 u} \cdot \frac{\sin^2 v}{\cos^2 v} = 1 - \tan^2 u \tan^2 v$$

Now, since $0 \le x \le 1$ and $0 \le y \le 1$, we have $0 \le \frac{\sin u}{\cos v} \cdot \frac{\sin v}{\cos u} \le 1$ or $0 \le \tan u \tan v \le 1$. Therefore, $0 \le \tan^2 u \, \tan^2 v \le 1$, hence

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = 1 - \tan^2 u \tan^2 v$$

We now identify a domain $\mathcal{D}_0$ in the uv-plane that is mapped by Φ onto $\mathcal{D}$ and Φ is one-to-one on $\mathcal{D}_0$.



We examine each segment on the boundary of $\mathcal{D}$ separately.

$y = 0$:

$$\frac{\sin v}{\cos u} = 0 \quad \Rightarrow \quad \sin v = 0 \quad \Rightarrow \quad v = \pi k$$

$x = 0$:

$$\frac{\sin u}{\cos v} = 0 \quad \Rightarrow \quad \sin u = 0 \quad \Rightarrow \quad u = \pi k$$

$y = 1$:

$$\frac{\sin v}{\cos u} = 1 \quad \Rightarrow \quad \sin v = \cos u \quad \Rightarrow \quad v + u = \frac{\pi}{2} + 2\pi k \quad \text{or} \quad v - u = \frac{\pi}{2} + 2\pi k \tag{1}$$

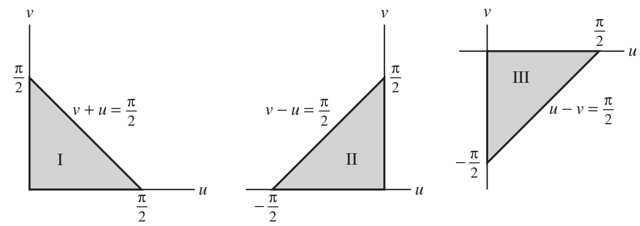
$x = 1$:

$$\frac{\sin u}{\cos v} = 1 \quad \Rightarrow \quad \sin u = \cos v \quad \Rightarrow \quad v + u = \frac{\pi}{2} + 2\pi k \quad \text{or} \quad u - v = \frac{\pi}{2} + 2\pi k \tag{2}$$

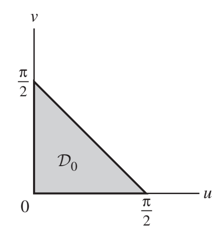
One of the possible regions $\mathcal{D}_0$ is obtained by choosing $k = 0$ in all solutions. We get

$$v = 0, \quad u = 0, \quad \left(v + u = \frac{\pi}{2} \text{ or } v - u = \frac{\pi}{2}\right), \quad \left(u + v = \frac{\pi}{2} \text{ or } u - v = \frac{\pi}{2}\right)$$

The corresponding regions are:



In II, $x = \frac{\sin u}{\cos v} < 0$ and in III $y = \frac{\sin v}{\cos u} < 0$, therefore these regions are not mapped to the unit square in the xy-plane. The appropriate region is I.



We now use the Change of Variables Formula and the result obtained previously to obtain the following integral:

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1 - x^2 y^2} = \iint_{\mathcal{D}_0} \frac{1}{1 - \tan^2 u \tan^2 v} \cdot (1 - \tan^2 u \tan^2 v)\,du\,dv$$
$$= \iint_{\mathcal{D}_0} 1\,du\,dv = \text{Area}(\mathcal{D}_0) = \frac{\frac{\pi}{2} \cdot \frac{\pi}{2}}{2} = \frac{\pi^2}{8}$$

46. Verify properties (1) and (2) for linear functions and show that any map satisfying these two properties is linear.

SOLUTION Let $\Phi(u, v) = (Au + Cv, Bu + Dv)$ be a linear function. We show that property (1) is satisfied:

$$\begin{aligned}\Phi(u + u', v + v') &= \big(A(u + u') + C(v + v'), B(u + u') + D(v + v')\big)\\ &= \big((Au + Cv) + (Au' + Cv'), (Bu + Dv) + (Bu' + Dv')\big)\\ &= (Au + Cv, Bu + Dv) + (Au' + Cv', Bu' + Dv')\\ &= \Phi(u, v) + \Phi(u', v')\end{aligned}$$

We now verify property (2):

$$\begin{aligned}\Phi(cu, cv) &= (A(cu) + C(cv), B(cu) + D(cv)) = (cAu + cCv, cBu + cDv)\\ &= c(Au + Cv, Bu + Dv) = c\Phi(u, v)\end{aligned}$$

We now suppose that $\Phi(u, v)$ is defined for all u and v and satisfies properties (1) and (2). We show that Φ is linear. We denote the following values:

$$\Phi(1, 0) = (A, B), \quad \Phi(0, 1) = (C, D) \tag{1}$$

We write

$$(u, v) = (u \cdot 1 + v \cdot 0, u \cdot 0 + v \cdot 1)$$

Using property (1) gives

$$\Phi(u, v) = \phi(u \cdot 1 + v \cdot 0, u \cdot 0 + v \cdot 1) = \phi(u \cdot 1, u \cdot 0) + \phi(v \cdot 0, v \cdot 1)$$

We now use property (2) and equation (3) to write

$$\Phi(u, v) = u\phi(1, 0) + v\phi(0, 1) = u(A, B) + v(C, D) = (Au, Bu) + (Cv, Dv) = (Au + Cv, Bu + Dv)$$

Hence Φ is linear.

47. Let P and Q be points in $\mathbf{R}^2$. Show that a linear map $\Phi(u, v) = (Au + Cv, Bu + Dv)$ maps the segment joining P and Q to the segment joining $\Phi(P)$ to $\Phi(Q)$. *Hint:* The segment joining P and Q has parametrization

$$(1-t)\overrightarrow{OP} + t\overrightarrow{OQ} \quad \text{for} \quad 0 \le t \le 1$$

SOLUTION First let $P(x_0, y_0)$ and $Q(x_1, y_1)$ so that we see if

$$\mathbf{r}(0) = \overrightarrow{OP} = (x_0, y_0), \quad \mathbf{r}(1) = \overrightarrow{OQ} = (x_1, y_1)$$

Then using the linear map we see:

$$\Phi(x_0, y_0) = (Ax_0 + Cy_0, Bx_0 + Dy_0) = \Phi(P)$$

and

$$\Phi(x_1, y_1) = (Ax_1 + Cy_1, Bx_1 + Dy_1) = \Phi(Q)$$

Hence this linear map take the endpoints P and Q to the new endpoints $\Phi(P)$ and $\Phi(Q)$. Now to determine the line segment mapping, consider the following:

$$\begin{aligned}
\Phi(\mathbf{r}(t)) &= \Phi((1-t)\overrightarrow{OP} + t\overrightarrow{OQ}) = \Phi((1-t)x_0 + tx_1, (1-t)y_0 + ty_1) \\
&= (A((1-t)x_0 + tx_1) + C((1-t)y_0 + ty_1), B((1-t)x_0 + tx_1 + D((1-t)y_0 + ty_1)) \\
&= ((1-t)Ax_0 + t(Ax_1) + (1-t)Cy_0 + t(Cy_1), (1-t)Bx_0 + t(Bx_1) + (1-t)Dy_0 + t(Dy_1)) \\
&= ((1-t)(Ax_0 + Cy_0) + t(Ax_1 + Cy_1), (1-t)(Bx_0 + Dy_0) + t(Bx_1 + Dy_1)) \\
&= (1-t)(Ax_0 + Cy_0, Bx_0 + Dy_0) + t(Ax_1 + Cy_1, Bx_1 + Dy_1) \\
&= (1-t)\Phi(P) + t\Phi(Q)
\end{aligned}$$

This is a parameterization for the line segment joining $\Phi(P)$ and $\Phi(Q)$. Therefore, the linear map maps the line segment joining P and Q to the line segment joining $\Phi(P)$ and $\Phi(Q)$.

48. Let Φ be a linear map. Prove Eq. (6) in the following steps.

(a) For any set $\mathcal{D}$ in the uv-plane and any vector $\mathbf{u}$, let $\mathcal{D} + \mathbf{u}$ be the set obtained by translating all points in $\mathcal{D}$ by $\mathbf{u}$. By linearity, Φ maps $\mathcal{D} + \mathbf{u}$ to the translate $\Phi(\mathcal{D}) + \Phi(\mathbf{u})$ [Figure 19(C)]. Therefore, if Eq. (6) holds for $\mathcal{D}$, it also holds for $\mathcal{D} + \mathbf{u}$.

(b) In the text, we verified Eq. (6) for the unit rectangle. Use linearity to show that Eq. (6) also holds for all rectangles with vertex at the origin and sides parallel to the axes. Then argue that it also holds for each triangular half of such a rectangle, as in Figure 19(A).

(c) Figure 19(B) shows that the area of a parallelogram is a difference of the areas of rectangles and triangles covered by steps (a) and (b). Use this to prove Eq. (6) for arbitrary parallelograms.

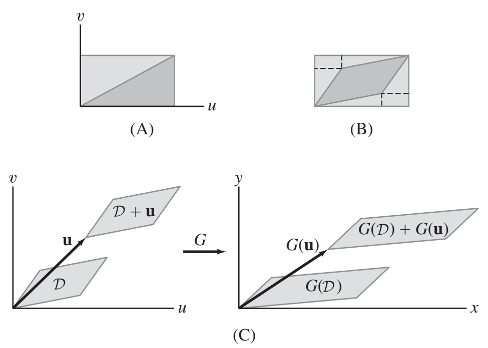


FIGURE 19

SOLUTION We must show that if Φ is a linear map, then

$$\text{Area}\,(\Phi(\mathcal{D})) = |\text{Jac}\,(\Phi)\,|\text{Area}(\mathcal{D}) \tag{1}$$

(a) For any vector $\mathbf{v} \in \mathcal{D}$, $\mathbf{v} + \mathbf{u}$ is in $\mathcal{D} + \mathbf{u}$. We show that $\Phi(\mathbf{v} + \mathbf{u})$ is in $\Phi(\mathcal{D}) + \Phi(\mathbf{u})$. By linearity, we have

$$\Phi(\mathbf{v} + \mathbf{u}) = \Phi(\mathbf{v}) + \Phi(\mathbf{u})$$

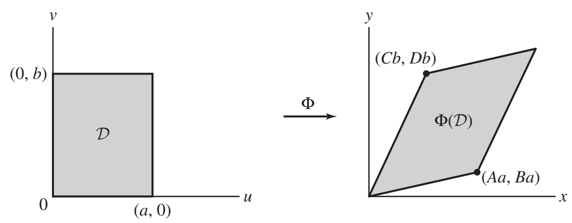
Since $\mathbf{v} \in \mathcal{D}$, $\Phi(\mathbf{v}) \in \Phi(\mathcal{D})$, hence $\Phi(\mathbf{v}) + \Phi(\mathbf{u}) \in \Phi(\mathcal{D}) + \Phi(\mathbf{u})$. Therefore, Φ maps $\mathcal{D} + \mathbf{u}$ to $\Phi(\mathcal{D}) + \Phi(\mathbf{u})$.

(b) Let $\mathcal{D}$ be the rectangle $\mathcal{D} = [0, a] \times [0, b]$ in the uv-plane, and $\Phi(\mathcal{D})$ be the image of $\mathcal{D}$ under the linear mapping Φ:

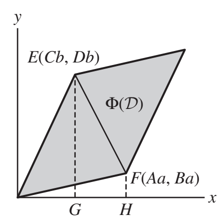
$$\Phi(u, v) = (Au + Cv, Bu + Dv)$$

Suppose that $a > 0$, $b > 0$, and $AD > BC$. For other cases, the proof is similar. To determine $\Phi(\mathcal{D})$, we compute the images of the vertices $(0, b)$ and $(a, 0)$ of $\mathcal{D}$:

$$\Phi(0, b) = (A \cdot 0 + Cb, B \cdot 0 + Db) = (Cb, Db)$$
$$\Phi(a, 0) = (Aa + C \cdot 0, B \cdot a + D \cdot 0) = (Aa, Ba)$$



Therefore, $\Phi(\mathcal{D})$ is the parallelogram spanned by the vectors $\langle Cb, Db\rangle$ and $\langle Aa, Ba\rangle$.



The area of the parallelogram is

$$\begin{aligned} 2\left(S_{\text{tri}(OEG)} + S_{\text{tra}(GEFH)} - S_{\text{tri}(OFH)}\right) &= 2\left(\frac{EG \cdot OG}{2} + \frac{(EG + FH)GH}{2} - \frac{FH \cdot OH}{2}\right) \\ &= 2\left(\frac{Db \cdot Cb}{2} + \frac{(Db + Ba)(Aa - Cb)}{2} - \frac{Ba \cdot Aa}{2}\right) \\ &= DCb^2 + ADab - DCb^2 + ABa^2 - BCab - ABa^2 \\ &= ab(AD - BC) \end{aligned}$$

That is,

$$\text{Area}\,(\Phi(\mathcal{D})) = (AD - BC)ab$$

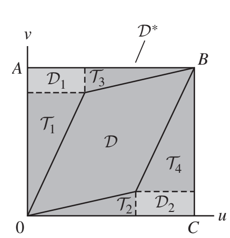
Since $|\text{Jac}(\Phi)| = AD - BC$ and $\text{Area}(\mathcal{D}) = ab$, we get

$$\text{Area}\,(\Phi(\mathcal{D})) = |\text{Jac}\,(\Phi)\,|\text{Area}(\mathcal{D}).$$

Now, let $\mathcal{D}_1$ be a triangular half of the parallelogram $\mathcal{D}$. Then, $\Phi(\mathcal{D}_1)$ is a triangular half of the parallelogram $\Phi(\mathcal{D})$. Using the result above, we have

$$\text{Area}\Phi(\mathcal{D}_1) = \frac{1}{2}\text{Area}\Phi(\mathcal{D}) = \frac{1}{2} \cdot |\text{Jac}(\Phi)|\text{Area}(\mathcal{D}) = |\text{Jac}(\Phi)|\frac{\text{Area}(\mathcal{D})}{2} = |\text{Jac}(\Phi)|\text{Area}(\mathcal{D}_1)$$

(c) By part (a), if we show that Eq. (4) holds for parallelograms with vertex at the origin, it holds for all other parallelograms (since each parallelogram can be translated to a parallelogram with a vertex at the origin). We consider a parallelogram with a vertex at the origin, and inscribe it in a rectangle $\mathcal{D}^*$; as shown in the figure:



We denote the triangles and rectangles as shown in the figure. By parts (a) and (b), it follows that Eq. (4) holds for each one of the rectangles and triangles. That is,

$$\begin{aligned}\text{Area}\,\Phi(\mathcal{D}^*) &= |\text{Jac}(\Phi)|\text{Area}(\mathcal{D}^*)\\ \text{Area}\,\Phi(\mathcal{D}_i) &= |\text{Jac}(\Phi)|\text{Area}(\mathcal{D}_i), \quad i = 1, 2\\ \text{Area}\,\Phi(T_i) &= |\text{Jac}(\Phi)|\text{Area}(T_i), \quad i = 1, 2, 3, 4 \end{aligned} \tag{2}$$

Since

$$\sum_{i=1}^{2} \text{Area}\,\Phi(\mathcal{D}_i) + \sum_{i=1}^{4} \text{Area}\,\Phi(T_i) + \text{Area}\,\Phi(\mathcal{D}) = \text{Area}\,\Phi(\mathcal{D}^*)$$

we have by (1),

$$|\text{Jac}(\Phi)| \left(\sum_{i=1}^{2} \text{Area}\,(\mathcal{D}_i) + \sum_{i=1}^{4} \text{Area}(T_i) \right) + \text{Area}\,\Phi(\mathcal{D}) = |\text{Jac}\,\Phi|\text{Area}(\mathcal{D}^*)$$

Translating sides gives

$$|\text{Jac}(\Phi)| \left(\text{Area}(\mathcal{D}^*) - \sum_{i=1}^{2} \text{Area}(\mathcal{D}_i) - \sum_{i=1}^{4} \text{Area}(T_i) \right) = \text{Area}\,\Phi(\mathcal{D})$$

But the difference in the brackets on the left-hand side is the area of the parallelogram $\mathcal{D}$. Therefore, we get

$$|\text{Jac}(\Phi)|\text{Area}(\mathcal{D}) = \text{Area}\,\Phi(\mathcal{D})$$

We thus showed that Eq. (4) holds for $\mathcal{D}$.

49. The product of 2×2 matrices A and B is the matrix AB defined by

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}}_{AB}$$

The (i, j)-entry of A is the **dot product** of the ith row of A and the jth column of B. Prove that $\det(AB) = \det(A)\det(B)$.

SOLUTION The determinants of A and B are

$$\begin{aligned}\det(A) &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,\\ \det(B) &= \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = a'd' - b'c' \end{aligned} \tag{1}$$

We now compute the determinant of AB:

$$\begin{aligned}\det(AB) = \begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix} &= (aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc')\\ &= aa'cb' + aa'dd' + bc'cb' + bc'dd' - ab'ca' - ab'dc' - bd'ca' - bd'dc'\\ &= (aa'dd' - bd'ca') + (bc'cb' - ab'dc') = a'd'(ad - bc) - b'c'(ad - bc)\\ &= (ad - bc)(a'd' - b'c') \end{aligned} \tag{2}$$

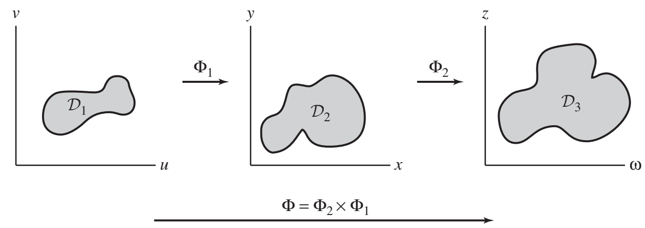
We combine (1) and (2) to conclude

$$\det(AB) = \det(A)\det(B).$$

50. Let $\Phi_1 : \mathcal{D}_1 \to \mathcal{D}_2$ and $\Phi_2 : \mathcal{D}_2 \to \mathcal{D}_3$ be C^1 maps, and let $\Phi_2 \circ \Phi_1 : \mathcal{D}_1 \to \mathcal{D}_3$ be the composite map. Use the Multivariable Chain Rule and Exercise 49 to show that

$$\text{Jac}(\Phi_2 \circ \Phi_1) = \text{Jac}(\Phi_2)\text{Jac}(\Phi_1)$$

SOLUTION



Let $\Phi = \Phi_2 \circ \Phi_1$. We have

$$\operatorname{Jac}(\Phi_1) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{1}$$

$$\operatorname{Jac}(\Phi_2) = \begin{vmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} \tag{2}$$

$$\operatorname{Jac}(\Phi) = \begin{vmatrix} \frac{\partial \omega}{\partial u} & \frac{\partial \omega}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \tag{3}$$

We use the multivariable Chain Rule to write

$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial \omega}{\partial v} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial v}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Substituting in (3) we obtain

$$\operatorname{Jac}(\Phi) = \begin{vmatrix} \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

We now use the definition of the product of two matrices, given in Exercise 49, equalities (1) and (2), and the equality proved in Exercise 49, to write

$$\operatorname{Jac}(\Phi) = \left| \begin{pmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \begin{vmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \operatorname{Jac}(\Phi_2)\operatorname{Jac}(\Phi_1)$$

That is,

$$\operatorname{Jac}(\Phi_2 \circ \Phi_1) = \operatorname{Jac}(\Phi_2)\operatorname{Jac}(\Phi_1)$$

51. Use Exercise 50 to prove that

$$\operatorname{Jac}(\Phi^{-1}) = \operatorname{Jac}(\Phi)^{-1}$$

Hint: Verify that $\operatorname{Jac}(I) = 1$, where I is the identity map $I(u, v) = (u, v)$.

SOLUTION Since $\Phi^{-1}(\Phi(u, v)) = (u, v)$, we have $(\Phi^{-1} \circ \Phi)(u, v) = (u, v)$. Therefore, $\Phi^{-1} \circ \Phi = I$. Using Exercise 50, we have

$$\operatorname{Jac}(I) = \operatorname{Jac}(\Phi^{-1} \circ \Phi) = \operatorname{Jac}(\Phi^{-1})\operatorname{Jac}(\Phi) \tag{1}$$

The Jacobian of the linear map $I(u, v) = (u, v) = (1 \cdot u + 0 \cdot v, 0 \cdot u + 1 \cdot v)$ is

$$\operatorname{Jac}(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$

Substituting in (1) gives

$$1 = \text{Jac}(\Phi^{-1})\text{Jac}(\Phi)$$

or

$$\text{Jac}(\Phi^{-1}) = (\text{Jac}(\Phi))^{-1}.$$

52. Let $(\overline{x}, \overline{y})$ be the centroid of a domain $\mathcal{D}$. For $\lambda > 0$, let $\lambda\mathcal{D}$ be the **dilate** of $\mathcal{D}$, defined by

$$\lambda\mathcal{D} = \{(\lambda x, \lambda y) : (x, y) \in \mathcal{D}\}$$

Use the Change of Variables Formula to prove that the centroid of $\lambda\mathcal{D}$ is $(\lambda\overline{x}, \lambda\overline{y})$.

SOLUTION The centroid of $\mathcal{D}$ has the following coordinates, where $S = \text{Area}(\mathcal{D})$:

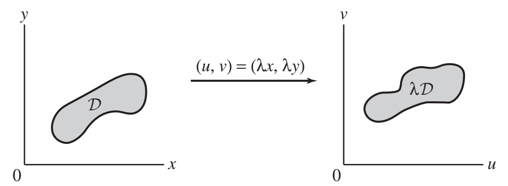
$$\overline{x} = \frac{1}{S}\iint_{\mathcal{D}} x\,dx\,dy, \quad \overline{y} = \frac{1}{S}\iint_{\mathcal{D}} y\,dx\,dy \tag{1}$$

The centroid of $\lambda\mathcal{D}$ is the following point:

$$\overline{u} = \frac{1}{\text{Area}(\lambda\mathcal{D})}\iint_{\lambda\mathcal{D}} u\,du\,dv, \quad \overline{v} = \frac{1}{\text{Area}(\lambda\mathcal{D})}\iint_{\lambda\mathcal{D}} v\,du\,dv \tag{2}$$

We compute the double integrals in (2) using change of variables with the following mapping:

$$u = \lambda x, \quad v = \lambda y$$



Therefore $(u, v) \in \lambda\mathcal{D}$ if and only if $(x, y) \in \mathcal{D}$, hence the image of $\lambda\mathcal{D}$ under this mapping is the domain $\mathcal{D}$ in the xy-plane. The Jacobian of the linear mapping $(u, v) = (\lambda x + 0y, 0x + \lambda y)$ is

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

We compute the integrals:

$$\iint_{\lambda\mathcal{D}} u\,du\,dv = \iint_{\mathcal{D}} \lambda x \cdot \lambda^2\,dx\,dy = \lambda^3\iint_{\mathcal{D}} x\,dx\,dy = \lambda^3 S\overline{x}$$
$$\iint_{\lambda\mathcal{D}} v\,du\,dv = \iint_{\mathcal{D}} \lambda y \cdot \lambda^2\,dx\,dy = \lambda^3\iint_{\mathcal{D}} y\,dx\,dy = \lambda^3 S\overline{y}$$

Substituting in (2) gives

$$\overline{u} = \frac{\lambda^3 S}{\text{Area}(\lambda\mathcal{D})}\overline{x}, \quad \overline{v} = \frac{\lambda^3 S}{\text{Area}(\lambda\mathcal{D})}\overline{y} \tag{3}$$

We now compute the area of $\lambda\mathcal{D}$ using the same mapping:

$$\text{Area}(\lambda\mathcal{D}) = \iint_{\lambda\mathcal{D}} 1\,du\,dv = \iint_{\mathcal{D}} \lambda^2\,dx\,dy = \lambda^2\iint_{\mathcal{D}} dx\,dy = \lambda^2\text{Area}(\mathcal{D}) = \lambda^2 S$$

Substituting in (3), we obtain the following centroid of $\lambda\mathcal{D}$:

$$\overline{u} = \frac{\lambda^3 S\overline{x}}{\lambda^2 S} = \lambda\overline{x}, \quad \overline{v} = \frac{\lambda^3 S\overline{y}}{\lambda^2 S} = \lambda\overline{y}$$

The centroid of $\lambda\mathcal{D}$ is $(\lambda\overline{x}, \lambda\overline{y})$.

CHAPTER REVIEW EXERCISES

1. Calculate the Riemann sum $S_{2,3}$ for $\int_1^4 \int_2^6 x^2 y \, dx \, dy$ using two choices of sample points:

(a) Lower-left vertex

(b) Midpoint of rectangle

Then calculate the exact value of the double integral.

SOLUTION

(a) The rectangle $[2, 6] \times [1, 3]$ is divided into 2×3 subrectangles. The lower-left vertices of the subrectangles are

$$\begin{array}{lll} P_{11} = (2, 1) & P_{21} = (2, 2) & P_{31} = (2, 3) \\ P_{12} = (3, 1) & P_{22} = (3, 2) & P_{32} = (3, 3) \end{array}$$

Also $\Delta x = \frac{6-2}{2} = 2$, $\Delta y = \frac{4-1}{3} = 1$, hence $\Delta A = 2 \cdot 1 = 2$. The Riemann sum $S_{3,4}$ is the following sum:

$$\begin{aligned} S_{2,3} &= 2\left(2^2 \cdot 1 + 2^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 1 + 3^2 \cdot 2 + 3^2 \cdot 3\right) \\ &= 2\,(4 + 8 + 12 + 9 + 18 + 27) = 156 \end{aligned}$$

(b) The midpoints of the subrectangles are

$$\begin{array}{lll} P_{11} = (3, 3/2) & P_{21} = (3, 5/2) & P_{31} = (3, 7/2) \\ P_{12} = (5, 3/2) & P_{22} = (5, 5/2) & P_{32} = (5, 7/2) \end{array}$$

Also $\Delta x = 2$, $\Delta y = 1$, hence $\Delta A = 2 \cdot 1 = 2$. The Riemann sum $S_{2,3}$ is

$$\begin{aligned} S_{2,3} &= 2\left(3^2 \cdot \frac{3}{2} + 3^2 \cdot \frac{5}{2} + 3^2 \cdot \frac{7}{2} + 5^2 \cdot \frac{3}{2} + 5^2 \cdot \frac{5}{2} + 5^2 \cdot \frac{7}{2}\right) \\ &= 2\left(\frac{27}{2} + \frac{45}{2} + \frac{63}{2} + \frac{75}{2} + \frac{125}{2} + \frac{175}{2}\right) \\ &= 510 \end{aligned}$$

We compute the exact value of the double integral, using an iterated integral of a product function. We get

$$\begin{aligned} \int_1^4 \int_2^6 x^2 y \, dx \, dy &= \left(\int_1^4 y \, dy\right)\left(\int_2^6 x^2 \, dx\right) = \left(\frac{y^2}{2}\Big|_1^4\right)\left(\frac{x^3}{3}\Big|_2^6\right) \\ &= \frac{16-1}{2} \cdot \frac{216-8}{3} = \frac{3120}{6} = 520 \end{aligned}$$

2. Let $S_{N,N}$ be the Riemann sum for $\int_0^1 \int_0^1 \cos(xy) \, dx \, dy$ using midpoints as sample points.

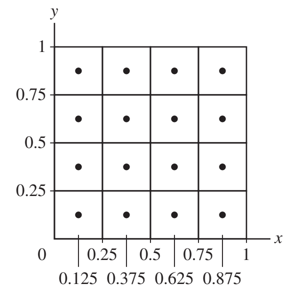
(a) Calculate $S_{4,4}$.

(b) CAS Use a computer algebra system to calculate $S_{N,N}$ for $N = 10, 50, 100$.

SOLUTION

(a) The midpoints of the 16 subrectangles are

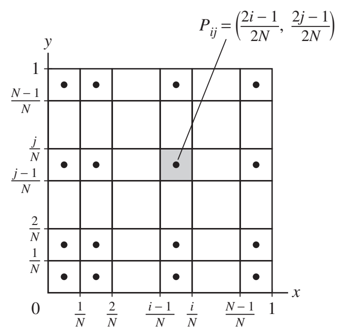
$$\begin{array}{ll} P_{11} = (0.125, 0.125) & P_{21} = (0.375, 0.125) \\ P_{31} = (0.625, 0.125) & P_{41} = (0.875, 0.125) \\ P_{12} = (0.125, 0.375) & P_{22} = (0.375, 0.375) \\ P_{32} = (0.625, 0.375) & P_{42} = (0.875, 0.375) \\ P_{13} = (0.125, 0.625) & P_{23} = (0.375, 0.625) \\ P_{33} = (0.625, 0.625) & P_{43} = (0.875, 0.625) \\ P_{14} = (0.125, 0.875) & P_{24} = (0.375, 0.875) \\ P_{34} = (0.625, 0.875) & P_{44} = (0.875, 0.875) \end{array}$$



Also $\Delta x = \Delta y = 0.25$, hence $\Delta A = 0.0625$. The Riemann Sum $S_{4,4}$ is

$$\begin{aligned} S_{4,4} = 0.0625 \cdot \Big(&\cos 0.125^2 + 2\cos(0.125 \cdot 0.375) + 2\cos(0.125 \cdot 0.625) + 2\cos(0.125 \cdot 0.875) \\ &+ \cos(0.375^2) + 2\cos(0.375 \cdot 0.625) + 2\cos(0.375 \cdot 0.875) + \cos(0.625^2) \\ &+ 2\cos(0.625 \cdot 0.875) + \cos(0.875^2)\Big) \\ = 0.947644& \end{aligned}$$

(b) The subrectangles on the rectangle $[0, 1] \times [0, 1]$ have sides of length $\Delta x = \Delta y = \frac{1}{N}$ and area $\Delta A = \frac{1}{N^2}$.



The midpoints are $P_{ij} = \left(\frac{2i-1}{2N}, \frac{2j-1}{2N}\right)$ for $i, j = 1, \ldots, N$. The corresponding Riemann sum is

$$S_{N,N} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \cos\left(\frac{2i-1}{2N} \cdot \frac{2j-1}{2N}\right)$$

We compute the sums for $N = 10$, 50, and 100 using a CAS:

$$S_{10,10} = 0.946334, \quad S_{50,50} = 0.946093, \quad S_{100,100} = 0.946086$$

3. Let $\mathcal{D}$ be the shaded domain in Figure 1.

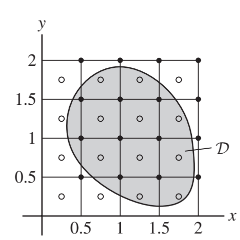
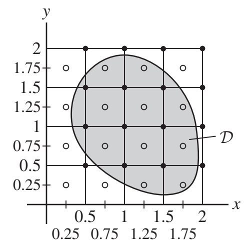


FIGURE 1

Estimate $\iint_{\mathcal{D}} xy\, dA$ by the Riemann sum whose sample points are the midpoints of the squares in the grid.

SOLUTION The subrectangles have sides of length $\Delta x = \Delta y = 0.5$ and area $\Delta A = 0.5^2 = 0.25$. Of sixteen sample points only ten lie in $\mathcal{D}$. The sample points that lie in $\mathcal{D}$ are

$$(0.75, 0.75),\ (0.75, 1.25),\ (0.75, 1.75),\ (1.25, 0.25),\ (1.25, 0.75),$$
$$(1.25, 1.25),\ (1.25, 1.75),\ (1.75, 0.25),\ (1.75, 0.75),\ (1.75, 1.25)$$



The Riemann sum S_{44} is thus

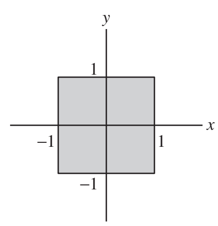
$$\begin{aligned}
S_{44} &= 0.25\,(f(0.75,0.75)+f(0.75,1.25)+f(0.75,1.75)+f(1.25,0.25)+f(1.25,0.75)\\
&\quad + f(1.25,1.25)+f(1.25,1.75)+f(1.75,0.25)+f(1.75,0.75)+f(1.75,1.25))\\
&= 0.25\left(0.75^2+0.75\cdot 1.25+0.75\cdot 1.75+1.25\cdot 0.25+1.25\cdot 0.75+1.25^2\right.\\
&\quad \left.+\,1.25\cdot 1.75+1.75\cdot 0.25+1.75\cdot 0.75+1.75\cdot 1.25\right)\\
&= 0.25\cdot 11.75 = 2.9375
\end{aligned}$$

4. Explain the following:

(a) $\int_{-1}^{1}\int_{-1}^{1}\sin(xy)\,dx\,dy = 0$ **(b)** $\int_{-1}^{1}\int_{-1}^{1}\cos(xy)\,dx\,dy > 0$

SOLUTION

(a) The double integral is the signed volume of the region between the graph of $f(x, y)$ and the xy-plane.



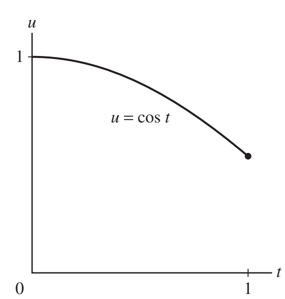
Region of integration

The function $f(x, y) = \sin(xy)$ satisfies $f(-x, y) = \sin(-xy) = -\sin(xy) = -f(x, y)$, hence the region left of the y-axis cancels with the region to the right of the y-axis. Therefore the double integral is zero.

(b) The function $f(x, y) = \cos(xy)$ satisfies $f(-x, y) = f(x, -y) = f(-x, -y) = f(x, y)$, hence

$$\int_{-1}^{1}\int_{-1}^{1}\cos(xy)\,dx = 4\int_{0}^{1}\int_{0}^{1}\cos(xy)\,dx.$$

Since $0 \le xy \le 1$ for the square $[0, 1] \times [0, 1]$, we have $\cos xy > 0$, and the double integral is positive.



$u = \cos t$

In Exercises 5–8, evaluate the iterated integral.

5. $\int_{0}^{2}\int_{3}^{5} y(x-y)\,dx\,dy$

SOLUTION First we evaluate the inner integral treating y as a constant:

$$\int_{3}^{5} y(x-y)\,dx = y\left(\frac{x^2}{2}-yx\right)\Big|_{x=3}^{5} = y\left(\left(\frac{25}{2}-5y\right)-\left(\frac{9}{2}-3y\right)\right) = y(8-2y) = 8y-2y^2$$

Now we integrate this result with respect to y:

$$\int_0^2 (8y - 2y^2)\,dy = 4y^2 - \frac{2}{3}y^3\Big|_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$$

Therefore,

$$\int_0^2 \int_3^5 y(x-y)\,dx\,dy = \frac{32}{3}.$$

6. $\displaystyle\int_{1/2}^0 \int_0^{\pi/6} e^{2y} \sin 3x\,dx\,dy$

SOLUTION We use an iterated integral of a product function to compute the double integral as the product of two single integrals. That is,

$$\int_{1/2}^0 \int_0^{\pi/6} e^{2y}\sin 3x\,dx\,dy = \left(\int_{1/2}^0 e^{2y}\,dy\right)\left(\int_0^{\pi/6}\sin 3x\,dx\right) = \left(\frac{1}{2}e^{2y}\Big|_{1/2}^0\right)\left(-\frac{1}{3}\cos 3x\Big|_0^{\pi/6}\right)$$

$$= \frac{1}{2}(1-e)\cdot\left(-\frac{1}{3}\right)\left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{1}{6}(1-e)$$

7. $\displaystyle\int_0^{\pi/3} \int_0^{\pi/6} \sin(x+y)\,dx\,dy$

SOLUTION We compute the inner integral treating y as a constant:

$$\int_0^{\pi/6}\sin(x+y)\,dx = -\cos(x+y)\Big|_{x=0}^{\pi/6} = -\cos\left(\frac{\pi}{6}+y\right) + \cos y = \cos y - \cos\left(y + \frac{\pi}{6}\right)$$

We now integrate the result with respect to y:

$$\int_0^{\pi/3}\int_0^{\pi/6}\sin(x+y)\,dx\,dy = \int_0^{\pi/3}\left(\cos y - \cos\left(y+\frac{\pi}{6}\right)\right)dy = \sin y - \sin\left(y + \frac{\pi}{6}\right)\Big|_0^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \left(\sin 0 - \sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - 1 + \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

8. $\displaystyle\int_1^2 \int_1^2 \frac{y\,dx\,dy}{x+y^2}$

SOLUTION We compute the inner integral with respect to x, then compute the outer integral of the result with respect to y. We obtain

$$\int_1^2 \frac{y}{x+y^2}\,dx = y\ln(x+y^2)\Big|_{x=1}^2 = y\ln(2+y^2) - y\ln(1+y^2)$$

$$\int_1^2\int_1^2 \frac{y}{x+y^2}\,dx\,dy = \int_1^2\left(y\ln(2+y^2) - y\ln(1+y^2)\right)dy = \int_1^2 y\ln(2+y^2)\,dy - \int_1^2 y\ln(1+y^2)\,dy$$

We compute the integrals using the substitutions $u = 2+y^2$, $du = 2y\,dy$, and $V = 1+y^2$, $dV = 2y\,dy$, respectively. We get

$$\int_1^2\int_1^2 \frac{y}{x+y^2}\,dx\,dy = \frac{1}{2}\int_3^6 \ln u\,du - \frac{1}{2}\int_2^5 \ln V\,dV = \frac{1}{2}u(\ln u - 1)\Big|_3^6 - \frac{1}{2}V(\ln V - 1)\Big|_2^5$$

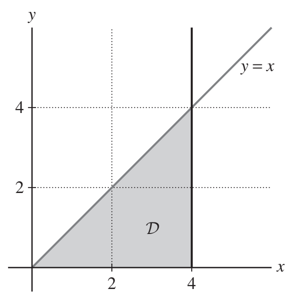
$$= \frac{1}{2}\cdot 6(\ln 6 - 1) - \frac{1}{2}\cdot 3(\ln 3 - 1) - \frac{1}{2}\cdot 5(\ln 5 - 1) + \frac{1}{2}\cdot 2(\ln 2 - 1)$$

$$= 3\ln 6 - \frac{3}{2}\ln 3 - \frac{5}{2}\ln 5 + \ln 2 \approx 0.396912$$

In Exercises 9–14, sketch the domain $\mathcal{D}$ and calculate $\displaystyle\iint_{\mathcal{D}} f(x,y)\,dA$.

9. $\mathcal{D} = \{0 \le x \le 4,\ 0 \le y \le x\}$, $\quad f(x,y) = \cos y$

SOLUTION The domain $\mathcal{D}$ is shown in the figure:



We compute the double integral, considering $\mathcal{D}$ as a vertically simple region. We describe $\mathcal{D}$ by the inequalities

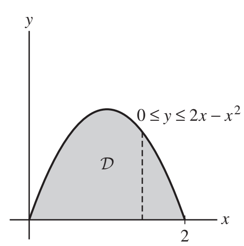
$$0 \le x \le 4, \quad 0 \le y \le x.$$

We now write the double integral as an iterated integral and compute:

$$\iint_{\mathcal{D}} \cos y \, dA = \int_0^4 \int_0^x \cos y \, dy \, dx = \int_0^4 \sin y \Big|_{y=0}^{x} dx$$
$$= \int_0^4 (\sin x - \sin 0) dx = \int_0^4 \sin x \, dx = -\cos x \Big|_0^4 = 1 - \cos 4$$

10. $\mathcal{D} = \{0 \le x \le 2, \; 0 \le y \le 2x - x^2\}, \quad f(x, y) = \sqrt{x}y$

SOLUTION The limits of the inner integral are $0 \le y \le 2x - x^2$, and the limits of outer integral are $0 \le x \le 2$.

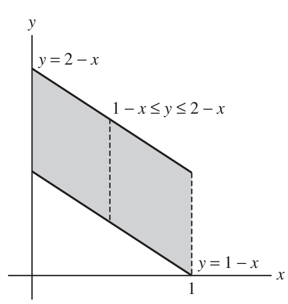


The region is vertically simple and the double integral is computed by the following iterated integral:

$$\iint_{\mathcal{D}} \sqrt{x} y \, dA = \int_0^2 \int_0^{2x-x^2} \sqrt{x} y \, dy \, dx = \int_0^2 \frac{\sqrt{x} y^2}{2} \Big|_{y=0}^{2x-x^2} dx = \int_0^2 \frac{\sqrt{x}(2x - x^2)^2}{2} \, dx$$
$$= \int_0^2 \left(2x^{5/2} - 2x^{7/2} + \frac{1}{2} x^{9/2} \right) dx = \frac{4}{7} x^{7/2} - \frac{4}{9} x^{9/2} + \frac{1}{11} x^{11/2} \Big|_0^2$$
$$= \frac{4}{7} \cdot 2^{7/2} - \frac{4}{9} \cdot 2^{9/2} + \frac{1}{11} \cdot 2^{11/2} = \frac{256\sqrt{2}}{693}$$

11. $\mathcal{D} = \{0 \le x \le 1, \; 1 - x \le y \le 2 - x\}, \quad f(x, y) = e^{x+2y}$

SOLUTION



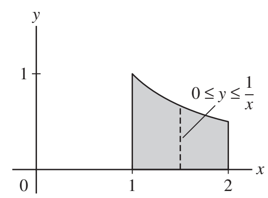
$\mathcal{D}$ is a vertically simple region, hence the double integral over $\mathcal{D}$ is the following iterated integral:

$$\iint_{\mathcal{D}} e^{x+2y} \, dA = \int_0^1 \int_{1-x}^{2-x} e^{x+2y} \, dy \, dx = \int_0^1 \frac{1}{2} e^{x+2y} \Big|_{y=1-x}^{2-x} dx = \int_0^1 \left(\frac{1}{2} e^{x+2(2-x)} - \frac{1}{2} e^{x+2(1-x)} \right) dx$$

$$= \int_0^1 \left(\frac{1}{2}e^{4-x} - \frac{1}{2}e^{2-x}\right) dx = -\frac{1}{2}e^{4-x} + \frac{1}{2}e^{2-x}\Big|_0^1 = -\frac{1}{2}e^3 + \frac{1}{2}e + \frac{1}{2}e^4 - \frac{1}{2}e^2$$
$$= \frac{1}{2}e(e^3 - e^2 - e + 1) = \frac{1}{2}e(e+1)(e-1)^2$$

12. $\mathcal{D} = \{1 \le x \le 2,\ 0 \le y \le 1/x\}, \quad f(x, y) = \cos(xy)$

SOLUTION

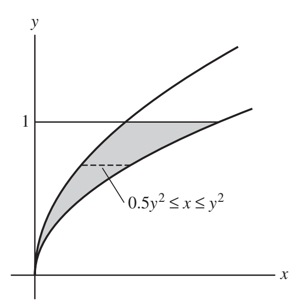


The region is vertically simple, hence the double integral is computed by the following iterated integral:

$$\iint_{\mathcal{D}} \cos(xy)\, dA = \int_1^2 \int_0^{1/x} \cos(xy)\, dy\, dx = \int_1^2 \frac{1}{x} \sin(xy)\Big|_{y=0}^{1/x} dx = \int_1^2 \frac{1}{x}\left(\sin\left(x \cdot \frac{1}{x}\right) - \sin 0\right) dx$$
$$= \int_1^2 \frac{1}{x} \sin 1\, dx = (\sin 1) \ln x\Big|_1^2 = (\sin 1)(\ln 2 - \ln 1) = (\sin 1) \ln 2$$

13. $\mathcal{D} = \{0 \le y \le 1,\ 0.5y^2 \le x \le y^2\}, \quad f(x, y) = ye^{1+x}$

SOLUTION



The region is horizontally simple, hence the double integral is equal to the following iterated integral:

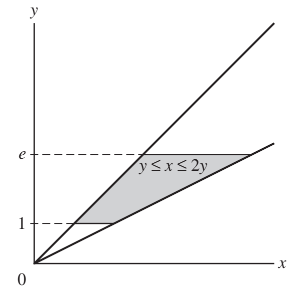
$$\iint_{\mathcal{D}} ye^{1+x}\, dA = \int_0^1 \int_{0.5y^2}^{y^2} ye^{1+x}\, dx\, dy = \int_0^1 ye^{1+x}\Big|_{x=0.5y^2}^{y^2} dy$$
$$= \int_0^1 y\left(e^{1+y^2} - e^{1+0.5y^2}\right) dy = \int_0^1 ye^{1+y^2}\, dy - \int_0^1 ye^{1+0.5y^2}\, dy$$

We compute the integrals using the substitutions $u = 1 + y^2$, $du = 2y\, dy$, and $v = 1 + 0.5y^2$, $dv = y\, dy$, respectively. We get

$$\iint_{\mathcal{D}} ye^{1+x}\, dA = \frac{1}{2}\int_1^2 e^u\, du - \int_1^{1.5} e^v dv = \frac{1}{2}e^u\Big|_1^2 - e^v\Big|_1^{1.5} = \frac{1}{2}(e^2 - e) - (e^{3/2} - e)$$
$$= \frac{1}{2}e^2 + \frac{1}{2}e - e^{3/2} = 0.5(e^2 - 2e^{1.5} + e)$$

14. $\mathcal{D} = \{1 \le y \le e,\ y \le x \le 2y\}, \quad f(x, y) = \ln(x + y)$

SOLUTION



The region is horizontally simple. We compute the double integral by the following iterated integral:

$$\begin{aligned}
\iint_{\mathcal{D}} \ln(x+y)\,dA &= \int_1^e \int_y^{2y} \ln(x+y)\,dx\,dy = \int_1^e (x+y)\left(\ln(x+y)-1\right)\Big|_{x=y}^{2y} dy \\
&= \int_1^e \left(3y(\ln 3y - 1) - 2y(\ln 2y - 1)\right)\,dy = \int_1^e (3y\ln 3y - 2y\ln 2y - y)\,dy \\
&= \int_1^e y\ln\frac{27y^3}{4y^2}\,dy - \int_1^e y\,dy = \int_1^e y\ln\frac{27y}{4}\,dy - \frac{y^2}{2}\Big|_1^e \\
&= \int_1^e y\ln\frac{27y}{4}\,dy - 0.5(e^2-1) \qquad (1)
\end{aligned}$$

We compute the integral using the substitution $u = \frac{27y}{4}$, $du = \frac{27}{4}\,dy$ and the integration formula:

$$\int u\ln u\,du = \frac{u^2}{2}\left(\ln u - \frac{1}{2}\right) + C$$

We get

$$\begin{aligned}
\int_1^e y\ln\frac{27y}{4}\,dy &= \int_{27/4}^{27e/4} \frac{4}{27}u\ln u \cdot \frac{4}{27}\,du = \left(\frac{4}{27}\right)^2 \int_{27/4}^{27e/4} u\ln u\,du = \left(\frac{4}{27}\right)^2 \cdot \frac{1}{2}u^2\left(\ln u - \frac{1}{2}\right)\Big|_{27/4}^{27e/4} \\
&= \left(\frac{4}{27}\right)^2 \cdot \frac{1}{2}\left(\left(\frac{27}{4}\right)^2 e^2\left(\ln\frac{27e}{4} - \frac{1}{2}\right) - \left(\frac{27}{4}\right)^2\left(\ln\frac{27}{4} - \frac{1}{2}\right)\right) \\
&= \frac{1}{2}\left(e^2\left(\ln\frac{27}{4} + \frac{1}{2}\right) - \left(\ln\frac{27}{4} - \frac{1}{2}\right)\right)
\end{aligned}$$

Substituting in (1) we obtain the following solution:

$$\begin{aligned}
\iint_{\mathcal{D}} \ln(x+y)\,dA &= 0.5\left(e^2\left(\ln\frac{27}{4} + \frac{1}{2}\right) - \left(\ln\frac{27}{4} - \frac{1}{2}\right) - (e^2-1)\right) \\
&= 0.5\left(\left(\ln\left(\frac{27}{4}\right) - 0.5\right)e^2 - \left(\ln\left(\frac{27}{4}\right) - 1.5\right)\right) \approx 5
\end{aligned}$$

15. Express $\int_{-3}^{3}\int_0^{9-x^2} f(x, y)\,dy\,dx$ as an iterated integral in the order $dx\,dy$.

SOLUTION The limits of integration correspond to the inequalities describing the domain $\mathcal{D}$:

$$-3 \le x \le 3, \quad 0 \le y \le 9 - x^2.$$

A quick sketch verifies that this is the region under the upper part of the parabola $y = 9 - x^2$, that is, the part that is above the x-axis. Therefore, the double integral can be rewritten as the following sum:

$$\int_{-3}^{3}\int_0^{9-x^2} f(x, y)\,dy\,dx = \int_0^9 \int_{-\sqrt{9-y}}^{\sqrt{9-y}} f(x, y)\,dx\,dy$$

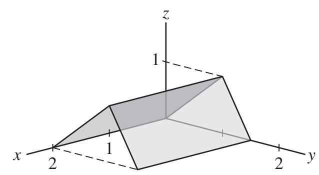
16. Let $\mathcal{W}$ be the region bounded by the planes $y = z$, $2y + z = 3$, and $z = 0$ for $0 \le x \le 4$.

(a) Express the triple integral $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ as an iterated integral in the order $dy\, dz\, dx$ (project $\mathcal{W}$ onto the yz-plane).

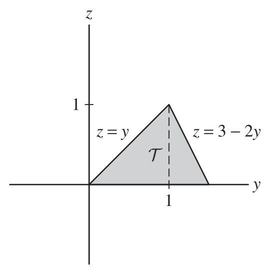
(b) Evaluate the triple integral for $f(x, y, z) = 1$.

(c) Compute the volume of $\mathcal{W}$ using geometry and check that the result coincides with the answer to (b).

SOLUTION The region $\mathcal{W}$ is the prism shown in the figure:



The projection of $\mathcal{W}$ onto the yz-plane is the triangle determined by the lines $z = y$, $2y + z = 3$ (or $z = 3 - 2y$), and the y-axis.



(a) First to express this triple integral as an iterated integral in the order $dy\, dz\, dx$:

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \int_{x=0}^{4} \int_{z=0}^{1} \int_{y=z}^{3/2-1/2z} f(x, y, z)\, dy\, dz\, dx$$

(b) Now evaluate this integral for $f(x, y, z) = 1$:

$$\begin{aligned}
\int_{x=0}^{4} \int_{z=0}^{1} \int_{y=z}^{3/2-1/2z} 1\, dy\, dz\, dx &= \int_{0}^{4} \int_{0}^{1} y \Big|_{y=z}^{3/2-1/2z} dz\, dx \\
&= \int_{0}^{4} \int_{0}^{1} \frac{3}{2} - \frac{1}{2}z - z\, dz\, dx \\
&= \int_{0}^{4} \int_{0}^{1} \frac{3}{2} - \frac{3}{2}z\, dz\, dx \\
&= \int_{0}^{4} \frac{3}{2}z - \frac{3}{4}z^2 \Big|_{0}^{1} dx \\
&= \int_{0}^{4} \left(\frac{3}{2} - \frac{3}{4}\right) dx \\
&= \frac{3}{4} \int_{0}^{4} dx = 3
\end{aligned}$$

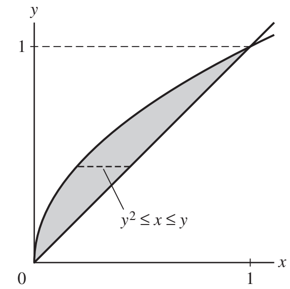
17. Let $\mathcal{D}$ be the domain between $y = x$ and $y = \sqrt{x}$. Calculate $\iint_{\mathcal{D}} xy\, dA$ as an iterated integral in the order $dx\, dy$ and $dy\, dx$.

SOLUTION In the order $dx\, dy$: The inequalities describing $\mathcal{D}$ as a horizontally simple region are obtained by first rewriting the equations of the curves with x as a function of y, that is, $x = y$ and $x = y^2$, respectively. The points of intersection are found solving the equation

$$y = y^2 \quad \Rightarrow \quad y(1 - y) = 0 \quad \Rightarrow \quad y = 0, \quad y = 1$$

We obtain the following inequalities for $\mathcal{D}$ (see figure):

$$\mathcal{D} : 0 \le y \le 1,\ y^2 \le x \le y$$



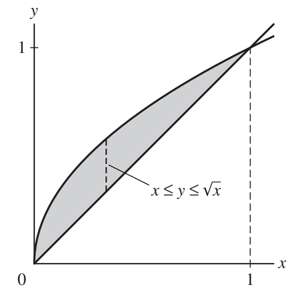
We now compute the double integral as the following iterated integral:

$$\iint_{\mathcal{D}} xy\,dA = \int_0^1 \int_{y^2}^{y} xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\bigg|_{x=y^2}^{x=y} dy = \int_0^1 \left(\frac{y\cdot y^2}{2} - \frac{y^4\cdot y}{2}\right) dy$$

$$= \int_0^1 \left(\frac{y^3}{2} - \frac{y^5}{2}\right) dy = \frac{y^4}{8} - \frac{y^6}{12}\bigg|_0^1 = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}$$

In the order $dy\,dx$: $\mathcal{D}$ is described as a vertically simple region by the following inequalities (see figure):

$$\mathcal{D}:\ 0 \le x \le 1,\ x \le y \le \sqrt{x}$$



The corresponding iterated integral is

$$\iint_{\mathcal{D}} xy\,dA = \int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\bigg|_{y=x}^{\sqrt{x}} dx = \int_0^1 \left(\frac{x\cdot x}{2} - \frac{x\cdot x^2}{2}\right) dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{2}\right) dx = \frac{x^3}{6} - \frac{x^4}{8}\bigg|_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

18. Find the double integral of $f(x, y) = x^3y$ over the region between the curves $y = x^2$ and $y = x(1-x)$.

SOLUTION The region $\mathcal{D}$ is a vertically simple region defined by the inequalities

$$0 \le x \le \frac{1}{2}, \quad x^2 \le y \le x(1-x)$$

We obtain the following integral:

$$\int_0^{1/2} \int_{x^2}^{x(1-x)} x^3y\,dy\,dx = \int_0^{1/2} \frac{x^3y^2}{2}\bigg|_{y=x^2}^{x(1-x)} dx = \int_0^{1/2} \left(\frac{x^3\cdot x^2(1-x)^2}{2} - \frac{x^3\cdot x^4}{2}\right) dx$$

$$= \int_0^{1/2} \left(\frac{x^5}{2} - x^6\right) dx = \frac{x^6}{12} - \frac{x^7}{7}\bigg|_0^{1/2} = \frac{\left(\frac{1}{2}\right)^6}{12} - \frac{\left(\frac{1}{2}\right)^7}{7} = \frac{1}{42}\cdot\left(\frac{1}{2}\right)^7 = \frac{1}{5376}$$

19. Change the order of integration and evaluate $\displaystyle\int_0^9 \int_0^{\sqrt{y}} \frac{x\,dx\,dy}{(x^2+y)^{1/2}}$.

SOLUTION The region here is described by the inequalities:

$$0 \le x \le \sqrt{y}, \quad 0 \le y \le 9$$

This region can also be described by writing these inequalities:

$$0 \le x \le 3, \quad x^2 \le y \le 9$$

Hence, changing the order of integration and evaluating we get:

$$\int_0^9 \int_0^{\sqrt{y}} \frac{x}{\sqrt{x^2+y}}\,dx\,dy = \int_0^3 \int_{x^2}^9 \frac{x}{\sqrt{x^2+y}}\,dy\,dx = \int_0^3 x\left(2\sqrt{x^2+y}\Big|_{x^2}^{9}\right)dx$$

$$= 2\int_0^3 x\sqrt{x^2+9} - x\sqrt{x^2+x^2}\,dx = 2\int_0^3 x\sqrt{x^2+9} - x^2\sqrt{2}\,dx$$

$$= 2\left(\frac{1}{3}(x^2+9)^{3/2} - \frac{\sqrt{2}}{3}x^3\Big|_0^3\right)$$

$$= \frac{2}{3}\cdot 18^{3/2} - \frac{2\sqrt{2}}{3}\cdot 27 - 2\cdot\frac{1}{3}\cdot 9^{3/2}$$

$$= 36\sqrt{2} - 18\sqrt{2} - 18 = 18\sqrt{2} - 18$$

20. Verify directly that

$$\int_2^3 \int_0^2 \frac{dy\,dx}{1+x-y} = \int_0^2 \int_2^3 \frac{dx\,dy}{1+x-y}$$

SOLUTION We compute the two iterated integrals:

$$I_1 = \int_2^3 \int_0^2 \frac{dy\,dx}{1+x-y} = \int_2^3 \left(\int_0^2 \frac{dy}{1+x-y}\right)dx = \int_2^3 -\ln(1+x-y)\Big|_{y=0}^{2} dx$$

$$= \int_2^3 (-\ln(1+x-2) + \ln(1+x-0))\,dx = \int_2^3 (\ln(1+x) - \ln(x-1))\,dx$$

$$= (1+x)\,(\ln(1+x)-1) - (x-1)\,(\ln(x-1)-1)\Big|_2^3$$

$$= 4(\ln 4 - 1) - 2(\ln 2 - 1) - (3(\ln 3 - 1) - (\ln 1 - 1)) = 6\ln 2 - 3\ln 3$$

$$I_2 = \int_0^2 \int_2^3 \frac{dx\,dy}{1+x-y} = \int_0^2 \left(\int_2^3 \frac{dx}{1+x-y}\right)dy = \int_0^2 \ln(1+x-y)\Big|_{x=2}^{3} dy$$

$$= \int_0^2 (\ln(1+3-y) - \ln(1+2-y))\,dy = \int_0^2 (\ln(4-y) - \ln(3-y))\,dy$$

$$= \int_{-2}^0 (\ln(4+u) - \ln(3+u))\,du = (4+u)\,(\ln(4+u)-1) - (3+u)\,(\ln(3+u)-1)\Big|_{u=-2}^{0}$$

$$= 4(\ln 4 - 1) - 3(\ln 3 - 1) - (2(\ln 2 - 1) - (\ln 1 - 1)) = 4\ln 4 - 3\ln 3 - 2\ln 2 = 6\ln 2 - 3\ln 3$$

The two integrals are equal.

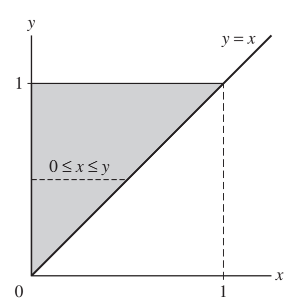
21. Prove the formula

$$\int_0^1 \int_0^y f(x)\,dx\,dy = \int_0^1 (1-x)f(x)\,dx$$

Then use it to calculate $\int_0^1 \int_0^y \frac{\sin x}{1-x}\,dx\,dy$.

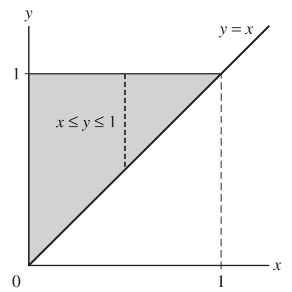
SOLUTION The region of integration of the double integral $\int_0^1 \int_0^y f(x)\,dx\,dy$ is described as horizontally simple by the inequalities

$$0 \le y \le 1, \quad 0 \le x \le y$$



The region can also be described as a vertically simple region, by the inequalities

$$0 \le x \le 1, \quad x \le y \le 1$$



Therefore,

$$\int_0^1 \int_0^y f(x)\,dx\,dy = \int_0^1 \int_x^1 f(x)\,dy\,dx = \int_0^1 f(x)y\Big|_{y=x}^{1} dx = \int_0^1 f(x)(1-x)\,dx$$

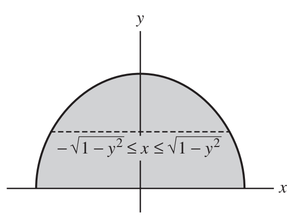
We use the formula with $f(x) = \frac{\sin x}{1-x}$. We get

$$\int_0^1 \int_0^y \frac{\sin x}{1-x}\,dx\,dy = \int_0^1 (1-x)\cdot\frac{\sin x}{1-x}\,dx = \int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = 1 - \cos 1$$

22. Rewrite $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{y\,dx\,dy}{(1+x^2+y^2)^2}$ by interchanging the order of integration, and evaluate.

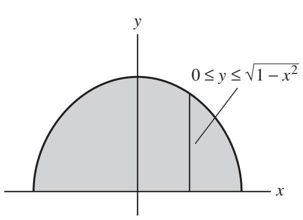
SOLUTION This integral gets simpler if we change the order of integration. We first identify the region $\mathcal{D}$ by the limits of integration. That is,

$$\mathcal{D}:\ 0 \le y \le 1,\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$



The semicircle $x^2 + y^2 = 1$, $y \ge 0$ can be rewritten as $y = \sqrt{1-x^2}$. The inequalities describing $\mathcal{D}$ as a vertically simple region are thus

$$\mathcal{D}:\ -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}$$



We obtain the following iterated integral:

$$\iint_{\mathcal{D}} \frac{y}{(1+x^2+y^2)^2}\,dx\,dy = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \frac{y}{(1+x^2+y^2)^2}\,dy\,dx \qquad (1)$$

We compute the inner integral with respect to y, using the substitution $u = 1 + x^2 + y^2$, $du = 2y\,dy$ (x is considered as a constant). This gives

$$\int_0^{\sqrt{1-x^2}} \frac{y\,dy}{(1+x^2+y^2)^2} = \int_{1+x^2}^{2} \frac{\frac{1}{2}\,du}{u^2} = -\frac{1}{2u}\Big|_{1+x^2}^{2} = -\frac{1}{4} + \frac{1}{2(1+x^2)}$$

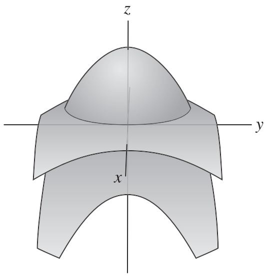
We compute the outer integral in (1):

$$\iint_{\mathcal{D}} \frac{y}{(1+x^2+y^2)}\,dx\,dy = \int_{-1}^{1}\left(-\frac{1}{4}+\frac{1}{2(1+x^2)}\right)dx = \int_0^1\left(-\frac{1}{2}+\frac{1}{1+x^2}\right)dx$$

$$= -\frac{x}{2}+\tan^{-1}x\Big|_0^1 = \left(-\frac{1}{2}+\tan^{-1}1\right)-0 = -\frac{1}{2}+\frac{\pi}{4} = \frac{\pi}{4}-\frac{1}{2}$$

23. Use cylindrical coordinates to compute the volume of the region defined by $4-x^2-y^2 \le z \le 10-4x^2-4y^2$.

SOLUTION



We first find the projection of $\mathcal{W}$ onto the xy-plane. The intersection curve of the upper and lower boundaries of $\mathcal{W}$ is obtained by solving

$$10-4x^2-4y^2 = 4-x^2-y^2$$
$$6 = 3(x^2+y^2) \quad\Rightarrow\quad x^2+y^2=2$$

Therefore, the projection of $\mathcal{W}$ onto the xy-plane is the circle $x^2+y^2 \le 2$. The upper surface is $z = 10-4(x^2+y^2)$ or $z = 10-4r^2$ and the lower surface is $z = 4-(x^2+y^2) = 4-r^2$. Therefore, the inequalities for $\mathcal{W}$ in cylindrical coordinates are

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{2}, \quad 4-r^2 \le z \le 10-4r^2$$

We use the volume as a triple integral and change of variables in cylindrical coordinates to write

$$V = \text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{4-r^2}^{10-4r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} rz\Big|_{z=4-r^2}^{10-4r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r\left(10-4r^2-\left(4-r^2\right)\right)dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}}\left(6r-3r^3\right)dr\,d\theta$$

$$= \int_0^{2\pi} 3r^2-\frac{3}{4}r^4\Big|_{r=0}^{\sqrt{2}} d\theta = \int_0^{2\pi}(6-3)\,d\theta = 6\pi$$

24. Evaluate $\iint_{\mathcal{D}} x\,dA$, where $\mathcal{D}$ is the shaded domain in Figure 2.

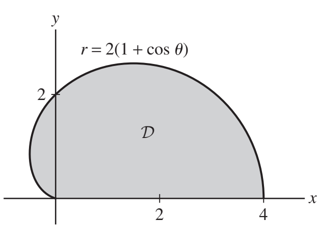
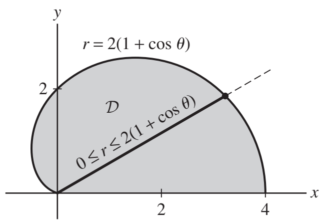


FIGURE 2

SOLUTION The domain $\mathcal{D}$ is defined by the inequalities

$$0 \le \theta \le \pi, \quad 0 \le r \le 2(1+\cos\theta).$$

Notice that the value of θ at the origin is found by solving $r = 2(1+\cos\theta) = 0$ for $0 \le \theta \le 2\pi$, that is, $\theta = \pi$.

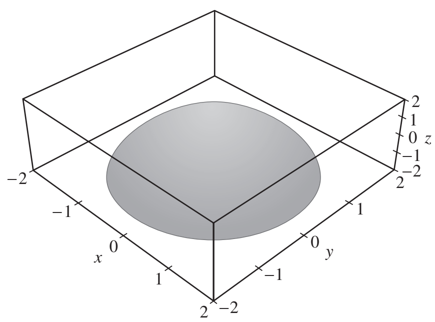


Using Double Integral in Polar Coordinates gives

$$\int_{\mathcal{D}} x\,dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} r\cos\theta\cdot r\,dr\,d\theta = \int_0^{\pi}\int_0^{2(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi} \frac{r^3\cos\theta}{3}\Big|_{r=0}^{2(1+\cos\theta)} d\theta = \int_0^{\pi}\frac{8(1+\cos\theta)^3\cos\theta}{3}\,d\theta$$
$$= \int_0^{\pi}\frac{8}{3}(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\,d\theta$$
$$= \frac{8}{3}\int_0^{\pi}\cos\theta\,d\theta + 8\int_0^{\pi}\cos^2\theta\,d\theta + 8\int_0^{\pi}\cos^3\theta\,d\theta + \frac{8}{3}\int_0^{\pi}\cos^4\theta\,d\theta = 5\pi$$

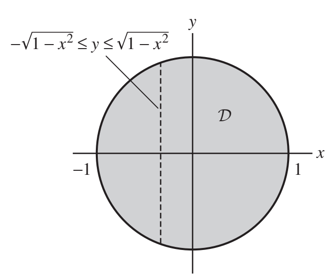
25. Find the volume of the region between the graph of the function $f(x, y) = 1 - (x^2 + y^2)$ and the xy-plane.

SOLUTION



The intersection of the surface $z = 1 - (x^2 + y^2)$ with the xy-plane is obtained by setting $z = 0$. That is, $1 - (x^2 + y^2) = 0$ or $x^2 + y^2 = 1$. Therefore, the projection of the solid onto the xy-plane is the disk $x^2 + y^2 \le 1$. We describe the disk as a vertically simple region:

$$\mathcal{D}:\ -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$



The volume V is the double integral of $z = 1 - (x^2 + y^2)$ over $\mathcal{D}$. That is,

$$V = \iint_{\mathcal{D}} \left(1 - (x^2 + y^2)\right) dA = \int_0^{2\pi}\int_0^1 (1 - r^2)\,r\,dr\,d\theta = 2\pi\int_0^1 (r - r^3)\,dr = \pi/2$$

26. Evaluate $\int_0^3\int_1^4\int_2^4 (x^3 + y^2 + z)\,dx\,dy\,dz$.

SOLUTION We evaluate to obtain

$$\int_0^3\int_1^4\int_2^4 (x^3 + y^2 + z)\,dx\,dy\,dz = \int_0^3\int_1^4 \left(\frac{x^4}{4} + y^2x + zx\right)\Big|_{x=2}^4 dy\,dz$$

$$= \int_0^3 \int_1^4 \left(\left(\frac{4^4}{4} + 4y^2 + 4z \right) - \left(\frac{16}{4} + 2y^2 + 2z \right) \right) dy\, dz$$

$$= \int_0^3 \int_1^4 \left(60 + 2y^2 + 2z\right) dy\, dz = \int_0^3 60y + \frac{2}{3}y^3 + 2yz \Big|_{y=1}^{4} dz$$

$$= \int_0^3 \left(240 + \frac{128}{3} + 8z\right) - \left(60 + \frac{2}{3} + 2z\right) dz$$

$$= \int_0^3 222 + 6z\, dz = 222z + 3z^2 \Big|_0^3 = 666 + 27 = 693$$

27. Calculate $\iiint_{\mathcal{B}} (xy + z)\, dV$, where

$$\mathcal{B} = \left\{0 \le x \le 2,\ 0 \le y \le 1,\ 1 \le z \le 3\right\}$$

as an iterated integral in two different ways.

SOLUTION The triple integral over the box may be evaluated in any order. For instance,

$$\iiint_{\mathcal{B}} (xy + z)\, dV = \int_0^2 \int_0^1 \int_1^3 (xy + z)\, dz\, dy\, dx = \int_0^1 \int_0^2 \int_1^3 (xy + z)\, dz\, dx\, dy$$
$$= \int_1^3 \int_0^2 \int_0^1 (xy + z)\, dy\, dx\, dz$$

We compute the integral in two of the possible orders:

$$\iiint_{\mathcal{B}} (xy + z)\, dV = \int_0^2 \int_0^1 \int_1^3 (xy + z)\, dz\, dy\, dx = \int_0^2 \int_0^1 xyz + \frac{z^2}{2} \Big|_{z=1}^{3} dy\, dx$$

$$= \int_0^2 \int_0^1 \left(\left(3xy + \frac{9}{2}\right) - \left(xy + \frac{1}{2}\right) \right) dy\, dx = \int_0^2 \int_0^1 (2xy + 4)\, dy\, dx$$

$$= \int_0^2 xy^2 + 4y \Big|_{y=0}^{1} dx = \int_0^2 (x + 4)\, dx = \frac{x^2}{2} + 4x \Big|_0^2 = \frac{4}{2} + 8 = 10$$

$$\iiint_{\mathcal{B}} (xy + z)\, dV = \int_0^1 \int_0^2 \int_1^3 (xy + z)\, dz\, dx\, dy = \int_0^1 \int_0^2 xyz + \frac{z^2}{2} \Big|_{z=1}^{3} dx\, dy$$

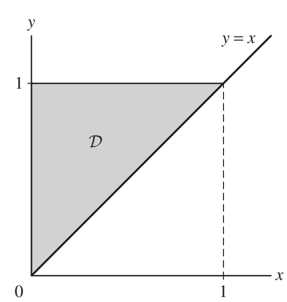
$$= \int_0^1 \int_0^2 \left(\left(3xy + \frac{9}{2}\right) - \left(xy + \frac{1}{2}\right) \right) dx\, dy = \int_0^1 \int_0^2 (2xy + 4)\, dx\, dy$$

$$= \int_0^1 x^2 y + 4x \Big|_{x=0}^{2} dy = \int_0^1 (4y + 8)\, dy = 2y^2 + 8y \Big|_0^1 = 2 + 8 = 10$$

28. Calculate $\iiint_{\mathcal{W}} xyz\, dV$, where

$$\mathcal{W} = \left\{0 \le x \le 1,\ x \le y \le 1,\ x \le z \le x + y\right\}$$

SOLUTION



$\mathcal{W}$ is the region between the two planes $z = x$ and $z = x + y$ lying over the triangle $\mathcal{D}$, defined by $0 \le x \le 1, x \le y \le 1$. Therefore, the triple integral is equal to the following iterated integral:

$$\iiint_{\mathcal{W}} xyz\,dV = \int_0^1 \int_x^1 \int_x^{x+y} xyz\,dz\,dy\,dx = \int_0^1 \int_x^1 \frac{xyz^2}{2}\bigg|_{z=x}^{x+y} dy\,dx = \int_0^1 \int_x^1 \left(\frac{xy}{2}\right)\left((x+y)^2 - x^2\right) dy\,dx$$

$$= \int_0^1 \int_x^1 \left(x^2y^2 + \frac{xy^3}{2}\right) dy\,dx = \int_0^1 \frac{x^2y^3}{3} + \frac{xy^4}{8}\bigg|_{y=x}^{1} dx$$

$$= \int_0^1 \left(\frac{x^2}{3} + \frac{x}{8} - \left(\frac{x^5}{3} + \frac{x^5}{8}\right)\right) dx = \int_0^1 \left(-\frac{11x^5}{24} + \frac{x^2}{3} + \frac{x}{8}\right) dx$$

$$= \frac{-11x^6}{144} + \frac{x^3}{9} + \frac{x^2}{16}\bigg|_0^1 = -\frac{11}{144} + \frac{1}{9} + \frac{1}{16} = \frac{7}{72}$$

29. Evaluate $I = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^1 (x + y + z)\,dz\,dy\,dx$.

SOLUTION We compute the triple integral:

$$I_1 = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^1 (x+y+z)\,dz\,dy\,dx = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x+y)z + \frac{z^2}{2}\bigg|_{y=0}^{1} dy\,dx$$

$$= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \left(x + y + \frac{1}{2}\right) dy\,dx = \int_{-1}^1 \left(x + \frac{1}{2}\right) y + \frac{y^2}{2}\bigg|_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^1 \left(x + \frac{1}{2}\right)\sqrt{1-x^2} + \frac{1-x^2}{2}\,dx = \int_{-1}^1 x\sqrt{1-x^2}\,dx + \int_{-1}^1 \frac{1}{2}\sqrt{1-x^2}\,dx + \int_{-1}^1 \frac{1-x^2}{2}\,dx \qquad (1)$$

The first integral is zero since the integrand is an odd function. Therefore, using Integration Formulas we get

$$I_1 = \int_0^1 \sqrt{1-x^2}\,dx + \int_0^1 (1-x^2)\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\bigg|_0^1 + \left(x - \frac{x^3}{3}\right)\bigg|_0^1$$

$$= \frac{1}{2}\sin^{-1} 1 + \frac{2}{3} = \frac{\pi}{4} + \frac{2}{3}$$

30. Describe a region whose volume is equal to:

(a) $\int_0^{2\pi} \int_0^{\pi/2} \int_4^9 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_{-2}^1 \int_{\pi/3}^{\pi/4} \int_0^2 r\,dr\,d\theta\,dz$

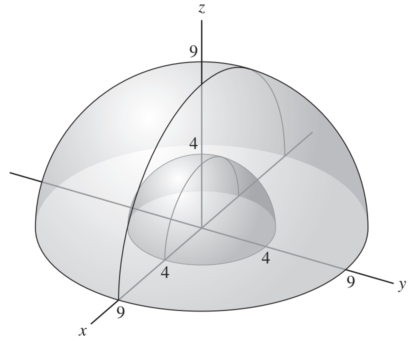
(c) $\int_0^{2\pi} \int_0^3 \int_{-\sqrt{9-r^2}}^0 r\,dz\,dr\,d\theta$

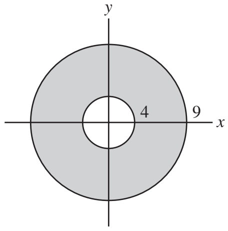
SOLUTION

(a) The limits of integration correspond to the inequalities in spherical coordinates, describing the region

$$4 \le \rho \le 9, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi$$

This is, the region between the upper hemispheres of radii 4 and 9.

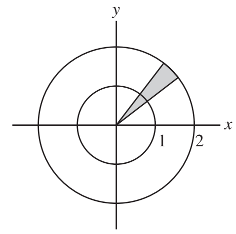




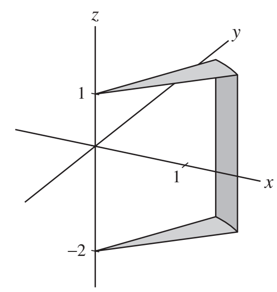
(b) The limits of integration correspond to the inequalities in cylindrical coordinates, describing the region $\mathcal{W}$:

$$0 \le r \le 2, \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{3}, \quad -2 \le z \le 1.$$

The projection of $\mathcal{W}$ onto the xy-plane is the sector $\mathcal{D}$ shown in the figure.



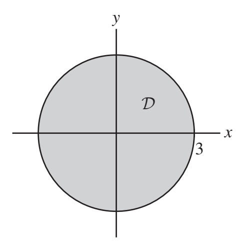
$\mathcal{W}$ is the region above and below $\mathcal{D}$, which is between the planes $z = -2$ and $z = 1$.



(c) The limits of integration correspond to the inequalities in cylindrical coordinates, describing the region $\mathcal{W}$,

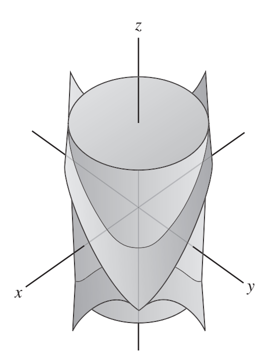
$$0 \le \theta \le 2\pi, \quad 0 \le r \le 3, \quad -\sqrt{9 - r^2} \le z \le 0.$$

The projection $\mathcal{D}$ of $\mathcal{W}$ onto the (x, y)-plane is the disk of radius 3. The lower surface is $z = -\sqrt{9 - r^2} = -\sqrt{9 - (x^2 + y^2)}$, which is the lower hemisphere $x^2 + y^2 + z^2 = 9$. Therefore, $\mathcal{W}$ is the lower half of the ball of radius 3 centered at the origin.



31. Find the volume of the solid contained in the cylinder $x^2 + y^2 = 1$ below the curve $z = (x + y)^2$ and above the curve $z = -(x - y)^2$.

SOLUTION



We rewrite the equations of the surfaces using cylindrical coordinates:

$$z = (x + y)^2 = x^2 + y^2 + 2xy = r^2 + 2(r\cos\theta)(r\sin\theta) = r^2(1 + \sin 2\theta)$$
$$z = -(x - y)^2 = -(x^2 + y^2 - 2xy) = -(r^2 - 2r^2\cos\theta\sin\theta) = -r^2(1 - \sin 2\theta)$$

The projection of the solid onto the xy-plane is the unit disk. Therefore, the solid is described by the following inequalities:

$$\mathcal{W}: \ 0 \le \theta \le 2\pi, \ 0 \le r \le 1, \ -r^2(1-\sin 2\theta) \le z \le r^2(1+\sin 2\theta)$$

Expressing the volume as a triple integral and converting the triple integral to cylindrical coordinates, we get

$$\begin{aligned}
V = \text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\,dv = \int_0^{2\pi}\int_0^1 \int_{-r^2(1-\sin 2\theta)}^{r^2(1+\sin 2\theta)} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 rz\Big|_{z=-r^2(1-\sin 2\theta)}^{r^2(1+\sin 2\theta)} dr\,d\theta = \int_0^{2\pi}\int_0^1 r\left(r^2(1+\sin 2\theta)+r^2(1-\sin 2\theta)\right) dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r^3\cdot 2\,dr\,d\theta = \left(\int_0^{2\pi} 2d\theta\right)\left(\int_0^1 r^3\,dr\right) = 4\pi\cdot\frac{r^4}{4}\Big|_0^1 = \pi
\end{aligned}$$

32. Use polar coordinates to evaluate $\iint_{\mathcal{D}} x\,dA$, where $\mathcal{D}$ is the shaded region between the two circles of radius 1 in Figure 3.

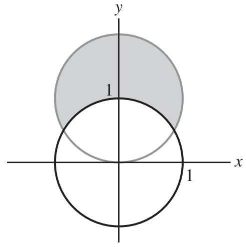
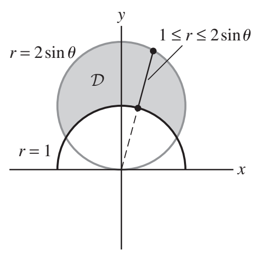


FIGURE 3

SOLUTION



To describe $\mathcal{D}$ in polar coordinates, we first find the polar equations of the circles.

- $x^2 + y^2 = 1$: $r = 1$
- $x^2 + (y-1)^2 = 1$:

$$\begin{aligned}
1 &= (r\cos\theta)^2 + (r\sin\theta - 1)^2 = r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 \\
&= r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta + 1 = r^2 - 2r\sin\theta + 1
\end{aligned}$$

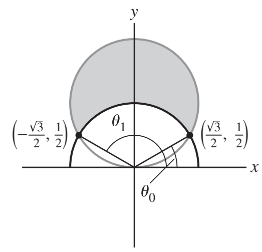
or

$$r^2 - 2r\sin\theta = 0 \quad \Rightarrow \quad r^2 = 2r\sin\theta \quad \Rightarrow \quad r = 2\sin\theta$$

To find the interval for θ, we notice that the two circles intersect at the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Hence,

$$\theta_0 = \tan^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\theta_1 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$



We are now able to write the polar inequalities for $\mathcal{D}$:

$$\mathcal{D}: \frac{\pi}{6} \le \theta \le \frac{5\pi}{6},\ 1 \le r \le 2\sin\theta$$

The function is $x = r\cos\theta$. Converting to polar coordinates, we get

$$\iint_{\mathcal{D}} x\,dA = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} (r\cos\theta) r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \frac{r^3\cos\theta}{3}\bigg|_{r=1}^{2\sin\theta} d\theta$$
$$= \int_{\pi/6}^{5\pi/6} \frac{8\sin^3\theta\cos\theta - \cos\theta}{3}\,d\theta = \frac{8}{3}\int_{\pi/6}^{5\pi/6} \sin^3\theta\cos\theta\,d\theta - \frac{1}{3}\int_{\pi/6}^{5\pi/6} \cos\theta\,d\theta$$
$$= \frac{8}{3}\int_{\pi/6}^{5\pi/6} \sin^3\theta\cos\theta\,d\theta - \frac{1}{3}\sin\theta\bigg|_{\theta=\frac{\pi}{6}}^{5\pi/6} = \frac{8}{3}\int_{\pi/6}^{5\pi/6} \sin^3\theta\cos\theta\,d\theta$$

We use Integration Formulas to obtain

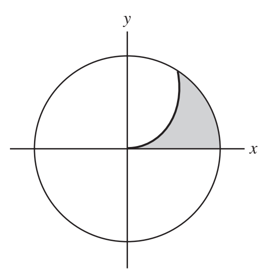
$$\iint_{\mathcal{D}} x\,dA = \frac{8}{3}\frac{\sin^4\theta}{4}\bigg|_{\pi/6}^{5\pi/6} = \frac{8}{12}\left(\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right) = 0$$

Notice that since the region $\mathcal{D}$ is symmetric with respect to the y-axis, we expect the integral $\iint_{\mathcal{D}} x\,dA$ to be zero.

33. Use polar coordinates to calculate $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\,dA$, where $\mathcal{D}$ is the region in the first quadrant bounded by the spiral $r = \theta$, the circle $r = 1$, and the x-axis.

SOLUTION The region of integration, shown in the figure, has the following description in polar coordinates:

$$\mathcal{D}:\ 0 \le \theta \le 1,\ \theta \le r \le 1$$



The function is $f(x, y) = \sqrt{x^2+y^2} = r$. We convert the double integral to polar coordinates and compute to obtain

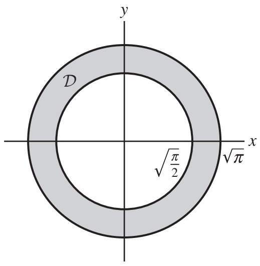
$$\iint_{\mathcal{D}} \sqrt{x^2+y^2}\,dA = \int_0^1 \int_\theta^1 r \cdot r\,dr\,d\theta = \int_0^1 \int_\theta^1 r^2\,dr\,d\theta = \int_0^1 \frac{r^3}{3}\bigg|_{r=\theta}^1 d\theta$$
$$= \int_0^1 \left(\frac{1}{3} - \frac{\theta^3}{3}\right) d\theta = \frac{\theta}{3} - \frac{\theta^4}{12}\bigg|_0^1 = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

34. Calculate $\iint_{\mathcal{D}} \sin(x^2+y^2)\,dA$, where

$$\mathcal{D} = \left\{\frac{\pi}{2} \le x^2+y^2 \le \pi\right\}$$

SOLUTION The annulus $\mathcal{D}$ is defined by the following inequalities in polar coordinates:

$$\mathcal{D}:\ 0 \le \theta \le 2\pi,\ \sqrt{\frac{\pi}{2}} \le r \le \sqrt{\pi}$$



The function in polar coordinates is $f(x, y) = \sin(x^2 + y^2) = \sin(r^2)$. We convert the double integral to polar coordinates and evaluate to obtain

$$\iint_{\mathcal{D}} \sin(x^2 + y^2)\,dA = \int_0^{2\pi} \int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \sin(r^2) r\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \sin(r^2) r\,dr\right)$$
$$= 2\pi \int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \sin(r^2) r\,dr$$

We evaluate the integral using the substitution $u = r^2$, $du = 2r\,dr$:

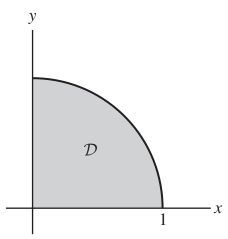
$$\iint_{\mathcal{D}} \sin(x^2 + y^2)\,dA = 2\pi \int_{\pi/2}^{\pi} \sin u \cdot \frac{du}{2} = \pi(-\cos u)\Big|_{u=\pi/2}^{\pi} = \pi(1 + 0) = \pi$$

35. Express in cylindrical coordinates and evaluate:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx$$

SOLUTION We evaluate the integral by converting it to cylindrical coordinates. The projection of the region of integration onto the xy-plane, as defined by the limits of integration, is

$$\mathcal{D}:\ 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}$$



That is, $\mathcal{D}$ is the part of the disk $x^2 + y^2 \le 1$ in the first quadrant. The inequalities defining $\mathcal{D}$ in polar coordinates are

$$\mathcal{D}:\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1$$

The upper surface is $z = \sqrt{x^2 + y^2} = r$ and the lower surface is $z = 0$. Therefore, the inequalities defining the region of integration in cylindrical coordinates are

$$\mathcal{W}:\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ 0 \le z \le r$$

Converting the double integral to cylindrical coordinates gives

$$I = \int_0^{\pi/2} \int_0^1 \int_0^r zr\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 \frac{z^2 r}{2}\Big|_{z=0}^{r} dr\,d\theta = \int_0^{\pi/2} \int_0^1 \frac{r^3}{2}\,dr\,d\theta$$
$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^1 \frac{r^3}{2}\,dr\right) = \frac{\pi}{2} \cdot \frac{r^4}{8}\Big|_0^1 = \frac{\pi}{16}$$

36. Use spherical coordinates to calculate the triple integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the region

$$1 \le x^2 + y^2 + z^2 \le 4$$

SOLUTION The region of integration is the region enclosed by the spheres $x^2 + y^2 + z^2 = 1$ or $\rho = 1$, and $x^2 + y^2 + z^2 = 4$ or $\rho = 2$. In this region, θ is changing between 0 and 2π, and ϕ is changing between 0 and π. Therefore, $\mathcal{W}$ is described by the following inequalities:

$$\mathcal{W}: \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi, \ 1 \le \rho \le 2$$

The function is $f(x, y, z) = x^2 + y^2 + z^2 = \rho^2$. Using triple integrals in spherical coordinates we get

$$\begin{aligned}\iiint_{\mathcal{W}} (x^2 + y^2 + z^2)\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\right)\left(\int_1^2 \rho^4 d\rho\right) = 2\pi \cdot (-\cos\phi)\Big|_0^{\pi} \cdot \left(\frac{\rho^5}{5}\Big|_1^2\right) \\ &= 2\pi \cdot 2 \cdot \frac{(2^5 - 1)}{5} = \frac{124\pi}{5}\end{aligned}$$

37. Convert to spherical coordinates and evaluate:

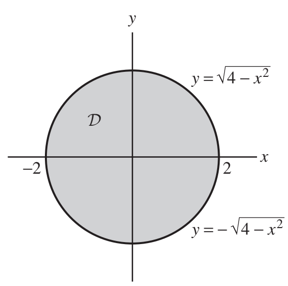
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}} \, dz\, dy\, dx$$

SOLUTION The region of integration as defined by the limits of integration is

$$\mathcal{W}: \ -2 \le x \le 2, \ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, \ 0 \le z \le \sqrt{4 - x^2 - y^2}$$

That is, $\mathcal{W}$ is the region enclosed by the sphere $x^2 + y^2 + z^2 = 4$ and the xy-plane. We see that the region of integration is the upper half-ball $x^2 + y^2 + z^2 \le 4$, hence the inequalities defining $\mathcal{W}$ in spherical coordinates are

$$\mathcal{W}: \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \frac{\pi}{2}, \ 0 \le \rho \le 2$$



The function is $f(x, y, z) = e^{-(x^2+y^2+z^2)^{3/2}} = e^{-(\rho^2)^{3/2}} = e^{-\rho^3}$, therefore the integral in spherical coordinates is

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 e^{-\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin\phi \, d\phi\right)\left(\int_0^2 e^{-\rho^3} \rho^2 d\rho\right) \\ &= 2\pi \left(-\cos\phi \Big|_0^{\pi/2}\right) \int_0^2 e^{-\rho^3} \rho^2 d\rho = 2\pi \int_0^2 e^{-\rho^3} \rho^2 d\rho \end{aligned}$$

We compute the integral using the substitution $u = \rho^3$, $du = 3\rho^2 d\rho$. We get

$$I = 2\pi \int_0^8 e^{-u} \frac{du}{3} = \frac{2\pi}{3} (-e^{-u}) \Big|_0^8 = \frac{2\pi}{3} (-e^{-8} + 1) = \frac{2\pi \left(-1 + e^8\right)}{3e^8}$$

38. Find the average value of $f(x, y, z) = xy^2z^3$ on the box $[0, 1] \times [0, 2] \times [0, 3]$.

SOLUTION The volume of the box is $V = 1 \cdot 2 \cdot 3 = 6$, hence the average value of $f(x, y, z) = xy^2z^3$ on the box $\mathcal{B}$ is

$$\overline{f} = \frac{1}{V}\iiint_{\mathcal{B}} f(x, y, z)\, dV = \frac{1}{6}\int_0^1 \int_0^2 \int_0^3 xy^2z^3\, dz\, dy\, dx = \frac{1}{6}\left(\int_0^1 x\, dx\right)\left(\int_0^2 y^2\, dy\right)\left(\int_0^3 z^3\, dz\right)$$

$$= \frac{1}{6} \cdot \left(\frac{x^2}{2}\bigg|_0^1\right)\left(\frac{y^3}{3}\bigg|_0^2\right)\left(\frac{z^4}{4}\bigg|_0^3\right) = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{8}{3} \cdot \frac{81}{4} = 4.5$$

39. Let $\mathcal{W}$ be the ball of radius R in $\mathbf{R}^3$ centered at the origin, and let $P = (0, 0, R)$ be the North Pole. Let $d_P(x, y, z)$ be the distance from P to (x, y, z). Show that the average value of d_P over the sphere $\mathcal{W}$ is equal to $\overline{d} = 6R/5$. *Hint:* Show that

$$\overline{d} = \frac{1}{\frac{4}{3}\pi R^3}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{R}\int_{\phi=0}^{\pi} \rho^2 \sin\phi\sqrt{R^2 + \rho^2 - 2\rho R\cos\phi}\, d\phi\, d\rho\, d\theta$$

and evaluate.

SOLUTION We know that the volume of the ball is $\frac{4}{3}\pi R^3$.

In spherical coordinates, the distance from P to a point on the ball is

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-0)^2 + (z-R)^2} &= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - R)^2} \\
&= \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi - 2\rho R\cos\phi + R^2} \\
&= \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho R\cos\phi + R^2} \\
&= \sqrt{\rho^2(\sin^2\phi + \cos^2\phi) - 2\rho R\cos\phi + R^2} \\
&= \sqrt{R^2 + \rho^2 - 2\rho R\cos\phi}
\end{aligned}$$

Now, to write the average value of d_P we have:

$$d_P = \frac{1}{\frac{4}{3}\pi R^3}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{R}\int_{\phi=0}^{\pi} \rho^2 \sin\phi\sqrt{R^2 + \rho^2 - 2\rho R\cos\phi}\, d\phi\, d\rho\, d\theta$$

Using substitution, and the fact that $0 \le \rho \le R$,

$$\begin{aligned}
\int_{\phi=0}^{\pi} \sin\phi\sqrt{R^2 + \rho^2 - 2\rho R\cos\phi}\, d\phi &= \frac{2\rho}{3R}(R^2 + \rho^2 - 2\rho R\cos\phi)^{3/2}\bigg|_0^{\pi} \\
&= \frac{2\rho}{3R}\Big((R + \rho^3) - (R - \rho)^3\Big) = \frac{2\rho}{3R}(\rho^3 + 3R^2\rho)
\end{aligned}$$

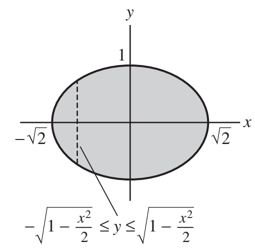
Now integrate with respect to θ and ρ:

$$\begin{aligned}
d_P &= \frac{1}{\frac{4}{3}\pi R^3}\int_0^{2\pi}\int_0^{R} \rho^2 \cdot \frac{2\rho}{3R}(\rho^3 + 3R^2\rho)\, d\rho\, d\theta \\
&= \frac{3}{4\pi R^3} \cdot \frac{2}{3R}\int_0^{2\pi}\int_0^{R} \rho^3(\rho^3 + 3R^2\rho)\, d\rho\, d\theta \\
&= \frac{1}{2\pi R^4}\int_0^{2\pi}\int_0^{R} \rho^6 + 3R^2\rho^4\, d\rho\, d\theta \\
&= \frac{1}{2\pi R^4}\int_0^{2\pi} \frac{1}{7}\rho^7 + \frac{3}{5}R^2\rho^5\bigg|_0^{R} d\theta \\
&= \frac{1}{2\pi R^4}\int_0^{2\pi} \frac{1}{7}R^7 + \frac{3}{5}R^7\, d\theta \\
&= \frac{1}{2\pi R^4}\left(\frac{26}{35}R^7\right) \cdot 2\pi = \frac{26}{35}R^3
\end{aligned}$$

to get $8\pi R^4/5$. Dividing by the volume of the sphere gives us $6R/5$

40. CAS Express the average value of $f(x, y) = e^{xy}$ over the ellipse $\frac{x^2}{2} + y^2 = 1$ as an iterated integral, and evaluate numerically using a computer algebra system.

SOLUTION



The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab, hence the area of the given ellipse is

$$\text{Area}(\mathcal{D}) = \pi \cdot \sqrt{2} \cdot 1 = \pi\sqrt{2}$$

The average value of $f(x, y) = e^{xy}$ inside the ellipse is

$$\overline{f} = \frac{1}{\pi\sqrt{2}} \iint_{\mathcal{D}} e^{xy}\, dA \tag{1}$$

$\mathcal{D}$ is described as a vertically simple region by the inequalities

$$\mathcal{D}: \; -\sqrt{2} \le x \le \sqrt{2}, \; -\sqrt{1-\frac{x^2}{2}} \le y \le \sqrt{1-\frac{x^2}{2}}$$

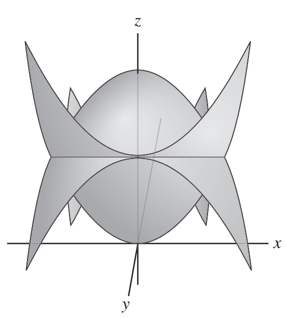
Therefore, the double integral (1) is equal to the following iterated integral, which we compute using a CAS:

$$\overline{f} = \frac{1}{\pi\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{1-\frac{x^2}{2}}}^{\sqrt{1-\frac{x^2}{2}}} e^{xy}\, dy\, dx \approx 1.0421$$

41. Use cylindrical coordinates to find the mass of the solid bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$, assuming a mass density of $f(x, y, z) = (x^2 + y^2)^{1/2}$.

SOLUTION The mass of the solid $\mathcal{W}$ is the following integral:

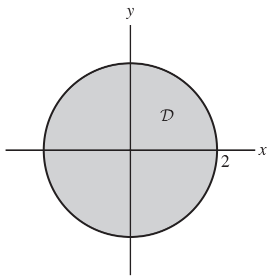
$$M = \iiint_{\mathcal{W}} \left(x^2 + y^2\right)^{1/2} dV$$



The projection of $\mathcal{W}$ on the xy-plane is obtained by equating the equations of the two surfaces:

$$\begin{aligned} 8 - x^2 - y^2 &= x^2 + y^2 \\ 2(x^2 + y^2) &= 8 \end{aligned} \quad \Rightarrow \quad x^2 + y^2 = 4$$

We conclude that the projection is the disk $\mathcal{D}: \; x^2 + y^2 \le 4$.



Therefore, $\mathcal{W}$ is described by

$$\mathcal{W}: x^2 + y^2 \le z \le 8 - (x^2 + y^2), \ (x, y) \in \mathcal{D}$$

Thus,

$$M = \iint_{\mathcal{D}} \int_{x^2+y^2}^{8-(x^2+y^2)} (x^2 + y^2)^{1/2} \, dz \, dx \, dy$$

We convert the integral to cylindrical coordinates. The inequalities for $\mathcal{W}$ are

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le 8 - r^2.$$

Also, $(x^2 + y^2)^{1/2} = r$, hence we obtain the following integral:

$$M = \int_0^2 \int_0^{2\pi} \int_{r^2}^{8-r^2} r \cdot r \, dz \, d\theta \, dr = \int_0^2 \int_0^{2\pi} \int_{r^2}^{8-r^2} r^2 \, dz \, d\theta \, dr = \int_0^2 \int_0^{2\pi} r^2 z \Big|_{z=r^2}^{8-r^2} d\theta \, dr$$

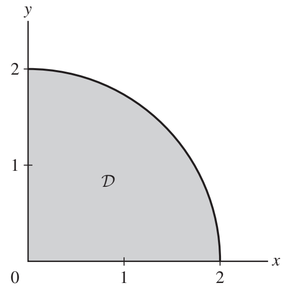
$$= \int_0^2 \int_0^{2\pi} r^2(8 - r^2 - r^2) \, d\theta \, dr = \int_0^2 \int_0^{2\pi} (8r^2 - 2r^4) \, d\theta \, dr = \left(\int_0^{2\pi} 1 \, d\theta \right) \left(\int_0^2 (8r^2 - 2r^4) \, dr \right)$$

$$= 2\pi \left(\frac{8r^3}{3} - \frac{2}{5} r^5 \Big|_0^2 \right) = \frac{256}{15}\pi \approx 53.62$$

42. Let $\mathcal{W}$ be the portion of the half-cylinder $x^2 + y^2 \le 4$, $x \ge 0$ such that $0 \le z \le 3y$. Use cylindrical coordinates to compute the mass of $\mathcal{W}$ if the mass density is $\rho(x, y, z) = z^2$.

SOLUTION Since $0 \le z \le 3y$, we have $y \ge 0$. Also $x \ge 0$, hence $\mathcal{W}$ projects onto the quarter circle $\mathcal{D}$ in the xy-plane of radius 2, where $x \ge 0$ and $y \ge 0$. In polar coordinates,

$$\mathcal{D}: 0 \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2$$



The upper boundary of $\mathcal{W}$ is the plane $z = 3y = 3r\sin\theta$ and the lower boundary is $z = 0$. Hence,

$$\mathcal{W}: 0 \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2, \ 0 \le z \le 3r\sin\theta$$

Using cylindrical coordinates, the total mass is the following integral:

$$M = \iiint_{\mathcal{W}} z^2 \, dv = \int_0^{\pi/2} \int_0^2 \int_0^{3r\sin\theta} z^2 r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 \frac{z^3 r}{3} \Big|_{z=0}^{3r\sin\theta} dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \frac{r(3r\sin\theta)^3}{3} \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 9r^4 \sin^3\theta \, dr \, d\theta$$

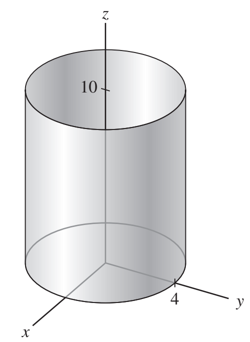
$$= \left(\int_0^{\pi/2} \sin^3\theta \, d\theta \right) \left(\int_0^2 9r^4 \, dr \right) = \left(\int_0^{\pi/2} \sin^3\theta \, d\theta \right) \frac{9r^5}{5} \Big|_{r=0}^2 = \frac{288}{5} \int_0^{\pi/2} \sin^3\theta \, d\theta$$

We compute the integral using an integration table to obtain

$$M = \frac{288}{5}\left(-\frac{\sin^2\theta\cos\theta}{3} - \frac{2}{3}\cos\theta\right)\Big|_{\theta=0}^{\pi/2} = \frac{288}{5}\left(0 - \left(-\frac{2}{3}\right)\right) = \frac{288}{5}\cdot\frac{2}{3} = 38.4$$

43. Use cylindrical coordinates to find the mass of a cylinder of radius 4 and height 10 if the mass density at a point is equal to the square of the distance from the cylinder's central axis.

SOLUTION

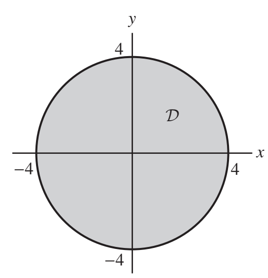


The mass density is $\rho(x, y, z) = x^2 + y^2 = r^2$, hence the mass of the cylinder is

$$M = \iiint_{\mathcal{W}} (x^2 + y^2)\, dV$$

The region $\mathcal{W}$ is described using cylindrical coordinates by the following inequalities:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 4,\ 0 \le z \le 10$$



Thus,

$$M = \iiint_{\mathcal{W}} (x^2 + y^2)\, dV = \int_0^{2\pi}\int_0^4\int_0^{10} r^2\cdot r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^4\int_0^{10} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r^3 z\Big|_{z=0}^{10} dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^4 r^3\cdot 10\, dr\, d\theta = \int_0^{2\pi} \frac{10r^4}{4}\Big|_{r=0}^{4} d\theta = \int_0^{2\pi} 640\, d\theta = 640\cdot 2\pi = 1280\pi$$

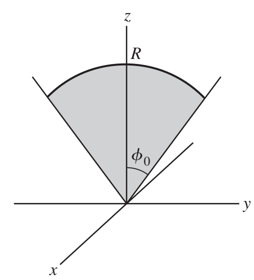
44. Find the centroid of the region $\mathcal{W}$ bounded, in spherical coordinates, by $\phi = \phi_0$ and the sphere $\rho = R$.

SOLUTION The centroid is the point $P = (\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{1}{V}\iiint_{\mathcal{W}} x\, dV, \quad \overline{y} = \frac{1}{V}\iiint_{\mathcal{W}} y\, dV, \quad \overline{z} = \frac{1}{V}\iiint_{\mathcal{W}} z\, dV \tag{1}$$

We first compute the volume V of $\mathcal{W}$. The region $\mathcal{W}$ has the following definition in spherical coordinates:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \phi_0,\ 0 \le \rho \le R$$



Hence,

$$V = \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi}\int_0^{\phi_0}\int_0^R \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\phi_0}\sin\phi\,d\phi\right)\left(\int_0^R \rho^2 d\rho\right)$$

$$= 2\pi\cdot\left(-\cos\phi\Big|_0^{\phi_0}\right)\left(\frac{\rho^3}{3}\Big|_0^R\right) = 2\pi\cdot(1-\cos\phi_0)\cdot\frac{R^3}{3} = \frac{2\pi R^3(1-\cos\phi_0)}{3} \tag{2}$$

Since the region $\mathcal{D}$ is symmetric with respect to the z-axis, its centroid lies on the z-axis. That is, $\overline{x} = \overline{y} = 0$. To compute $\overline{z}$, we evaluate the following integral:

$$\iiint_{\mathcal{W}} z\,dV = \int_0^{2\pi}\int_0^{\phi_0}\int_0^R (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\phi_0}\int_0^R \rho^3\cdot\frac{\sin 2\phi}{2}\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\phi_0}\frac{\sin 2\phi}{2}\,d\phi\right)\left(\int_0^R \rho^3\,d\rho\right) = 2\pi\cdot\left(-\frac{\cos 2\phi}{4}\Big|_0^{\phi_0}\right)\left(\frac{\rho^4}{4}\Big|_0^R\right)$$

$$= 2\pi\cdot\frac{1-\cos 2\phi}{4}\cdot\frac{R^4}{4} = \frac{R^4(1-\cos 2\phi_0)\pi}{8}$$

Combining with (1) and (2) gives

$$\overline{z} = \frac{3}{2\pi R^3(1-\cos\phi_0)}\cdot\frac{R^4(1-\cos 2\phi_0)\pi}{8} = \frac{3R(1-\cos 2\phi_0)}{16(1-\cos\phi_0)}$$

The centroid is thus

$$P = \left(0, 0, \frac{3R(1-\cos 2\phi_0)}{16(1-\cos\phi_0)}\right).$$

45. Find the centroid of the solid bounded by the xy-plane, the cylinder $x^2+y^2=R^2$, and the plane $x/R + z/H = 1$.

SOLUTION First to find the volume of this solid. The first equation lends itself well to cylindrical coordinates:

$$x^2+y^2=R^2 \quad\Rightarrow\quad r = R, 0\le\theta\le 2\pi$$

and

$$\frac{x}{R}+\frac{z}{H} = 1 \quad\Rightarrow\quad z = H\left(1-\frac{x}{R}\right) = H\left(1-\frac{r\cos\theta}{R}\right)$$

The volume is:

$$\begin{aligned} Volume &= \int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} 1\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^R H\left(1-\frac{r\cos\theta}{R}\right)dr\,d\theta \\ &= H\int_0^{2\pi} r - \frac{1}{2}\cdot\frac{r^2\cos\theta}{R}\Big|_{r=0}^R d\theta \\ &= H\int_0^{2\pi} R - \frac{1}{2}R\cos\theta\,d\theta \\ &= H\left(R\theta - \frac{1}{2}R\sin\theta\Big|_0^{2\pi}\right) \\ &= 2\pi HR \end{aligned}$$

Now to compute the coordinates of the centroid:

$$\begin{aligned} \overline{x} &= \frac{1}{V}\iiint_{\mathcal{W}} x\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} r\cos\theta\,dz\,dr\,d\theta \\ &= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R r\cos\theta\cdot z\Big|_0^{H(1-r\cos\theta/R)} dr\,d\theta \\ &= \frac{H}{2\pi HR}\int_0^{2\pi}\int_0^R r\cos\theta\left(1-\frac{r\cos\theta}{R}\right)dr\,d\theta \end{aligned}$$

$$= \frac{1}{2\pi R}\int_0^{2\pi}\int_0^R r\cos\theta - \frac{1}{R}r^2\cos^2\theta\,dr\,d\theta$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} \frac{1}{2}r^2\cos\theta - \frac{1}{3R}r^3\cos^2\theta\Big|_0^R\,d\theta$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} \frac{1}{2}R^2\cos\theta - \frac{R^2}{6}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2\pi R}\left(\frac{1}{2}R^2\sin\theta - \frac{R^2}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right) = \frac{1}{2\pi R}\cdot -\frac{R^2}{6}(2\pi) = -\frac{R}{6}$$

$$\overline{y} = \frac{1}{V}\iiint_{\mathcal{W}} y\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} r\sin\theta\,dz\,dr\,d\theta$$

$$= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R r\sin\theta\cdot z\Big|_0^{H(1-r\cos\theta/R)}\,dr\,d\theta$$

$$= \frac{H}{2\pi HR}\int_0^{2\pi}\int_0^R r\sin\theta\left(1-\frac{r\cos\theta}{R}\right)dr\,d\theta$$

$$= \frac{1}{2\pi R}\int_0^{2\pi}\int_0^R r\sin\theta - \frac{1}{R}r^2\sin\theta\cos\theta\,dr\,d\theta$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} \frac{1}{2}r^2\sin\theta - \frac{1}{3R}r^3\sin\theta\cos\theta\Big|_0^R\,d\theta$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} \frac{1}{2}R^2\sin\theta - \frac{R^2}{3}\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2\pi R}\left(-\frac{1}{2}R^2\cos\theta - \frac{R^2}{6}\sin^2\theta\Big|_0^{2\pi}\right) = 0$$

$$\overline{z} = \frac{1}{V}\iiint_{\mathcal{W}} z\,dV = \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R\int_0^{H(1-r\cos\theta/R)} z\,dz\,dr\,d\theta$$

$$= \frac{1}{2\pi HR}\int_0^{2\pi}\int_0^R \frac{1}{2}z^2\Big|_0^{H(1-r\cos\theta/R)}\,dr\,d\theta$$

$$= \frac{H^2}{4\pi HR}\int_0^{2\pi}\int_0^R \left(1-\frac{r\cos\theta}{R}\right)^2 dr\,d\theta$$

$$= \frac{H}{4\pi R}\int_0^{2\pi}\int_0^R 1 - \frac{2r\cos\theta}{R} + \frac{r^2\cos^2\theta}{R^2}\,dr\,d\theta$$

$$= \frac{H}{4\pi R}\int_0^{2\pi} r - \frac{r^2\cos\theta}{R} + \frac{r^3\cos^2\theta}{3R^2}\Big|_0^R\,d\theta$$

$$= \frac{H}{4\pi R}\int_0^{2\pi} R - R\cos\theta + \frac{1}{6}R(1+\cos 2\theta)\,d\theta$$

$$= \frac{H}{4\pi R}\left(R\theta - R\sin\theta + \frac{1}{6}R\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right)$$

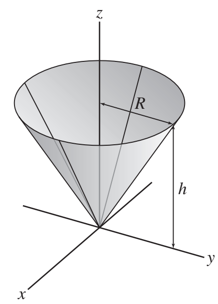
$$= \frac{H}{4\pi R}\left(2\pi R + \frac{1}{3}\pi R\right) = \frac{H}{4\pi R}\cdot\frac{7\pi R}{3} = \frac{7}{12}H$$

The coordinates of the centroid are $(-R/6, 0, 7H/12)$.

46. Using cylindrical coordinates, prove that the centroid of a right circular cone of height h and radius R is located at height $\frac{h}{4}$ on the central axis.

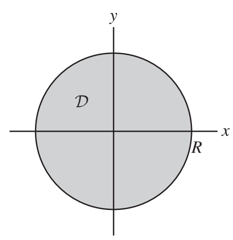
SOLUTION The volume of the cone is $V = \frac{\pi R^2 h}{3}$, therefore the coordinates of the centroid are

$$\overline{x} = \frac{3}{\pi R^2 h}\iiint_{\mathcal{W}} x\,dV, \quad \overline{y} = \frac{3}{\pi R^2 h}\iiint_{\mathcal{W}} y\,dV, \quad \overline{z} = \frac{3}{\pi R^2 h}\iiint_{\mathcal{W}} z\,dV$$



Since the cone $\mathcal{W}$ is symmetric with respect to the z-axis, its centroid lies on the z-axis. That is,

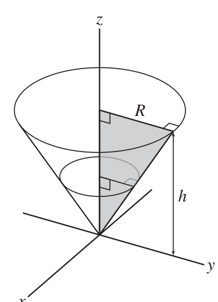
$$\overline{x} = \overline{y} = 0$$



Thus we need to find the z-coordinate. The projection of $\mathcal{W}$ on to the xy-plane is the disk $x^2 + y^2 \le R^2$, or in polar coordinates,

$$\mathcal{D} : 0 \le \theta \le 2\pi, \ 0 \le r \le R$$

The upper surface is the plane $z = h$. To find the equation of the lower surface we use proportion in similar triangles (see figure).



We get

$$\frac{r}{R} = \frac{z}{h} \quad \Rightarrow \quad z = \frac{rh}{R}$$

The inequalities defining $\mathcal{W}$ in cylindrical coordinates are thus

$$\mathcal{W} : 0 \le \theta \le 2\pi, \ 0 \le r \le R, \ \frac{rh}{R} \le z \le h$$

We now compute $\overline{z}$ using Triple Integral in Cylindrical coordinates:

$$\begin{aligned}
\overline{z} &= \frac{3}{\pi R^2 h} \int_0^{2\pi} \int_0^R \int_{\frac{rh}{R}}^h zr\,dz\,dr\,d\theta = \frac{3}{\pi R^2 h} \int_0^{2\pi} \int_0^R \frac{z^2 r}{2} \bigg|_{z=\frac{rh}{R}}^h dr\,d\theta \\
&= \frac{3}{2\pi R^2 h} \int_0^{2\pi} \int_0^R \left(h^2 r - \frac{h^2 r^3}{R^2} \right) dr\,d\theta = \frac{3}{2\pi R^2 h} \int_0^{2\pi} \frac{h^2 r^2}{2} - \frac{h^2 r^4}{4R^2} \bigg|_{r=0}^R d\theta \\
&= \frac{3}{2\pi R^2 h} \int_0^{2\pi} \left(\frac{h^2 R^2}{2} - \frac{h^2 R^4}{4R^2} \right) d\theta = \frac{3}{2\pi R^2 h} \cdot 2\pi \left(\frac{h^2 R^2}{2} - \frac{h^2 R^2}{4} \right) = \frac{3h}{4}
\end{aligned}$$

Therefore the centroid is the following point:

$$\left(0, 0, \frac{3h}{4}\right).$$

That is, the height (measured from the base) is $\frac{h}{4}$.

47. Find the centroid of solid (A) in Figure 4 defined by $x^2 + y^2 \le R^2$, $0 \le z \le H$, and $\frac{\pi}{6} \le \theta \le 2\pi$, where θ is the polar angle of (x, y).

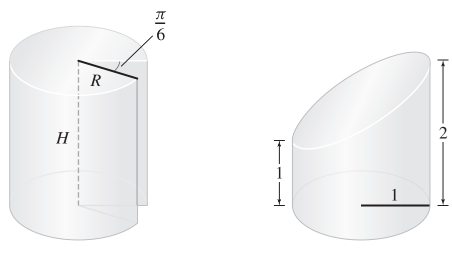


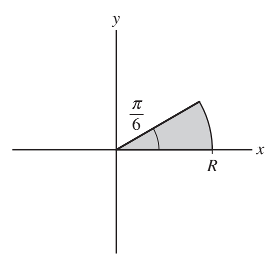
FIGURE 4

SOLUTION Since the mass distribution is uniform, we may assume that $\rho(x, y, z) = 1$, hence the center of mass is

$$x_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} x\, dV, \quad y_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} y\, dV, \quad z_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} z\, dV$$

The inequalities describing $\mathcal{W}$ in cylindrical coordinates are

$$\mathcal{W}: 0 \le \theta \le \frac{\pi}{6},\ 0 \le r \le R,\ 0 \le z \le H$$



The entire cylinder has a total volume $\pi R^2 H$. The region $\mathcal{W}$ has the fraction $(2\pi - \frac{\pi}{6})/(2\pi)$ of this total volume, so

$$V = \frac{(2\pi - \frac{\pi}{6})}{2\pi}(\pi R^2 H) = \frac{11\pi R^2 H}{12}$$

We use cylindrical coordinates to compute the triple integrals:

$$x_{CM} = \frac{1}{V}\int_{\pi/6}^{2\pi}\int_0^R\int_0^H (r\cos\theta) r\, dz\, dr\, d\theta = \frac{12}{11\pi R^2 H}\left(\int_{\pi/6}^{2\pi}\cos\theta\, d\theta\right)\left(\int_0^R r^2\, dr\right)\left(\int_0^H dz\right)$$

$$= \frac{12}{11\pi R^2 H}\left(-\frac{1}{2}\right)\left(\frac{R^3}{3}\right)(H) = -\frac{2R}{11\pi}$$

$$y_{CM} = \frac{1}{V}\int_{\pi/6}^{2\pi}\int_0^R\int_0^H (r\sin\theta) r\, dz dr d\theta = \frac{12}{11\pi R^2 H}\left(\int_{\pi/6}^{2\pi}\sin\theta\, d\theta\right)\left(\int_0^R r^2\, dr\right)\left(\int_0^H dz\right)$$

$$= \frac{12}{11\pi R^2 H}\left(\frac{-2+\sqrt{3}}{2}\right)\left(\frac{R^3}{3}\right)(H) = \frac{2R}{11\pi}(\sqrt{3} - 2)$$

$$z_{CM} = \frac{1}{V}\int_{\pi/6}^{2\pi}\int_0^R\int_0^H zr\, dz\, dr\, d\theta = \frac{12}{11\pi R^2 H}\left(\int_{\pi/6}^{2\pi} d\theta\right)\left(\int_0^R r\, dr\right)\left(\int_0^H z\, dz\right)$$

$$= \frac{12}{11\pi R^2 H}\left(\frac{11\pi}{6}\right)\left(\frac{R^2}{2}\right)\left(\frac{H^2}{2}\right) = \frac{H}{2}$$

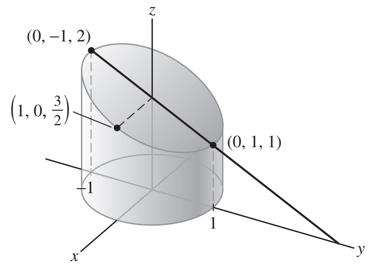
Therefore, the center of mass is the following point:

$$\left(-\frac{2R}{11\pi}, \frac{2R}{11\pi}(\sqrt{3} - 2), \frac{H}{2}\right)$$

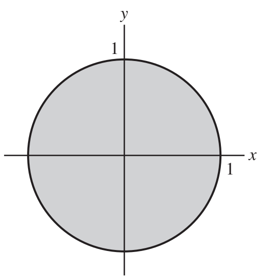
48. Calculate the coordinate y_{CM} of the centroid of solid (B) in Figure 4 defined by $x^2 + y^2 \le 1$ and $0 \le z \le \frac{1}{2}y + \frac{3}{2}$.

SOLUTION Notice that our picture here is slightly different from the one in the book; we've arranged our picture so that the top slopes down in the positive y direction. Since the mass distribution is uniform, we may assume that $\rho(x, y, z) = 1$, hence the center of mass is

$$x_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} x\,dV, \quad y_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} y\,dV, \quad z_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} z\,dV$$



We first must find the equation of the upper plane. This plane is passing through the points $(0, -1, 2)$, $(0, 1, 1)$, and $\left(1, 0, \frac{3}{2}\right)$, hence it has the equation $y + 2z = 3$ or $z = \frac{3-y}{2} = \frac{3-r\sin\theta}{2}$.



The region $\mathcal{W}$ has the following definition in cylindrical coordinates:

$$\mathcal{W}:\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le \frac{3 - r\sin\theta}{2}$$

We first find the volume of $\mathcal{W}$:

$$\begin{aligned}
V &= \iiint_{\mathcal{W}} 1\,dV = \int_0^{2\pi}\int_0^1\int_0^{(3-r\sin\theta)/2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 rz\Big|_0^{(3-r\sin\theta)/2} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r\frac{3-r\sin\theta}{2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(\frac{3r}{2} - \frac{r^2}{2}\sin\theta\right) dr\,d\theta = \int_0^{2\pi} \frac{3r^2}{4} - \frac{r^3\sin\theta}{6}\Big|_{r=0}^1 d\theta \\
&= \int_0^{2\pi}\left(\frac{3}{4} - \frac{\sin\theta}{6}\right) d\theta = \frac{3}{4}\theta + \frac{\cos\theta}{6}\Big|_0^{2\pi} = \frac{3\pi}{2} + \frac{1}{6} - \frac{1}{6} = \frac{3\pi}{2}
\end{aligned}$$

$V = \frac{3\pi}{2}$: Since the region is symmetric with respect to the yz-plane, we have

$$x_{\text{CM}} = \frac{1}{V}\iiint_{\mathcal{W}} x\,dV = 0$$

We compute y_{CM}:

$$\begin{aligned}
y_{\text{CM}} &= \frac{2}{3\pi}\int_0^{2\pi}\int_0^1\int_0^{(3-r\sin\theta)/2} (r\sin\theta)r\,dz\,dr\,d\theta = \frac{2}{3\pi}\int_0^{2\pi}\int_0^1\int_0^{(3-r\sin\theta)/2} r^2\sin\theta\,dz\,dr\,d\theta \\
&= \frac{2}{3\pi}\int_0^{2\pi}\int_0^1 r^2\sin\theta\left(\frac{3-r\sin\theta}{2}\right) dr\,d\theta = \frac{2}{3\pi}\int_0^{2\pi}\int_0^1\left(\frac{3r^2\sin\theta}{2} - \frac{r^3\sin^2\theta}{2}\right) dr\,d\theta \\
&= \frac{2}{3\pi}\int_0^{2\pi}\left(\frac{3\sin\theta}{2}\frac{r^3}{3} - \frac{r^4\sin^2\theta}{8}\Big|_{r=0}^1\right) d\theta = \frac{2}{3\pi}\int_0^{2\pi}\left(\frac{\sin\theta}{2} - \frac{\sin^2\theta}{8}\right) d\theta \\
&= \frac{1}{3\pi}\int_0^{2\pi}\sin\theta\,d\theta - \frac{1}{12\pi}\int_0^{2\pi}\sin^2\theta\,d\theta = -\frac{1}{12\pi}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_{\theta=0}^{2\pi}\right) = -\frac{1}{12\pi}\cdot\pi = -\frac{1}{12}
\end{aligned}$$

Finally we find z_{CM}:

$$\begin{aligned} z_{CM} &= \frac{2}{3\pi}\int_0^{2\pi}\int_0^1\int_0^{(3-r\sin\theta)/2} zr\,dz\,dr\,d\theta = \frac{2}{3\pi}\int_0^{2\pi}\int_0^1 \frac{z^2r}{2}\bigg|_{z=0}^{(3-r\sin\theta)/2} dr\,d\theta \\ &= \frac{2}{3\pi}\int_0^{2\pi}\int_0^1 \frac{r}{2}\left(\frac{3-r\sin\theta}{2}\right)^2 dr\,d\theta = \frac{2}{8\cdot 3\pi}\int_0^{2\pi}\int_0^1 \left(r^3\sin^2\theta - 6r^2\sin\theta + 9r\right) dr\,d\theta \\ &= \frac{1}{12\pi}\int_0^{2\pi} \frac{r^4\sin^2\theta}{4} - 2r^3\sin\theta + \frac{9}{2}r^2\bigg|_0^1 d\theta = \frac{1}{12\pi}\int_0^{2\pi}\left(\frac{\sin^2\theta}{4} - 2\sin\theta + \frac{9}{2}\right) d\theta \\ &= \frac{1}{48\pi}\int_0^{2\pi}\sin^2\theta\,d\theta - \frac{1}{6\pi}\int_0^{2\pi}\sin\theta\,d\theta + \frac{9}{24\pi}\int_0^{2\pi} d\theta = \frac{1}{48\pi}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\bigg|_0^{2\pi}\right) - 0 + \frac{9}{24\pi}\cdot 2\pi \\ &= \frac{1}{48} + \frac{3}{4} = \frac{37}{48} \end{aligned}$$

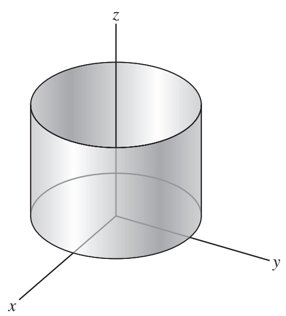
The center of mass is the following point:

$$\left(0, -\frac{1}{12}, \frac{37}{48}\right)$$

If you had arranged the axes differently, you could have computed the answer as $\left(0, \frac{1}{12}, \frac{37}{48}\right)$ (depending on orientation).

49. Find the center of mass of the cylinder $x^2 + y^2 = 1$ for $0 \le z \le 1$, assuming a mass density of $\rho(x, y, z) = z$.

SOLUTION By symmetry, we can note that the center of mass lies on the z-axis.

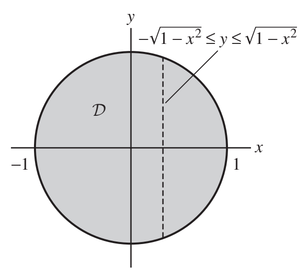


The coordinates of the center of mass are defined as,

$$\begin{aligned} x_{CM} &= \frac{\iiint_{\mathcal{W}} x\left(x^2+y^2\right) dV}{M} \\ y_{CM} &= \frac{\iiint_{\mathcal{W}} y\left(x^2+y^2\right) dV}{M} \\ z_{CM} &= \frac{\iiint_{\mathcal{W}} z\left(x^2+y^2\right) dV}{M} \end{aligned} \qquad (1)$$

where M is the total mass of $\mathcal{W}$. The cylinder $\mathcal{W}$ is defined by the inequalities

$$-1 \le x \le 1, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad 0 \le z \le 1$$



We compute the total mass of $\mathcal{W}$:

$$\begin{aligned} M &= \iiint_{\mathcal{W}} z\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 z\,dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}z^2\Big|_{z=0}^{1} dy\,dx \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 1\,dy\,dx = \int_{-1}^{1} y\Big|_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1}\sqrt{1-x^2}\,dx \end{aligned}$$

This integral is the area of a half of the unit circle, hence the total mass is

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{\pi}{2} = M$$

We now compute the numerators in (1). Using (2), we get

$$\begin{aligned} \iiint_{\mathcal{W}} xz\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 xz\,dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}xz^2\Big|_{z=0}^{1} dy\,dx \\ &= 2\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \frac{1}{2}x\,dy\,dx = \int_{-1}^{1} xy\Big|_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} x\sqrt{1-x^2}\,dx = -\frac{2}{3}(1-x^2)^{3/2}\Big|_{-1}^{1} = 0 \end{aligned} \tag{2}$$

Now to compute the next numerator:

$$\begin{aligned} \iiint_{\mathcal{W}} yz\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 yz\,dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}yz^2\Big|_{z=0}^{1} dy\,dx \\ &= 2\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \frac{1}{2}y\,dy\,dx = \int_{-1}^{1} \frac{1}{2}y^2\Big|_{y=0}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1}(1-x^2)\,dx = \left(x-x^3\right)\Big|_{-1}^{1} = 0 \end{aligned} \tag{3}$$

Thus far we have:

$$\iiint_{\mathcal{W}} yz\,dV = \iiint_{\mathcal{W}} xz\,dV = 0 \tag{4}$$

We compute the numerator of z_{CM} in (1):

$$\begin{aligned} \iiint_{\mathcal{W}} z\cdot z\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 z^2\,dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{3}z^3\Big|_{z=0}^{1} dy\,dx \\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{3}\,dy\,dx = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \frac{2}{3}\,dy\,dx \\ &= \int_{-1}^{1} \frac{2}{3}y\Big|_0^{\sqrt{1-x^2}} dx \\ &= \frac{2}{3}\int_{-1}^{1}\sqrt{1-x^2}\,dx \end{aligned}$$

This is $2/3$ times half of the area of the circle, centered at the origin, having radius 1, so the integral is $1/3\pi$.

Finally, we substitute $M = \frac{\pi}{2}$ and the computed integrals for (1) to obtain the following solution:

$$(x_{\text{CM}}, y_{\text{CM}}, z_{\text{CM}}) = \left(0, 0, \frac{\frac{\pi}{3}}{\frac{\pi}{2}}\right) = \left(0, 0, \frac{2}{3}\right).$$

50. Find the center of mass of the sector of central angle $2\theta_0$ (symmetric with respect to the y-axis) in Figure 5, assuming that the mass density is $\rho(x, y) = x^2$.

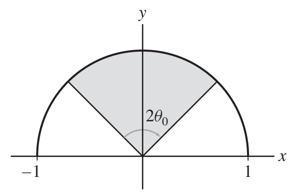


FIGURE 5

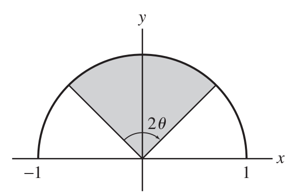
SOLUTION Since the region is symmetric with respect to the y-axis, the x-coordinate of the center of mass is zero, hence the center of mass is located on the y-axis. We first compute the mass:

$$M = \iint \rho\, dA = \int_0^1 \int_{\pi/2-\theta_0}^{\pi/2+\theta_0} r^2 \cos^2\alpha \cdot r\, dr\, d\alpha = \frac{1}{4}r^4\Big|_0^1 \cdot \left[\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right]\Big|_{\pi/2-\theta_0}^{\pi/2+\theta_0}$$

$$= \frac{1}{4}\left[\theta_0 + \frac{1}{4}\left(\sin(\pi + 2\theta_0) - \sin(\pi - 2\theta_0)\right)\right] = \frac{1}{4}\left[\theta_0 + \frac{1}{4}\left(-\sin(2\theta_0) - \sin(2\theta_0)\right)\right]$$

$$= \frac{1}{4}\left[\theta_0 - \frac{1}{2}\sin(2\theta_0)\right]$$

We now compute y_{CM}.



We obtain the integral

$$y_{CM} = \frac{1}{M}\int_0^1 \int_{\frac{\pi}{2}-\theta_0}^{\pi/2+\theta_0} r\sin\alpha \cdot r^2\cos^2\alpha \cdot r\, d\alpha\, dr = \frac{1}{M}\frac{1}{5}r^5\Big|_0^1 \cdot \left[\frac{-1}{3}\cos^3\alpha\right]\Big|_{\pi/2-\theta_0}^{\pi/2+\theta_0}$$

$$= \frac{1}{15M}\left[\cos^3(\pi/2 - \theta_0) - \cos^3(\pi/2 + \theta_0)\right] = \frac{1}{15M}\left[\sin^3(\theta_0) + \sin^3(\theta_0)\right] = \frac{2\sin^3(\theta_0)}{15M}$$

Substituting in the previous value for the mass M, we obtain

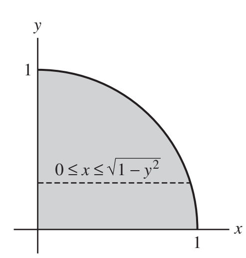
$$y_{CM} = \frac{8\sin^3\theta_0}{15(\theta_0 - \frac{1}{2}\sin 2\theta_0)}$$

51. Find the center of mass of the first octant of the ball $x^2 + y^2 + z^2 = 1$, assuming a mass density of $\rho(x, y, z) = x$.

SOLUTION

(a) The solid is the part of the unit sphere in the first octant. The inequalities defining the projection of the solid onto the xy-plane are

$$\mathcal{D} : 0 \le y \le 1,\ 0 \le x \le \sqrt{1 - y^2}$$



$\mathcal{W}$ is the region bounded by $\mathcal{D}$ and the sphere $z = \sqrt{1 - x^2 - y^2}$, hence $\mathcal{W}$ is defined by the inequalities

$$\mathcal{W} : 0 \le y \le 1,\ 0 \le x \le \sqrt{1 - y^2},\ 0 \le z \le \sqrt{1 - x^2 - y^2} \qquad (1)$$

We first must compute the mass of the solid. Using the mass as a triple integral, we have

$$M = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} x\,dz\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} xz\Big|_{z=0}^{\sqrt{1-x^2-y^2}} dx\,dy$$
$$= \int_0^1 \int_0^{\sqrt{1-y^2}} x\sqrt{1-x^2-y^2}\,dx\,dy$$

We compute the inner integral using the substitution $u = \sqrt{1-x^2-y^2}$, $du = -\frac{x}{u}\,dx$, or $x\,dx = -u\,du$. We get

$$\int_0^{\sqrt{1-y^2}} x\sqrt{1-x^2-y^2}\,dx = \int_{\sqrt{1-y^2}}^0 u(-u\,du) = \int_0^{\sqrt{1-y^2}} u^2\,du = \frac{u^3}{3}\Big|_0^{\sqrt{1-y^2}} = \frac{(1-y^2)^{3/2}}{3} \tag{2}$$

We substitute in (2) and compute the resulting integral substituting $y = \sin t$, $dy = \cos t\,dt$:

$$M = \int_0^1 \frac{(1-y^2)^{3/2}}{3}\,dy = \frac{1}{3}\int_0^{\pi/2} (1-\sin^2 t)^{3/2}\cos t\,dt = \frac{1}{3}\int_0^{\pi/2} \cos^4 t\,dt$$
$$= \frac{1}{3}\left(\frac{\cos^3 t\sin t}{4} + \frac{3}{4}\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{\pi/2}\right) = \frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{16}$$

That is, $M = \frac{\pi}{16}$. We now find the coordinates of the center of mass. To compute $x_{\rm CM}$ we use the definition of $\mathcal{D}$ as a vertically simple region to obtain a simpler integral. That is, we write the inequalities for $\mathcal{W}$ as

$$\mathcal{W}: 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le \sqrt{1-x^2-y^2} \tag{3}$$

Thus,

$$x_{\rm CM} = \frac{1}{M}\iiint_{\mathcal{W}} x\rho\,dV = \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} x^2\,dz\,dy\,dx = \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 z\Big|_{z=0}^{\sqrt{1-x^2-y^2}} dy\,dx$$
$$= \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-x^2}} x^2\sqrt{1-x^2-y^2}\,dy\,dx = \frac{16}{\pi}\int_0^1 x^2\left(\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\right)dx \tag{4}$$

We compute the inner integral using Integration Formulas:

$$\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy = \frac{y}{2}\sqrt{1-x^2-y^2} + \frac{1-x^2}{2}\sin^{-1}\frac{y}{\sqrt{1-x^2}}\Big|_{y=0}^{\sqrt{1-x^2}}$$
$$= \frac{1-x^2}{2}\sin^{-1} 1 = \frac{1-x^2}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}(1-x^2)$$

Substituting in (4) gives

$$x_{\rm CM} = \frac{16}{\pi}\int_0^1 x^2\cdot\frac{\pi}{4}(1-x^2)\,dx = 4\int_0^1 (x^2-x^4)\,dx = 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}$$

(b) We compute the y-coordinate of the center of mass, using (1):

$$y_{\rm CM} = \frac{1}{M}\iiint_{\mathcal{W}} y\rho\,dV = \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} yx\,dz\,dx\,dy = \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-y^2}} yxz\Big|_{z=0}^{\sqrt{1-x^2-y^2}} dx\,dy$$
$$= \frac{16}{\pi}\int_0^1 \int_0^{\sqrt{1-y^2}} yx\sqrt{1-x^2-y^2}\,dx\,dy = \frac{16}{\pi}\int_0^1 y\left(\int_0^{\sqrt{1-y^2}} x\sqrt{1-x^2-y^2}\,dx\right)dy$$

The inner integral was computed in (2), therefore,

$$y_{\rm CM} = \frac{16}{\pi}\int_0^1 y\cdot\frac{(1-y^2)^{3/2}}{3}\,dy = \frac{16}{3\pi}\int_0^1 y(1-y^2)^{3/2}\,dy$$

We compute the integral using the substitution $u = 1-y^2$, $du = -2y\,dy$. We get

$$y_{\rm CM} = \frac{16}{3\pi}\int_1^0 u^{3/2}\cdot\left(-\frac{du}{2}\right) = \frac{8}{3\pi}\int_0^1 u^{3/2}\,du = \frac{8}{3\pi}\cdot\frac{2}{5}\cdot u^{5/2}\Big|_0^1 = \frac{16}{15\pi}$$

Finally we find the z-coordinate of the center of mass, using (1):

$$z_{\mathrm{CM}} = \frac{1}{M}\iiint_{\mathcal{W}} z\rho\,dV = \frac{16}{\pi}\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} zx\,dz\,dx\,dy = \frac{16}{\pi}\int_0^1\int_0^{\sqrt{1-y^2}} \frac{z^2x}{2}\Big|_{z=0}^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \frac{16}{\pi}\int_0^1\int_0^{\sqrt{1-y^2}} \frac{x}{2}(1-x^2-y^2)\,dx\,dy = \frac{8}{\pi}\int_0^1\int_0^{\sqrt{1-y^2}}(x-x^3-xy^2)\,dx\,dy$$

$$= \frac{8}{\pi}\int_0^1 \frac{x^2}{2}-\frac{x^4}{4}-\frac{x^2y^2}{2}\Big|_{x=0}^{\sqrt{1-y^2}} dy = \frac{8}{\pi}\int_0^1\left(\frac{1-y^2}{2}-\frac{(1-y^2)^2}{4}-\frac{(1-y^2)y^2}{2}\right)dy$$

$$= \frac{2}{\pi}\int_0^1 (y^4-2y^2+1)\,dy = \frac{2}{\pi}\left(\frac{y^5}{5}-\frac{2y^3}{3}+y\right)\Big|_{y=0}^1 = \frac{2}{\pi}\left(\frac{1}{5}-\frac{2}{3}+1\right) = \frac{16}{15\pi}$$

The center mass is the following point:

$$P = \left(\frac{8}{15}, \frac{16}{15\pi}, \frac{16}{15\pi}\right).$$

52. Find a constant C such that

$$p(x,y) = \begin{cases} C(4x-y+3) & \text{if } 0\le x\le 2 \text{ and } 0\le y\le 3 \\ 0 & \text{otherwise}\end{cases}$$

is a probability distribution and calculate $P(X\le 1; Y\le 2)$.

SOLUTION For $p(x,y)$ to be a probability distribution, we need:

$$\int_{x=0}^2\int_{y=0}^3 p(x,y)\,dy\,dx = 1$$

Solving we see:

$$\int_{x=0}^2\int_{y=0}^3 p(x,y)\,dy\,dx = \int_0^2\int_0^3 C(4x-y+3)\,dy\,dx$$

$$= C\int_0^2\int_0^3 4x-y+3\,dy\,dx = C\int_0^2 4xy-\frac{1}{2}y^2+3y\Big|_0^3 dx$$

$$= C\int_0^2 12x-\frac{9}{2}+9\,dx = C\int_0^2 12+\frac{9}{2}\,dx$$

$$= C\left(6x^2+\frac{9}{2}x\right)\Big|_0^2 = C\,(24+9) = 33C$$

Therefore we have $C = 1/33$.

Now to compute $P(X\le 1; Y\le 2)$ we have to write

$$\int_0^1\int_0^2 \frac{1}{33}(4x-y+3)\,dy\,dx = \frac{1}{33}\int_0^1 4xy-\frac{1}{2}y^2+3y\Big|_0^2 dx$$

$$= \frac{1}{33}\int_0^1 8x-2+6\,dx = \frac{1}{33}\left(4x^2+4x\right)\Big|_0^1 = \frac{8}{33}$$

53. Calculate $P(3X+2Y\ge 6)$ for the probability density in Exercise 52.

SOLUTION Previously we found $p(x,y)=\frac{1}{33}(4x-y+3)$. Then using $P(3X+2Y\ge 6)$ we want to find $1-P(3X+2Y\le 6)$. Hence we need to integrate the following:

$$P(3X+2Y\le 6) = \int_{x=0}^2\int_{y=0}^{3-3/2x}\frac{1}{33}(4x-y+3)\,dy\,dx$$

$$= \frac{1}{33}\int_0^2 4xy-\frac{1}{2}y^2+3y\Big|_0^{3-3/2x} dx$$

$$= \frac{1}{33}\int_0^2 4x\left(3-\frac{3}{2}x\right)-\frac{1}{2}\left(3-\frac{3}{2}x\right)^2+3\left(3-\frac{3}{2}x\right)\,dx$$

$$= \frac{1}{33}\int_0^2 -\frac{57}{8}x^2 + 12x + \frac{9}{2}\,dx$$
$$= \frac{1}{33}\left(-\frac{57}{24}x^3 + 6x^2 + \frac{9}{2}x\right)\Big|_0^2$$
$$= \frac{1}{33}(-19 + 24 + 9) = \frac{14}{33}$$

Thus we have:

$$P(3X + 2Y \geq 6) = 1 - P(3X + 2Y \leq 6) = 1 - \frac{14}{33} = \frac{19}{33}$$

54. The lifetimes X and Y (in years) of two machine components have joint probability density

$$p(x, y) = \begin{cases} \frac{6}{125}(5 - x - y) & \text{if } 0 \leq x \leq 5 - y \text{ and } 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that both components are still functioning after 2 years?

SOLUTION This problem asks for $P(X \geq 2; Y \geq 2)$. We will find $P(X \geq 2; Y \geq 2)$ using the following inequalities on x and y:

$$2 \leq x \leq 5 - y, \quad 2 \leq y \leq 5$$

$$P(X \geq 2; Y \geq 2) = \int_{y=2}^{5}\int_{x=2}^{5-y} p(x, y)\,dx\,dy = \frac{6}{125}\int_{y=2}^{5}\int_{x=2}^{5-y} 5 - x - y\,dx\,dy$$
$$= \frac{6}{125}\int_2^5 5x - \frac{1}{2}x^2 - xy\Big|_2^{5-y}\,dy$$
$$= \frac{6}{125}\int_2^5 5(5 - y) - \frac{1}{2}(5 - y)^2 - (5 - y)y - [10 - 2 - 2y]\,dy$$
$$= \frac{6}{125}\int_2^5 \frac{9}{2} - 3y + \frac{1}{2}y^2\,dy$$
$$= \frac{6}{125}\left(\frac{9}{2}y - \frac{3}{2}y^2 + \frac{1}{6}y^3\right)\Big|_2^5$$
$$= \frac{6}{125}\left(\frac{45}{2} - \frac{75}{2} + \frac{125}{6}\right) - \frac{6}{125}\left(9 - 6 + \frac{4}{3}\right) = \frac{9}{125}$$

Hence,

$$P(X \geq 2; Y \geq 2) = \frac{9}{125}$$

55. An insurance company issues two kinds of policies A and B. Let X be the time until the next claim of type A is filed, and let Y be the time (in days) until the next claim of type B is filed. The random variables have joint probability density

$$p(x, y) = 12e^{-4x-3y}$$

Find the probability that $X \leq Y$.

SOLUTION We must compute

$$P(X \leq Y) = \int_{x=0}^{\infty}\int_{y=x}^{\infty} p(x, y)\,dy\,dx$$

Now evaluating we get:

$$P(X \leq Y) = \int_{x=0}^{\infty}\int_{y=x}^{\infty} 12e^{-4x-3y}\,dy\,dx = 12\int_0^{\infty}\int_x^{\infty} e^{-4x}e^{-3y}\,dy\,dx$$
$$= 12\int_0^{\infty} e^{-4x}\left(-\frac{1}{3}e^{-3y}\right)\Big|_x^{\infty}\,dx = -4\int_0^{\infty} e^{-4x}\left(\lim_{t\to\infty} e^{-3t} - e^{-3x}\right)dx$$
$$= 4\int_0^{\infty} e^{-4x}\cdot e^{-3x}\,dx = 4\int_0^{\infty} e^{-7x}\,dx$$
$$= -\frac{4}{7}\left(e^{-7x}\right)\Big|_0^{\infty} = -\frac{4}{7}\lim_{t\to\infty}\left(e^{-7t} - 1\right) = \frac{4}{7}$$

56. Compute the Jacobian of the map

$$\Phi(r,s) = \left(e^r\cosh(s), e^r\sinh(s)\right)$$

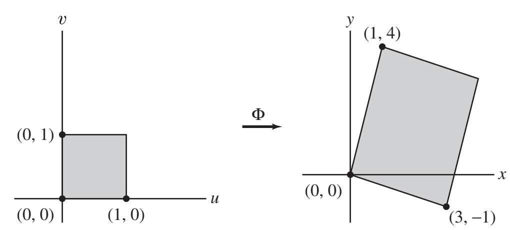
SOLUTION We have $x = e^r\cosh(s)$ and $y = e^r\sinh(s)$. Therefore,

$$\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} e^r\cosh(s) & e^r\sinh(s) \\ e^r\sinh(s) & e^r\cosh(s) \end{vmatrix}$$
$$= e^{2r}\cosh^2(s) - e^{2r}\sinh^2(s) = e^{2r}(\cosh^2(s) - \sinh^2(s)) = e^{2r}$$

57. Find a linear mapping $\Phi(u,v)$ that maps the unit square to the parallelogram in the xy-plane spanned by the vectors $\langle 3,-1\rangle$ and $\langle 1,4\rangle$. Then, use the Jacobian to find the area of the image of the rectangle $\mathcal{R} = [0,4]\times[0,3]$ under Φ.

SOLUTION We denote the linear map by

$$G(u,v) = (Au + Cv, Bu + Dv) \tag{1}$$



The image of the unit square is the quadrangle whose vertices are the images of the vertices of the square. Therefore we ask that

$$\begin{aligned} G(0,0) &= (A\cdot 0 + C\cdot 0, B\cdot 0 + D\cdot 0) = (0,0) & & (0,0) = (0,0) \\ G(1,0) &= (A\cdot 1 + C\cdot 0, B\cdot 1 + D\cdot 0) = (3,-1) & \Rightarrow\quad & (A,B) = (3,-1) \\ G(0,1) &= (A\cdot 0 + C\cdot 1, B\cdot 0 + D\cdot 1) = (1,4) & & (C,D) = (1,4) \end{aligned}$$

These equations imply that $A = 3$, $B = -1$, $C = 1$, and $D = 4$. Substituting in (1) we obtain the following map:

$$G(u,v) = (3u + v, -u + 4v)$$

The area of the rectangle $R = [0,4]\times[0,3]$ is $4\cdot 3 = 12$, therefore the transformed area is

$$\text{Area} = |\mathrm{Jac}(G)|\cdot 12$$

The Jacobian of the linear map G is

$$\mathrm{Jac}\,(G) = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 12 - (-1) = 13$$

Therefore,

$$\text{Area} = 13\cdot 12 = 156.$$

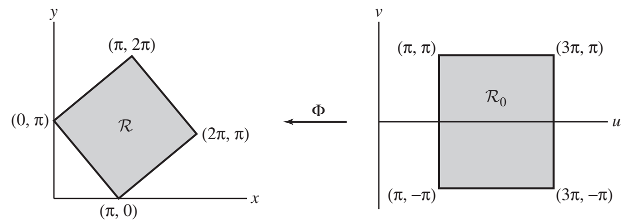
58. Use the map

$$\Phi(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

to compute $\iint_{\mathcal{R}} \left((x-y)\sin(x+y)\right)^2\,dx\,dy$, where $\mathcal{R}$ is the square with vertices $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, and $(0,\pi)$.

SOLUTION We express $f(x,y) = ((x-y)\sin(x+y))^2$ in terms of u and v. Since $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, we have $x - y = v$ and $x + y = u$. Hence, $f(x,y) = v^2\sin^2 u$. We find the Jacobian of the linear transformation:

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



To compute the vertices of the quadrangle $\mathcal{R}$ mapped by Φ onto $\mathcal{R}$, we first find the inverse of Φ by solving the following equations for u, v in terms of x and y:

$$\begin{array}{l} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{array} \quad \Rightarrow \quad \begin{array}{l} u+v = 2x \\ u-v = 2y \end{array} \quad \Rightarrow \quad u = x+y, \quad v = x-y$$

Hence,

$$\Phi^{-1}(x, y) = (x+y, x-y)$$

We now compute the vertices of P as the following images:

$$\begin{aligned} G^{-1}(\pi, 0) &= (\pi, \pi) \\ G^{-1}(2\pi, \pi) &= (3\pi, \pi) \\ G^{-1}(\pi, 2\pi) &= (3\pi, -\pi) \\ G^{-1}(0, \pi) &= (\pi, -\pi) \end{aligned}$$

Finally, we apply the change of variable formula to compute the integral:

$$\begin{aligned} \iint_{\mathcal{R}} (x-y)^2 \sin^2(x+y)\, dx\, dy &= \iint_{R_0} v^2 \sin^2 u |\text{Jac}(G)|\, du\, dv = \frac{1}{2} \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} v^2 \sin^2 u\, dv\, du \\ &= \frac{1}{2} \left(\int_{\pi}^{3\pi} (\sin^2 u)\, du \right) \left(\int_{-\pi}^{\pi} v^2\, dv \right) = \left(\frac{u}{4} - \frac{\sin 2u}{8} \Big|_{\pi}^{3\pi} \right) \left(\frac{v^3}{3} \Big|_{-\pi}^{\pi} \right) \\ &= \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \cdot \frac{2\pi^3}{3} = \frac{\pi^4}{3} \end{aligned}$$

59. Let $\mathcal{D}$ be the shaded region in Figure 6, and let F be the map

$$u = y + x^2, \qquad v = y - x^3$$

(a) Show that F maps $\mathcal{D}$ to a rectangle $\mathcal{R}$ in the uv-plane.

(b) Apply Eq. (7) in Section 15.6 with $P = (1, 7)$ to estimate Area($\mathcal{D}$).

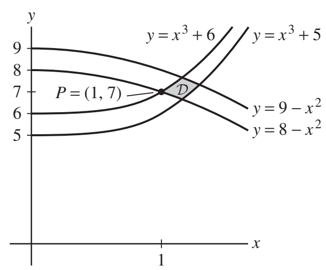
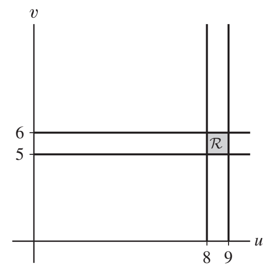


FIGURE 6

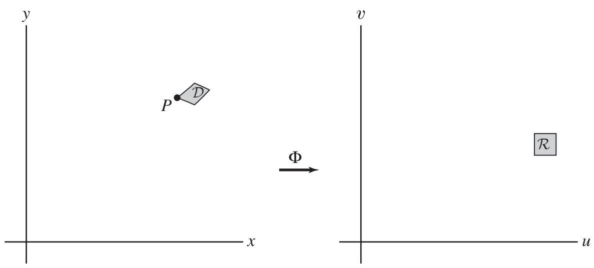
SOLUTION

(a) Note that the appropriate map should be $u = y + x^2$ rather than $u = -y + x^2$. We examine the images of the boundary curves of $\mathcal{D}$ under the map $(u, v) = \Phi(x, y) = (x^2 + y, y - x^3)$. The curves $y = x^3 + 6$ and $y = x^3 + 5$ can be rewritten as $y - x^3 = 6$ and $y - x^3 = 5$. Since $v = y - x^3$, these curves are mapped to the horizontal lines $v = 6$ and $v = 5$, respectively. The curves $y = 8 - x^2$ and $y = 9 - x^2$ can be rewritten as $y + x^2 = 8$ and $y + x^2 = 9$. Since $u = y + x^2$, these curves are mapped to the vertical lines $u = 8$ and $u = 9$, respectively. We conclude that $\mathcal{D}$ is mapped to the rectangle $\mathcal{R} = [8, 9] \times [5, 6]$ in the (u, v)-plane.



(b) We use Eq. (5) in section 16.5, where this time Φ is a mapping from the (x, y)-plane to the (u, v)-plane, and $P = (1, 7)$ is a point in $\mathcal{D}$:

$$\text{Area}\Phi(\mathcal{D}) \approx |\text{Jac}(\Phi)(P)|\text{Area}(\mathcal{D})$$



Here, $\text{Area}\Phi(\mathcal{D}) = \text{Area}(\mathcal{R}) = 1^2 = 1$, therefore we get

$$1 \approx |\text{Jac}(\Phi)(P)|\text{Area}(\mathcal{D})$$

or

$$\text{Area}(\mathcal{D}) \approx |\text{Jac}(\Phi)(P)|^{-1} \tag{1}$$

We compute the Jacobian of $\Phi(x, y) = (u, v) = (y + x^2, y - x^3)$ at $P = (1, 7)$:

$$\text{Jac}(\Phi) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 1 \\ -3x^2 & 1 \end{vmatrix} = 2x + 3x^2 \quad \Rightarrow \quad \text{Jac}(\Phi)(P) = 2 \cdot 1 + 3 \cdot 7^2 = 149$$

Combining with (1) gives

$$\text{Area}(\mathcal{D}) \approx (149)^{-1} = \frac{1}{149}.$$

60. Calculate the integral of $f(x, y) = e^{3x-2y}$ over the parallelogram in Figure 7.

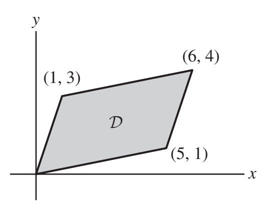
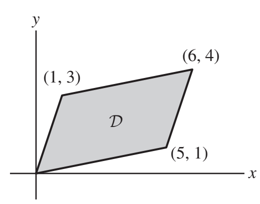


FIGURE 7

SOLUTION The equation of the boundary lines are $y = \frac{1}{5}x$, $y = \frac{1}{5}x + \frac{14}{5}$, $y = 3x$, and $y = 3x - 14$. These equations may be written as

$$x - 5y = 0, \quad x - 5y = -14, \quad 3x - y = 0, \quad 3x - y = 14$$

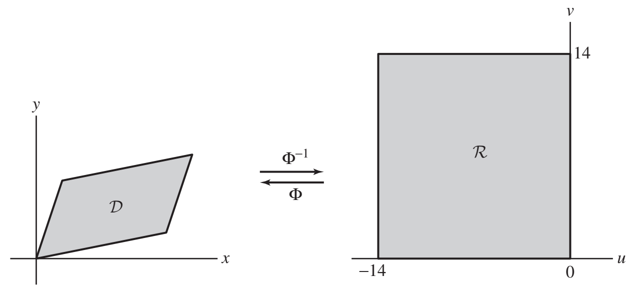


We define the following map:

$$\Phi^{-1}(x, y) = (u, v) = (x - 5y, 3x - y)$$

Φ^{-1} maps the boundary lines to the lines $u = 0$, $u = -14$, $v = 0$, and $v = 14$ in the (u, v)-plane. Therefore the image of $\mathcal{D}$ under Φ^{-1} is the rectangle $\mathcal{R} = [-14, 0] \times [0, 14]$ in the (u, v)-plane. Using the Change of variables Formula, we have

$$\iint_{\mathcal{D}} f(x, y)\, dx\, dy = \iint_{\mathcal{R}} f\left(x(u, v), y(u, v)\right) |\text{Jac}(\Phi)|\, du\, dv \qquad (1)$$



We compute the inverse Φ of Φ^{-1} by solving the equations $u = x - 5y$, $v = 3x - y$ for x, y in terms of u, v. We get

$$\begin{aligned} u &= x - 5y \\ v &= 3x - y \end{aligned} \quad \Rightarrow \quad x = \frac{-u + 5v}{14}, \quad y = \frac{-3u + v}{14}$$

Therefore the function $f(x, y) = e^{3x-2y}$ in terms of u and v is

$$f\left(x(u, v), y(u, v)\right) = e^{3\left(\frac{-u+5v}{14}\right) - 2\left(\frac{-3u+v}{14}\right)} = e^{\frac{3u+13v}{14}} \qquad (2)$$

The Jacobian of $\Phi(u, v) = (x, y) = \left(\frac{-u+5v}{14}, \frac{-3u+v}{14}\right)$ is

$$\text{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{14} & \frac{5}{14} \\ -\frac{3}{14} & \frac{1}{14} \end{vmatrix} = \frac{1}{14} \qquad (3)$$

Substituting (2) and (3) in (1) gives

$$\begin{aligned} \iint_{\mathcal{D}} e^{3x-2y}\, dx\, dy &= \int_0^{14} \int_{-14}^{0} e^{\frac{3u+13v}{14}} \cdot \frac{1}{14}\, du\, dv = \frac{1}{14} \int_0^{14} \int_{-14}^{0} e^{\frac{3u}{14}} \cdot e^{\frac{13v}{14}}\, du\, dv \\ &= \frac{1}{14} \left(\int_{-14}^{0} e^{\frac{3u}{14}}\, du \right) \left(\int_0^{14} e^{\frac{13v}{14}}\, dv \right) = \frac{1}{14} \left(\frac{14}{3} e^{\frac{3u}{14}} \Big|_{u=-14}^{0} \right) \left(\frac{14}{13} e^{\frac{13v}{14}} \Big|_{v=0}^{14} \right) \\ &= \frac{1}{3}(1 - e^{-3}) \cdot \frac{14}{13}(e^{13} - 1) = \frac{14}{39}(1 - e^{-3})(e^{13} - 1) \end{aligned}$$

61. Sketch the region $\mathcal{D}$ bounded by the curves $y = 2/x$, $y = 1/(2x)$, $y = 2x$, $y = x/2$ in the first quadrant. Let F be the map $u = xy$, $v = y/x$ from the xy-plane to the uv-plane.

(a) Find the image of $\mathcal{D}$ under F.

(b) Let $\Phi = F^{-1}$. Show that $|\text{Jac}(\Phi)| = \dfrac{1}{2|v|}$.

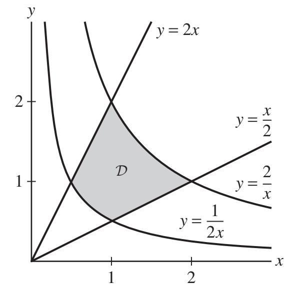
(c) Apply the Change of Variables Formula to prove the formula

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right) dx\, dy = \frac{3}{4} \int_{1/2}^{2} \frac{f(v)\, dv}{v}$$

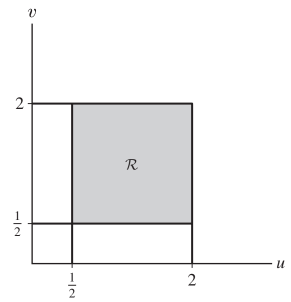
(d) Apply (c) to evaluate $\displaystyle\iint_{\mathcal{D}} \frac{ye^{y/x}}{x}\, dx\, dy$.

SOLUTION

(a) The region $\mathcal{D}$ is shown in the figure:



We rewrite the equations of the boundary curves as $xy = 2$, $xy = \frac{1}{2}$, $\frac{y}{x} = 2$, and $\frac{y}{x} = \frac{1}{2}$. These curves are mapped by Φ to the lines $u = 2$, $u = \frac{1}{2}$, $v = 2$, and $v = \frac{1}{2}$. Therefore, the image of $\mathcal{D}$ is the rectangle $\mathcal{R} = \left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right]$ in the (u, v)-plane.



(b) We use the Jacobian of the inverse map:

$$\text{Jac}(F^{-1}) = (\text{Jac}(F))^{-1}$$

We compute the Jacobian of $F(x, y) = (u, v) = \left(xy, \frac{y}{x}\right)$:

$$\text{Jac}(F) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{yx}{x^2} = \frac{2y}{x} = 2v$$

(Note that everything is positive, so we don't need absolute values!) Thus,

$$\text{Jac}(F^{-1}) = (\text{Jac}(F))^{-1} = \frac{1}{2v} = \frac{1}{|2v|}.$$

(c) The general change of variables formula is

$$\iint_{\mathcal{D}} f(x, y)\, dA = \iint_{\mathcal{R}} f\left(x(u, v), y(u, v)\right) |\text{Jac}(F^{-1})(u, v)|\, du\, dv$$

Here, $f\left(\frac{y}{x}\right) = f(v)$, $\mathcal{R} = \left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right]$ in the (u, v)-plane and $|\text{Jac}(F^{-1})(u, v)| = |\frac{1}{2v}| = \frac{1}{2v}$ ($v > 0$ in $\mathcal{R}$). Therefore, we have

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right) dA = \int_{1/2}^{2} \int_{1/2}^{2} f(v) \cdot \frac{1}{2v}\, du\, dv = \left(\int_{1/2}^{2} 1\, du\right) \left(\int_{1/2}^{2} \frac{f(v)}{2v}\, dv\right) = \frac{3}{4} \int_{1/2}^{2} \frac{f(v)}{v}\, dv$$

(d) We use part (c) with $f\left(\frac{y}{x}\right) = \frac{y}{x} \cdot e^{y/x}$. We have $f(v) = v \cdot e^v$, hence

$$\iint_{\mathcal{D}} \frac{ye^{y/x}}{x}\, dx\, dy = \frac{3}{4} \int_{1/2}^{2} \frac{ve^v}{v}\, dv = \frac{3}{4} \int_{1/2}^{2} e^v\, dv = \frac{3}{4} e^v \Big|_{1/2}^{2} = \frac{3}{4}(e^2 - e^{1/2}) = \frac{3}{4}\left(e^2 - \sqrt{e}\right)$$

16 LINE AND SURFACE INTEGRALS

16.1 Vector Fields (LT Section 17.1)

Preliminary Questions

1. Which of the following is a unit vector field in the plane?

(a) $\mathbf{F} = \langle y, x\rangle$

(b) $\mathbf{F} = \left\langle \dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$

(c) $\mathbf{F} = \left\langle \dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

SOLUTION

(a) The length of the vector $\langle y, x\rangle$ is

$$\|\langle y, x\rangle\| = \sqrt{y^2+x^2}$$

This value is not 1 for all points, hence it is not a unit vector field.

(b) We have

$$\left\|\left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle\right\| = \sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}$$

$$= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} = \sqrt{\frac{y^2+x^2}{x^2+y^2}} = 1$$

Hence the field is a unit vector field, for $(x, y) \neq (0, 0)$.

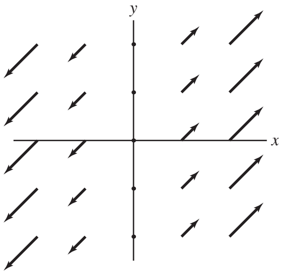
(c) We compute the length of the vector:

$$\left\|\left\langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle\right\| = \sqrt{\left(\frac{y}{x^2+y^2}\right)^2 + \left(\frac{x}{x^2+y^2}\right)^2} = \sqrt{\frac{y^2+x^2}{\left(x^2+y^2\right)^2}} = \sqrt{\frac{1}{x^2+y^2}}$$

Since the length is not identically 1, the field is not a unit vector field.

2. Sketch an example of a nonconstant vector field in the plane in which each vector is parallel to $\langle 1, 1\rangle$.

SOLUTION The non-constant vector $\langle x, x\rangle$ is parallel to the vector $\langle 1, 1\rangle$.



3. Show that the vector field $\mathbf{F} = \langle -z, 0, x\rangle$ is orthogonal to the position vector $\overrightarrow{OP}$ at each point P. Give an example of another vector field with this property.

SOLUTION The position vector at $P = (x, y, z)$ is $\langle x, y, z\rangle$. We must show that the following dot product is zero:

$$\langle x, y, z\rangle \cdot \langle -z, 0, x\rangle = x \cdot (-z) + y \cdot 0 + z \cdot x = 0$$

Therefore, the vector field $\mathbf{F} = \langle -z, 0, x\rangle$ is orthogonal to the position vector. Another vector field with this property is $\mathbf{F} = \langle 0, -z, y\rangle$, since

$$\langle 0, -z, y\rangle \cdot \langle x, y, z\rangle = 0 \cdot x + (-z) \cdot y + y \cdot z = 0$$

4. Give an example of a potential function for $\langle yz, xz, xy\rangle$ other than $f(x, y, z) = xyz$.

SOLUTION Since any two potential functions of a gradient vector field differ by a constant, a potential function for the given field other than $V(x, y, z) = xyz$ is, for instance, $V_1(x, y, z) = xyz + 1$.

Exercises

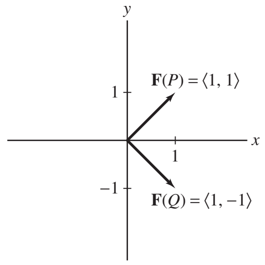
1. Compute and sketch the vector assigned to the points $P = (1, 2)$ and $Q = (-1, -1)$ by the vector field $\mathbf{F} = \langle x^2, x\rangle$.

SOLUTION The vector assigned to $P = (1, 2)$ is obtained by substituting $x = 1$ in $\mathbf{F}$, that is,

$$\mathbf{F}(1, 2) = \langle 1^2, 1\rangle = \langle 1, 1\rangle$$

Similarly,

$$\mathbf{F}(-1, -1) = \langle (-1)^2, -1\rangle = \langle 1, -1\rangle$$



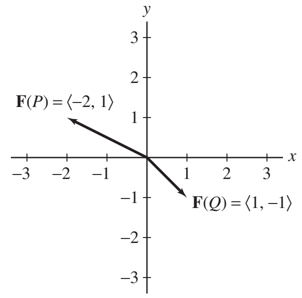
2. Compute and sketch the vector assigned to the points $P = (1, 2)$ and $Q = (-1, -1)$ by the vector field $\mathbf{F} = \langle -y, x\rangle$.

SOLUTION To find the vector assigned to the point $P = (1, 2)$, we substitute $x = 1$, $y = 2$ in $\mathbf{F} = \langle -y, x\rangle$, obtaining

$$\mathbf{F}(P) = \langle -2, 1\rangle$$

Similarly, the vector assigned to $Q = (-1, -1)$ is

$$\mathbf{F}(Q) = \langle -(-1), -1\rangle = \langle 1, -1\rangle$$



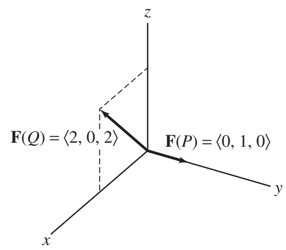
3. Compute and sketch the vector assigned to the points $P = (0, 1, 1)$ and $Q = (2, 1, 0)$ by the vector field $\mathbf{F} = \langle xy, z^2, x\rangle$.

SOLUTION To find the vector assigned to the point $P = (0, 1, 1)$, we substitute $x = 0$, $y = 1$, $z = 1$ in $\mathbf{F} = \langle xy, z^2, x\rangle$. We get

$$\mathbf{F}(P) = \langle 0 \cdot 1, 1^2, 0\rangle = \langle 0, 1, 0\rangle$$

Similarly, $\mathbf{F}(Q)$ is obtained by substituting $x = 2$, $y = 1$, $z = 0$ in $\mathbf{F}$. That is,

$$\mathbf{F}(Q) = \langle 2 \cdot 1, 0^2, 2\rangle = \langle 2, 0, 2\rangle$$



4. Compute the vector assigned to the points $P = (1, 1, 0)$ and $Q = (2, 1, 2)$ by the vector fields $\mathbf{e}_r$, $\frac{\mathbf{e}_r}{r}$, and $\frac{\mathbf{e}_r}{r^2}$.

SOLUTION The unit radial vector is

$$\mathbf{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$$

Hence,

$$\frac{\mathbf{e}_r}{r} = \left\langle \frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right\rangle \quad \text{and} \quad \frac{\mathbf{e}_r}{r^2} = \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle.$$

For $P = (1, 1, 0)$ we have $r = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$, and for $Q = (2, 1, 2)$ we have $r = \sqrt{2^2 + 1^2 + 2^2} = 3$. Therefore,

$$\mathbf{e}_r(P) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{0}{\sqrt{2}} \right\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$\mathbf{e}_r(Q) = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

$$\frac{\mathbf{e}_r}{r}(P) = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{0}{2} \right\rangle = \left\langle \frac{1}{2}, \frac{1}{2}, 0 \right\rangle$$

$$\frac{\mathbf{e}_r}{r}(Q) = \left\langle \frac{2}{9}, \frac{1}{9}, \frac{2}{9} \right\rangle$$

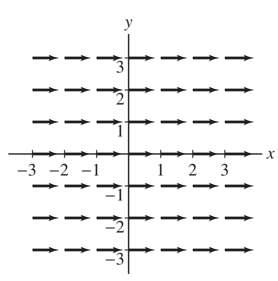
$$\frac{\mathbf{e}_r}{r^2}(P) = \left\langle \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{0}{2\sqrt{2}} \right\rangle = \left\langle \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0 \right\rangle$$

$$\frac{\mathbf{e}_r}{r^2}(Q) = \left\langle \frac{2}{27}, \frac{1}{27}, \frac{2}{27} \right\rangle$$

In Exercises 5–12, sketch the following planar vector fields by drawing the vectors attached to points with integer coordinates in the rectangle $-3 \le x \le 3$, $-3 \le y \le 3$. Instead of drawing the vectors with their true lengths, scale them if necessary to avoid overlap.

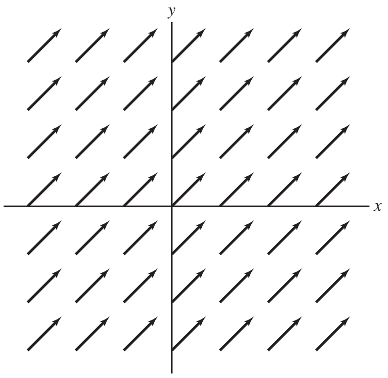
5. $\mathbf{F} = \langle 1, 0\rangle$

SOLUTION The constant vector field $\langle 1, 0\rangle$ is shown in the figure:



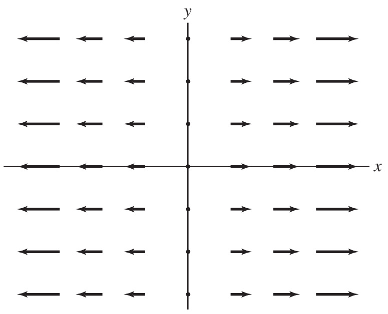
6. $\mathbf{F} = \langle 1, 1\rangle$

SOLUTION We sketch the graph of the constant vector field $\mathbf{F}(x, y) = \langle 1, 1\rangle$:



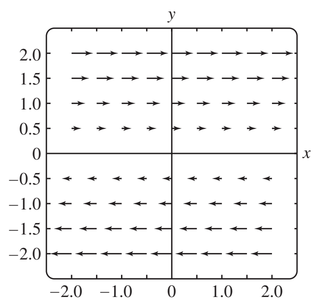
7. $\mathbf{F} = x\mathbf{i}$

SOLUTION The vector field $\mathbf{F}(x, y) = x\mathbf{i} = (x, 0)$ is sketched in the following figure:



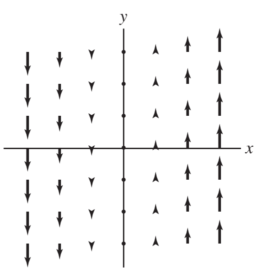
8. $\mathbf{F} = y\mathbf{i}$

SOLUTION



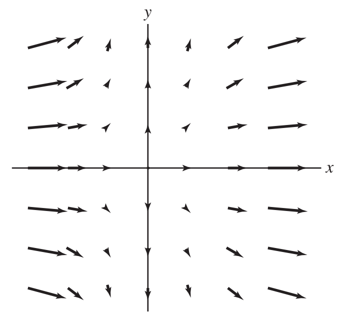
9. $\mathbf{F} = \langle 0, x\rangle$

SOLUTION We sketch the vector field $\mathbf{F}(x, y) = \langle 0, x\rangle$:



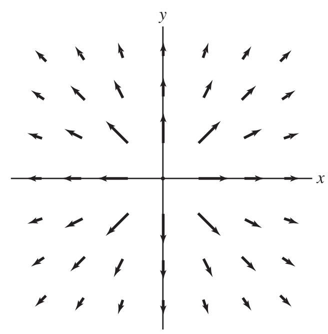
10. $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$

SOLUTION The graph of the vector field $\mathbf{F}(x, y) = x^2\mathbf{i} + y\mathbf{j}$ is shown in the following figure:



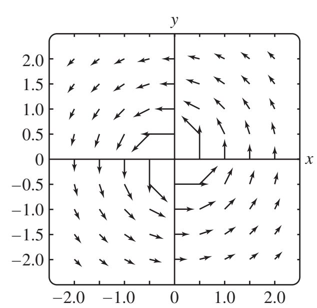
11. $\mathbf{F} = \left\langle \dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2} \right\rangle$

SOLUTION



12. $\mathbf{F} = \left\langle \dfrac{-y}{\sqrt{x^2 + y^2}}, \dfrac{x}{\sqrt{x^2 + y^2}} \right\rangle$

SOLUTION



In Exercises 13–16, match each of the following planar vector fields with the corresponding plot in Figure 10.

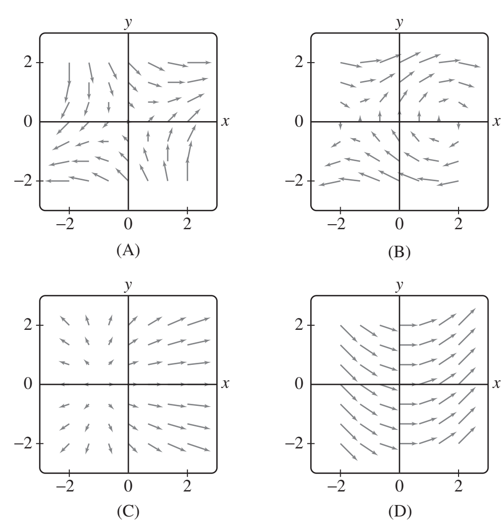


FIGURE 10

13. $\mathbf{F} = \langle 2, x \rangle$

SOLUTION The x coordinate of the vector field $\langle 2, x \rangle$ is always 2. This matches only with Plot (D).

14. $\mathbf{F} = \langle 2x + 2, y \rangle$

SOLUTION We compute the images of the point $(0, 2)$, for instance, and identify the corresponding graph accordingly:

$$\mathbf{F}(x, y) = \langle 2x + 2, y \rangle \quad \Rightarrow \quad \mathbf{F}(0, 2) = \langle 2, 2 \rangle \quad \Rightarrow \quad \text{Plot(C)}$$

15. $\mathbf{F} = \langle y, \cos x \rangle$

SOLUTION We compute the images of the point $(0, 2)$, for instance, and identify the corresponding graph accordingly:

$$\mathbf{F}(x, y) = \langle y, \cos x \rangle \quad \Rightarrow \quad \mathbf{F}(0, 2) = \langle 2, 1 \rangle \quad \Rightarrow \quad \text{Plot(B)}$$

16. $\mathbf{F} = \langle x + y, x - y \rangle$

SOLUTION We compute the images of the point $(0, 2)$, for instance, and identify the corresponding graph accordingly:

$$\mathbf{F}(x, y) = \langle x + y, x - y \rangle \quad \Rightarrow \quad \mathbf{F}(0, 2) = \langle 2, -2 \rangle \quad \Rightarrow \quad \text{Plot(A)}$$

In Exercises 17–20, match each three-dimensional vector field with the corresponding plot in Figure 11.

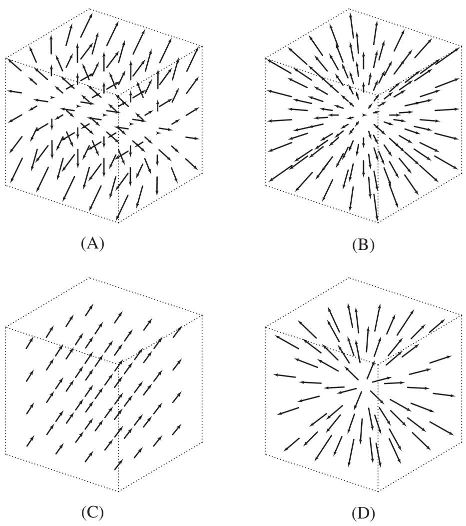


FIGURE 11

17. $\mathbf{F} = \langle 1, 1, 1 \rangle$

SOLUTION The constant vector field $\langle 1, 1, 1 \rangle$ is shown in plot (C).

18. $\mathbf{F} = \langle x, 0, z \rangle$

SOLUTION This vector field is shown in (A) (by process of elimination).

19. $\mathbf{F} = \langle x, y, z \rangle$

SOLUTION $\langle x, y, z \rangle$ is shown in plot (B). Note that the vectors are pointing away from the origin and are of increasing magnitude.

20. $\mathbf{F} = \mathbf{e}_r$

SOLUTION The unit radial vector field is shown in plot (D), as these vectors are radial and of uniform length.

21. Find (by inspection) a potential function for $\mathbf{F} = \langle x, 0 \rangle$ and prove that $\mathbf{G} = \langle y, 0 \rangle$ is not conservative.

SOLUTION For $f(x, y) = \frac{1}{2}x^2$ we have $\nabla f = \langle x, 0 \rangle$.

$$\frac{\partial G_1}{\partial y} = 1 \neq \frac{\partial G_2}{\partial x} = 0$$

Thus $\mathbf{G}$ is not conservative.

22. Prove that $\mathbf{F} = \langle yz, xz, y \rangle$ is not conservative.

SOLUTION

$$\frac{\partial F_2}{\partial z} = x \neq \frac{\partial F_3}{\partial y} = 1$$

Thus $\mathbf{F}$ is not conservative.

In Exercises 23–26, find a potential function for the vector field $\mathbf{F}$ *by inspection.*

23. $\mathbf{F} = \langle x, y \rangle$

SOLUTION We must find a function $\varphi(x, y)$ such that $\frac{\partial \varphi}{\partial x} = x$ and $\frac{\partial \varphi}{\partial y} = y$. We choose the following function:

$$\varphi(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

24. $\mathbf{F} = \left\langle ye^{xy}, xe^{xy} \right\rangle$

SOLUTION The function $\varphi(x, y) = e^{xy}$ satisfies $\frac{\partial \varphi}{\partial x} = ye^{xy}$ and $\frac{\partial \varphi}{\partial y} = xe^{xy}$, hence φ is a potential function for the given vector field.

25. $\mathbf{F} = \left\langle yz^2, xz^2, 2xyz \right\rangle$

SOLUTION We choose a function $\varphi(x, y, z)$ such that

$$\frac{\partial \varphi}{\partial x} = yz^2, \quad \frac{\partial \varphi}{\partial y} = xz^2, \quad \frac{\partial \varphi}{\partial z} = xyz$$

The function $\varphi(x, y, z) = xyz^2$ is a potential function for the given field.

26. $\mathbf{F} = \left\langle 2xze^{x^2}, 0, e^{x^2} \right\rangle$

SOLUTION The function $\varphi(x, y, z) = ze^{x^2}$ satisfies $\frac{\partial \varphi}{\partial y} = 0$, $\frac{\partial \varphi}{\partial x} = 2xze^{x^2}$ and $\frac{\partial \varphi}{\partial z} = e^{x^2}$, hence φ is a potential function for the given vector field.

27. Find potential functions for $\mathbf{F} = \frac{\mathbf{e}_r}{r^3}$ and $\mathbf{G} = \frac{\mathbf{e}_r}{r^4}$ in $\mathbf{R}^3$. *Hint:* See Example 6.

SOLUTION We use the gradient of r, $\nabla r = \mathbf{e}_r$, and the Chain Rule for Gradients to write

$$\nabla\left(-\frac{1}{2}r^{-2}\right) = r^{-3}\nabla r = r^{-3}\mathbf{e}_r = \frac{\mathbf{e}_r}{r^3} = \mathbf{F}$$

$$\nabla\left(-\frac{1}{3}r^{-3}\right) = r^{-4}\nabla r = r^{-4}\mathbf{e}_r = \frac{\mathbf{e}_r}{r^4} = \mathbf{G}$$

Therefore $\varphi_1(r) = -\frac{1}{2r^2}$ and $\varphi_2(r) = -\frac{1}{3r^3}$ are potential functions for $\mathbf{F}$ and $\mathbf{G}$, respectively.

28. Show that $\mathbf{F} = \langle 3, 1, 2 \rangle$ is conservative. Then prove more generally that any constant vector field $\mathbf{F} = \langle a, b, c \rangle$ is conservative.

SOLUTION $\mathbf{F} = \nabla\varphi$ for $\varphi(x, y, z) = 3x + y + 2z$. Further for $\mathbf{F} = \langle a, b, c \rangle$, $\mathbf{F} = \nabla\varphi$ for $\varphi(x, y, z) = ax + by + cz$.

29. Let $\varphi = \ln r$, where $r = \sqrt{x^2 + y^2}$. Express $\nabla\varphi$ in terms of the unit radial vector $\mathbf{e}_r$ in $\mathbf{R}^2$.

SOLUTION Since $r = (x^2 + y^2 + z^2)^{1/2}$, we have $\varphi = \ln (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2}\ln(x^2 + y^2 + z^2)$. We compute the partial derivatives:

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2}\frac{2x}{x^2 + y^2 + z^2} = \frac{x}{r^2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{2}\frac{2y}{x^2 + y^2 + z^2} = \frac{y}{r^2}$$

$$\frac{\partial \varphi}{\partial z} = \frac{1}{2}\frac{2z}{x^2 + y^2 + z^2} = \frac{z}{r^2}$$

Therefore, the gradient of φ is the following vector:

$$\nabla\varphi = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = \left\langle \frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right\rangle = \frac{1}{r}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \frac{\mathbf{e}_r}{r}$$

30. For $P = (a, b)$, we define the unit radial vector field based at P:

$$\mathbf{e}_P = \frac{\langle x-a, y-b\rangle}{\sqrt{(x-a)^2+(y-b)^2}}$$

(a) Verify that $\mathbf{e}_P$ is a unit vector field.
(b) Calculate $\mathbf{e}_P(1, 1)$ for $P = (3, 2)$.
(c) Find a potential function for $\mathbf{e}_P$.

SOLUTION
(a)

$$\mathbf{e}_P \cdot \mathbf{e}_P = \frac{(x-a)^2+(y-b)^2}{\left(\sqrt{(x-a)^2+(y-b)^2}\right)^2} = 1$$

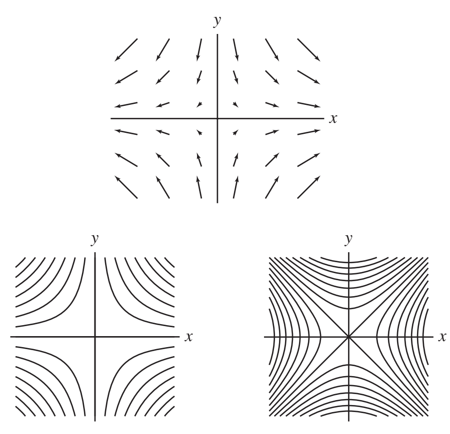
(b)

$$\mathbf{e}_{(3,2)}(1, 1) = \frac{\langle 1-3, 1-2\rangle}{\sqrt{(1-3)^2+(1-2)^2}} = \frac{\langle -2, -1\rangle}{\sqrt{5}}$$

(c) Let $\varphi(x, y) = \sqrt{(x-a)^2+(y-b)^2}$. Then,

$$\nabla\varphi = \frac{1}{2}((x-a)^2+(y-b)^2)^{-1/2}\langle 2(x-a), 2(y-b)\rangle = \mathbf{e}_P$$

31. Which of (A) or (B) in Figure 12 is the contour plot of a potential function for the vector field $\mathbf{F}$? Recall that the gradient vectors are perpendicular to the level curves.

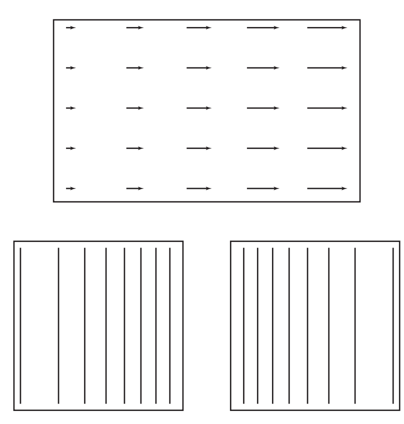


(A) (B)

FIGURE 12

SOLUTION By the equality $\nabla\varphi = \mathbf{F}$ and since the gradient vectors are perpendicular to the level curves, it follows that the vectors $\mathbf{F}$ are perpendicular to the corresponding level curves of φ. This property is satisfied in (B) and not satisfied in (A). Therefore (B) is the contour plot of φ.

32. Which of (A) or (B) in Figure 13 is the contour plot of a potential function for the vector field $\mathbf{F}$?



(A) (B)

FIGURE 13

SOLUTION Since $\nabla\varphi = \mathbf{F}$, $\mathbf{F}$ is perpendicular to the level curves of φ, as seen in both the contour plots (A) and (B). The plot of $\mathbf{F}$ shows that $\mathbf{F}$ has the form $\mathbf{F} = \langle f(x), 0\rangle$ for an increasing function $f(x)$. Therefore, $\frac{\partial\varphi}{\partial x} = f(x)$ is increasing, implying that the rate of change of φ with respect to x is increasing. Hence, the density of the vertical lines is greater in the direction of growing x. We conclude that (A) is the contour plot of φ.

33. Match each of these descriptions with a vector field in Figure 14:

(a) The gravitational field created by two planets of equal mass located at P and Q.

(b) The electrostatic field created by two equal and opposite charges located at P and Q (representing the force on a negative test charge; opposite charges attract and like charges repel).

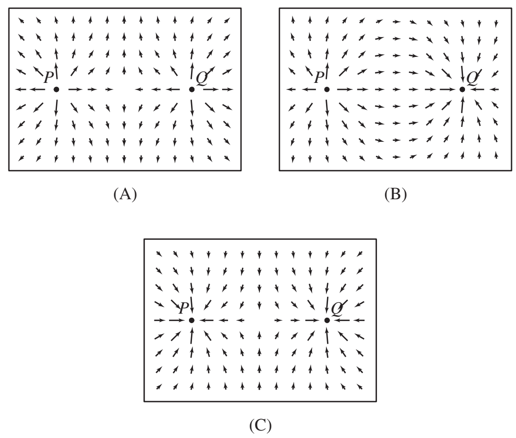


FIGURE 14

SOLUTION

(a) There will be an equilibrium point half way between the two planets. The vector field should pull objects near one planet toward that planet. (C)

(b) A test charge at the midpoint between the two charges will be drawn by one, and repelled by the other. Therefore no equilibrium. (B)

34. In this exercise, we show that the vector field $\mathbf{F}$ in Figure 15 is not conservative. Explain the following statements:

(a) If a potential function V for $\mathbf{F}$ exists, then the level curves of V must be vertical lines.

(b) If a potential function V for $\mathbf{F}$ exists, then the level curves of V must grow farther apart as y increases.

(c) Explain why (a) and (b) are incompatible, and hence V cannot exist.

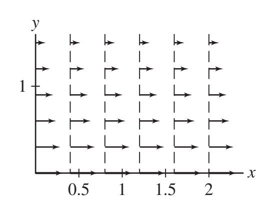


FIGURE 15

SOLUTION

(a) If V is a potential function for $\mathbf{F}$, then $\nabla V = \mathbf{F}$. Therefore, $\mathbf{F}$ is orthogonal to the level curves of V. The plot shows that the vectors $\mathbf{F}(x, y)$ are horizontal, hence the level curves of V are vertical lines.

(b) As indicated by the graph of the vector field $\mathbf{F} = \langle F_1, F_2\rangle$, $F_2 = 0$ and F_1 is decreasing as y increases. Since $\mathbf{F} = \frac{\partial V}{\partial x}$, it follows that $\frac{\partial V}{\partial x}$ is decreasing as y increases, or the rate of change of V with respect to x is decreasing as y increases. Therefore, as y increases, the level curves of V are getting farther apart.

(c) By (a), the level curves of V are vertical lines, hence the distance between any two level curves is constant rather than increasing with y as concluded in (b).

Further Insights and Challenges

35. Show that any vector field of the form

$$\mathbf{F} = \langle f(x), g(y), h(z) \rangle$$

has a potential function. Assume that f, g, and h are continuous.

SOLUTION Let $F(x)$, $G(y)$, and $H(z)$ be antiderivatives of $f(x)$, $g(y)$, and $h(z)$, respectively. That is, $F'(x) = f(x)$, $G'(y) = g(y)$, and $H'(y) = h(z)$. We define the function

$$\varphi(x, y, z) = F(x) + G(y) + H(z)$$

Then,

$$\frac{\partial \varphi}{\partial x} = F'(x) = f(x), \quad \frac{\partial \varphi}{\partial x} = G'(y) = g(y), \quad \frac{\partial \varphi}{\partial z} = H'(z) = h(z)$$

Therefore, $\nabla \varphi = \mathbf{F}$, or φ is a potential function for $\mathbf{F}$.

36. Let $\mathcal{D}$ be a disk in $\mathbf{R}^2$. This exercise shows that if

$$\nabla V(x, y) = \mathbf{0}$$

for all (x, y) in $\mathcal{D}$, then V is constant. Consider points $P = (a, b)$, $Q = (c, d)$ and $R = (c, b)$ as in Figure 16.

(a) Use single-variable calculus to show that V is constant along the segments $\overline{PR}$ and $\overline{RQ}$.

(b) Conclude that $V(P) = V(Q)$ for any two points $P, Q \in \mathcal{D}$.

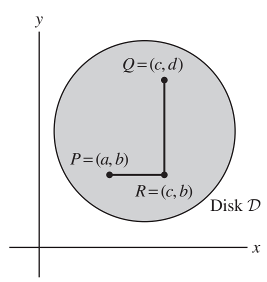
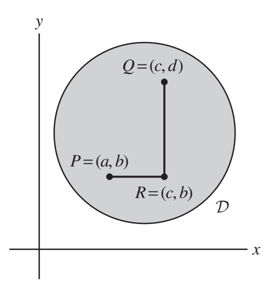


FIGURE 16

SOLUTION Given any two points $P = (a, b)$ and $Q = (c, d)$ in $\mathcal{D}$, we must show that

$$V(P) = V(Q)$$

We consider the point $R = (c, b)$ and the segments $\overline{PR}$ and $\overrightarrow{RQ}$. (We assume that (c, b) is in $\mathcal{D}$; if not, just use $R' = (a, d)$.)



Since $\frac{\partial V}{\partial x}(x, y) = 0$ in $\mathcal{D}$, in particular $\frac{\partial V}{\partial x}(x, b) = 0$ for $a \le x \le c$. Therefore, for $a \le x \le c$ we have

$$V(x, b) = \int_a^x \frac{\partial V}{\partial u}(u, b)\, du + V(a, b) = \int_a^x 0\, du + V(a, b) = k + V(a, b)$$

Substituting $x = a$ determines $k = 0$. Hence,

$$V(x, b) = V(a, b) \quad \text{for} \quad a \le x \le c$$

In particular,

$$V(c, b) = V(a, b) \quad \Rightarrow \quad V(R) = V(P) \tag{1}$$

Similarly, since $\frac{\partial V}{\partial y}(x, y) = 0$ in $\mathcal{D}$, we have $\frac{\partial V}{\partial y}(c, y) = 0$ for $b \le y \le d$. Therefore for $b \le y \le d$ we have

$$V(c, y) = \int_b^y \frac{\partial V}{\partial v}(c, v)\, dv + V(c, b) = \int_b^y 0\, dv + V(c, b) = k + V(c, b)$$

Substituting $y = b$ gives $V(c, b) = k + V(c, b)$ or $k = 0$. Therefore,

$$V(c, y) = V(c, b) \quad \text{for} \quad b \le y \le d$$

In particular,

$$V(c, d) = V(c, b) \quad \Rightarrow \quad V(Q) = V(R) \tag{2}$$

Combining (1) and (2) we obtain the desired equality $V(P) = V(Q)$. Since P and Q are any two points in $\mathcal{D}$, we conclude that V is constant on $\mathcal{D}$.

16.2 Line Integrals (LT Section 17.2)

Preliminary Questions

1. What is the line integral of the constant function $f(x, y, z) = 10$ over a curve $\mathcal{C}$ of length 5?

SOLUTION Since the length of $\mathcal{C}$ is the line integral $\int_{\mathcal{C}} 1\, ds = 5$, we have

$$\int_{\mathcal{C}} 10\, ds = 10 \int_{\mathcal{C}} 1\, ds = 10 \cdot 5 = 50$$

2. Which of the following have a zero line integral over the vertical segment from (0, 0) to (0, 1)?

(a) $f(x, y) = x$
(b) $f(x, y) = y$
(c) $\mathbf{F} = \langle x, 0 \rangle$
(d) $\mathbf{F} = \langle y, 0 \rangle$
(e) $\mathbf{F} = \langle 0, x \rangle$
(f) $\mathbf{F} = \langle 0, y \rangle$

SOLUTION The vertical segment from (0, 0) to (0, 1) has the parametrization

$$\mathbf{c}(t) = (0, t), \quad 0 \le t \le 1$$

Therefore, $\mathbf{c}'(t) = \langle 0, 1 \rangle$ and $\|\mathbf{c}'(t)\| = 1$. The line integrals are thus computed by

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_0^1 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt \tag{1}$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot ds = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \tag{2}$$

(a) We have $f(\mathbf{c}(t)) = x = 0$. Therefore by (1) the line integral is zero.

(b) By (1), the line integral is

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_0^1 t \cdot 1\, dt = \frac{1}{2}t^2 \Big|_0^1 = \frac{1}{2} \ne 0$$

(c) This vector line integral is computed using (2). Since $\mathbf{F}(\mathbf{c}(t)) = \langle x, 0 \rangle = \langle 0, 0 \rangle$, the vector line integral is zero.

(d) By (2) we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \langle t, 0 \rangle \cdot \langle 0, 1 \rangle\, dt = \int_0^1 0\, dt = 0$$

(e) The vector integral is computed using (2). Since $\mathbf{F}(\mathbf{c}(t)) = \langle 0, x \rangle = \langle 0, 0 \rangle$, the line integral is zero.

(f) For this vector field we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 \langle 0, t \rangle \cdot \langle 0, 1 \rangle\, dt = \int_0^1 t\, dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2} \ne 0$$

So, we conclude that (a), (c), (d), and (e) have an integral of zero.

3. State whether each statement is true or false. If the statement is false, give the correct statement.

(a) The scalar line integral does not depend on how you parametrize the curve.

(b) If you reverse the orientation of the curve, neither the vector line integral nor the scalar line integral changes sign.

SOLUTION

(a) True: It can be shown that any two parametrizations of the curve yield the same value for the scalar line integral, hence the statement is true.

(b) False: For the definition of the scalar line integral, there is no need to specify a direction along the path, hence reversing the orientation of the curve does not change the sign of the integral. However, reversing the orientation of the curve changes the sign of the vector line integral.

4. Suppose that $\mathcal{C}$ has length 5. What is the value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ if:

(a) $\mathbf{F}(P)$ is normal to $\mathcal{C}$ at all points P on $\mathcal{C}$?

(b) $\mathbf{F}(P)$ is a unit vector pointing in the negative direction along the curve?

SOLUTION

(a) The vector line integral is the integral of the tangential component of the vector field along the curve. Since $\mathbf{F}(P)$ is normal to $\mathcal{C}$ at all points P on $\mathcal{C}$, the tangential component is zero, hence the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ is zero.

(b) In this case we have

$$\mathbf{F}(P) \cdot \mathbf{T}(P) = \mathbf{T}(P) \cdot \mathbf{T}(P) = \|\mathbf{T}(P)\|^2 = 1$$

Therefore,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_{\mathcal{C}} 1\, ds = \text{Length of } \mathcal{C} = 5.$$

Exercises

1. Let $f(x, y, z) = x + yz$, and let $\mathcal{C}$ be the line segment from $P = (0, 0, 0)$ to $(6, 2, 2)$.

(a) Calculate $f(\mathbf{c}(t))$ and $ds = \|\mathbf{c}'(t)\|\, dt$ for the parametrization $\mathbf{c}(t) = (6t, 2t, 2t)$ for $0 \le t \le 1$.

(b) Evaluate $\int_{\mathcal{C}} f(x, y, z)\, ds$.

SOLUTION

(a) We substitute $x = 6t$, $y = 2t$, $z = 2t$ in the function $f(x, y, z) = x + yz$ to find $f(\mathbf{c}(t))$:

$$f(\mathbf{c}(t)) = 6t + (2t)(2t) = 6t + 4t^2$$

We differentiate the vector $c(t)$ and compute the length of the derivative vector:

$$\mathbf{c}'(t) = \frac{d}{dt}\langle 6t, 2t, 2t\rangle = \langle 6, 2, 2\rangle \quad \Rightarrow \quad \mathbf{c}'(t) = \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11}$$

Hence,

$$ds = \|\mathbf{c}'(t)\|\, dt = 2\sqrt{11}\, dt$$

(b) Computing the scalar line integral, we obtain

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_0^1 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^1 (6t + 4t^2) \cdot 2\sqrt{11}\, dt$$

$$= 2\sqrt{11}\left(3t^2 + \frac{4}{3}t^3\right)\Big|_0^1 = 2\sqrt{11}\left(3 + \frac{4}{3}\right) = \frac{26\sqrt{11}}{3}$$

2. Repeat Exercise 1 with the parametrization $\mathbf{c}(t) = (3t^2, t^2, t^2)$ for $0 \le t \le \sqrt{2}$.

SOLUTION

(a) We substitute $x = 3t^2$, $y = t^2$, $z = t^2$ in the function $f(x, y, z) = x + yz$ to find $f(\mathbf{c}(t))$:

$$f(\mathbf{c}(t)) = 3t^2 + (t^2)(t^2) = 3t^2 + t^4$$

We differentiate the vector $c(t)$ and compute the length of the derivative vector:

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle 3t^2, t^2, t^2\right\rangle = \langle 6t, 2t, 2t\rangle \quad \Rightarrow \quad \mathbf{c}'(t) = \sqrt{(6t)^2 + (2t)^2 + (2t)^2} = \sqrt{44t^2} = 2\sqrt{11}t$$

Hence,

$$ds = \|\mathbf{c}'(t)\|\, dt = 2\sqrt{11}t\, dt$$

(b) Computing the scalar line integral, we obtain

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_0^{\sqrt{2}} (3t^2 + t^4) \cdot 2\sqrt{11}t\, dt = 2\sqrt{11} \int_0^{\sqrt{2}} (3t^3 + t^5)\, dt$$
$$= 2\sqrt{11} \left(\frac{3t^4}{4} + \frac{t^6}{6} \right) \Big|_0^{\sqrt{2}} = 2\sqrt{11} \left(3 + \frac{8}{6} \right) = \frac{26\sqrt{11}}{3}$$

3. Let $\mathbf{F} = \langle y^2, x^2 \rangle$, and let $\mathcal{C}$ be the curve $y = x^{-1}$ for $1 \le x \le 2$, oriented from left to right.

(a) Calculate $\mathbf{F}(\mathbf{c}(t))$ and $d\mathbf{s} = \mathbf{c}'(t)\, dt$ for the parametrization of $\mathcal{C}$ given by $\mathbf{c}(t) = (t, t^{-1})$.

(b) Calculate the dot product $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$ and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$.

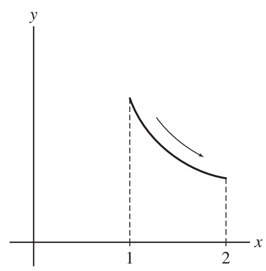
SOLUTION

(a) We calculate $\mathbf{F}(\mathbf{c}(t))$ by substituting $x = t$, $y = t^{-1}$ in $\mathbf{F} = \langle y^2, x^2 \rangle$. We get

$$\mathbf{F}(\mathbf{c}(t)) = \langle (t^{-1})^2, t^2 \rangle = \langle t^{-2}, t^2 \rangle$$

We compute $\mathbf{c}'(t)$:

$$\mathbf{c}'(t) = \frac{d}{dt} \langle t, t^{-1} \rangle = \langle 1, -t^{-2} \rangle \quad \Rightarrow \quad d\mathbf{s} = \langle 1, -t^{-2} \rangle\, dt$$



(b) We compute the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle t^{-2}, t^2 \rangle \cdot \langle 1, -t^{-2} \rangle = t^{-2} \cdot 1 + t^2 \cdot (-t^{-2}) = t^{-2} - 1$$

Computing the vector line integral, we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_1^2 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_1^2 (t^{-2} - 1)\, dt = -t^{-1} - t \Big|_1^2 = \left(-\frac{1}{2} - 2 \right) - (-1 - 1) = -\frac{1}{2}$$

4. Let $\mathbf{F} = \langle z^2, x, y \rangle$ and let $\mathcal{C}$ be the path $\mathbf{c}(t) = \langle 3 + 5t^2, 3 - t^2, t \rangle$ for $0 \le t \le 2$.

(a) Calculate $\mathbf{F}(\mathbf{c}(t))$ and $d\mathbf{s} = \mathbf{c}'(t)\, dt$.

(b) Calculate the dot product $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$ and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$.

SOLUTION

(a) We compute $\mathbf{F}(\mathbf{c}(t))$ by substituting $x = 3 + 5t^2$, $y = 3 - t^2$, $z = t$ in $\mathbf{F} = \langle z^2, x, y \rangle$. We get

$$\mathbf{F}(\mathbf{c}(t)) = \langle t^2, 3 + 5t^2, 3 - t^2 \rangle$$

We differentiate $\mathbf{c}(t)$:

$$\mathbf{c}'(t) = \frac{d}{dt} \langle 3 + 5t^2, 3 - t^2, t \rangle = \langle 10t, -2t, 1 \rangle \quad \Rightarrow \quad d\mathbf{s} = \langle 10t, -2t, 1 \rangle\, dt$$

(b) We compute the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \langle t^2, 3 + 5t^2, 3 - t^2 \rangle \cdot \langle 10t, -2t, 1 \rangle\, dt$$
$$= \left(t^2 \cdot 10t + (3 + 5t^2)(-2t) + (3 - t^2) \cdot 1 \right) dt$$
$$= (10t^3 - 6t - 10t^3 + 3 - t^2)\, dt = (-t^2 - 6t + 3)\, dt$$

Computing the vector line integral gives

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^2 (-t^2 - 6t + 3)\, dt = -\frac{t^3}{3} - 3t^2 + 3t \Big|_0^2 = -\frac{26}{3} = -8\frac{2}{3}$$

In Exercises 5–8, compute the integral of the scalar function or vector field over $\mathbf{c}(t) = (\cos t, \sin t, t)$ *for* $0 \le t \le \pi$.

5. $f(x, y, z) = x^2 + y^2 + z^2$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$. We differentiate $\mathbf{c}(t)$:

$$\mathbf{c}'(t) = \frac{d}{dt}\langle \cos t, \sin t, t\rangle = \langle -\sin t, \cos t, 1\rangle$$

Hence,

$$\|\mathbf{c}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$
$$ds = \|\mathbf{c}'(t)\|\, dt = \sqrt{2}\, dt$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = \cos t$, $y = \sin t$, $z = t$ in $f(x, y, z) = x^2 + y^2 + z^2$ to obtain

$$f(\mathbf{c}(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$$

Step 3. Compute the line integral. Using the Theorem on Scalar Line Integrals we obtain

$$\int_{\mathcal{C}} (x^2 + y^2 + z^2)\, ds = \int_0^{\pi} f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^{\pi} (1 + t^2)\sqrt{2}\, dt = \sqrt{2}\left(t + \frac{t^3}{3}\right)\Big|_0^{\pi} = \sqrt{2}\left(\pi + \frac{\pi^3}{3}\right)$$

6. $f(x, y, z) = xy + z$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$. We differentiate $\mathbf{c}(t)$:

$$\mathbf{c}'(t) = \frac{d}{dt}\langle \cos t, \sin t, t\rangle = \langle -\sin t, \cos t, 1\rangle$$

Hence,

$$\|\mathbf{c}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$
$$ds = \|\mathbf{c}'(t)\|\, dt = \sqrt{2}\, dt$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = \cos t$, $y = \sin t$, $z = t$ in $f(x, y, z) = xy + z$ to obtain

$$f(\mathbf{c}(t)) = \cos t \sin t + t$$

Step 3. Compute the line integral. Using the Theorem on Scalar Line Integrals we obtain

$$\int_{\mathcal{C}} (xy + z)\, ds = \int_0^{\pi} (\cos t \sin t + t)\sqrt{2}\, dt = \sqrt{2}\left(\frac{\sin^2 t}{2} + \frac{t^2}{2}\right)\Big|_0^{\pi} = \sqrt{2}\frac{\pi^2}{2} = \frac{\pi^2}{\sqrt{2}}$$

7. $\mathbf{F} = \langle x, y, z\rangle$

SOLUTION

Step 1. Calculate the integrand. We write out the vectors:

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$
$$\mathbf{F}(\mathbf{c}(t)) = \langle x, y, z\rangle = \langle \cos t, \sin t, t\rangle$$
$$\mathbf{c}'(t) = \langle -\sin t, \cos t, 1\rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle \cos t, \sin t, t\rangle \cdot \langle -\sin t, \cos t, 1\rangle = -\cos t \sin t + \sin t \cos t + t = t$$

Step 2. Evaluate the integral. We use the Theorem on Vector Line Integrals to evaluate the integral:

$$\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{s} = \int_0^{\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{\pi} t\, dt = \frac{1}{2}t^2\Big|_0^{\pi} = \frac{\pi^2}{2}$$

8. $\mathbf{F} = \langle xy, 2, z^3 \rangle$

SOLUTION

Step 1. Calculate the integrand. We write out the vectors:

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$
$$\mathbf{c}'(t) = (-\sin t, \cos t, 1)$$
$$F(\mathbf{c}(t)) = \left\langle \cos t \sin t, 2, t^3 \right\rangle$$

The integrand is the dot product:

$$F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle \cos t \sin t, 2, t^3 \right\rangle \cdot \langle -\sin t, \cos t, 1 \rangle = -\cos t \sin^2 t + 2\cos t + t^3$$

Step 2. Evaluate the integral. We have

$$\int_{\mathcal{C}} \mathbf{F} d\mathbf{s} = \int_0^{\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{\pi} \left(-\cos t \ \sin^2 t + 2\cos t + t^3 \right) dt$$
$$= -\int_0^{\pi} \cos t \ \sin^2 t\, dt + 2\int_0^{\pi} \cos t\, dt + \int_0^{\pi} t^3\, dt$$

Since $\int_0^{\pi} \cos t\, dt = 0$ and $\int_0^{\pi} \cos t \sin^2 t = 0$,

$$\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{s} = \int_0^{\pi} t^3\, dt = \frac{t^4}{4}\bigg|_0^{\pi} = \frac{\pi^4}{4}$$

In Exercises 9–16, compute $\int_{\mathcal{C}} f\, ds$ *for the curve specified.*

9. $f(x, y) = \sqrt{1 + 9xy}, \quad y = x^3$ for $0 \le x \le 1$

SOLUTION The curve is parametrized by $\mathbf{c}(t) = \left(t, t^3\right)$ for $0 \le t \le 1$

Step 1. Compute $\|\mathbf{c}'(t)\|$. We have

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle t, t^3 \right\rangle = \left\langle 1, 3t^2 \right\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1 + 9t^4}$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = t$, $y = t^3$ in $f(x, y) = \sqrt{1 + 9xy}$ to obtain

$$f(\mathbf{c}(t)) = \sqrt{1 + 9t \cdot t^3} = \sqrt{1 + 9t^4}$$

Step 3. Compute the line integral. We use the Theorem on Scalar Line Integrals to write

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_0^1 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^1 \sqrt{1 + 9t^4}\sqrt{1 + 9t^4}\, dt = \int_0^1 \left(1 + 9t^4\right) dt$$
$$= t + \frac{9t^5}{5}\bigg|_0^1 = \frac{14}{5} = 2.8$$

10. $f(x, y) = \dfrac{y^3}{x^7}, \quad y = \frac{1}{4}x^4$ for $1 \le x \le 2$

SOLUTION We parametrize the curve by $\mathbf{c}(t) = \left\langle t, \frac{t^4}{4} \right\rangle$ for $1 \le t \le 2$.

Step 1. Compute $\|\mathbf{c}'(t)\|$. We have

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle t, \frac{t^4}{4} \right\rangle = \left\langle 1, t^3 \right\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1 + t^6}$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = t$, $y = \frac{t^4}{4}$ in $f(x, y) = \frac{y^3}{x^7}$ to obtain:

$$f(\mathbf{c}(t)) = \frac{\left(\frac{t^4}{4}\right)^3}{t^7} = \frac{t^5}{64}$$

Step 3. Compute the line integral. We have

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_1^2 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_1^2 \frac{t^5}{64}\sqrt{1+t^6}\, dt$$

We substitute $u = 1 + t^6$, $du = 6t^5\, dt$:

$$\int_{\mathcal{C}} f(x, y)\, ds = \frac{1}{384}\int_2^{65} \sqrt{u}\, du = \frac{1}{384}\frac{2}{3}u^{3/2}\Big|_2^{65} = \frac{1}{576}(65^{3/2} - 2^{3/2}) \approx 0.9049$$

11. $f(x, y, z) = z^2, \quad \mathbf{c}(t) = (2t, 3t, 4t)$ for $0 \le t \le 2$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$ We have

$$\mathbf{c}'(t) = \frac{d}{dt}\langle 2t, 3t, 4t\rangle = \langle 2, 3, 4\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

Step 2. Write out $f(\mathbf{c}(t))$ We substitute $z = 4t$ in $f(x, y, z) = z^2$ to obtain:

$$f(\mathbf{c}(t)) = 16t^2$$

Step 3. Compute the line integral. By the Theorem on Scalar Line Integrals we have

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_0^2 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^2 16t^2 \cdot \sqrt{29}\, dt = \sqrt{29} \cdot \frac{16}{3}t^3\Big|_0^2 = \frac{128\sqrt{29}}{3} \approx 229.8$$

12. $f(x, y, z) = 3x - 2y + z, \quad \mathbf{c}(t) = (2 + t, 2 - t, 2t)$
for $-2 \le t \le 1$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$. We differentiate $\mathbf{c}(t) = (2 + t, 2 - t, 2t)$ and compute the length of the derivative vector:

$$\mathbf{c}'(t) = (1, -1, 2) \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = 2 + t$, $y = 2 - t$, $z = 2t$ in $f(x, y, z) = 3x - 2y + z$ to obtain:

$$f(\mathbf{c}(t)) = 3(2 + t) - 2(2 - t) + 2t = 7t + 2$$

Step 3. Compute the line integral. We have

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_{-2}^1 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_{-2}^1 (7t + 2)\sqrt{6}\, dt = \sqrt{6}\left(\frac{7t^2}{2} + 2t\right)\Big|_{-2}^1$$

$$= \sqrt{6}\left(\left(\frac{7}{2} + 2\right) - \left(\frac{28}{2} - 4\right)\right) = -\frac{9\sqrt{6}}{2}$$

13. $f(x, y, z) = xe^{z^2}$, piecewise linear path from $(0, 0, 1)$ to $(0, 2, 0)$ to $(1, 1, 1)$

SOLUTION Let $\mathcal{C}_1$ be the segment joining the points $(0, 0, 1)$ and $(0, 2, 0)$ and $\mathcal{C}_2$ be the segment joining the points $(0, 2, 0)$ and $(1, 1, 1)$. We parametrize $\mathcal{C}_1$ and $\mathcal{C}_2$ by the following parametrization:

$$\mathcal{C}_1 : \mathbf{c}_1(t) = (0, 2t, 1 - t),\ 0 \le t \le 1$$
$$\mathcal{C}_2 : \mathbf{c}_2(t) = (t, 2 - t, t),\ 0 \le t \le 1$$

For $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_1$ we have

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_{\mathcal{C}_1} f(x, y, z)\, ds + \int_{\mathcal{C}_2} f(x, y, z)\, ds \tag{1}$$

We compute the integrals on the right hand side.

- The integral over $\mathcal{C}_1$: We have

$$\mathbf{c}_1'(t) = \frac{d}{dt}\langle 0, 2t, 1 - t\rangle = \langle 0, 2, -1\rangle \quad \Rightarrow \quad \|\mathbf{c}_1'(t)\| = \sqrt{0 + 4 + 1} = \sqrt{5}$$
$$f(\mathbf{c}(t)) = xe^{z^2} = 0 \cdot e^{(1-t)^2} = 0$$

Hence,

$$\int_{\mathcal{C}_1} f(x,y,z)\,ds = \int_0^1 f(\mathbf{c}_1(t))\,\|\mathbf{c}_1'(t)\|\,dt = \int_0^1 0\,dt = 0 \qquad (2)$$

- The integral over $\mathcal{C}_2$: We have

$$\mathbf{c}_2'(t) = \frac{d}{dt}\langle t, 2-t, t\rangle = \langle 1, -1, 1\rangle \quad \Rightarrow \quad \|\mathbf{c}_2'(t)\| = \sqrt{1+1+1} = \sqrt{3}$$

$$f(\mathbf{c}_2(t)) = xe^{z^2} = te^{t^2}$$

Hence,

$$\int_{\mathcal{C}_2} f(x,y,z)\,ds = \int_0^1 te^{t^2}\sqrt{3}\,dt \qquad (3)$$

Using the substitution $u = t^2$ we find that

$$\int_{\mathcal{C}_2} f(x,y,z)\,ds = \int_0^1 \frac{\sqrt{3}}{2}e^u\,du = \frac{\sqrt{3}}{2}(e-1) \approx 1.488$$

Hence,

$$\int_{\mathcal{C}} f(x,y,z)\,ds \approx 1.488$$

14. $f(x,y,z) = x^2z, \quad \mathbf{c}(t) = (e^t, \sqrt{2}t, e^{-t})$ for $0 \le t \le 1$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$. We have

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle e^t, \sqrt{2}t, e^{-t}\right\rangle = \left\langle e^t, \sqrt{2}, -e^{-t}\right\rangle$$

Hence,

$$\|\mathbf{c}'(t)\| = \sqrt{\left(e^t\right)^2 + \left(\sqrt{2}\right)^2 + \left(-e^{-t}\right)^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = e^t$, $y = \sqrt{2}t$, $z = e^{-t}$ in $f(x,y,z) = x^2z$ to obtain

$$f(\mathbf{c}(t)) = e^{2t}\cdot e^{-t} = e^t$$

Step 3. Compute the integral. We use the Theorem on Scalar Line Integral to obtain the following integral:

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_0^1 f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt = \int_0^1 e^t\left(e^t + e^{-t}\right)dt = \int_0^1 \left(e^{2t} + 1\right)dt$$

$$= \frac{1}{2}e^{2t} + t\Big|_0^1 = \left(\frac{1}{2}e^2 + 1\right) - \left(\frac{1}{2}\right) = \frac{1}{2}\left(e^2 + 1\right)$$

15. $f(x,y,z) = 2x^2 + 8z, \quad \mathbf{c}(t) = (e^t, t^2, t), \quad 0 \le t \le 1$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$.

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle e^t, t^2, t\right\rangle = \left\langle e^t, 2t, 1\right\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{e^{2t} + 4t^2 + 1}$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = e^t$, $y = t^2$, $z = t$ in $f(x,y,z) = 2x^2 + 8z$ to obtain:

$$f(\mathbf{c}(t)) = 2e^{2t} + 8t$$

Step 3. Compute the line integral. We have

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_0^1 f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt = \int_0^1 (2e^{2t} + 8t)\sqrt{e^{2t} + 4t^2 + 1}\,dt$$

We compute the integral using the substitution $u = e^{2t} + 4t^2 + 1$, $du = 2e^{2t} + 8t\,dt$. We get:

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_2^{e^2+5} u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_2^{e^2+5} = \frac{2}{3}\left((e^2+5)^{3/2} - 2^{3/2}\right)$$

16. $f(x, y, z) = 6xz - 2y^2$, $\mathbf{c}(t) = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\right)$, $0 \le t \le 2$

SOLUTION

Step 1. Compute $\|\mathbf{c}'(t)\|$.

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\right\rangle = \left\langle 1, \sqrt{2}t, t^2\right\rangle$$

$$\Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1^2 + (\sqrt{2}t)^2 + (t^2)^2} = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1 + t^2)^2} = 1 + t^2$$

Step 2. Write out $f(\mathbf{c}(t))$. We substitute $x = t$, $y = \frac{t^2}{\sqrt{2}}$, $z = \frac{t^3}{3}$ in $f(x, y, z) = 6xz - 2y^2$ to obtain:

$$f(\mathbf{c}(t)) = 6t\left(\frac{t^3}{3}\right) - 2\left(\frac{t^2}{\sqrt{2}}\right)^2 = 2t^4 - t^4 = t^4$$

Step 3. Compute the line integral. We have

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_0^2 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^2 t^4(1 + t^2)\, dt = \int_0^2 t^4 + t^6\, dt$$

$$= \left.\frac{t^5}{5} + \frac{t^7}{7}\right|_0^2 = \frac{32}{5} + \frac{128}{7} = \frac{864}{35}$$

17. Calculate $\int_{\mathcal{C}} 1\, ds$, where the curve $\mathcal{C}$ is parametrized by $\mathbf{c}(t) = (4t, -3t, 12t)$ for $2 \le t \le 5$. What does this integral represent?

SOLUTION Compute $\|\mathbf{c}'(t)\|$.

$$\mathbf{c}'(t) = \frac{d}{dt} < 4t, -3t, 12t > = < 4, -3, 12 > \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{4^2 + (-3)^2 + (12)^2} = 13$$

Compute the line integral. We have

$$\int_{\mathcal{C}} 1\, ds = \int_2^5 \|\mathbf{c}'(t)\|\, dt = \int_2^5 13\, dt = 13(5 - 2) = 39$$

This represents the distance from the point $(8, -6, 24)$ to the point $(20, -15, 60)$.

18. Calculate $\int_{\mathcal{C}} 1\, ds$, where the curve $\mathcal{C}$ is parametrized by $\mathbf{c}(t) = (e^t, \sqrt{2}t, e^{-t})$ for $0 \le t \le 2$.

SOLUTION Compute $\|\mathbf{c}'(t)\|$.

$$\mathbf{c}'(t) = \frac{d}{dt} < e^t, \sqrt{2}t, e^{-t} > = < e^t, \sqrt{2}, -e^{-t} >$$

$$\Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

Compute the line integral. We have

$$\int_{\mathcal{C}} 1\, ds = \int_0^2 \|\mathbf{c}'(t)\|\, dt = \int_0^2 e^t + e^{-t}\, dt = \left. e^t - e^{-t}\right|_0^2 = e^2 - e^{-2}$$

In Exercises 19–26, compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ *for the oriented curve specified.*

19. $\mathbf{F} = \langle x^2, xy\rangle$, line segment from $(0, 0)$ to $(2, 2)$

SOLUTION The oriented line segment is parametrized by

$$\mathbf{c}(t) = (t, t), \quad t \text{ varies from 0 to 2.}$$

Therefore,

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle x^2, xy\right\rangle = \left\langle t^2, t \cdot t\right\rangle = \left\langle t^2, t^2\right\rangle$$

$$\mathbf{c}'(t) = \frac{d}{dt}\langle t, t\rangle = \langle 1, 1\rangle$$

The integrand is the dot product:

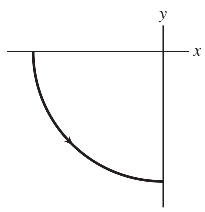
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle t^2, t^2 \right\rangle \cdot \langle 1, 1\rangle = t^2 + t^2 = 2t^2$$

We now use the Theorem on vector line integral to compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^2 2t^2\, dt = \frac{2t^3}{3}\bigg|_0^2 = \frac{16}{3}$$

20. $\mathbf{F} = \langle 4, y\rangle$, quarter circle $x^2 + y^2 = 1$ with $x \le 0$, $y \le 0$, oriented counterclockwise

SOLUTION



The oriented path is parametrized by:

$$\mathbf{c}(t) = (\cos t, \sin t), \quad \pi \le t \le \frac{3\pi}{2}$$

We compute the integrand:

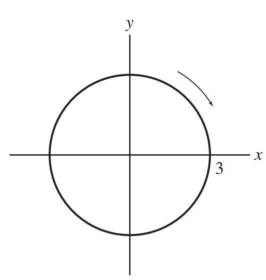
$$\begin{aligned} \mathbf{F}(\mathbf{c}(t)) &= \langle 4, \sin t\rangle \\ \mathbf{c}'(t) &= \langle -\sin t, \cos t\rangle \\ \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) &= \langle 4, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle = -4\sin t + \sin t\cos t = -4\sin t + \frac{1}{2}\sin 2t \end{aligned}$$

The vector line integral is the following integral:

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\pi}^{3\pi/2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{\pi}^{3\pi/2} \left(-4\sin t + \frac{1}{2}\sin 2t\right) dt = 4\cos t - \frac{1}{4}\cos 2t\bigg|_{\pi}^{3\pi/2} \\ &= \left(4\cos\frac{3\pi}{2} - \frac{1}{4}\cos 3\pi\right) - \left(4\cos\pi - \frac{1}{4}\cos 2\pi\right) = \frac{1}{4} + 4 + \frac{1}{4} = 4.5 \end{aligned}$$

21. $\mathbf{F} = \left\langle x^2, xy \right\rangle$, part of circle $x^2 + y^2 = 9$ with $x \le 0$, $y \ge 0$, oriented clockwise

SOLUTION



The oriented path is parametrized by

$$\mathbf{c}(t) = (-3\cos t, 3\sin t); \quad t \text{ is changing from } 0 \text{ to } \frac{\pi}{2}.$$

Note: $\mathbf{c}(0) = (-3, 0)$ and $\mathbf{c}\left(\frac{\pi}{2}\right) = (0, 3)$. $\cos t$ and $\sin t$ are both positive in this range, so $x = -3\cos t \le 0$ and $y = 3\sin t \ge 0$. We compute the integrand:

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle x^2, xy \right\rangle = \left\langle 9\cos^2 t, -9\cos t \sin t \right\rangle$$

$$\mathbf{c}'(t) = \langle 3\sin t, 3\cos t\rangle$$
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle 9\cos^2 t, -9\cos t \sin t\right\rangle \cdot \langle 3\sin t, 3\cos t\rangle = 27\cos^2 t\sin t - 27\cos^2 t\sin t = 0$$

Hence,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{\frac{\pi}{2}} 0\, dt = 0$$

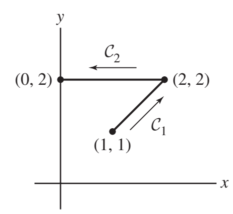
22. $\mathbf{F} = \left\langle e^{y-x}, e^{2x}\right\rangle$, piecewise linear path from (1, 1) to (2, 2) to (0, 2)

SOLUTION Let $\mathcal{C}_1$ be the linear path from (1, 1) to (2, 2), $\mathcal{C}_2$ the linear path from (2, 2) to (0, 2) and

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$$

We use the following parametrizations:

- $\mathcal{C}_1 : \mathbf{c}_1(t) = (t, t)$ t is changing from $t = 1$ to $t = 2$
- $\mathcal{C}_2 : \mathbf{c}_2(t) = (t, 2)$ t is changing from $t = 2$ to $t = 0$



By properties of line integrals,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We compute each integral on the right-hand side:

- The integral over $\mathcal{C}_1$:

$$\mathbf{F}(\mathbf{c}_1(t)) = \left\langle e^{y-x}, e^{2x}\right\rangle = \left\langle e^{t-t}, e^{2t}\right\rangle = \left\langle 1, e^{2t}\right\rangle$$
$$\mathbf{c}_1'(t) = \langle 1, 1\rangle$$
$$\mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t) = \left\langle 1, e^{2t}\right\rangle \cdot \langle 1, 1\rangle = 1 + e^{2t}$$

Hence,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_1^2 \left(1 + e^{2t}\right) dt = t + \frac{1}{2}e^{2t}\Big|_1^2 = \left(2 + \frac{1}{2}e^4\right) - \left(1 + \frac{1}{2}e^2\right) = 1 + \frac{1}{2}e^4 - \frac{1}{2}e^2 \tag{2}$$

- The integral over $\mathcal{C}_2$:

$$\mathbf{F}(\mathbf{c}_2(t)) = \left\langle e^{y-x}, e^{2x}\right\rangle = \left\langle e^{2-t}, e^{2t}\right\rangle$$
$$\mathbf{c}_2'(t) = \langle 1, 0\rangle$$
$$\mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t) = \left\langle e^{2-t}, e^{2t}\right\rangle \cdot \langle 1, 0\rangle = e^{2-t} \cdot 1 + e^{2t} \cdot 0 = e^{2-t}$$

Hence,

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_2^0 \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_2^0 e^{2-t}\, dt = -e^{2-t}\Big|_2^0 = -e^{2-0} + e^{2-2} = -e^2 + 1 \tag{3}$$

We combine (1), (2), and (3) to obtain the following solution:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \left(1 + \frac{1}{2}e^4 - \frac{1}{2}e^2\right) + \left(-e^2 + 1\right) = \frac{1}{2}e^4 - \frac{3}{2}e^2 + 2$$

23. $\mathbf{F} = \langle 3zy^{-1}, 4x, -y \rangle$, $\mathbf{c}(t) = (e^t, e^t, t)$ for $-1 \le t \le 1$

SOLUTION

Step 1. Calculate the integrand. We write out the vectors and compute the integrand:

$$\mathbf{c}(t) = \left(e^t, e^t, t\right)$$
$$\mathbf{F}(\mathbf{c}(t)) = \left\langle 3zy^{-1}, 4x, -y \right\rangle = \left\langle 3te^{-t}, 4e^t, -e^t \right\rangle$$
$$\mathbf{c}'(t) = \left\langle e^t, e^t, 1 \right\rangle$$

The integrand is the dot product:

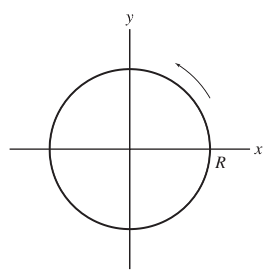
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle 3te^{-t}, 4e^t, -e^t \right\rangle \cdot \left\langle e^t, e^t, 1 \right\rangle = 3te^{-t} \cdot e^t + 4e^t \cdot e^t - e^t \cdot 1 = 3t + 4e^{2t} - e^t$$

Step 2. Evaluate the integral. The vector line integral is:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^{1} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{-1}^{1} \left(3t + 4e^{2t} - e^t\right) dt = 0 + \int_{-1}^{1} \left(4e^{2t} - e^t\right) dt = 2e^{2t} - e^t \Big|_{-1}^{1}$$
$$= \left(2e^2 - e\right) - \left(2e^{-2} - e^{-1}\right) = 2\left(e^2 - e^{-2}\right) - \left(e - e^{-1}\right) \approx 12.157$$

24. $\mathbf{F} = \left\langle \dfrac{-y}{(x^2+y^2)^2}, \dfrac{x}{(x^2+y^2)^2} \right\rangle$, circle of radius R with center at the origin oriented counterclockwise

SOLUTION



The path has the following parametrization:

$$\mathbf{c}(t) = \langle R\cos t, R\sin t \rangle, \quad 0 \le t \le 2\pi$$

Step 1. Calculate the integrand. Since $x^2 + y^2 = R^2$ on the circle, we have

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle \frac{-y}{(x^2+y^2)^2}, \frac{x}{(x^2+y^2)^2} \right\rangle = \left\langle \frac{-R\sin t}{R^4}, \frac{R\cos t}{R^4} \right\rangle = \frac{1}{R^3}\langle -\sin t, \cos t \rangle$$
$$\mathbf{c}'(t) = \frac{d}{dt}\langle R\cos t, R\sin t \rangle = R\langle -\sin t, \cos t \rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{1}{R^3}\langle -\sin t, \cos t \rangle \cdot R\langle -\sin t, \cos t \rangle = \frac{1}{R^2}(\cos^2 t + \sin^2 t) = \frac{1}{R^2}$$

Step 2. Evaluate the integral. We obtain the following integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} \frac{dt}{R^2} = \frac{2\pi}{R^2}$$

25. $\mathbf{F} = \left\langle \dfrac{1}{y^3+1}, \dfrac{1}{z+1}, 1 \right\rangle$, $\mathbf{c}(t) = (t^3, 2, t^2)$ for $0 \le t \le 1$

SOLUTION

Step 1. Calculate the integrand. We have

$$\mathbf{c}(t) = \left(t^3, 2, t^2\right)$$
$$\mathbf{F}(\mathbf{c}(t)) = \left\langle \frac{1}{y^3+1}, \frac{1}{z+1}, 1 \right\rangle = \left\langle \frac{1}{2^3+1}, \frac{1}{t^2+1}, 1 \right\rangle = \left\langle \frac{1}{9}, \frac{1}{t^2+1}, 1 \right\rangle$$

$$\mathbf{c}'(t) = \left\langle 3t^2, 0, 2t \right\rangle$$

Hence,

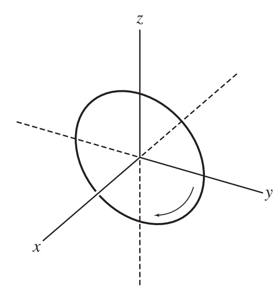
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle \frac{1}{9}, \frac{1}{t^2+1}, 1 \right\rangle \cdot \left\langle 3t^2, 0, 2t \right\rangle = \frac{3t^2}{9} + 0 + 2t$$

Step 2. Evaluate the integral. Using the Theorem on vector line integrals we obtain:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 \frac{t^2}{3}\, dt + \int_0^1 2t\, dt$$
$$= \left.\frac{t^3}{9}\right|_0^1 + \left.t^2\right|_0^1 = \frac{10}{9}$$

26. $\mathbf{F} = \left\langle z^3, yz, x \right\rangle$, quarter of the circle of radius 2 in the yz-plane with center at the origin where $y \geq 0$ and $z \geq 0$, oriented clockwise when viewed from the positive x-axis

SOLUTION



The oriented path has the following parametrization:

$$\mathbf{c}(t) = (0, 2\cos t, 2\sin t)$$

t is changing from $\frac{\pi}{2}$ to 0.

Step 1. Calculate the integrand. We write out the vectors and compute the integrand:

$$\mathbf{c}(t) = (0, 2\cos t, 2\sin t)$$
$$\mathbf{F}(\mathbf{c}(t)) = \left\langle z^3, yz, x \right\rangle = \left\langle 8\,\sin^3 t, 4\cos t \sin t, 0 \right\rangle$$
$$\mathbf{c}'(t) = \langle 0, -2\sin t, 2\cos t \rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle 8\sin^3 t, 4\cos t \sin t, 0 \right\rangle \cdot \langle 0, -2\sin t, 2\cos t \rangle = -8\cos t \sin^2 t$$

Step 2. Evaluate the integral. We obtain the following vector line integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\frac{\pi}{2}}^0 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{\frac{\pi}{2}}^0 -8\cos t \sin^2 t\, dt = \int_0^{\frac{\pi}{2}} 8\sin^2 t \cos t\, dt = 8\left(\left.\frac{\sin^3 t}{3}\right|_0^{\frac{\pi}{2}} \right) = \frac{8}{3}$$

In Exercises 27–32, evaluate the line integral.

27. $\int_{\mathcal{C}} y\, dx - x\, dy$, parabola $y = x^2$ for $0 \leq x \leq 2$

SOLUTION

Step 1. Calculate the integrand.

$$\mathbf{c}(t) = (t, t^2)$$
$$\mathbf{F}(\mathbf{c}(t)) = \langle y, -x \rangle = \left\langle t^2, -t \right\rangle$$
$$\mathbf{c}'(t) = \langle 1, 2t \rangle$$

The integrand is the dot product

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle t^2, -t \right\rangle \cdot \langle 1, 2t \rangle = t^2 - 2t^2 = -t^2$$

Step 2.

$$\int_{\mathcal{C}} y\,dx - x\,dy = \int_0^2 -t^2\,dt = -\frac{t^3}{3}\bigg|_0^2 = -\frac{8}{3}$$

28. $\int_{\mathcal{C}} y\,dx + z\,dy + x\,dz$, $\quad \mathbf{c}(t) = (2 + t^{-1}, t^3, t^2)$ for $0 \le t \le 1$

SOLUTION

Step 1. Calculate the integrand.

$$\mathbf{c}(t) = (2 + t^{-1}, t^3, t^2)$$
$$\mathbf{F}(\mathbf{c}(t)) = \langle y, z, x \rangle = \left\langle t^3, t^2, 2 + t^{-1} \right\rangle$$
$$\mathbf{c}'(t) = \left\langle -t^{-2}, 3t^2, 2t \right\rangle$$

The integrand is the dot product

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle t^3, t^2, 2 + t^{-1} \right\rangle \cdot \left\langle -t^{-2}, 3t^2, 2t \right\rangle = (t^3)(-t^{-2}) + t^2(3t^2) + (2 + t^{-1})2t = 3t^4 + 3t + 2$$

Step 2.

$$\int_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \int_0^1 3t^4 + 3t + 2\,dt = \frac{3t^5}{5} + \frac{3t^2}{2} + 2t\bigg|_0^1 = \frac{41}{10}$$

29. $\int_{\mathcal{C}} (x - y)\,dx + (y - z)\,dy + z\,dz$, $\quad$ line segment from $(0, 0, 0)$ to $(1, 4, 4)$

SOLUTION The oriented line segment from $(0, 0, 0)$ to $(1, 4, 4)$ has the parametrization:

$$\mathbf{c}(t) = (t, 4t, 4t),\ 0 \le t \le 1$$

Step 1. Calculate the integrand. We have

$$\mathbf{F}(\mathbf{c}(t)) = \langle x - y, y - z, z \rangle = \langle t - 4t, 4t - 4t, 4t \rangle = \langle -3t, 0, 4t \rangle$$
$$\mathbf{c}'(t) = \frac{d}{dt}\langle t, 4t, 4t \rangle = \langle 1, 4, 4 \rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle -3t, 0, 4t \rangle \cdot \langle 1, 4, 4 \rangle = -3t \cdot 1 + 0 \cdot 4 + 4t \cdot 4 = 13t$$

Step 2. Evaluate the integral. The vector line integral is:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = \int_0^1 13t\,dt = \frac{13}{2}t^2\bigg|_0^1 = 6.5$$

30. $\int_{\mathcal{C}} z\,dx + x^2\,dy + y\,dz$, $\quad \mathbf{c}(t) = (\cos t, \tan t, t)$ for $0 \le t \le \frac{\pi}{4}$

SOLUTION

Step 1. Calculate the integrand.

$$\mathbf{c}(t) = (\cos t, \tan t, t)$$
$$\mathbf{F}(\mathbf{c}(t)) = \left\langle z, x^2, y \right\rangle = \left\langle t, \cos^2 t, \tan t \right\rangle$$
$$\mathbf{c}'(t) = \left\langle -\sin t, \sec^2 t, 1 \right\rangle$$

The integrand is the dot product

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle t, \cos^2 t, \tan t \right\rangle \cdot \left\langle -\sin t, \sec^2 t, 1 \right\rangle = t(-\sin t) + \cos^2 t \sec^2 t + \tan t = -t\sin t + 1 + \tan t$$

Step 2.

$$\int_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \int_0^{\frac{\pi}{4}} -t\sin t + 1 + \tan t\,dt$$
$$= t\cos t - \sin t + t - \ln(\cos t)\bigg|_0^{\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{2}}{2} + \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

31. $\displaystyle\int_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2+y^2}$, segment from (1, 0) to (0, 1).

SOLUTION

Step 1. Calculate the integrand.

$$\mathbf{c}(t) = (1-t, t) \quad (0 \le t \le 1)$$
$$\mathbf{F}(\mathbf{c}(t)) = \frac{1}{x^2+y^2}\langle -y, x\rangle = \frac{1}{(1-t)^2+t^2}\langle -t, 1-t\rangle$$
$$\mathbf{c}'(t) = \langle -1, 1\rangle$$

The integrand is the dot product

$$\mathbf{F}(\mathbf{c}(t))\cdot \mathbf{c}'(t) = \frac{1}{(1-t)^2+t^2}\langle -t, 1-t\rangle \cdot \langle -1, 1\rangle = \frac{t+1-t}{(1-t)^2+t^2} = \frac{1}{2t^2-2t+1}$$

Step 2.

$$\int_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^1 \frac{dt}{2t^2-2t+1} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

We use the trigonometric substitution $t = \frac{1}{2} + \frac{1}{2}\tan\theta \Rightarrow dt = \frac{1}{2}\sec^2\theta\,d\theta$.

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{4}(\tan^2\theta+1)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{\pi}{2}$$

32. $\displaystyle\int_{\mathcal{C}} y^2\,dx + z^2\,dy + (1-x^2)\,dz$, quarter of the circle of radius 1 in the xz-plane with center at the origin in the quadrant $x \ge 0$, $z \le 0$, oriented counterclockwise when viewed from the positive y-axis.

SOLUTION

Step 1. Calculate the integrand.

$$\mathbf{c}(t) = (\cos t, 0, \sin t) \quad \left(\frac{3\pi}{2} \le t \le 2\pi\right)$$
$$\mathbf{F}(\mathbf{c}(t)) = \left\langle y^2, z^2, 1-x^2\right\rangle = \left\langle 0^2, \sin^2 t, 1-\cos^2 t\right\rangle = \left\langle 0, \sin^2 t, \sin^2 t\right\rangle$$
$$\mathbf{c}'(t) = \langle -\sin t, 0, \cos t\rangle$$

The integrand is the dot product

$$\mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t) = \left\langle 0, \sin^2 t, \sin^2 t\right\rangle \cdot \langle -\sin t, 0, \cos t\rangle$$
$$= \sin^2 t\ \cos t$$

Step 2.

$$\int_{\mathcal{C}} y^2\,dx + z^2\,dy + (1-x^2)\,dz = \int_{\frac{3\pi}{2}}^{2\pi} \sin^2 t\ \cos t\,dt$$
$$= \left.\frac{\sin^3 t}{3}\right|_{\frac{3\pi}{2}}^{2\pi}$$
$$= \frac{1}{3}(0-(-1)^3) = \frac{1}{3}$$

33. CAS Let $f(x,y,z) = x^{-1}yz$, and let $\mathcal{C}$ be the curve parametrized by $\mathbf{c}(t) = (\ln t, t, t^2)$ for $2 \le t \le 4$. Use a computer algebra system to calculate $\int_{\mathcal{C}} f(x,y,z)\,ds$ to four decimal places.

SOLUTION Note that $\mathbf{c}'(t) = \langle 1/t, 1, 2t\rangle$, so $\|\mathbf{c}'(t)\| = \sqrt{1/t^2+1+4t^2}$. Our line integral is

$$\int_2^4 f(\ln t, t, t^2)\sqrt{1/t^2+1+4t^2}\,dt,$$

which we calculate to be 339.5587.

34. CAS Use a CAS to calculate $\int_{\mathcal{C}} \langle e^{x-y}, e^{x+y}\rangle \cdot d\mathbf{s}$ to four decimal places, where $\mathcal{C}$ is the curve $y = \sin x$ for $0 \le x \le \pi$, oriented from left to right.

SOLUTION Using the parameterization $\mathbf{c}(t) = \langle t, \sin t\rangle$, our integral becomes $\int_0^{\pi} \left\langle e^{t-\sin t}, e^{t+\sin t}\right\rangle \cdot \langle 1, \cos t\rangle\, dt$, which is calculated to be -4.5088.

In Exercises 35 and 36, calculate the line integral of $\mathbf{F} = \left\langle e^z, e^{x-y}, e^y\right\rangle$ *over the given path.*

35. The blue path from P to Q in Figure 14

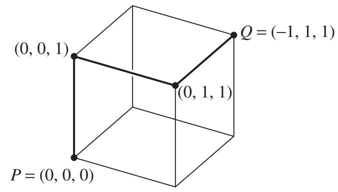
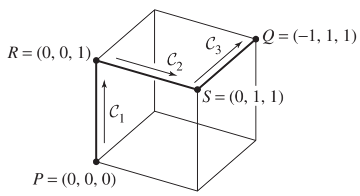


FIGURE 14

SOLUTION



Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ denote the oriented segments from P to R, from R to S and S to Q respectively. These paths have the following parametrizations (see figure):

$$\begin{array}{llll} \mathcal{C}_1: & \mathbf{c}_1(t) = (0,0,t) & 0 \le t \le 1 & & \mathbf{c}_1'(t) = \langle 0,0,1\rangle \\ \mathcal{C}_2: & \mathbf{c}_2(t) = (0,t,1) & 0 \le t \le 1 & \Rightarrow & \mathbf{c}_2'(t) = \langle 0,1,0\rangle \\ \mathcal{C}_3: & \mathbf{c}_3(t) = (-t,1,1) & 0 \le t \le 1 & & \mathbf{c}_3'(t) = \langle -1,0,0\rangle \end{array}$$

Since $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We compute each integral on the right hand side separately.

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt = \int_0^1 \left\langle e^t, e^{0-0}, e^0\right\rangle \cdot \langle 0,0,1\rangle\, dt = \int_0^1 1\, dt = 1$$

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_0^1 \left\langle e^1, e^{0-t}, e^t\right\rangle \cdot \langle 0,1,0\rangle\, dt = \int_0^1 e^{-t}\, dt = -e^{-t}\Big|_0^1 = 1 - e^{-1}$$

$$\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_3(t)) \cdot \mathbf{c}_3'(t)\, dt = \int_0^1 \left\langle e^1, e^{t-1}, e^1\right\rangle \cdot \langle -1,0,0\rangle\, dt = \int_0^1 -e\, dt = -e$$

Substituting these integrals in (1) gives

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 1 + (1 - e^{-1}) - e = 2 - e^{-1} - e$$

36. The closed path *ABCA* in Figure 15

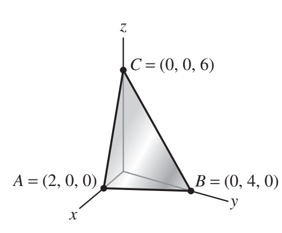
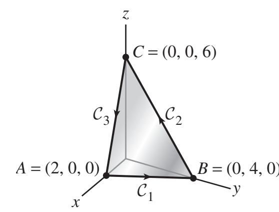


FIGURE 15

SOLUTION



We denote by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ the oriented segments from A to B, from B to C and from C to A. We parametrize these paths by,

$$\begin{aligned}
&\mathcal{C}_1 : \mathbf{c}_1(t) = (1-t)(2,0,0) + t(0,4,0) = (2-2t, 4t, 0),\ 0 \le t \le 1 && \mathbf{c}_1'(t) = \langle -2, 4, 0\rangle \\
&\mathcal{C}_2 : \mathbf{c}_2(t) = (1-t)(0,4,0) + t(0,0,6) = (0, 4-4t, 6t),\ 0 \le t \le 1 \quad \Rightarrow && \mathbf{c}_2'(t) = \langle 0, -4, 6\rangle \\
&\mathcal{C}_3 : \mathbf{c}_3(t) = (1-t)(0,0,6) + t(2,0,0) = (2t, 0, 6-6t),\ 0 \le t \le 1 && \mathbf{c}_3'(t) = \langle 2, 0, -6\rangle
\end{aligned}$$

Since $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ we have,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} \int_{\mathcal{C}_i} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We compute the integrals on the right-hand side:

$$\begin{aligned}
\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} &= \int \left\langle e^0, e^{2-6t}, e^{4t} \right\rangle \cdot \langle -2, 4, 0\rangle\, dt = \int_0^1 \left\langle 1, e^{2-6t}, e^{4t} \right\rangle \cdot \langle -2, 4, 0\rangle\, dt \\
&= \int_0^1 \left(-2 + 4e^{2-6t}\right) dt = -2t - \frac{2}{3}e^{2-6t}\Big|_0^1 = \frac{2}{3}e^2 - \frac{2}{3}e^{-4} - 2 \\
\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left\langle e^{6t}, e^{-4+4t}, e^{4-4t} \right\rangle \cdot \langle 0, -4, 6\rangle\, dt = \int_0^1 \left(-4e^{-4+4t} + 6e^{4-4t}\right) dt \\
&= -e^{-4+4t} - \frac{3}{2}e^{4-4t}\Big|_0^1 = \frac{3}{2}e^4 + e^{-4} - \frac{5}{2} \\
\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left\langle e^{6-6t}, e^{2t}, e^0 \right\rangle \cdot \langle 2, 0, -6\rangle\, dt = \int_0^1 \left(2e^{6-6t} - 6\right) dt \\
&= -\frac{1}{3}e^{6-6t} - 6t\Big|_0^1 = \frac{1}{3}e^6 - \frac{19}{3}
\end{aligned}$$

We substitute these values in (1) to obtain the solution:

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \left(\frac{2}{3}e^2 - \frac{2}{3}e^{-4} - 2\right) + \left(\frac{3}{2}e^4 + e^{-4} - \frac{5}{2}\right) + \left(\frac{1}{3}e^6 - \frac{19}{3}\right) \\
&= \frac{1}{3}e^6 + \frac{3}{2}e^4 + \frac{2}{3}e^2 - \frac{65}{6} + \frac{1}{3}e^{-4}
\end{aligned}$$

In Exercises 37 and 38, $\mathcal{C}$ is the path from P to Q in Figure 16 that traces $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ in the orientation indicated, and **F** *is a vector field such that*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 5, \qquad \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 8, \qquad \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = 8$$

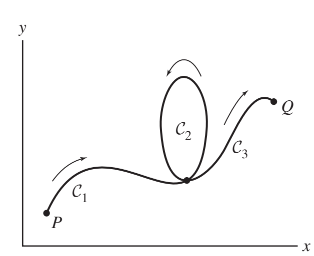
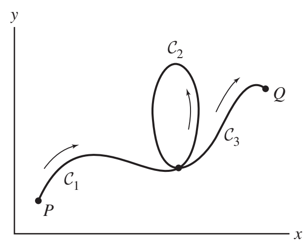


FIGURE 16

37. Determine:

(a) $\displaystyle\int_{-\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s}$ **(b)** $\displaystyle\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s}$ **(c)** $\displaystyle\int_{-\mathcal{C}_1-\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s}$

SOLUTION



(a) If the orientation of the path is reversed, the line integral changes sign, thus:

$$\int_{-\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} = -8$$

(b) By additivity of line integrals, we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s}$$

Substituting the given values we obtain

$$5 = 8 + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} + 8$$

or

$$\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} = 5 - 16 = -11$$

(c) Using properties of line integrals gives

$$\int_{-\mathcal{C}_1-\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} = \int_{-\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{-\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} - \int_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{s} = -8 - 8 = -16$$

38. Find the value of $\displaystyle\int_{\mathcal{C}'} \mathbf{F}\cdot d\mathbf{s}$, where $\mathcal{C}'$ is the path that traverses the loop $\mathcal{C}_2$ four times in the clockwise direction.

SOLUTION Using additivity and the integral over the curve with the reversed orientation, the line integral of **F** over the path that traverses the loop $\mathcal{C}_2$ four times in the clockwise direction is:

$$4\int_{-\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} = 4\cdot\left(-\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s}\right) = -4\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{s} = -4\cdot(-11) = 44$$

39. The values of a function $f(x, y, z)$ and vector field $\mathbf{F}(x, y, z)$ are given at six sample points along the path ABC in Figure 17. Estimate the line integrals of f and **F** along ABC.

Point	$f(x, y, z)$	$\mathbf{F}(x, y, z)$
$\left(1, \frac{1}{6}, 0\right)$	3	$\langle 1, 0, 2\rangle$
$\left(1, \frac{1}{2}, 0\right)$	3.3	$\langle 1, 1, 3\rangle$
$\left(1, \frac{5}{6}, 0\right)$	3.6	$\langle 2, 1, 5\rangle$
$\left(1, 1, \frac{1}{6}\right)$	4.2	$\langle 3, 2, 4\rangle$
$\left(1, 1, \frac{1}{2}\right)$	4.5	$\langle 3, 3, 3\rangle$
$\left(1, 1, \frac{5}{6}\right)$	4.2	$\langle 5, 3, 3\rangle$

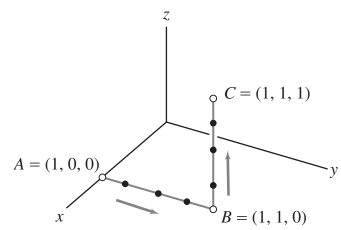
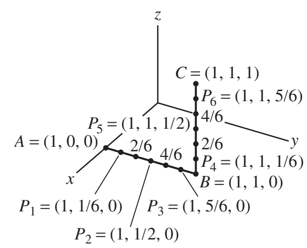


FIGURE 17

SOLUTION



We write the integrals as sum of integrals and estimate each integral by a Riemann Sum. That is,

$$\int_{ABC} f(x, y, z)\,ds = \int_{AB} f(x, y, z)\,ds + \int_{BC} f(x, y, z)\,d\mathbf{s} \approx \sum_{i=1}^{3} f(P_i)\,\Delta s_i + \sum_{i=4}^{6} f(P_i)\Delta s_i \qquad (1)$$

$$\int_{ABC} \mathbf{F}\cdot d\mathbf{s} = \int_{AB} \mathbf{F}\cdot d\mathbf{s} + \int_{BC} \mathbf{F}\cdot d\mathbf{s} = \int_{AB} (\mathbf{F}\cdot\mathbf{T})ds + \int_{BC} (\mathbf{F}\cdot\mathbf{T})ds$$

On AB, the unit tangent vector is $\mathbf{T} = \langle 0, 1, 0\rangle$, hence $\mathbf{F}\cdot\mathbf{T} = F_2$. On BC, the unit tangent vector is $\mathbf{T} = \langle 0, 0, 1\rangle$, hence $\mathbf{F}\cdot\mathbf{T} = F_3$. Therefore,

$$\int_{ABC} \mathbf{F}\,ds = \int_{AB} F_1\,ds + \int_{BC} F_3\,ds \approx \sum_{i=1}^{3} F_1(P_i)\,\Delta s_i + \sum_{i=4}^{6} F_3(P_i)\,\Delta s_i \qquad (2)$$

We consider the partitions of AB and BC to three subarcs with equal length $\Delta s_i = \frac{1}{3}$, therefore (1) and (2) give

$$\int_{ABC} f(x, y, z)\,ds \approx \frac{1}{3}\left(f(P_1) + f(P_2) + f(P_3) + f(P_4) + f(P_5) + f(P_6)\right)$$

$$\int_{ABC} \mathbf{F}\,d\mathbf{s} \approx \frac{1}{3}\left(F_2(P_1) + F_2(P_2) + F_2(P_3) + F_3(P_4) + F_3(P_5) + F_3(P_6)\right)$$

We now substitute the values of the functions at the sample points to obtain the following approximations:

$$\int_{ABC} f(x, y, z)\,ds \approx \frac{1}{3}(3 + 3.3 + 3.6 + 4.2 + 4.5 + 4.2) = 7.6$$

$$\int_{ABC} \mathbf{F}\cdot d\mathbf{s} \approx \frac{1}{3}(0 + 1 + 1 + 4 + 3 + 3) = 4$$

40. Estimate the line integrals of $f(x, y)$ and $\mathbf{F}(x, y)$ along the quarter circle (oriented counterclockwise) in Figure 18 using the values at the three sample points along each path.

Point	$f(x, y)$	$\mathbf{F}(x, y)$
A	1	$\langle 1, 2\rangle$
B	-2	$\langle 1, 3\rangle$
C	4	$\langle -2, 4\rangle$

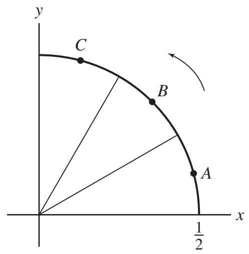


FIGURE 18

SOLUTION We estimate the line integral of $f(x, y)$ along the quarter circle $\mathcal{C}$ by the Riemann sum:

$$\int_{\mathcal{C}} f(x, y)\, d\mathbf{s} \approx f(A)\Delta s_1 + f(B)\Delta s_2 + f(C)\Delta s_3$$

We consider the partition of $\mathcal{C}$ to three arcs with equal length $\Delta s_i = \frac{\pi}{12}$. Using the values of f at the sample points, we get

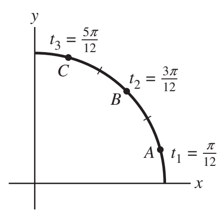
$$\int_{\mathcal{C}} f(x, y) d\mathbf{s} \approx \frac{\pi}{12}(1 - 2 + 4) = \frac{\pi}{4}$$

To estimate the vector line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ we use the parametrization

$$\mathcal{C} : \ \mathbf{c}(t) = \frac{1}{2}(\cos t, \sin t), \ 0 \le t \le \frac{\pi}{2}$$

Then,

$$\mathbf{c}'(t) = \frac{1}{2}(-\sin t, \cos t)$$



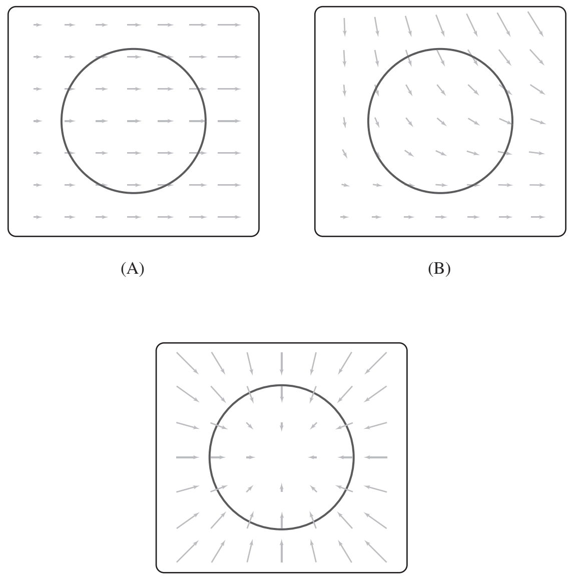
The partition points correspond to $t_1 = \frac{\pi}{12}, t_2 = \frac{3\pi}{12}, t_3 = \frac{5\pi}{12}$. By the theorem on vector line integrals, we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{\pi/2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

We estimate the integral using the Riemann sum, corresponding to the partition of the interval $0 \le t \le \frac{\pi}{2}$ to three intervals with equal length $\Delta t = \frac{\pi}{6}$. We get

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds &= \frac{\pi}{6}\left(\mathbf{F}(\mathbf{c}(t_1)) \cdot \mathbf{c}'(t_1) + \mathbf{F}(\mathbf{c}(t_2)) \cdot \mathbf{c}'(t_2) + \mathbf{F}(\mathbf{c}(t_3)) \cdot \mathbf{c}'(t_3)\right) \\
&= \frac{\pi}{6}\left(\mathbf{F}(A) \cdot \mathbf{c}'\left(\frac{\pi}{12}\right) + \mathbf{F}(B) \cdot \mathbf{c}'\left(\frac{3\pi}{12}\right) + \mathbf{F}(C) \cdot \mathbf{c}'\left(\frac{5\pi}{12}\right)\right) \\
&= \frac{\pi}{6}\left(\langle 1, 2\rangle \cdot \frac{1}{2}\left\langle -\sin\frac{\pi}{12}, \cos\frac{\pi}{12}\right\rangle + \langle 1, 3\rangle \cdot \frac{1}{2}\left\langle -\sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right\rangle + \langle -2, 4\rangle \cdot \frac{1}{2}\left\langle -\sin\frac{5\pi}{12}, \cos\frac{5\pi}{12}\right\rangle\right) \\
&= \frac{\pi}{12}\left(-\sin\frac{\pi}{12} + 2\cos\frac{\pi}{12} - \sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} + 2\sin\frac{5\pi}{12} + 4\cos\frac{5\pi}{12}\right) = 0.505\pi
\end{aligned}$$

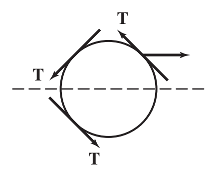
41. Determine whether the line integrals of the vector fields around the circle (oriented counterclockwise) in Figure 19 are positive, negative, or zero.



(A) (B) (C)

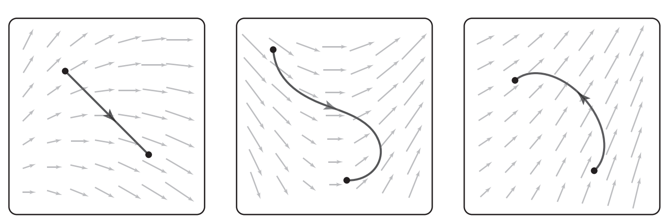
FIGURE 19

SOLUTION The vector line integral of F is the integral of the tangential component of F along the curve. The positive direction of a curve is counterclockwise.



For the vector field in (A), the line integral around the circle is zero because the contribution of the negative tangential components from the upper part of the circle is the same as the contribution of the positive tangential components from the lower part. For the vector in (B) the contribution of the negative tangential component appear to dominate over the positive contribution, hence the line integral is negative. In (C), the vector field is orthogonal to the unit tangent vector at each point, hence the line integral is zero.

42. Determine whether the line integrals of the vector fields along the oriented curves in Figure 20 are positive or negative.



(A) (B) (C)

FIGURE 20

SOLUTION

(A) Positive: The direction of the path is initially perpendicular to the vector field, becoming more and more oriented along the vector field.

(B) Positive: The path is oriented along the vector field when the vectors have a large magnitude. A short section of the path near the end is oriented against the vector field, but the magnitude of these vectors is small. The negative contribution of this section should not cancel out the earlier, strongly positive section.

(C) Positive: As before, the vector field has larger magnitude vectors in the section where the path is oriented along the vector field than the section where it is oriented against the vector field.

43. Calculate the total mass of a circular piece of wire of radius 4 cm centered at the origin whose mass density is $\rho(x, y) = x^2$ g/cm.

SOLUTION The total mass is the following integral:

$$M = \int_{\mathcal{C}} x^2 \, ds$$

We use the following parametrization of the wire:

$$\mathbf{c}(t) = (4\cos t, 4\sin t), \quad 0 \le t \le 2\pi$$

Hence,

$$\mathbf{c}'(t) = \langle -4\sin t,\ 4\cos t\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$$

We compute the line integral using the Theorem on Scalar Line Integrals. We get

$$\begin{aligned} M &= \int_0^{2\pi} \rho\left(\mathbf{c}(t)\right) \|\mathbf{c}'(t)\| \, dt = \int_0^{2\pi} (4\cos t)^2 \cdot 4 \, dt \\ &= 64 \int_0^{2\pi} \cos^2 t \, dt = 64 \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) \Big|_0^{2\pi} = 64 \cdot \frac{2\pi}{2} = 64\pi \text{ g} \end{aligned}$$

44. Calculate the total mass of a metal tube in the helical shape $\mathbf{c}(t) = (\cos t, \sin t, t^2)$ (distance in centimeters) for $0 \le t \le 2\pi$ if the mass density is $\rho(x, y, z) = \sqrt{z}$ g/cm.

SOLUTION The total mass is the following integral:

$$M = \int_{\mathcal{C}} \sqrt{z} \, ds$$

We have

$$\mathbf{c}'(t) = \frac{d}{dt} \left\langle \cos t, \sin t, t^2 \right\rangle = \langle -\sin t, \cos t, 2t\rangle$$

Hence,

$$\|\mathbf{c}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

Using the Theorem on Scalar Line Integrals, we get

$$M = \int_{\mathcal{C}} \sqrt{z} \, ds = \int_0^{2\pi} \rho\left(\mathbf{c}(t)\right) \|\mathbf{c}'(t)\| \, dt = \int_0^{2\pi} \sqrt{t^2}\sqrt{1 + 4t^2} \, dt = \int_0^{2\pi} t\sqrt{1 + 4t^2} \, dt$$

We compute the integral using the substitution $u = 1 + 4t^2$, $du = 8t \, dt$. This gives

$$M = \int_1^{1+16\pi^2} \frac{u^{1/2}}{8} \, du = \frac{u^{3/2}}{12} \Big|_1^{1+16\pi^2} = \frac{1}{12} \left(\left(1 + 16\pi^2\right)^{3/2} - 1 \right) \approx 166.86 \text{ g}$$

45. Find the total charge on the curve $y = x^{4/3}$ for $1 \le x \le 8$ (in cm) assuming a charge density of $\rho(x, y) = x/y$ (in units of 10^{-6} C/cm).

SOLUTION We parametrize the curve by $\mathbf{c}(t) = (t, t^{\frac{4}{3}}) \quad (1 \le t \le 8)$. Then

$$\mathbf{c}'(t) = \left\langle 1, \frac{4}{3} t^{\frac{1}{3}} \right\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1 + \frac{16}{9} t^{\frac{2}{3}}}$$

$$\rho(\mathbf{c}(t)) = \frac{x}{y} = \frac{t}{t^{\frac{4}{3}}}$$

Therefore the total charge will be

$$\int_{\mathcal{C}} \frac{x}{y} \, ds = \int_1^8 \frac{t}{t^{\frac{4}{3}}} \sqrt{1 + \frac{16}{9} t^{\frac{2}{3}}} \, dt = \int_1^8 \sqrt{1 + \frac{16}{9} t^{\frac{2}{3}}} \, t^{-\frac{1}{3}} \, dt$$

Using the substitution $u = 1 + \frac{16}{9} t^{\frac{2}{3}} \Rightarrow du = \frac{32}{27} t^{-\frac{1}{3}} \, dt$, we calculate the total charge as

$$\int_{\frac{25}{9}}^{\frac{73}{9}} \sqrt{u} \, \frac{27}{32} \, du = \frac{27}{32} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{\frac{25}{9}}^{\frac{73}{9}} = \frac{1}{48} \left(73^{\frac{3}{2}} - 25^{\frac{3}{2}} \right) \approx 10.39$$

Thus the total charge is 10.39×10^{-6} C.

46. Find the total charge on the curve $\mathbf{c}(t) = (\sin t, \cos t, \sin^2 t)$ in centimeters for $0 \le t \le \frac{\pi}{8}$ assuming a charge density of $\rho(x, y, z) = xy(y^2 - z)$ (in units of 10^{-6} C/cm).

SOLUTION Using the trigonometric identities $\sin 2t = 2 \sin t \cos t$ and $\cos 2t = \cos^2 t - \sin^2 t$, we first calculate the integrand

$$\mathbf{c}'(t) = \langle \cos t, -\sin t, 2\sin t \cos t\rangle = \langle \cos t, -\sin t, \sin 2t\rangle$$

$$\|\mathbf{c}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + \sin^2 2t} = \sqrt{1 + \sin^2 2t}$$

$$\rho(\mathbf{c}(t)) = xy(y^2 - z) = \sin t \cos t(\cos^2 t - \sin^2 t) = \frac{1}{2}\sin 2t \cos 2t$$

Total charge is thus,

$$\int_{\mathcal{C}} xy(y^2 - z)\, ds = \int_0^{\frac{\pi}{8}} \frac{1}{2}\sin 2t \cos 2t\sqrt{1 + \sin^2 2t}\, dt$$

Using the substitution $u = 1 + \sin^2 2t \Rightarrow du = 4 \sin 2t \cos 2t\, dt$, we have $u(0) = 1 + \sin 0 = 1$ and $u(\frac{\pi}{8}) = 1 + \sin^2(2\frac{\pi}{8}) = \frac{3}{2}$. Thus,

$$\int_{\mathcal{C}} xy(y^2 - z)\, ds = \int_1^{\frac{3}{2}} \sqrt{u}\frac{1}{8}\, du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}}\Big|_1^{\frac{3}{2}} = \frac{1}{12}\left(\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right) \approx 0.0698$$

Thus the total charge is 0.0698×10^{-6} C.

In Exercises 47–50, use Eq. (6) to compute the electric potential $V(P)$ at the point P for the given charge density (in units of 10^{-6} C).

47. Calculate $V(P)$ at $P = (0, 0, 12)$ if the electric charge is distributed along the quarter circle of radius 4 centered at the origin with charge density $\rho(x, y, z) = xy$.

SOLUTION We parametrize the curve by $\mathbf{c}(t) = (4\cos t, 4\sin t, 0)$, $(0 \le t \le \frac{\pi}{2})$. Then $\mathbf{c}'(t) = (-4\sin t, 4\cos t, 0) \Rightarrow \|\mathbf{c}'(t)\| = 4$. The distance from the point $(0, 0, 12)$ to $\mathbf{c}(t)$ is

$$r_P(t) = \sqrt{(0 - 4\cos t)^2 + (0 - 4\sin t)^2 + (12 - 0)^2} = \sqrt{16 + 144} = 4\sqrt{10}$$

while the charge density along the curve is

$$\rho(\mathbf{c}(t)) = xy = 4\cos t\, 4\sin t = 16 \sin t \cos t = 8 \sin 2t$$

Therefore

$$V(P) = k\int_{\mathcal{C}} \frac{\rho}{r_P}\, ds = k\int_0^{\frac{\pi}{2}} \frac{8\sin 2t}{4\sqrt{10}} 4\, dt = \frac{8k}{\sqrt{10}} \cdot \frac{-\cos 2t}{2}\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{4k}{\sqrt{10}}(-\cos\pi + \cos 0) = \frac{8k}{\sqrt{10}}$$

Thus the electric potential is $\frac{8k}{\sqrt{10}} \times 10^{-6}$ C ≈ 22743.1 volts

48. Calculate $V(P)$ at the origin $P = (0, 0)$ if the negative charge is distributed along $y = x^2$ for $1 \le x \le 2$ with charge density $\rho(x, y) = -y\sqrt{x^2 + 1}$.

SOLUTION A parametrization for the curve is $\mathbf{c}(t) = (t, t^2)$ $(1 \le t \le 2)$. Then

$$\mathbf{c}'(t) = \langle 1, 2t\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1 + 4t^2}$$

The charge density along the curve is

$$\rho(\mathbf{c}(t)) = -y\sqrt{x^2 + 1} = -t^2\sqrt{t^2 + 1}$$

The distance from the origin to $\mathbf{c}(t)$ is

$$r_P(t) = \sqrt{(t - 0)^2 + (t^2 - 0)^2} = \sqrt{t^2 + t^4} = |t|\sqrt{1 + t^2}$$

Therefore,

$$V(P) = k\int_{\mathcal{C}} \frac{\rho}{r_P}\, ds = k\int_1^2 \frac{-t^2\sqrt{t^2 + 1}}{|t|\sqrt{1 + t^2}}\sqrt{1 + 4t^2}\, dt = -k\int_1^2 t\sqrt{1 + 4t^2}\, dt$$

Using the substitution $u = 4t^2 + 1 \Rightarrow du = 8t\,dt$, we have

$$V(P) = -\frac{k}{8}\int_5^{17} \sqrt{u}\,du = -\frac{k}{8}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_5^{17} = -\frac{k}{12}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

Thus the electric potential is $-\frac{k}{12}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) \times 10^{-6}$ C ≈ -44135 volts

49. Calculate $V(P)$ at $P = (2, 0, 2)$ if the negative charge is distributed along the y-axis for $1 \le y \le 3$ with charge density $\rho(x, y, z) = -y$.

SOLUTION A parametrization for the curve is $\mathbf{c}(t) = (0, t, 0)$ $(1 \le t \le 3)$. Then $\mathbf{c}'(t) = \langle 0, 1, 0\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = 1$, and the charge density along the curve is $\rho(\mathbf{c}(t)) = -y = -t$. The distance from the origin to $\mathbf{c}(t)$ is

$$r_P(t) = \sqrt{(2-0)^2 + (0-t)^2 + (2-0)^2} = \sqrt{8 + t^2}$$

Therefore,

$$V(P) = k\int_{\mathcal{C}} \frac{\rho}{r_P}\,ds = k\int_1^3 \frac{-t}{\sqrt{8+t^2}}\cdot 1\,dt$$

Using the substitution $u = 8 + t^2 \Rightarrow du = 2t\,dt$, we have

$$V(P) = -k\int_9^{17} u^{-\frac{1}{2}}\frac{1}{2}\,du = -\frac{k}{2}\cdot 2u^{\frac{1}{2}}\Big|_9^{17} = -\frac{k}{2}\left(17^{\frac{1}{2}} - 9^{\frac{1}{2}}\right)$$

Thus the electric potential is $-\frac{k}{2}\left(17^{\frac{1}{2}} - 9^{\frac{1}{2}}\right) \times 10^{-6}$ C ≈ -10097 volts

50. Calculate $V(P)$ at the origin $P = (0, 0)$ if the electric charge is distributed along $y = x^{-1}$ for $\frac{1}{2} \le x \le 2$ with charge density $\rho(x, y) = x^3 y$.

SOLUTION A parametrization for the curve is $\mathbf{c}(t) = (t, t^{-1})$ $(\frac{1}{2} \le t \le 2)$. Then

$$\mathbf{c}'(t) = \left\langle 1, -t^{-2}\right\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1 + t^{-4}} = \frac{1}{t^2}\sqrt{1 + t^4}$$

The charge density along the curve is

$$\rho(\mathbf{c}(t)) = x^3 y = t^3 t^{-1} = t^2$$

The distance from the origin to $\mathbf{c}(t)$ is

$$r_P(t) = \sqrt{(t-0)^2 + (t^{-1} - 0)^2} = \sqrt{t^2 + t^{-2}} = \frac{1}{|t|}\sqrt{1 + t^4}$$

Therefore,

$$V(P) = k\int_{\mathcal{C}} \frac{\rho}{r_P}\,ds = k\int_{\frac{1}{2}}^2 \frac{t^2}{\frac{1}{|t|}\sqrt{1+t^4}}\frac{1}{t^2}\sqrt{1+t^4}\,dt = k\int_{\frac{1}{2}}^2 t\,dt$$

$$= k\cdot\frac{t^2}{2}\Big|_{\frac{1}{2}}^2 = \frac{k}{2}\left(4 - \frac{1}{4}\right) = \frac{15k}{8}$$

Thus the electric potential is $\frac{15k}{8} \times 10^{-6}$ C ≈ 16856 volts

51. Calculate the work done by a field $\mathbf{F} = \langle x + y, x - y\rangle$ when an object moves from $(0, 0)$ to $(1, 1)$ along each of the paths $y = x^2$ and $x = y^2$.

SOLUTION We calculate the work done by $\mathbf{F} = \langle x + y, x - y\rangle$ along the path $y = x^2$ from $(0, 0)$ to $(1, 1)$. We use the parametrization:

$$\mathbf{c}_1(t) = (t, t^2), \quad 0 \le t \le 1$$

We have

$$\mathbf{F}(\mathbf{c}_1(t)) = \left\langle t + t^2, t - t^2\right\rangle$$

$$\mathbf{c}_1'(t) = \langle 1, 2t\rangle$$

$$\mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t) = \left\langle t + t^2, t - t^2\right\rangle\cdot\langle 1, 2t\rangle = t + t^2 + 2t^2 - 2t^3 = -2t^3 + 3t^2 + t$$

The work is the following line integral:

$$W = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt = \int_0^1 \left(-2t^3 + 3t^2 + t\right) dt = -\frac{1}{2}t^4 + t^3 + \frac{1}{2}t^2 \Big|_0^1 = 1$$

We now compute the work along the path $x = y^2$. We parametrize the path by:

$$\mathbf{c}_2(t) = (t^2, t), \quad 0 \le t \le 1$$

Then

$$\begin{aligned} \mathbf{F}(\mathbf{c}_2(t)) &= \left\langle t^2 + t, t^2 - t \right\rangle \\ \mathbf{c}_2'(t) &= \langle 2t, 1 \rangle \\ \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t) &= \left\langle t^2 + t, t^2 - t \right\rangle \cdot \langle 2t, 1 \rangle = 2t^3 + 2t^2 + t^2 - t = 2t^3 + 3t^2 - t \end{aligned}$$

The work is the line integral

$$W = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_0^1 \left(2t^3 + 3t^2 - t\right) dt = \frac{1}{2}t^4 + t^3 - \frac{1}{2}t^2 \Big|_0^1 = \frac{1}{2} + 1 - \frac{1}{2} = 1$$

We obtain the same work along the two paths.

52. Calculate the work done by the force field $\mathbf{F} = \langle x, y, z \rangle$ along the path $(\cos t, \sin t, t)$ for $0 \le t \le 3\pi$.

SOLUTION The work done by the force field $\mathbf{F}$ is the line integral:

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

We compute the integrand:

$$\begin{aligned} \mathbf{F}(\mathbf{c}(t)) &= \langle x, y, z \rangle = \langle \cos t, \sin t, t \rangle \\ \mathbf{c}'(t) &= \frac{d}{dt} \langle \cos t, \sin t, t \rangle = \langle -\sin t, \cos t, 1 \rangle \\ \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) &= \langle \cos t, \sin t, t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle = -\cos t \sin t + \sin t \cos t + t = t \end{aligned}$$

We obtain the following integral:

$$W = \int_0^{3\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{3\pi} t\, dt = \frac{t^2}{2}\Big|_0^{3\pi} = \frac{9\pi^2}{2}$$

53. Figure 21 shows a force field $\mathbf{F}$.

(a) Over which of the two paths, ADC or ABC, does $\mathbf{F}$ perform less work?

(b) If you have to work against $\mathbf{F}$ to move an object from C to A, which of the paths, CBA or CDA, requires less work?

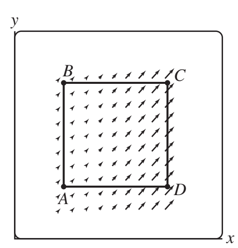
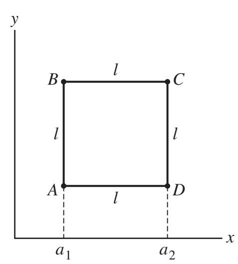


FIGURE 21

SOLUTION

(a) Since x is constant on $\overline{AB}$ and $\overline{DC}$, $\mathbf{F}(x, y) = \langle x, x \rangle$ is also constant on these segments.



Let a_1 and a_2 denote the constant values of x on the segments $\overline{AB}$ and $\overline{DC}$ respectively, and l denote the lengths of these segments. By Exercise 55 we have

$$\int_{AB} \mathbf{F} \cdot d\mathbf{s} = \langle a_1, a_1 \rangle \cdot \langle 0, l \rangle = a_1 \cdot 0 + a_1 \cdot l = a_1 l$$
$$\int_{DC} \mathbf{F} \cdot d\mathbf{s} = \langle a_2, a_2 \rangle \cdot \langle 0, l \rangle = a_2 \cdot 0 + a_2 \cdot l = a_2 l$$

Since $a_1 < a_2$ we have $\int_{AB} \mathbf{F} \cdot d\mathbf{s} < \int_{DC} \mathbf{F} \cdot d\mathbf{s}$.

(b) We compute the integral over BC. This segment is parametrized by:

$$\mathbf{c}(t) = (a_1 + lt, b)\,,\ 0 \le t \le 1.$$

Hence,

$$\mathbf{F}(\mathbf{c}(t)) = \langle x, x \rangle = \langle a_1 + lt, a_1 + lt \rangle\,,\ \mathbf{c}'(t) = \langle l, 0 \rangle$$
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle a_1 + lt, a_1 + lt \rangle \cdot \langle l, 0 \rangle = a_1 l + l^2 t$$

Thus,

$$\int_{BC} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \left(a_1 l + l^2 t \right) dt = a_1 lt + \frac{l^2 t^2}{2} \bigg|_{t=0}^{1} = a_1 l + \frac{l^2}{2}$$

We see that the line integral does not depend on b, therefore,

$$\int_{AD} \mathbf{F} \cdot d\mathbf{s} = \int_{BC} \mathbf{F} \cdot d\mathbf{s} \qquad (1)$$

In part (a) we showed that:

$$\int_{AB} \mathbf{F} \cdot d\mathbf{s} < \int_{DC} \mathbf{F} \cdot d\mathbf{s} \qquad (2)$$

Combining (1) and (2) gives:

$$\int_{ABC} \mathbf{F} \cdot d\mathbf{s} = \int_{AB} \mathbf{F} \cdot d\mathbf{s} + \int_{BC} \mathbf{F} \cdot d\mathbf{s} < \int_{DC} \mathbf{F} \cdot d\mathbf{s} + \int_{AD} \mathbf{F} \cdot d\mathbf{s} = \int_{ADC} \mathbf{F} \cdot d\mathbf{s}$$

54. Verify that the work performed along the segment $\overline{PQ}$ by the constant vector field $\mathbf{F} = \langle 2, -1, 4 \rangle$ is equal to $\mathbf{F} \cdot \overrightarrow{PQ}$ in these cases:

(a) $P = (0, 0, 0)$, $Q = (4, 3, 5)$

(b) $P = (3, 2, 3)$, $Q = (4, 8, 12)$

SOLUTION

(a) The segment $\overline{PQ}$, where $P = (0, 0, 0)$ and $Q = (4, 3, 5)$ has the parametrization,

$$\mathbf{c}(t) = (4t, 3t, 5t), \quad 0 \le t \le 1$$

Therefore, $\mathbf{c}'(t) = \langle 4, 3, 5 \rangle$ and we obtain the following integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 \langle 2, -1, 4 \rangle \cdot \langle 4, 3, 5 \rangle\, dt = \int_0^1 25\, dt = 25$$

This equals $\mathbf{F} \cdot \overrightarrow{PQ} = \langle 2, -1, 4 \rangle \cdot \langle 4, 3, 5 \rangle = 25$.

(b) The segment $\overline{PQ}$, where $P = (3, 2, 3)$ and $Q = (4, 8, 12)$ has the parametrization,

$$\mathbf{c}(t) = (3 + t, 2 + 6t, 3 + 9t), \quad 0 \le t \le 1$$

Therefore, $\mathbf{c}'(t) = \langle 1, 6, 9 \rangle$ and we obtain the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 \langle 2, -1, 4 \rangle \cdot \langle 1, 6, 9 \rangle\, dt = \int_0^1 32\, dt = 32$$

Note that $\overrightarrow{PQ} = \langle 1, 6, 9 \rangle$, so $\mathbf{F} \cdot \overrightarrow{PQ} = \langle 2, -1, 4 \rangle \cdot \langle 1, 6, 9 \rangle = 32$.

55. Show that work performed by a constant force field $\mathbf{F}$ over any path $\mathcal{C}$ from P to Q is equal to $\mathbf{F} \cdot \overrightarrow{PQ}$.

SOLUTION We denote by $\mathbf{c}(t) = (x(t), y(t), c(t))$, $t_0 \le t \le t_1$ a parametrization of the oriented path from P to Q (then $\mathbf{c}(t_0) = P$ and $\mathbf{c}(t_1) = Q$). Let $\mathbf{F} = \langle a, b, c \rangle$ be a constant vector field. Then,

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle a, b, c \rangle \cdot \left\langle x'(t), y'(t), z'(t) \right\rangle = ax'(t) + by'(t) + cz'(t)$$

The vector line integral is, thus,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_{t_0}^{t_1} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{t_0}^{t_1} \left(ax'(t) + by'(t) + cz'(t)\right) dt \\
&= a \int_{t_0}^{t_1} x'(t)\, dt + b \int_{t_0}^{t_1} y'(t)\, dt + c \int_{t_0}^{t_1} z'(t)\, dt = ax(t)\Big|_{t=t_0}^{t_1} + by(t)\Big|_{t=t_0}^{t_1} + cz(t)\Big|_{t=t_0}^{t_1} \\
&= a\,(x(t_1) - x(t_0)) + b\,(y(t_1) - y(t_0)) + c\,(z(t_1) - z(t_0)) \\
&= \langle a, b, c \rangle \cdot \langle x(t_1) - x(t_0), y(t_1) - y(t_0), z(t_1) - z(t_0) \rangle
\end{aligned}$$

Since $P = \langle x(t_0), y(t_0), z(t_0) \rangle$ and $Q = \langle x(t_1), y(t_1), z(t_1) \rangle$ we conclude that,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \langle a, b, c \rangle \cdot \overrightarrow{PQ} = \mathbf{F} \cdot \overrightarrow{PQ}.$$

56. Note that a curve $\mathcal{C}$ in polar form $r = f(\theta)$ is parametrized by $\mathbf{c}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$ because the x- and y-coordinates are given by $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Show that $\|\mathbf{c}'(\theta)\| = \sqrt{f(\theta)^2 + f'(\theta)^2}$.

(b) Evaluate $\int_{\mathcal{C}} (x - y)^2\, ds$, where $\mathcal{C}$ is the semicircle in Figure 22 with polar equation $r = 2\cos\theta$, $0 \le \theta \le \frac{\pi}{2}$.

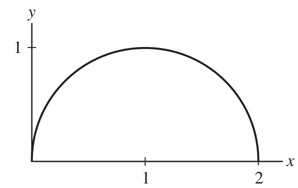


FIGURE 22 Semicircle $r = 2\cos\theta$.

SOLUTION

(a) Finding the magnitude of the tangent vector

$$\begin{aligned}
\mathbf{c}(\theta) &= \langle f(\theta)\cos(\theta), f(\theta)\sin(\theta) \rangle \\
\Rightarrow \quad \mathbf{c}'(\theta) &= \left\langle f'(\theta)\cos(\theta) - f(\theta)\sin(\theta), f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \right\rangle \\
\Rightarrow \quad \|\mathbf{c}'(\theta)\|^2 &= (f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))^2 + (f'(\theta)\sin(\theta) + f(\theta)\cos(\theta))^2 \\
&= (f'(\theta))^2\cos^2(\theta) - 2f'(\theta)f(\theta)\cos(\theta)\sin(\theta) + (f(\theta))^2\sin^2(\theta) \\
&\quad + (f'(\theta))^2\sin^2(\theta) + 2f'(\theta)f(\theta)\cos(\theta)\sin(\theta) + (f(\theta))^2\cos^2(\theta) \\
&= (f'(\theta))^2 + (f(\theta))^2 \\
\Rightarrow \quad \|\mathbf{c}'(\theta)\| &= \sqrt{(f'(\theta))^2 + (f(\theta))^2}
\end{aligned}$$

(b) Using the previous part we have $f(\theta) = 2\cos(\theta)$ and $f'(\theta) = -2\sin(\theta)$. And so

$$\|\mathbf{c}'(\theta)\| = \sqrt{(2\cos(\theta))^2 + (-2\sin(\theta))^2} = 2$$

We have that $x = f(\theta)\cos(\theta) = 2\cos(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta) = 2\cos(\theta)\sin(\theta)$. Integrating,

$$\begin{aligned}
\int_{\mathcal{C}} (x - y)^2\, ds &= \int_0^{\frac{\pi}{2}} (2\cos^2(\theta) - 2\cos(\theta)\sin(\theta))^2 2\, d\theta \\
&= 8 \int_0^{\frac{\pi}{2}} \cos^4(\theta) - 2\cos^3(\theta)\sin(\theta) + \sin^2(\theta)\cos^2(\theta)\, d\theta \\
&= 8 \int_0^{\frac{\pi}{2}} \cos^4(\theta) - 2\cos^3(\theta)\sin(\theta) + (1 - \cos^2(\theta))\cos^2(\theta)\, d\theta
\end{aligned}$$

$$= 8\int_0^{\frac{\pi}{2}} \cos^2(\theta) - 2\cos^3(\theta)\sin(\theta)\,d\theta$$
$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta - 16\int_0^{\frac{\pi}{2}} \cos^3(\theta)\sin(\theta)\,d\theta$$
$$= 4\theta + 2\sin(2\theta)\Big|_0^{\frac{\pi}{2}} + 4\cos^4(\theta)\Big|_0^{\frac{\pi}{2}} = 2\pi - 4$$

57. Charge is distributed along the spiral with polar equation $r = \theta$ for $0 \le \theta \le 2\pi$. The charge density is $\rho(r,\theta) = r$ (assume distance is in centimeters and charge in units of 10^{-6} C/cm). Use the result of Exercise 56(a) to compute the total charge.

SOLUTION Following Exercise 56(a), $f(\theta) = \theta$, and $f'(\theta) = 1$. Thus $\|\mathbf{c}'(\theta)\| = \sqrt{\theta^2+1}$. The total charge will be

$$\int_{\mathcal{C}} \rho\,ds = \int_0^{2\pi} \theta\sqrt{\theta^2+1}\,d\theta$$

Substituting $u = \theta^2 + 1 \Rightarrow du = 2\theta\,d\theta$, we have

$$\int_{\mathcal{C}} \rho\,ds = \int_1^{4\pi^2+1} \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{4\pi^2+1}$$
$$= \frac{1}{2}\left((4\pi^2+1)^{\frac{3}{2}} - 1\right) \approx 85.5$$

Thus the total charge is 85.5×10^{-6} C.

In Exercises 58–61, let **F** *be the* **vortex field** *(so-called because it swirls around the origin as in Figure 23):*

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

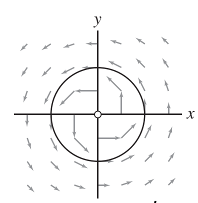
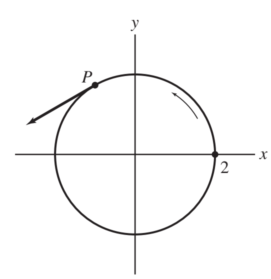


FIGURE 23 Vortex field $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

58. Calculate $I = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$, where $\mathcal{C}$ is the circle of radius 2 centered at the origin. Verify that I changes sign when $\mathcal{C}$ is oriented in the clockwise direction.

SOLUTION

(a)



The circle of radius 2 oriented counterclockwise has the parametrization:

$$\mathbf{c}(t) = (2\cos t, 2\sin t), \quad 0 \le t \le 2\pi$$

Hence,

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle \frac{-2\sin t}{4\cos^2 t + 4\sin^2 t}, \frac{2\cos t}{4\cos^2 t + 4\sin^2 t} \right\rangle = \frac{1}{2}\langle -\sin t, \cos t\rangle$$

$$\mathbf{c}'(t) = \langle -2\sin t, 2\cos t\rangle$$

Therefore, the integrand is the dot product,

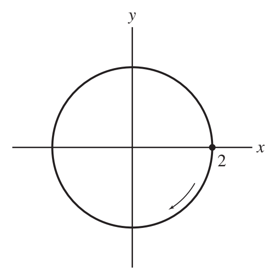
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{1}{2}\langle -\sin t, \cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle = \sin^2 t + \cos^2 t = 1$$

We obtain the following integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

(b) When $\mathcal{C}$ is oriented in the clockwise direction, the parameter t is changing from 2π to 0, therefore, the line integral is,

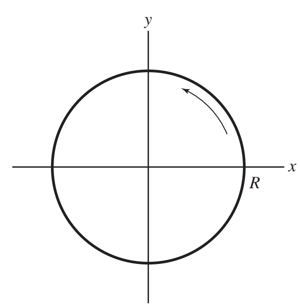
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{2\pi}^{0} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = -\int_0^{2\pi} 1\, dt = -2\pi$$



59. Show that the value of $\int_{\mathcal{C}_R} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}_R$ is the circle of radius R centered at the origin and oriented counterclockwise, does not depend on R.

SOLUTION We parametrize $\mathcal{C}_R$ by:

$$\mathbf{c}(t) = (R\cos t, R\sin t), \quad 0 \le t < 2\pi.$$



Step 1. Calculate the integrand:

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle = \left\langle -\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2}\right\rangle = \frac{1}{R}\langle -\sin t, \cos t\rangle$$

$$\mathbf{c}'(t) = \frac{d}{dt}\langle R\cos t, R\sin t\rangle = \langle -R\sin t, R\cos t\rangle = R\langle -\sin t, \cos t\rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{1}{R}\langle -\sin t, \cos t\rangle \cdot R\langle -\sin t, \cos t\rangle = \sin^2 t + \cos^2 t = 1$$

Step 2. Evaluate the integral.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

60. Let $a > 0$, $b < c$. Show that the integral of $\mathbf{F}$ along the segment [Figure 24(A)] from $P = (a, b)$ to $Q = (a, c)$ is equal to the angle $\angle POQ$.

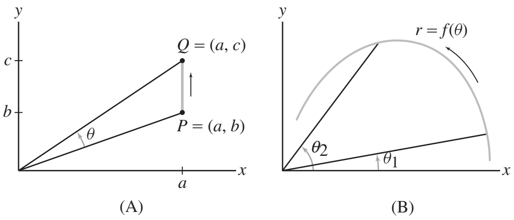


FIGURE 24

SOLUTION Note that the points P and Q are on the vertical line $x = a$. Now, a nice parametrization of this line would be $\mathbf{c}(t) = \langle a, b + (c - b)t \rangle$ for t from 0 to 1, with $\mathbf{c}'(t) = \langle 0, c - b \rangle$. We find that:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{a(c - b)}{a^2 + (b + (c - b)t)^2}$$

and so our integral becomes:

$$\int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = \int_0^1 \frac{a(c - b)}{a^2 + (b + (c - b)t)^2}\,dt = a \int_b^c \frac{du}{a^2 + u^2}\,du$$

where the last integral was done with the substitution $u = b + (c - b)t$. This gives us:

$$\int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = \tan^{-1} \frac{u}{a}\Big|_b^c = \tan^{-1} \frac{c}{a} - \tan^{-1} \frac{b}{a}$$

Of course, $\tan^{-1} \frac{c}{a} - \tan^{-1} \frac{b}{a}$ is just the angle $\angle POQ$.

61. Let $\mathcal{C}$ be a curve in polar form $r = f(\theta)$ for $\theta_1 \le \theta \le \theta_2$ [Figure 24(B)], parametrized by $\mathbf{c}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta))$ as in Exercise 56.

(a) Show that the vortex field in polar coordinates is written $\mathbf{F} = r^{-1}\langle -\sin\theta, \cos\theta \rangle$.

(b) Show that $\mathbf{F} \cdot \mathbf{c}'(\theta)\,d\theta = d\theta$.

(c) Show that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \theta_2 - \theta_1$.

SOLUTION

(a) Letting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ we have

$$\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle = \left\langle \frac{-r\sin(\theta)}{(r\cos(\theta))^2 + (r\sin(\theta))^2}, \frac{r\cos(\theta)}{(r\cos(\theta))^2 + (r\sin(\theta))^2} \right\rangle$$
$$= \frac{r}{r^2}\left\langle \frac{-\sin(\theta)}{\cos^2(\theta) + \sin^2(\theta)}, \frac{\cos(\theta)}{\cos^2(\theta) + \sin^2(\theta)} \right\rangle = r^{-1}\langle -\sin\theta, \cos\theta \rangle$$

(b) From the solution to Exercise 56(a) we have

$$\mathbf{c}'(\theta) = \langle f'(\theta)\cos(\theta) - f(\theta)\sin(\theta), f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \rangle$$

Substituting $r = f(\theta)$ into the previous part, we have

$$\mathbf{F} \cdot \mathbf{c}'(\theta)\,d\theta = \frac{1}{f(\theta)}\langle -\sin(\theta), \cos(\theta) \rangle \cdot \langle f'(\theta)\cos(\theta) - f(\theta)\sin(\theta), f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) \rangle\,d\theta$$
$$= \frac{1}{f(\theta)}(-f'(\theta)\sin(\theta)\cos(\theta) + f(\theta)\sin^2(\theta) + f'(\theta)\cos(\theta)\sin(\theta) + f(\theta)\cos^2(\theta))\,d\theta$$
$$= \frac{1}{f(\theta)} f(\theta)(\sin^2(\theta) + \cos^2(\theta))\,d\theta = d\theta$$

(c)

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\theta_1}^{\theta_2} \mathbf{F} \cdot \mathbf{c}'(\theta)\,d\theta = \int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1$$

In Exercises 62–65, use Eq. (10) to calculate the flux of the vector field across the curve specified.

62. $\mathbf{F} = \langle -y, x \rangle$; upper half of the unit circle, oriented clockwise

SOLUTION The curve is parametrized by $\mathbf{c}(t) = (-\cos t, \sin t)$ $(0 \le t \le \pi)$. Then

$$\mathbf{c}'(t) = \langle \sin t, \cos t \rangle$$

$$\mathbf{F}(\mathbf{c}(t)) = \langle -y, x \rangle = \langle -\sin t, -\cos t \rangle$$

Therefore the flux is

$$\int_{\mathcal{C}} F_1\,dy - F_2\,dx = \int_0^{\pi} (-\sin t)(\cos t) - (-\cos t)(\sin t)\,dt = 0$$

63. $\mathbf{F} = \left\langle x^2, y^2 \right\rangle$; segment from $(3, 0)$ to $(0, 3)$, oriented upward

SOLUTION The curve is parametrized by $\mathbf{c}(t) = (3 - t, t)$ $(0 \le t \le 3) \Rightarrow \mathbf{c}'(t) = \langle -1, 1 \rangle$. Then

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle x^2, y^2 \right\rangle = \left\langle (3-t)^2, t^2 \right\rangle$$

Therefore the flux is

$$\int_{\mathcal{C}} F_1\,dy - F_2\,dx = \int_0^3 (3-t)^2(1) - (t^2)(-1)\,dt$$

$$= \int_0^3 2t^2 - 6t + 9\,dt = \frac{2t^3}{3} - 3t^2 + 9t\Big|_0^3 = 18$$

64. $\mathbf{v} = \left\langle \frac{x+1}{(x+1)^2 + y^2}, \frac{y}{(x+1)^2 + y^2} \right\rangle$; segment $1 \le y \le 4$ along the y-axis, oriented upward

SOLUTION The curve is parametrized by $\mathbf{c}(t) = (0, t)$ $(1 \le t \le 4) \Rightarrow \mathbf{c}'(t) = \langle 0, 1 \rangle$. Then

$$\mathbf{v}(\mathbf{c}(t)) = \left\langle \frac{x+1}{(x+1)^2 + y^2}, \frac{y}{(x+1)^2 + y^2} \right\rangle = \left\langle \frac{1}{1+t^2}, \frac{t}{1+t^2} \right\rangle$$

Therefore the flux is

$$\int_{\mathcal{C}} v_1\,dy - v_2\,dx = \int_1^4 \left(\frac{1}{1+t^2}\right)(1) - \left(\frac{t}{1+t^2}\right)(0)\,dt = \tan^{-1} t\Big|_1^4 = \tan^{-1}(4) - \tan^{-1}(1)$$

65. $\mathbf{v} = \left\langle e^y, 2x - 1 \right\rangle$; parabola $y = x^2$ for $0 \le x \le 1$, oriented left to right

SOLUTION The curve is parametrized by $\mathbf{c}(t) = (t, t^2)$ $(0 \le t \le 1) \Rightarrow \mathbf{c}'(t) = \langle 1, 2t \rangle$. Then

$$\mathbf{v}(\mathbf{c}(t)) = \left\langle e^y, 2x - 1 \right\rangle = \left\langle e^{t^2}, 2t - 1 \right\rangle$$

Therefore the flux is

$$\int_{\mathcal{C}} v_1\,dy - v_2\,dx = \int_0^1 \left(e^{t^2}\right)(2t) - (2t-1)(1)\,dt = e^{t^2} - t^2 + t\Big|_0^1 = e - 1$$

66. Let $I = \int_{\mathcal{C}} f(x, y, z)\,ds$. Assume that $f(x, y, z) \ge m$ for some number m and all points (x, y, z) on $\mathcal{C}$. Which of the following conclusions is correct? Explain.

(a) $I \ge m$

(b) $I \ge mL$, where L is the length of $\mathcal{C}$

SOLUTION Since $f(\mathbf{c}(t)) \ge m$ for all points on $\mathcal{C}$, also $f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\| \ge m\|\mathbf{c}'(t)\|$. By properties of integrals we have

$$I = \int_a^b f(\mathbf{c}(t)) \cdot \|\mathbf{c}'(t)\|\,dt \ge \int_a^b m\|\mathbf{c}'(t)\|\,dt = m\int_a^b \|\mathbf{c}'(t)\|\,dt$$

Since $L = \int_a^b \|\mathbf{c}'(t)\|\,dt$, we get:

$$I \ge mL$$

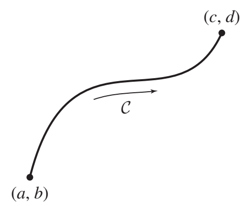
Therefore conclusion (b) is correct.

Further Insights and Challenges

67. Let $\mathbf{F} = \langle x, 0 \rangle$. Prove that if $\mathcal{C}$ is any path from (a, b) to (c, d), then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2}(c^2 - a^2)$$

SOLUTION



We denote the parametrization of the path by,

$$\mathbf{c}(t) = (x(t), y(t)), \quad t_0 \le t \le t_1, \quad \mathbf{c}(t_0) = (a, b), \quad \mathbf{c}(t_1) = (c, d)$$

By the Theorem on vector line integrals we have,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{t_0}^{t_1} \langle x(t), 0 \rangle \cdot \langle x'(t), y'(t) \rangle\, dt = \int_{t_0}^{t_1} x(t)x'(t)\, dt$$

We use the hint to compute the integral, obtaining

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2} x(t)^2 \Big|_{t_0}^{t_1} = \frac{1}{2}\left(x(t_1)^2 - x(t_0)^2\right) = \frac{1}{2}\left(c^2 - a^2\right)$$

Proof of the hint: By the Chain Rule for differentiation we have

$$\frac{d}{dt} f^2(t) = 2f(t)f'(t) \quad \Rightarrow \quad f(t)f'(t) = \frac{1}{2}\frac{d}{dt} f^2(t)$$

Applying the Fundamental Theorem of calculus we obtain

$$\int_{t_0}^{t_1} f(t)f'(t)\, dt = \frac{1}{2}\int_{t_0}^{t_1} \frac{d}{dt}\left(f^2(t)\right) dt = \frac{1}{2}\left(f^2(t_1) - f^2(t_0)\right)$$

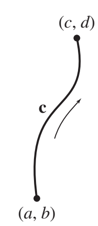
Alternatively we can evaluate the integral $\int f(t)f'(t)\, dt$ using the substitution $u = f(t)$, $du = f'(t)\, dt$. We get

$$\int f(t)f'(t)\, dt = \int u\, du = \frac{1}{2}u^2 + c = \frac{1}{2} f^2(t) + c.$$

68. Let $\mathbf{F} = \langle y, x \rangle$. Prove that if $\mathcal{C}$ is any path from (a, b) to (c, d), then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = cd - ab$$

SOLUTION



We denote a parametrization of the path by:

$$\mathbf{c}(t) = (x(t), y(t)), \quad t_0 \le t \le t_1$$
$$\mathbf{c}(t_0) = (a, b), \quad \mathbf{c}(t_1) = (c, d)$$

By the Theorem on vector line integrals we have

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{t_0}^{t_1} \langle y(t), x(t) \rangle \cdot \langle x'(t), y'(t) \rangle\, dt$$

$$= \int_{t_0}^{t_1} \left(y(t)x'(t) + x(t)y'(t) \right) dt = \int_{t_0}^{t_1} \frac{d}{dt} \left(x(t)y(t) \right) dt$$

The last equality follows from the Product Rule for differentiation. We now use the Fundamental Theorem of Calculus to obtain:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = x(t)y(t) \Big|_{t=t_0}^{t_1} = x(t_1)\, y(t_1) - x(t_0)\, y(t_0) = cd - ab$$

69. We wish to define the **average value** $\mathrm{Av}(f)$ of a continuous function f along a curve $\mathcal{C}$ of length L. Divide $\mathcal{C}$ into N consecutive arcs $\mathcal{C}_1, \ldots, \mathcal{C}_N$, each of length L/N, and let P_i be a sample point in $\mathcal{C}_i$ (Figure 25). The sum

$$\frac{1}{N} \sum_{i=1} f(P_i)$$

may be considered an approximation to $\mathrm{Av}(f)$, so we define

$$\mathrm{Av}(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1} f(P_i)$$

Prove that

$$\mathrm{Av}(f) = \frac{1}{L} \int_{\mathcal{C}} f(x, y, z)\, ds \qquad \boxed{11}$$

Hint: Show that $\frac{L}{N} \sum_{i=1} f(P_i)$ is a Riemann sum approximation to the line integral of f along $\mathcal{C}$.

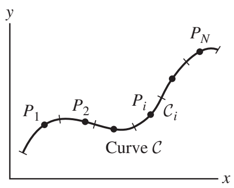


FIGURE 25

SOLUTION The Riemann sum approximation to the line integral is:

$$\sum_{i=1}^{N} f(P_i)\, \Delta S_i$$

If the consecutive arcs $\mathcal{C}_1, \ldots, \mathcal{C}_2$ have equal lengths $\frac{L}{N}$, the corresponding Riemann sum is,

$$\sum_{i=1}^{N} f(P_i) \cdot \frac{L}{N} = \frac{L}{N} \sum_{i=1}^{N} f(P_i)$$

We let $N \to \infty$,

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \lim_{N \to \infty} \frac{L}{N} \sum_{i=1}^{N} f(P_i) = L \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(P_i) = L\mathrm{Av}(f)$$

That is,

$$\mathrm{Av}(f) = \frac{1}{L} \int_{\mathcal{C}} f(x, y, z)\, ds.$$

70. Use Eq. (11) to calculate the average value of $f(x, y) = x - y$ along the segment from $P = (2, 1)$ to $Q = (5, 5)$.

SOLUTION We can parametrize this line segment by

$$\mathbf{c}(t) = (2 + 3t, 1 + 4t), \quad 0 \le t \le 1$$

Therefore,

$$\mathbf{c}'(t) = \langle 3, 4 \rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{9 + 16} = 5$$

We compute the length of the curve,

$$L = \int_0^1 \|\mathbf{c}'(t)\|\,dt = \int_0^1 5\,dt = 5$$

Thus, using our values for x and y given above, we find that

$$\mathrm{Av}(f) = \frac{1}{L}\int_{\mathcal{C}} x - y\,dt = \frac{1}{5}\int_0^1 (2+3t) - (1+4t)\,dt = \frac{1}{5}\int_0^1 1 - t\,dt = \frac{1}{10}$$

71. Use Eq. (11) to calculate the average value of $f(x, y) = x$ along the curve $y = x^2$ for $0 \le x \le 1$.

SOLUTION The average value is

$$\mathrm{Av}(f) = \frac{1}{L}\int_{\mathbf{c}} x\,ds \tag{1}$$

We parametrize the curve by the parametrization,

$$\mathbf{c}(t) = \left(t, t^2\right), \quad 0 \le t \le 1.$$

Hence,

$$\mathbf{c}'(t) = \langle 1, 2t\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1+4t^2}$$

We first must calculate the length of the path. That is,

$$L = \int_{\mathbf{c}} \|\mathbf{c}'(t)\|\,dt = \int_0^1 \sqrt{1+4t^2}\,dt = \frac{1}{2}t\sqrt{1+4t^2} + \frac{1}{4}\ln\left(2t + \sqrt{1+4t^2}\right)\Big|_0^1$$

$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2+\sqrt{5}\right) = \frac{2\sqrt{5} + \ln\left(2+\sqrt{5}\right)}{4}$$

We compute the line integral in (1):

$$\int_{\mathbf{c}} x\,ds = \int_0^1 t\|\mathbf{c}'(t)\|\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt$$

We compute the integral using the substitution $u = 1 + 4t^2$, $du = 8t\,dt$.

$$\int_{\mathbf{c}} x\,ds = \int_0^1 \sqrt{1+4t^2}\cdot t\,dt = \int_1^5 u^{1/2}\cdot\frac{du}{8}$$

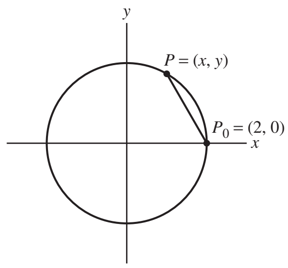
$$= \frac{1}{8}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^5 = \frac{1}{12}\left(5^{\frac{3}{2}} - 1\right)$$

Combining gives the following solution:

$$\mathrm{Av}(f) = \frac{4}{2\sqrt{5} + \ln\left(2+\sqrt{5}\right)}\cdot\frac{5^{\frac{3}{2}} - 1}{12} = \frac{5\sqrt{5} - 1}{\left(6\sqrt{5} + 3\ln\left(2+\sqrt{5}\right)\right)}$$

72. The temperature (in degrees centigrade) at a point P on a circular wire of radius 2 cm centered at the origin is equal to the square of the distance from P to $P_0 = (2, 0)$. Compute the average temperature along the wire.

SOLUTION



The temperature at a point $P(x, y)$ on the wire is given by the function,

$$T(x, y) = (x-2)^2 + y^2$$

The length of the wire is the length of the circle of radius 2, $L = 2\pi \cdot 2 = 4\pi$. Therefore, the average temperature along the wire is,

$$\text{Av}(T) = \frac{1}{L}\int_{\mathcal{C}} T\,ds = \frac{1}{4\pi}\int_{\mathcal{C}} \left((x-2)^2 + y^2\right) ds$$

To compute the line integral, we parametrize the circle by:

$$\mathbf{c}(t) = (2\cos t, 2\sin t), \quad 0 \le t \le 2\pi.$$

Then,

$$\mathbf{c}(t) = \langle -2\sin t, 2\cos t\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

We express T in terms of the parameter:

$$\begin{aligned} T\left(\mathbf{c}(t)\right) &= (x-2)^2 + y^2 = (2\cos t - 2)^2 + (2\sin t)^2 = 4\cos^2 t - 8\cos t + 4 + 4\sin^2 t \\ &= 4\left(\cos^2 + \sin^2 t\right) + 4 - 8\cos t = 8(1-\cos t) \end{aligned}$$

We obtain the integral,

$$\text{Av}(T) = \frac{1}{4\pi}\int_0^{2\pi} T\left(\mathbf{c}(t)\right)\|\mathbf{c}'(t)\|\,dt = \frac{1}{4\pi}\int_0^{2\pi} 16(1-\cos t)\,dt = \frac{4}{\pi}\left(t - \sin t\Big|_0^{2\pi}\right) = \frac{4\cdot 2\pi}{\pi} = 8$$

73. The value of a scalar line integral does not depend on the choice of parametrization (because it is defined without reference to a parametrization). Prove this directly. That is, suppose that $\mathbf{c}_1(t)$ and $\mathbf{c}(t)$ are two parametrizations such that $\mathbf{c}_1(t) = \mathbf{c}(\varphi(t))$, where $\varphi(t)$ is an increasing function. Use the Change of Variables Formula to verify that

$$\int_c^d f(\mathbf{c}_1(t))\|\mathbf{c}_1'(t)\|\,dt = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,dt$$

where $a = \varphi(c)$ and $b = \varphi(d)$.

SOLUTION We compute the integral $\int_a^b f\left(\mathbf{c}(t)\right)\|\mathbf{c}'(t)\|\,dt$ using the substitution $t = \varphi(u)$, $a = \varphi(c)$, $b = \varphi(d)$. We get:

$$\int_a^b f\left(\mathbf{c}_1(t)\right)\|\mathbf{c}'(t)\|\,dt = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f\left(\mathbf{c}\left(\varphi(t)\right)\right)\|\mathbf{c}'\left(\varphi(t)\right)\|\varphi'(u)\,du \tag{1}$$

Since φ is an increasing function, $\varphi'(u) > 0$ for all u, therefore:

$$\left\|\mathbf{c}'\left(\varphi(u)\right)\right\|\varphi'(u) = \left\|\mathbf{c}'\left(\varphi(u)\right)\varphi'(u)\right\| \tag{2}$$

By the Chain Rule for vector valued functions, we have,

$$\frac{d}{du}\mathbf{c}\left(\varphi(u)\right) = \varphi'(u)\mathbf{c}'\left(\varphi(u)\right) \tag{3}$$

Combining (2) and (3) gives:

$$\left\|\mathbf{c}'\left(\varphi(u)\right)\right\|\varphi'(u) = \left\|\frac{d}{du}\mathbf{c}\left(\varphi(u)\right)\right\| = \left\|\frac{d}{du}\mathbf{c}_1(u)\right\| = \|\mathbf{c}_1'(u)\| \tag{4}$$

We substitute (4) in (1) to obtain:

$$\int_a^b f\left(\mathbf{c}(t)\right)\|\mathbf{c}'(t)\|\,dt = \int_c^d f\left(\mathbf{c}_1(u)\right)\|\mathbf{c}_1'(u)\|\,du = \int_c^d f\left(\mathbf{c}_1(t)\right)\|\mathbf{c}_1'(t)\|\,dt$$

The last step is simply replacing the dummy variable of integration u by t.

16.3 Conservative Vector Fields (LT Section 17.3)

Preliminary Questions

1. The following statement is false. *If* **F** *is a gradient vector field, then the line integral of* **F** *along every curve is zero.* Which single word must be added to make it true?

SOLUTION The missing word is "closed" (curve). The line integral of a gradient vector field along every closed curve is zero.

2. Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

(a) The line integral along a path from P to Q does not depend on which path is chosen.

(b) The line integral over an oriented curve $\mathcal{C}$ does not depend on how $\mathcal{C}$ is parametrized.

(c) The line integral around a closed curve is zero.

(d) The line integral changes sign if the orientation is reversed.

(e) The line integral is equal to the difference of a potential function at the two endpoints.

(f) The line integral is equal to the integral of the tangential component along the curve.

(g) The cross-partials of the components are equal.

SOLUTION

(a) This statement is true only for conservative vector fields.

(b) This statement is true for all vector fields.

(c) This statement holds only for conservative vector fields.

(d) This is a property of all vector fields.

(e) Only conservative vector fields have a potential function, and the line integral is computed by using the potential function as stated.

(f) All vector fields' line integrals share this property.

(g) The cross-partial of the components of a conservative field are equal. For other fields, the cross-partial of the components may or may not equal.

3. Let **F** be a vector field on an open, connected domain $\mathcal{D}$. Which of the following statements are always true, and which are true under additional hypotheses on $\mathcal{D}$?

(a) If **F** has a potential function, then **F** is conservative.

(b) If **F** is conservative, then the cross-partials of **F** are equal.

(c) If the cross-partials of **F** are equal, then **F** is conservative.

SOLUTION

(a) This statement is always true, since every gradient vector field is conservative.

(b) If **F** is conservative on a connected domain $\mathcal{D}$, then **F** has a potential function $\mathcal{D}$ and consequently the cross partials of **F** are equal in $\mathcal{D}$.

(c) If the cross partials of **F** are equal in a simply-connected region $\mathcal{D}$, then **F** is a gradient vector field in $\mathcal{D}$.

4. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be the oriented curves in Figure 16 and let $\mathbf{F} = \nabla V$ be a gradient vector field such that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 4$. What are the values of the following integrals?

(a) $\int_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{s}$ **(b)** $\int_{\mathcal{E}} \mathbf{F} \cdot d\mathbf{s}$

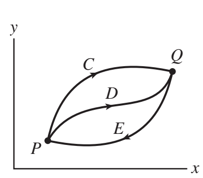


FIGURE 16

SOLUTION Since **F** is a gradient vector field the integrals over closed paths are zero. Therefore, by the equivalent conditions for path independence we have:

(a) $\int_D \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 4$

(b) $\int_E \mathbf{F} \cdot d\mathbf{s} = \int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -4$

Exercises

1. Let $V(x, y, z) = xy\sin(yz)$ and $\mathbf{F} = \nabla V$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}$ is any path from $(0, 0, 0)$ to $(1, 1, \pi)$.

SOLUTION By the Fundamental Theorem for Gradient Vector Fields, we have:

$$\int_{\mathbf{c}} \nabla V \cdot d\mathbf{s} = V(1, 1, \pi) - V(0, 0, 0) = 1 \cdot 1 \sin \pi - 0 = 0$$

2. Let $\mathbf{F} = \langle x^{-1}z, y^{-1}z, \log(xy) \rangle$.

(a) Verify that $\mathbf{F} = \nabla V$, where $V(x, y, z) = z\ln(xy)$.

(b) Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = \langle e^t, e^{2t}, t^2 \rangle$ for $1 \le t \le 3$.

(c) Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ for any path $\mathbf{c}$ from $P = (\frac{1}{2}, 4, 2)$ to $Q = (2, 2, 3)$ contained in the region $x > 0$, $y > 0$.

(d) Why is it necessary to specify that the path lie in the region where x and y are positive?

SOLUTION

(a)

$$\frac{\partial V}{\partial x} = z\frac{1}{xy}y = x^{-1}z = F_1$$

$$\frac{\partial V}{\partial y} = z\frac{1}{xy}x = y^{-1}z = F_2$$

$$\frac{\partial V}{\partial z} = \ln(xy) = F_3$$

(b) $\mathbf{c}(1) = (e, e^2, 1)$, while $\mathbf{c}(3) = (e^3, e^6, 9)$. Therefore,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(e^3, e^6, 9) - V(e, e^2, 1) = 9\ln(e^3 \cdot e^6) - \ln(e \cdot e^2) = 78$$

(c)

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(2, 2, 3) - V\left(\frac{1}{2}, 4, 2\right) = 3\ln(2 \cdot 2) - 2\ln\left(\frac{1}{2} \cdot 4\right) = 4\ln(2)$$

(d) $\mathbf{F}$ is not defined for $x = 0$ or $y = 0$. A continuous path which leaves the region $x > 0$, $y > 0$ would necessarily cross one of these lines where the line integral is not defined.

In Exercises 3–6, verify that $\mathbf{F} = \nabla V$ *and evaluate the line integral of* $\mathbf{F}$ *over the given path.*

3. $\mathbf{F} = \langle 3, 6y \rangle$, $\quad V(x, y, z) = 3x + 3y^2$; $\quad \mathbf{c}(t) = (t, 2t^{-1})$ for $1 \le t \le 4$

SOLUTION The gradient of $V = 3x + 3y^2$ is:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle = \langle 3, 6y \rangle = \mathbf{F}$$

Using the Fundamental Theorem for Gradient Vector Fields, we have:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(\mathbf{c}(4)) - V(\mathbf{c}(1)) = V\left(4, \frac{1}{2}\right) - V(1, 2) = \left(3 \cdot 4 + 3 \cdot \frac{1}{4}\right) - (3 \cdot 1 + 3 \cdot 4) = -\frac{9}{4}$$

4. $\mathbf{F} = \langle \cos y, -x\sin y \rangle$, $V(x, y) = x\cos y$; upper half of the unit circle centered at the origin, oriented counterclockwise

SOLUTION

(a) Verifying that $\mathbf{F} = \nabla V$,

$$\frac{\partial V}{\partial x} = \cos y = F_1, \quad \frac{\partial V}{\partial y} = x(-\sin y) = F_2$$

(b)

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(-1, 0) - V(1, 0) = -1\cos(0) - 1\cos(0) = -2$$

5. $\mathbf{F} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$, $V(x, y, z) = xye^z$; $\mathbf{c}(t) = (t^2, t^3, t-1)$ for $1 \le t \le 2$

SOLUTION We verify that $\mathbf{F}$ is the gradient of V:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \left\langle ye^z, xe^z, xye^z \right\rangle = \mathbf{F}$$

We use the Fundamental Theorem for Gradient Vectors with the initial point $\mathbf{c}(1) = (1, 1, 0)$ and terminal point $\mathbf{c}(2) = (4, 8, 1)$, to obtain:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = V(4, 8, 1) - V(1, 1, 0) = 32e - 1$$

6. $\mathbf{F} = \frac{z}{x}\mathbf{i} + \mathbf{j} + \ln x\mathbf{k}$, $V(x, y, z) = y + z\ln x$;
circle $(x-4)^2 + y^2 = 1$ in the clockwise direction

SOLUTION

(a) Verifying that $\mathbf{F} = \nabla V$,

$$\frac{\partial V}{\partial x} = \frac{z}{x} = F_1, \quad \frac{\partial V}{\partial y} = 1 = F_2, \quad \frac{\partial V}{\partial z} = \ln x = F_3$$

(b) Since $\mathbf{c}$ is closed curve,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$$

In Exercises 7–16, find a potential function for $\mathbf{F}$ *or determine that* $\mathbf{F}$ *is not conservative.*

7. $\mathbf{F} = \langle z, 1, x \rangle$

SOLUTION We check whether the vector field $\mathbf{F} = \langle z, 1, x \rangle$ satisfies the cross partials condition:

$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(z) = 0 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(1) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}(1) = 0 \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}(x) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}(x) = 1 \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}(z) = 1 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

$\mathbf{F}$ satisfies the cross partials condition everywhere. Hence, $\mathbf{F}$ is conservative. We find a potential function $V(x, y, z)$.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. V is an antiderivative of $F_1 = z$ when y and z are fixed, therefore:

$$V(x, y, z) = \int z\,dx = zx + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. By (1) we have:

$$\frac{\partial}{\partial y}(zx + g(y, z)) = 1$$
$$g_y(y, z) = 1$$

Integrating with respect to y, while holding z fixed, gives:

$$g(y, z) = \int 1\,dy = y + h(z)$$

We substitute in (1) to obtain:

$$V(x, y, z) = zx + y + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. Using (2) we get:

$$\frac{\partial}{\partial z}(zx + y + h(z)) = x$$
$$x + h'(z) = x$$
$$h'(z) = 0 \quad \Rightarrow \quad h(z) = c$$

Substituting in (2) gives the following potential functions:

$$V(x, y, z) = zx + y + c.$$

One of the potential functions is obtained by choosing $c = 0$:

$$V(x, y, z) = zx + y$$

8. $\mathbf{F} = x\mathbf{j} + y\mathbf{k}$

SOLUTION Since $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(0) = 0$ and $\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x) = 1$, we have $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$. Therefore **F** does not satisfy the cross-partials condition, hence **F** is not conservative.

9. $\mathbf{F} = y^2\mathbf{i} + (2xy + e^z)\mathbf{j} + ye^z\mathbf{k}$

SOLUTION We examine whether **F** satisfies the cross partials condition:

$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right) = 2y \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(2xy + e^z\right) = 2y \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}\left(2xy + e^z\right) = e^z \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}\left(ye^z\right) = e^z \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}\left(ye^z\right) = 0 \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}\left(y^2\right) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

We see that **F** satisfies the cross partials condition everywhere, hence **F** is conservative. We find a potential function for **F**.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. V is an antiderivative of $F_1 = y^2$ when y and z are fixed. Hence:

$$V(x, y, z) = \int y^2\,dx = y^2x + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. By (1) we have:

$$\frac{\partial}{\partial y}\left(y^2x + g(y, z)\right) = 2xy + e^z$$
$$2yx + g_y(y, z) = 2xy + e^z \quad \Rightarrow \quad g_y(y, z) = e^z$$

We integrate with respect to y, holding z fixed:

$$g(y, z) = \int e^z\,dy = e^z y + h(z)$$

Substituting in (1) gives:

$$V(x, y, z) = y^2x + e^z y + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2), we get:

$$\frac{\partial}{\partial z}\left(y^2x + e^z y + h(z)\right) = ye^z$$
$$e^z y + h'(z) = ye^z \quad \Rightarrow \quad h'(z) = 0$$

Therefore $h(z) = c$. Substituting in (2) we get:

$$V(x, y, z) = y^2 x + e^z y + c$$

The potential function corresponding to $c = 0$ is:

$$V(x, y, z) = y^2 x + e^z y.$$

10. $\mathbf{F} = \langle y, x, z^3 \rangle$

SOLUTION We examine whether the field $\mathbf{F} = \left\langle y, x, z^3 \right\rangle$ satisfies the cross partials condition.

$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(y) = 1 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(x) = 1 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}(x) = 0 \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}\left(z^3\right) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}\left(z^3\right) = 0 \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}(y) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

Since $\mathbf{F}$ satisfies the cross partials condition everywhere, $\mathbf{F}$ is conservative. We find a potential function for $\mathbf{F}$.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. V is an antiderivative of $F_1 = y$ when y and z are fixed. Therefore:

$$V(x, y, z) = \int y\, dx = yx + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. By (1) we have:

$$\frac{\partial}{\partial y}(yx + g(y, z)) = x$$

$$x + g_y(y, z) = x \quad \Rightarrow \quad g_y(y, z) = 0$$

Therefore, $g(y, z) = g(z)$. Substituting in (1) gives:

$$V(x, y, z) = yx + g(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. Using (2) we have:

$$\frac{\partial}{\partial z}(yx + g(z)) = z^3$$

$$g'(z) = z^3 \quad \Rightarrow \quad g(z) = \frac{1}{4}z^4 + c$$

Substituting in (2) gives the following general potential function:

$$V(x, y, z) = yx + \frac{1}{4}z^4 + c$$

Choosing $c = 0$ we obtain the potential:

$$V(x, y, z) = yx + \frac{z^4}{4}.$$

11. $\mathbf{F} = \langle \cos(xz), \sin(yz), xy \sin z \rangle$

SOLUTION Since $\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}(\sin(yz)) = y\cos(yz)$ and $\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}(xy \sin z) = x \sin z$, we have $\frac{\partial F_2}{\partial z} \neq \frac{\partial F_3}{\partial y}$. The cross partials condition is not satisfied, therefore the vector field is not conservative.

12. $\mathbf{F} = \langle \cos z, 2y, -x \sin z \rangle$

SOLUTION We examine whether **F** satisfies the cross partials condition:

$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(\cos z) = 0 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(2y) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}(2y) = 0 \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}(-x \sin z) = 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}(-x \sin z) = -\sin z \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}(\cos z) = -\sin z \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

We see that the conditions are satisfied, therefore **F** is conservative. We find a potential function for **F**.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. $V(x, y, z)$ is an antiderivative of $F_1 = \cos z$ when y and z are fixed, therefore:

$$V(x, y, z) = \int \cos z \, dx = x \cos z + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Using (1) we get:

$$\frac{\partial}{\partial y}\left(x \cos z + g(y, z)\right) = 2y$$

$$g_y(y, z) = 2y$$

We integrate with respect to y, holding z fixed:

$$g(y, z) = \int 2y \, dy = y^2 + g(z)$$

Substituting in (1) gives

$$V(x, y, z) = x \cos z + y^2 + g(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2) we have

$$\frac{\partial}{\partial z}\left(x \cos z + y^2 + g(z)\right) = -x \sin z$$

$$-x \sin z + g'(z) = -x \sin z$$

$$g'(z) = 0 \quad \Rightarrow \quad g(z) = c$$

Substituting in (2) we obtain the general potential function:

$$V(x, y, z) = x \cos z + y^2 + c$$

Choosing $c = 0$ gives the potential function:

$$V(x, y, z) = x \cos z + y^2.$$

13. $\mathbf{F} = \langle z \sec^2 x, z, y + \tan x \rangle$

SOLUTION

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. $V(x, y, z)$ is an antiderivative of $F_1 = z \sec^2 x$ when y and z are fixed, therefore:

$$V(x, y, z) = \int z \sec^2 x \, dx = z \tan x + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Using (1) we get:

$$\frac{\partial}{\partial y}\left(z \tan x + g(y, z)\right) = z$$

$$g_y(y, z) = z$$

We integrate with respect to y, holding z fixed:

$$g(y,z) = \int z\,dy = yz + h(z)$$

Substituting in (1) gives

$$V(x,y,z) = z\tan x + yz + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2) we have

$$\begin{aligned} \frac{\partial}{\partial z}\left(z\tan x + yz + h(z)\right) &= y + \tan x \\ \tan x + y + h'(z) &= y + \tan x \\ h'(z) = 0 \quad &\Rightarrow \quad h(z) = c \end{aligned}$$

Substituting in (2) we obtain the general potential function:

$$V(x,y,z) = z\tan x + yz + c$$

Choosing $c = 0$ gives the potential function:

$$V(x,y,z) = z\tan x + yz$$

14. $\mathbf{F} = \left\langle e^x(z+1), -\cos y, e^x \right\rangle$

SOLUTION

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. $V(x,y,z)$ is an antiderivative of $F_1 = e^x(z+1)$ when y and z are fixed, therefore:

$$V(x,y,z) = \int e^x(z+1)\,dx = e^x(z+1) + g(y,z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Using (1) we get:

$$\begin{aligned} \frac{\partial}{\partial y}\left(e^x(z+1) + g(y,z)\right) &= -\cos y \\ g_y(y,z) &= -\cos y \end{aligned}$$

We integrate with respect to y, holding z fixed:

$$g(y,z) = \int -\cos y\,dy = -\sin y + h(z)$$

Substituting in (1) gives

$$V(x,y,z) = e^x(z+1) - \sin y + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2) we have

$$\begin{aligned} \frac{\partial}{\partial z}\left(e^x(z+1) - \sin y + h(z)\right) &= e^x \\ e^x + h'(z) &= e^x \\ h'(z) = 0 \quad &\Rightarrow \quad h(z) = c \end{aligned}$$

Substituting in (2) we obtain the general potential function:

$$V(x,y,z) = e^x(z+1) - \sin y + c$$

Choosing $c = 0$ gives the potential function:

$$V(x,y,z) = e^x(z+1) - \sin y$$

15. $\mathbf{F} = \left\langle 2xy + 5, x^2 - 4z, -4y \right\rangle$

SOLUTION We find a potential function $V(x,y,z)$ for $\mathbf{F}$, using the following steps.

Step 1. Use the condition $\frac{\partial V}{\partial x} = \mathbf{F}_1$. V is an antiderivative of $\mathbf{F}_1 = 2xy + 5$ when y and z are fixed, therefore,

$$V(x,y,z) = \int (2xy+5)\,dx = x^2y + 5x + g(y,z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = \mathbf{F}_2$. We have,

$$\frac{\partial}{\partial y}\left(x^2 y + 5x + g(y,z)\right) = x^2 - 4z$$

$$x^2 + g_y(y,z) = x^2 - 4z \quad \Rightarrow \quad g_y(y,z) = -4z$$

We integrate with respect to y, holding z fixed:

$$g(y,z) = \int -4z\,dy = -4zy + h(z)$$

Combining with (1) gives:

$$V(x,y,z) = x^2 y + 5x - 4zy + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = \mathbf{F}_3$. We have,

$$\frac{\partial}{\partial z}\left(x^2 y + 5x - 4zy + h(z)\right) = -4y$$

$$-4y + h'(z) = -4y$$

$$h'(z) = 0 \quad \Rightarrow \quad h(z) = c$$

Substituting in (2) we obtain the general potential function:

$$V(x,y,z) = x^2 y + 5x - 4zy + c$$

To compute the line integral we need one of the potential functions. We choose $c = 0$ to obtain the function,

$$V(x,y,z) = x^2 y + 5x - 4zy$$

16. $\mathbf{F} = \langle yze^{xy}, xze^{xy} - z, e^{xy} - y\rangle$

SOLUTION

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. $V(x,y,z)$ is an antiderivative of $F_1 = yze^{xy}$ when y and z are fixed, therefore:

$$V(x,y,z) = \int yze^{xy}\,dx = ze^{xy} + g(y,z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Using (1) we get:

$$\frac{\partial}{\partial y}\left(ze^{xy} + g(y,z)\right) = xze^{xy} - z$$

$$xze^{xy} + g_y(y,z) = xze^{xy} - z$$

$$g_y(y,z) = -z$$

We integrate with respect to y, holding z fixed:

$$g(y,z) = \int -z\,dy = -yz + h(z)$$

Substituting in (1) gives

$$V(x,y,z) = ze^{xy} - yz + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2) we have

$$\frac{\partial}{\partial z}\left(ze^{xy} - yz + h(z)\right) = e^{xy} - y$$

$$e^{xy} - y + h'(z) = e^{xy} - y$$

$$h'(z) = 0 \quad \Rightarrow \quad h(z) = c$$

Substituting in (2) we obtain the general potential function:

$$V(x,y,z) = ze^{xy} - yz + c$$

Choosing $c = 0$ gives the potential function:

$$V(x,y,z) = ze^{xy} - yz$$

17. Evaluate

$$\int_{\mathbf{c}} 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

over the path $\mathbf{c}(t) = (t^2, \sin(\pi t/4), e^{t^2-2t})$ for $0 \le t \le 2$.

SOLUTION A potential function is

$$V(x, y, z) = x^2yz$$

The path begins at $\mathbf{c}(0) = (0, 0, 1)$ and ends at $\mathbf{c}(2) = (4, 1, 1)$ so the line integral is

$$V(4, 1, 1) - V(0, 0, 1) = 16 - 0 = 16$$

18. Evaluate

$$\oint_{\mathcal{C}} \sin x\,dx + z\cos y\,dy + \sin y\,dz$$

where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

SOLUTION First we find a potential function for the vector field $\mathbf{F} = \langle \sin x, z\cos y, \sin y\rangle$.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$.

$$V(x, y, z) = \int \sin x\,dx = -\cos x + g(y, z) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Using (1) we get:

$$\begin{aligned} \frac{\partial}{\partial y}\left(-\cos x + g(y, z)\right) &= z\cos y \\ g_y(y, z) &= z\cos y \end{aligned}$$

We integrate with respect to y, holding z fixed:

$$g(y, z) = \int z\cos y\,dy = z\sin y + h(z)$$

Substituting in (1) gives

$$V(x, y, z) = -\cos x + z\sin y + h(z) \tag{2}$$

Step 3. Use the condition $\frac{\partial V}{\partial z} = F_3$. By (2) we have

$$\begin{aligned} \frac{\partial}{\partial z}\left(-\cos x + z\sin y + h(z)\right) &= \sin y \\ \sin y + h'(z) &= \sin y \\ h'(z) &= 0 \quad \Rightarrow \quad h(z) = c \end{aligned}$$

Substituting in (2) we obtain the general potential function:

$$V(x, y, z) = -\cos x + z\sin y + c$$

Choosing $c = 0$ gives the potential function:

$$V(x, y, z) = -\cos x + z\sin y$$

Since this is a conservative vector field integrated around a closed curve we have

$$\oint_{\mathcal{C}} \sin x\,dx + z\cos y\,dy + \sin y\,dz = 0$$

19. A vector field $\mathbf{F}$ and contour lines of a potential function for $\mathbf{F}$ are shown in Figure 17. Calculate the common value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ for the curves shown in Figure 17 oriented in the direction from P to Q.

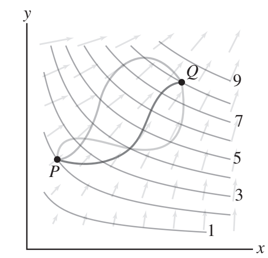


FIGURE 17

SOLUTION

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \nabla V \cdot d\mathbf{s} = V(Q) - V(P) = 8 - 2 = 6$$

20. Give a reason why the vector field $\mathbf{F}$ in Figure 18 is not conservative.

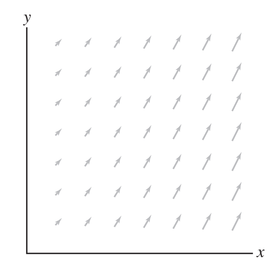


FIGURE 18

SOLUTION The line integral from the lower left corner to the upper right corner would clearly be larger if the path passed through the lower right region (where the vector field has large magnitude) than if it passed through the upper left region (where the vector field has small magnitude). If the vector field were conservative, the line integral would not depend on the path. Thus the vector field is not conservative.

21. Calculate the work expended when a particle is moved from O to Q along segments $\overline{OP}$ and $\overline{PQ}$ in Figure 19 in the presence of the force field $\mathbf{F} = \langle x^2, y^2 \rangle$. How much work is expended moving in a complete circuit around the square?

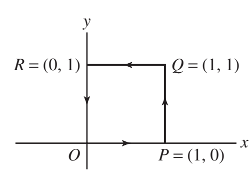
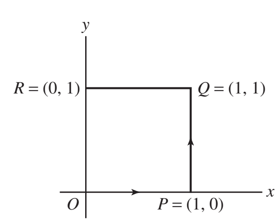


FIGURE 19

SOLUTION



Since $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(x^2) = 0$ and $\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(y^2) = 0$, we have $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. That is, $\mathbf{F}$ satisfies the cross partials condition, therefore $\mathbf{F}$ is conservative. We choose the function $\frac{x^3}{3} + \frac{y^3}{3}$, such that $\mathbf{F}$ is the gradient of the function. The potential energy is, thus, $V = -\frac{x^3}{3} - \frac{y^3}{3}$. The work done against $\mathbf{F}$ is computed by the Fundamental Theorem for Gradient vectors:

$$\text{Work against } \mathbf{F} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(O) = V(1, 1) - V(0) = -\frac{2}{3} - 0 = -\frac{2}{3}$$

(The negative sign is to be expected, as our force field is actually helping us move along OP and OQ. The line integral of a conservative field along a closed curve is zero, therefore the integral of $\mathbf{F}$ along the complete square is zero, and we get:

$$W = -\int_{OPQR} \mathbf{F} \cdot d\mathbf{s} = 0$$

22. Let $\mathbf{F} = \left\langle \frac{1}{x}, \frac{-1}{y} \right\rangle$. Calculate the work against F required to move an object from $(1, 1)$ to $(3, 4)$ along any path in the first quadrant.

SOLUTION $\mathbf{F}$ is a conservative force, since $\mathbf{F} = -\nabla V$ with potential energy $V(x, y) = \ln y - \ln x$. The work required to move an object from $(1, 1)$ to $(3, 4)$ along any path $\mathcal{C}$ is equal to the change in potential energy:

$$\text{Work against } \mathbf{F} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = V(3, 4) - V(1, 1) = (\ln 4 - \ln 3) - (\ln 1 - \ln 1) = \ln 4 - \ln 3$$

23. Compute the work W against the earth's gravitational field required to move a satellite of mass $m = 1000$ kg along any path from an orbit of altitude 4000 km to an orbit of altitude 6000 km.

SOLUTION Work against gravity is calculated with the integral

$$W = -\int_{\mathcal{C}} m\mathbf{F} \cdot d\mathbf{s} = 1000 \int_{\mathcal{C}} \nabla V \cdot d\mathbf{s} = 1000(V(r_2) - V(r_1))$$

Since r_1 and r_2 are measured from the center of the earth,

$$r_1 = 4 \times 10^6 + 6.4 \times 10^6 = 10.4 \times 10^6 \text{ meters}$$
$$r_2 = 6 \times 10^6 + 6.4 \times 10^6 = 12.4 \times 10^6 \text{ meters}$$

$$V(r) = -\frac{k}{r} \quad \Rightarrow \quad W = -\frac{1000k}{10^6} \left(\frac{1}{12.4} - \frac{1}{10.4} \right) \approx 6.2 \times 10^9 \text{ J}$$

24. An electric dipole with dipole moment $p = 4 \times 10^{-5}$ C-m sets up an electric field (in newtons per coulomb)

$$\mathbf{F}(x, y, z) = \frac{kp}{r^5} \left\langle 3xz, 3yz, 2z^2 - x^2 - y^2 \right\rangle$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ with distance in meters and $k = 8.99 \times 10^9$ N-m^2/C^2. Calculate the work against $\mathbf{F}$ required to move a particle of charge $q = 0.01$ C from $(1, -5, 0)$ to $(3, 4, 4)$. *Note:* The force on q is $q\mathbf{F}$ newtons.

SOLUTION We first calculate the potential function.

$$\begin{aligned} -\frac{\partial V}{\partial x} &= \frac{kpq3xz}{r^5} = qF_1 \\ \Rightarrow \quad V(x, y, z) &= -3kpqz \int \frac{x}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}\, dx \\ &= -3kpqz \frac{1}{2} \left(-\frac{2}{3} \right) \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + g(y, z) = \frac{kpqz}{r^3} + g(y, z) \end{aligned}$$

We may let $g(y, z) = 0$ since,

$$\begin{aligned} -\frac{\partial}{\partial y} \left(\frac{kpqz}{r^3} \right) &= -kpqz \left(-\frac{3}{2} \right) \frac{2y}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{kpq3yz}{r^5} = qF_2 \\ -\frac{\partial}{\partial z} \left(\frac{kpqz}{r^3} \right) &= -\frac{kpq}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - kpqz \left(-\frac{3}{2} \right) \frac{2z}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= kpq \frac{-(x^2 + y^2 + z^2) + 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{kpq(2z^2 - x^2 - y^2)}{r^5} = qF_3 \end{aligned}$$

Hence $V = \frac{kpqz}{r^3}$, so

$$V(3,4,4) - V(1,-5,0) = V(3,4,4) = \frac{(0.01)(4\times 10^{-5})(8.99\times 10^9)(4)}{(3^2+4^2+4^2)^{\frac{3}{2}}} \approx 54.8 \text{ J}$$

25. On the surface of the earth, the gravitational field (with z as vertical coordinate measured in meters) is $\mathbf{F} = \langle 0, 0, -g\rangle$.

(a) Find a potential function for $\mathbf{F}$.

(b) Beginning at rest, a ball of mass $m = 2$ kg moves under the influence of gravity (without friction) along a path from $P = (3, 2, 400)$ to $Q = (-21, 40, 50)$. Find the ball's velocity when it reaches Q.

SOLUTION

(a) By inspection $\mathbf{F} = -\nabla V$ for $V(x, y, z) = gz$.

(b) The force of gravity is $m\mathbf{F} = \langle 0, 0, -mg\rangle$, therefore $m\mathbf{F} = -\nabla V$ for $V(x, y, z) = mgz$. The work performed moving the ball from P to Q is the line integral of $m\mathbf{F}$ over the path. Since $m\mathbf{F}$ is conservative, the energy is independent of the path connecting the two points. Using the Fundamental Theorem for Gradient Vector Fields we have:

$$W = -\int_{\mathbf{c}} m\mathbf{F}\cdot d\mathbf{s} = V(-21,40,50) - V(3,2,400) = 2\cdot 9.8(50-400) = -6860 \text{ joules}$$

By conservation of energy, the kinetic energy of the ball will be 6860 joules, so

$$\frac{mv^2}{2} = 6860 \Rightarrow v = \sqrt{\frac{2\cdot 6860}{2}} \approx 82.8 \text{ m/s}$$

26. An electron at rest at $P = (5, 3, 7)$ moves along a path ending at $Q = (1, 1, 1)$ under the influence of the electric field (in newtons per coulomb)

$$\mathbf{F}(x, y, z) = 400(x^2 + z^2)^{-1}\langle x, 0, z\rangle$$

(a) Find a potential function for $\mathbf{F}$.

(b) What is the electron's speed at point Q? Use Conservation of Energy and the value $q_e/m_e = -1.76\times 10^{11}$ C/kg, where q_e and m_e are the charge and mass on the electron, respectively.

SOLUTION

(a)

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{400x}{x^2+z^2} = F_1\\ \Rightarrow \quad V(x,y,z) &= \int \frac{400x}{x^2+z^2}\,dx\\ &= 200\ln(x^2+z^2) + g(y,z)\end{aligned}$$

We may let $g(y, z) = 0$ since,

$$\begin{aligned}\frac{\partial}{\partial y}(200\ln(x^2+z^2)) &= 0 = F_2\\ \frac{\partial}{\partial z}(200\ln(x^2+z^2)) &= \frac{400z}{x^2+z^2} = F_3\end{aligned}$$

Hence $V = 200\ln(x^2 + z^2)$ volts.

(b) The energy at the point Q will be $q_e(V(1, 1, 1) - V(5, 3, 7))$. By conservation of energy,

$$\begin{aligned}\frac{m_e v^2}{2} &= q_e(V(1,1,1) - V(5,3,7))\\ \Rightarrow \quad v &= \sqrt{\frac{2q_e(V(1,1,1)-V(5,3,7))}{m_e}} = \sqrt{400\frac{q_e}{m_e}(\ln(1^2+1^2) - \ln(5^2+7^2))} \approx 1.59\times 10^7 \text{ m/s}\end{aligned}$$

27. Let $\mathbf{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ be the vortex field. Determine $\displaystyle\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ for each of the paths in Figure 20.

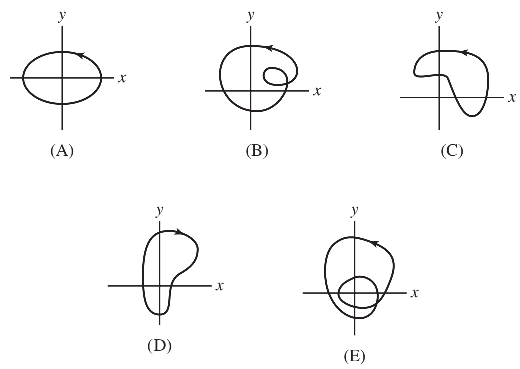


FIGURE 20

SOLUTION Since the cross partials of **F** are equal, **F** has the property,

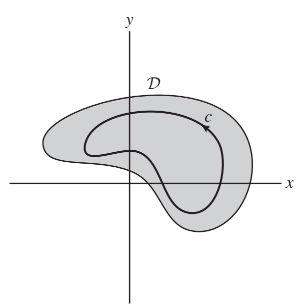
$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 2\pi n$$

where **c** is a closed curve not passing through the origin, and n is the number of times **c** winds around the origin (n is negative if n winds in the clockwise direction). We use this property to compute the line integrals of **F** over the paths in Figure 18:

(A) The path (A) winds around the origin one time in the counterclockwise direction hence the line integral is $2\pi \cdot 1 = 2\pi$.

(B) The point (B) winds around the origin one time in the counterclockwise direction hence the line integral is $2\pi \cdot 1 = 2\pi$.

(C) The path (C) does not encounter the origin, hence the line integral is $2\pi \cdot 0 = 0$. Notice that there exists a simply connected domain $\mathcal{D}$, not including the origin, so that the path c and the region inside **c** are in $\mathcal{D}$. Therefore, Theorem 4 applies in $\mathcal{D}$ and **F** is a gradient vector in $\mathcal{D}$. Consequently, the line integral of **F** over **c** is zero.



(D) This path winds around the origin one time in the clockwise direction, hence $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 2\pi \cdot (-1) = -2\pi$.

(E) The path winds around the origin twice in the counterclockwise direction, hence the line integral is $2\pi \cdot 2 = 4\pi$.

28. The vector field $\mathbf{F} = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2} \right\rangle$ is defined on the domain $\mathcal{D} = \{(x, y) \neq (0, 0)\}$.

(a) Is $\mathcal{D}$ simply-connected?

(b) Show that **F** satisfies the cross-partial condition. Does this guarantee that **F** is conservative?

(c) Show that **F** is conservative on $\mathcal{D}$ by finding a potential function.

(d) Do these results contradict Theorem 4?

SOLUTION

(a) $\mathcal{D}$ is not simply-connected since it has a “hole" at the origin.

(b) We compute the partials of **F**:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2yx}{\left(x^2+y^2\right)^2}$$

The cross partials are equal in $\mathcal{D}$, however this does not guarantee **F** is conservative since $\mathcal{D}$ is not simply connected.

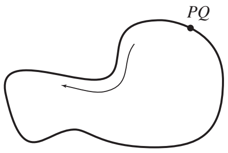
(c) We compute the gradient of $V(x, y) = \frac{1}{2}\ln\left(x^2 + y^2\right)$:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial x} \right\rangle = \frac{1}{2}\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle = \mathbf{F}$$

(d) The requirement in Theorem 4 (that the domain be simply connected) is a sufficient condition for a vector field with equal cross-partial to have a potential function. It is not necessary, since as in our example, even if the domain is not simply-connected the field may have a gradient function. Moreover, for any closed curve in $\mathcal{D}$, V have the same value after completing one round along c. This is perhaps best seen by noting that $V = \log(r)$ in polar coordinates, which will be independent of θ. Therefore,

$$\int_c \mathbf{F} \cdot d\mathbf{s} = 0$$

Hence, **F** is conservative.



Further Insights and Challenges

29. Suppose that **F** is defined on $\mathbf{R}^3$ and that $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$ for all closed paths **c** in $\mathbf{R}^3$. Prove:

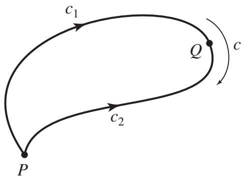
(a) **F** is path-independent; that is, for any two paths $\mathbf{c}_1$ and $\mathbf{c}_2$ in $\mathcal{D}$ with the same initial and terminal points,

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$

(b) **F** is conservative.

SOLUTION

(a) Choose two distinct points P and Q, and let $\mathbf{c}_1$ and $\mathbf{c}_2$ be paths from P to Q. We construct a path from P to P by first using $\mathbf{c}_1$ to reach Q, then using $\mathbf{c}_2$ with its orientation reversed to return to P. (This reversed path is designated $-\mathbf{c}_2$.) Such a closed path **c** can be represented as a difference $\mathbf{c} = \mathbf{c}_1 - \mathbf{c}_2$. (See figure below)



The closed loop **c** is represented as $\mathbf{c}_1 - \mathbf{c}_2$.

Thus,

$$\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{-\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$
$$= \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$

Since the problem states that the integral around any closed path is zero, we have

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s} = 0 \quad \Rightarrow \quad \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$$

(b) Since **F** is defined for all of $\mathbb{R}^3$, it is certainly defined in a simply connected domain $\mathcal{D}$. Since we have just established that **F** is also path independent, **F** is conservative by Theorem 2.

16.4 Parametrized Surfaces and Surface Integrals (LT Section 17.4)

Preliminary Questions

1. What is the surface integral of the function $f(x, y, z) = 10$ over a surface of total area 5?

SOLUTION Using Surface Integral and Surface Area we have:

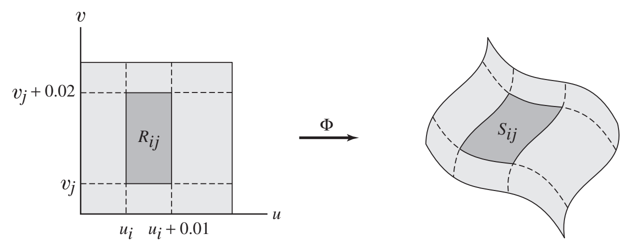
$$\iint_S f(x, y, z)\, dS = \iint_{\mathcal{D}} f(\Phi(u, v))\, \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} 10\|\mathbf{n}(u, v)\|\, du\, dv$$
$$= 10 \iint_{\mathcal{D}} \|\mathbf{n}(u, v)\|\, du\, dv = 10\,\text{Area}(S) = 10 \cdot 5 = 50$$

2. What interpretation can we give to the length $\|\mathbf{n}\|$ of the normal vector for a parametrization $G(u, v)$?

SOLUTION The approximation:

$$\text{Area}\left(S_{ij}\right) \approx \|\mathbf{n}\left(u_{ij}, v_{ij}\right)\| \text{Area}\left(R_{ij}\right)$$

tells that $\|\mathbf{n}\|$ is a distortion factor that indicates how much the area of a small rectangle R_{ij} is altered under the map ϕ.



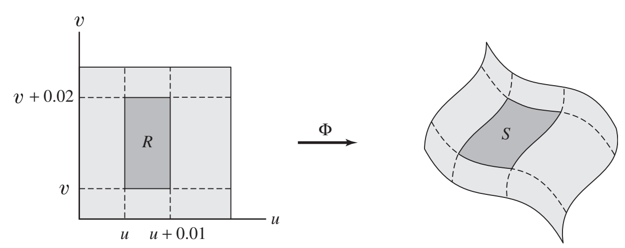
3. A parametrization maps a rectangle of size 0.01×0.02 in the uv-plane onto a small patch $\mathcal{S}$ of a surface. Estimate Area($\mathcal{S}$) if $\mathbf{T}_u \times \mathbf{T}_v = \langle 1, 2, 2\rangle$ at a sample point in the rectangle.

SOLUTION We use the estimation

$$\text{Area}(S) \approx \|\mathbf{n}(u, v)\|\text{Area}(R)$$

where $\mathbf{n}(u, v) = \mathbf{T}_u \times \mathbf{T}_v$ at a sample point in R. We get:

$$\text{Area}(S) \approx \|\,\langle 1, 2, 2\rangle\,\| \cdot 0.01 \cdot 0.02 = \sqrt{1^2 + 2^2 + 2^2} \cdot 0.0002 = 0.0006$$



4. A small surface $\mathcal{S}$ is divided into three small pieces, each of area 0.2. Estimate $\iint_{\mathcal{S}} f(x, y, z)\, dS$ if $f(x, y, z)$ takes the values 0.9, 1, and 1.1 at sample points in these three pieces.

SOLUTION We use the approximation obtained by the Riemann Sum:

$$\iint_S f(x, y, z)\, dS \approx \sum_{ij} f\left(P_{ij}\right) \text{Area}\left(S_{ij}\right) = 0.9 \cdot 0.2 + 1 \cdot 0.2 + 1.1 \cdot 0.2 = 0.6$$

5. A surface $\mathcal{S}$ has a parametrization whose domain is the square $0 \le u, v \le 2$ such that $\|\mathbf{n}(u, v)\| = 5$ for all (u, v). What is Area($\mathcal{S}$)?

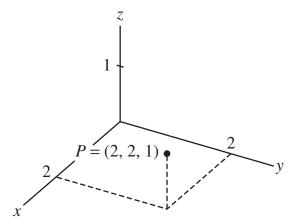
SOLUTION Writing the surface area as a surface integral where $\mathcal{D}$ is the square $[0, 2] \times [0, 2]$ in the uv-plane, we have:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} 5\, du\, dv = 5 \iint_{\mathcal{D}} 1\, du\, dv = 5\text{Area}(\mathcal{D}) = 5 \cdot 2^2 = 20$$

6. What is the outward-pointing unit normal to the sphere of radius 3 centered at the origin at $P = (2, 2, 1)$?

SOLUTION The outward-pointing normal to the sphere of radius $R = 3$ centered at the origin is the following vector:

$$\langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle \qquad (1)$$



We compute the values in (1) corresponding to $P = (2, 2, 1)$: $x = y = 2$, $z = 1$ hence $0 \le \theta \le \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$. We get:

$$\cos\phi = \frac{z}{\rho} = \frac{1}{3} \quad \Rightarrow \quad \sin\phi = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$$

$$\cos\theta = \frac{x}{\rho \sin\phi} = \frac{2}{3 \cdot \frac{2\sqrt{2}}{3}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \sin\theta = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Substituting in (1) we get the following unit normal:

$$\left\langle \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{3}, \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{3}, \frac{1}{3} \right\rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

Exercises

1. Match each parametrization with the corresponding surface in Figure 16.

(a) $(u, \cos v, \sin v)$
(b) $(u, u + v, v)$
(c) (u, v^3, v)
(d) $(\cos u \sin v, 3 \cos u \sin v, \cos v)$
(e) $(u, u(2 + \cos v), u(2 + \sin v))$

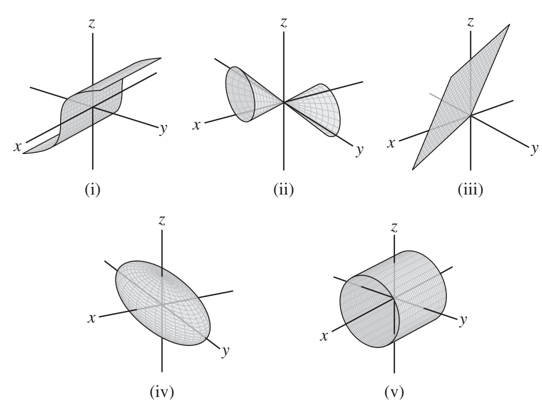


FIGURE 16

SOLUTION (a) = (v), because the y and z coordinates describe a circle with fixed radius.
(b) = (iii), because the coordinates are all linear in u and v.
(c) = (i), because the parametrization gives $y = z^3$.
(d) = (iv), an ellipsoid.
(e) = (ii), because the y and z coordinates describe a circle with varying radius.

2. Show that $G(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$ parametrizes the paraboloid $z = 1 - x^2 - y^2$. Describe the grid curves of this parametrization.

SOLUTION We substitute $x = r\cos\theta$ and $y = r\sin\theta$ in the right-hand side of the equation of the cone, and verify that the result is $1 - r^2$:

$$1 - x^2 - y^2 = 1 - (r\cos\theta)^2 - (r\sin\theta)^2 = 1 - r^2\left(\cos^2\theta + \sin^2\theta\right) = 1 - r^2 \cdot 1 = 1 - r^2 = z$$

Moreover, for every x, y, z satisfying $z = 1 - x^2 - y^2$ there are suitable values of r and θ so that $x = r\cos\theta$, $y = r\sin\theta$ and $z = 1 - r^2$. We conclude that $G(r, \theta)$ parametrizes the whole paraboloid $z = 1 - x^2 - y^2$. The grid curves on the paraboloid through $P = (r_0, \theta_0)$ are:

- r-grid curve:

$$\left(r\cos\theta_0, r\sin\theta_0, 1 - r^2\right) = \text{parabola } z = 1 - r^2 \text{ in the plane } \sin(\theta_0)x - \cos(\theta_0)y = 0.$$

- θ-grid curve:

$$\left(r_0\cos\theta, r_0\sin\theta, 1 - r_0^2\right) = \text{circle of radius } r_0 \text{ at height } 1 - r_0^2.$$

3. Show that $G(u, v) = (2u + 1, u - v, 3u + v)$ parametrizes the plane $2x - y - z = 2$. Then:

(a) Calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{n}(u, v)$.

(b) Find the area of $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u, v) : 0 \le u \le 2, 0 \le v \le 1\}$.

(c) Express $f(x, y, z) = yz$ in terms of u and v, and evaluate $\iint_{\mathcal{S}} f(x, y, z)\, dS$.

SOLUTION We show that $x = 2u + 1$, $y = u - v$, and $z = 3u + v$ satisfy the equation of the plane,

$$2x - y - z = 2(2u + 1) - (u - v) - (3u + v) = 4u + 2 - u + v - 3u - v = 2$$

Moreover, for any x, y, z satisfying $2x - y - z = z$, there are values of u and v such that $x = 2u + 1$, $y = u - v$, and $z = 3u + v$, since the following equations can be solved for u and v:

$$\begin{aligned} x &= 2u + 1 \\ y &= u - v \\ z &= 3u + v \\ 2x - y - z &= 2 \end{aligned} \quad \Rightarrow \quad u = \frac{x - 1}{2}, \quad v = \frac{x - 1}{2} - y$$

We conclude that $\Phi(u, v)$ parametrizes the whole plane $2x - y - z = 2$.

(a) The tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are:

$$\mathbf{T}_u = \frac{\partial\phi}{\partial u} = \frac{\partial}{\partial u}(2u + 1, u - v, 3u + v) = \langle 2, 1, 3\rangle$$

$$\mathbf{T}_v = \frac{\partial\phi}{\partial v} = \frac{\partial}{\partial v}(2u + 1, u - v, 3u + v) = \langle 0, -1, 1\rangle$$

The normal vector is the following cross product:

$$\mathbf{n}(u, v) = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{k}$$

$$= 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = \langle 4, -2, -2\rangle$$

(b) That area of $S = \Phi(\mathcal{D})$ is the following surface integral:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} \|\langle 4, -2, -2\rangle\|\, du\, dv = \sqrt{24}\iint_{\mathcal{D}} 1\, du\, dv$$

$$= \sqrt{24}\,\text{Area}(\mathcal{D}) = \sqrt{24} \cdot 2 \cdot 1 = 4\sqrt{6}$$

(c) We express $f(x, y, z) = yz$ in terms of the parameters u and v:

$$f(\phi(u, v)) = (u - v)(3u + v) = 3u^2 - 2uv - v^2$$

Using the Theorem on Surface Integrals we have:

$$\iint_S f(x, y, z)\, dS = \iint_{\mathcal{D}} f(\phi(u, v))\, \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} \left(3u^2 - 2uv - v^2\right) \|\langle 4, -2, -2\rangle\|\, du\, dv$$

$$= \sqrt{24}\int_0^1 \int_0^2 \left(3u^2 - 2uv - v^2\right) du\, dv = \sqrt{24}\int_0^1 \left(u^3 - u^2 v - v^2 u\right)\Big|_{u=0}^2 dv$$

$$= \sqrt{24}\int_0^1 \left(8 - 4v - 2v^2\right) dv = \sqrt{24}\left(8v - 2v^2 - \frac{2}{3}v^3\right)\Big|_0^1 = \frac{32\sqrt{6}}{3}$$

4. Let $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u, v) : u^2 + v^2 \le 1, u \ge 0, v \ge 0\}$ and G is as defined in Exercise 3.

(a) Calculate the surface area of $\mathcal{S}$.

(b) Evaluate $\iint_{\mathcal{S}} (x - y)\, dS$. *Hint:* Use polar coordinates.

SOLUTION The surface S is given by the parametrization:

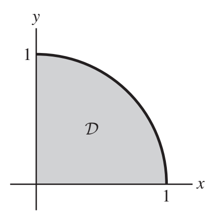
$$\Phi(u, v) = (2u + 1, u - v, 3u + v).$$

In Exercise 3 we computed the normal vector:

$$\mathbf{n}(u, v) = \langle 4, -2, 2\rangle \quad \Rightarrow \quad \|\mathbf{n}(u, v)\| = \sqrt{24}.$$

The surface area for u and v in the domain $\mathcal{D} = \left\{(u, v) : u^2 + v^2 \le 1, u \ge 0, v \ge 0\right\}$ is the surface integral:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} \sqrt{24}\, du\, dv = \sqrt{24} \iint_{\mathcal{D}} 1\, du\, dv = \sqrt{24}\, \text{Area}(\mathcal{D}) \tag{1}$$



$\mathcal{D}$ is the quarter disc $u^2 + v^2 \le 1$ in the first quadrant, hence it has the area $\frac{\pi \cdot 1^2}{4} = \frac{\pi}{4}$. Substituting in (1) gives:

$$\text{Area}(S) = \sqrt{24} \cdot \frac{\pi}{4} = \frac{\pi\sqrt{6}}{2}.$$

To compute the surface integral of $f(x, y, z) = x - y$, we first express f in terms of the parameters u and v:

$$f(\Phi(u, v)) = (2u + 1) - (u - v) = u + v + 1.$$

We now use the Theorem on surface integrals to compute the integral:

$$\iint_S f(x, y, z)\, dS = \iint_{\mathcal{D}} f(\Phi(u, v))\, \|\mathbf{n}(u, v)\|\, du\, dv = \iint_{\mathcal{D}} (u + v + 1)\sqrt{24}\, du\, dv$$

We convert the integral to polar coordinates:

$$u = r\cos\theta, \quad v = r\sin\theta$$

Then, the region of integration is:

$$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

We get:

$$\begin{aligned}
\iint_S f(x, y, z)\, dS &= \int_0^{\pi/2} \int_0^1 (r\cos\theta + r\sin\theta + 1)\sqrt{24}\, r\, dr\, d\theta = \sqrt{24} \int_0^{\pi/2} \int_0^1 \left(r^2(\cos\theta + \sin\theta) + r\right) dr\, d\theta \\
&= \sqrt{24} \int_0^{\pi/2} \frac{r^3(\cos\theta + \sin\theta)}{3} + \frac{r^2}{2}\bigg|_{r=0}^{1} d\theta = \sqrt{24} \int_0^{\pi/2} \left(\frac{\cos\theta + \sin\theta}{3} + \frac{1}{2}\right) d\theta \\
&= \sqrt{24} \left(\frac{\sin\theta - \cos\theta}{3} + \frac{\theta}{2}\right)\bigg|_0^{\pi/2} = \sqrt{24}\left(\frac{2}{3} + \frac{\pi}{4}\right) = \frac{8 + 3\pi}{\sqrt{6}}
\end{aligned}$$

5. Let $G(x, y) = (x, y, xy)$.

(a) Calculate $\mathbf{T}_x$, $\mathbf{T}_y$, and $\mathbf{n}(x, y)$.

(b) Let S be the part of the surface with parameter domain $\mathcal{D} = \{(x, y) : x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$. Verify the following formula and evaluate using polar coordinates:

$$\iint_S 1\, dS = \iint_{\mathcal{D}} \sqrt{1 + x^2 + y^2}\, dx\, dy$$

(c) Verify the following formula and evaluate:

$$\iint_S z\,dS = \int_0^{\pi/2}\int_0^1 (\sin\theta\cos\theta) r^3\sqrt{1+r^2}\,dr\,d\theta$$

SOLUTION

(a) The tangent vectors are:

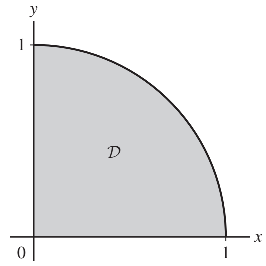
$$\mathbf{T}_x = \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x, y, xy) = \langle 1, 0, y\rangle$$
$$\mathbf{T}_y = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x, y, xy) = \langle 0, 1, x\rangle$$

The normal vector is the cross product:

$$\mathbf{n}(x,y) = \mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \begin{vmatrix} 0 & y \\ 1 & x \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & y \\ 0 & x \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= -y\mathbf{i} - x\mathbf{j} + \mathbf{k} = \langle -y, -x, 1\rangle$$

(b) Using the Theorem on evaluating surface integrals we have:

$$\iint_S 1\,dS = \iint_{\mathcal{D}} \|\mathbf{n}(x,y)\|\,dx\,dy = \iint_{\mathcal{D}} \|\langle -y, -x, 1\rangle\|\,dx\,dy = \iint_{\mathcal{D}} \sqrt{y^2+x^2+1}\,dx\,dy$$



We convert the integral to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The new region of integration is:

$$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

We get:

$$\iint_S 1\,dS = \int_0^{\pi/2}\int_0^1 \sqrt{r^2+1}\cdot r\,dr\,d\theta = \int_0^{\pi/2}\left(\int_0^1 \sqrt{r^2+1}\cdot r\,dr\right)d\theta$$
$$= \int_0^{\pi/2}\left(\int_1^2 \frac{\sqrt{u}}{2}\,du\right)d\theta = \int_0^{\pi/2}\frac{2\sqrt{2}-1}{3}\,d\theta = \frac{\left(2\sqrt{2}-1\right)\pi}{6}$$

(c) The function z expressed in terms of the parameters x, y is $f(\Phi(x,y)) = xy$. Therefore,

$$\iint_S z\,dS = \iint_{\mathcal{D}} xy\cdot\|\mathbf{n}(x,y)\|\,dx\,dy = \iint_{\mathcal{D}} xy\sqrt{1+x^2+y^2}\,dx\,dy$$

We compute the double integral by converting it to polar coordinates. We get:

$$\iint_S z\,dS = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (\sin\theta\cos\theta) r^3\sqrt{1+r^2}\,dr\,d\theta$$
$$= \left(\int_0^{\pi/2}(\sin\theta\cos\theta)\,d\theta\right)\left(\int_0^1 r^3\sqrt{1+r^2}\,dr\right) \tag{1}$$

We compute each integral in (1). Using the substitution $u = 1+r^2$, $du = 2r\,dr$ we get:

$$\int_0^1 r^3\sqrt{1+r^2}\,dr = \int_0^1 r^2\sqrt{1+r^2}\cdot r\,dr = \int_1^2 \left(u^{3/2}-u^{1/2}\right)\frac{du}{2} = \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\Big|_1^2 = \frac{2\left(\sqrt{2}+1\right)}{15}$$

Also,

$$\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta = -\frac{\cos 2\theta}{4}\bigg|_0^{\pi/2} = \frac{1}{2}$$

We substitute the integrals in (1) to obtain the following solution:

$$\iint_S z\, dS = \frac{1}{2} \cdot \frac{2\left(\sqrt{2}+1\right)}{15} = \frac{\sqrt{2}+1}{15}$$

6. A surface $\mathcal{S}$ has a parametrization $G(u,v)$ whose domain $\mathcal{D}$ is the square in Figure 17. Suppose that G has the following normal vectors:

$$\mathbf{n}(A) = \langle 2, 1, 0\rangle, \quad \mathbf{n}(B) = \langle 1, 3, 0\rangle$$
$$\mathbf{n}(C) = \langle 3, 0, 1\rangle, \quad \mathbf{n}(D) = \langle 2, 0, 1\rangle$$

Estimate $\iint_{\mathcal{S}} f(x,y,z)\, dS$, where f is a function such that $f(G(u,v)) = u + v$.

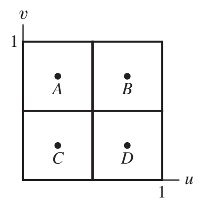


FIGURE 17

SOLUTION We estimate the surface integral by the following Riemann sum:

$$\iint_S f(x,y,z)\, dS = \Delta u \Delta v\, (f(A)\|\mathbf{n}(A)\| + f(B)\|\mathbf{n}(B)\| + f(C)\|\mathbf{n}(C)\| + f(D)\|\mathbf{n}(D)\|) \tag{1}$$

In the given grid, we have $\Delta u = \Delta v = \frac{1}{2}$. We compute the lengths of the normal vectors:

$$\|\mathbf{n}(A)\| = \|\langle 2,1,0\rangle\| = \sqrt{4+1+0} = \sqrt{5}$$
$$\|\mathbf{n}(B)\| = \|\langle 1,3,0\rangle\| = \sqrt{1+9+0} = \sqrt{10}$$
$$\|\mathbf{n}(C)\| = \|\langle 3,0,1\rangle\| = \sqrt{9+0+1} = \sqrt{10}$$
$$\|\mathbf{n}(D)\| = \|\langle 2,0,1\rangle\| = \sqrt{4+0+1} = \sqrt{5}$$

The sample points are $A = \left(\frac{1}{4}, \frac{3}{4}\right)$, $B = \left(\frac{3}{4}, \frac{3}{4}\right)$, $C = \left(\frac{1}{4}, \frac{1}{4}\right)$, $D = \left(\frac{3}{4}, \frac{1}{4}\right)$, hence the values of f at the sample points are:

$$f(A) = \frac{1}{4} + \frac{3}{4} = 1, \quad f(B) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}, \quad f(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad f(D) = \frac{3}{4} + \frac{1}{4} = 1$$

Substituting the values in (1) gives the following estimation:

$$\iint_S f(x,y,z)\, dS = \frac{1}{2} \cdot \frac{1}{2}\left(1 \cdot \sqrt{5} + \frac{3}{2} \cdot \sqrt{10} + \frac{1}{2} \cdot \sqrt{10} + 1 \cdot \sqrt{5}\right) = \frac{\sqrt{5}+\sqrt{10}}{2}$$

In Exercises 7–10, calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{n}(u,v)$ for the parametrized surface at the given point. Then find the equation of the tangent plane to the surface at that point.

7. $G(u,v) = (2u+v, u-4v, 3u)$; $\quad u = 1, \quad v = 4$

SOLUTION The tangent vectors are the following vectors,

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}(2u+v, u-4v, 3u) = \langle 2, 1, 3\rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v}(2u+v, u-4v, 3u) = \langle 1, -4, 0\rangle$$

The normal is the cross product:

$$\mathbf{n}(u,v) = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}\mathbf{k}$$

$$= 12\mathbf{i} + 3\mathbf{j} - 9\mathbf{k} = 3\langle 4, 1, -3\rangle$$

The equation of the plane passing through the point $P : \Phi(1, 4) = (6, -15, 3)$ with the normal vector $\langle 4, 1, -3\rangle$ is:

$$\langle x - 6, y + 15, z - 3\rangle \cdot \langle 4, 1, -3\rangle = 0$$

or

$$\begin{aligned} 4(x - 6) + y + 15 - 3(z - 3) &= 0 \\ 4x + y - 3z &= 0 \end{aligned}$$

8. $G(u, v) = (u^2 - v^2, u + v, u - v);\qquad u = 2,\quad v = 3$

SOLUTION We compute the tangent vectors:

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}\left(u^2 - v^2, u + v, u - v\right) = \langle 2u, 1, 1\rangle \\ \mathbf{T}_v &= \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v}\left(u^2 - v^2, u + v, u - v\right) = \langle -2v, 1, -1\rangle \end{aligned}$$

The normal is the cross product of $\mathbf{T}_u$ and $\mathbf{T}_v$. That is:

$$\begin{aligned} \mathbf{n}(u, v) = \mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 1 & 1 \\ -2v & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2u & 1 \\ -2v & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2u & 1 \\ -2v & 1 \end{vmatrix}\mathbf{k} \\ &= -2\mathbf{i} - 2(v - u)\mathbf{j} + 2(u + v)\mathbf{k} = 2\langle -1, u - v, v + u\rangle \end{aligned}$$

We compute the tangency point and the normal $\mathbf{n}(u, v)$ at this point:

$$\begin{aligned} P = \Phi(2, 3) &= \left(2^2 - 3^2, 2 + 3, 2 - 3\right) = (-5, 5, -1) \\ \mathbf{n}(2, 3) &= 2\langle -1, 2 - 3, 3 + 2\rangle = 2\langle -1, -1, 5\rangle \end{aligned}$$

The equation of the plane passing through the point $P = (-5, 5, -1)$ with the normal vector $\langle -1, -1, 5\rangle$ is:

$$\langle x + 5, y - 5, z + 1\rangle \cdot \langle -1, -1, 5\rangle = 0$$

or

$$\begin{aligned} -(x + 5) - (y - 5) + 5(z + 1) &= 0 \\ -x - y + 5z + 5 &= 0 \end{aligned}$$

9. $G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi);\qquad \theta = \frac{\pi}{2},\quad \phi = \frac{\pi}{4}$

SOLUTION We compute the tangent vectors:

$$\begin{aligned} \mathbf{T}_\theta &= \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0\rangle \\ \mathbf{T}_\phi &= \frac{\partial \Phi}{\partial \phi} = \frac{\partial}{\partial \phi}(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi\rangle \end{aligned}$$

The normal vector is the cross product:

$$\begin{aligned} \mathbf{n}(\theta, \phi) = \mathbf{T}_\theta \times \mathbf{T}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta \sin\phi & \cos\theta \sin\phi & 0 \\ \cos\theta \cos\phi & \sin\theta \cos\phi & -\sin\phi \end{vmatrix} \\ &= \left(-\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\sin\theta\ \sin^2\phi\right)\mathbf{j} + \left(-\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \cos\phi \sin\phi\right)\mathbf{k} \\ &= -\left(\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\sin\theta\ \sin^2\phi\right)\mathbf{j} - (\sin\phi \cos\phi)\mathbf{k} \end{aligned}$$

The tangency point and the normal at this point are,

$$\begin{aligned} P = \Phi\left(\frac{\pi}{2}, \frac{\pi}{4}\right) &= \left(\cos\frac{\pi}{2} \sin\frac{\pi}{4}, \sin\frac{\pi}{2} \sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\ \mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) &= -\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k} = -\frac{1}{2}(\mathbf{j} + \mathbf{k}) = -\frac{1}{2}\langle 0, 1, 1\rangle \end{aligned}$$

The equation of the plane orthogonal to the vector $\langle 0,1,1\rangle$ and passing through $P=\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ is:

$$\left\langle x, y-\frac{\sqrt{2}}{2}, z-\frac{\sqrt{2}}{2}\right\rangle\cdot\langle 0,1,1\rangle=0$$

or

$$y-\frac{\sqrt{2}}{2}+z-\frac{\sqrt{2}}{2}=0$$
$$y+z=\sqrt{2}$$

10. $G(r,\theta)=(r\cos\theta, r\sin\theta, 1-r^2);\qquad r=\frac{1}{2},\quad \theta=\frac{\pi}{4}$

SOLUTION The tangent vectors are:

$$\mathbf{T}_r=\frac{\partial\Phi}{\partial r}=\frac{\partial}{\partial r}\left(r\cos\theta, r\sin\theta, 1-r^2\right)=\langle\cos\theta,\sin\theta,-2r\rangle$$
$$\mathbf{T}_\theta=\frac{\partial\Phi}{\partial\theta}=\frac{\partial}{\partial\theta}\left(r\cos\theta, r\sin\theta, 1-r^2\right)=\langle -r\sin\theta, r\cos\theta, 0\rangle$$

The normal vector is the cross product of $\mathbf{T}_r$ and $\mathbf{T}_\theta$. That is,

$$\mathbf{n}(r,\theta)=\mathbf{T}_r\times\mathbf{T}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & -2r\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}=\left(2r^2\cos\theta\right)\mathbf{i}+\left(2r^2\sin\theta\right)\mathbf{j}+\left(r\cos^2\theta+r\sin^2\theta\right)\mathbf{k}$$
$$=\left(2r^2\cos\theta\right)\mathbf{i}+\left(2r^2\sin\theta\right)\mathbf{j}+r\mathbf{k}=r\,\langle 2r\cos\theta, 2r\sin\theta, 1\rangle$$

We compute the tangency point and the normal vector at this point. We get:

$$P=\Phi\left(\frac{1}{2},\frac{\pi}{4}\right)=\left(\frac{1}{2}\cos\frac{\pi}{4},\frac{1}{2}\sin\frac{\pi}{4},1-\left(\frac{1}{2}\right)^2\right)=\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{3}{4}\right)$$
$$\mathbf{N}\left(\frac{1}{2},\frac{\pi}{4}\right)=\frac{1}{2}\left\langle 2\cdot\frac{1}{2}\cos\frac{\pi}{4}, 2\cdot\frac{1}{2}\sin\frac{\pi}{4}, 1\right\rangle=\frac{1}{2}\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right\rangle$$

The equation of the plane through P, with normal vector $\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right\rangle$ is:

$$\left\langle x-\frac{1}{2\sqrt{2}}, y-\frac{1}{2\sqrt{2}}, z-\frac{3}{4}\right\rangle\cdot\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right\rangle=0$$

or

$$\frac{1}{\sqrt{2}}\left(x-\frac{1}{2\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\left(y-\frac{1}{2\sqrt{2}}\right)+z-\frac{3}{4}=0$$
$$\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y+z=\frac{5}{4}$$
$$2\sqrt{2}x+2\sqrt{2}y+4=5$$

11. Use the normal vector computed in Exercise 8 to estimate the area of the small patch of the surface $G(u,v)=(u^2-v^2, u+v, u-v)$ defined by

$$2\le u\le 2.1,\qquad 3\le v\le 3.2$$

SOLUTION We denote the rectangle $\mathcal{D}=\{(u,v): 2\le u\le 2.1,\ 3\le v\le 3.2\}$. Using the sample point corresponding to $u=2$, $v=3$ we obtain the following estimation for the area of $S=\Phi(\mathcal{D})$:

$$\text{Area}(S)\approx\|\mathbf{n}(2,3)\|\text{Area}(\mathcal{D})=\|\mathbf{n}(2,3)\|\cdot 0.1\cdot 0.2=0.02\|\mathbf{n}(2,3)\| \tag{1}$$

In Exercise 8 we found that $\mathbf{n}(2,3)=2\,\langle -1,-1,5\rangle$. Therefore,

$$\|\mathbf{n}(2,3)\|=2\sqrt{1^2+1^2+5^2}=2\sqrt{27}$$

Substituting in (1) gives the following estimation:

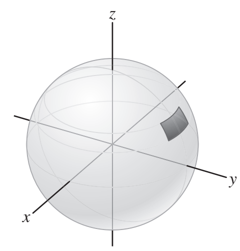
$$\text{Area}(S)\approx 0.02\cdot 2\cdot\sqrt{27}\approx 0.2078.$$

12. Sketch the small patch of the sphere whose spherical coordinates satisfy

$$\frac{\pi}{2} - 0.15 \le \theta \le \frac{\pi}{2} + 0.15, \qquad \frac{\pi}{4} - 0.1 \le \phi \le \frac{\pi}{4} + 0.1$$

Use the normal vector computed in Exercise 9 to estimate its area.

SOLUTION The small patch of the sphere is shown in the following figure:



Let $\mathcal{D}$ denote the rectangle in the (θ, ϕ)-plane. Using the sample point corresponding to the parameters $\theta = \frac{\pi}{2}, \phi = \frac{\pi}{4}$, we obtain the following estimation for the area of $S = \phi(\mathcal{D})$:

$$\text{Area}(S) = \left\| \mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right\| \cdot \text{Area}(\mathcal{D}) = \left\| \mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right\| \cdot 0.3 \cdot 0.2 = \left\| \mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right\| \cdot 0.06 \tag{1}$$

In Exercise 9 we found that:

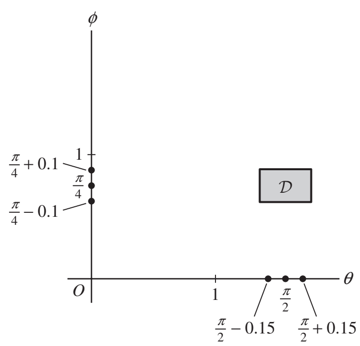
$$\mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = -\frac{1}{2}\langle 0, 1, 1\rangle$$

hence:

$$\left\| \mathbf{n}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \right\| = \frac{1}{2}\sqrt{0+1+1} = \frac{1}{\sqrt{2}}$$

Substituting in (1) gives the following estimation:

$$\text{Area}(S) \approx \frac{1}{\sqrt{2}} \cdot 0.06 \approx 0.0424$$



In Exercises 13–26, calculate $\iint_S f(x, y, z)\, dS$ *for the given surface and function.*

13. $G(u, v) = (u\cos v, u\sin v, u), \quad 0 \le u \le 1, \qquad 0 \le v \le 1;\ f(x, y, z) = z(x^2 + y^2)$

SOLUTION

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}(u\cos v, u\sin v, u) = \langle \cos v, \sin v, 1\rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v}(u\cos v, u\sin v, u) = \langle -u\sin v, u\cos v, 0\rangle$$

The normal vector is the cross product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$
$$= (-u\cos v)\mathbf{i} - (u\sin v)\mathbf{j} + \left(u\ \cos^2 v + u\ \sin^2 v\right)\mathbf{k}$$
$$= (-u\cos v)\mathbf{i} - (u\sin v)\mathbf{j} + u\mathbf{k} = \langle -u\cos v, -u\sin v, u\rangle$$

We compute the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = \sqrt{(-u\cos v)^2 + (-u\sin v)^2 + u^2} = \sqrt{u^2\left(\cos^2 v + \sin^2 v + 1\right)} = \sqrt{u^2 \cdot 2} = \sqrt{2}|u| = \sqrt{2}u$$

Notice that in the region of integration $u \ge 0$, therefore $|u| = u$.

Step 2. Calculate the surface integral. We express the function $f(x, y, z) = z\left(x^2 + y^2\right)$ in terms of the parameters u, v:

$$f(\Phi, (u, v)) = u\left(u^2 \cos^2 v + u^2 \sin^2 v\right) = u \cdot u^2 = u^3$$

We obtain the following integral:

$$\iint_S f(x, y, z)\,dS = \int_0^1 \int_0^1 f(\Phi, (u, v))\,\|\mathbf{n}\|\,du\,dv = \int_0^1 \int_0^1 u^3 \cdot \sqrt{2}u\,du\,dv$$
$$= \left(\int_0^1 \sqrt{2}\,dv\right)\left(\int_0^1 u^4\,du\right) = \sqrt{2} \cdot \left.\frac{u^5}{5}\right|_0^1 = \frac{\sqrt{2}}{5}$$

14. $G(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi; \qquad f(x, y, z) = \sqrt{x^2 + y^2}$

SOLUTION

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta, r\sin\theta, \theta) = \langle \cos\theta, \sin\theta, 0\rangle$$
$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta, r\sin\theta, \theta) = \langle -r\sin\theta, r\cos\theta, 1\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k}$$
$$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k} = \langle \sin\theta, -\cos\theta, r\rangle$$

We compute the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1 + r^2}$$

Step 2. Calculate the surface integral. The surface integral is computed by the following double integral:

$$\iint_S f(x, y, z)\,dS = \iint_{\mathcal{D}} f(\Phi(r, \theta))\,\|\mathbf{n}\|\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\sqrt{1 + r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^1 r\sqrt{1 + r^2}\,dr\,d\theta = \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^1 r\sqrt{1 + r^2}\,dr\right) = 2\pi \int_0^1 r\sqrt{1 + r^2}\,dr$$

We compute the integral using the substitution $t = 1 + r^2, \quad dt = 2r\,dr$. We get:

$$\iint_S f(x, y, z)\,dS = 2\pi \int_1^2 \frac{1}{2} t^{1/2}\,dt = \frac{2\pi\left(2\sqrt{2} - 1\right)}{3}$$

15. $y = 9 - z^2, \quad 0 \le x \le 3, 0 \le z \le 3; \qquad f(x, y, z) = z$

SOLUTION We use the formula for the surface integral over a graph $y = g(x, z)$:

$$\iint_S f(x, y, z)\,dS = \iint_{\mathcal{D}} f(x, g(x, z), z)\sqrt{1 + g_x^2 + g_z^2}\,dx\,dz \tag{1}$$

Since $y = g(x, z) = 9 - z^2$, we have $g_x = 0$, $g_z = -2z$, hence:

$$\sqrt{1 + g_x^2 + g_z^2} = \sqrt{1 + 4z^2}$$
$$f(x, g(x, z), z) = z$$

The domain of integration is the square $[0, 3] \times [0, 3]$ in the xz-plane. By (1) we get:

$$\iint_S f(x, y, z)\, dS = \int_0^3 \int_0^3 z\sqrt{1+4z^2}\, dz\, dx = \left(\int_0^3 1\, dx\right)\left(\int_0^3 z\sqrt{1+4z^2}\, dz\right) = 3\int_0^3 z\sqrt{1+4z^2}\, dz$$

We use the substitution $u = 1 + 4z^2,\ du = 8z\, dz$ to compute the integral. This gives:

$$\iint_S f(x, y, z)\, dS = 3\int_0^3 z\sqrt{1+4z^2}\, dz = 3\int_1^{37} \frac{u^{1/2}}{8}\, du = \frac{37\sqrt{37}-1}{4} \approx 56$$

16. $y = 9 - z^2, \quad 0 \le x \le z \le 3; \qquad f(x, y, z) = 1$

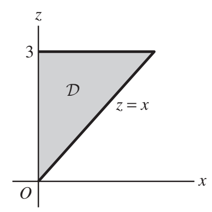
SOLUTION We use the formula for the surface integral over a graph $y = g(x, z)$:

$$\iint_S f(x, y, z)\, dS = \iint_{\mathcal{D}} f\,(x, g(x, z), z)\sqrt{1 + g_x^2 + g_z^2}\, dx\, dz \tag{1}$$

Since $y = g(x, z) = 9 - z^2$ we have $g_x = 0$, $g_z = -2z$, hence:

$$\sqrt{1 + g_x^2 + g_z^2} = \sqrt{1+4z^2}$$

The domain of integration $\mathcal{D}$ is the triangle in the xz-plane shown in the figure.



By (1) we get:

$$\begin{aligned}\iint_S f(x, y, z)\, dS &= \iint_{\mathcal{D}} \sqrt{1+4z^2}\, dx\, dz = \int_0^3 \int_0^z \sqrt{1+4z^2}\, dx\, dz \\ &= \int_0^3 \sqrt{1+4z^2}\, x\Big|_{x=0}^{z} dz = \int_0^3 \sqrt{1+4z^2} \cdot z\, dz\end{aligned}$$

We compute the integral using the substitution $u = 1 + 4z^2,\ du = 8z\, dz$. This gives:

$$\iint_S f(x, y, z)\, dS = \int_0^3 \sqrt{1+4z^2} \cdot z\, dz = \int_1^{37} \frac{u^{1/2}}{8}\, du = \frac{37\sqrt{37}-1}{12} \approx 18.672$$

17. $x^2 + y^2 + z^2 = 1,\ \ x, y, z \ge 0; \quad f(x, y, z) = x^2.$

SOLUTION The octant of the unit sphere centered at the origin, where $x, y, z \ge 0$ has the following parametrization in spherical coordinates:

$$\Phi(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi), \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2}$$

The length of the normal vector is:

$$\|\mathbf{n}\| = \sin\phi$$

The function x^2 expressed in terms of the parameters is $\cos^2\theta\ \sin^2\phi$. Using the theorem on computing surface integrals we obtain,

$$\begin{aligned}\iint_S x^2\, dS &= \int_0^{\pi/2}\int_0^{\pi/2} \left(\cos^2\theta\ \sin^2\phi\right)(\sin\phi)\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \cos^2\theta\ \sin^3\phi\, d\phi\, d\theta \\ &= \left(\int_0^{\pi/2} \cos^2\theta\, d\theta\right)\left(\int_0^{\pi/2} \sin^3\phi\, d\phi\right) = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\theta=0}^{\pi/2} \cdot \left(-\frac{\sin^2\phi\cos\phi}{3} - \frac{2}{3}\cos\phi\right)\Big|_{\phi=0}^{\pi/2} \\ &= \frac{\pi}{4} \cdot \frac{2}{3} = \frac{\pi}{6}\end{aligned}$$

18. $z = 4 - x^2 - y^2, \quad 0 \le z \le 3; \qquad f(x, y, z) = x^2/(4 - z)$

SOLUTION Use the formula for the surface integral over the graph of $z = g(x, y)$.

$$\begin{aligned} z &= g(x, y) = 4 - x^2 - y^2 \\ \Rightarrow \quad g_x(x, y) &= -2x \quad g_y(x, y) = -2y \\ \|\mathbf{n}\| &= \sqrt{1 + (-2x)^2 + (-2y)^2} = \sqrt{1 + 4(x^2 + y^2)} \end{aligned}$$

The domain of integration, $\mathcal{D}$, can be determined by the restrictions on z,

$$\begin{aligned} 0 \le z \le 3 \quad &\Rightarrow \quad 0 \le 4 - x^2 - y^2 \le 3 \\ &\Rightarrow \quad x^2 + y^2 \le 4 \text{ and } 4 - 3 \le x^2 + y^2 \\ &\Rightarrow \quad 1 \le x^2 + y^2 \le 4 \end{aligned}$$

$\mathcal{D}$ is the annulus for $1 \le r \le 2$. Changing into polar coordinates, the integral may be written,

$$\begin{aligned} \iint_S \frac{x^2}{4 - z}\, dS &= \iint_{\mathcal{D}} \frac{x^2}{x^2 + y^2} \sqrt{1 + 4(x^2 + y^2)}\, dx\, dy \\ &= \int_0^{2\pi} \int_1^2 \frac{r^2 \cos^2\theta}{r^2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \left(\int_0^{2\pi} \cos^2\theta\, d\theta \right) \left(\int_1^2 \sqrt{1 + 4r^2}\, r\, dr \right) \end{aligned}$$

Substituting $u = 1 + 4r^2 \Rightarrow du = 8r\, dr$ into the second integral, we have

$$\begin{aligned} &= \left(\int_0^{2\pi} \frac{1 + \cos 2\theta}{2}\, d\theta \right) \left(\int_1^{17} \frac{\sqrt{u}}{8}\, du \right) \\ &= \pi \cdot \frac{1}{8} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_5^{17} = \frac{\pi}{12}(17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

19. $x^2 + y^2 = 4, \quad 0 \le z \le 4; \qquad f(x, y, z) = e^{-z}$

SOLUTION The cylinder has the following parametrization in cylindrical coordinates:

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z), \; 0 \le \theta \le 2\pi, \; 0 \le z \le 4$$

Step 1. Compute the tangent and normal vectors. The tangent vectors are the partial derivatives:

$$\begin{aligned} \mathbf{T}_\theta &= \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(2\cos\theta, 2\sin\theta, z) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle \\ \mathbf{T}_z &= \frac{\partial}{\partial z}(2\cos\theta, 2\sin\theta, z) = \langle 0, 0, 1 \rangle \end{aligned}$$

The normal vector is their cross product:

$$\mathbf{n}(\theta, z) = \mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$$

The length of the normal vector is thus

$$\|\mathbf{n}(\theta, z)\| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2 + 0} = \sqrt{4\left(\cos^2\theta + \sin^2\theta\right)} = \sqrt{4} = 2$$

Step 2. Calculate the surface integral. The surface integral equals the following double integral:

$$\begin{aligned} \iint_S f(x, y, z)\, dS &= \iint_{\mathcal{D}} f\left(\Phi(\theta, z)\right) \|\mathbf{n}\|\, d\theta\, dz = \int_0^{2\pi} \int_0^4 e^{-z} \cdot 2\, d\theta\, dz \\ &= \left(\int_0^{2\pi} 2\, d\theta \right) \left(\int_0^4 e^{-z}\, dz \right) = 4\pi \cdot \left(-e^{-z}\right) \Big|_0^4 = 4\pi\left(1 - e^{-4}\right) \end{aligned}$$

20. $G(u,v) = (u, v^3, u+v), \quad 0 \le u \le 1, 0 \le v \le 1; \qquad f(x,y,z) = y$

SOLUTION

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}\left(u, v^3, u+v\right) = \langle 1, 0, 1\rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v}\left(u, v^3, u+v\right) = \left\langle 0, 3v^2, 1\right\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = \left(-3v^2\right)\mathbf{i} - \mathbf{j} + \left(3v^2\right)\mathbf{k} = \left\langle -3v^2, -1, 3v^2\right\rangle$$

The length of the normal vector is, thus:

$$\|\mathbf{n}\| = \sqrt{9v^4 + 1 + 9v^4} = \sqrt{1 + 18v^4}$$

Step 2. Calculate the surface integral. We express $f(x,y,z) = y$ in terms of the parameters:

$$f\left(\Phi(u,v)\right) = v^3$$

We compute the surface integral by the following double integral:

$$\iint_S f(x,y,z)\,dS = \int_0^1 \int_0^1 f\left(\Phi(u,v)\right)\|\mathbf{n}\|\,dv\,du = \int_0^1 \int_0^1 v^3\sqrt{1+18v^4}\,dv\,du$$
$$= \left(\int_0^1 du\right)\left(\int_0^1 v^3\sqrt{1+18v^4}\,dv\right) = \int_0^1 v^3\sqrt{1+18v^4}\,dv$$

We compute the integral using the substitution $t = 1 + 18v^4, \quad dt = 72v^3\,dv$. We get:

$$\iint_S f(x,y,z)\,dS = \int_0^1 \sqrt{1+18v^4}\cdot v^3\,dv = \int_1^{19} \frac{t^{1/2}}{72}\,dt = \frac{19\sqrt{19}-1}{108} \approx 0.758$$

21. Part of the plane $x + y + z = 1$, where $x, y, z \ge 0$;
$f(x,y,z) = z$

SOLUTION We let $z = g(x,y) = 1 - x - y$ and use the formula for the surface integral over the graph of $z = g(x,y)$, where $\mathcal{D}$ is the parameter domain in the xy-plane. That is:

$$\iint_S f(x,y,z)\,dS = \iint_{\mathcal{D}} f\left(x,y,g(x,y)\right)\sqrt{1+g_x^2+g_y^2}\,dx\,dy \tag{1}$$

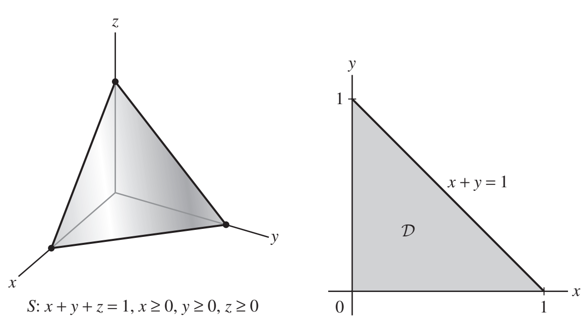
We have, $g_x = -1$ and $g_y = -1$ therefore:

$$\sqrt{1+g_x^2+g_y^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3}$$

We express the function $f(x,y,z) = z$ in terms of the parameters x and y:

$$f\left(x,y,g(x,y)\right) = z = 1 - x - y$$

The domain of integration is the triangle $\mathcal{D}$ in the xy-plane shown in the figure.



By (1) we get:

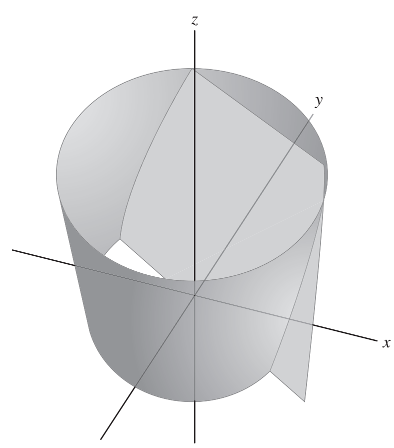
$$\iint_S f(x,y,z)\,dS = \int_0^1 \int_0^{1-y} (1-x-y)\sqrt{3}\,dx\,dy = \sqrt{3}\int_0^1 x - \frac{x^2}{2} - yx\Big|_{x=0}^{1-y} dy$$
$$= \sqrt{3}\int_0^1 \left((1-y)^2 - \frac{(1-y)^2}{2}\right) dy = \frac{\sqrt{3}}{2}\int_0^1 \left(1-2y+y^2\right) dy$$
$$= \frac{\sqrt{3}}{2}\left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1 = \frac{\sqrt{3}}{6}$$

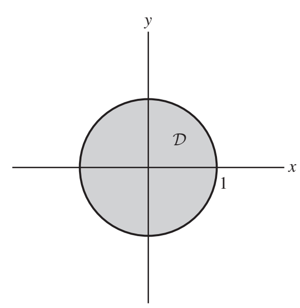
22. Part of the plane $x+y+z=0$ contained in the cylinder $x^2+y^2=1$; $\quad f(x,y,z)=z^2$

SOLUTION We let $z=-x-y$ and use the formula for the surface integral over the graph of $z=g(x,y)$. We have, $g_x=-1$ and $g_y=-1$ therefore:

$$\|\mathbf{n}\| = \sqrt{1+g_x^2+g_y^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3}$$

The domain of integration, $\mathcal{D}: x^2+y^2 \le 1$, is the unit disk.





Changing to polar coordinates the integral may be evaluated

$$\iint_S z^2\,dS = \iint_{\mathcal{D}} (-x-y)^2\sqrt{3}\,dx\,dy = \int_0^{2\pi}\int_0^1 (-r\cos\theta - r\sin\theta)^2\sqrt{3}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 (r^2)(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta)\cdot\sqrt{3}\,r\,dr\,d\theta$$
$$= \left(\int_0^{2\pi} 1+\sin(2\theta)\,d\theta\right)\left(\sqrt{3}\int_0^1 r^3\,dr\right)$$
$$= \left(\theta - \frac{\cos(2\theta)}{2}\Big|_0^{2\pi}\right)\left(\sqrt{3}\frac{r^4}{4}\Big|_0^1\right) = \frac{\pi\sqrt{3}}{2}$$

23. $x^2 + y^2 + z^2 = 4$, $1 \le z \le 2$; $\quad f(x, y, z) = z^2(x^2 + y^2 + z^2)^{-1}$

SOLUTION We use spherical coordinates to parametrize the cap S.

$$\Phi(\theta, \phi) = (2\cos\theta \sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$$
$$\mathcal{D} : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \phi_0$$

The angle ϕ_0 is determined by $\cos\phi_0 = \frac{1}{2}$, that is, $\phi_0 = \frac{\pi}{3}$. The length of the normal vector in spherical coordinates is:

$$\|\mathbf{n}\| = R^2 \sin\phi = 4\sin\phi$$

We express the function $f(x, y, z) = z^2\left(x^2 + y^2 + z^2\right)^{-1}$ in terms of the parameters:

$$f(\Phi(\theta, \phi)) = (2\cos\phi)^2 4^{-1} = \cos^2\phi$$

Using the theorem on computing the surface integral we get:

$$\begin{aligned}\iint_S f(x, y, z)\, dS &= \iint_{\mathcal{D}} f(\Phi(\theta, \phi))\, \|\mathbf{n}\|\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(\cos^2\phi\right) \cdot 4\sin\phi\, d\phi\, d\theta \\ &= \left(\int_0^{2\pi} 4\, d\theta\right)\left(\int_0^{\pi/3} \cos^2\phi \sin\phi\, d\phi\right) = 8\pi\left(-\frac{\cos\phi}{3}\right)\Big|_0^{\pi/3} \\ &= \frac{8\pi}{3}\left(-\left(\frac{1}{2}\right)^3 - (-1)\right) = \frac{8\pi}{3} \cdot \frac{7}{8} = \frac{7\pi}{3}\end{aligned}$$

24. $x^2 + y^2 + z^2 = 4$, $0 \le y \le 1$; $\quad f(x, y, z) = y$

SOLUTION Since $y \ge 0$, we may consider the surface the graph $y = \sqrt{4 - x^2 - z^2}$. Using the formula for the surface integral over the graph

$$\begin{aligned} y = g(x, z) &= \sqrt{4 - x^2 - z^2} \\ \Rightarrow \quad g_x(x, z) &= \frac{-x}{\sqrt{4 - x^2 - z^2}} \\ \Rightarrow \quad g_z(x, z) &= \frac{-z}{\sqrt{4 - x^2 - z^2}} \\ \|\mathbf{n}\| &= \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - z^2}}\right)^2 + \left(\frac{-z}{\sqrt{4 - x^2 - z^2}}\right)^2} \\ &= \sqrt{1 + \frac{x^2 + z^2}{4 - x^2 - z^2}} = \frac{2}{\sqrt{4 - x^2 - z^2}}\end{aligned}$$

The domain of integration, $\mathcal{D}$, can be determined by the restrictions on y,

$$\begin{aligned} 0 \le y \le 1 \quad &\Rightarrow \quad 0 \le \sqrt{4 - x^2 - z^2} \le 1 \\ &\Rightarrow \quad x^2 + z^2 \le 4 \text{ and } 4 - 1 \le x^2 + z^2 \\ &\Rightarrow \quad 3 \le x^2 + z^2 \le 4\end{aligned}$$

$\mathcal{D}$ is the annulus for $\sqrt{3} \le r \le 2$. Changing into polar coordinates, the integral may be written,

$$\begin{aligned}\iint_S y\, dS &= \iint_{\mathcal{D}} \sqrt{4 - x^2 - z^2}\, \frac{2}{\sqrt{4 - x^2 - z^2}}\, dx\, dz \\ &= \int_0^{2\pi}\int_{\sqrt{3}}^2 2r\, dr\, d\theta = 2\pi \cdot r^2\Big|_{\sqrt{3}}^2 = 2\pi(4 - 3) = 2\pi\end{aligned}$$

25. Part of the surface $z = x^3$, where $0 \le x \le 1$, $0 \le y \le 1$; $\quad f(x, y, z) = z$

SOLUTION Use the formula for the surface integral over the graph of $z = g(x, y)$. We have, $g_x = 3x^2$ and $g_y = 0$ therefore:

$$\|\mathbf{n}\| = \sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + (3x^2)^2 + (0)^2} = \sqrt{1 + 9x^4}$$

The integral then is

$$\iint_S z\,dS = \int_0^1 \int_0^1 x^3\sqrt{1+9x^4}\,dx\,dy = \left(\int_0^1 dy\right)\left(\int_0^1 x^3\sqrt{1+9x^4}\,dx\right)$$

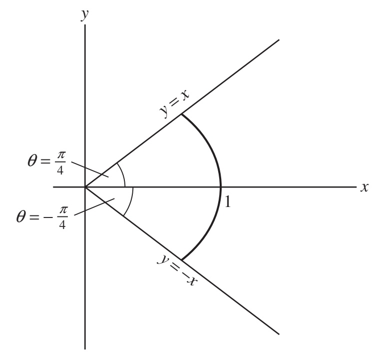
Substituting $u = 1 + 9x^4$, $du = 36x^3\,dx$

$$= 1 \cdot \int_1^{10} u^{\frac{1}{2}} \frac{du}{36} = \frac{1}{36} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}}\Big|_1^{10} = \frac{1}{54}(10\sqrt{10} - 1)$$

26. Part of the unit sphere centered at the origin, where $x \ge 0$ and $|y| \le x$; $\quad f(x, y, z) = x$

SOLUTION We parametrize the surface as follows:

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad 0 \le \phi \le \pi$$



The length of the normal vector is:

$$\|\mathbf{n}\| = 1^2 \sin\phi = \sin\phi$$

The function expressed in terms of the parameters is:

$$f\left(\Phi(\theta, \phi)\right) = x = \cos\theta \sin\phi$$

We obtain the following integral:

$$\iint_S f(x, y, z)\,dS = \iint_{\mathcal{D}} f\left(\Phi(\theta, \phi)\right) \|\mathbf{n}\|\,d\theta\,d\phi = \int_0^{\pi} \int_{-\pi/4}^{\pi/4} (\cos\theta \sin\phi) \sin\phi\,d\theta\,d\phi$$

$$= \int_0^{\pi} \int_{-\pi/4}^{\pi/4} \cos\theta\, \sin^2\phi\,d\theta\,d\phi = \left(\int_0^{\pi} \sin^2\phi\,d\phi\right)\left(\int_{-\pi/4}^{\pi/4} \cos\theta\,d\theta\right)$$

$$= \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\Big|_{\phi=0}^{\pi}\right)\left(\sin\theta\Big|_{\theta=-\frac{\pi}{4}}^{\pi/4}\right) = \frac{\pi}{2} \cdot \sqrt{2} = \frac{\pi}{\sqrt{2}}$$

27. A surface $\mathcal{S}$ has a parametrization $G(u, v)$ with domain $0 \le u \le 2$, $0 \le v \le 4$ such that the following partial derivatives are constant:

$$\frac{\partial G}{\partial u} = \langle 2, 0, 1 \rangle, \qquad \frac{\partial G}{\partial v} = \langle 4, 0, 3 \rangle$$

What is the surface area of $\mathcal{S}$?

SOLUTION Since the partial derivatives are constant, the normal vector is also constant. We find it by computing the cross product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 4 & 0 & 3 \end{vmatrix} = -2\mathbf{j} = \langle 0, -2, 0 \rangle \quad \Rightarrow \quad \|\mathbf{n}\| = 2$$

We denote the rectangle $\mathcal{D} = \{(u, v) : 0 \le u \le 2,\ 0 \le v \le 4\}$, and use the surface area to compute the area of $S = \Phi(\mathcal{D})$. We obtain:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}\|\,du\,dv = \iint_{\mathcal{D}} 2\,du\,dv = 2\iint_{\mathcal{D}} 1\,du\,dv = 2 \cdot \text{Area}(\mathcal{D}) = 2 \cdot 2 \cdot 4 = 16$$

28. Let S be the sphere of radius R centered at the origin. Explain using symmetry:

$$\iint_S x^2\,dS = \iint_S y^2\,dS = \iint_S z^2\,dS$$

Then show that $\iint_S x^2\,dS = \frac{4}{3}\pi R^4$ by adding the integrals.

SOLUTION

(a) Since the sphere is symmetric with respect to the yz-plane, the surface integrals of x over the hemispheres on the two sides of the plane cancel each other and the result is zero. The two other integrals are zero due to the symmetry of the sphere with respect to the xz and xy-planes.

(b) Since the sphere is symmetric with respect to the xy, xz and yz-planes, interchanging x and y in the integral for $\iint_S x^2\,dS$ does not change the value of the integral and the result is $\iint_S y^2\,dS$. The equality for $\iint_S z^2\,dS$ is explained similarly.

On the sphere, we have $x^2 + y^2 + z^2 = R^2$ so, using properties of integrals, the integral for surface area, and the surface area of the sphere of radius R we obtain:

$$\begin{aligned}\iint_S x^2\,dS + \iint_S y^2\,dS + \iint_S z^2\,dS &= \iint_S \left(x^2 + y^2 + z^2\right)\,dS = R^2 \iint_S 1\,dS \\ &= R^2 \cdot \text{Area}(S) = R^2 \cdot 4\pi R^2 = 4\pi R^4\end{aligned}$$

Combining with (b) we conclude that the value of each of the integrals is $\frac{4}{3}\pi R^4$. That is:

$$\iint_S x^2\,dS = \iint_S y^2\,dS = \iint_S z^2\,dS = \frac{4}{3}\pi R^4.$$

29. Calculate $\iint_{\mathcal{S}} (xy + e^z)\,dS$, where $\mathcal{S}$ is the triangle in Figure 18 with vertices $(0, 0, 3)$, $(1, 0, 2)$, and $(0, 4, 1)$.

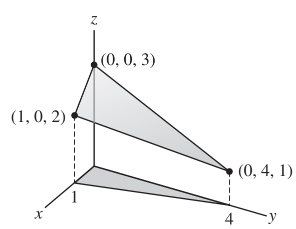
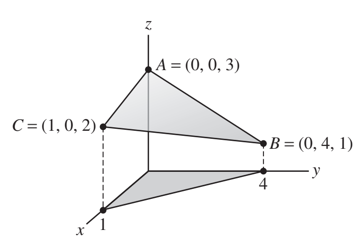


FIGURE 18

SOLUTION We find the equation of the plane through the points $A = (0, 0, 3)$, $B = (0, 4, 1)$ and $C = (1, 0, 2)$.



A normal to the plane is the cross product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, 4, -2\rangle \times \langle 1, 0, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & -2 \\ 1 & 0 & -1 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} = -2\langle 2, 1, 2\rangle$$

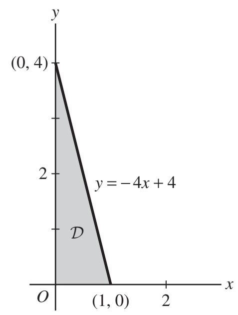
The equation of the plane passing through $A = (0, 0, 3)$ and perpendicular to the vector $\langle 2, 1, 2\rangle$ is:

$$\begin{aligned}\langle x - 0, y - 0, z - 3\rangle \cdot \langle 2, 1, 2\rangle &= 0 \\ 2x + y + 2(z - 3) &= 0 \\ 2x + y + 2z &= 6\end{aligned}$$

or

$$z = g(x, y) = -x - \frac{1}{2}y + 3$$

We compute the surface integral of $f(x, y, z) = xy + e^z$ over the triangle ABC using the formula for the surface integral over a graph. The parameter domain $\mathcal{D}$ is the projection of the triangle ABC onto the xy-plane (see figure).



We have:

$$\iint_S f(x, y, z)\, dS = \iint_{\mathcal{D}} f\big(x, y, g(x, y)\big)\sqrt{1 + g_x^2 + g_z^2}\, dx\, dy \tag{1}$$

We compute the functions in the integrand. Since $z = g(x, y) = -x - \frac{y}{2} + 3$, we have:

$$g_x = -1,\ g_y = -\frac{1}{2} \quad \Rightarrow \quad \sqrt{1 + g_x^2 + g_z^2} = \sqrt{1 + (-1)^2 + \left(-\frac{1}{2}\right)^2} = \frac{3}{2}$$

$$f\big(x, y, g(x, y, z)\big) = xy + e^z = xy + e^{-x-\frac{y}{2}+3}$$

Substituting in (1) gives:

$$\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^{-4x+4} \left(xy + e^{-x-y/2+3}\right) \cdot \frac{3}{2}\, dy\, dx = \int_0^1 \frac{3xy^2}{4} - 3e^{-x-y/2+3}\Big|_{y=0}^{-4x+4} dx$$

$$= \int_0^1 \left(\frac{3x(-4x+4)^2}{4} - 3e^{-x-(-4x+4)/2+3} + 3e^{-x+3}\right) dx$$

$$= \int_0^1 \left(12x^3 - 24x^2 + 12x - 3e^{x+1} + 3e^{-x+3}\right) dx = 3x^4 - 8x^3 + 6x^2 - 3e^{x+1} - 3e^{-x+3}\Big|_0^1$$

$$= (1 - 3e^2 - 3e^2) - (-3e - 3e^3) = 3e^3 - 6e^2 + 3e + 1 \approx 25.08$$

30. Use spherical coordinates to compute the surface area of a sphere of radius R.

SOLUTION The sphere of radius R centered at the origin has the following parametrization in spherical coordinates:

$$\Phi(\theta, \phi) = (R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

The length of the normal vector is:

$$\|\mathbf{n}\| = R^2 \sin\phi$$

Using the integral for surface area gives:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}\|\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi\, d\phi\, d\theta = \left(\int_0^{2\pi} R^2\, d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)$$

$$= 2\pi R^2 \cdot \left(-\cos\phi\Big|_0^{\pi}\right) = 2\pi R^2 \cdot 2 = 4\pi R^2$$

31. Use cylindrical coordinates to compute the surface area of a sphere of radius R.

SOLUTION As $z = \pm\sqrt{R^2 - (x^2 + y^2)}$ we may parametrize the upper hemisphere by the map

$$G(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{R^2 - r^2})$$

To compute the surface area of the hemisphere S, we first must find the tangent vectors and the normal vector. That is,

$$\mathbf{T}_r = \frac{\partial G}{\partial r} = \frac{\partial}{\partial r}\left\langle r\cos\theta, r\sin\theta, \sqrt{R^2-r^2}\right\rangle = \left\langle \cos\theta, \sin\theta, -\frac{r}{\sqrt{R^2-r^2}}\right\rangle$$

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta}\left\langle r\cos\theta, r\sin\theta, \sqrt{R^2-r^2}\right\rangle = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

The normal vector is the cross product:

$$\begin{aligned}
\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{R^2-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= \left(\frac{r^2\cos\theta}{\sqrt{R^2-r^2}}\right)\mathbf{i} + \left(\frac{r^2\sin\theta}{\sqrt{R^2-r^2}}\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} \\
&= \left(\frac{r^2\cos\theta}{\sqrt{R^2-r^2}}\right)\mathbf{i} + \left(\frac{r^2\sin\theta}{\sqrt{R^2-r^2}}\right)\mathbf{j} + r\mathbf{k}
\end{aligned}$$

The length of the normal vector is thus

$$\|\mathbf{n}\| = \sqrt{\frac{r^4\cos^2\theta}{R^2-r^2} + \frac{r^4\sin^2\theta}{R^2-r^2} + r^2} = \sqrt{\frac{r^4}{R^2-r^2}\left(\cos^2\theta + \sin^2\theta\right) + r^2} = \sqrt{\frac{r^4}{R^2-r^2} + r^2} = \frac{rR}{\sqrt{R^2-r^2}}$$

We now compute the surface area as the following surface integral:

$$\begin{aligned}
\text{Area}(S) &= \iint_{\mathcal{D}} \|\mathbf{n}\|\,dr\,d\theta = \int_0^{2\pi}\int_0^R \frac{rR}{\sqrt{R^2-r^2}}\,dr\,d\theta \\
&= \left(\int_0^{2\pi} R\,d\theta\right)\left(\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr\right) = 2\pi R\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr
\end{aligned}$$

We compute the integral using the substitution $t = R^2 - r^2$, $dt = -2r\,dr$. We get:

$$\text{Area}(S) = 2\pi R\int_{R^2}^0 \frac{-1}{2t^{1/2}}\,dt = 2\pi R^2$$

The area of the entire sphere is twice this or $4\pi R^2$.

32. CAS Let $\mathcal{S}$ be the surface with parametrization

$$G(u, v) = \big((3+\sin v)\cos u,\ (3+\sin v)\sin u,\ v\big)$$

for $0 \le u \le 2\pi$, $0 \le v \le 2\pi$. Using a computer algebra system:

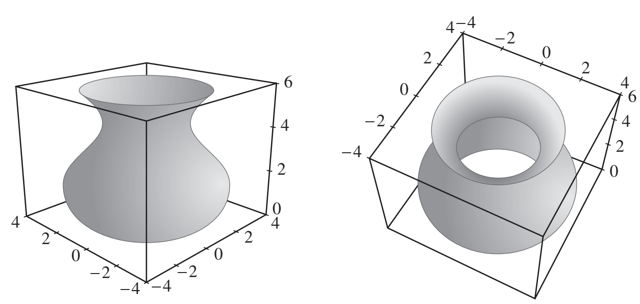
(a) Plot $\mathcal{S}$ from several different viewpoints. Is $\mathcal{S}$ best described as a "vase that holds water" or a "bottomless vase"?

(b) Calculate the normal vector $\mathbf{n}(u, v)$.

(c) Calculate the surface area of $\mathcal{S}$ to four decimal places.

SOLUTION

(a) We show the graph of $\mathcal{S}$ here.



Note that it is best described as a "bottomless vase."

(b) We compute the tangent and normal vectors:

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \langle (3+\sin v)(-\sin u), (3+\sin v)(\cos u), 0\rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \langle \cos v \cos u, \cos v \sin u, 1\rangle$$

The normal vector is the cross product:

$$\mathbf{n}(u,v) = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (3+\sin v)(-\sin u) & (3+\sin v)(\cos u) & 0 \\ \cos v \cos u & \cos v \sin u & 1 \end{vmatrix}$$
$$= ((3+\sin v)\cos u)\mathbf{i} + ((3+\sin v)\sin u)\mathbf{j} - ((3+\sin v)\cos v)\mathbf{k}$$

Hence,

$$\|\mathbf{n}(u,v)\| = (3+\sin v)\sqrt{1+\cos^2 v}$$

We obtain the following area:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}\|\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} (3+\sin v)\sqrt{1+\cos^2 v}\, du\, dv \approx 144.0181$$

33. CAS Let $\mathcal{S}$ be the surface $z = \ln(5 - x^2 - y^2)$ for $0 \le x \le 1$, $0 \le y \le 1$. Using a computer algebra system:
(a) Calculate the surface area of $\mathcal{S}$ to four decimal places.
(b) Calculate $\iint_{\mathcal{S}} x^2 y^3\, dS$ to four decimal places.

SOLUTION
(a) Using that $z_x = -2x/(5-x^2-y^2)$ and $z_y = -2y/(5-x^2-y^2)$, we calculate $\|n\|$ to be

$$\|n\| = \sqrt{1+(z_x)^2+(z_y)^2} = \frac{\sqrt{(5-x^2-y^2)^2+4x^2+4y^2}}{5-x^2-y^2}$$

Thus, the surface area is

$$\text{Area}(S) = \int_0^1\int_0^1 \frac{\sqrt{(5-x^2-y^2)^2+4x^2+4y^2}}{5-x^2-y^2}\, dx\, dy \approx 1.078$$

(b) We calculate $\iint_{\mathcal{S}} x^2 y^3\, dS$ as follows:

$$\iint_{\mathcal{S}} x^2 y^3\, dS = \int_0^1\int_0^1 x^2 y^3 \frac{\sqrt{(5-x^2-y^2)^2+4x^2+4y^2}}{5-x^2-y^2}\, dx\, dy \approx 0.09814$$

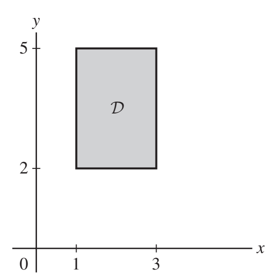
34. Find the area of the portion of the plane $2x + 3y + 4z = 28$ lying above the rectangle $1 \le x \le 3$, $2 \le y \le 5$ in the xy-plane.

SOLUTION We rewrite the equation of the plane as:

$$z = g(x,y) = -\frac{x}{2} - \frac{3}{4}y + 7 \tag{1}$$

The domain of the parameters is the rectangle $\mathcal{D} = [1,3] \times [2,5]$ in the xy-plane. Using the integral for surface area and the surface integral over a graph we have:

$$\text{Area}(S) = \iint_S 1\, dS = \iint_{\mathcal{D}} \sqrt{1+g_x^2+g_z^2}\, dx\, dy \tag{2}$$



By (1) we have:

$$g_x = -\frac{1}{2},\, g_y = -\frac{3}{4} \quad \Rightarrow \quad \sqrt{1 + g_x^2 + g_z^2} = \sqrt{1 + \frac{1}{4} + \frac{9}{16}} = \frac{\sqrt{29}}{4}$$

We substitute in (2) to obtain:

$$\text{Area}(S) = \iint_{\mathcal{D}} \frac{\sqrt{29}}{4}\, dx\, dy = \frac{\sqrt{29}}{4} \iint_{\mathcal{D}} 1\, dx\, dy = \frac{\sqrt{29}}{4}\text{Area}(\mathcal{D}) = \frac{\sqrt{29}}{4} \cdot 3 \cdot 2 = \frac{3\sqrt{29}}{2}$$

35. What is the area of the portion of the plane $2x + 3y + 4z = 28$ lying above the domain $\mathcal{D}$ in the xy-plane in Figure 19 if $\text{Area}(\mathcal{D}) = 5$?

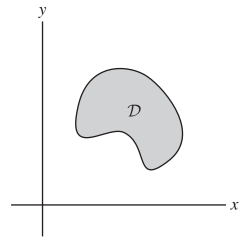


FIGURE 19

SOLUTION We rewrite the equation of the plane as:

$$z = g(x, y) = -\frac{x}{2} - \frac{3}{4}y + 7$$

Hence:

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{3}{4}\right)^2} = \frac{\sqrt{29}}{4}$$

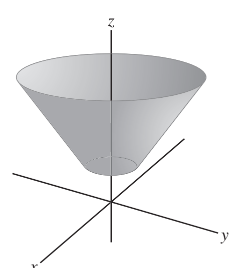
We use the integral for surface area and the surface integral over a graph to write:

$$\text{Area}(S) = \iint_S 1\, dS = \iint_{\mathcal{D}} \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy = \iint_{\mathcal{D}} \frac{\sqrt{29}}{4}\, dx\, dy$$

$$= \frac{\sqrt{29}}{4} \iint_{\mathcal{D}} 1\, dx\, dy = \frac{\sqrt{29}}{4}\text{Area}(\mathcal{D}) = \frac{\sqrt{29}}{4} \cdot 5 = \frac{5\sqrt{29}}{4} \approx 6.73$$

36. Find the surface area of the part of the cone $x^2 + y^2 = z^2$ between the planes $z = 2$ and $z = 5$.

SOLUTION



We use the following parametrization of the surface S:

$$\Phi(u, v) = (u \cos v, u \sin v, u)$$

$$\mathcal{D} : \; 0 \le v \le 2\pi, \; 2 \le u \le 5$$

Step 1. Compute the tangent and normal vectors. We have,

$$\mathbf{T}_u = \frac{\partial \varphi}{\partial u} = \frac{\partial}{\partial u}(u \cos v, u \sin v, u) = \langle \cos v, \sin v, 1 \rangle$$

$$\mathbf{T}_v = \frac{\partial \varphi}{\partial v} = \frac{\partial}{\partial v}(u \cos v, u \sin v, u) = \langle -u \sin v, u \cos v, 0 \rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-u\cos v)\mathbf{i} - (u\sin v)\mathbf{j} + \left(u\cos^2 v + u\sin^2 v\right)\mathbf{k}$$

$$= (-u\cos v)\mathbf{i} - (u\sin v)\mathbf{j} + u\mathbf{k} = \langle -u\cos v, -u\sin v, u\rangle$$

We compute the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{u^2\left(\cos^2 v + \sin^2 v\right) + u^2} = \sqrt{2u^2} = u\sqrt{2}$$

Step 2. Calculate the surface area. Using the formula for the surface area as a double integral gives:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}\|\,du\,dv = \int_0^{2\pi}\int_2^5 \sqrt{2}u\,du\,dv = \left(\int_0^{2\pi}\sqrt{2}\,dv\right)\left(\int_2^5 u\,du\right)$$

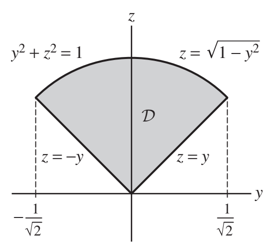
$$= 2\sqrt{2}\pi\left(\frac{u^2}{2}\Big|_2^5\right) = 2\sqrt{2}\pi\,\frac{25-4}{2} = 21\sqrt{2}\pi \approx 93.3$$

37. Find the surface area of the portion S of the cone $z^2 = x^2 + y^2$, where $z \ge 0$, contained within the cylinder $y^2 + z^2 \le 1$.

SOLUTION We rewrite the equation of the cone as $x = \pm\sqrt{z^2 - y^2}$. The projection of the cone onto the yz-plane is obtained by setting $x = 0$ in the equation of the cone, that is,

$$x = 0 = \sqrt{z^2 - y^2} \quad\Rightarrow\quad z = \pm y$$

Since on S, $z \ge 0$, we get $z = |y|$. We conclude that the projection of the upper part of the cone $x^2 + y^2 = z^2$ onto the yz-plane is the region between the lines $z = y$ and $z = -y$ on the upper part of the yz-plane. Therefore, the projection $\mathcal{D}$ of S onto the yz-plane is the region shown in the figure:



There are two identical portions of the surface parametrized by this region—one for $x \ge 0$, and one for $x \le 0$. Therefore the area of S is twice the integral over the domain $\mathcal{D}$:

$$\text{Area}(S) = \iint_S dS = 2\iint_{\mathcal{D}} \sqrt{1 + g_y^2 + g_z^2}\,dy\,dz$$

We compute the integral using a surface integral over a graph. Since $x = g(y,z) = \pm\sqrt{z^2 - y^2}$ we have,

$$g_z = \pm\frac{z}{\sqrt{z^2 - y^2}}, \quad g_y = \pm\frac{y}{\sqrt{z^2 - y^2}}$$

Hence, (notice that $z \ge 0$ on S):

$$\sqrt{1 + g_y^2 + g_z^2} = \sqrt{1 + \frac{z^2}{z^2 - y^2} + \frac{y^2}{z^2 - y^2}} = \sqrt{\frac{2z^2}{z^2 - y^2}} = \frac{z\sqrt{2}}{\sqrt{z^2 - y^2}}$$

We obtain the following integral:

$$\text{Area}(S) = 2\iint_{\mathcal{D}} \sqrt{1 + g_y^2 + g_z^2}\,dy\,dz = 2\iint_{\mathcal{D}} \frac{z\sqrt{2}}{\sqrt{z^2 - y^2}}\,dz\,dy$$

Using symmetry gives:

$$\text{Area}(S) = 4\int_0^{1/(\sqrt{2})}\int_y^{\sqrt{1-y^2}} \frac{z\sqrt{2}}{\sqrt{z^2 - y^2}}\,dz\,dy = 4\sqrt{2}\int_0^{1/(\sqrt{2})}\left(\int_y^{\sqrt{1-y^2}} \frac{z\,dz}{\sqrt{z^2 - y^2}}\right)dy \qquad (1)$$

We compute the inner integral using the substitution $u = \sqrt{z^2 - y^2}$, $du = \frac{z}{u}\,dz$. We get:

$$\int_y^{\sqrt{1-y^2}} \frac{z\,dz}{\sqrt{z^2 - y^2}} = \int_0^{\sqrt{1-y^2}} \frac{u\,du}{u} = \int_0^{\sqrt{1-2y^2}} du = \sqrt{1 - 2y^2}$$

We substitute in (1) and compute the resulting integral using the substitution $t = \sqrt{2}y$. We get:

$$\text{Area}(S) = 4\sqrt{2}\int_0^{1/(\sqrt{2})} \sqrt{1 - 2y^2}\,dy = 4\sqrt{2}\int_0^1 \sqrt{1 - t^2}\,\frac{dt}{\sqrt{2}} = 4\int_0^1 \sqrt{1 - t^2}\,dt = 4 \cdot \frac{\pi}{4} = \pi$$

38. Calculate the integral of ze^{2x+y} over the surface of the box in Figure 20.

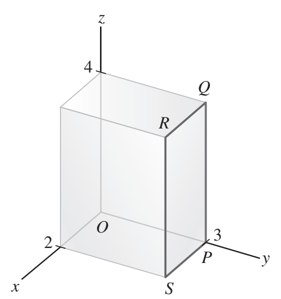


FIGURE 20

SOLUTION The cube may be parametrized by six functions for the six faces of the cube.

$$\phi_1(x, y) = (x, y, 0)\ \phi_2(x, y) = (x, y, 4)\ (0 \le x \le 3)(0 \le y \le 2)$$
$$\phi_3(x, z) = (x, 0, z)\ \phi_4(x, z) = (x, 2, z)\ (0 \le x \le 3)(0 \le z \le 4)$$
$$\phi_5(y, z) = (0, y, z)\ \phi_6(x, y) = (3, y, z)\ (0 \le y \le 2)(0 \le z \le 4)$$

We calculate the outward normal vector for the face parametrized by ϕ_1.

$$\frac{\partial \phi_1}{\partial x} = \langle 1, 0, 0\rangle, \frac{\partial \phi_1}{\partial y} = \langle 0, 1, 0\rangle \Rightarrow \mathbf{n}_1 = -\langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \langle 0, 0, -1\rangle$$

We have that $\|\mathbf{n}_1\| = 1$. Clearly this will be true for all the other normal vectors, $\|n_i\|$. The integrals over the six faces proceed as follows

$$\iint_{S_1} ze^{2x+y}\,dS_1 = \int_0^3\int_0^2 0 \cdot e^{2x+y}\,dx\,dy = 0$$
$$\iint_{S_2} ze^{2x+y}\,dS_2 = \int_0^2\int_0^3 4 \cdot e^{2x+y}\,dx\,dy = 4\frac{e^{2x}}{2}\bigg|_0^3 e^y\bigg|_0^2 = 2(e^6 - 1)(e^2 - 1)$$
$$\iint_{S_3} ze^{2x+y}\,dS_3 = \int_0^4\int_0^3 z \cdot e^{2x}\,dx\,dz = \frac{e^{2x}}{2}\bigg|_0^3 \frac{z^2}{2}\bigg|_0^4 = 4(e^6 - 1)$$
$$\iint_{S_4} ze^{2x+y}\,dS_4 = \int_0^4\int_0^3 z \cdot e^{2x+2}\,dx\,dz = \frac{e^{2x+2}}{2}\bigg|_0^3 \frac{z^2}{2}\bigg|_0^4 = 4(e^8 - e^2)$$
$$\iint_{S_5} ze^{2x+y}\,dS_5 = \int_0^4\int_0^2 z \cdot e^y\,dy\,dz = e^y\bigg|_0^2 \frac{z^2}{2}\bigg|_0^4 = 8(e^2 - 1)$$
$$\iint_{S_6} ze^{2x+y}\,dS_6 = \int_0^4\int_0^2 z \cdot e^{6+y}\,dy\,dz = e^{y+6}\bigg|_0^2 \frac{z^2}{2}\bigg|_0^4 = 8(e^8 - e^6)$$

The integral over the surface is just the sum of the integrals over the faces.

$$\iint_S ze^{2x+y}\,dS = 0 + 2(e^6 - 1)(e^2 - 1) + 4(e^6 - 1) + 4(e^8 - e^2) + 8(e^2 - 1) + 8(e^8 - e^6)$$
$$= 14e^8 - 6e^6 + 2e^2 - 10$$

39. Calculate $\iint_G x^2 z\, dS$, where G is the cylinder (including the top and bottom) $x^2 + y^2 = 4$, $0 \le z \le 3$.

SOLUTION We calculate the surface integral for each of the three surfaces. We begin with the bottom.

$$S_1 : \phi(x, y) = (x, y, 0)$$

$$\iint_{S_1} x^2 z\, dS_1 = \iint_{\mathcal{D}} x^2(0)\|\mathbf{n}_1\|\, dx\, dy = 0$$

Then the top

$$S_2 : \phi(x, y) = (x, y, 3)$$

$$\mathbf{T}_x = \langle 1, 0, 0\rangle, \mathbf{T}_y\ \langle 0, 1, 0\rangle \Rightarrow \mathbf{n}_2 = \langle 0, 0, 1\rangle$$

$$\iint_{S_2} x^2 z\, dS_2 = \iint_{\mathcal{D}} x^2(3)\|\mathbf{n}_2\|\, dx\, dy = 3\iint_{\mathcal{D}} x^2\, dx\, dy$$

The domain, $\mathcal{D}$, is the disk of radius 2. Changing to polar coordinates,

$$= 3\int_0^{2\pi}\int_0^2 (r\cos\theta)^2\, r\, dr\, d\theta = 3\int_0^{2\pi}\cos^2\theta\, d\theta \cdot \int_0^2 r^3\, dr$$

$$= 3\left(\frac{1}{2} + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} \cdot \frac{r^4}{4}\Big|_0^2 = 12\pi$$

Finally the side,

$$S_3 : \phi(r, \theta) = (2\cos\theta, 2\sin\theta, z)$$

$$\mathbf{T}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle, \mathbf{T}_z\ \langle 0, 0, 1\rangle$$

$$\Rightarrow \quad \mathbf{n}_3 = \langle 2\cos\theta, 2\sin\theta, 0\rangle \Rightarrow \quad \|\mathbf{n}_3\| = 2$$

$$\iint_{S_3} x^2 z\, dS_2 = \int_0^{2\pi}\int_0^3 (2\cos\theta)^2\, z\, 2\, dz\, d\theta$$

$$= 8\int_0^{2\pi}\cos^2\theta\, d\theta \cdot \int_0^3 z\, dz = 8\left(\frac{1}{2} + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} \cdot \frac{z^2}{2}\Big|_0^3 = 36\pi$$

The total surface integral is thus

$$\iint_G x^2 z\, dS = 0 + 12\pi + 36\pi = 48\pi$$

40. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 9$, where $1 \le x^2 + y^2 \le 4$ and $z \ge 0$ (Figure 21). Find a parametrization of S in polar coordinates and use it to compute:

(a) The area of S **(b)** $\iint_S z^{-1}\, dS$

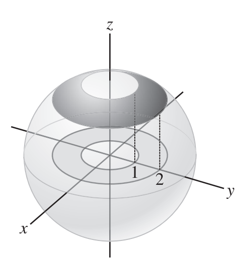
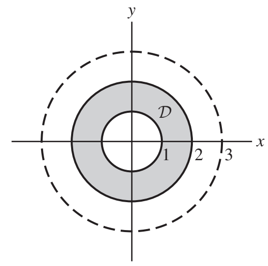


FIGURE 21

SOLUTION

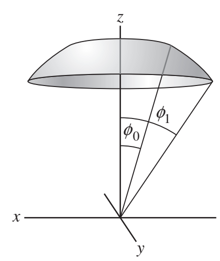


We parametrize S by spherical coordinates as follows:

$$\Phi(\theta,\phi) = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi)$$
$$\mathcal{D}:\ 0 \le \theta \le 2\pi,\ \phi_0 \le \phi \le \phi_1$$

The angles ϕ_0 and ϕ_1 are determined by,

$$\sin\phi_0 = \frac{1}{3} \quad\Rightarrow\quad \phi_0 = \sin^{-1}\frac{1}{3}$$
$$\sin\phi_1 = \frac{2}{3} \quad\Rightarrow\quad \phi_1 = \sin^{-1}\frac{2}{3}$$

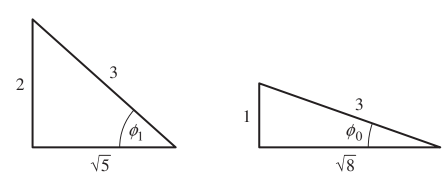


The length of the normal is:

$$\|\mathbf{n}\| = R^2\sin\phi = 9\sin\phi$$

(a) Using the integral for the surface area we have,

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}\|\,d\phi\,d\theta = \int_0^{2\pi}\int_{\sin^{-1}(1/3)}^{\sin^{-1}(2/3)} 9\sin\phi\,d\phi\,d\theta = \left(\int_0^{2\pi} 9\,d\phi\right)\left(\int_{\sin^{-1}(1/3)}^{\sin^{-1}(2/3)}\sin\phi\,d\phi\right)$$
$$= 18\pi\left(-\cos\phi\Big|_{\phi_0=\sin^{-1}(1/3)}^{\phi_1=\sin^{-1}(2/3)}\right) = 18\pi\left(-\frac{\sqrt{5}}{3}+\frac{\sqrt{8}}{3}\right) = 6\pi\left(\sqrt{8}-\sqrt{5}\right) \approx 11.166$$



(b) We express the function $f(x,y,z) = z^{-1}$ in the terms of the parameters:

$$f(\Phi(\theta,\phi)) = (3\cos\phi)^{-1} = \frac{\sec\phi}{3}$$

Using the surface integral as a double integral we obtain:

$$\iint_S z^{-1}\,dS = \iint_{\mathcal{D}} f(\Phi(\theta,\phi))\,\|\mathbf{n}\|\,d\phi\,d\theta = \int_0^{2\pi}\int_{\phi_0}^{\phi_1}\frac{\sec\phi}{3}\cdot 9\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_{\phi_0}^{\phi_1} 3\tan\phi\,d\phi\,d\theta$$
$$= \left(\int_0^{2\pi} 3\,d\theta\right)\left(\int_{\phi_0}^{\phi_1}\tan\phi\,d\phi\right) = 6\pi\left(\ln(\sec\phi)\Big|_{\phi_0=\sin^{-1}(1/3)}^{\phi_1=\sin^{-1}(2/3)}\right)$$
$$= 6\pi\left(\ln\frac{3}{\sqrt{5}} - \ln\frac{3}{\sqrt{8}}\right) = 6\pi\ln\sqrt{\frac{8}{5}} = 3\pi\ln 1.6 \approx 4.43$$

41. Prove a famous result of Archimedes: The surface area of the portion of the sphere of radius R between two horizontal planes $z = a$ and $z = b$ is equal to the surface area of the corresponding portion of the circumscribed cylinder (Figure 22).

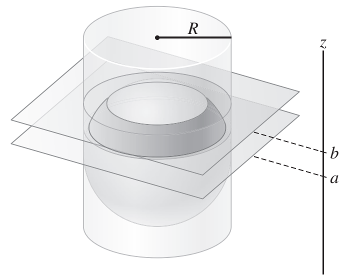


FIGURE 22

SOLUTION We compute the area of the portion of the sphere between the planes a and b. The portion S_1 of the sphere has the parametrization,

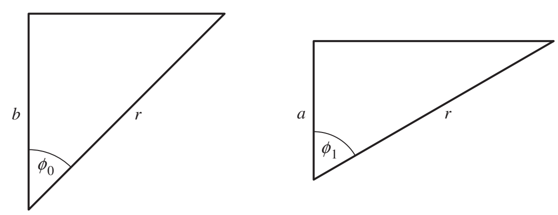
$$\Phi(\theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$$

where,

$$\mathcal{D}_1 : 0 \le \theta \le 2\pi,\ \phi_0 \le \phi \le \phi_1$$

If we assume $0 < a < b$, then the angles ϕ_0 and ϕ_1 are determined by,

$$\cos\phi_0 = \frac{b}{r} \quad \Rightarrow \quad \phi_0 = \cos^{-1}\frac{b}{r}$$
$$\cos\phi_1 = \frac{a}{r} \quad \Rightarrow \quad \phi_1 = \cos^{-1}\frac{a}{r}$$



The length of the normal vector is $\|\mathbf{n}\| = r^2\sin\phi$. We obtain the following integral:

$$\text{Area }(S_1) = \iint_{\mathcal{D}_1} \|\mathbf{n}\| d\phi\, d\theta = \int_0^{2\pi}\int_{\phi_0}^{\phi_1} r^2\sin\phi\, d\phi\, d\theta = \left(\int_0^{2\pi} r^2 d\phi\right)\left(\int_{\phi_1}^{\phi_2}\sin\phi\, d\phi\right)$$
$$= 2\pi r^2\left(-\cos\phi\Big|_{\phi=\cos^{-1}\frac{b}{r}}^{\cos^{-1}\frac{a}{r}}\right) = 2\pi r^2\left(-\frac{a}{r}+\frac{b}{r}\right) = 2\pi r(b-a)$$

The area of the part S_2 of the cylinder of radius r between the planes $z = a$ and $z = b$ is:

$$\text{Area }(S_2) = 2\pi r \cdot (b-a)$$

We see that the two areas are equal:

$$\text{Area }(S_1) = \text{Area }(S_2)$$

Further Insights and Challenges

42. Surfaces of Revolution Let $\mathcal{S}$ be the surface formed by rotating the region under the graph $z = g(y)$ in the yz-plane for $c \le y \le d$ about the z-axis, where $c \ge 0$ (Figure 23).

(a) Show that the circle generated by rotating a point $(0, a, b)$ about the z-axis is parametrized by

$$(a\cos\theta, a\sin\theta, b), \quad 0 \le \theta \le 2\pi$$

(b) Show that $\mathcal{S}$ is parametrized by

$$G(y, \theta) = (y\cos\theta, y\sin\theta, g(y))$$ 13

for $c \le y \le d$, $0 \le \theta \le 2\pi$.

(c) Use Eq. (13) to prove the formula

$$\text{Area}(\mathcal{S}) = 2\pi \int_c^d y\sqrt{1 + g'(y)^2}\,dy \qquad \boxed{14}$$

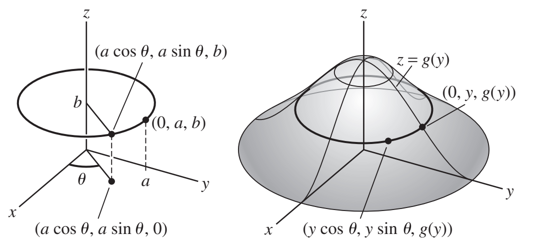


FIGURE 23

SOLUTION

(a) The circle generated by rotating a point $(0, a, b)$ about the z-axis is a circle of radius a centered at the point $(0, 0, b)$ on the z-axis. Therefore it is parametrized by,

$$(a\cos\theta, a\sin\theta, b), \quad 0 \le \theta \le 2\pi$$

(b) An arbitrary point $(x, y, g(y))$ on the surface S lies on the circle generated by rotating the point $(0, y, g(y))$ about the z-axis. Using part (a), a parametrization of this circle is:

$$(y\cos\theta, y\sin\theta, g(y)), \quad 0 \le \theta \le 2\pi$$

Therefore, the following parametrization parametrizes the surface S:

$$\Phi(y, \theta) = (y\cos\theta, y\sin\theta, g(y)), \quad 0 \le \theta \le 2\pi, \quad c \le y \le d.$$

(c) To compute the area of S we first find the tangent and normal vectors. We have:

$$\mathbf{T}_y = \frac{\partial\Phi}{\partial y} = \frac{\partial}{\partial y}(y\cos\theta, y\sin\theta, g(y)) = \langle \cos\theta, \sin\theta, g'(y)\rangle$$
$$\mathbf{T}_\theta = \frac{\partial\Phi}{\partial\theta} = \frac{\partial}{\partial\theta}(y\cos\theta, y\sin\theta, g(y)) = \langle -y\sin\theta, y\cos\theta, 0\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_y \times \mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & g'(y) \\ -y\sin\theta & y\cos\theta & 0 \end{vmatrix}$$
$$= \left(-y\cos\theta g'(y)\right)\mathbf{i} - \left(y\sin\theta g'(y)\right)\mathbf{j} + \left(y\ \cos^2\theta + y\ \sin^2\theta\right)\mathbf{k}$$
$$= y\left\langle -\cos\theta g'(y), -\sin\theta g'(y), 1\right\rangle$$

We compute the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = |y|\sqrt{\cos^2\theta g'(y)^2 + \sin^2\theta g'(y)^2 + 1} = |y|\sqrt{g'(y)^2\left(\cos^2\theta + \sin^2\theta\right) + 1} = |y|\sqrt{1 + g'(y)^2}$$

We obtain the following integral for the surface area:

$$\text{Area}(S) = \iint_{\mathcal{D}} 1\,dS = \iint_{\mathcal{D}} \|\mathbf{n}\|\,dy\,d\theta = \int_0^{2\pi}\int_c^d |y|\sqrt{1 + g'(y)^2}\,dy\,d\theta$$
$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_c^d |y|\sqrt{1 + g'(y)^2}\,dy\right) = 2\pi\int_c^d |y|\sqrt{1 + g'(y)^2}\,dy$$

43. Use Eq. (14) to compute the surface area of $z = 4 - y^2$ for $0 \le y \le 2$ rotated about the z-axis.

SOLUTION Since $g(y) = 4 - y^2$, we have $g'(y) = -2y$. By Eq. (14) we obtain the following integral,

$$\text{Area}(S) = 2\pi\int_0^2 |y|\sqrt{1 + (-2y)^2}\,dy = 2\pi\int_0^2 y\cdot\sqrt{1 + 4y^2}\,dy$$

We compute the integral using the substitution $u = 1 + 4y^2$, $du = 8y\,dy$. We get:

$$\text{Area}(S) = 2\pi \int_1^{17} u^{1/2} \cdot \frac{du}{8} = 2\pi \frac{2}{3} \cdot \frac{u^{3/2}}{8}\bigg|_1^{17} = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.18$$

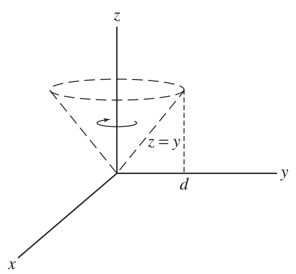
44. Describe the upper half of the cone $x^2 + y^2 = z^2$ for $0 \le z \le d$ as a surface of revolution (Figure 2) and use Eq. (14) to compute its surface area.

SOLUTION We find the graph $z = g(y)$ in the (yz)-plane that when revolved about the z-axis describes the upper part of the cone. Substituting $x = 0$ in the equation of the cone we have,

$$0^2 + y^2 = z^2 \quad \Rightarrow \quad z = \pm y$$

The generating curve is $z = g(y) = y$ for $0 \le y \le d$. Therefore $g'(y) = 1$ and Eq. (14) gives:

$$\text{Area}(S) = 2\pi \int_0^d |y|\sqrt{1 + g'(y)^2}\,dy = 2\pi \int_0^d y\sqrt{1 + 1^2}\,dy = 2\pi\sqrt{2}\int_0^d y\,dy = 2\pi\sqrt{2} \cdot \frac{y^2}{2}\bigg|_0^d = \sqrt{2}\pi d^2$$



45. Area of a Torus Let T be the torus obtained by rotating the circle in the yz-plane of radius a centered at $(0, b, 0)$ about the z-axis (Figure 24). We assume that $b > a > 0$.

(a) Use Eq. (14) to show that

$$\text{Area(T)} = 4\pi \int_{b-a}^{b+a} \frac{ay}{\sqrt{a^2 - (b - y)^2}}\,dy$$

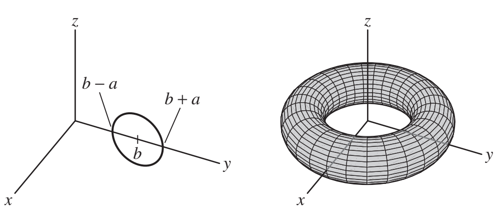
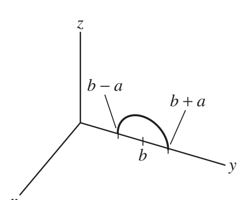


FIGURE 24 The torus obtained by rotating a circle of radius a.

(b) Show that $\text{Area(T)} = 4\pi^2 ab$.

SOLUTION

(a) Using symmetry, the area of the surface obtained by rotating the upper part of the circle is half the area of the torus.



The rotated graph is $z = g(y) = \sqrt{a^2 - (y - b)^2}$, $b - a \le y \le b + a$. So, we have,

$$g'(y) = \frac{-2(y - b)}{2\sqrt{a^2 - (y - b)^2}} = -\frac{y - b}{\sqrt{a^2 - (y - b)^2}}$$

$$\sqrt{1 + g'(y)^2} = \sqrt{1 + \frac{(y - b)^2}{a^2 - (y - b)^2}} = \sqrt{\frac{a^2 - (y - b)^2 + (y - b)^2}{a^2 - (y - b)^2}} = \frac{a}{\sqrt{a^2 - (y - b)^2}}$$

We now use symmetry and Eq. (14) to obtain the following area of the torus (we assume that $b-a>0$, hence $y>0$):

$$\text{Area}\,(\mathbf{T}) = 2\cdot 2\pi \int_{b-a}^{b+a} |y|\sqrt{1+g'(y)^2}\,dy = 4\pi \int_{b-a}^{b+a} \frac{ay}{\sqrt{a^2-(y-b)^2}}\,dy \tag{1}$$

(b) We compute the integral using the substitution $u=\frac{y-b}{a}$, $du=\frac{1}{a}\,dy$. We get:

$$\int_{b-a}^{b+a} \frac{ay}{\sqrt{a^2-(y-b)^2}}\,dy = \int_{-1}^{1} \frac{a^2u+ab}{\sqrt{a^2-a^2u^2}}a\,du = \int_{-1}^{1} \frac{a^2u+ab}{\sqrt{1-u^2}}\,du = \int_{-1}^{1} \frac{a^2u}{\sqrt{1-u^2}}\,du + \int_{-1}^{1} \frac{ab}{\sqrt{1-u^2}}\,du$$

The first integral is zero since the integrand is an odd function. We get:

$$\int_{b-a}^{b+a} \frac{ay}{\sqrt{a^2-(y-b)^2}}\,dy = 2\int_{0}^{1} \frac{ab}{\sqrt{1-u^2}}\,du = 2ab\sin^{-1}u\Big|_0^1 = 2ab\left(\frac{\pi}{2}-0\right) = \pi ab$$

Substituting in (1) gives the following area:

$$\text{Area}\,(\mathbf{T}) = 4\pi\cdot\pi ab = 4\pi^2 ab$$

46. Pappus's Theorem (also called **Guldin's Rule**) states that the area of a surface of revolution $\mathcal{S}$ is equal to the length L of the generating curve times the distance traversed by the center of mass. Use Eq. (14) to prove Pappus's Theorem. If $\mathcal{C}$ is the graph $z=g(y)$ for $c\le y\le d$, then the center of mass is defined as the point $(\overline{y},\overline{z})$ with

$$\overline{y} = \frac{1}{L}\int_{\mathcal{C}} y\,ds, \qquad \overline{z} = \frac{1}{L}\int_{\mathcal{C}} z\,ds$$

SOLUTION We may assume that the generating curve $z=g(y)$ lies in the region $y\ge 0$ (otherwise we translate the axes so that this condition is satisfied). The curve $z=g(y)$, $c\le y\le d$ is parametrized by:

$$\mathcal{C}:\ \mathbf{c}(y) = (y, g(y)), \quad c\le y\le d$$

Hence,

$$\mathbf{c}'(y) = \langle 1, g'(y)\rangle \quad\Rightarrow\quad \|\mathbf{c}'(y)\| = \sqrt{1+g'(y)^2}$$

Using the theorem on computing scalar line integrals, we obtain:

$$\int_{\mathcal{C}} y\,ds = \int_c^d y\|\mathbf{c}'(y)\|\,dy = \int_c^d y\sqrt{1+g'(y)^2}\,dy \tag{1}$$

The center of mass $(\overline{y},\overline{z})$ of the generating curve traverses a circle of radius $\overline{y}$. Therefore the distance traversed by the center of mass is $2\pi\overline{y}$. The length L of the generating curve, times the distance traversed by the center of mass, is:

$$L\cdot 2\pi\overline{y} = L\cdot 2\pi\cdot\frac{1}{L}\int_{\mathcal{C}} y\,ds = 2\pi\int_{\mathcal{C}} y\,ds$$

Combining with (1) we get:

$$L\cdot 2\pi\overline{y} = 2\pi\cdot\int_c^d y\sqrt{1+g'(y)^2}\,dy$$

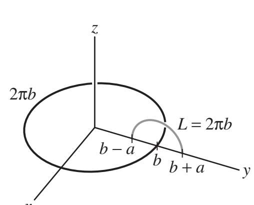
Since $y\ge 0$, the right hand-side is the area of the surface of revolution S, as stated in Eq. (14). Therefore we get:

$$L\cdot 2\pi\overline{y} = \text{Area}(S)$$

This proves Pappus' Theorem.

47. Compute the surface area of the torus in Exercise 45 using Pappus's Theorem.

SOLUTION The generating curve is the circle of radius a in the (y,z)-plane centered at the point $(0,b,0)$. The length of the generating curve is $L=\pi a$.



The center of mass of the circle is at the center $(\overline{y}, \overline{z}) = (b, 0)$, and it traverses a circle of radius b centered at the origin. Therefore, the center of mass makes a distance of $2\pi b$. Using Pappus' Theorem, the area of the torus is:

$$L \cdot 2\pi a = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

48. **Potential Due to a Uniform Sphere** Let $\mathcal{S}$ be a hollow sphere of radius R with center at the origin with a uniform mass distribution of total mass m [since $\mathcal{S}$ has surface area $4\pi R^2$, the mass density is $\rho = m/(4\pi R^2)$]. The gravitational potential $V(P)$ due to $\mathcal{S}$ at a point $P = (a, b, c)$ is equal to

$$-G \iint_{\mathcal{S}} \frac{\rho \, dS}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

(a) Use symmetry to conclude that the potential depends only on the distance r from P to the center of the sphere. Therefore, it suffices to compute $V(P)$ for a point $P = (0, 0, r)$ on the z-axis (with $r \neq R$).

(b) Use spherical coordinates to show that $V(0, 0, r)$ is equal to

$$\frac{-Gm}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\sin\phi \, d\theta \, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

(c) Use the substitution $u = R^2 + r^2 - 2Rr\cos\phi$ to show that

$$V(0, 0, r) = \frac{-mG}{2Rr}\big(|R + r| - |R - r|\big)$$

(d) Verify Eq. (12) for V.

SOLUTION

(a) The gravitational potential due to S at a point $P = (a, b, c)$ is given by:

$$\varphi(P) = -G \iint_S \frac{\rho \, dS}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

Due to the symmetry of the sphere, the "sum" of the distances of a point $P = (a, b, c)$ from the points $Q = (x, y, z)$ on the sphere is equal for all the points P located at the same distance r from the center of the sphere. That is, the integral $\iint_S \frac{dS}{|Q-P|}$ depends only on r. Since ρ is constant, the integral for $\varphi(P)$ also depends only on r. Thus, we might as well assume that P is on the z-axis at the point $(0, 0, r)$, with r being $\sqrt{a^2 + b^2 + c^2}$.

(b) For $a = b = 0$, and $c = r$ we have,

$$\varphi(0, 0, r) = -G \iint_S \frac{\rho \, dS}{\sqrt{x^2 + y^2 + (z-r)^2}} \tag{1}$$

We parametrize the sphere S by the spherical parametrization:

$$\Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

Then, the length of the normal vector is:

$$\|\mathbf{n}\| = R^2 \sin\phi$$

We express the function in the integrand in terms of the parameters:

$$\begin{aligned} x^2 + y^2 + (z-r)^2 &= R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta\sin^2\phi + (R\cos\phi - r)^2 \\ &= R^2\sin^2\phi + R^2\cos^2\phi - 2Rr\cos\phi + r^2 = R^2 - 2Rr\cos\phi + r^2 \end{aligned}$$

The surface integral in (1) is equal to the following double integral:

$$\begin{aligned} \varphi(0, 0, r) &= -G \int_0^{\pi} \int_0^{2\pi} \frac{\rho \|\mathbf{n}\| \, d\theta \, d\phi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}} = -G \int_0^{\pi} \int_0^{2\pi} \frac{\frac{m}{4\pi R^2} \cdot R^2 \sin\phi \, d\theta \, d\phi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}} \\ &= -\frac{Gm}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\sin\phi \, d\theta \, d\phi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}} \end{aligned}$$

(c) We compute the double integral for $\varphi(0, 0, r)$. Since the integrand does not depend on θ, we have,

$$\varphi(0, 0, r) = -\frac{Gm}{4\pi} \cdot 2\pi \int_0^{\pi} \frac{\sin\phi \, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}} = -\frac{Gm}{2} \int_0^{\pi} \frac{\sin\phi \, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

We compute the integral using the substitution $u = R^2 + r^2 - 2Rr\cos\phi$, $du = 2Rr\sin\phi\, d\phi$. We obtain:

$$\varphi(0,0,r) = -\frac{Gm}{2}\int_{(R-r)^2}^{(R+r)^2} \frac{\frac{du}{2Rr}}{\sqrt{u}} = -\frac{Gm}{4Rr}\int_{(R-r)^2}^{(R+r)^2} u^{-1/2}\,du = -\frac{Gm}{4Rr}\cdot 2u^{1/2}\Big|_{u=(R-r)^2}^{(R+r)^2}$$

$$= -\frac{Gm}{2Rr}\left(\left((R+r)^2\right)^{1/2} - \left((R-r)^2\right)^{1/2}\right) = -\frac{Gm}{2Rr}(|R+r| - |R-r|)$$

(d) For $r > R$:

$$|R+r| - |R-r| = R + r - (r - R) = 2R$$

Thus,

$$\phi(0,0,r) = -\frac{Gm}{2Rr}\cdot 2R = -\frac{Gm}{r}.$$

Similarly, for $r < R$ we obtain,

$$\phi(0,0,r) = -\frac{Gm}{2Rr}\cdot 2r = -\frac{Gm}{R}.$$

49. Calculate the gravitational potential V for a hemisphere of radius R with uniform mass distribution.

SOLUTION In Exercise 48(b) we expressed the potential φ for a sphere of radius R. To find the potential for a hemisphere of radius R, we need only to modify the limits of the angle ϕ to $0 \le \phi \le \frac{\pi}{2}$. This gives the following integral:

$$\varphi(0,0,r) = \varphi(r) = -\frac{Gm}{4\pi}\int_0^{\pi/2}\int_0^{2\pi}\frac{\sin\phi\, d\theta\, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}} = -\frac{Gm}{4\pi}\cdot 2\pi\int_0^{\pi/2}\frac{\sin\phi\, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

$$= -\frac{Gm}{2}\int_0^{\pi/2}\frac{\sin\phi\, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

We compute the integral using the substitution $u = R^2 + r^2 - 2Rr\cos\phi$, $du = 2Rr\sin\phi\, d\phi$. We obtain:

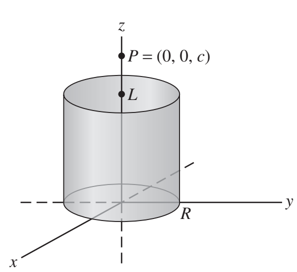
$$\varphi(r) = -\frac{Gm}{2}\int_{(R-r)^2}^{R^2+r^2}\frac{\frac{du}{2Rr}}{\sqrt{u}} = -\frac{Gm}{4Rr}\int_{(R-r)^2}^{R^2+r^2} u^{-1/2}\,du = -\frac{Gm}{4Rr}\cdot 2u^{1/2}\Big|_{u=(R-r)^2}^{R^2+r^2}$$

$$= -\frac{Gm}{2Rr}\left(\left(R^2 + r^2\right)^{1/2} - \left((R-r)^2\right)^{1/2}\right) = -\frac{Gm}{2Rr}\left(\sqrt{R^2 + r^2} - |R - r|\right)$$

50. The surface of a cylinder of radius R and length L has a uniform mass distribution ρ (the top and bottom of the cylinder are excluded). Use Eq. (11) to find the gravitational potential at a point P located along the axis of the cylinder.

SOLUTION By Eq. (11) the gravitational potential at a point $P = (0,0,c)$ along the axis of the cylinder is:

$$\varphi(0,0,c) = -G\iint_S \frac{\rho\, dS}{\sqrt{x^2 + y^2 + (z-c)^2}}$$



We parametrize the cylinder S by the following parametrization:

$$\Phi(\theta, z) = (R\cos\theta, R\sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le L$$

We compute the tangent and normal vectors:

$$\mathbf{T}_\theta = \frac{\partial\Phi}{\partial\theta} = \langle -R\sin\theta, R\cos\theta, 0\rangle$$

$$\mathbf{T}_z = \frac{\partial\Phi}{\partial z} = \langle 0, 0, 1\rangle$$

Hence,

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos\theta)\mathbf{i} + (R\sin\theta)\mathbf{j} = R\,\langle\cos\theta, \sin\theta, 0\rangle$$

Hence,

$$\|\mathbf{n}\| = R\sqrt{\cos^2\theta + \sin^2\theta} = R$$

We express the function in the integrand in terms of the parameters:

$$\sqrt{x^2 + y^2 + (z-c)^2} = \sqrt{R^2\,\cos^2\theta + R^2\,\sin^2\theta + (z-c)^2} = \sqrt{R^2 + (z-c)^2}$$

We obtain the following integral:

$$\varphi(0,0,c) = -G\int_0^{2\pi}\int_0^L \frac{\rho\|\mathbf{n}\|\,dz\,d\theta}{\sqrt{R^2+(z-c)^2}} = -G\int_0^{2\pi}\int_0^L \frac{\rho R\,dz\,d\theta}{\sqrt{R^2+(z-c)^2}} = -G\rho R\int_0^{2\pi}\int_0^L \frac{dz\,d\theta}{\sqrt{R^2+(z-c)^2}}$$

$$= -G\rho R\cdot 2\pi\int_0^L \frac{dz}{\sqrt{R^2+(z-c)^2}} = -2G\rho R\pi\,\ln\left|z-c+\sqrt{(z-c)^2+R^2}\right|\Big|_{z=0}^{L}$$

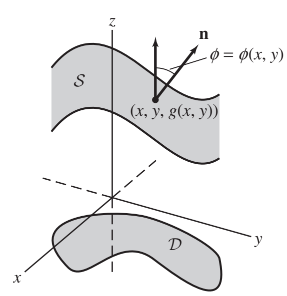
$$= -2G\rho R\pi\left(\ln\left(L-c+\sqrt{(L-c)^2+R^2}\right)\right) - \ln\left(-c+\sqrt{c^2+R^2}\right)$$

$$= -2G\rho R\pi\,\ln\left(\frac{L-c+\sqrt{(L-c)^2+R^2}}{-c+\sqrt{c^2+R^2}}\right)$$

51. Let S be the part of the graph $z = g(x,y)$ lying over a domain $\mathcal{D}$ in the xy-plane. Let $\phi = \phi(x,y)$ be the angle between the normal to S and the vertical. Prove the formula

$$\text{Area}(S) = \iint_{\mathcal{D}} \frac{dA}{|\cos\phi|}$$

SOLUTION



Using the Surface Integral over a Graph we have:

$$\text{Area}(S) = \iint_S 1\,dS = \iint_{\mathcal{D}} \sqrt{1 + g_x^2 + g_y^2}\,dA \tag{1}$$

In parametrizing the surface by $\phi(x,y) = (x, y, g(x,y))$, $(x,y) = \mathcal{D}$, we have:

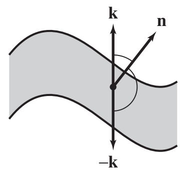
$$\mathbf{T}_x = \frac{\partial\Phi}{\partial x} = \langle 1, 0, g_x\rangle$$

$$\mathbf{T}_y = \frac{\partial\Phi}{\partial y} = \langle 0, 1, g_y\rangle$$

Hence,

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k} = \langle -g_x, -g_y, 1\rangle$$

$$\|\mathbf{n}\| = \sqrt{g_x^2 + g_y^2 + 1}$$



There are two adjacent angles between the normal **n** and the vertical, and the cosines of these angles are opposite numbers. Therefore we take the absolute value of $\cos\phi$ to obtain a positive value for Area(S). Using the Formula for the cosine of the angle between two vectors we get:

$$|\cos\phi| = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{|\langle -g_x, -g_y, 1\rangle \cdot \langle 0,0,1\rangle|}{\sqrt{1+g_x^2+g_y^2}\cdot 1} = \frac{1}{\sqrt{1+g_x^2+g_y^2}}$$

Substituting in (1) we get:

$$\text{Area}(S) = \iint_{\mathcal{D}} \frac{dA}{|\cos\phi|}$$

16.5 Surface Integrals of Vector Fields (LT Section 17.5)

Preliminary Questions

1. Let **F** be a vector field and $G(u,v)$ a parametrization of a surface $\mathcal{S}$, and set $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$. Which of the following is the normal component of **F**?

(a) $\mathbf{F}\cdot\mathbf{n}$ **(b)** $\mathbf{F}\cdot\mathbf{e_n}$

SOLUTION The normal component of **F** is $\mathbf{F}\cdot\mathbf{e}_n$ rather than $\mathbf{F}\cdot\mathbf{n}$.

2. The vector surface integral $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$ is equal to the scalar surface integral of the function (choose the correct answer):

(a) $\|\mathbf{F}\|$

(b) $\mathbf{F}\cdot\mathbf{n}$, where **n** is a normal vector

(c) $\mathbf{F}\cdot\mathbf{e_n}$, where $\mathbf{e_n}$ is the unit normal vector

SOLUTION The vector surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is defined as the scalar surface integral of the normal component of **F** on the oriented surface. That is, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (\mathbf{F}\cdot\mathbf{e}_n)\, dS$ as stated in (c).

3. $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$ is zero if (choose the correct answer):

(a) **F** is tangent to $\mathcal{S}$ at every point.

(b) **F** is perpendicular to $\mathcal{S}$ at every point.

SOLUTION Since $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is equal to the scalar surface integral of the normal component of **F** on S, this integral is zero when the normal component is zero at every point, that is, when **F** is tangent to S at every point as stated in (a).

4. If $\mathbf{F}(P) = \mathbf{e_n}(P)$ at each point on $\mathcal{S}$, then $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$ is equal to (choose the correct answer):

(a) Zero **(b)** Area$(\mathcal{S})$ **(c)** Neither

SOLUTION If $\mathbf{F}(P) = \mathbf{e}_n(P)$ at each point on S, then,:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (\mathbf{e}_n\cdot\mathbf{e}_n)\, dS = \iint_S \|\mathbf{e}_n\|^2\, dS = \iint_S 1\, dS = \text{Area}(S)$$

Therefore, (b) is the correct answer.

5. Let $\mathcal{S}$ be the disk $x^2+y^2 \le 1$ in the xy-plane oriented with normal in the positive z-direction. Determine $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$ for each of the following vector constant fields:

(a) $\mathbf{F} = \langle 1,0,0\rangle$ **(b)** $\mathbf{F} = \langle 0,0,1\rangle$ **(c)** $\mathbf{F} = \langle 1,1,1\rangle$

SOLUTION The unit normal vector to the oriented disk is $\mathbf{e}_n = \langle 0,0,1\rangle$.

(a) Since $\mathbf{F}\cdot\mathbf{e}_n = \langle 1,0,0\rangle\cdot\langle 0,0,1\rangle = 0$, **F** is perpendicular to the unit normal vector at every point on S, therefore $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$.

(b) Since $\mathbf{F} = \mathbf{e}_n$ at every point on S, we have:

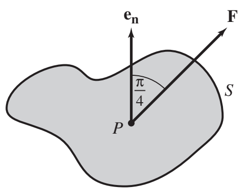
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (\mathbf{e}_n\cdot\mathbf{e}_n)\, dS = \iint_S \|\mathbf{e}_n\|^2\, dS = \iint_S 1\, dS = \text{Area}(S) = \pi$$

(c) For $\mathbf{F} = \langle 1, 1, 1 \rangle$ we have:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{e}_n)\, dS = \iint_S \langle 1, 1, 1 \rangle \cdot \langle 0, 0, 1 \rangle\, dS = \iint_S 1\, dS = \text{Area}(S) = \pi$$

6. Estimate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathcal{S}$ is a tiny oriented surface of area 0.05 and the value of $\mathbf{F}$ at a sample point in $\mathcal{S}$ is a vector of length 2 making an angle $\frac{\pi}{4}$ with the normal to the surface.

SOLUTION



Since S is a tiny surface, we may assume that the dot product $\mathbf{F} \cdot \mathbf{e}_n$ on S is equal to the dot product at the sample point. This gives the following approximation:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{e}_n)\, d\mathbf{S} \approx \iint_S (\mathbf{F}(P) \cdot \mathbf{e}_n(P))\, d\mathbf{S} = \mathbf{F}(P) \cdot \mathbf{e}_n(P) \iint_S 1 d\mathbf{S} = \mathbf{F}(P) \cdot \mathbf{e}_n \text{Area}(S)$$

That is,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \approx \mathbf{F}(P) \cdot \mathbf{e}_n(P)\text{Area}(S) \tag{1}$$

We are given that $\text{Area}(S) = 0.05$. We compute the dot product:

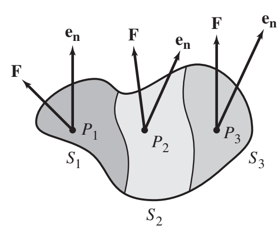
$$\mathbf{F}(P) \cdot \mathbf{e}_n(P) = \|\mathbf{F}(P)\| \|\mathbf{e}_n(P)\| \cos\frac{\pi}{4} = 2 \cdot 1 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$$

Combining with (1) gives the following estimation:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \approx 0.05\sqrt{2} \approx 0.0707.$$

7. A small surface $\mathcal{S}$ is divided into three pieces of area 0.2. Estimate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}$ is a unit vector field making angles of $85°$, $90°$, and $95°$ with the normal at sample points in these three pieces.

SOLUTION



We estimate the vector surface integral by the following sum:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \mathbf{F}(P_1) \cdot \mathbf{e}_n(P_1)\,\text{Area}(S_1) + \mathbf{F}(P_2) \cdot \mathbf{e}_n(P_2)\,\text{Area}(S_2) + \mathbf{F}(P_3) \cdot \mathbf{e}_n(P_3)\,\text{Area}(S_3) \\ &= 0.2\,(\mathbf{F}(P_1) \cdot \mathbf{e}_n(P_1) + \mathbf{F}(P_2) \cdot \mathbf{e}_n(P_2) + \mathbf{F}(P_3) \cdot \mathbf{e}_n(P_3))\end{aligned}$$

We compute the dot product. Since $\mathbf{F}$ and $\mathbf{e}_n$ are unit vectors, we have:

$$\begin{aligned}\mathbf{F}(P_1) \cdot \mathbf{e}_n(P_1) &= \cos 85° \approx 0.0872 \\ \mathbf{F}(P_2) \cdot \mathbf{e}_n(P_2) &= \cos 90° = 0 \\ \mathbf{F}(P_3) \cdot \mathbf{e}_n(P_3) &= \cos 95° \approx -0.0872\end{aligned}$$

Substituting gives the following estimation:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \approx 0.2(0.0872 + 0 - 0.0872) = 0.$$

Exercises

1. Let $\mathbf{F} = \langle z, 0, y\rangle$ and let $\mathcal{S}$ be the oriented surface parametrized by $G(u, v) = (u^2 - v, u, v^2)$ for $0 \le u \le 2$, $-1 \le v \le 4$. Calculate:

(a) $\mathbf{n}$ and $\mathbf{F}\cdot\mathbf{n}$ as functions of u and v

(b) The normal component of $\mathbf{F}$ to the surface at $P = (3, 2, 1) = G(2, 1)$

(c) $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$

SOLUTION

(a) The tangent vectors are,

$$\mathbf{T}_u = \frac{\partial \mathrm{G}}{\partial u} = \frac{\partial}{\partial u}\left(u^2 - v, u, v^2\right) = \langle 2u, 1, 0\rangle$$
$$\mathbf{T}_v = \frac{\partial \mathrm{G}}{\partial v} = \frac{\partial}{\partial v}\left(u^2 - v, u, v^2\right) = \langle -1, 0, 2v\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 1 & 0 \\ -1 & 0 & 2v \end{vmatrix} = v\mathbf{i} - 4uv\mathbf{j} + \mathbf{k} = \langle 2v, -4uv, 1\rangle$$

We write $\mathbf{F} = \langle z, 0, y\rangle$ in terms of the parameters $x = u^2 - v$, $y = u$, $z = v^2$ and then compute $\mathbf{F}\cdot\mathbf{n}$:

$$\mathbf{F}(\Phi(u, v)) = \langle z, 0, y\rangle = \left\langle v^2, 0, u\right\rangle$$
$$\mathbf{F}(\Phi(u, v))\cdot\mathbf{n}(u, v) = \left\langle v^2, 0, u\right\rangle\cdot\langle 2v, -4uv, 1\rangle$$
$$= 2v^3 + u$$

(b) At the point $P = (3, 2, 1) = \Phi(2, 1)$ we have:

$$\mathbf{F}(P) = \langle 1, 0, 2\rangle$$
$$\mathbf{n}(P) = \langle 2, -8, 1\rangle$$
$$\mathbf{e}_n(P) = \frac{\mathbf{n}(P)}{\|\mathbf{n}(P)\|} = \frac{\langle 2, -8, 1\rangle}{\sqrt{4 + 64 + 1}} = \frac{1}{\sqrt{69}}\langle 2, -8, 1\rangle$$

Hence, the normal component of $\mathbf{F}$ to the surface at P is the dot product:

$$\mathbf{F}(P)\cdot\mathbf{e}_n(P) = \langle 1, 0, 2\rangle\cdot\frac{1}{\sqrt{69}}\langle 2, -8, 1\rangle = \frac{4}{\sqrt{69}}$$

(c) Using the definition of the vector surface integral and the dot product in part (a), we have:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\phi(u, v))\cdot\mathbf{n}(u, v)\, du\, dv = \int_0^2\int_{-1}^4 \left(2v^3 + u\right) dv\, du$$
$$= \int_0^2 \frac{2v^4}{4} + uv\Big|_{v=-1}^{4} du$$
$$= \int_0^2 \left(128 - \frac{1}{2}\right) + (4 - (-1))u\, du$$
$$= \int_0^2 \frac{255}{2} + 5u\, du = \frac{255u}{2} + \frac{5u^2}{2}\Big|_0^2 = 265$$

2. Let $\mathbf{F} = \langle y, -x, x^2 + y^2\rangle$ and let $\mathcal{S}$ be the portion of the paraboloid $z = x^2 + y^2$ where $x^2 + y^2 \le 3$.

(a) Show that if $\mathcal{S}$ is parametrized in polar variables $x = r\cos\theta$, $y = r\sin\theta$, then $\mathbf{F}\cdot\mathbf{n} = r^3$.

(b) Show that $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\sqrt{3}} r^3\, dr\, d\theta$ and evaluate.

SOLUTION

(a) The parametrization of this surface in these coordinates is:

$$\phi(r, \theta) = (x, y, x^2 + y^2) = (r\cos\theta, r\sin\theta, r^2) \quad (0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{3})$$

Calculating the normal vector,

$$\mathbf{T}_r = \langle \cos\theta, \sin\theta, 2r \rangle$$
$$\mathbf{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$
$$\Rightarrow \quad \mathbf{n} = \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle$$

Then

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n} &= \left\langle y, -x, x^2+y^2 \right\rangle \cdot \mathbf{n} = \left\langle r\sin\theta, -r\cos\theta, r^2 \right\rangle \cdot \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle \\ &= -2r^3\sin\theta\cos\theta + 2r^3\cos\theta\sin\theta + r^3 = r^3\end{aligned}$$

(b) The integral over this surface is

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\sqrt{3}} \mathbf{F}\cdot\mathbf{n}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r^3\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{3}} r^3\,dr = 2\pi \left.\frac{r^4}{4}\right|_0^{\sqrt{3}} = \frac{9\pi}{2}\end{aligned}$$

3. Let $\mathcal{S}$ be the unit square in the xy-plane shown in Figure 14, oriented with the normal pointing in the positive z-direction. Estimate

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}$ is a vector field whose values at the labeled points are

$$\mathbf{F}(A) = \langle 2, 6, 4 \rangle, \qquad \mathbf{F}(B) = \langle 1, 1, 7 \rangle$$
$$\mathbf{F}(C) = \langle 3, 3, -3 \rangle, \qquad \mathbf{F}(D) = \langle 0, 1, 8 \rangle$$

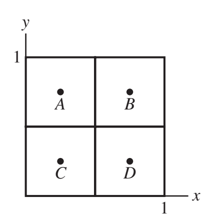
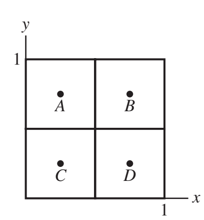


FIGURE 14

SOLUTION The unit normal vector to S is $\mathbf{e}_n = \langle 0, 0, 1 \rangle$. We estimate the vector surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ using the division and sample points given in Figure 12.



Each subsquare has area $\frac{1}{4}$, therefore we obtain the following estimation:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &\approx (\mathbf{F}(A)\cdot\mathbf{e}_n + \mathbf{F}(B)\cdot\mathbf{e}_n + \mathbf{F}(C)\cdot\mathbf{e}_n + \mathbf{F}(D)\cdot\mathbf{e}_n)\cdot\frac{1}{4} \\ &= (\langle 2,6,4\rangle\cdot\langle 0,0,1\rangle + \langle 1,1,7\rangle\cdot\langle 0,0,1\rangle + \langle 3,3,-3\rangle\cdot\langle 0,0,1\rangle + \langle 0,1,8\rangle\cdot\langle 0,0,1\rangle)\cdot\frac{1}{4} \\ &= (4+7-3+8)\cdot\frac{1}{4} = 4\end{aligned}$$

4. Suppose that $\mathcal{S}$ is a surface in $\mathbf{R}^3$ with a parametrization G whose domain $\mathcal{D}$ is the square in Figure 14. The values of a function f, a vector field $\mathbf{F}$, and the normal vector $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ at $G(P)$ are given for the four sample points in $\mathcal{D}$ in the following table. Estimate the surface integrals of f and $\mathbf{F}$ over $\mathcal{S}$.

Point P in $\mathcal{D}$	f	$\mathbf{F}$	$\mathbf{n}$
A	3	$\langle 2, 6, 4\rangle$	$\langle 1, 1, 1\rangle$
B	1	$\langle 1, 1, 7\rangle$	$\langle 1, 1, 0\rangle$
C	2	$\langle 3, 3, -3\rangle$	$\langle 1, 0, -1\rangle$
D	5	$\langle 0, 1, 8\rangle$	$\langle 2, 1, 0\rangle$

SOLUTION The area of each subrectangle is $\frac{1}{4}$. We estimate the surface integral of f over $\mathcal{S}$ by the following sum:

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = (f(A)\|\mathbf{n}(A)\| + f(B)\|\mathbf{n}(B)\| + f(C)\|\mathbf{n}(C)\| + f(D)\|\mathbf{n}(D)\|) \cdot \frac{1}{4} \tag{1}$$

We use the given data to compute the length of the normal vectors:

$$\|\mathbf{n}(A)\| = \|\,\langle 1, 1, 1\rangle\,\| = \sqrt{3}$$
$$\|\mathbf{n}(B)\| = \|\,\langle 1, 1, 0\rangle\,\| = \sqrt{2}$$
$$\|\mathbf{n}(C)\| = \|\,\langle 1, 0, -1\rangle\,\| = \sqrt{2}$$
$$\|\mathbf{n}(D)\| = \|\,\langle 2, 1, 0\rangle\,\| = \sqrt{5}$$

Substituting the values in (1) we obtain the following estimation:

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \left(3\sqrt{3} + 1 \cdot \sqrt{2} + 2\sqrt{2} + 5\sqrt{5}\right) \cdot \frac{1}{4} = \frac{3\sqrt{3} + 3\sqrt{2} + 5\sqrt{5}}{4} \approx 5.155$$

We now estimate the vector surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ by the following sum:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = (\mathbf{F}(A) \cdot \mathbf{n} + \mathbf{F}(B) \cdot \mathbf{n} + \mathbf{F}(C) \cdot \mathbf{n} + \mathbf{F}(D) \cdot \mathbf{n}) \cdot \frac{1}{4}$$
$$= (\langle 2, 6, 4\rangle \cdot \langle 1, 1, 1\rangle + \langle 1, 1, 7\rangle \cdot \langle 1, 1, 0\rangle + \langle 3, 3, -3\rangle \cdot \langle 1, 0, -1\rangle + \langle 0, 1, 8\rangle \cdot \langle 2, 1, 0\rangle) \cdot \frac{1}{4}$$
$$= (12 + 2 + 6 + 1) \cdot \frac{1}{4} = \frac{21}{4}$$

In Exercises 5–17, compute $\iint_{\mathcal{S}} \mathbf{F} \cdot\, d\mathbf{S}$ *for the given oriented surface.*

5. $\mathbf{F} = \langle y, z, x\rangle$, plane $3x - 4y + z = 1$, $0 \le x \le 1$, $0 \le y \le 1$, upward-pointing normal

SOLUTION We rewrite the equation of the plane as $z = 1 - 3x + 4y$, and parametrize the plane by:

$$\Phi(x, y) = (x, y, 1 - 3x + 4y)$$

Here, the parameter domain is the square $\mathcal{D} = \{(x, y) : 0 \le x,\ y \le 1\}$ in the xy-plane.

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_x = \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}(x, y, 1 - 3x + 4y) = \langle 1, 0, -3\rangle$$
$$\mathbf{T}_y = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(x, y, 1 - 3x + 4y) = \langle 0, 1, 4\rangle$$
$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 1 & 4 \end{vmatrix} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} = \langle 3, -4, 1\rangle$$

Since the plane is oriented with upward pointing normal, the normal vector $\mathbf{n}$ is:

$$\mathbf{n} = \langle 3, -4, 1\rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We write $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}(\Phi(x, y)) = \langle y, z, x\rangle = \langle y, 1 - 3x + 4y, x\rangle$$

The dot product $\mathbf{F} \cdot \mathbf{n}$ is thus

$$\mathbf{F}(\Phi(x, y)) \cdot \mathbf{n} = \langle y, 1 - 3x + 4y, x\rangle \cdot \langle 3, -4, 1\rangle = 3y - 4(1 - 3x + 4y) + x = 13x - 13y - 4$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(x,y))\cdot \mathbf{n}(x,y)\,dx\,dy = \int_0^1\int_0^1 (13x - 13y - 4)\,dx\,dy$$
$$= \int_0^1 \frac{13x^2}{2} - 13yx - 4x\Big|_{x=0}^1 dy = \int_0^1 \left(\frac{13}{2} - 13y - 4\right) dy = \frac{5y}{2} - \frac{13y^2}{2}\Big|_0^1 = -4$$

6. $\mathbf{F} = \langle e^z, z, x\rangle$, $\quad G(r,s) = (rs, r+s, r), 0 \le r \le 1, 0 \le s \le 1,\quad$ oriented by $\mathbf{T}_r \times \mathbf{T}_s$

SOLUTION

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}(rs, r+s, r) = \langle s, 1, 1\rangle$$
$$\mathbf{T}_s = \frac{\partial \Phi}{\partial s} = \frac{\partial}{\partial s}(rs, r+s, r) = \langle r, 1, 0\rangle$$
$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_s = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s & 1 & 1 \\ r & 1 & 0 \end{vmatrix} = -\mathbf{i} + r\mathbf{j} + (s-r)\mathbf{k} = \langle -1, r, s-r\rangle$$

Step 2. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We write $\mathbf{F}$ in terms of the parameters $x = rs$, $y = r+s$, $z = r$:

$$\mathbf{F}(\phi(r,s)) = \langle e^z, z, x\rangle = \langle e^r, r, rs\rangle$$

We compute the dot product $\mathbf{F}\cdot\mathbf{n}$:

$$\mathbf{F}(\phi(r,s))\cdot \mathbf{n}(r,s) = \langle e^r, r, rs\rangle \cdot \langle -1, r, s-r\rangle = -e^r + r^2 + rs(s-r) = -e^r + r^2(1-s) + rs^2$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(r,s))\cdot\mathbf{n}(r,s)\,dr\,ds = \int_0^1\int_0^1 \left(-e^r + r^2(1-s) + rs^2\right) dr\,ds$$
$$= \int_0^1 -e^r + \frac{r^3(1-s)}{3} + \frac{r^2s^2}{2}\Big|_{r=0}^1 ds = \int_0^1 \left(-e + \frac{1-s}{3} + \frac{s^2}{2} + 1\right) ds$$
$$= (1-e)s + \frac{s - \frac{s^2}{2}}{3} + \frac{s^3}{6}\Big|_0^1 = 1 - e + \frac{1}{6} + \frac{1}{6} = \frac{4}{3} - e \approx -1.385$$

7. $\mathbf{F} = \langle 0, 3, x\rangle$, $\quad$ part of sphere $x^2 + y^2 + z^2 = 9$, where $x \ge 0$, $y \ge 0$, $z \ge 0$ $\quad$ outward-pointing normal

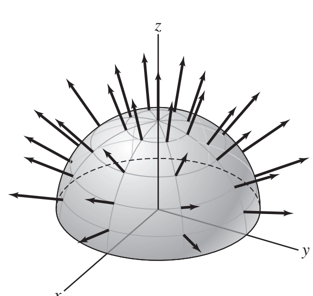
SOLUTION We parametrize the octant S by:

$$\Phi(\theta,\phi) = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi),\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}$$

Step 1. Compute the normal vector. As seen in the text, the normal vector that points to the outside of the sphere is:

$$\mathbf{n} = \mathbf{T}_\phi \times \mathbf{T}_\theta = 9\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

For $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$, all trigonometric functions are positive. Therefore all components of $\mathbf{n}$ are positive, so $\mathbf{n}$ points to the outside of the sphere.



Step 2. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We express the vector field in terms of the parameters:

$$\mathbf{F}(\Phi(\theta,\phi)) = \langle 0, 3, x\rangle = \langle 0, 3, 3\cos\theta\sin\phi\rangle$$

Hence:

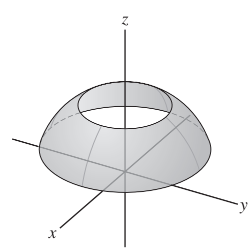
$$\begin{aligned}\mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n}(\theta,\phi) &= \langle 0, 3, 3\cos\theta\sin\phi\rangle \cdot 9\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\\ &= 27\sin\theta\sin^2\phi + 27\cos\theta\sin^2\phi\cos\phi\end{aligned}$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n}(\theta,\phi)\,d\theta\,d\phi\\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(27\sin\theta\sin^2\phi + 27\cos\theta\sin^2\phi\cos\phi\right)d\theta\,d\phi\\ &= 27\left(\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\cdot\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi + \int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^{\frac{\pi}{2}}\sin^2\phi\cos\phi\,d\phi\right)\\ &= 27\left(-\cos\theta\Big|_0^{\frac{\pi}{2}}\cdot\left(\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right)\Big|_0^{\frac{\pi}{2}} + \sin\theta\Big|_0^{\frac{\pi}{2}}\cdot\frac{\sin^3\phi}{3}\Big|_0^{\frac{\pi}{2}}\right)\\ &= 27\left(\frac{\pi}{4}\cdot 1 + \frac{1}{3}\cdot 1\right) = \frac{27}{12}(3\pi+4)\end{aligned}$$

8. $\mathbf{F} = \langle x, y, z\rangle$, part of sphere $x^2+y^2+z^2=1$, where $\frac{1}{2}\le z\le\frac{\sqrt{3}}{2}$, inward-pointing normal

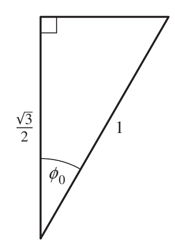
SOLUTION



We parametrize S by the following parametrization:

$$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

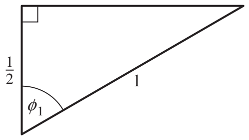
$$\mathcal{D}\,:\,0\le\theta\le 2\pi,\ \phi_0\le\phi\le\phi_1$$



The angles ϕ_0 and ϕ_1 are determined by:

$$\cos\phi_0 = \frac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi_0 = \frac{\pi}{6}$$

$$\cos\phi_1 = \frac{1}{2} \quad\Rightarrow\quad \phi_1 = \frac{\pi}{3}$$



Step 1. Determine the normal vector. The normal vector pointing to the inside of the sphere is:

$$\mathbf{n} = \mathbf{T}_\theta\times\mathbf{T}_\phi = -\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

(Notice that for $\frac{\pi}{6}\le\phi\le\frac{\pi}{3}$, $-\sin\phi\cos\phi < 0$, therefore the z-component is negative and the normal points to the inside of the sphere).

Step 2. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We express $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}(\Phi(\theta,\phi)) = \langle x, y, z\rangle = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

Hence:

$$\begin{aligned}\mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n}(\theta,\phi) &= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\cdot(-\sin\phi)\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\\ &= -\sin\phi\left(\cos^2\phi\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi\right)\\ &= -\sin\phi\cdot 1 = -\sin\phi\end{aligned}$$

Step 3. Evaluate the surface integral. We have:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n}(\theta,\phi)\,d\theta\,d\phi = \int_0^{2\pi}\int_{\pi/6}^{\pi/3} -\sin\phi\,d\theta\,d\phi = 2\pi\int_{\pi/6}^{\pi/3} -\sin\phi\,d\theta\\ &= 2\pi(\cos\phi)\Big|_{\phi=\pi/6}^{\pi/3} = 2\pi\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right) = 2\pi\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \pi\left(1-\sqrt{3}\right) \approx -2.3\end{aligned}$$

9. $\mathbf{F} = \langle z, z, x\rangle$, $\quad z = 9 - x^2 - y^2$, $x \ge 0$, $y \ge 0$, $z \ge 0$ upward-pointing normal

SOLUTION

Step 1. Find a parametrization. We use x and y as parameters and parametrize the surface by:

$$\Phi(x,y) = \left(x, y, 9 - x^2 - y^2\right)$$

The parameter domain $\mathcal{D}$ is determined by the conditions $z = 9 - x^2 - y^2 \ge 0 \Rightarrow x^2 + y^2 \le 9$ and $x, y \ge 0$. That is:

$$\mathcal{D} = \left\{(x,y) : x^2 + y^2 \le 9,\ x, y \ge 0\right\}$$

$\mathcal{D}$ is the portion of the disk of radius 3 in the first quadrant.

Step 2. Compute the tangent and normal vectors. We have:

$$\begin{aligned}\mathbf{T}_x &= \frac{\partial\Phi}{\partial x} = \frac{\partial}{\partial x}\left(x, y, 9 - x^2 - y^2\right) = \langle 1, 0, -2x\rangle\\ \mathbf{T}_y &= \frac{\partial\Phi}{\partial y} = \frac{\partial}{\partial y}\left(x, y, 9 - x^2 - y^2\right) = \langle 0, 1, -2y\rangle\end{aligned}$$

We compute the cross product of the tangent vectors:

$$\mathbf{T}_x\times\mathbf{T}_y = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -2x\\ 0 & 1 & -2y\end{vmatrix} = (2x)\mathbf{i} + (2y)\mathbf{j} + \mathbf{k} = \langle 2x, 2y, 1\rangle$$

Since the z-component is positive, the vector points upward, and we have:

$$\mathbf{n} = \langle 2x, 2y, 1\rangle$$

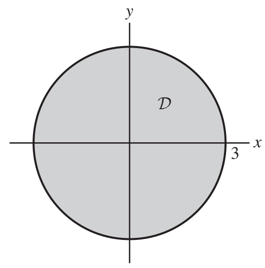
Step 3. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We first express the vector field in terms of the parameters x and y, by setting $z = 9 - x^2 - y^2$. We get:

$$\mathbf{F}(\Phi(x,y)) = \langle z, z, x\rangle = \left\langle 9 - x^2 - y^2, 9 - x^2 - y^2, x\right\rangle$$

We now compute the dot product:

$$\begin{aligned}\mathbf{F}(\Phi(x,y))\cdot\mathbf{n}(x,y) &= \left\langle 9 - x^2 - y^2, 9 - x^2 - y^2, x\right\rangle\cdot\langle 2x, 2y, 1\rangle\\ &= 2x(9 - x^2 - y^2) + 2y(9 - x^2 - y^2) + x\\ &= 19x + 18y - 2xy(x^2 + y^2)\end{aligned}$$

Step 4. Evaluate the surface integral.



The surface integral is equal to the following double integral:

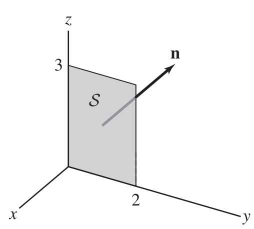
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(x, y)) \cdot \mathbf{n}(x, y)\, dx\, dy = \iint_{\mathcal{D}} \left(19x + 18y - 2xy(x^2 + y^2)\right) dx\, dy$$

We convert the integral to polar coordinates and use the identity $\sin 2\theta = 2\cos\theta \sin\theta$ to obtain:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{\pi/2} \left(19r\cos\theta + 18r\sin\theta - 2r^2\cos\theta\sin\theta\right) r\, d\theta\, dr \\
&= \left(\int_0^3 r^2\, dr\right) \cdot \left(\int_0^{\pi/2} 19\cos\theta + 18\sin\theta\, d\theta\right) + \left(\int_0^3 -r^3\, dr\right) \cdot \left(\int_0^{\pi/2} \sin 2\theta\, d\theta\right) \\
&= \left(\frac{r^3}{3}\bigg|_0^3\right) \cdot \left(19\sin\theta - 18\cos\theta\bigg|_0^{\pi/2}\right) + \left(-\frac{r^4}{4}\bigg|_0^3\right) \cdot \left(-\frac{1}{2}\cos 2\theta\bigg|_0^{\pi/2}\right) \\
&= 9 \cdot 37 - \frac{81}{4} \cdot (1) = 312.75
\end{aligned}$$

10. $\mathbf{F} = \langle \sin y, \sin z, yz \rangle$, rectangle $0 \le y \le 2$, $0 \le z \le 3$ in the (y, z)-plane, normal pointing in negative x-direction

SOLUTION



The surface is the plane $x = 0$ over the rectangle $0 \le y \le 2$, $0 \le z \le 3$ in the (y, z)-plane, hence it is parametrized by:

$$\Phi(y, z) = (0, y, z), \quad 0 \le y \le 2, \quad 0 \le z \le 3$$

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_y = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(0, y, z) = \langle 0, 1, 0 \rangle = \mathbf{j}$$

$$\mathbf{T}_z = \frac{\partial \Phi}{\partial z} = \frac{\partial}{\partial z}(0, y, z) = \langle 0, 0, 1 \rangle = \mathbf{k}$$

$$\mathbf{T}_y \times \mathbf{T}_z = \mathbf{j} \times \mathbf{k} = \mathbf{i} = \langle 1, 0, 0 \rangle$$

Since the normal points to the negative x-direction, the x-component must be negative. Hence:

$$\mathbf{n} = \langle -1, 0, 0 \rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We compute the dot product:

$$\mathbf{F}(\Phi(y, z)) \cdot \mathbf{n} = \langle \sin y, \sin z, yz \rangle \cdot \langle -1, 0, 0 \rangle = -\sin y$$

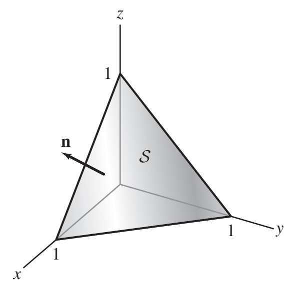
Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(y, z)) \cdot \mathbf{n}\, dy\, dz = \int_0^3 \int_0^2 (-\sin y)\, dy\, dz = \int_0^2 \int_0^3 -\sin y\, dz\, dy$$

$$= 3\int_0^2 (-\sin y)\, dy = 3\cos y\bigg|_0^2 = 3(\cos 2 - 1) \approx -4.25$$

11. $\mathbf{F} = y^2\mathbf{i} + 2\mathbf{j} - x\mathbf{k}$, portion of the plane $x + y + z = 1$ in the octant $x, y, z \geq 0$, upward-pointing normal

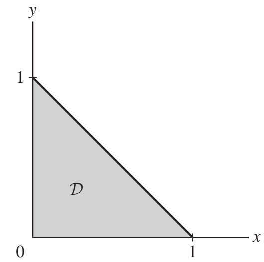
SOLUTION



We parametrize the surface by:

$$\Phi(x, y) = (x, y, 1 - x - y),$$

using the parameter domain $\mathcal{D}$ shown in the figure.



Step 1. Compute the tangent and normal vectors. We have:

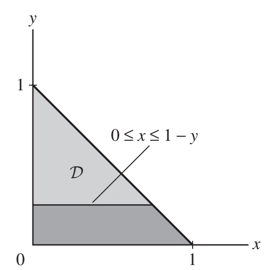
$$\mathbf{T}_x = \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}(x, y, 1 - x - y) = \langle 1, 0, -1 \rangle$$

$$\mathbf{T}_y = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(x, y, 1 - y) = \langle 0, 1, -1 \rangle$$

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle$$

Note that $\mathbf{n}$ points upward.

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$.



We compute the dot product:

$$\mathbf{F}(\Phi(x, y)) \cdot \mathbf{n} = \left\langle y^2, 2, -x \right\rangle \cdot \langle 1, 1, 1 \rangle = y^2 + 2 - x$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(x, y)) \cdot \mathbf{n}\, dx\, dy = \int_0^1 \int_0^{1-y} \left(y^2 + 2 - x\right) dx\, dy = \int_0^1 y^2 x + 2x - \frac{x^2}{2} \bigg|_{x=0}^{1-y} dy$$

$$= \int_0^1 \left(y^2(1 - y) + 2(1 - y) - \frac{(1 - y)^2}{2}\right) dy = \int_0^1 \left(y^2 - y^3 + 2(1 - y) - \frac{(y - 1)^2}{2}\right) dy$$

$$= \frac{y^3}{3} - \frac{y^4}{4} - (1 - y)^2 - \frac{(y - 1)^3}{6} \bigg|_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{6}\right) = \frac{11}{12}$$

12. $\mathbf{F} = \langle x, y, e^z \rangle$, cylinder $x^2 + y^2 = 4$, $1 \le z \le 5$, outward-pointing normal

SOLUTION We parametrize the surface by

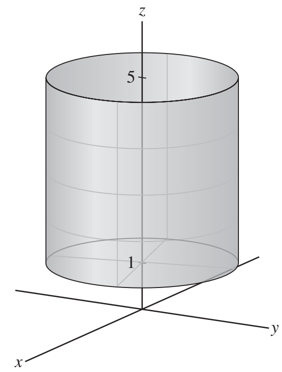
$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

where the parameter domain is:

$$\mathcal{D}: \ 0 \le \theta \le 2\pi, \ 1 \le z \le 5$$

For this parametrization it holds that:

$$\mathbf{T}_\theta \times \mathbf{T}_z = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

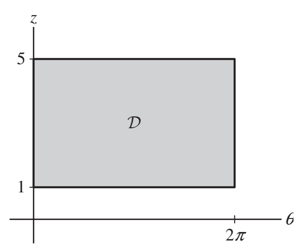


This vector is horizontal and points out of the cylinder (this can be verified by setting $\theta = 0$; the vector $\langle 2, 0, 0\rangle$ points out of the cylinder). Therefore:

$$\mathbf{n} = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

We compute the dot product $\mathbf{F} \cdot \mathbf{n}$:

$$\mathbf{F}(\Phi(\theta, z)) \cdot \mathbf{n}(\theta, z) = \left\langle 2\cos\theta, 2\sin\theta, e^z \right\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle = 4\cos^2\theta + 4\sin^2\theta = 4$$

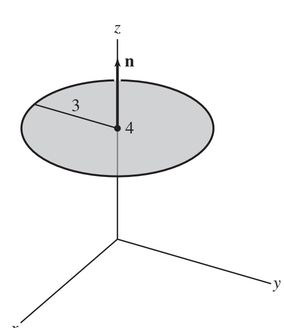


The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(\theta, z)) \cdot \mathbf{n}(\theta, z)\, dz\, d\theta = \iint_{\mathcal{D}} 4\, dz\, d\theta = 4\text{Area}(\mathcal{D}) = 4 \cdot 4 \cdot 2\pi = 32\pi$$

13. $\mathbf{F} = \left\langle xz, yz, z^{-1} \right\rangle$, disk of radius 3 at height 4 parallel to the xy-plane, upward-pointing normal

SOLUTION



We parametrize the surface S by:

$$\Phi(\theta, r) = (r\cos\theta, r\sin\theta, 4)$$

with the parameter domain:

$$\mathcal{D} = \{(\theta, r) : 0 \le \theta \le 2\pi, 0 \le r \le 3\}$$

Step 1. Compute the tangent and normal vectors. We have:

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta, r\sin\theta, 4) = \langle -r\sin\theta, r\cos\theta, 0\rangle$$
$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta, r\sin\theta, 4) = \langle \cos\theta, \sin\theta, 0\rangle$$
$$\mathbf{T}_\theta \times \mathbf{T}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = \left(-r\ \sin^2\theta - r\cos^2\theta\right)\mathbf{k} = -r\mathbf{k} = \langle 0, 0, -r\rangle$$

Since the orientation of S is with an upward pointing normal, the z-coordinate of $\mathbf{n}$ must be positive. Hence:

$$\mathbf{n} = \langle 0, 0, r\rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We first express $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}(\Phi(\theta, r)) = \left\langle xz, yz, z^{-1}\right\rangle = \left\langle r\cos\theta \cdot 4, r\sin\theta \cdot 4, 4^{-1}\right\rangle = \left\langle 4r\cos\theta, 4r\sin\theta, \frac{1}{4}\right\rangle$$

We now compute the dot product:

$$\mathbf{F}(\Phi(\theta, r)) \cdot \mathbf{n}(\theta, r) = \left\langle 4r\cos\theta, 4r\sin\theta, \frac{1}{4}\right\rangle \cdot \langle 0, 0, r\rangle = \frac{r}{4}$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(\theta, r)) \cdot \mathbf{n}(\theta, r)\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \frac{r}{4}\, dr\, d\theta = 2\pi \int_0^3 \frac{r}{4}\, dr = 2\pi \cdot \left.\frac{r^2}{8}\right|_0^3 = \frac{9\pi}{4}$$

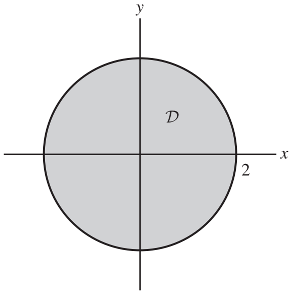
14. $\mathbf{F} = \langle xy, y, 0\rangle$, cone $z^2 = x^2 + y^2$, $x^2 + y^2 \le 4$, $z \ge 0$, downward-pointing normal

SOLUTION We parametrize the surface S by:

$$\Phi(\theta, t) = (t\cos\theta, t\sin\theta, t)$$

with the parameter domain:

$$\mathcal{D} = \{(\theta, t) : 0 \le \theta \le 2\pi,\ 0 \le t \le 2\}$$



In this parametrization it holds that (see Example 4 in Section 17.4).

$$\mathbf{T}_\theta \times \mathbf{T}_t = \langle t\cos\theta, t\sin\theta, -t\rangle$$

Since the normal is pointing downward, the z-coordinate must be negative. Therefore the normal is:

$$\mathbf{n} = \langle t\cos\theta, t\sin\theta, -t\rangle$$

We express $\mathbf{F}$ in terms of the parameters and then compute the dot product $\mathbf{F} \cdot \mathbf{n}$. We obtain:

$$\mathbf{F}(\Phi(\theta, t)) = \langle xy, y, 0 \rangle = \langle (t\cos\theta)(t\sin\theta), t\sin\theta, 0 \rangle = \left\langle t^2\cos\theta\sin\theta, t\sin\theta, 0 \right\rangle$$

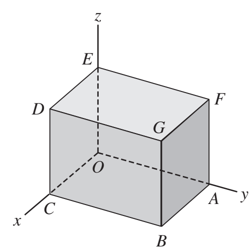
$$\mathbf{F}(\Phi(\theta, t)) \cdot \mathbf{n}(\theta, t) = t\sin\theta \langle t\cos\theta, 1, 0 \rangle \cdot t\langle \cos\theta, \sin\theta, -1 \rangle = t^2\sin\theta\left(t\cos^2\theta + \sin\theta\right)$$
$$= t^3\cos^2\theta\sin\theta + t^2\sin^2\theta$$

The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(\theta, t)) \cdot \mathbf{n}(\theta, t)\, dt\, d\theta = \int_0^{2\pi}\int_0^2 \left(t^3 \cos^2\theta\sin\theta + t^2 \sin^2\theta\right) dt\, d\theta$$
$$= \left(\int_0^2 t^3\, dt\right)\left(\int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta\right) + \left(\int_0^2 t^2\, dt\right)\left(\int_0^{2\pi} \sin^2\theta\, d\theta\right)$$
$$= \left(\frac{t^4}{4}\bigg|_{t=0}^{2}\right)\left(-\frac{\cos^3\theta}{3}\bigg|_{\theta=0}^{2\pi}\right) + \left(\frac{t^3}{3}\bigg|_{t=0}^{2}\right)\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\bigg|_{\theta=0}^{2\pi}\right) = 4 \cdot 0 + \frac{8}{3} \cdot \pi = \frac{8\pi}{3}$$

15. $\mathbf{F} = \left\langle 0, 0, e^{y+z} \right\rangle$, boundary of unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, outward-pointing normal

SOLUTION



We denote the faces of the cube by:

$$S_1 = \text{Face } OABC \quad S_2 = \text{Face } DGEF \quad S_3 = \text{Face } ABGF$$
$$S_4 = \text{Face } OCDE \quad S_5 = \text{Face } BCDG \quad S_6 = \text{Face } OAFE$$

- On S_1

$$\Phi_1(x, y) = (x, y, 0)$$

and $\mathbf{n}_1 = \langle 0, 0, -1 \rangle$. Thus,

$$\mathbf{F}(\Phi_1(x, y)) \cdot \mathbf{n}_1 = \left\langle 0, 0, e^y \right\rangle \cdot \langle 0, 0, -1 \rangle = -e^y$$

- On S_2

$$\Phi_2(x, y) = (x, y, 1)$$

and $\mathbf{n}_2 = \langle 0, 0, 1 \rangle$. Thus,

$$\mathbf{F}(\Phi_2(x, y)) \cdot \mathbf{n}_2 = \left\langle 0, 0, e^{y+1} \right\rangle \cdot \langle 0, 0, 1 \rangle = e^{y+1}$$

- On any other surface S_i, $3 \le i \le 6$, we have

$$\mathbf{F}(\Phi_1(x, y)) \cdot \mathbf{n}_i = 0,$$

because the z-component of $\mathbf{n}_i = 0$ and the x, y components of $\mathbf{F}$ equal 0. Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1\int_0^1 -e^y\, dx\, dy + \int_0^1\int_0^1 e^{y+1}\, dx\, dy$$
$$= \int_0^1\int_0^1 \left(e^{y+1} - e^y\right) dx\, dy = \int_0^1 \left(e^{y+1} - e^y\right) dy$$
$$= \int_0^1 e^y(e-1)\, dy = (e-1)e^y\bigg|_0^1 = (e-1)^2$$

16. $\mathbf{F} = \langle 0, 0, z^2 \rangle$, $G(u, v) = (u \cos v, u \sin v, v)$, $0 \le u \le 1$, $0 \le v \le 2\pi$, upward-pointing normal

SOLUTION For this parametrization it holds that (see Example 3 in Section 17.4):

$$\mathbf{T}_u \times \mathbf{T}_v = \langle \sin v, -\cos v, u \rangle$$

Since $u \ge 0$, the upward pointing normal is,

$$\mathbf{n} = \langle \sin v, -\cos v, u \rangle$$

We express $\mathbf{F}$ in terms of the parameters and then compute $\mathbf{F} \cdot \mathbf{n}$ we get:

$$\mathbf{F}(\Phi(u, v)) = \left\langle 0, 0, z^2 \right\rangle = \left\langle 0, 0, v^2 \right\rangle$$

$$\mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v) = \left\langle 0, 0, v^2 \right\rangle \cdot \langle \sin v, -\cos v, u \rangle = v^2 u$$

The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v)\, du\, dv = \int_0^{2\pi} \int_0^1 v^2 u\, du\, dv = \left(\int_0^{2\pi} v^2\, dv \right) \left(\int_0^1 u\, du \right)$$

$$= \left(\frac{v^3}{3} \bigg|_{v=0}^{2\pi} \right) \left(\frac{u^2}{2} \bigg|_{u=0}^{1} \right) = \frac{8\pi^3}{3} \cdot \frac{1}{2} = \frac{4\pi^3}{3}$$

17. $\mathbf{F} = \langle y, z, 0 \rangle$, $G(u, v) = (u^3 - v, u + v, v^2)$, $0 \le u \le 2$, $0 \le v \le 3$, downward-pointing normal

SOLUTION

Step 1. Compute the tangent and normal vectors. We have,

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u} \left(u^3 - v, u + v, v^2 \right) = \left\langle 3u^2, 1, 0 \right\rangle$$

$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v} \left(u^3 - v, u + v, v^2 \right) = \langle -1, 1, 2v \rangle$$

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3u^2 & 1 & 0 \\ -1 & 1 & 2v \end{vmatrix} = (2v)\mathbf{i} - \left(6u^2 v\right)\mathbf{j} + \left(3u^2 + 1\right)\mathbf{k} = \left\langle 2v, -6u^2 v, 3u^2 + 1 \right\rangle$$

Since the normal is pointing downward, the z-coordinate is negative, hence,

$$\mathbf{n} = \left\langle -2v, 6u^2 v, -3u^2 - 1 \right\rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We first express $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}(\Phi(u, v)) = \langle y, z, 0 \rangle = \left\langle u + v, v^2, 0 \right\rangle$$

We compute the dot product:

$$\mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v) = \left\langle u + v, v^2, 0 \right\rangle \cdot \left\langle -2v, 6u^2 v, -3u^2 - 1 \right\rangle$$

$$= -2v(u + v) + 6u^2 v \cdot v^2 + 0 = -2vu - 2v^2 + 6u^2 v^3$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v)\, du\, dv = \int_0^3 \int_0^2 \left(-2uv - 2v^2 + 6u^2 v^3 \right) du\, dv$$

$$= \int_0^3 -u^2 v - 2v^2 u + 2u^3 v^3 \bigg|_{u=0}^{2} dv = \int_0^3 \left(16v^3 - 4v^2 - 4v \right) dv = 4v^4 - \frac{4}{3}v^3 - 2v^2 \bigg|_0^3 = 270$$

18. Let $\mathcal{S}$ be the oriented half-cylinder in Figure 15. In (a)–(f), determine whether $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ is positive, negative, or zero. Explain your reasoning.

(a) $\mathbf{F} = \mathbf{i}$ **(b)** $\mathbf{F} = \mathbf{j}$

(c) $\mathbf{F} = \mathbf{k}$ **(d)** $\mathbf{F} = y\mathbf{i}$

(e) $\mathbf{F} = -y\mathbf{j}$ **(f)** $\mathbf{F} = x\mathbf{j}$

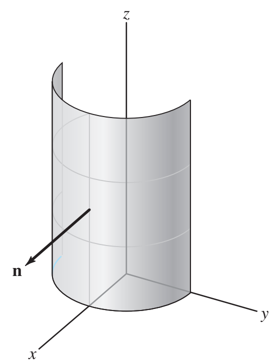
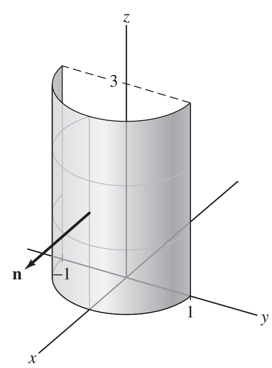


FIGURE 15

SOLUTION



S is parametrized by:

$$\Phi(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \le z \le 3, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Hence,

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \langle -\sin\theta, \cos\theta, 0\rangle$$

$$\mathbf{T}_z = \frac{\partial \Phi}{\partial z} = \langle 0, 0, 1\rangle$$

$$\mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} = \langle \cos\theta, \sin\theta, 0\rangle$$

The normal to S is pointing in the outward direction, hence the x-coordinate of $\mathbf{n}$ is positive. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we have $\cos\theta \ge 0$, hence,

$$\mathbf{n} = \langle \cos\theta, \sin\theta, 0\rangle$$

(a) Since $\mathbf{F} \cdot \mathbf{n} = \langle 1, 0, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \cos\theta$, and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we have $\mathbf{F} \cdot \mathbf{n} \ge 0$ therefore:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} > 0.$$

(b) We compute the dot product:

$$\mathbf{F} \cdot \mathbf{n} = \langle 0, 1, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \sin\theta$$

The part of the surface integral for $-\frac{\pi}{2} \le \theta \le 0$ is canceled by the part for $0 \le \theta \le \frac{\pi}{2}$, therefore the surface integral is zero.

(c) The normal vector $\mathbf{n}$ is horizontal at each point on S, therefore it is orthogonal to the field $\mathbf{F} = \mathbf{k}$. We conclude that the surface integral is zero.
(d) We compute the dot product:

$$\mathbf{F} \cdot \mathbf{n} = \langle y, 0, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \langle \sin\theta, 0, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \sin\theta\cos\theta$$

The surface integral over the part of S where $-\frac{\pi}{2} \le \theta \le 0$ is canceled by the integral over the part where $0 \le \theta \le \frac{\pi}{2}$, hence the surface integral over S is zero.
(e) We have,

$$\mathbf{F} \cdot \mathbf{n} = \langle 0, -y, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \langle 0, -\sin\theta, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = -\sin^2\theta \le 0$$

Since the dot product is nonpositive, the surface integral is negative.
(f) We compute the dot product:

$$\mathbf{F} \cdot \mathbf{n} = \langle 0, x, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \langle 0, \cos\theta, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = \cos\theta\sin\theta$$

The surface integral is zero by the same reasoning given in part (d).

19. Let $\mathbf{e_r} = \langle x/r, y/r, z/r\rangle$ be the unit radial vector, where $r = \sqrt{x^2 + y^2 + z^2}$. Calculate the integral of $\mathbf{F} = e^{-r}\mathbf{e_r}$ over:
(a) The upper hemisphere of $x^2 + y^2 + z^2 = 9$, outward-pointing normal.
(b) The octant $x \ge 0$, $y \ge 0$, $z \ge 0$ of the unit sphere centered at the origin.

SOLUTION
(a) We parametrize the upper-hemisphere by,

$$\Phi : x = 3\cos\theta\sin\phi,\ y = 3\sin\theta\sin\phi,\ z = 3\cos\phi$$

with the parameter domain:

$$\mathcal{D} = \left\{(\theta, \phi) : 0 \le \theta < 2\pi, 0 \le \phi < \frac{\pi}{2}\right\}$$

The outward pointing normal is (see Eq. (4) in sec. 17.4):

$$\mathbf{n} = 9\sin\phi\mathbf{e}_r$$

We compute the dot product $\mathbf{F} \cdot \mathbf{n}$ on the sphere. On the sphere $r = 3$, hence,

$$\mathbf{F} \cdot \mathbf{n} = e^{-r}\,\mathbf{e}_r \cdot \mathbf{n} = e^{-3}\mathbf{e}_r \cdot 9\sin\phi\,\mathbf{e}_r = 9e^{-3}\sin\phi\,\mathbf{e}_r \cdot \mathbf{e}_r = 9\,e^{-3}\sin\phi$$

We obtain the following integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{F} \cdot \mathbf{n})\ d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 9e^{-3}\sin\phi\, d\phi\, d\theta$$
$$= 18\pi e^{-3}\int_0^{\pi/2}\sin\phi\, d\phi = 18\pi e^{-3}\left(-\cos\phi\Big|_0^{\pi/2}\right) = 18\pi e^{-3}$$

(b) We parametrize the first octant of the sphere by,

$$\Phi : x = \cos\theta\sin\phi,\ y = \sin\theta\sin\phi,\ z = \cos\phi$$

with the parameter domain:

$$\mathcal{D} = \left\{(\theta, \phi) : 0 \le \theta < \frac{\pi}{2}, 0 \le \phi < \frac{\pi}{2}\right\}$$

The outward pointing normal is (as seen above):

$$\mathbf{n} = 1\sin\phi\mathbf{e}_r$$

We compute the dot product $\mathbf{F} \cdot \mathbf{n}$ on the sphere. On the sphere $r = 1$, hence,

$$\mathbf{F} \cdot \mathbf{n} = e^{-r}\,\mathbf{e}_r \cdot \mathbf{n} = e^{-1}\mathbf{e}_r \cdot \sin\phi\,\mathbf{e}_r = e^{-1}\sin\phi\,\mathbf{e}_r \cdot \mathbf{e}_r = e^{-1}\sin\phi$$

We obtain the following integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{F} \cdot \mathbf{n})\ d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} e^{-1}\sin\phi\, d\phi\, d\theta$$
$$= \frac{\pi}{2}e^{-1}\int_0^{\pi/2}\sin\phi\, d\phi = \frac{\pi}{2}e^{-1}\left(-\cos\phi\Big|_0^{\pi/2}\right) = \frac{\pi}{2}e^{-1}$$

20. Show that the flux of $\mathbf{F} = \dfrac{\mathbf{e}_r}{r^2}$ through a sphere centered at the origin does not depend on the radius of the sphere.

SOLUTION We parametrize the sphere of radius R centered at the origin by,

$$\Phi : x = R\cos\theta\sin\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\phi,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi$$

The outward pointing normal is (See Eq. (4) in sec. 17.4):

$$\mathbf{n} = R^2 \sin\phi\, \mathbf{e}_r$$

We compute the product $\mathbf{F}\cdot\mathbf{n}$ on the sphere. On the sphere $r = R$, therefore we get:

$$\mathbf{F}\cdot\mathbf{n} = \frac{\mathbf{e}_r}{r^2}\cdot R^2\sin\phi\,\mathbf{e}_r = \frac{\mathbf{e}_r}{R^2}\cdot R^2\sin\phi\mathbf{e}_r = (\sin\phi)\mathbf{e}_r\cdot\mathbf{e}_r = \sin\phi$$

hence,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{F}\cdot\mathbf{n})\ d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}(\sin\phi)d\phi\,d\theta$$
$$= 2\pi\int_0^{\pi}\sin\phi\,d\phi = 2\pi\left(-\cos\phi\big|_0^{\pi}\right) = 4\pi$$

We see that the surface integral of $\mathbf{F}$ does not depend on the radius R of the sphere.

21. The electric field due to a point charge located at the origin in $\mathbf{R}^3$ is $\mathbf{E} = k\dfrac{\mathbf{e}_r}{r^2}$, where $r = \sqrt{x^2+y^2+z^2}$ and k is a constant. Calculate the flux of $\mathbf{E}$ through the disk D of radius 2 parallel to the xy-plane with center $(0, 0, 3)$.

SOLUTION Let $r = \sqrt{x^2+y^2+z^2}$ and $\hat{r} = \sqrt{x^2+y^2}$. We parametrize the disc by:

$$\Phi(\hat{r},\theta) = (\hat{r}\cos\theta, \hat{r}\sin\theta, 3)$$
$$\mathbf{T}_{\hat{r}} = \frac{\partial\Phi}{\partial\hat{r}} = \langle\cos\theta, \sin\theta, 0\rangle$$
$$\mathbf{T}_{\theta} = \frac{\partial\Phi}{\partial\theta} = \left\langle -\hat{r}\sin\theta, \hat{r}\cos\theta, 0\right\rangle$$
$$\mathbf{n} = \mathbf{T}_{\hat{r}}\times\mathbf{T}_{\theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\hat{r}\sin\theta & \hat{r}\cos\theta & 0 \end{vmatrix} = \left\langle 0, 0, \hat{r}\right\rangle$$

Now,

$$\mathbf{E}\cdot\mathbf{n} = k\frac{\mathbf{e}_r}{r^2}\cdot\left\langle 0,0,\hat{r}\right\rangle = \frac{k\hat{r}}{r^3}\langle x,y,z\rangle\cdot\langle 0,0,1\rangle = \frac{zk\hat{r}}{r^3}$$

Since on the disk $z = 3$, we get:

$$\mathbf{E}\cdot\mathbf{n} = 3k\frac{\hat{r}}{r^3} \text{ and } r = \sqrt{\hat{r}^2+9}$$

so $\mathbf{E}\cdot\mathbf{n} = 3k\dfrac{\hat{r}}{\left(\sqrt{\hat{r}^2+9}\right)^3}$.

$$\iint_{\mathcal{D}} \mathbf{E}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 \frac{3k\hat{r}}{(\hat{r}^2+9)^{3/2}}\,d\hat{r}\,d\theta = 6\pi k\int_0^2\frac{\hat{r}}{(\hat{r}^2+9)^{3/2}}\,d\hat{r}$$

Substituting $u = \hat{r}^2 + 9$ and $\frac{1}{2}\,du = \hat{r}\,d\hat{r}$, we get:

$$\iint_{\mathcal{D}} \mathbf{E}\cdot d\mathbf{S} = 3\pi k\int_9^{13}\frac{du}{u^{3/2}} = -6\pi k u^{-1/2}\Big|_9^{13} = \left(2 - \frac{6}{\sqrt{13}}\right)\pi k$$

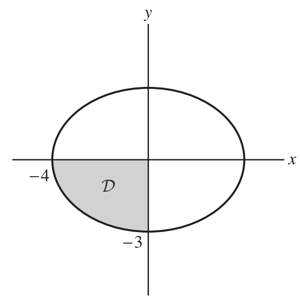
22. Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z\mathbf{i}$ over the portion of $\mathcal{S}$ where $x, y, z \le 0$ with upward-pointing normal. *Hint:* Parametrize $\mathcal{S}$ using a modified form of spherical coordinates (θ, ϕ).

SOLUTION We parametrize the ellipsoid by a modified form of spherical coordinates. That is,

$$\Phi(\theta,\phi) = (4\cos\theta\sin\phi, 3\sin\theta\sin\phi, 2\cos\phi)$$

with the parameter domain

$$\mathcal{D} = \left\{(\theta,\phi) : \pi \le \theta < \frac{3\pi}{2}, \frac{\pi}{2} \le \phi < \pi\right\}$$



One can easily verify that $\Phi(u, v)$ satisfies the equation of the ellipsoid.

Step 1. Compute the tangent and normal vectors. We have,

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \langle -4\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0\rangle$$

$$\mathbf{T}_\phi = \frac{\partial \Phi}{\partial \phi} = \langle 4\cos\theta\cos\phi, 3\sin\theta\cos\phi, -2\sin\phi\rangle$$

$$\begin{aligned}\mathbf{T}_\theta \times \mathbf{T}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin\theta\sin\phi & 3\cos\theta\sin\phi & 0 \\ 4\cos\theta\cos\phi & 3\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} \\ &= \left(-6\cos\theta\sin^2\phi\right)\mathbf{i} - \left(8\sin\theta\sin^2\phi\right)\mathbf{j} - \left(12\ \sin^2\theta\cos\phi\sin\phi + 12\ \cos^2\theta\cos\phi\sin\phi\right)\mathbf{k} \\ &= \left\langle -6\cos\theta\sin^2\phi, -8\sin\theta\sin^2\phi, -12\cos\phi\sin\phi\right\rangle\end{aligned}$$

In $\mathcal{D}$, $\frac{\pi}{2} \le \phi < \pi$ hence $-12\cos\phi\sin\phi \ge 0$. This is the upward pointing normal on this portion of the ellipse since the z-coordinate is positive. Therefore,

$$\mathbf{n} = \left\langle -6\cos\theta\sin^2\phi, -8\sin\theta\sin^2\phi, -12\cos\phi\sin\phi\right\rangle = -2\sin\phi\,\langle 3\cos\theta\sin\phi, 4\sin\theta\sin\phi, 6\cos\phi\rangle$$

Step 2. Compute the dot product $\mathbf{F}\cdot\mathbf{n}$. We first express $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}\left(\Phi(\theta, \phi)\right) = \langle z, 0, 0\rangle = \langle 2\cos\phi, 0, 0\rangle$$

Hence,

$$\begin{aligned}\mathbf{F}\left(\Phi(\theta, \phi)\right)\cdot\mathbf{n}(\theta, \phi) &= -2\sin\phi(6\cos\theta\sin\phi\cos\phi) \\ &= -4\ \sin^2\phi(3\cos\theta\cos\phi)\end{aligned}$$

Step 3. Calculate the surface integral. The surface integral is equal to the following double integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}\left(\Phi(\theta, \phi)\right)\cdot\mathbf{n}(\theta, \phi)\,d\phi\,d\theta &= -\int_\pi^{3\pi/2}\int_{\pi/2}^{\pi}\left(12\sin^2\phi\cos\phi\cos\theta\right)d\phi\,d\theta \\ &= -\left(\int_\pi^{3\pi/2} 12\ \cos\theta\,d\theta\right)\left(\int_{\pi/2}^{\pi}\sin^2\phi\cos\phi\,d\phi\right) \\ &= -\left(12\sin\theta\Big|_{\theta=\pi}^{3\pi/2}\right)\left(\frac{\sin^3\phi}{3}\Big|_{\phi=\pi/2}^{\pi}\right) \\ &= 12\cdot\left(-\frac{1}{3}\right) = -4\end{aligned}$$

23. Let $\mathbf{v} = z\mathbf{k}$ be the velocity field (in meters per second) of a fluid in $\mathbf{R}^3$. Calculate the flow rate (in cubic meters per second) through the upper hemisphere ($z \ge 0$) of the sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION We use the spherical coordinates:

$$x = \cos\theta\sin\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\phi$$

with the parameter domain

$$0 \le \theta < 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$$

The normal vector is (see Eq. (4) in Section 17.4):

$$\mathbf{n} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \sin\phi \,\langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

We express the function in terms of the parameters:

$$\mathbf{v} = \langle 0, 0, z \rangle = \langle 0, 0, \cos\phi \rangle$$

Hence,

$$\mathbf{v} \cdot \mathbf{n} = \langle 0, 0, \cos\phi \rangle \cdot \sin\phi \,\langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle = \sin\phi \cos^2\phi$$

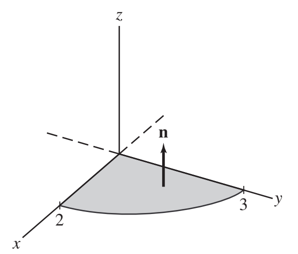
The flow rate of the fluid through the upper hemisphere S is equal to the flux of the velocity vector through S. That is,

$$\begin{aligned}\iint_S \mathbf{v} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\phi \cos^2\phi \, d\theta \, d\phi \\ &= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\phi \cos^2\phi \, d\phi = 2\pi \cdot \frac{-\cos^3\phi}{3}\Big|_0^{\frac{\pi}{2}} \\ &= \frac{2\pi}{3} \mathrm{m}^3/\mathrm{s}\end{aligned}$$

24. Calculate the flow rate of a fluid with velocity field $\mathbf{v} = \langle x, y, x^2 y \rangle$ (in m/s) through the portion of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ in the xy-plane, where $x, y \geq 0$, oriented with the normal in the positive z-direction.

SOLUTION We use the following parametrization for the surface (see remark at the end of the solution):

$$\begin{aligned}&\Phi : x = 2r\cos\theta, \; y = 3r\sin\theta, \; z = 0 \\ &0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1\end{aligned} \qquad (1)$$



Step 1. Compute the tangent and normal vectors. We have,

$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}(2r\cos\theta, 3r\sin\theta, 0) = \langle 2\cos\theta, 3\sin\theta, 0 \rangle$$

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(2r\cos\theta, 3r\sin\theta, 0) = \langle -2r\sin\theta, 3r\cos\theta, 0 \rangle$$

$$\mathbf{T}_r \times \mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\theta & 3\sin\theta & 0 \\ -2r\sin\theta & 3r\cos\theta & 0 \end{vmatrix} = \left(6r\,\cos^2\theta + 6r\sin^2\theta\right)\mathbf{k} = 6r\mathbf{k}$$

Since the normal points to the positive z-direction, the normal vector is,

$$\mathbf{n} = 6r\mathbf{k} = \langle 0, 0, 6r \rangle$$

Step 2. Compute the dot product $\mathbf{v} \cdot \mathbf{n}$. We write the velocity vector in terms of the parameters:

$$\begin{aligned}\mathbf{v} &= \left\langle x, y, x^2 y \right\rangle = \left\langle 2r\cos\theta, 3r\sin\theta, 4r^2\cos^2\theta \cdot 3r\sin\theta \right\rangle \\ &= \left\langle 2r\cos\theta, 3r\sin\theta, 12r^3\cos^2\theta\sin\theta \right\rangle\end{aligned}$$

Hence,

$$\mathbf{v} \cdot \mathbf{n} = 12r^3\cos^2\theta\sin\theta \cdot 6r = 72r^4\cos^2\theta\sin\theta$$

Step 3. Compute the flux. The flow rate of the fluid is the flux of the velocity vector through S. That is,

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^1 72r^4\cos^2\theta\sin\theta\,dr\,d\theta = \left(\int_0^1 72r^4\,dr\right)\left(\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right)$$

$$= \left(\frac{72}{5}r^5\Big|_0^1\right)\left(-\frac{\cos^3\theta}{3}\Big|_{\theta=0}^{\pi/2}\right) = \frac{72}{5}\cdot\left(0+\frac{1}{3}\right) = \frac{24}{5} = 4.8\ \text{ft}^3/\text{s}$$

Remark: We explain why (1) parametrizes the given portion of the ellipse. At any point (x, y) which satisfies (1) we have,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^3 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$$

Therefore (x, y) is inside the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. The limits of θ determine the part of the region inside the ellipse in the first quadrant.

In Exercises 25–26, let $\mathcal{T}$ be the triangular region with vertices $(1, 0, 0)$, $(0, 1, 0)$, *and* $(0, 0, 1)$ *oriented with upward-pointing normal vector (Figure 16). Assume distances are in meters.*

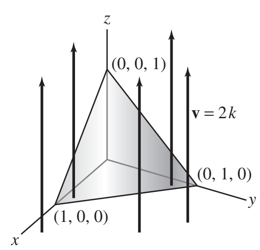
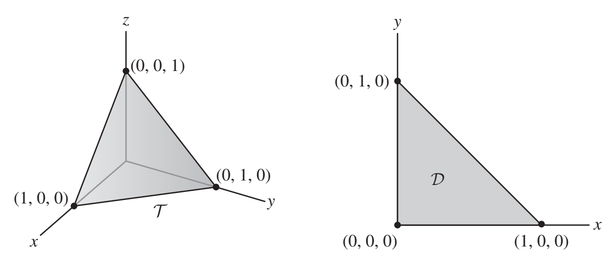


FIGURE 16

25. A fluid flows with constant velocity field $\mathbf{v} = 2\mathbf{k}$ (m/s). Calculate:
(a) The flow rate through $\mathcal{T}$
(b) The flow rate through the projection of $\mathcal{T}$ onto the xy-plane [the triangle with vertices (0, 0, 0), (1, 0, 0), and (0, 1, 0)]

SOLUTION



The equation of the plane through the three vertices is $x + y + z = 1$, hence the upward pointing normal vector is:

$$\mathbf{n} = \langle 1, 1, 1\rangle$$

and the unit normal is:

$$\mathbf{e_n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$$

We compute the dot product $\mathbf{v}\cdot\mathbf{e_n}$:

$$\mathbf{v}\cdot\mathbf{e_n} = \langle 0, 0, 2\rangle\cdot\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle = \frac{2}{\sqrt{3}}$$

The flow rate through $\mathcal{T}$ is equal to the flux of $\mathbf{v}$ through $\mathcal{T}$. That is,

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \iint_S (\mathbf{v}\cdot\mathbf{e_n})\,dS = \iint_S \frac{2}{\sqrt{3}}\,dS = \frac{2}{\sqrt{3}}\iint_S 1\,dS = \frac{2}{\sqrt{3}}\cdot\text{Area}(S)$$

The area of the equilateral triangle $\mathcal{T}$ is $\frac{\left(\sqrt{2}\right)^2\cdot\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. Therefore,

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = 1$$

Let $\mathcal{D}$ denote the projection of $\mathcal{T}$ onto the xy-plane. Then the upward pointing normal is $\mathbf{n} = \langle 0, 0, 1\rangle$. We compute the dot product $\mathbf{v} \cdot \mathbf{n}$:

$$\mathbf{v} \cdot \mathbf{n} = \langle 0, 0, 2\rangle \cdot \langle 0, 0, 1\rangle = 2$$

The flow rate through $\mathcal{D}$ is equal to the flux of $\mathbf{v}$ through $\mathcal{D}$. That is,

$$\iint_{\mathcal{D}} \mathbf{v} \cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{v} \cdot \mathbf{n})\, dS = \iint_{\mathcal{D}} 2\, dS = 2 \iint_{\mathcal{D}} 1\, dS = 2 \cdot \text{Area}(\mathcal{D}) = 2 \cdot \frac{1 \cdot 1}{2} = 1$$

26. Calculate the flow rate through $\mathcal{T}$ if $\mathbf{v} = -\mathbf{j}$ m/s.

SOLUTION We compute the flow rate through $\mathcal{T}$. Since the unit normal vector is $\mathbf{e_n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ we have,

$$\mathbf{v} \cdot \mathbf{e_n} = \langle 0, -1, 0\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{-1}{\sqrt{3}}$$

Therefore, the flow rate through $\mathcal{T}$ is the following flux:

$$\iint_S \mathbf{v} \cdot d\mathbf{S} = \iint_S (\mathbf{v} \cdot \mathbf{e_n})\, dS = \iint_S \frac{-1}{\sqrt{3}}\, dS = -\text{Area}(S)/\sqrt{3} = -\frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}} = \frac{-1}{2}$$

The upward pointing normal to the projection $\mathcal{D}$ of $\mathcal{T}$ onto the xy-plane is $\mathbf{n} = \langle 0, 0, 1\rangle$. Since $\mathbf{v} = \langle 0, -1, 0\rangle$ is orthogonal to $\mathbf{n}$, the flux of $\mathbf{v}$ through $\mathcal{D}$ is zero.

27. Prove that if $\mathcal{S}$ is the part of a graph $z = g(x, y)$ lying over a domain $\mathcal{D}$ in the xy-plane, then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \right) dx\, dy$$

SOLUTION

Step 1. Find a parametrization. We parametrize the surface by

$$\Phi(x, y) = (x, y, g(x, y)), \ (x, y) \in \mathcal{D}$$

Step 2. Compute the tangent and normal vectors. We have,

$$\mathbf{T}_x = \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}(x, y, g(x, y)) = \left\langle 1, 0, \frac{\partial g}{\partial x} \right\rangle$$

$$\mathbf{T}_y = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(x, y, g(x, y)) = \left\langle 0, 1, \frac{\partial g}{\partial y} \right\rangle$$

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial x}\mathbf{j} + \mathbf{k} = \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle$$

Step 3. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$.

$$\mathbf{F} \cdot \mathbf{n} = \langle F_1, F_2, F_3\rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle = -F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3$$

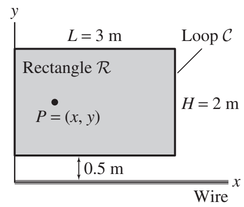
Step 4. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{F} \cdot \mathbf{n})\, dx\, dy = \iint_{\mathcal{D}} \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \right) dx\, dy$$

In Exercises 28–29, a varying current $i(t)$ flows through a long straight wire in the xy-plane as in Example 5. The current produces a magnetic field $\mathbf{B}$ *whose magnitude at a distance r from the wire is $B = \frac{\mu_0 i}{2\pi r}$ T, where $\mu_0 = 4\pi \cdot 10^{-7}$* T-m/A. *Furthermore,* $\mathbf{B}$ *points into the page at points P in the xy-plane.*

28. Assume that $i(t) = t(12 - t)$ A (t in seconds). Calculate the flux $\Phi(t)$, at time t, of $\mathbf{B}$ through a rectangle of dimensions $L \times H = 3 \times 2$ m whose top and bottom edges are parallel to the wire and whose bottom edge is located $d = 0.5$ m above the wire, similar to Figure 13(B). Then use Faraday's Law to determine the voltage drop around the rectangular loop (the boundary of the rectangle) at time t.

SOLUTION



We choose the coordinate system as shown in the figure. Therefore the rectangle $\mathcal{R}$ is the region:

$$\mathcal{R} = \{(x, y) : 0 \le x \le 3, 0.5 \le y \le 2.5\}$$

Since the magnetic field points into the page and $\mathcal{R}$ is oriented with normal vector pointing out of the page (as in Example 5) we have $\mathbf{B} = -\|\mathbf{B}\|\mathbf{k}$ and $\mathbf{n} = \mathbf{e_n} = \mathbf{k}$. Hence:

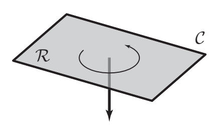
$$\mathbf{B} \cdot \mathbf{n} = \|\mathbf{B}\| (-\mathbf{k}) \cdot \mathbf{k} = -\|\mathbf{B}\| = -\frac{\mu_0 i}{2\pi r}$$

The distance from $P = (x, y)$ in $\mathcal{R}$ to the wire is $r = y$, hence, $\mathbf{B} \cdot \mathbf{n} = -\frac{\mu_0 i}{2\pi y}$. We now compute the flux $\Phi(t)$ of $\mathbf{B}$ through the rectangle $\mathcal{R}$, by evaluating the following double integral:

$$\begin{aligned}\Phi(t) &= \iint_{\mathcal{R}} \mathbf{B} \cdot d\mathbf{S} = \iint_{\mathcal{R}} \mathbf{B} \cdot \mathbf{n}\, dy\, dx = \int_0^3 \int_{0.5}^{2.5} -\frac{\mu_0 i}{2\pi y}\, dy\, dx = -\frac{\mu_0 i}{2\pi} \int_0^3 \int_{0.5}^{2.5} \frac{1}{y}\, dy\, dx \\ &= -\frac{3\mu_0 i}{2\pi} \int_{0.5}^{2.5} \frac{dy}{y} = -\frac{3\mu_0 i}{2\pi} (\ln 2.5 - \ln 0.5) = -\frac{3\mu_0 i}{2\pi} \ln \frac{2.5}{0.5} \\ &= \frac{-3 \cdot 4\pi \cdot 10^{-7} \ln 5}{2\pi} t(12 - t) = -9.65 \times 10^{-7} t(12 - t) \text{ T/m}^2\end{aligned}$$

We now use Faraday's Law to determine the voltage drop around the boundary $\mathcal{C}$ of the rectangle. By Faraday's Law, the voltage drop around $\mathcal{C}$, when $\mathcal{C}$ is oriented according to the orientation of $\mathcal{R}$ and the Right Hand Rule (that is, counterclockwise) is,

$$\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{S} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(-9.65 \cdot 10^{-7} t(12 - t)\right) = 9.65 \cdot 10^{-7} \cdot 2(6 - t) = 1.93 \cdot 10^{-6}(6 - t) \text{ volts}$$



29. Assume that $i = 10e^{-0.1t}$ A (t in seconds). Calculate the flux $\Phi(t)$, at time t, of $\mathbf{B}$ through the isosceles triangle of base 12 cm and height 6 cm whose bottom edge is 3 cm from the wire, as in Figure 17. Assume the triangle is oriented with normal vector pointing out of the page. Use Faraday's Law to determine the voltage drop around the triangular loop (the boundary of the triangle) at time t.

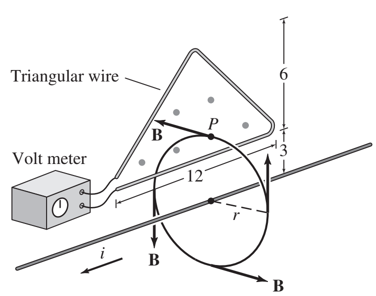
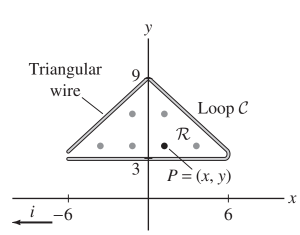


FIGURE 17

SOLUTION The magnetic field is $\mathbf{B} = \frac{-\mu_0 i}{2\pi r}\mathbf{k}$ and the unit normal on the triangle points out of the page, hence $\mathbf{n} = \mathbf{e_n} = \mathbf{k}$.

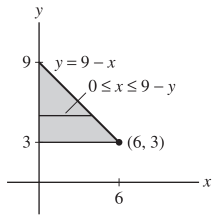


Also, the distance from a point $P = (x, y)$ in $\mathcal{R}$ to the wire is $r = y$. Hence:

$$\mathbf{B} \cdot \mathbf{n} = \frac{-\mu_0 i}{2\pi y}\mathbf{k} \cdot \mathbf{k} = \frac{-\mu_0 i}{2\pi y}$$

The flux $\Phi(t)$ of $\mathbf{B}$ through $\mathcal{R}$ is the following integral:

$$\Phi(t) = \iint_{\mathcal{R}} \mathbf{B} \cdot \mathbf{n}\, dx\, dy = \frac{-\mu_0 i}{2\pi} \iint_{\mathcal{R}} \frac{1}{y}\, dx\, dy$$



Using symmetry we have:

$$\Phi(t) = \frac{-\mu_0 i}{\pi} \int_3^9 \int_0^{9-y} \frac{1}{y}\, dx\, dy = \frac{-\mu_0 i}{\pi} \int_3^9 \left.\frac{x}{y}\right|_{x=0}^{9-y} dy = \frac{-\mu_0 i}{\pi} \int_3^9 \frac{9-y}{y}\, dy$$

$$= \frac{-\mu_0 i}{\pi} \int_3^9 \left(\frac{9}{y} - 1\right) dy = \frac{-\mu_0 i}{\pi} \left(9 \ln y - y\Big|_{y=3}^{9}\right) = \frac{-\mu_0 i}{\pi} \left((9 \ln 9 - 9) - (9 \ln 3 - 3)\right)$$

$$= \frac{-\mu_0 i}{\pi}(9 \ln 3 - 6) = \frac{-4\pi \cdot 10^{-7}}{\pi}(9 \ln 3 - 6) i = -1.56 \cdot 10^{-6} i = -1.56 \cdot 10^{-5} \cdot e^{-0.1t}$$

Using Faraday's Law, the voltage drop around the triangular loop $\mathcal{C}$ (oriented counterclockwise):

$$\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{S} = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(-1.56 \cdot 10^{-5} \cdot e^{-0.1t}\right) = -1.56 \cdot 10^{-6} \cdot e^{-0.1t} \text{ Volts}$$

Further Insights and Challenges

30. A point mass m is located at the origin. Let Q be the flux of the gravitational field $\mathbf{F} = -Gm\dfrac{\mathbf{e}_r}{r^2}$ through the cylinder $x^2 + y^2 = R^2$ for $a \le z \le b$, including the top and bottom (Figure 18). Show that $Q = -4\pi Gm$ if $a < 0 < b$ (m lies inside the cylinder) and $Q = 0$ if $0 < a < b$ (m lies outside the cylinder).

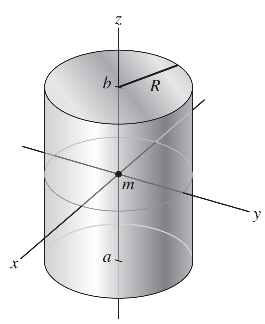
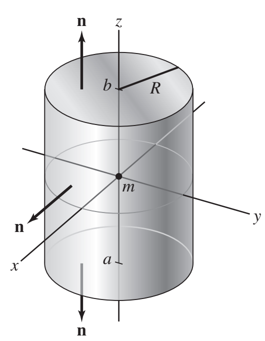


FIGURE 18

SOLUTION Let the surface be oriented with normal vector pointing outward.



We denote by S_1, S_2 and S_3 the cylinder, the top and the bottom respectively. These surfaces are parametrized by:

- S_1:

$$\Phi_1(\theta, z) = (R\cos\theta, R\sin\theta, z),\ 0 \le \theta < 2\pi,\ a \le z \le b,\ \mathbf{n} = R\langle\cos\theta, \sin\theta, 0\rangle$$

- S_2:

$$\Phi_2(\theta, r) = (r\cos\theta, r\sin\theta, b),\ 0 \le \theta < 2\pi,\ 0 \le r \le R,\ \mathbf{n} = \langle 0, 0, r\rangle$$

- S_3:

$$\Phi_3(\theta, r) = (r\cos\theta, r\sin\theta, a),\ 0 \le \theta < 2\pi,\ 0 \le r \le R,\ \mathbf{n} = \langle 0, 0, -r\rangle$$

Using properties of integrals we have,

$$Q = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{F}\cdot d\mathbf{S} \tag{1}$$

Let us assume that $a < 0$. We compute the integrals over each part of the surface S separately.

- S_1: On S_1, we have:

$$\mathbf{F}(\Phi_1(\theta, z)) = -Gm\frac{\mathbf{e}_r}{r^2} = -\frac{Gm}{\left(R^2+z^2\right)^{3/2}}\langle R\cos\theta, R\sin\theta, z\rangle$$

Hence,

$$\mathbf{F}(\Phi_1(\theta, z))\cdot\mathbf{n}(\theta, z) = -\frac{Gm}{\left(R^2+z^2\right)^{3/2}}\langle R\cos\theta, R\sin\theta, z\rangle\cdot R\langle\cos\theta, \sin\theta, 0\rangle = -\frac{GmR^2}{\left(R^2+z^2\right)^{3/2}}$$

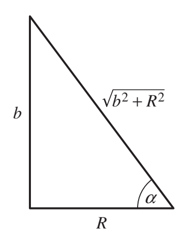
We obtain the following integral:

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_a^b -\frac{GmR^2}{\left(R^2+z^2\right)^{3/2}}\,dz\,d\theta = -2\pi GmR^2\int_a^b \frac{dz}{\left(R^2+z^2\right)^{3/2}}$$

We compute the integral using the substitution $z = R\tan t$. This gives:

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = -2\pi GmR^2\int_{\tan^{-1}\frac{a}{R}}^{\tan^{-1}\frac{b}{R}} \frac{\cos t}{R^2}\,dt = -2\pi Gm\sin t\Big|_{t=\tan^{-1}\frac{a}{R}}^{\tan^{-1}\frac{b}{R}}$$

$$= -2\pi Gm\left(\frac{b}{\sqrt{b^2+R^2}} - \frac{a}{\sqrt{a^2+R^2}}\right) \tag{2}$$



$\sin\left(\tan^{-1}\frac{b}{R}\right) = \frac{b}{\sqrt{b^2+R^2}}$

- S_2:

$$\mathbf{F}(\Phi_2(\theta, r)) = -Gm\frac{\mathbf{e}_r}{r^2} = -\frac{Gm}{\left(r^2+b^2\right)^{3/2}}\langle r\cos\theta, r\sin\theta, b\rangle$$

Hence,

$$\mathbf{F}(\Phi_2(\theta, r))\cdot\mathbf{n}(\theta, r) = -\frac{Gm}{\left(r^2+b^2\right)^{3/2}}\langle r\cos\theta, r\sin\theta, b\rangle\cdot\langle 0, 0, r\rangle = -\frac{Gmbr}{\left(r^2+b^2\right)^{3/2}}$$

We obtain the following integral:

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^R -\frac{Gmbr}{\left(r^2+b^2\right)^{3/2}}\,dr\,d\theta = -2\pi Gmb\int_0^R \frac{r\,dr}{\left(r^2+b^2\right)^{3/2}}$$

We compute the integral using the substitution $t = r^2 + b^2$, $dt = 2r\,dr$, and we get:

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\pi Gmb \int_{b^2}^{R^2+b^2} \frac{dt}{t^{3/2}} = 2\pi Gmb \frac{1}{\sqrt{t}}\bigg|_{t=b^2}^{R^2+b^2} = 2\pi Gmb \left(\frac{1}{\sqrt{b^2+R^2}} - \frac{1}{b} \right) \tag{3}$$

- S_3:

$$\mathbf{F}\left(\Phi_3(\theta, r)\right) = -Gm\frac{\mathbf{e}_r}{r^2} = -\frac{Gm}{\left(r^2+a^2\right)^{3/2}} \langle r\cos\theta, r\sin\theta, a\rangle$$

$$\mathbf{F}\left(\Phi_3(\theta, r)\right) \cdot \mathbf{n}(\theta, r) = -\frac{Gm}{\left(r^2+a^2\right)^{3/2}} \langle r\cos\theta, r\sin\theta, a\rangle \cdot \langle 0, 0, -r\rangle = \frac{Gmar}{\left(r^2+a^2\right)^{3/2}}$$

Hence, by the same computation as for S_2 we get (notice that since $a < 0$, we have $\sqrt{a^2} = -a$):

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^R \frac{Gmar}{\left(r^2+a^2\right)^{3/2}}\,dr\,d\theta = -2\pi Gma \frac{1}{\sqrt{t}}\bigg|_{t=a^2}^{R^2+a^2}$$

$$= -2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{\sqrt{a^2}} \right) = -2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} + \frac{1}{a} \right) \tag{4}$$

Substituting (2), (3), and (4) in (1) we get:

$$Q = -2\pi Gm \left(\frac{b}{\sqrt{b^2+R^2}} - \frac{a}{\sqrt{a^2+R^2}} \right) + 2\pi Gmb \left(\frac{1}{\sqrt{b^2+R^2}} - \frac{1}{b} \right) - 2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} + \frac{1}{a} \right)$$
$$= -2\pi Gm - 2\pi Gm = -4\pi Gm$$

If $0 < a < b$ the only difference is in the integral in (4). In this case $\sqrt{a} = a$ therefore,

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = -2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{\sqrt{a^2}} \right) = -2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{a} \right).$$

Therefore, adding the integrals gives:

$$Q = -2\pi Gm \left(\frac{b}{\sqrt{b^2+R^2}} - \frac{a}{\sqrt{a^2+R^2}} \right) + 2\pi Gmb \left(\frac{1}{\sqrt{b^2+R^2}} - \frac{1}{b} \right) - 2\pi Gma \left(\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{a} \right)$$
$$= -2\pi Gm + 2\pi Gm = 0$$

In Exercises 31 and 32, let $\mathcal{S}$ be the surface with parametrization

$$G(u, v) = \left(\left(1 + v\cos\frac{u}{2}\right)\cos u, \left(1 + v\cos\frac{u}{2}\right)\sin u, v\sin\frac{u}{2} \right)$$

for $0 \le u \le 2\pi$, $-\frac{1}{2} \le v \le \frac{1}{2}$.

31. CAS Use a computer algebra system.

(a) Plot $\mathcal{S}$ and confirm visually that $\mathcal{S}$ is a Möbius strip.

(b) The intersection of $\mathcal{S}$ with the xy-plane is the unit circle $G(u, 0) = (\cos u, \sin u, 0)$. Verify that the normal vector along this circle is

$$\mathbf{n}(u, 0) = \left\langle \cos u \sin\frac{u}{2}, \sin u \sin\frac{u}{2}, -\cos\frac{u}{2} \right\rangle$$

(c) As u varies from 0 to 2π, the point $G(u, 0)$ moves once around the unit circle, beginning and ending at $G(0, 0) = G(2\pi, 0) = (1, 0, 0)$. Verify that $\mathbf{n}(u, 0)$ is a unit vector that varies continuously but that $\mathbf{n}(2\pi, 0) = -\mathbf{n}(0, 0)$. This shows that $\mathcal{S}$ is not orientable—that is, it is not possible to choose a nonzero normal vector at each point on $\mathcal{S}$ in a continuously varying manner (if it were possible, the unit normal vector would return to itself rather than to its negative when carried around the circle).

SOLUTION

(a) We use a computer algebra system to graph the plot of $\mathcal{S}$, and it is indeed a Möbius strip.

(b) To find the normal vector along the unit circle, we use our computer to first find the cross product $\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v}$ and simplify, we get the very ugly expression

$$\begin{aligned}\mathbf{n}(u, v) = \frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v} = \Bigg\langle & \frac{1}{2}\left(-v\cos\left(\frac{u}{2}\right) + 2\cos u + v\cos\left(\frac{3u}{2}\right)\right)\sin\left(\frac{u}{2}\right), \\ & \frac{1}{4}\left(v + 2\cos\left(\frac{u}{2}\right) + 2v\cos(u) - 2\cos\left(\frac{3u}{2}\right) - v\cos(2u)\right), \\ & -\cos\left(\frac{u}{2}\right)\left(1 + v\cos\left(\frac{u}{2}\right)\right)\Bigg\rangle\end{aligned}$$

(Different computer algebra systems may produce different simplifications.) When we replace v with 0 and simplify, we find that:

$$\mathbf{n}(u, 0) = \left\langle \cos u \sin\frac{u}{2}, \frac{1}{2}\left(\cos\frac{u}{2} - \cos\frac{3u}{2}\right), -\cos\frac{u}{2}\right\rangle$$

This is almost, but not quite, what we want. Let's examine that middle term a bit more.

$$\begin{aligned}\frac{1}{2}\left(\cos\frac{u}{2} - \cos\frac{3u}{2}\right) &= \frac{1}{2}\left(\cos\frac{u}{2} - (\cos u \cos\frac{u}{2} - \sin u \sin\frac{u}{2})\right) = \frac{1}{2}\left(\cos\frac{u}{2}(1 - \cos u) + \sin u \sin\frac{u}{2}\right) \\ &= \frac{1}{2}\left(\cos\frac{u}{2}\cdot 2\sin^2\frac{u}{2} + \sin u \sin\frac{u}{2}\right) = \frac{1}{2}\sin\frac{u}{2}\left(2\sin\frac{u}{2}\cos\frac{u}{2} + \sin u\right) \\ &= \frac{1}{2}\sin\frac{u}{2}(\sin u + \sin u) = \sin u \sin\frac{u}{2}\end{aligned}$$

which is what we expect. Thus, we see that

$$\mathbf{n}(u, 0) = \left\langle \cos u \sin\frac{u}{2}, \sin u \sin\frac{u}{2}, -\cos\frac{u}{2}\right\rangle$$

(c) To verify that $\mathbf{n}(u, 0)$ is a unit vector, we note that

$$\begin{aligned}\|\mathbf{n}(u, 0)\| &= \sqrt{\left(\cos u \sin\frac{u}{2}\right)^2 + \left(\sin u \sin\frac{u}{2}\right)^2 + \left(\cos\frac{u}{2}\right)^2} \\ &= \sqrt{\cos^2 u \sin^2\frac{u}{2} + \sin^2 u \sin^2\frac{u}{2} + \cos^2\frac{u}{2}} = \sqrt{\sin^2\frac{u}{2} + \cos^2\frac{u}{2}} = \sqrt{1} = 1\end{aligned}$$

It is clear that $\mathbf{n}(u, 0)$ varies continuously with u, as each of its three components are non-constant continuous functions of u. Finally, we note that $\mathbf{n}(0, 0) = \langle 0, 0, -1\rangle$ but $\mathbf{n}(2\pi, 0) = \langle 0, 0, 1\rangle$, so indeed $\mathbf{n}(2\pi, 0) = -\mathbf{n}(0, 0)$.

32. CAS We cannot integrate vector fields over $\mathcal{S}$ because $\mathcal{S}$ is not orientable, but it is possible to integrate functions over $\mathcal{S}$. Using a computer algebra system:

(a) Verify that

$$\|\mathbf{n}(u, v)\|^2 = 1 + \frac{3}{4}v^2 + 2v\cos\frac{u}{2} + \frac{1}{2}v^2\cos u$$

(b) Compute the surface area of $\mathcal{S}$ to four decimal places.

(c) Compute $\iint_{\mathcal{S}} (x^2 + y^2 + z^2)\, dS$ to four decimal places.

SOLUTION

(a) Using a CAS, we discover that

$$\begin{aligned}\mathbf{n}(u, v) = \frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v} = \Bigg\langle & \frac{1}{2}\left(-v\cos\left(\frac{u}{2}\right) + 2\cos u + v\cos\left(\frac{3u}{2}\right)\right)\sin\left(\frac{u}{2}\right), \\ & \frac{1}{4}\left(v + 2\cos(u/2) + 2v\cos(u) - 2\cos\left(\frac{3u}{2}\right) - v\cos(2u)\right), \\ & -\cos\left(\frac{u}{2}\right)\left(1 + v\cos\left(\frac{u}{2}\right)\right)\Bigg\rangle\end{aligned}$$

and after taking the norm of this, we find that

$$\|\mathbf{n}(u, v)\|^2 = 1 + \frac{3}{4}v^2 + 2v\cos\frac{u}{2} + \frac{1}{2}v^2\cos u$$

(b) We calculate the area of $\mathcal{S}$ as follows:

$$A(\mathcal{S}) = \int\int \|\mathbf{n}(u, v)\|\, du\, dv = \int_{-1/2}^{1/2}\int_0^{2\pi} \sqrt{1 + \frac{3}{4}v^2 + 2v\cos\frac{u}{2} + \frac{1}{2}v^2\cos u}\, du\, dv \approx 6.3533$$

(c) We proceed as follows. Since

$$x^2 + y^2 + z^2 = \left(\left(1 + v\cos\frac{u}{2}\right)\cos u\right)^2 + \left(\left(1 + v\cos\frac{u}{2}\right)\sin u\right)^2 + \left(v\sin\frac{u}{2}\right)^2$$

and

$$\|\mathbf{n}(u, v)\| = \sqrt{1 + \frac{3}{4}v^2 + 2v\cos\frac{u}{2} + \frac{1}{2}v^2\cos u}$$

then, substituting these expressions into the double integral $\iint_{\mathcal{S}}(x^2 + y^2 + z^2)\, dS = \iint_{\mathcal{S}}(x^2 + y^2 + z^2)\|\mathbf{n}(u, v)\|\, du\, dv$, and integrating over $0 \le u \le 2\pi$, $-\frac{1}{2} \le v \le \frac{1}{2}$, we find that

$$\iint_{\mathcal{S}}(x^2 + y^2 + z^2)\, dS \approx 7.4003$$

CHAPTER REVIEW EXERCISES

1. Compute the vector assigned to the point $P = (-3, 5)$ by the vector field:

(a) $\mathbf{F} = \langle xy, y - x\rangle$

(b) $\mathbf{F} = \langle 4, 8\rangle$

(c) $\mathbf{F} = \left\langle 3^{x+y}, \log_2(x + y)\right\rangle$

SOLUTION

(a) Substituting $x = -3$, $y = 5$ in $\mathbf{F} = \langle xy, y - x\rangle$ we obtain:

$$\mathbf{F} = \langle -3\cdot 5, 5 - (-3)\rangle = \langle -15, 8\rangle$$

(b) The constant vector field $\mathbf{F} = \langle 4, 8\rangle$ assigns the vector $\langle 4, 8\rangle$ to all the vectors. Thus:

$$\mathbf{F}(-3, 5) = \langle 4, 8\rangle$$

(c) Substituting $x = -3$, $y = 5$ in $\mathbf{F} = \left\langle 3^{x+y}, \log_2(x + y)\right\rangle$ we obtain

$$\mathbf{F} = \left\langle 3^{-3+5}, \log_2(-3 + 5)\right\rangle = \left\langle 3^2, \log_2(2)\right\rangle = \langle 9, 1\rangle$$

2. Find a vector field $\mathbf{F}$ in the plane such that $\|\mathbf{F}(x, y)\| = 1$ and $\mathbf{F}(x, y)$ is orthogonal to $\mathbf{G}(x, y) = \langle x, y\rangle$ for all x, y.

SOLUTION The vector field $\langle y, -x\rangle$ is orthogonal to $\mathbf{G}(x, y) = \langle x, y\rangle$ since the dot product of the two vectors is zero.

$$\langle y, -x\rangle \cdot \langle x, y\rangle = yx - xy = 0$$

We now normalize the vector $\langle y, -x\rangle$ to obtain a unit vector orthogonal to $\mathbf{G}$:

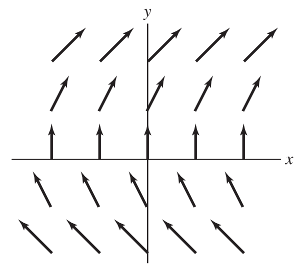
$$\mathbf{F}(x, y) = \frac{\langle y, -x\rangle}{\|\langle y, -x\rangle\|} = \frac{\langle y, -x\rangle}{\sqrt{y^2 + (-x)^2}} = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}}\right\rangle$$

However, this is only for $(x, y) \neq (0, 0)$! We can assign any unit vector for $\mathbf{F}(0, 0)$, as since $\mathbf{G}(0, 0) = \langle 0, 0\rangle$, then anything is perpendicular to $\mathbf{G}(0, 0)$. So, let's define $\mathbf{F}$ to be as above for $(x, y) \neq (0, 0)$, and let's define $\mathbf{F}(0, 0)$ to be, say, $\langle 1, 0\rangle$. Now we have a vector field $\mathbf{F}$ such that $\mathbf{F}(x, y)$ is always a unit vector and always perpendicular to $\mathbf{G}(x, y)$.

In Exercises 3–6, sketch the vector field.

3. $\mathbf{F}(x, y) = \langle y, 1\rangle$

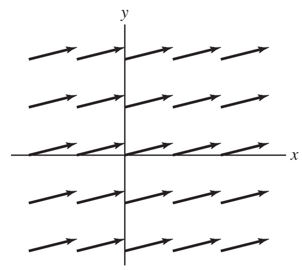
SOLUTION Notice that the vector field is constant along horizontal lines.



$\mathbf{F} = \langle y, 1\rangle$

4. $\mathbf{F}(x, y) = \langle 4, 1\rangle$

SOLUTION $\mathbf{F}$ is the constant field $\langle 4, 1\rangle$ shown in the figure.



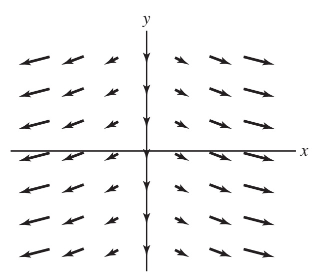
$\mathbf{F} = \langle 4, 1\rangle$

5. ∇V, where $V(x, y) = x^2 - y$

SOLUTION The gradient of $V(x, y) = x^2 - y$ is the following vector:

$$\mathbf{F}(x, y) = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle = \langle 2x, -1\rangle$$

This vector is sketched in the following figure:



$\nabla\varphi = \langle 2x, -1\rangle$

6. $\mathbf{F}(x, y) = \left\langle \dfrac{4y}{\sqrt{x^2 + 4y^2}}, \dfrac{-x}{\sqrt{x^2 + 16y^2}} \right\rangle$

Hint: Show that $\mathbf{F}$ is a unit vector field tangent to the family of ellipses $x^2 + 4y^2 = c^2$.

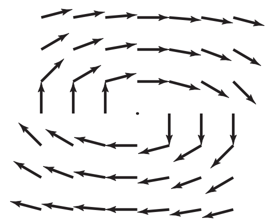
SOLUTION First, $\mathbf{F}$ is a unit vector since:

$$\|\mathbf{F}(x, y)\|^2 = \left(\frac{4y}{\sqrt{x^2 + 16y^2}}\right)^2 + \left(\frac{-x}{\sqrt{x^2 + 16y^2}}\right)^2 = \frac{16y^2}{x^2 + 16y^2} + \frac{x^2}{x^2 + 16y^2}$$
$$= \frac{16y^2 + x^2}{x^2 + 16y^2} = 1 \quad \Rightarrow \quad \|\mathbf{F}(x, y)\| = 1$$

Next, we show that $\mathbf{F}$ is tangent to the ellipses $x^2 + 4y^2 = c^2$. We find $\frac{dy}{dx}$ using implicit differentiation of $x^2 + 4y^2 = c^2$:

$$2x + 8y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{4y}$$

Therefore the vector $\langle 4y, -x\rangle$ is tangent to the ellipses. Since $\mathbf{F}(x, y)$ is a scalar multiple of this vector, i.e, $\mathbf{F}(x, y) = \frac{1}{\sqrt{x^2+16y^2}} \langle 4y, -x\rangle$, $\mathbf{F}$ is parallel to $\langle 4y, -x\rangle$ hence $\mathbf{F}$ is tangent to the ellipses $x^2 + 4y^2 = c^2$. We find that $\mathbf{F}$ can be sketched by first sketching ellipses in the family $x^2 + 4y^2 = c^2$, and then sketching the unit tangents at points on the ellipses. The field $\mathbf{F}(x, y)$ is shown in the figure:



In Exercises 7–15, determine whether the vector field is conservative, and if so, find a potential function.

7. $\mathbf{F}(x, y) = \langle x^2 y, y^2 x \rangle$

SOLUTION If $\mathbf{F}$ is conservative, the cross partials must be equal. We compute the cross partials:

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y\right) = x^2$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(y^2 x\right) = y^2$$

Since the cross-partials are not equal, $\mathbf{F}$ is not conservative.

8. $\mathbf{F}(x, y) = \langle 4x^3 y^5, 5x^4 y^4 \rangle$

SOLUTION

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(4x^3 y^5\right) = 20x^3 y^4$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(5x^4 y^4\right) = 20x^3 y^4$$

Since the cross partials are equal F is conservative.

$$F_1 = \frac{\partial V}{\partial x} = 4x^3 y^5$$
$$\Rightarrow \quad V = \int 4x^3 y^5 \, dx = x^4 y^5 + h(y)$$
$$F_2 = \frac{\partial}{\partial y}(x^4 y^5 + h(y)) = 5x^4 y^4 + h'(y) = 5x^4 y^4$$
$$\Rightarrow \quad h'(y) = 0 \Rightarrow h(y) = c$$

We may choose the constant $c = 0$, giving us the potential function,

$$V(x, y) = x^4 y^5$$

9. $\mathbf{F}(x, y, z) = \langle \sin x, e^y, z \rangle$

SOLUTION We examine the cross partials of $\mathbf{F}$. Since $\mathbf{F}_1 = \sin x$, $\mathbf{F}_2 = e^y$, $\mathbf{F}_3 = z$ we have:

$$\begin{array}{lll} \dfrac{\partial \mathbf{F}_1}{\partial y} = 0 & \dfrac{\partial \mathbf{F}_2}{\partial z} = 0 & \dfrac{\partial \mathbf{F}_3}{\partial x} = 0 \\ \dfrac{\partial \mathbf{F}_2}{\partial x} = 0 & \dfrac{\partial \mathbf{F}_3}{\partial y} = 0 & \dfrac{\partial \mathbf{F}_1}{\partial z} = 0 \end{array} \quad \Rightarrow \quad \frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}, \quad \frac{\partial \mathbf{F}_2}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial y}, \quad \frac{\partial \mathbf{F}_3}{\partial x} = \frac{\partial \mathbf{F}_1}{\partial z}$$

Since the cross partials are equal, $\mathbf{F}$ is conservative. We denote the potential field by $V(x, y, z)$. So we have:

$$V_x = \sin x \quad V_y = e^y \quad V_z = z$$

By integrating we get:

$$V(x, y, z) = \int \sin x \, dx = -\cos x + C(y, z)$$
$$V_y = C_y = e^y \quad \Rightarrow \quad C(y, z) = e^y + D(z)$$
$$V(x, y, z) = -\cos x + e^y + D(z)$$
$$V_z = D_z = z \quad \Rightarrow \quad D(z) = \frac{z^2}{2}$$

We conclude that $V(x, y, z) = -\cos x + e^y + \frac{z^2}{2}$. Indeed:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle \sin x, e^y, z \rangle = \mathbf{F}$$

10. $\mathbf{F}(x, y, z) = \langle 2, 4, e^z \rangle$

SOLUTION We examine the cross partials of $\mathbf{F}$. We have, $\mathbf{F}_1 = 2, \mathbf{F}_2 = 4, \mathbf{F}_3 = e^z$ hence:

$$\left.\begin{array}{l} \frac{\partial \mathbf{F}_1}{\partial y} = 0 \\ \frac{\partial \mathbf{F}_2}{\partial x} = 0 \end{array}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}$$

$$\left.\begin{array}{l} \frac{\partial \mathbf{F}_2}{\partial z} = 0 \\ \frac{\partial \mathbf{F}_3}{\partial y} = 0 \end{array}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_2}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial y}$$

$$\left.\begin{array}{l} \frac{\partial \mathbf{F}_3}{\partial x} = 0 \\ \frac{\partial \mathbf{F}_1}{\partial z} = 0 \end{array}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_3}{\partial x} = \frac{\partial \mathbf{F}_1}{\partial z}$$

Since the cross-partials are equal, $\mathbf{F}$ is conservative. We denote the potential field by $V(x, y, z)$. We have:

$$V_x = 2 \quad V_y = 4 \quad V_z = e^z$$

By integrating we get:

$$\begin{aligned} V(x, y, z) &= \int 2\, dx = 2x + C(y, z) \\ V_y = C_y &= 4 \quad \Rightarrow \quad C(y, z) = 4y + D(z) \\ V(x, y, z) &= 2x + 4y + D(z) \\ V_z = D_z &= e^z \quad \Rightarrow \quad D = e^z \end{aligned}$$

We conclude that $V(x, y, z) = 2x + 4y + e^z$. We verify:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle 2, 4, e^z \rangle = \mathbf{F}$$

11. $\mathbf{F}(x, y, z) = \langle xyz, \frac{1}{2}x^2 z, 2z^2 y \rangle$

SOLUTION No. We show that the cross partials for x and z are not equal. Since the equality of the cross partials is a necessary condition for a field to be a gradient vector field, we conclude that $\mathbf{F}$ is not a gradient field. We have:

$$\left.\begin{array}{l} \frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}(xyz) = xy \\ \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}(2z^2 y) = 0 \end{array}\right. \quad \Rightarrow \quad \frac{\partial F_1}{\partial z} \neq \frac{\partial F_3}{\partial x}$$

Therefore the cross partials condition is not satisfied, hence $\mathbf{F}$ is not a gradient vector field.

12. $\mathbf{F}(x, y) = \langle y^4 x^3, x^4 y^3 \rangle$

SOLUTION YES. We examine the cross-partials of $\mathbf{F}$:

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(y^4 x^3\right) = 4y^3 x^3 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(y^4 x^3\right) = 4y^3 x^3 \end{aligned}$$

Since the cross-partials are equal, $\mathbf{F}$ is conservative. We compute the potential function $V(x, y)$ of $\mathbf{F}$.

Step 1. Use the condition $\frac{\partial V}{\partial x} = F_1$. Since V is an antiderivative of $F_1 = y^4 x^3$ when y is fixed, we have:

$$V(x, y) = \int y^4 x^3\, dx = y^4 \cdot \frac{x^4}{4} + g(y) \tag{1}$$

Step 2. Use the condition $\frac{\partial V}{\partial y} = F_2$. Differentiating V with respect to y gives:

$$4y^3 \cdot \frac{x^4}{4} + g'(y) = F_2 = x^4y^4$$
$$y^3x^4 + g'(y) = x^4y^3$$
$$g'(y) = 0 \quad \Rightarrow \quad g(y) = c$$

Substituting in (1) gives:

$$V(x, y) = \frac{1}{4}x^4y^4 + c$$

Choosing $c = 0$, we obtain one of the potential functions, which is $V(x, y) = \frac{1}{4}x^4y^4$.

13. $\mathbf{F}(x, y, z) = \left\langle \frac{y}{1+x^2}, \tan^{-1} x, 2z \right\rangle$

SOLUTION We examine the cross partials of $\mathbf{F}$. Since $\mathbf{F}_1 = \frac{y}{1+x^2}$, $\mathbf{F}_2 = \tan^{-1} x$, $\mathbf{F}_3 = 2z$ we have:

$$\left.\begin{aligned} \frac{\partial \mathbf{F}_1}{\partial y} &= \frac{1}{1+x^2} \\ \frac{\partial \mathbf{F}_2}{\partial x} &= \frac{1}{1+x^2} \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial \mathbf{F}_2}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial \mathbf{F}_2}{\partial z} &= 0 \\ \frac{\partial \mathbf{F}_3}{\partial y} &= 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_2}{\partial z} = \frac{\partial \mathbf{F}_3}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial \mathbf{F}_3}{\partial x} &= 0 \\ \frac{\partial \mathbf{F}_1}{\partial z} &= 0 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{F}_3}{\partial x} = \frac{\partial \mathbf{F}_1}{\partial z}$$

Since the cross partials are equal, $\mathbf{F}$ is conservative. We denote the potential function by $V(x, y, z)$. We have:

$$V_x = \frac{y}{1+x^2}, \quad V_y = \tan^{-1}(x), \quad V_z = 2z$$

By integrating we get:

$$V(x, y, z) = \int \frac{y}{1+x^2}\, dx = y \tan^{-1}(x) + c(y, z)$$
$$V_y = \tan^{-1}(x) + c_y(y, z) = \tan^{-1}(x) \quad \Rightarrow \quad c_y(y, z) = 0 \quad \Rightarrow \quad c(y, z) = c(z)$$

Hence $V(x, y, z) = y \tan^{-1}(x) + c(z)$. $V_z = c'(z) = 2z \Rightarrow c(z) = z^2$. We conclude that $V(x, y, z) = y \tan^{-1}(x) + z^2$. Indeed:

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \left\langle \frac{y}{1+x^2}, \tan^{-1} x, 2z \right\rangle = \mathbf{F}$$

14. $\mathbf{F}(x, y, z) = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle$

SOLUTION

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2xy}{x^2+z}\right) = \frac{2x}{x^2+z}$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x^2+z)\right) = \frac{2x}{x^2+z}$$
$$\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}\left(\frac{2xy}{x^2+z}\right) = \frac{-2xy}{(x^2+z)^2}$$
$$\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+z}\right) = \frac{-2xy}{(x^2+z)^2}$$
$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+z}\right) = \frac{1}{x^2+z}$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\left(\ln(x^2+z)\right) = \frac{1}{x^2+z}$$

Since the cross partials are equal F is conservative.

$$F_1 = \frac{\partial V}{\partial x} = \frac{2xy}{x^2+z}$$
$$\Rightarrow \quad V = \int \frac{2xy}{x^2+z}\,dx = y\ln(x^2+z) + g(y,z)$$
$$F_2 = \frac{\partial}{\partial y}(y\ln(x^2+z) + g(y,z)) = \ln(x^2+z) + g_y(y,z) = \ln(x^2+z)$$
$$\Rightarrow \quad g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$$
$$F_3 = \frac{\partial}{\partial z}(y\ln(x^2+z) + h(z)) = \frac{y}{x^2+z} + h'(z) = \frac{y}{x^2+z}$$
$$\Rightarrow \quad h(z) = 0 \quad \Rightarrow \quad h(y) = c$$

We may choose the constant $c = 0$, giving us the potential function,

$$V(x,y) = y\ln(x^2+z)$$

15. $\mathbf{F}(x,y,z) = \left\langle xe^{2x}, ye^{2z}, ze^{2y}\right\rangle$

SOLUTION We have:

$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\left(ze^{2y}\right) = 2ze^{2y}$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\left(ye^{2z}\right) = 2ye^{2y}$$

Since $\frac{\partial F_3}{\partial y} \neq \frac{\partial F_2}{\partial z}$, the cross-partials condition is not satisfied , hence $\mathbf{F}$ is not conservative.

16. Find a conservative vector field of the form $\mathbf{F} = \langle g(y), h(x)\rangle$ such that $\mathbf{F}(0,0) = \langle 1, 1\rangle$, where $g(y)$ and $h(x)$ are differentiable functions. Determine all such vector fields.

SOLUTION We need to find a scalar function $V(x,y)$ such that $\mathbf{F} = \nabla V$. That is:

$$\frac{\partial V}{\partial x} = g(y) \tag{1}$$
$$\frac{\partial V}{\partial y} = h(x) \tag{2}$$

Integrating (1) with respect to x, treating y as a constant, we get:

$$V(x,y) = xg(y) + f(y) \tag{3}$$

We differentiate $V(x,y)$ with respect to y and equate with (2). This gives:

$$\frac{\partial V}{\partial y} = xg'(y) + f'(y) = h(x)$$

This equation can hold only when $g'(y)$ and $f'(y)$ are constants. That is, $g'(y) = c_1$ and $f'(y) = c_2$, yielding $g(y) = c_1 y + d_1$ and $f(y) = c_2 y + d_2$. Substituting in (3) gives:

$$V(x,y) = x(c_1 y + d_1) + c_2 y + d_2$$

Or

$$V(x,y) = d_1 x + c_2 y + c_1 xy + d_2 \quad \Rightarrow \quad \nabla V = \langle d_1 + c_1 y, c_2 + c_1 x\rangle$$

We conclude that $\mathbf{F} = \langle g(y), h(x)\rangle$ are gradient vector fields if and only if $g(y) = a + by$ and $h(x) = c + bx$ for any constants a, b, c. That is:

$$\mathbf{F} = \langle a + by, c + bx\rangle$$

We also want that $\mathbf{F}(0,0) = \langle 1, 1\rangle$, hence:

$$\mathbf{F}(0,0) = \langle a, c\rangle = \langle 1, 1\rangle \quad \Rightarrow \quad a = c = 1$$

Therefore, all the gradient fields of the form $\mathbf{F} = \langle g(y), h(x)\rangle$ such that $\mathbf{F}(0,0) = \langle 1, 1\rangle$ are:

$$\mathbf{F} = \langle 1 + by, 1 + bx\rangle$$

In Exercises 17–20, compute the line integral $\int_{\mathcal{C}} f(x, y)\, ds$ *for the given function and path or curve.*

17. $f(x, y) = xy$, the path $\mathbf{c}(t) = (t, 2t - 1)$ for $0 \le t \le 1$

SOLUTION

Step 1. Compute $ds = \|\mathbf{c}'(t)\|\, dt$. We differentiate $\mathbf{c}(t) = (t, 2t - 1)$ and compute the length of the derivative vector:

$$\mathbf{c}'(t) = \langle 1, 2\rangle \quad \Rightarrow \quad \|\mathbf{c}'(t)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Hence,

$$ds = \|\mathbf{c}'(t)\|\, dt = \sqrt{5}\, dt$$

Step 2. Write out $f(\mathbf{c}(t))$ and evaluate the line integral. We have:

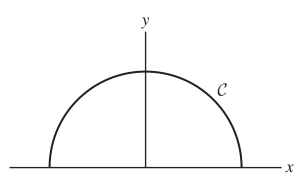
$$f(\mathbf{c}(t)) = xy = t(2t - 1) = 2t^2 - t$$

Using the Theorem on Scalar Line Integral we have:

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_0^1 f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^1 \left(2t^2 - t\right)\sqrt{5}\, dt = \sqrt{5}\left(\frac{2}{3}t^3 - \frac{1}{2}t^2\right)\bigg|_0^1 = \sqrt{5}\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{\sqrt{5}}{6}$$

18. $f(x, y) = x - y$, the unit semicircle $x^2 + y^2 = 1$, $y \ge 0$

SOLUTION



The semi unit circle above the x-axis is parametrized by:

$$c(\theta) = (\cos\theta, \sin\theta),\ 0 \le \theta \le \pi$$

Step 1. Compute $ds = \|\mathbf{c}'(\theta)\|\, d\theta$. We have:

$$\mathbf{c}'(\theta) = \frac{d}{d\theta}\langle\cos\theta, \sin\theta\rangle = \langle -\sin\theta, \cos\theta\rangle$$

$$\|\mathbf{c}'(\theta)\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

Hence:

$$ds = \|\mathbf{c}'(\theta)\|\, d\theta = d\theta$$

Step 2. Write out $f(\mathbf{c}(\theta))$ and evaluate the integral.

$$f(\mathbf{c}(\theta)) = x - y = \cos\theta - \sin\theta$$

We use the Theorem on Scalar Line Integrals to obtain:

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_0^{\pi} f(\mathbf{c}(\theta))\, \|\mathbf{c}'(\theta)\|\, dt = \int_0^{\pi} (\cos\theta - \sin\theta)\, d\theta$$

$$= \sin\theta + \cos\theta\bigg|_0^{\pi} = (0 - 1) - (0 + 1) = -2$$

19. $f(x, y, z) = e^x - \dfrac{y}{2\sqrt{2}z}$, the path $\mathbf{c}(t) = \left(\ln t, \sqrt{2}t, \frac{1}{2}t^2\right)$ for $1 \le t \le 2$

SOLUTION

Step 1. Compute $ds = \|\mathbf{c}'(t)\|\, dt$. We have:

$$\mathbf{c}'(t) = \frac{d}{dt}\left\langle \ln t, \sqrt{2}t, \frac{1}{2}t^2\right\rangle = \left\langle \frac{1}{t}, \sqrt{2}, t\right\rangle$$

$$\|\mathbf{c}'(t)\| = \sqrt{\left(\frac{1}{t}\right)^2 + \left(\sqrt{2}\right)^2 + t^2} = \sqrt{\frac{1}{t^2} + 2 + t^2} = \sqrt{\left(\frac{1}{t} + t\right)^2} = \frac{1}{t} + t$$

Hence:

$$ds = \|\mathbf{c}'(t)\|\,dt = \left(t + \frac{1}{t}\right)dt$$

Step 2. Write out $f(\mathbf{c}(t))$ and evaluate the integral.

$$f(\mathbf{c}(t)) = e^x - \frac{y}{2\sqrt{2}z} = e^{\ln t} - \frac{\sqrt{2}t}{2\sqrt{2}\cdot\frac{1}{2}t^2} = t - \frac{1}{t}$$

We use the Theorem on Scalar Line Integrals to compute the line integral:

$$\int_{\mathcal{C}} f(x,y)\,ds = \int_1^2 f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt = \int_1^2 \left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right)dt$$

$$= \int_1^2 \left(t^2 - \frac{1}{t^2}\right)dt = \frac{t^3}{3} + \frac{1}{t}\Big|_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}$$

20. $f(x,y,z) = x + 2y + z$, the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $-1 \le t \le 3$

SOLUTION We have:

$$\mathbf{c}'(t) = \frac{d}{dt}\langle \cos t, \sin t, t\rangle = \langle -\sin t, \cos t, 1\rangle$$

$$\|\mathbf{c}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

We write out $f(\mathbf{c}(t))$:

$$f(\mathbf{c}(t)) = x + 2y + z = \cos t + 2\sin t + t$$

The scalar Line Integral is the following integral:

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_{-1}^3 f(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt = \int_{-1}^3 (\cos t + 2\sin t + t)\sqrt{2}\,dt$$

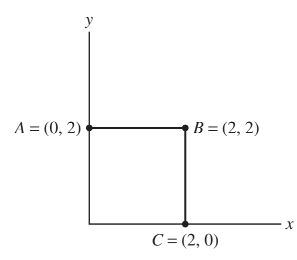
$$= \sqrt{2}\left(\sin t - 2\cos t + \frac{t^2}{2}\right)\Big|_{-1}^3$$

$$= \sqrt{2}\left(\left(\sin 3 - 2\cos 3 + \frac{9}{2}\right) - \left(\sin(-1) - 2\cos(-1) + \frac{1}{2}\right)\right)$$

$$= \sqrt{2}(\sin 3 - 2\cos 3 + \sin 1 + 2\cos 1 + 4) \approx 11.375$$

21. Find the total mass of an L-shaped rod consisting of the segments $(2t, 2)$ and $(2, 2-2t)$ for $0 \le t \le 1$ (length in centimeters) with mass density $\rho(x,y) = x^2y$ g/cm.

SOLUTION



The total mass of the rod is the following sum:

$$M = \int_{\overline{AB}} x^2y\,ds + \int_{\overline{BC}} x^2y\,ds \qquad (1)$$

The segment $\overline{AB}$ is parametrized by $\mathbf{c}_1(t) = (2t, 2)$, $0 \le t \le 1$. Hence

$$\mathbf{c}_1'(t) = \langle 2, 0\rangle, \ \|\mathbf{c}_1'(t)\| = 2$$

and

$$f(\mathbf{c}_1(t)) = x^2y = (2t)^2 \cdot 2 = 8t^2.$$

The segment $\overline{BC}$ is parametrized by $\mathbf{c}_2(t) = (2, 2-2t)$, $0 \le t \le 1$. Hence

$$\mathbf{c}_2'(t) = \langle 0, -2 \rangle, \ \|\mathbf{c}_2'(t)\| = 2$$

and

$$f(\mathbf{c}_2(t)) = x^2 y = 2^2(2-2t) = 8 - 8t.$$

Using these values, the Theorem on Scalar Line Integrals and (1) we get:

$$M = \int_0^1 8t^2 \cdot 2\,dt + \int_0^1 (8-8t)\cdot 2\,dt = \frac{16t^3}{3}\bigg|_0^1 + 16t - 8t^2\bigg|_0^1 = \frac{40}{3} = 13\frac{1}{3}$$

22. Calculate $\mathbf{F} = \nabla V$, where $V(x, y, z) = xye^z$, and compute $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$, where:

(a) $\mathcal{C}$ is any curve from $(1, 1, 0)$ to $(3, e, -1)$.

(b) $\mathcal{C}$ is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$ oriented counterclockwise.

SOLUTION The gradient of $V(x, y, z) = xye^z$ is the following vector:

$$\mathbf{F} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \left\langle ye^z, xe^z, xye^z \right\rangle$$

(a) By the Fundamental Theorem for Gradient Vector Fields, we have:

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathcal{C}} \nabla V \cdot d\mathbf{s} = V(3, e, -1) - V(1, 1, 0) = 3\cdot e \cdot e^{-1} - 1\cdot 1e^0 = 3 - 1 = 2$$

(b) Since $\mathbf{F}$ is the gradient of a function, $\mathbf{F}$ is conservative. That is, the line integral of $\mathbf{F}$ over any closed curve is zero. Therefore, the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ is zero, where $\mathcal{C}$ is the boundary of a square.

23. Calculate $\int_{\mathcal{C}_1} y\,dx + x^2 y\,dy$, where $\mathcal{C}_1$ is the oriented curve in Figure 1(A).

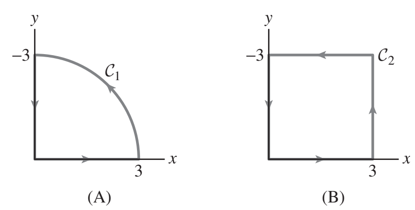
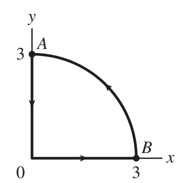


FIGURE 1

SOLUTION We compute the line integral as the sum of the line integrals over the segments $\overline{AO}$, $\overline{OB}$ and the circular arc BA.



The vector field is $\mathbf{F} = \left\langle y, x^2 y \right\rangle$. We have:

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{s} = \int_{\overline{AO}} \mathbf{F}\cdot d\mathbf{s} + \int_{\overline{OB}} \mathbf{F}\cdot d\mathbf{s} + \int_{\text{arc } BA} \mathbf{F}\cdot d\mathbf{s} \tag{1}$$

We compute each integral separately.

- The line integral over $\overline{AO}$. The segment $\overline{AO}$ is parametrized by $\mathbf{c}(t) = (0, -t)$, $-3 \le t \le 0$. Hence:

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle y, x^2 y \right\rangle = \langle -t, 0 \rangle$$

$$\mathbf{c}'(t) = \langle 0, -1 \rangle$$

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle -t, 0 \rangle \cdot \langle 0, -1 \rangle = 0$$

Therefore:

$$\int_{\overline{AO}} \mathbf{F}\cdot d\mathbf{s} = \int_{-3}^{0} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = 0 \tag{2}$$

- The line integral over $\overline{OB}$. We parametrize the segment $\overline{OB}$ by $\mathbf{c}(t) = (t, 0), \quad 0 \le t \le 3$. Hence:

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle y, x^2 y \right\rangle = \langle 0, 0 \rangle$$
$$\mathbf{c}'(t) = \langle 1, 0 \rangle$$
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0$$

Therefore:

$$\int_{\overline{OB}} \mathbf{F} \cdot d\mathbf{s} = \int_0^3 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = 0 \qquad (3)$$

- The line integral over the circular arc BA. We parametrize the circular arc by $\mathbf{c}(t) = (3\cos t, 3\sin t)$, $0 \le t \le \frac{\pi}{2}$. Then $\mathbf{c}'(t) = \langle -3\sin t, 3\cos t \rangle$ and $\mathbf{F}(\mathbf{c}(t)) = \left\langle y, x^2 y \right\rangle = \left\langle 3\sin t, 27\cos^2 t \sin t \right\rangle$. We compute the dot product:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \left\langle 3\sin t, 27\cos^2 t \sin t \right\rangle \cdot \langle -3\sin t, 3\cos t \rangle = -9\sin^2 t + 81\cos^3 t \sin t$$

We obtain the integral:

$$\begin{aligned}\int_{\text{arc } BA} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\pi/2} -9\sin^2 t + 81\cos^3 t \sin t\, dt \\ &= -9\left(\frac{t}{2} - \frac{\sin 2t}{4}\right) - 81\left(\frac{\cos^4 t}{4}\right)\Bigg|_0^{\pi/2} \\ &= -\frac{9\pi}{4} + \frac{81}{4} = \frac{81 - 9\pi}{4}\end{aligned}$$

Combining (1), (2), (3), and (4) gives:

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 0 + 0 + \frac{81 - 9\pi}{4} \approx 13.18$$

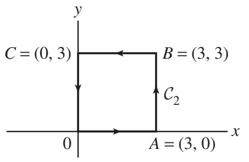
24. Let $\mathbf{F}(x, y) = \left\langle 9y - y^3, e^{\sqrt{y}}(x^2 - 3x) \right\rangle$ and let $\mathcal{C}_2$ be the oriented curve in Figure 1(B).

(a) Show that $\mathbf{F}$ is not conservative.

(b) Show that $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 0$ without explicitly computing the integral. *Hint:* Show that $\mathbf{F}$ is orthogonal to the edges along the square.

SOLUTION

(a) We show that the cross-partials of $\mathbf{F}(x, y) = \left\langle 9y - y^3, e^{\sqrt{y}}\left(x^2 - 3x\right) \right\rangle$ are not equal.



We have $\mathbf{F}_1 = 9y - y^3$ and $\mathbf{F}_2 = e^{\sqrt{y}}\left(x^2 - 3x\right)$, therefore:

$$\frac{\partial \mathbf{F}_1}{\partial y} = 9 - 3y^2$$
$$\frac{\partial \mathbf{F}_2}{\partial x} = e^{\sqrt{y}}(2x - 3)$$

The cross-partials are not equal, hence $\mathbf{F}$ is not conservative.

(b) On $\overline{OA}$, $y = 0$ hence

$$\mathbf{F} = \mathbf{F}(x, 0) = \left\langle 0, x^2 - 3x \right\rangle.$$

Therefore the tangential component of $\mathbf{F}$ along the segment $\overline{OA}$ is zero, resulting in $\int_{\overline{OA}} \mathbf{F} \cdot d\mathbf{s} = 0$. On $\overline{AB}$, $x = 3$ hence

$$\mathbf{F}(3, y) = \left\langle 9y - y^3, e^{\sqrt{y}}\left(3^2 - 3 \cdot 3\right) \right\rangle = \left\langle 9y - y^3, 0 \right\rangle.$$

We see that $\mathbf{F}$ is orthogonal to the segment AB, resulting in $\int_{\overline{AB}} \mathbf{F} \cdot d\mathbf{s} = 0$. On $\overline{BC}$, $y = 3$ hence

$$\mathbf{F}(x,3) = \left\langle 9 \cdot 3 - 3^3, e^{\sqrt{3}}\left(x^2 - 3x\right)\right\rangle = \left\langle 0, e^{\sqrt{3}}\left(x^2 - 3x\right)\right\rangle.$$

We see that $\mathbf{F}$ is orthogonal to the segment $\overline{BC}$, therefore $\int_{\overline{BC}} \mathbf{F} \cdot d\mathbf{s} = 0$. Finally, on CO we have $x = 0$, hence $\mathbf{F}(0, y) = \left\langle 9y - y^3, 0\right\rangle$. It follows that $\mathbf{F}$ is orthogonal to the segment $\overline{CO}$, therefore the tangential component of $\mathbf{F}$ along $\overline{CO}$ is zero. We conclude that $\int_{\overline{CO}} \mathbf{F} \cdot d\mathbf{s} = 0$. Combining these integrals we conclude that:

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\overline{OA}} \mathbf{F} \cdot d\mathbf{s} + \int_{\overline{AB}} \mathbf{F} \cdot d\mathbf{s} + \int_{\overline{BC}} \mathbf{F} \cdot d\mathbf{s} + \int_{\overline{CO}} \mathbf{F} \cdot d\mathbf{s} = 0 + 0 + 0 + 0 = 0$$

In Exercises 25–28, compute the line integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ *for the given vector field and path.*

25. $\mathbf{F}(x, y) = \left\langle \dfrac{2y}{x^2 + 4y^2}, \dfrac{x}{x^2 + 4y^2}\right\rangle$,

the path $\mathbf{c}(t) = \left(\cos t, \frac{1}{2}\sin t\right)$ for $0 \le t \le 2\pi$

SOLUTION

Step 1. Calculate the integrand $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$.

$$\mathbf{c}(t) = \left(\cos t, \frac{1}{2}\sin t\right)$$

$$\mathbf{F}(\mathbf{c}(t)) = \left\langle \frac{2y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}\right\rangle = \left\langle \frac{2 \cdot \frac{1}{2} \cdot \sin t}{\cos^2 t + 4 \cdot \frac{1}{4}\sin^2 t}, \frac{\cos t}{\cos^2 t + 4 \cdot \frac{1}{4}\sin^2 t}\right\rangle$$

$$= \left\langle \frac{\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right\rangle = \langle \sin t, \cos t\rangle$$

$$\mathbf{c}'(t) = \left\langle -\sin t, \frac{1}{2}\cos t\right\rangle$$

The integrand is the dot product:

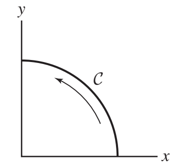
$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle \sin t, \cos t\rangle \cdot \left\langle -\sin t, \frac{1}{2}\cos t\right\rangle = -\sin^2 t + \frac{1}{2}\cos^2 t = \frac{1}{2}\cos 2t - \frac{1}{2}\sin^2 t$$

Step 2. Evaluate the line integral.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} \left(\frac{1}{2}\cos 2t - \frac{1}{2}\sin^2 t\right) dt = \left.\frac{\sin 2t}{4} - \frac{t}{4} + \frac{\sin 2t}{8}\right|_0^{2\pi} = -\frac{\pi}{2}$$

26. $\mathbf{F}(x, y) = \left\langle 2xy, x^2 + y^2\right\rangle$, the part of the unit circle in the first quadrant oriented counterclockwise.

SOLUTION



The path is parametrized by:

$$\mathbf{c}(\theta) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le \frac{\pi}{2}$$

Step 1. Calculate the integrand $\mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta)$.

$$\mathbf{c}'(\theta) = \langle -\sin\theta, \cos\theta\rangle$$

$$\mathbf{F}(\mathbf{c}(\theta)) = \left\langle 2xy, x^2 + y^2\right\rangle = \left\langle 2\cos\theta\sin\theta, \cos^2\theta + \sin^2\theta\right\rangle = \langle 2\cos\theta\sin\theta, 1\rangle$$

The integrand is the dot product:

$$\mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) = \langle 2\cos\theta\sin\theta, 1\rangle \cdot \langle -\sin\theta, \cos\theta\rangle = -2\cos\theta\sin^2\theta + \cos\theta$$

Step 2. Evaluate the line integral. The vector line integral is:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} \mathbf{F}\left(\mathbf{c}(\theta)\right) \cdot \mathbf{c}'(\theta)\, d\theta = \int_0^{\pi/2} \left(-2\cos\theta \sin^2\theta + \cos\theta\right) d\theta$$

$$= -2\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta + \int_0^{\pi/2} \cos\theta\, d\theta = \frac{-2\sin^3\theta}{3}\bigg|_0^{\pi/2} + \sin\theta\bigg|_0^{\pi/2} = -\frac{2}{3} + 1 = \frac{1}{3}$$

27. $\mathbf{F}(x, y) = \langle x^2 y, y^2 z, z^2 x\rangle$, the path $\mathbf{c}(t) = \left(e^{-t}, e^{-2t}, e^{-3t}\right)$ for $0 \le t < \infty$

SOLUTION

Step 1. Calculate the integrand $\mathbf{F}\left(\mathbf{c}(t)\right) \cdot \mathbf{c}'(t)$.

$$\mathbf{c}(t) = \left(e^{-t}, e^{-2t}, e^{-3t}\right)$$

$$\mathbf{c}'(t) = \left\langle e^{-t}, -2e^{-2t}, -3e^{-3t}\right\rangle$$

$$\mathbf{F}\left(\mathbf{c}(t)\right) = \left\langle x^2 y, y^2 z, z^2 x\right\rangle = \left\langle e^{-2t}\cdot e^{-2t}, e^{-4t}\cdot e^{-3t}, e^{-6t}\cdot e^{-t}\right\rangle = \left\langle e^{-4t}, e^{-7t}, e^{-7t}\right\rangle$$

The integrand is the dot product:

$$\mathbf{F}\left(\mathbf{c}(t)\right) \cdot \mathbf{c}'(t) = \left\langle e^{-4t}, e^{-7t}, e^{-7t}\right\rangle \cdot \left\langle e^{-t}, -2e^{-2t}, -3e^{-3t}\right\rangle = -e^{-5t} - 2e^{-9t} - 3e^{-10t}$$

Step 2. Evaluate the line integral.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\infty} \mathbf{F}\left(\mathbf{c}(t)\right) \cdot \mathbf{c}'(t)\, dt = \int_0^{\infty} \left(-e^{-5t} - 2e^{-9t} - 3e^{-10t}\right) dt$$

$$= \lim_{R\to\infty} \left(\frac{1}{5}e^{-5R} + \frac{2}{9}e^{-9R} + \frac{3}{10}e^{-10R}\right) - \left(\frac{1}{5} + \frac{2}{9} + \frac{3}{10}\right) = 0 - \frac{13}{18} = -\frac{13}{18}$$

28. $\mathbf{F} = \nabla V$, where $V(x, y, z) = 4x^2 \ln(1 + y^4 + z^2)$, the path $\mathbf{c}(t) = \left(t^3, \ln(1 + t^2), e^t\right)$ for $0 \le t \le 1$

SOLUTION We use the Fundamental Theorem for Gradient Vector Field to write:

$$\int_c \nabla V \cdot d\mathbf{s} = V\left(c(1)\right) - V\left(c(0)\right) \tag{1}$$

We compute the values on the right-hand side:

$$c(1) = \left(1^3, \ln\left(1 + 1^2\right), e^1\right) = (1, \ln 2, e)$$

$$c(0) = \left(0^3, \ln\left(1 + 0^2\right), e^0\right) = (0, 0, 1)$$

Hence:

$$V\left(c(1)\right) = 4 \cdot 1^2 \ln\left(1 + \ln^4 2 + e^2\right) = 4\ln\left(1 + e^2 + \ln^4 2\right) \approx 8.616$$

$$V\left(c(0)\right) = 0$$

Combining with (1) gives:

$$\int_c \nabla V \cdot d\mathbf{s} = 8.616$$

29. Consider the line integrals $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ for the vector fields $\mathbf{F}$ and paths $\mathbf{c}$ in Figure 2. Which two of the line integrals appear to have a value of zero? Which of the other two appears to have a negative value?

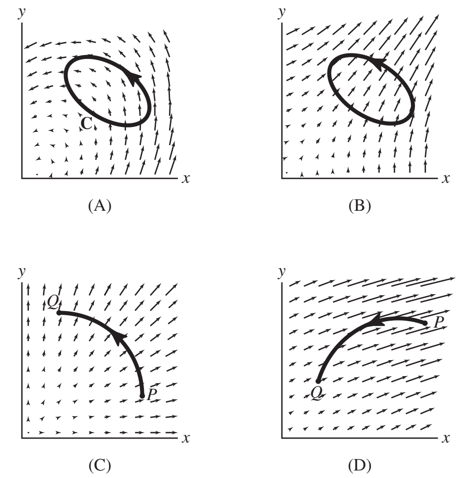


FIGURE 2

SOLUTION In (A), the line integral around the ellipse appears to be positive, because the negative tangential components from the lower part of the curve appears to be smaller than the positive contribution of the tangential components from the upper part.

In (B), the line integral around the ellipse appears to be zero, since $\mathbf{F}$ is orthogonal to the ellipse at all points except for two points where the tangential components of $\mathbf{F}$ cancel each other.

In (C), $\mathbf{F}$ is orthogonal to the path, hence the tangential component is zero at all points on the curve. Therefore the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is zero.

In (D), the direction of $\mathbf{F}$ is opposite to the direction of the curve. Therefore the dot product $\mathbf{F} \cdot \mathbf{T}$ is negative at each point along the curve, resulting in a negative line integral.

30. Calculate the work required to move an object from $P = (1, 1, 1)$ to $Q = (3, -4, -2)$ against the force field $\mathbf{F}(x, y, z) = -12r^{-4}\langle x, y, z\rangle$ (distance in meters, force in newtons), where $r = \sqrt{x^2 + y^2 + z^2}$. *Hint:* Find a potential function for $\mathbf{F}$.

SOLUTION The work performed against $\mathbf{F}$ is given by the line integral:

$$W = -\int_{PQ} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We notice that $\mathbf{F} = -\frac{12}{\left(x^2+y^2+z^2\right)^2}\langle x, y, z\rangle$ is the gradient of the function $V(x, y, z) = \frac{6}{x^2+y^2+z^2}$ since:

$$\begin{aligned} \nabla V &= \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \left\langle \frac{6 \cdot (-2x)}{\left(x^2 + y^2 + z^2\right)^2}, \frac{6 \cdot (-2y)}{\left(x^2 + y^2 + z^2\right)^2}, \frac{6 \cdot (-2z)}{\left(x^2 + y^2 + z^2\right)^2} \right\rangle \\ &= -\frac{12}{\left(x^2 + y^2 + z^2\right)^2}\langle x, y, z\rangle = \mathbf{F} \end{aligned}$$

We now use the Fundamental Theorem for Gradient vector Field to compute the line integral (1):

$$\begin{aligned} W &= -\int_{PQ} \mathbf{F} \cdot d\mathbf{s} = -\int_{PQ} \nabla V \cdot d\mathbf{s} = -\left(V(Q) - V(P)\right) = -\left(V(3, -4, -2) - V(1, 1, 1)\right) \\ &= -\left(\frac{6}{9 + 16 + 4} - \frac{6}{1 + 1 + 1}\right) = \frac{52}{29} \approx 1.79 \end{aligned}$$

31. Find constants a, b, c such that

$$G(u, v) = (u + av, bu + v, 2u - c)$$

parametrizes the plane $3x - 4y + z = 5$. Calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{n}(u, v)$.

SOLUTION We substitute $x = u + av$, $y = bu + v$ and $z = 2u - c$ in the equation of the plane $3x - 4y + z = 5$, to obtain:

$$5 = 3x - 4y + z = 3(u + av) - 4(bu + v) + 2u - c = (5 - 4b)u + (3a - 4)v - c$$

or

$$(5-4b)u+(3a-4)v-(5+c)=0$$

This equation must be satisfied for all u and v, therefore the following must hold:

$$\begin{aligned} 5-4b&=0 & & b=\frac{5}{4}\\ 3a-4&=0 &\Rightarrow\quad & a=\frac{4}{3}\\ 5+c&=0 & & c=-5 \end{aligned}$$

We obtain the following parametrization for the plane $3x-4y+z=5$:

$$\phi(u,v)=\left(u+\frac{4}{3}v,\ \frac{5}{4}u+v,\ 2u+5\right)$$

We compute the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$:

$$\mathbf{T}_u=\frac{\partial\phi}{\partial u}=\left\langle 1,\frac{5}{4},2\right\rangle;\ \mathbf{T}_v=\frac{\partial\phi}{\partial v}=\left\langle\frac{4}{3},1,0\right\rangle$$

The normal vector is their cross product:

$$\begin{aligned}\mathbf{n}=\mathbf{T}_u\times\mathbf{T}_v&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 1&\frac{5}{4}&2\\ \frac{4}{3}&1&0\end{vmatrix}=\begin{vmatrix}\frac{5}{4}&2\\ 1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&2\\ \frac{4}{3}&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&\frac{5}{4}\\ \frac{4}{3}&1\end{vmatrix}\mathbf{k}\\ &=-2\mathbf{i}+\frac{8}{3}\mathbf{j}+\left(1-\frac{5}{3}\right)\mathbf{k}=\left\langle -2,\frac{8}{3},-\frac{2}{3}\right\rangle\end{aligned}$$

32. Calculate the integral of $f(x,y,z)=e^z$ over the portion of the plane $x+2y+2z=3$, where $x,y,z\ge 0$.

SOLUTION We consider the surface a graph of the form $x=g(y,z)=3-2y-2z$. The requirement that $x\ge 0$ means $3-2y-2z\ge 0\Rightarrow y+z\le\frac{3}{2}$. Combined with the requirements that $y,z\ge 0$ we have that our parameter domain, $\mathcal{D}$, is the triangle bounded by the $y-$axis, the $z-$axis, and the line $y+z=\frac{3}{2}$.

Calculating the magnitude of the normal vector,

$$\begin{aligned}\Phi(y,z)&=(3-2y-2z,y,z)\ (x,y)\in\mathcal{D}\\ \mathbf{T}_y&=\langle -2,1,0\rangle\\ \mathbf{T}_z&=\langle -2,0,1\rangle\\ \mathbf{n}&=\langle 1,2,2\rangle\quad\Rightarrow\quad\|\mathbf{n}\|=\sqrt{1^2+2^2+2^2}=3\end{aligned}$$

Our integral is

$$\begin{aligned}\iint_{\mathcal{S}}f(x,y,z)\,dS&=\iint_{\mathcal{D}}e^z\|n\|\,dz\,dy=\int_0^{3/2}\int_0^{-y+3/2}e^z\,3\,dz\,dy\\ &=\int_0^{3/2}\left(e^{-y+3/2}-1\right)3\,dy=-3e^{-y+3/2}-3y\Big|_0^{3/2}\\ &=-3+3e^{3/2}-\frac{9}{2}=3e^{3/2}-\frac{15}{2}\approx 5.945\end{aligned}$$

33. Let $\mathcal{S}$ be the surface parametrized by

$$G(u,v)=\left(2u\sin\frac{v}{2},2u\cos\frac{v}{2},3v\right)$$

for $0\le u\le 1$ and $0\le v\le 2\pi$.

(a) Calculate the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ and the normal vector $\mathbf{n}(u,v)$ at $P=G(1,\frac{\pi}{3})$.

(b) Find the equation of the tangent plane at P.

(c) Compute the surface area of $\mathcal{S}$.

SOLUTION

(a) The tangent vectors are the partial derivatives:

$$\mathbf{T}_u = \frac{\partial G}{\partial u} = \frac{\partial}{\partial u}\left\langle 2u\sin\frac{v}{2}, 2u\cos\frac{v}{2}, 3v\right\rangle = \left\langle 2\sin\frac{v}{2}, 2\cos\frac{v}{2}, 0\right\rangle$$
$$\mathbf{T}_v = \frac{\partial G}{\partial v} = \frac{\partial}{\partial v}\left\langle 2u\sin\frac{v}{2}, 2u\cos\frac{v}{2}, 3v\right\rangle = \left\langle u\cos\frac{v}{2}, -u\sin\frac{v}{2}, 3\right\rangle$$

The normal vector is their cross-product:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sin\frac{v}{2} & 2\cos\frac{v}{2} & 0 \\ u\cos\frac{v}{2} & -u\sin\frac{v}{2} & 3 \end{vmatrix} = \begin{vmatrix} 2\cos\frac{v}{2} & 0 \\ -u\sin\frac{v}{2} & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2\sin\frac{v}{2} & 0 \\ u\cos\frac{v}{2} & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2\sin\frac{v}{2} & 2\cos\frac{v}{2} \\ u\cos\frac{v}{2} & -u\sin\frac{v}{2} \end{vmatrix}\mathbf{k}$$
$$= \left(6\cos\frac{v}{2}\right)\mathbf{i} - \left(6\sin\frac{v}{2}\right)\mathbf{j} + \left(-2u\sin^2\frac{v}{2} - 2u\cos^2\frac{v}{2}\right)\mathbf{k}$$
$$= \left(6\cos\frac{v}{2}\right)\mathbf{i} - \left(6\sin\frac{v}{2}\right)\mathbf{j} - 2u\mathbf{k} = \left\langle 6\cos\frac{v}{2}, -6\sin\frac{v}{2}, -2u\right\rangle$$

At the point $P = G\left(1, \frac{\pi}{3}\right)$, $u = 1$ and $v = \frac{\pi}{3}$. The tangents and the normal vector at this point are,

$$\mathbf{T}_u\left(1, \frac{\pi}{3}\right) = \left\langle 2\sin\frac{\pi}{6}, 2\cos\frac{\pi}{6}, 0\right\rangle = \left\langle 1, \sqrt{3}, 0\right\rangle$$
$$\mathbf{T}_v\left(1, \frac{\pi}{3}\right) = \left\langle 1\cdot\cos\frac{\pi}{6}, -1\cdot\sin\frac{\pi}{6}, 3\right\rangle = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 3\right\rangle$$
$$\mathbf{n}\left(1, \frac{\pi}{3}\right) = \left\langle 6\cos\frac{\pi}{6}, -6\sin\frac{\pi}{6}, -2\cdot 1\right\rangle = \left\langle 3\sqrt{3}, -3, -2\right\rangle$$

(b) A normal to the plane is $\mathbf{n}\left(1, \frac{\pi}{3}\right) = \left\langle 3\sqrt{3}, -3, -2\right\rangle$ found in part (a). We find the tangency point:

$$P = \phi\left(1, \frac{\pi}{3}\right) = \left(2\cdot 1\sin\frac{\pi}{6}, 2\cdot 1\cos\frac{\pi}{6}, 3\cdot\frac{\pi}{3}\right) = \left(1, \sqrt{3}, \pi\right)$$

The equation of the tangent plane is, thus,

$$\left\langle x-1, y-\sqrt{3}, z-\pi\right\rangle \cdot \left\langle 3\sqrt{3}, -3, -2\right\rangle = 0$$

or

$$3\sqrt{3}(x-1) - 3\left(y-\sqrt{3}\right) - 2(z-\pi) = 0$$
$$3\sqrt{3}x - 3y - 2z + 2\pi = 0$$

(c) In part (a) we found the normal vector:

$$\mathbf{n} = \left\langle 6\cos\frac{v}{2}, -6\sin\frac{v}{2}, -2u\right\rangle$$

We compute the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = \sqrt{36\cos^2\frac{v}{2} + 36\sin^2\frac{v}{2} + 4u^2} = \sqrt{36+4u^2} = 2\sqrt{9+u^2}$$

Using the Integral for the Surface Area we get:

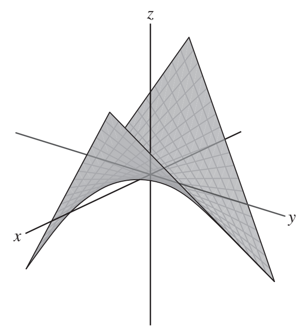
$$\text{Area}(S) = \iint_{\mathcal{D}} \|n(u,v)\|\,du\,dv = \int_0^{2\pi}\int_0^1 2\sqrt{9+u^2}\,du\,dv = 4\pi\int_0^1 \sqrt{9+u^2}\,du$$
$$= 4\pi\left(\frac{u}{2}\sqrt{u^2+9} + \frac{9}{2}\ln\left(u+\sqrt{9+u^2}\right)\Big|_{u=0}^{1}\right) = 4\pi\left(\frac{1}{2}\sqrt{10} + \frac{9}{2}\ln\left(1+\sqrt{10}\right) - \frac{9}{2}\ln 3\right)$$
$$= 2\sqrt{10}\pi + 18\pi\ln\left(1+\sqrt{10}\right) - 18\pi\ln 3 = 2\sqrt{10}\pi + 18\pi\ln\frac{1+\sqrt{10}}{3} \approx 38.4$$

34. CAS Plot the surface with parametrization

$$G(u,v) = (u+4v, 2u-v, 5uv)$$

for $-1 \le v \le 1, -1 \le u \le 1$. Express the surface area as a double integral and use a computer algebra system to compute the area numerically.

SOLUTION The surface is shown in the following plot:



We compute the area of the surface, using the Integral for the surface Area. We find the tangent and normal vectors:

$$\mathbf{T}_u = \frac{\partial \phi}{\partial u} = \frac{\partial}{\partial u}\langle u+4v, 2u-v, 5uv\rangle = \langle 1, 2, 5v\rangle$$

$$\mathbf{T}_v = \frac{\partial \phi}{\partial v} = \frac{\partial}{\partial v}\langle u+4v, 2u-v, 5uv\rangle = \langle 4, -1, 5u\rangle$$

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 5v \\ 4 & -1 & 5u \end{vmatrix} = \begin{vmatrix} 2 & 5v \\ -1 & 5u \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 5v \\ 4 & 5u \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}\mathbf{k}$$

$$= (10u+5v)\mathbf{i} - (5u-20v)\mathbf{j} + (-1-8)\mathbf{k} = \langle 10u+5v, -5u+20v, -9\rangle$$

Next, we calculate the length of the normal vector:

$$\|\mathbf{n}\| = \sqrt{(10u+5v)^2 + (-5u+20v)^2 + (-9)^2}$$
$$= \sqrt{100u^2 + 100uv + 25v^2 + 25u^2 - 200uv + 400v^2 + 81}$$
$$= \sqrt{125u^2 - 100uv + 425v^2 + 81}$$

We obtain the following integral, which we compute by a CAS:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{n}(u,v)\|\,du\,dv = \int_{-1}^{1}\int_{-1}^{1}\sqrt{125u^2 - 100uv + 425v^2 + 81}\,du\,dv = 62.911$$

35. CAS Express the surface area of the surface $z = 10 - x^2 - y^2$ for $-1 \le x \le 1$, $-3 \le y \le 3$ as a double integral. Evaluate the integral numerically using a CAS.

SOLUTION We use the Surface Integral over a graph. Let $g(x,y) = 10 - x^2 - y^2$. Then $g_x = -2x$, $g_y = -2y$ hence $\sqrt{1 + {g_x}^2 + {g_y}^2} = \sqrt{1 + 4x^2 + 4y^2}$. The area at the surface is the following integral which we compute using a CAS:

$$\text{Area}(S) = \iint_{\mathcal{D}} \sqrt{1 + {g_x}^2 + {g_y}^2}\,dx\,dy = \int_{-3}^{3}\int_{-1}^{1}\sqrt{1 + 4x^2 + 4y^2}\,dx\,dy \approx 41.8525$$

36. Evaluate $\iint_{\mathcal{S}} x^2 y\,dS$, where $\mathcal{S}$ is the surface $z = \sqrt{3}x + y^2$, $-1 \le x \le 1$, $0 \le y \le 1$.

SOLUTION We use the Surface Integral over a Graph with $g(x,y) = \sqrt{3}x + y^2$. The partial derivatives are $g_x = \sqrt{3}$ and $g_y = 2y$. Therefore

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \left(\sqrt{3}\right)^2 + (2y)^2} = \sqrt{4 + 4y^2} = 2\sqrt{1 + y^2}$$

We obtain the following integral:

$$\iint_S x^2 y \cdot dS = \iint_{\mathcal{D}} x^2 y\sqrt{1 + g_x^2 + g_y^2}\,dx\,dy = \int_0^1\int_{-1}^{1} x^2 y \cdot 2\sqrt{1+y^2}\,dx\,dy = \left(\int_{-1}^{1} x^2\,dx\right)\left(\int_0^1 \sqrt{1+y^2}\cdot 2y\,dy\right)$$

$$= \left(\frac{x^3}{3}\bigg|_{x=-1}^{1}\right)\left(\int_0^1 \sqrt{1+y^2}\cdot 2y\,dy\right) = \frac{2}{3}\int_0^1 \sqrt{1+y^2}\cdot 2y\,dy$$

We compute the integral using the substitution $u = 1 + y^2$, $du = 2y\,dy$. We get:

$$\iint_S x^2 y \cdot dS = \frac{2}{3}\int_1^2 u^{1/2} du = \frac{2}{3} \cdot \frac{2}{3} u^{3/2}\Big|_1^2 = \frac{4}{9}\left(2^{3/2} - 1\right) = \frac{4}{9}\left(2\sqrt{2} - 1\right)$$

37. Calculate $\iint_{\mathcal{S}} \left(x^2 + y^2\right) e^{-z}\, dS$, where $\mathcal{S}$ is the cylinder with equation $x^2 + y^2 = 9$ for $0 \le z \le 10$.

SOLUTION We parametrize the cylinder $\mathcal{S}$ by,

$$G(\theta, z) = (3\cos\theta, 3\sin\theta, z)$$

with the parameter domain:

$$0 \le \theta \le 2\pi, \quad 0 \le z \le 10.$$

We compute the tangent and normal vectors:

$$\mathbf{T}_\theta = \frac{\partial\phi}{\partial\theta} = \frac{\partial}{\partial\theta}\langle 3\cos\theta, 3\sin\theta, z\rangle = \langle -3\sin\theta, 3\cos\theta, 0\rangle$$
$$\mathbf{T}_z = \frac{\partial\phi}{\partial\theta} = \frac{\partial}{\partial\theta}\langle 3\cos\theta, 3\sin\theta, z\rangle = \langle 0, 0, 1\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3\cos\theta & 0 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3\sin\theta & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3\sin\theta & 3\cos\theta \\ 0 & 0 \end{vmatrix}\mathbf{k}$$
$$= (3\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j} = 3\langle\cos\theta, \sin\theta, 0\rangle$$

We compute the length of the normal vector:

$$\|\mathbf{n}\| = 3\sqrt{\cos^2\theta + \sin^2\theta + 0} = 3$$

We now express the function $f(x, y, z) = \left(x^2 + y^2\right) e^{-z}$ in terms of the parameters:

$$f(\phi(\theta, z)) = \left(x^2 + y^2\right) e^{-z} = \left(9\cos^2\theta + 9\sin^2\theta\right) e^{-z} = 9e^{-z}$$

Using the Theorem on Surface Integrals, we obtain:

$$\iint_{\mathcal{S}} \left(x^2 + y^2\right) e^{-z} dS = \int_0^{10}\int_0^{2\pi} 9e^{-z} 3\, d\theta\, dz = 27 \cdot 2\pi \int_0^{10} e^{-z} dz = 54\pi\left(-e^{-z}\right)\Big|_{z=0}^{10}$$
$$= 54\pi\left(-e^{-10} + 1\right) \approx 54\pi$$

38. Let $\mathcal{S}$ be the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$. For each of the functions (a)–(d), determine whether $\iint_{\mathcal{S}} f\, dS$ is positive, zero, or negative (without evaluating the integral). Explain your reasoning.

(a) $f(x, y, z) = y^3$

(b) $f(x, y, z) = z^3$

(c) $f(x, y, z) = xyz$

(d) $f(x, y, z) = z^2 - 2$

SOLUTION

(a) Since $f(x, y, z) = y^3$ is an odd function of y, and the upper hemisphere is symmetric with respect to the (x, z)-plane, the surface integrals over the parts where $y \ge 0$ and $y \le 0$ cancel each other. Therefore the surface integral is zero.

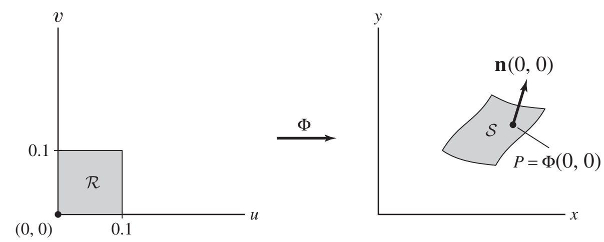
(b) The function $f(x, y, z) = z^3$ is non-negative on $\mathcal{S}$, hence the surface integral is positive.

(c) Since $f(-x, y, z) = -xyz = -f(x, y, z)$ and since the upper hemisphere is symmetric with respect to (y, z)-plane, the surface integrals over the parts of $\mathcal{S}$ where $x \ge 0$ and $x \le 0$ cancel each other to obtain a zero surface integral.

(d) On $\mathcal{S}$ we have $z^2 = 1 - x^2 - y^2 \le 1$, hence $z^2 - 2 < 0$. That is, $f(x, y, z) = z^2 - 2$ is negative on $\mathcal{S}$, therefore the surface integral is negative.

39. Let $\mathcal{S}$ be a small patch of surface with a parametrization $G(u,v)$ for $0 \le u \le 0.1$, $0 \le v \le 0.1$ such that the normal vector $\mathbf{n}(u,v)$ for $(u,v) = (0,0)$ is $\mathbf{n} = \langle 2, -2, 4\rangle$. Use Eq. (3) in Section 16.4 to estimate the surface area of $\mathcal{S}$.

SOLUTION



We use Eq. (3) in section 16.4 with $(u_{ij}, v_{ij}) = (0,0)$, $\mathcal{R}_{ij} = \mathcal{R} = [0, 0.1] \times [0, 0.1]$ in the (u,v)-plane and $\mathcal{S}_{ij} = \mathcal{S} = G(\mathcal{R})$, in the (x,y)-plane to obtain the following estimation for the area of $\mathcal{S}$:

$$\text{Area}(\mathcal{S}) \approx \|\mathbf{n}(0,0)\| \text{Area}(\mathcal{R})$$

That is:

$$\text{Area}(\mathcal{S}) \approx \|\langle 2, -2, 4\rangle\| 0.1^2 = \sqrt{2^2 + (-2)^2 + 4^2} \cdot (0.1)^2 = 0.02\sqrt{6} \approx 0.049$$

40. The upper half of the sphere $x^2 + y^2 + z^2 = 9$ has parametrization $G(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{9 - r^2})$ in cylindrical coordinates (Figure 3).

(a) Calculate the normal vector $\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta$ at the point $G\left(2, \frac{\pi}{3}\right)$.

(b) Use Eq. (3) in Section 16.4 to estimate the surface area of $G(\mathcal{R})$, where $\mathcal{R}$ is the small domain defined by

$$2 \le r \le 2.1, \qquad \frac{\pi}{3} \le \theta \le \frac{\pi}{3} + 0.05$$

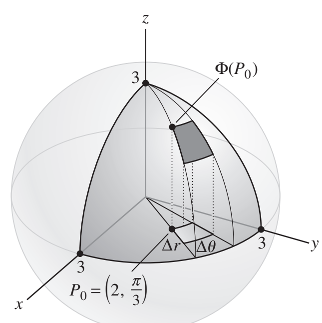


FIGURE 3

SOLUTION

(a) We first find the tangent vectors at the given point:

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta}\left\langle r\cos\theta, r\sin\theta, \sqrt{9 - r^2}\right\rangle = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\Rightarrow \quad \mathbf{T}_\theta\left(2, \frac{\pi}{3}\right) = \left\langle -2\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}, 0\right\rangle = \left\langle -\sqrt{3}, 1, 0\right\rangle$$

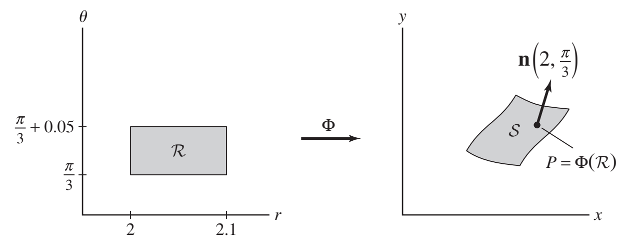
$$\mathbf{T}_r = \frac{\partial G}{\partial r} = \frac{\partial}{\partial r}\left\langle r\cos\theta, r\sin\theta, \sqrt{9 - r^2}\right\rangle = \left\langle \cos\theta, \sin\theta, -\frac{r}{\sqrt{9 - r^2}}\right\rangle$$

$$\Rightarrow \quad \mathbf{T}_r\left(2, \frac{\pi}{3}\right) = \left\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}, -\frac{2}{\sqrt{9 - 2^2}}\right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{2}{\sqrt{5}}\right\rangle$$

The normal vector is the cross product:

$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{2}{\sqrt{5}} \\ -\sqrt{3} & 1 & 0 \end{vmatrix} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{2\sqrt{3}}{\sqrt{5}}\mathbf{j} + \left(\frac{1}{2} + \frac{3}{2}\right)\mathbf{k} = \left\langle \frac{2}{\sqrt{5}}, \frac{2\sqrt{3}}{\sqrt{5}}, 2\right\rangle$$

(b) We use Eq. (3) with the sample point $(u, v) = \left(2, \frac{\pi}{3}\right)$.



In part (a) we found that the normal to $G(\mathcal{R})$ at this point is $\mathbf{n} = \left\langle \frac{2}{\sqrt{5}}, \frac{2\sqrt{3}}{\sqrt{5}}, 2 \right\rangle$. Therefore, $\|\mathbf{n}\| = \sqrt{\frac{4}{5} + \frac{12}{5} + 4} = \frac{6}{\sqrt{5}}$. We get:

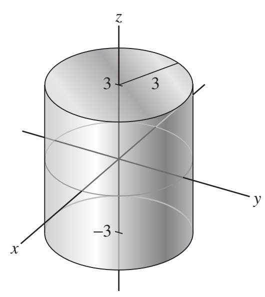
$$\text{Area}(G(\mathcal{R})) \approx \left\| \mathbf{n}\left(2, \frac{\pi}{3}\right) \right\| \text{Area}(\mathcal{R}) = \frac{6}{\sqrt{5}} \cdot 0.1 \cdot 0.05 = \frac{0.03}{\sqrt{5}} \approx 0.0134$$

In Exercises 41–46, compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ *for the given oriented surface or parametrized surface.*

41. $\mathbf{F}(x, y, z) = \langle y, x, e^{xz} \rangle$, $\quad x^2 + y^2 = 9$, $x \ge 0$, $y \ge 0$, $\quad -3 \le z \le 3$, $\quad$ outward-pointing normal

SOLUTION The part of the cylinder is parametrized by:

$$G(\theta, z) = (3\cos\theta, 3\sin\theta, z), \quad 0 \le \theta \le \frac{\pi}{2}, \quad -3 \le z \le 3$$



Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta} \langle 3\cos\theta, 3\sin\theta, z \rangle = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$$
$$\mathbf{T}_z = \frac{\partial G}{\partial z} = \frac{\partial}{\partial z} \langle 3\cos\theta, 3\sin\theta, z \rangle = \langle 0, 0, 1 \rangle$$

We compute the cross product:

$$\mathbf{T}_\theta \times \mathbf{T}_z = ((-3\sin\theta)\mathbf{i} + (3\cos\theta)\mathbf{j}) \times \mathbf{k} = (3\sin\theta)\mathbf{j} + (3\cos\theta)\mathbf{i} = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$$

The outward pointing normal is (when $\theta = 0$, the x-component must be positive):

$$\mathbf{n} = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We write $\mathbf{F}(x, y, z) = \langle y, x, e^{xz} \rangle$ in terms of the parameters by substituting $x = 3\cos\theta$, $y = 3\sin\theta$. We get:

$$\mathbf{F}(G(\theta, z)) = \left\langle 3\sin\theta, 3\cos\theta, e^{3z\cos\theta} \right\rangle$$

Hence:

$$\mathbf{F}(G(\theta, z)) \cdot \mathbf{n} = \left\langle 3\sin\theta, 3\cos\theta, e^{3z\cos\theta} \right\rangle \cdot \langle 3\cos\theta, 3\sin\theta, 0 \rangle$$
$$= 18\sin\theta\cos\theta$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral (we use the trigonometric identities $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$ and $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$):

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_{-3}^{3} \mathbf{F}\left(G(\theta,z)\right)\cdot \mathbf{n}(\theta,z)\,dz\,d\theta = \int_0^{\pi/2}\int_{-3}^{3} 18\sin\theta\cos\theta\,d\theta$$

$$= 18\int_0^{\pi/2} \frac{\sin 2\theta}{2}\,d\theta\cdot\int_{-3}^{3} dz = -9\frac{\cos 2\theta}{2}\Big|_0^{\pi/2}\cdot z\Big|_{-3}^{3}$$

$$= -\frac{9}{2}(-1-1)\cdot(3-(-3)) = 54$$

42. $\mathbf{F}(x,y,z) = \langle -y, z, -x\rangle$, $\quad G(u,v) = (u+3v, v-2u, 2v+5), 0\le u\le 1, 0\le v\le 1$, $\quad$ upward-pointing normal

SOLUTION

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_u = \frac{\partial G}{\partial u} = \frac{\partial}{\partial u}\langle u+3v, v-2u, 2v+5\rangle = \langle 1,-2,0\rangle$$

$$\mathbf{T}_v = \frac{\partial G}{\partial v} = \frac{\partial}{\partial v}\langle u+3v, v-2u, 2v+5\rangle = \langle 3,1,2\rangle$$

$$\mathbf{T}_u\times\mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 0\\ 3 & 1 & 2\end{vmatrix} = \begin{vmatrix} -2 & 0\\ 1 & 2\end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0\\ 3 & 2\end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2\\ 3 & 1\end{vmatrix}\mathbf{k} = -4\mathbf{i}-2\mathbf{j}+7\mathbf{k} = \langle -4,-2,7\rangle$$

Since the normal is pointing upward, the z-component is positive. Therefore the normal vector is $\mathbf{n} = \langle -4,-2,7\rangle$.

Step 2. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We write $\mathbf{F}$ in terms of the parameters. Since $x = u+3v$, $\quad y = v-2u$, $z = 2v+5$, we have:

$$\mathbf{F}\left(G(u,v)\right) = \langle -y, z, -x\rangle = \langle 2u-v, 2v+5, -u-3v\rangle$$

The dot product is, thus:

$$\mathbf{F}\left(G(u,v)\right)\cdot\mathbf{n} = \langle 2u-v, 2v+5, -u-3v\rangle\cdot\langle -4,-2,7\rangle$$

$$= -4(2u-v) - 2(2v+5) + 7(-u-3v) = -15u-21v-10$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_{S} \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{D}} \mathbf{F}\left(G(u,v)\right)\cdot\mathbf{n}\,du\,dv = \int_0^1\int_0^1 (-15u-21v-10)\,du\,dv$$

$$= \int_0^1 -\frac{15u^2}{2} - 21vu - 10u\Big|_{u=0}^{1} dv = \int_0^1\left(-\frac{35}{2}-21v\right)dv$$

$$= -\frac{35v}{2} - \frac{21v^2}{2}\Big|_0^1 = -\frac{35}{2} - \frac{21}{2} = -28$$

43. $\mathbf{F}(x,y,z) = \left\langle 0,0,x^2+y^2\right\rangle$, $\quad x^2+y^2+z^2 = 4$, $\quad z\ge 0$, outward-pointing normal

SOLUTION The upper hemisphere is parametrized by:

$$G(\theta,\phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi),\quad 0\le\theta\le 2\pi,\quad 0\le\phi\le\frac{\pi}{2}$$

As seen in Section 17.4, since $0\le\phi\le\frac{\pi}{2}$ then the outward-pointing normal is:

$$\mathbf{n} = 4\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

We express $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}\left(G(\theta,\phi)\right) = \left\langle 0,0,x^2+y^2\right\rangle = \left\langle 0,0,4\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\right\rangle$$

$$= \left\langle 0,0,4\sin^2\phi\right\rangle$$

The dot product $\mathbf{F}\cdot\mathbf{n}$ is thus

$$\mathbf{F}\left(G(\theta,\phi)\right)\cdot\mathbf{n}(\theta,\phi) = 16\sin^3\phi\,\cos\phi$$

We obtain the following integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{s} &= \iint_{\mathcal{D}} \mathbf{F}\left(G(\theta,\phi)\right)\cdot \mathbf{n}(\theta,\phi)\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi} 16\sin^3\phi\cos\phi\,d\theta\,d\phi = 16\int_0^{2\pi} d\theta\cdot\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi\\ &= 16\cdot 2\pi\cdot \frac{\sin^4\phi}{4}\bigg|_0^{\pi/2} = 8\pi\end{aligned}$$

44. $\mathbf{F}(x,y,z) = \left\langle z, 0, z^2\right\rangle$, $G(u,v) = (v\cosh u, v\sinh u, v)$, $0\le u\le 1$, $0\le v\le 1$, upward-pointing normal

SOLUTION

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_u = \frac{\partial G}{\partial u} = \frac{\partial}{\partial u}\langle v\cosh(u), v\sinh(u), v\rangle = \langle v\sinh(u), v\cosh(u), 0\rangle$$

$$\mathbf{T}_v = \frac{\partial G}{\partial v} = \frac{\partial}{\partial v}\langle v\cosh(u), v\sinh(u), v\rangle = \langle \cosh(u), \sinh(u), 1\rangle$$

$$\begin{aligned}\mathbf{T}_u\times\mathbf{T}_v &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v\sinh(u) & v\cosh(u) & 0\\ \cosh(u) & \sinh(u) & 1\end{vmatrix}\\ &= \begin{vmatrix} v\cosh(u) & 0\\ \sinh(u) & 1\end{vmatrix}\mathbf{i} - \begin{vmatrix} v\sinh(u) & 0\\ \cosh(u) & 1\end{vmatrix}\mathbf{j} + \begin{vmatrix} v\sinh(u) & v\cosh(u)\\ \cosh(u) & \sinh(u)\end{vmatrix}\mathbf{k}\\ &= (v\cosh(u))\,\mathbf{i} - (v\sinh(u))\,\mathbf{j} + \left(v\sinh^2(u) - v\cosh^2(u)\right)\mathbf{k}\\ &= \langle v\cosh(u), -v\sinh(u), -v\rangle\end{aligned}$$

Since the normal points upward, the z-component is positive. Therefore the normal vector is, (notice that $v\ge 0$):

$$\mathbf{n} = \langle -v\cosh(u), v\sinh(u), v\rangle$$

Step 2. Evaluate the dot product $\mathbf{F}\cdot\mathbf{n}$. We write $\mathbf{F}$ in terms of the parameters:

$$\mathbf{F}\left(G(u,v)\right) = \left\langle z, 0, z^2\right\rangle = \left\langle v, 0, v^2\right\rangle$$

Hence:

$$\mathbf{F}\left(G(u,v)\right)\cdot\mathbf{n}(u,v) = \left\langle v, 0, v^2\right\rangle\cdot\langle -v\cosh(u), v\sinh(u), v\rangle = -v^2\cosh(u) + v^3$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{s} &= \iint_{\mathcal{D}} \mathbf{F}\left(G(u,v)\right)\cdot\mathbf{n}\,du\,dv = \int_0^1\int_0^1\left(-v^2\cosh(u)+v^3\right)du\,dv\\ &= \int_0^1\int_0^1 -v^2\cosh(u)\,du\,dv + \int_0^1\int_0^1 v^3\,du\,dv = \left(\int_0^1 -v^2\,dv\right)\left(\int_0^1\cosh(u)\,du\right) + \int_0^1 v^3\,dv\\ &= \left(-\frac{v^3}{3}\Big|_{v=0}^1\right)\left(\sinh(u)\Big|_{u=0}^1\right) + \frac{v^4}{4}\bigg|_{v=0}^1 = -\frac{1}{3}\sinh(1) + \frac{1}{4}\approx -0.1417\end{aligned}$$

45. $\mathbf{F}(x,y,z) = \left\langle 0, 0, xze^{xy}\right\rangle$, $z = xy$, $0\le x\le 1$, $0\le y\le 1$, upward-pointing normal

SOLUTION We parametrize the surface by:

$$G(x,y) = (x, y, xy)$$

Where the parameter domain is the square:

$$\mathcal{D} = \{(x,y) : 0\le x\le 1, 0\le y\le 1\}$$

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_x = \frac{\partial G}{\partial x} = \frac{\partial}{\partial x}\langle x, y, xy\rangle = \langle 1, 0, y\rangle$$

$$\mathbf{T}_y = \frac{\partial G}{\partial y} = \frac{\partial}{\partial y}\langle x, y, xy\rangle = \langle 0, 1, x\rangle$$

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \begin{vmatrix} 0 & y \\ 1 & x \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & y \\ 0 & x \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -y\mathbf{i} - x\mathbf{j} + \mathbf{k} = \langle -y, -x, 1\rangle$$

Since the normal points upwards, the z-coordinate is positive. Therefore the normal vector is:

$$\mathbf{n} = \langle -y, -x, 1\rangle$$

Step 2. Evaluate the dot product $\mathbf{F} \cdot \mathbf{n}$. We express $\mathbf{F}$ in terms of x and y:

$$\mathbf{F}\left(G(x, y)\right) = \left\langle 0, 0, xze^{xy}\right\rangle = \left\langle 0, 0, x(xy)e^{xy}\right\rangle = \left\langle 0, 0, x^2ye^{xy}\right\rangle$$

Hence:

$$\mathbf{F}\left(G(x, y)\right) \cdot \mathbf{n}(x, y) = \left\langle 0, 0, x^2ye^{xy}\right\rangle \cdot \langle -y, -x, 1\rangle = x^2ye^{xy}$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{s} &= \iint_{\mathcal{D}} \mathbf{F}\left(G(x, y)\right) \cdot \mathbf{n}(x, y)\, dx\, dy \\ &= \int_0^1 \int_0^1 x^2ye^{xy}\, dy\, dx = \int_0^1 x^2 \left(\int_0^1 ye^{xy}\, dy\right) dx \end{aligned} \tag{1}$$

We evaluate the inner integral using integration by parts:

$$\int_0^1 ye^{xy}\, dy = \frac{y}{x}e^{xy}\Big|_{y=0}^1 - \int_0^1 \frac{1}{x}e^{xy}\, dy = \frac{e^x}{x} - \frac{1}{x^2}e^{xy}\Big|_{y=0}^1 = \frac{e^x}{x} - \frac{1}{x^2}\left(e^x - 1\right)$$

Substituting this integral in (1) gives:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left(xe^x - \left(e^x - 1\right)\right) dx = \int_0^1 xe^x\, dx - \int_0^1 \left(e^x - 1\right) dx \\ &= \int_0^1 xe^x\, dx - \left(e^x - x\right)\Big|_0^1 = \int_0^1 xe^x\, dx - (e - 2)\end{aligned}$$

Using integration by parts we have:

$$\iint_S F \cdot dS = xe^x - e^x\Big|_0^1 - (e - 2) = 1 - (e - 2) = 3 - e$$

46. $\mathbf{F}(x, y, z) = \langle 0, 0, z\rangle, \quad 3x^2 + 2y^2 + z^2 = 1, \quad z \geq 0,$
upward-pointing normal

SOLUTION We use modified spherical coordinates to parametrize the ellipsoid:

$$G(\theta, \phi) = \left(\frac{1}{\sqrt{3}}\cos\theta\sin\phi, \frac{1}{\sqrt{2}}\sin\theta\sin\phi, \cos\phi\right)$$

where the parameter domain is:

$$\mathcal{D} = \left\{(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}\right\}$$

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \left\langle -\frac{1}{\sqrt{3}}\sin\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\theta\sin\phi, 0\right\rangle$$

$$\mathbf{T}_\phi = \frac{\partial G}{\partial \phi} = \left\langle \frac{1}{\sqrt{3}}\cos\theta\cos\phi, \frac{1}{\sqrt{2}}\sin\theta\cos\phi, -\sin\phi\right\rangle$$

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{1}{\sqrt{3}}\sin\theta\sin\phi & \frac{1}{\sqrt{2}}\cos\theta\sin\phi & 0 \\ \frac{1}{\sqrt{3}}\cos\theta\cos\phi & \frac{1}{\sqrt{2}}\sin\theta\cos\phi & -\sin\phi \end{vmatrix}$$

$$= \left(-\frac{1}{\sqrt{2}}\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\frac{1}{\sqrt{3}}\sin\theta\ \sin^2\phi\right)\mathbf{j} - \left(\frac{1}{\sqrt{6}}\sin^2\theta\sin\phi\cos\phi + \frac{1}{\sqrt{6}}\cos^2\theta\sin\phi\cos\phi\right)\mathbf{k}$$

$$= \left(-\frac{1}{\sqrt{2}}\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\frac{1}{\sqrt{3}}\sin\theta\ \sin^2\phi\right)\mathbf{j} - \left(\frac{1}{\sqrt{6}}\sin\phi\cos\phi\right)\mathbf{k}$$

Since the normal points upward, the z-component is positive. For $0 \le \phi \le \frac{\pi}{2}$, $\sin\phi\cos\phi \ge 0$ therefore the normal vector is:

$$\mathbf{n} = \left\langle \frac{1}{\sqrt{2}}\cos\theta\ \sin^2\phi,\ \frac{1}{\sqrt{3}}\sin\theta\ \sin^2\phi,\ \frac{1}{\sqrt{6}}\sin\phi\cos\phi \right\rangle$$

Step 2. Compute the dot product $\mathbf{F}\cdot\mathbf{n}$. We have:

$$\mathbf{F}(G(\theta,\phi)) = \langle 0, 0, \cos\phi\rangle \cdot \left\langle \frac{1}{\sqrt{2}}\cos\theta\sin^2\phi,\ \frac{1}{\sqrt{3}}\sin\theta\sin^2\phi,\ \frac{1}{\sqrt{6}}\sin\phi\cos\phi \right\rangle = \frac{1}{\sqrt{6}}\sin\phi\cos^2\phi$$

Step 3. Evaluate the surface integral. The surface integral is equal to the following double integral:

$$\iint_S \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{D}} \mathbf{F}(G(\theta,\phi))\cdot\mathbf{n}(\theta,\phi)\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{\sqrt{6}}\sin\phi\ \cos^2\phi\,d\phi\,d\theta$$

$$= \frac{2\pi}{\sqrt{6}}\int_0^{\pi/2}\cos^2\phi\ \sin\phi\,d\phi = \frac{2\pi}{\sqrt{6}}\left(-\frac{\cos^3\phi}{3}\Big|_{\phi=0}^{\pi/2}\right) = \frac{2\pi}{\sqrt{6}}\left(\frac{0+1}{3}\right) = \frac{2\pi}{3\sqrt{6}} = \frac{\sqrt{2}}{3\sqrt{3}}\pi$$

47. Calculate the total charge on the cylinder

$$x^2 + y^2 = R^2, \qquad 0 \le z \le H$$

if the charge density in cylindrical coordinates is $\rho(\theta, z) = Kz^2\cos^2\theta$, where K is a constant.

SOLUTION The total change on the surface $\mathcal{S}$ is $\iint_{\mathcal{S}} \rho\, dS$. We parametrize the surface by,

$$\mathrm{G}(\theta, z) = (R\cos\theta, R\sin\theta, Hz)$$

with the parameter domain,

$$0 \le \theta \le 2\pi,\ 0 \le z \le 1.$$

We compute the tangent and normal vectors:

$$\mathbf{T}_\theta = \frac{\partial G}{\partial\theta} = \frac{\partial}{\partial\theta}\langle R\cos\theta, R\sin\theta, Hz\rangle = \langle -R\sin\theta, R\cos\theta, 0\rangle$$

$$\mathbf{T}_z = \frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\langle R\cos\theta, R\sin\theta, Hz\rangle = \langle 0, 0, H\rangle$$

The normal vector is their cross product:

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & H \end{vmatrix}$$

$$= \begin{vmatrix} R\cos\theta & 0 \\ 0 & H \end{vmatrix}\mathbf{i} - \begin{vmatrix} -R\sin\theta & 0 \\ 0 & H \end{vmatrix}\mathbf{j} + \begin{vmatrix} -R\sin\theta & R\cos\theta \\ 0 & 0 \end{vmatrix}\mathbf{k}$$

$$= (RH\cos\theta)\mathbf{i} + (RH\sin\theta)\mathbf{j} = RH\langle\cos\theta, \sin\theta, 0\rangle$$

We find the length of $\mathbf{n}$:

$$\|\mathbf{n}\| = RH\sqrt{\cos^2\theta + \sin^2\theta} = RH$$

We compute $\rho(G(\theta, z))$:

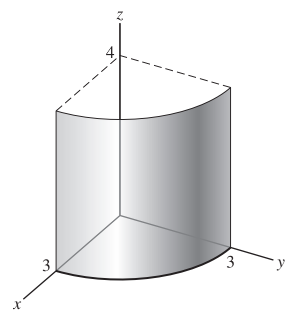
$$\rho(G(\theta, z)) = K(Hz)^2\cos^2\theta = KH^2z^2\cos^2\theta$$

Using the Theorem on Surface Integrals we obtain:

$$\iint_{\mathcal{S}} \rho \cdot dS = \iint_{\mathcal{D}} \rho\,(G(\theta, z)) \cdot \|\mathbf{n}(\theta, z)\|\, dz\, d\theta = \int_0^{2\pi} \int_0^1 KH^2 z^2 \cos^2\theta \cdot HR\, dz\, d\theta$$
$$= \left(\int_0^1 KH^3Rz^2 dz\right)\left(\int_0^{2\pi} \cos^2\theta\, d\theta\right) = \left(\frac{KH^3Rz^3}{3}\Big|_0^1\right)\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi}\right)$$
$$= \frac{KH^3R}{3} \cdot \pi = \frac{\pi}{3} KH^3R$$

48. Find the flow rate of a fluid with velocity field $\mathbf{v} = \langle 2x, y, xy\rangle$ m/s across the part of the cylinder $x^2 + y^2 = 9$ where $x \ge 0$, $y \ge 0$, and $0 \le z \le 4$ (distance in meters).

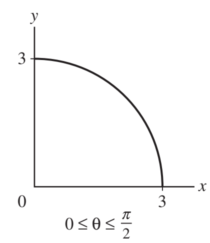
SOLUTION The flow rate of a fluid with velocity field $\mathbf{v}$ through the part $\mathcal{S}$ of the cylinder is the surface integral:



$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} \tag{1}$$

We parametrize $\mathcal{S}$ by,

$$G(\theta, z) = (3\cos\theta, 3\sin\theta, z), \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le z \le 4.$$



Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta}\langle 3\cos\theta, 3\sin\theta, z\rangle = \langle -3\sin\theta, 3\cos\theta, 0\rangle = (-3\sin\theta)\,\mathbf{i} + (3\cos\theta)\mathbf{j}$$
$$\mathbf{T}_z = \frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\langle 3\cos\theta, 3\sin\theta, z\rangle = \langle 0, 0, 1\rangle = \mathbf{k}$$
$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = ((-3\sin\theta)\mathbf{i} + (3\cos\theta)\mathbf{j}) \times \mathbf{k} = (3\sin\theta)\mathbf{j} + (3\cos\theta)\mathbf{i} = \langle 3\cos\theta, 3\sin\theta, 0\rangle$$

Step 2. Compute the dot product $\mathbf{v} \cdot \mathbf{n}$.

$$\mathbf{v}\,(G(\theta, z)) \cdot \mathbf{n} = \langle 2 \cdot 3\cos\theta, 3\sin\theta, 9\cos\theta\sin\theta\rangle \cdot \langle 3\cos\theta, 3\sin\theta, 0\rangle$$
$$= 18\cos^2\theta + 9\sin^2\theta = 9\left(\cos^2\theta + \sin^2\theta\right) + 9\cos^2\theta = 9\cos^2\theta + 9$$

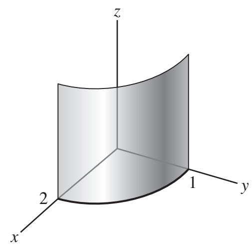
Step 3. Evaluate the flux of $\mathbf{v}$. The flux of $\mathbf{v}$ in (1) is equal to the following double integral (we use the equality $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$):

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{v}\,(G(\theta, z)) \cdot \mathbf{n}\, d\theta\, dz = \int_0^4 \int_0^{\pi/2} \left(9\cos^2\theta + 9\right) d\theta\, dz$$
$$= 4\int_0^{\pi/2} \left(9\cos^2\theta + 9\right) d\theta = 36\int_0^{\pi/2} \left(\cos^2\theta + 1\right) d\theta$$
$$= 36\int_0^{\pi/2} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = 36\left(\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta\Big|_{\theta=0}^{\pi/2}\right) = 27\pi$$

49. With $\mathbf{v}$ as in Exercise 48, calculate the flow rate across the part of the elliptic cylinder $\frac{x^2}{4} + y^2 = 1$ where $x \ge 0$, $y \ge 0$, and $0 \le z \le 4$.

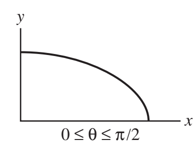
SOLUTION The flow rate of a fluid with velocity field $\mathbf{v} = \langle 2x, y, xy \rangle$ through the elliptic cylinder S is the surface integral:

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} \tag{1}$$



To compute this integral, we parametrize $\mathcal{S}$ by,

$$G(\theta, z) = (2\cos\theta, \sin\theta, z), \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le z \le 4$$



Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta} \langle 2\cos\theta, \sin\theta, z \rangle = \langle -2\sin\theta, \cos\theta, 0 \rangle$$

$$\mathbf{T}_z = \frac{\partial G}{\partial z} = \frac{\partial}{\partial z} \langle 2\cos\theta, \sin\theta, z \rangle = \langle 0, 0, 1 \rangle$$

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j} = \langle \cos\theta, 2\sin\theta, 0 \rangle$$

Step 2. Compute the dot product $\mathbf{v} \cdot \mathbf{n}$

$$\begin{aligned} \mathbf{v}(G(\theta, z)) \cdot \mathbf{n} &= \langle 4\cos\theta, \sin\theta, 2\cos\theta\sin\theta \rangle \cdot \langle \cos\theta, 2\sin\theta, 0 \rangle = 4\cos^2\theta + 2\sin^2\theta \\ &= 2\cos^2\theta + 2\left(\cos^2\theta + \sin^2\theta\right) = 2\cos^2\theta + 2 \end{aligned}$$

Step 3. Evaluate the flux of $\mathbf{v}$. The flux of $\mathbf{v}$ in (1) is equal to the following double integral (we use the equality $2\cos^2\theta = 1 + \cos 2\theta$ in our calculation):

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{v}(G(\theta, z)) \cdot \mathbf{n}\, d\theta\, dz = \int_0^4 \int_0^{\pi/2} \left(2\cos^2\theta + 2\right) d\theta\, dz \\ &= 4\int_0^{\pi/2} \left(2\cos^2\theta + 2\right) d\theta = 4\int_0^{\pi/2} (3 + \cos 2\theta)\, d\theta = 4\left(3\theta + \frac{\sin 2\theta}{2}\Big|_{\theta=0}^{\pi/2}\right) = 6\pi \end{aligned}$$

50. Calculate the flux of the vector field $\mathbf{E}(x, y, z) = \langle 0, 0, x \rangle$ through the part of the ellipsoid

$$4x^2 + 9y^2 + z^2 = 36$$

where $z \ge 3$, $x \ge 0$, $y \ge 0$. *Hint:* Use the parametrization

$$G(r, \theta) = \left(3r\cos\theta, 2r\sin\theta, 6\sqrt{1 - r^2}\right)$$

SOLUTION The flux of the vector field $\mathbf{E}$ through the part $\mathcal{S}$ of the ellipsoid is the surface integral:

$$\iint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} \tag{1}$$

We use the following parametrization for $\mathcal{S}$:

$$G(\theta, r) = \left(3r\cos\theta, 2r\sin\theta, 6\sqrt{1-r^2}\right)$$

With the parameter domain:

$$\mathcal{D} = \left\{(\theta, r) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \frac{\sqrt{3}}{2}\right\}$$

We justify this parametrization. First, we verify that $x = 3r\cos\theta$, $y = 2r\sin\theta$ and $z = 6\sqrt{1-r^2}$ satisfy the equation of the ellipsoid $4x^2 + 9y^2 + z^2 = 36$.

$$4x^2 + 9y^2 + z^2 = 4(3r\cos\theta)^2 + 9(2r\sin\theta)^2 + \left(6\sqrt{1-r^2}\right)^2 = 36r^2\cos^2\theta + 36r^2\sin^2\theta + 36\left(1-r^2\right)$$
$$= 36r^2\left(\cos^2\theta + \sin^2\theta\right) + 36 - 36r^2 = 36r^2 + 36 - 36r^2 = 36$$

Next, the part $z \ge 3$ of the ellipsoid corresponds to the values of r such that $6\sqrt{1-r^2} \ge 3$. We solve for $r \ge 0$:

$$6\sqrt{1-r^2} \ge 3 \quad \Rightarrow \quad \sqrt{1-r^2} \ge \frac{1}{2} \quad \Rightarrow \quad 1-r^2 \ge \frac{1}{4} \quad \Rightarrow \quad r^2 \le \frac{3}{4} \quad \Rightarrow \quad 0 \le r \le \frac{\sqrt{3}}{2}$$

Finally,

$$x \ge 0,\ y \ge 0 \quad \Rightarrow \quad \cos\theta \ge 0,\ \sin\theta \ge 0 \quad \Rightarrow \quad 0 \le \theta \le \frac{\pi}{2}$$

Step 1. Compute the tangent and normal vectors.

$$\mathbf{T}_\theta = \frac{\partial G}{\partial\theta} = \frac{\partial}{\partial\theta}\left\langle 3r\cos\theta, 2r\sin\theta, 6\sqrt{1-r^2}\right\rangle = \langle -3r\sin\theta, 2r\cos\theta, 0\rangle$$
$$\mathbf{T}_r = \frac{\partial G}{\partial r} = \frac{\partial}{\partial r}\left\langle 3r\cos\theta, 2r\sin\theta, 6\sqrt{1-r^2}\right\rangle = \left\langle 3\cos\theta, 2\sin\theta, -\frac{6r}{\sqrt{1-r^2}}\right\rangle$$

The outward pointing normal is:

$$\mathbf{n} = -\mathbf{T}_\theta \times \mathbf{T}_r = ((3r\sin\theta)\mathbf{i} - (2r\cos\theta)\mathbf{j}) \times \left((3\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j} - \frac{6r}{\sqrt{1-r^2}}\mathbf{k}\right)$$
$$= 6r\sin^2\theta\mathbf{k} + \frac{18r^2\sin\theta}{\sqrt{1-r^2}}\mathbf{j} + \left(6r\cos^2\theta\right)\mathbf{k} + \frac{12r^2\cos\theta}{\sqrt{1-r^2}}\mathbf{i} = \frac{12r^2\cos\theta}{\sqrt{1-r^2}}\mathbf{i} + \frac{18r^2\sin\theta}{\sqrt{1-r^2}}\mathbf{j} + 6r\mathbf{k}$$

Step 2. Compute the dot product $\mathbf{E}\cdot\mathbf{n}$.

$$\mathbf{E}\left(G(\theta, r)\right)\cdot\mathbf{n} = \langle 0, 0, 3r\cos\theta\rangle \cdot \left\langle \frac{12r^2\cos\theta}{\sqrt{1-r^2}}, \frac{18r^2\sin\theta}{\sqrt{1-r^2}}, 6r\right\rangle = 18r^2\cos\theta$$

Step 3. Evaluate the flux of $\mathbf{E}$. The flux of $\mathbf{E}$ in (1) is equal to the following double integral:

$$\iint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{E}\left(G(\theta, r)\right)\cdot\mathbf{n}\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\sqrt{3}/2} 18r^2\cos\theta\, dr\, d\theta$$
$$= 18\int_0^{\pi/2}\cos\theta\, d\theta \cdot \int_0^{\sqrt{3}/2} r^2\, dr = 18\sin\theta\Big|_0^{\pi/2} \cdot \frac{r^3}{3}\Big|_0^{\sqrt{3}/2}$$
$$= \frac{9\sqrt{3}}{4}$$

17 FUNDAMENTAL THEOREMS OF VECTOR ANALYSIS

17.1 Green's Theorem (LT Section 18.1)

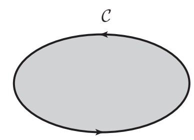
Preliminary Questions

1. Which vector field **F** is being integrated in the line integral $\oint x^2\,dy - e^y\,dx$?

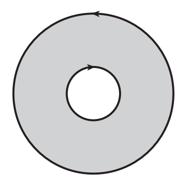
SOLUTION The line integral can be rewritten as $\oint -e^y\,dx + x^2\,dy$. This is the line integral of $\mathbf{F} = \left\langle -e^y, x^2 \right\rangle$ along the curve.

2. Draw a domain in the shape of an ellipse and indicate with an arrow the boundary orientation of the boundary curve. Do the same for the annulus (the region between two concentric circles).

SOLUTION The orientation on $\mathcal{C}$ is counterclockwise, meaning that the region enclosed by $\mathcal{C}$ lies to the left in traversing $\mathcal{C}$.



For the annulus, the inner boundary is oriented clockwise and the outer boundary is oriented counterclockwise. The region between the circles lies to the left while traversing each circle.



3. The circulation of a conservative vector field around a closed curve is zero. Is this fact consistent with Green's Theorem? Explain.

SOLUTION Green's Theorem asserts that

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathcal{C}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{1}$$

If **F** is conservative, the cross partials are equal, that is,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \tag{2}$$

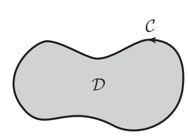
Combining (1) and (2) we obtain $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = 0$. That is, Green's Theorem implies that the integral of a conservative vector field around a simple closed curve is zero.

4. Indicate which of the following vector fields possess the following property: For every simple closed curve $\mathcal{C}$, $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ is equal to the area enclosed by $\mathcal{C}$.

(a) $\mathbf{F} = \langle -y, 0\rangle$ **(b)** $\mathbf{F} = \langle x, y\rangle$ **(c)** $\mathbf{F} = \left\langle \sin(x^2), x + e^{y^2}\right\rangle$

SOLUTION By Green's Theorem,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{1}$$



(a) Here, $P = -y$ and $Q = 0$, hence $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$. Therefore, by (1),

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} 1\,dx\,dy = \text{Area}(\mathcal{D})$$

(b) We have $P = x$ and $Q = y$, therefore $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 0 = 0$. By (1) we get

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} 0\,dx\,dy = 0 \neq \text{Area}(\mathcal{D})$$

(c) In this vector field we have $P = \sin(x^2)$ and $Q = x + e^{y^2}$. Therefore,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1.$$

By (1) we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} 1\,dx\,dy = \text{Area}(\mathcal{D}).$$

Exercises

1. Verify Green's Theorem for the line integral $\oint_{\mathcal{C}} xy\,dx + y\,dy$, where $\mathcal{C}$ is the unit circle, oriented counterclockwise.

SOLUTION

Step 1. Evaluate the line integral. We use the parametrization $\gamma(\theta) = \langle \cos\theta, \sin\theta \rangle$, $0 \le \theta \le 2\pi$ of the unit circle. Then

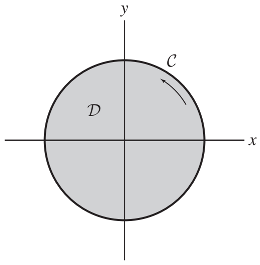
$$dx = -\sin\theta\,d\theta, \quad dy = \cos\theta\,d\theta$$

and

$$xy\,dx + y\,dy = \cos\theta\sin\theta(-\sin\theta\,d\theta) + \sin\theta\cos\theta\,d\theta = \left(-\cos\theta\sin^2\theta + \sin\theta\cos\theta\right)d\theta$$

The line integral is thus

$$\begin{aligned}\int_{\mathcal{C}} xy\,dx + y\,dy &= \int_0^{2\pi} \left(-\cos\theta\sin^2\theta + \sin\theta\cos\theta\right)d\theta \\ &= \int_0^{2\pi} -\cos\theta\sin^2\theta\,d\theta + \int_0^{2\pi} \sin\theta\cos\theta\,d\theta = -\frac{\sin^3\theta}{3}\Big|_0^{2\pi} - \frac{\cos 2\theta}{4}\Big|_0^{2\pi} = 0 \qquad (1)\end{aligned}$$



Step 2. Evaluate the double integral. Since $P = xy$ and $Q = y$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x = -x$$

We compute the double integral in Green's Theorem:

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{\mathcal{D}} -x\,dx\,dy = -\iint_{\mathcal{D}} x\,dx\,dy$$

The integral of x over the disk $\mathcal{D}$ is zero, since by symmetry the positive and negative values of x cancel each other. Therefore,

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0 \qquad (2)$$

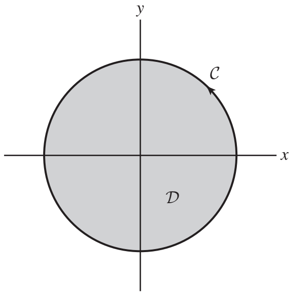
Step 3. Compare. The line integral in (1) is equal to the double integral in (2), as stated in Green's Theorem.

2. Let $I = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \left\langle y + \sin x^2, x^2 + e^{y^2} \right\rangle$ and $\mathcal{C}$ is the circle of radius 4 centered at the origin.

(a) Which is easier, evaluating I directly or using Green's Theorem?

(b) Evaluate I using the easier method.

SOLUTION



(a) Using the parametrization $\gamma(\theta) = \langle 4\cos\theta, 4\sin\theta \rangle$ for the circle, we have

$$dx = -4\sin\theta\, d\theta, \quad dy = 4\cos\theta\, d\theta$$

and

$$(y + \sin x^2)dx + (x^2 + e^{y^2})dy = \left(4\sin\theta + \sin(16\cos^2\theta)\right)(-4\sin\theta)\,d\theta + (16\cos^2\theta + e^{16\sin^2\theta}) \cdot 4\cos\theta\, d\theta$$

$$= \left(-16\sin^2\theta - 4\sin\theta\sin(16\cos^2\theta) + 64\cos^3\theta + 4\cos\theta e^{16\sin^2\theta}\right) d\theta$$

The line integral is thus

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \left(-16\sin^2\theta - 4\sin\theta\sin(16\cos^2\theta) + 64\cos^3\theta + 4\cos\theta e^{16\sin^2\theta}\right) d\theta \tag{1}$$

We examine the double integral in Green's Theorem. Since $P = y + \sin x^2$ and $Q = x^2 + e^{y^2}$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 1$$

The double integral is thus

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (2x - 1)\, dx\, dy \tag{2}$$

Clearly, the double integral in (2) is much easier to evaluate than the line integral in (1).

(b) To evaluate the double integral in (2), we notice that by symmetry the integral of $2x$ over $\mathcal{D}$ is zero, since the positive and negative values of x cancel each other. Hence,

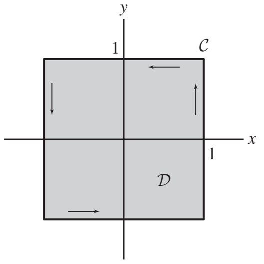
$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (2x - 1)\, dx\, dy = \iint_{\mathcal{D}} 2x\, dx\, dy - \iint_{\mathcal{D}} 1\, dx\, dy$$

$$= 0 - \text{Area}(\mathcal{D}) = -\pi \cdot 4^2 = -16\pi$$

In Exercises 3–10, use Green's Theorem to evaluate the line integral. Orient the curve counterclockwise unless otherwise indicated.

3. $\oint_{\mathcal{C}} y^2\, dx + x^2\, dy$, where $\mathcal{C}$ is the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$

SOLUTION



We have $P = y^2$ and $Q = x^2$, therefore

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$$

Using Green's Theorem we obtain

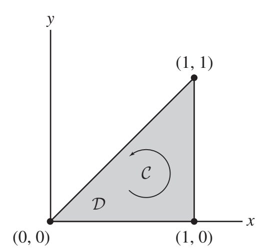
$$\int_{\mathcal{C}} y^2\,dx + x^2\,dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_{\mathcal{D}} (2x - 2y)\,dx\,dy = 2\iint_{\mathcal{D}} x\,dx\,dy - 2\iint_{\mathcal{D}} y\,dx\,dy$$

By symmetry, the positive and negative values of x cancel each other in the first integral, so this integral is zero. The second double integral is zero by similar reasoning. Therefore,

$$\int_{\mathcal{C}} y^2\,dx + x^2\,dy = 0 - 0 = 0$$

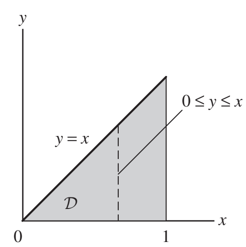
4. $\oint_{\mathcal{C}} e^{2x+y}\,dx + e^{-y}\,dy$, where $\mathcal{C}$ is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

SOLUTION



We have $P = e^{2x+y}$ and $Q = e^{-y}$, hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - e^{2x+y} = -e^{2x+y}$$



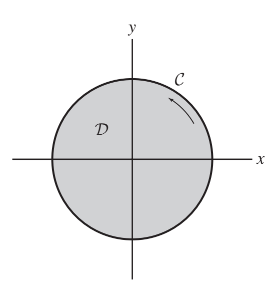
Using Green's Theorem we get

$$\int_{\mathcal{C}} e^{2x+y}\,dx + e^{-y}\,dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_{\mathcal{D}} -e^{2x+y}\,dA = \int_0^1 \int_0^x -e^{2x+y}\,dy\,dx$$

$$= \int_0^1 -e^{2x+y}\Big|_{y=0}^{x}\,dx = \int_0^1 \left(-e^{3x} + e^{2x}\right)\,dx = -\frac{e^{3x}}{3} + \frac{e^{2x}}{2}\Big|_0^1 = \frac{e^2}{2} - \frac{e^3}{3} - \frac{1}{6}$$

5. $\oint_{\mathcal{C}} x^2\,y\,dx$, where $\mathcal{C}$ is the unit circle centered at the origin

SOLUTION



In this function $P = x^2 y$ and $Q = 0$. Therefore,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x^2 = -x^2$$

We obtain the following integral:

$$I = \int_{\mathcal{C}} x^2 y\, dx = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} -x^2\, dA$$

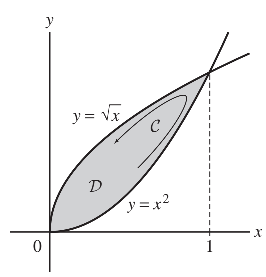
We convert the integral to polar coordinates. This gives

$$I = \int_0^{2\pi} \int_0^1 -r^2 \cos^2\theta \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 -r^3 \cos^2\theta\, dr\, d\theta$$
$$= \left(\int_0^{2\pi} \cos^2\theta\, d\theta \right) \left(\int_0^1 -r^3\, dr \right) = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_{\theta=0}^{2\pi} \right) \left(-\frac{r^4}{4} \Big|_{r=0}^{1} \right) = \pi \cdot \left(-\frac{1}{4} \right) = -\frac{\pi}{4}$$

6. $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle x + y, x^2 - y \rangle$ and $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$

SOLUTION By Green's Theorem we have

$$I = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Since $P = x + y$ and $Q = x^2 - y$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 1$$

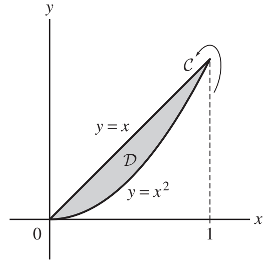
Therefore,

$$I = \iint_{\mathcal{D}} (2x - 1)\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (2x - 1)\, dy\, dx = \int_0^1 (2x - 1) y \Big|_{y=x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 (2x - 1) \left(\sqrt{x} - x^2 \right) dx = \int_0^1 \left(2x^{3/2} - 2x^3 - x^{1/2} + x^2 \right) dx$$
$$= \frac{4}{5} x^{5/2} - \frac{1}{2} x^4 - \frac{2}{3} x^{3/2} + \frac{x^3}{3} \Big|_0^1 = \frac{4}{5} - \frac{1}{2} - \frac{2}{3} + \frac{1}{3} = -\frac{1}{30}$$

7. $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle x^2, x^2 \rangle$ and $\mathcal{C}$ consists of the arcs $y = x^2$ and $y = x$ for $0 \le x \le 1$

SOLUTION By Green's Theorem,

$$I = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



We have $P = Q = x^2$, therefore

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 0 = 2x$$

Hence,

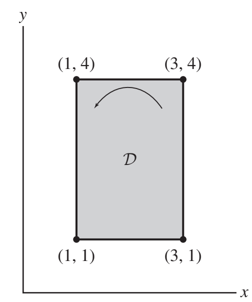
$$I = \iint_{\mathcal{D}} 2x\, dA = \int_0^1 \int_{x^2}^{x} 2x\, dy\, dx = \int_0^1 2xy \Big|_{y=x^2}^{x} dx = \int_0^1 2x(x - x^2)\, dx = \int_0^1 (2x^2 - 2x^3)\, dx$$

$$= \frac{2}{3}x^3 - \frac{1}{2}x^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

8. $\oint_{\mathcal{C}} (\ln x + y)\, dx - x^2\, dy$, where $\mathcal{C}$ is the rectangle with vertices $(1, 1)$, $(3, 1)$, $(1, 4)$, and $(3, 4)$

SOLUTION By Green's Theorem,

$$I = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



We have $P = \ln x + y$ and $Q = -x^2$, therefore

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x - 1$$

Hence,

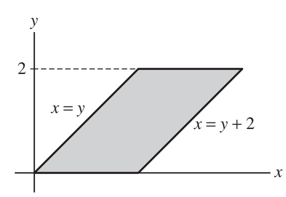
$$I = \iint_{\mathcal{D}} (-2x - 1)\, dA = \int_1^3 \int_1^4 (-2x - 1)\, dy\, dx = \int_1^3 (-2x - 1)y \Big|_{y=1}^{4} dx = \int_1^3 -3(2x + 1)\, dx$$

$$= -3(x^2 + x) \Big|_1^3 = -3(12 - 2) = -30$$

9. The line integral of $\mathbf{F} = \langle e^{x+y}, e^{x-y} \rangle$ along the curve (oriented clockwise) consisting of the line segments by joining the points $(0, 0)$, $(2, 2)$, $(4, 2)$, $(2, 0)$, and back to $(0, 0)$ (note the orientation).

SOLUTION Consider $\mathbf{F} = \langle e^{x+y}, e^{x-y} \rangle$. Here, $P = e^{x+y}$ and $Q = e^{x-y}$, hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^{x-y} - e^{x+y} = e^x(e^{-y} - e^y).$$



Using Green's Theorem we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} e^x(e^{-y} - e^y)\, dx\, dy = \int_0^2 \int_y^{y+2} e^x(e^{-y} - e^y)\, dx\, dy = \int_0^2 e^x(e^{-y} - e^y) \Big|_{x=y}^{y+2} dy$$

$$= \int_0^2 (e^{y+2} - e^y)(e^{-y} - e^y)\, dy = \int_0^2 (e^2 - 1)(1 - e^{2y})\, dy = (e^2 - 1)\left(y - \frac{e^{2y}}{2} \right) \Big|_{y=0}^{2}$$

$$= (e^2 - 1)\left(2 - \frac{e^4}{2} - \left(-\frac{1}{2} \right) \right) = \frac{(e^2 - 1)(5 - e^4)}{2}$$

10. $\int_{\mathcal{C}} xy\,dx + (x^2 + x)\,dy$, where $\mathcal{C}$ is the path in Figure 16

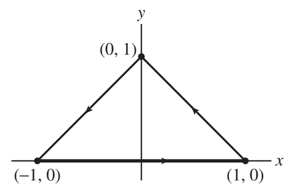
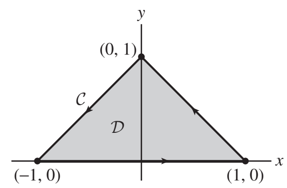


FIGURE 16

SOLUTION



In the given function, $P = xy$ and $Q = x^2 + x$. Therefore,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x + 1 - x = x + 1$$

By Green's Theorem we obtain the following integral:

$$\int_{\mathcal{C}} xy\,dx + (x^2 + x)\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (x+1)\,dA = \iint_{\mathcal{D}} x\,dA + \iint_{\mathcal{D}} 1\,dA$$

By symmetry, the positive and negative values of x cancel each other, causing the first integral to be zero. Thus,

$$\int_{\mathcal{C}} xy\,dx + (x^2 + x)\,dy = 0 + \iint_{\mathcal{D}} dA = \text{Area}(\mathcal{D}) = \frac{2 \cdot 1}{2} = 1.$$

11. Let $\mathbf{F} = \left\langle 2xe^y, x + x^2 e^y \right\rangle$ and let $\mathcal{C}$ be the quarter-circle path from A to B in Figure 17. Evaluate $I = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ as follows:

(a) Find a function $V(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla V$, where $\mathbf{G} = \langle 0, x \rangle$.

(b) Show that the line integrals of $\mathbf{G}$ along the segments $\overline{OA}$ and $\overline{OB}$ are zero.

(c) Evaluate I. *Hint:* Use Green's Theorem to show that

$$I = V(B) - V(A) + 4\pi$$

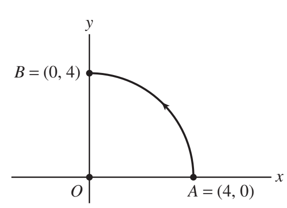


FIGURE 17

SOLUTION

(a) We need to find a potential function $V(x, y)$ for the difference

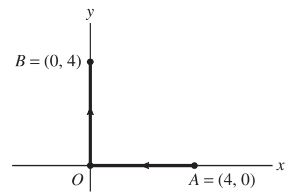
$$\mathbf{F} - \mathbf{G} = \left\langle 2xe^y, x + x^2 e^y \right\rangle - \langle 0, x \rangle = \left\langle 2xe^y, x^2 e^y \right\rangle$$

We let $V(x, y) = x^2 e^y$.

(b) We use the parametrizations $\overline{AO} : \langle t, 0 \rangle, 0 \le t \le 4$ and $\overline{OB} : \langle 0, t \rangle, 0 \le t \le 4$ to evaluate the integrals of $\mathbf{G} = \langle 0, x \rangle$. We get

$$\int_{\overline{OA}} \mathbf{G} \cdot d\mathbf{s} = \int_0^4 \langle 0, t \rangle \cdot \langle 1, 0 \rangle \; dt = \int_0^4 0\,dt = 0$$

$$\int_{\overline{OB}} \mathbf{G} \cdot d\mathbf{s} = \int_0^4 \langle 0, 0 \rangle \cdot \langle 0, 1 \rangle \; dt = \int_0^4 0\,dt = 0$$



(c) Since $\mathbf{F} - \mathbf{G} = \nabla V$, we have

$$\int_{\mathcal{C}} (\mathbf{F} - \mathbf{G}) \cdot d\mathbf{s} = V(B) - V(A) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s} = I - \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s}$$

That is,

$$I = V(B) - V(A) + \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s} \tag{1}$$

To compute the line integral on the right-hand side, we rewrite it as

$$\int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s} = \int_{\overline{BO}+\overline{OA}+C} \mathbf{G} \cdot d\mathbf{s} - \int_{\overline{BO}} \mathbf{G} \cdot d\mathbf{s} - \int_{\overline{OA}} \mathbf{G} \cdot d\mathbf{s}$$

Using part (b) we may write

$$\int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s} = \int_{\overline{BO}+\overline{OA}+C} \mathbf{G} \cdot d\mathbf{s} \tag{2}$$

We now use Green's Theorem. Since $\mathbf{G} = \langle 0, x \rangle$, we have $P = 0$ and $Q = x$, hence $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$. Thus,

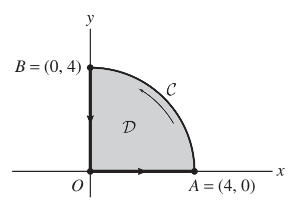
$$\int_{\overline{BO}+\overline{OA}+C} \mathbf{G} \cdot d\mathbf{s} = \iint_{\mathcal{D}} 1\, dA = \text{Area}(\mathcal{D}) = \frac{\pi \cdot 4^2}{4} = 4\pi \tag{3}$$

Combining (1), (2), and (3), we obtain

$$I = V(B) - V(A) + 4\pi$$

Since $V(x, y) = x^2 e^y$, we conclude that

$$I = V(0, 4) - V(4, 0) + 4\pi = 0 - 4^2 e^0 + 4\pi = 4\pi - 16.$$



12. Compute the line integral of $\mathbf{F} = \langle x^3, 4x \rangle$ along the path from A to B in Figure 18. To save work, use Green's Theorem to relate this line integral to the line integral along the vertical path from B to A.

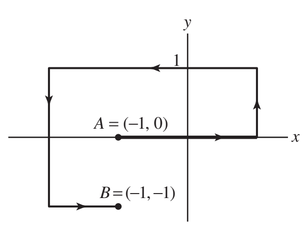
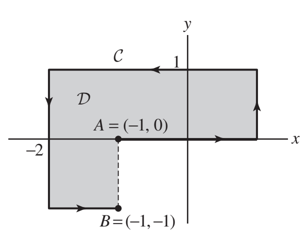


FIGURE 18

SOLUTION We denote by $\mathcal{C}$ the path from A to B, and $\mathcal{D}$ is the region enclosed by $\mathcal{C}$ and the segment $\overline{BA}$.



By Green's Theorem, we have

$$\int_{\mathcal{C}+\overline{BA}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} + \int_{\overline{BA}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

or

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \int_{\overline{AB}} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We compute the integrals on the right-hand side. We parametrize the segment $\overline{AB}$ by $\langle -1, t \rangle$, with t from 0 to -1. We get

$$\int_{\overline{AB}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{-1} \langle -1, -4 \rangle \cdot \langle 0, 1 \rangle \, dt = \int_0^{-1} -4 \, dt = \int_{-1}^{0} 4 \, dt = 4 \tag{2}$$

Since $Q = 4x$ and $P = x^3$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4 - 0 = 4$. Hence,

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} 4 \, dA = 4 \iint_{\mathcal{D}} 1 \, dA = 4\text{Area}(\mathcal{D}) = 4(1 \cdot 3 + 1 \cdot 1) = 16 \tag{3}$$

Substituting (2) and (3) in (1), we get

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 4 + 16 = 20$$

13. Evaluate $I = \int_{\mathcal{C}} (\sin x + y) \, dx + (3x + y) \, dy$ for the nonclosed path $ABCD$ in Figure 19. Use the method of Exercise 12.

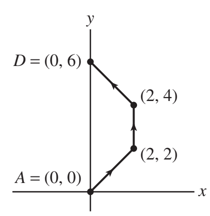
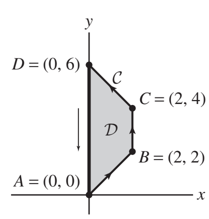


FIGURE 19

SOLUTION

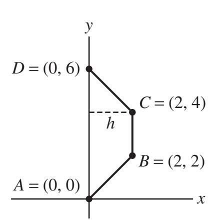


Let $\mathbf{F} = \langle \sin x + y, 3x + y \rangle$, hence $P = \sin x + y$ and $Q = 3x + y$. We denote by $\mathcal{C}_1$ the closed path determined by $\mathcal{C}$ and the segment $\overline{DA}$. Then by Green's Theorem,

$$\int_{\mathcal{C}_1} P \, dx + Q \, dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} (3 - 1) \, dA = 2 \iint_{\mathcal{D}} dA = 2 \,\text{Area}(\mathcal{D}) \tag{1}$$

The area of $\mathcal{D}$ is the area of the trapezoid $ABCD$, that is,

$$\text{Area}(\mathcal{D}) = \frac{\left(\overline{BC} + \overline{AD}\right) h}{2} = \frac{(2+6) \cdot 2}{2} = 8.$$



Combining with (1) we get

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = 2 \cdot 8 = 16$$

Using properties of line integrals, we have

$$\int_{\mathcal{C}} P\,dx + Q\,dy + \int_{\overline{DA}} P\,dx + Q\,dy = 16 \qquad (2)$$

We compute the line integral over $\overline{DA}$, using the parametrization

$$\overline{DA} : x = 0,\ \ y = t, t \text{ varies from 6 to 0.}$$

We get

$$\int_{\overline{DA}} P\,dx + Q\,dy = \int_6^0 F(0,t) \cdot \frac{d}{dt}\langle 0, t\rangle\,dt = \int_6^0 \langle \sin 0 + t, 3 \cdot 0 + t\rangle \cdot \langle 0, 1\rangle\,dt$$
$$= \int_6^0 \langle t, t\rangle \cdot \langle 0, 1\rangle\,dt = \int_6^0 t\,dt = \left.\frac{t^2}{2}\right|_{t=6}^0 = -18$$

We substitute in (2) and solve for the required integral:

$$\int_{\mathcal{C}} P\,dx + Q\,dy - 18 = 16 \quad \text{or} \quad \int_{\mathcal{C}} P\,dx + Q\,dy = 34.$$

14. Show that if $\mathcal{C}$ is a simple closed curve, then

$$\oint_{\mathcal{C}} -y\,dx = \oint_{\mathcal{C}} x\,dy$$

and both integrals are equal to the area enclosed by $\mathcal{C}$.

SOLUTION We show that $\int_{\mathcal{C}} y\,dx + x\,dy = 0$ by showing that the vector field $\mathbf{F} = \langle y, x\rangle$ is conservative. Indeed, since $P = y$ and $Q = x$, we have $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 1$. Therefore, the cross partials are equal and therefore $\mathbf{F}$ is conservative. By the formula for the area enclosed by a simple closed curve, the area enclosed by $\mathcal{C}$ is

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_{\mathcal{C}} x\,dy + \frac{1}{2}\int_{\mathcal{C}} -y\,dx$$

Using the equality obtained above, we have

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy + \frac{1}{2}\int_{\mathcal{C}} x\,dy = \int_{\mathcal{C}} x\,dy = \int_{\mathcal{C}} -y\,dx.$$

In Exercises 15–18, use Eq. (6) to calculate the area of the given region.

15. The circle of radius 3 centered at the origin

SOLUTION By Eq. (6), we have

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx$$

We parametrize the circle by $x = 3\cos\theta$, $y = 3\sin\theta$, hence,

$$x\,dy - y\,dx = 3\cos\theta \cdot 3\cos\theta\,d\theta - 3\sin\theta(-3\sin\theta)\,d\theta = (9\cos^2\theta + 9\sin^2\theta)\,d\theta = 9\,d\theta$$

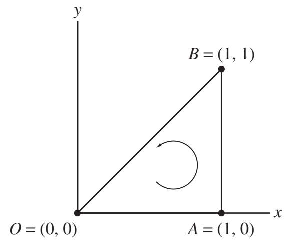
Therefore,

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} 9\,d\theta = \frac{9}{2} \cdot 2\pi = 9\pi.$$

16. The triangle with vertices (0, 0), (1, 0), and (1, 1)

SOLUTION We parametrize the segments by

$$\overline{OA} : \langle t, 0\rangle,\ \ t = 0 \text{ to } t = 1$$
$$\overline{AB} : \langle 1, t\rangle,\ \ t = 0 \text{ to } t = 1$$
$$\overline{BO} : \langle t, t\rangle,\ \ t = 1 \text{ to } t = 0$$



Using Eq. (6), we obtain the following area of the triangle:

$$\begin{aligned} A &= \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_{\overline{OA}} \langle -y, x\rangle \cdot ds + \frac{1}{2}\int_{\overline{AB}} \langle -y, x\rangle \cdot ds + \frac{1}{2}\int_{\overline{BO}} \langle -y, x\rangle \cdot ds \\ &= \frac{1}{2}\int_0^1 \langle 0, t\rangle \cdot \langle 1, 0\rangle\, dt + \frac{1}{2}\int_0^1 \langle -t, 1\rangle \cdot \langle 0, 1\rangle\, dt + \frac{1}{2}\int_1^0 \langle -t, t\rangle \cdot \langle 1, 1\rangle\, dt \\ &= \frac{1}{2}\int_0^1 0\,dt + \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^0 0\,dt = \frac{1}{2}\int_0^1 dt = \frac{1}{2}\cdot 1 = \frac{1}{2} \end{aligned}$$

17. The region between the x-axis and the cycloid parametrized by $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ for $0 \le t \le 2\pi$ (Figure 20)

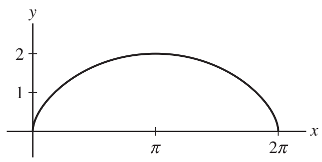
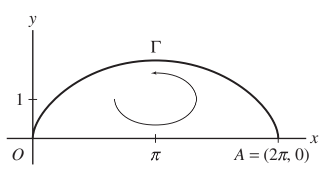


FIGURE 20 Cycloid.

SOLUTION By Eq. (6), the area is the following integral:

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx$$

where $\mathcal{C}$ is the closed curve determined by the segment OA and the cycloid Γ.



Therefore,

$$A = \frac{1}{2}\int_{\overline{OA}} x\,dy - y\,dx + \frac{1}{2}\int_{\Gamma} x\,dy - y\,dx \tag{1}$$

We compute the two integrals. The segment $\overline{OA}$ is parametrized by $\langle t, 0\rangle$, $t = 0$ to $t = 2\pi$. Hence, $x = t$ and $y = 0$. Therefore,

$$x\,dy - y\,dx = t\cdot 0\,dt - 0\cdot\,dt = 0$$

$$\int_{\overline{OA}} x\,dy - y\,dx = 0 \tag{2}$$

On Γ we have $x = t - \sin t$ and $y = 1 - \cos t$, therefore

$$\begin{aligned} x\,dy - y\,dx &= (t - \sin t)\sin t\,dt - (1 - \cos t)(1 - \cos t)\,dt \\ &= (t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t)\,dt = (t\sin t + 2\cos t - 2)\,dt \end{aligned}$$

Hence,

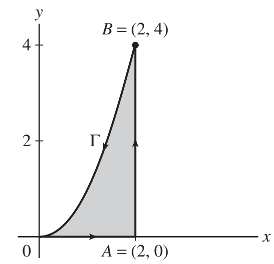
$$\begin{aligned} \int_{\Gamma} x\,dy - y\,dx &= \int_{2\pi}^0 (t\sin t + 2\cos t - 2)\,dt = \int_0^{2\pi} (2 - 2\cos t - t\sin t)\,dt \\ &= 2t - 2\sin t + t\cos t - \sin t\Big|_0^{2\pi} = 2t - 3\sin t + t\cos t\Big|_0^{2\pi} = 6\pi \end{aligned} \tag{3}$$

Substituting (2) and (3) in (1) we get

$$A = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 6\pi = 3\pi.$$

18. The region between the graph of $y = x^2$ and the x-axis for $0 \le x \le 2$

SOLUTION The boundary of the region consists of the curve Γ and the segments $\overline{OA}$ and $\overline{AB}$ shown in the figure.



By Eq. (6), the area A of the region is given by,

$$A = \frac{1}{2} \int_{\overline{OA}} x\,dy - y\,dx + \frac{1}{2} \int_{\overline{AB}} x\,dy - y\,dx + \frac{1}{2} \int_{\Gamma} x\,dy - y\,dx \tag{1}$$

We compute each integral separately. We use the following parametrizations:

$$\overline{OA} : c_1(t) = (t, 0), \text{ for } 0 \le t \le 2 \quad \Rightarrow \quad c_1'(t) = \langle 1, 0 \rangle$$
$$\overline{AB} : c_2(t) = (2, t), \text{ for } 0 \le t \le 4 \quad \Rightarrow \quad c_2'(t) = \langle 0, 1 \rangle$$
$$\Gamma : c_3(t) = (t, t^2) \text{ for } t \text{ from 2 to 0} \quad \Rightarrow \quad c_3'(t) = \langle 1, 2t \rangle$$

The line integrals in (1) are thus

$$\int_{\overline{OA}} x\,dy - y\,dx = \int_{\overline{OA}} \langle -y, x \rangle \cdot ds = \int_0^2 \langle 0, t \rangle \cdot \langle 1, 0 \rangle \, dt = 0$$
$$\int_{\overline{AB}} x\,dy - y\,dx = \int_{\overline{AB}} \langle -y, x \rangle \cdot ds = \int_0^4 \langle -t, 2 \rangle \cdot \langle 0, 1 \rangle \, dt = \int_0^4 2\,dt = 8$$
$$\int_{\Gamma} x\,dy - y\,dx = \int_{\Gamma} \langle -y, x \rangle \cdot ds = \int_2^0 \left\langle -t^2, t \right\rangle \cdot \langle 1, 2t \rangle \, dt = \int_2^0 (-t^2 + 2t^2) \cdot dt$$
$$= \int_0^2 -t^2\,dt = -\frac{t^3}{3}\Big|_0^2 = -\frac{8}{3}$$

Substituting the integrals in (1) we obtain the following area:

$$A = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot \left(-\frac{8}{3}\right) = \frac{8}{3}.$$

19. Let $x^3 + y^3 = 3xy$ be the **folium of Descartes** (Figure 21).

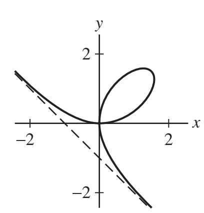


FIGURE 21 Folium of Descartes.

(a) Show that the folium has a parametrization in terms of $t = y/x$ given by

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3} \quad (-\infty < t < \infty) \quad (t \ne -1)$$

(b) Show that

$$x\,dy - y\,dx = \frac{9t^2}{(1+t^3)^2}\,dt$$

Hint: By the Quotient Rule,

$$x^2\,d\left(\frac{y}{x}\right) = x\,dy - y\,dx$$

(c) Find the area of the loop of the folium.

SOLUTION

(a) We show that $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$ satisfy the equation $x^3 + y^3 - 3xy = 0$ of the folium:

$$x^3 + y^3 - 3xy = \left(\frac{3t}{1+t^3}\right)^3 + \left(\frac{3t^2}{1+t^3}\right)^3 - 3\cdot\frac{3t}{1+t^3}\cdot\frac{3t^2}{1+t^3}$$

$$= \frac{27t^3 + 27t^6}{(1+t^3)^3} - \frac{27t^3(1+t^3)}{(1+t^3)^3} = \frac{27t^3\left(1+t^3-(1+t^3)\right)}{(1+t^3)^3} = \frac{0}{(1+t^3)^3} = 0$$

This proves that the curve parametrized by $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$ lies on the folium of Descartes. This parametrization parametrizes the whole folium since the two equations can be solved for t in terms of x and y. That is,

$$\left.\begin{aligned} x &= \frac{3t}{1+t^3} \\ y &= \frac{3t^2}{1+t^3} \end{aligned}\right. \quad \Rightarrow \quad t = \frac{y}{x}$$

A glance at the graph of the folium shows that any line $y = tx$, with slope t, intersects the folium exactly once. Thus, there is a one-to-one relationship between the values of t and the points on the graph.

(b) We differentiate the two sides of $t = \frac{y}{x}$ with respect to t. Using the Quotient Rule gives

$$1 = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

or

$$x\frac{dy}{dt} - y\frac{dx}{dt} = x^2 = \left(\frac{3t}{1+t^3}\right)^2$$

This equality can be written in the form

$$x\,dy - y\,dx = \frac{9t^2}{(1+t^3)^2}\,dt$$

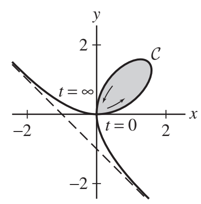
(c) We use the formula for the area enclosed by a closed curve and the result of part (b) to find the required area. That is,

$$A = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_0^{\infty} \frac{9t^2}{(1+t^3)^2}\,dt$$

From our earlier discussion on the parametrization of the folium, we see that the loop is traced when the parameter t is increasing along the interval $0 \le t < \infty$. We compute the improper integral using the substitution $u = 1 + t^3$, $du = 3t^2\,dt$. This gives

$$A = \frac{1}{2}\lim_{R\to\infty}\int_0^R \frac{9t^2}{(1+t^3)^2}\,dt = \frac{1}{2}\lim_{R\to\infty}\int_1^{1+R^3}\frac{3\,du}{u^2} = \frac{3}{2}\lim_{R\to\infty} -\frac{1}{u}\Big|_{u=1}^{1+R^3}$$

$$= \frac{3}{2}\lim_{R\to\infty}\left(1 - \frac{1}{1+R^3}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$$



20. Find a parametrization of the lemniscate $(x^2 + y^2)^2 = xy$ (see Figure 22) by using $t = y/x$ as a parameter (see Exercise 19). Then use Eq. (6) to find the area of one loop of the lemniscate.

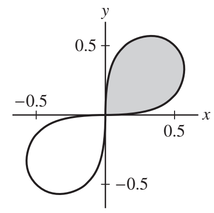


FIGURE 22 Lemniscate.

SOLUTION We first find the parametrization of the lemniscate, determined by the parameter $t = \frac{y}{x}$. We substitute $y = tx$ in the equation of the lemniscate and solve for x in terms of t. This gives

$$(x^2 + y^2)^2 = (x^2 + t^2x^2)^2 = (1 + t^2)^2 x^4$$
$$xy = x \cdot tx = tx^2$$

We obtain the equation

$$(1 + t^2)^2 x^4 = tx^2$$
$$(1 + t^2)^2 x^2 = t$$

or (taking x positive)

$$x = \frac{t^{1/2}}{1 + t^2}$$

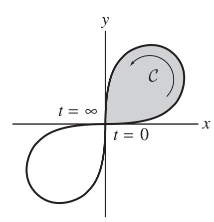
hence

$$y = tx = \frac{t^{3/2}}{1 + t^2}.$$

We obtain the parametrization

$$x = \frac{t^{1/2}}{1 + t^2}, \quad y = \frac{t^{3/2}}{1 + t^2}.$$

One loop is traced as $0 \le t < \infty$.



The area enclosed by one loop is given by the following line integral:

$$A = \frac{1}{2} \int_{\mathcal{C}} x\,dy - y\,dx \tag{1}$$

In Exercise 19 we showed the relation $t = \frac{y}{x}$ implies that

$$x\,dy - y\,dx = x^2\,dt$$

Now $x = \frac{t^{1/2}}{1+t^2}$, hence

$$x\,dy - y\,dx = \frac{t}{(1 + t^2)^2}\,dt$$

We substitute in (1) and compute the resulting improper integral using the substitution $u = 1 + t^2$, $du = 2t\,dt$. We get

$$A = \frac{1}{2} \int_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2} \int_0^{\infty} \frac{t}{(1 + t^2)^2}\,dt = \frac{1}{2} \lim_{R \to \infty} \int_0^R \frac{t\,dt}{(1 + t^2)^2} = \frac{1}{2} \lim_{R \to \infty} \int_1^{1+R^2} \frac{\frac{1}{2}\,du}{u^2}\,du$$
$$= \frac{1}{4} \lim_{R \to \infty} -\frac{1}{u}\bigg|_{u=1}^{1+R^2} = \frac{1}{4} \lim_{R \to \infty} \left(1 - \frac{1}{1 + R^2}\right) = \frac{1}{4}(1 - 0) = \frac{1}{4}$$

21. The Centroid via Boundary Measurements The centroid (see Section 15.5) of a domain $\mathcal{D}$ enclosed by a simple closed curve $\mathcal{C}$ is the point with coordinates $(\overline{x}, \overline{y}) = (M_y/M, M_x/M)$, where M is the area of $\mathcal{D}$ and the moments are defined by

$$M_x = \iint_{\mathcal{D}} y\, dA, \qquad M_y = \iint_{\mathcal{D}} x\, dA$$

Show that $M_x = \oint_{\mathcal{C}} xy\, dy$. Find a similar expression for M_y.

SOLUTION Consider the moment $M_x = \iint_{\mathcal{D}} y\, dA$, we know from Green's Theorem:

$$\iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_{\mathcal{C}} F_1 dx + F_2 dy$$

So then we need

$$\frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y} = y$$

If we set $F_2 = xy$ and $F_1 = 0$, then $\frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y} = y$ and

$$M_x = \iint_{\mathcal{D}} y\, dA = \oint_{\mathcal{C}} xy\, dy$$

Similarly, consider the moment $M_y = \iint_{\mathcal{D}} x\, dA$. We will now use Green's Theorem, stated above. Here we need

$$\frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y} = x$$

If we set $F_1 = -xy$ and $F_2 = 0$ then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x$ and

$$M_y = \iint_{\mathcal{D}} x\, dA = \oint_{\mathcal{C}} -xy\, dx$$

22. Use the result of Exercise 21 to compute the moments of the semicircle $x^2 + y^2 = R^2$, $y \geq 0$ as line integrals. Verify that the centroid is $(0, 4R/(3\pi))$.

SOLUTION Firstly, let us compute M which is the area of the semicircular region, here $\frac{1}{2}\pi R^2$. Then computing M_x and M_y as line integrals we must first parametrize the semicircle from $(1, 0)$ to $(-1, 0)$ as:

$$\mathbf{r}(\theta) = \langle R\cos\theta, R\sin\theta \rangle, \quad 0 \leq \theta \leq \pi$$

and the line segment from $(-1, 0)$ to $(1, 0)$ as:

$$\mathbf{r}(t) = \langle 2t - 1, 0 \rangle, \quad 0 \leq t \leq 1$$

Thus we have

$$\begin{aligned} M_x &= \oint_{\mathcal{C}} xy\, dy \\ &= \int_0^{\pi} (R\cos\theta)(R\sin\theta)(R\cos\theta)\, d\theta + \int_0^1 (2t-1)\cdot(0)\cdot(0)\, dt \\ &= \int_0^{\pi} R^3 \cos^2\theta \sin\theta\, d\theta \\ &= -\frac{R^3\cos^3\theta}{3}\bigg|_0^{\pi} = \frac{R^3}{3} + \frac{R^3}{3} = \frac{2}{3}R^3 \end{aligned}$$

and

$$\begin{aligned} M_y &= \oint_{\mathcal{C}} -xy\, dx \\ &= \int_0^{\pi} -(R\cos\theta)(R\sin\theta)(-R\sin\theta)\, d\theta + \int_0^1 (2t-1)\cdot(0)\cdot(2)\, dt \\ &= \int_0^{\pi} R^3 \cos\theta \sin^2\theta\, d\theta \\ &= \frac{R^3\sin^3\theta}{3}\bigg|_0^{\pi} = 0 \end{aligned}$$

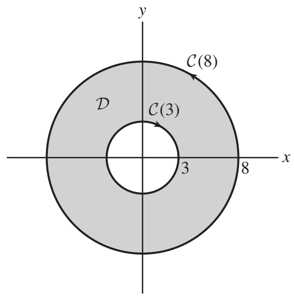
Now to compute the coordinates of the centroid we have:

$$\overline{x} = \frac{M_y}{M} = 0, \quad \overline{y} = \frac{M_x}{M} = \frac{2R^3/3}{\pi R^2/2} = \frac{2R^3}{3} \cdot \frac{2}{\pi R^2} = \frac{4R}{3\pi}$$

Thus the centroid is $(\overline{x}, \overline{y}) = (0, 4R/(3\pi))$.

23. Let $\mathcal{C}_R$ be the circle of radius R centered at the origin. Use the general form of Green's Theorem to determine $\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}$ is a vector field such that $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 9$ and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2$ for (x, y) in the annulus $1 \le x^2 + y^2 \le 4$.

SOLUTION We use Green's Theorem for the annulus $\mathcal{D}$ between the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ oriented as shown in the figure.



That is,

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

Substituting the given information, we get

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} - 9 = \iint_{\mathcal{D}} (x^2 + y^2)\,dx\,dy$$

or

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 9 + \iint_{\mathcal{D}} (x^2 + y^2)\,dx\,dy$$

We compute the double integral by converting it to polar coordinates:

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 9 + \int_0^{2\pi} \int_1^2 r^2 \cdot r\,dr\,d\theta = 9 + 2\pi \int_1^2 r^3\,dr = 9 + 2\pi \cdot \left.\frac{r^4}{4}\right|_1^2 = 9 + 2\pi \left(\frac{2^4 - 1^4}{4} \right) = 9 + \frac{15\pi}{2}$$

24. Referring to Figure 23, suppose that $\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 12$. Use Green's Theorem to determine $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s}$, assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} = -3$ in $\mathcal{D}$.

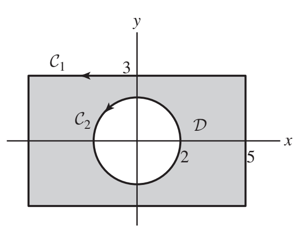


FIGURE 23

SOLUTION By Green's Theorem,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \operatorname{curl}(\mathbf{F})\,dx\,dy$$

Substituting the given information gives

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - 12 = \iint_{\mathcal{D}} -3\,dx\,dy = -3 \iint_{\mathcal{D}} 1\,dx\,dy = -3\,\text{Area}(\mathcal{D})$$

Hence,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - 12 = -3\,\text{Area}(\mathcal{D}) \tag{1}$$

We compute the area of $\mathcal{D}$ as the difference between the area of the rectangle and the area of the inner disk. That is,

$$\text{Area}(\mathcal{D}) = 6 \cdot 10 - \pi \cdot 2^2 = 60 - 4\pi$$

Substituting in (1) we get

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 12 - 3(60 - 4\pi) = 12\pi - 168 \approx -130.3.$$

25. Referring to Figure 24, suppose that

$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 3\pi, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = 4\pi$$

Use Green's Theorem to determine the circulation of $\mathbf{F}$ around $\mathcal{C}_1$, assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} = 9$ on the shaded region.

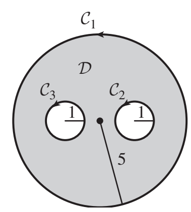


FIGURE 24

SOLUTION We must calculate $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s}$. We use Green's Theorem for the region $\mathcal{D}$ between the three circles $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. Because of orientation, the line integrals $\int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{-\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s}$ must be used in applying Green's Theorem. That is,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \text{curl}(\mathbf{F})\, dA$$

We substitute the given information to obtain

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - 3\pi - 4\pi = \iint_{\mathcal{D}} 9\, dA = 9 \iint_{\mathcal{D}} 1 \cdot dA = 9\,\text{Area}(\mathcal{D}) \tag{1}$$

The area of $\mathcal{D}$ is computed as the difference of areas of discs. That is,

$$\text{Area}(\mathcal{D}) = \pi \cdot 5^2 - \pi \cdot 1^2 - \pi \cdot 1^2 = 23\pi$$

We substitute in (1) and compute the desired circulation:

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - 7\pi = 9 \cdot 23\pi$$

or

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 214\pi.$$

26. Let $\mathbf{F}$ be the vortex vector field

$$\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

In Section 16.3 we verified that $\int_{\mathcal{C}_R} \mathbf{F} \cdot d\mathbf{s} = 2\pi$, where $\mathcal{C}_R$ is the circle of radius R centered at the origin. Prove that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 2\pi$ for any simple closed curve $\mathcal{C}$ whose interior contains the origin (Figure 25). *Hint:* Apply the general form of Green's Theorem to the domain between $\mathcal{C}$ and $\mathcal{C}_R$, where R is so small that $\mathcal{C}_R$ is contained in $\mathcal{C}$.

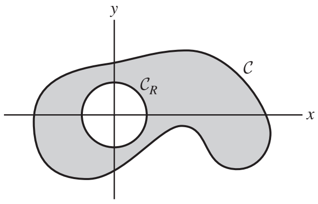
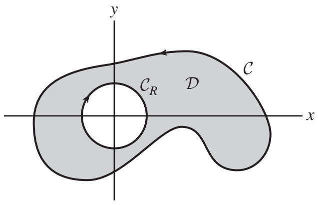


FIGURE 25

SOLUTION Let $R > 0$ be sufficiently small so that the circle $\mathcal{C}_R$ is contained in $\mathcal{C}$.



Let $\mathcal{D}$ denote the region between $\mathcal{C}_R$ and $\mathcal{C}$. We apply Green's Theorem to the region $\mathcal{D}$. The curve $\mathcal{C}$ is oriented counterclockwise and $\mathcal{C}_R$ is oriented clockwise. We have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_R} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \text{curl}(\mathbf{F})\, dA \tag{1}$$

From Exercise 16.3.27 we know that $\int_{\mathcal{C}_R} \mathbf{F} \cdot d\mathbf{s} = -2\pi$. Since $\mathcal{D}$ does not contain the origin, we have by part (a) of Exercise 16.3.28, curl $(\mathbf{F}) = 0$ on $\mathcal{D}$. Substituting in (1) we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} - 2\pi = \iint_{\mathcal{D}} 0\, dA = 0$$

or

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 2\pi.$$

In Exercises 27–30, refer to the Conceptual Insight that discusses the curl, defined by

$$\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

27. For the vector fields (A)–(D) in Figure 26, state whether the curl_z at the origin appears to be positive, negative, or zero.

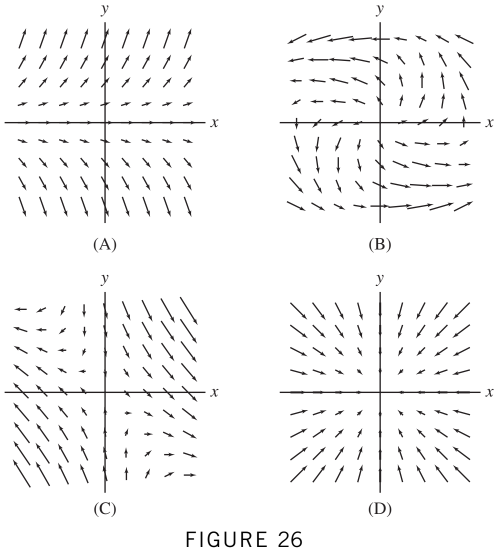


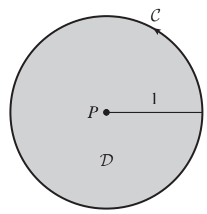
FIGURE 26

SOLUTION The vector field (A) does not have spirals, nor is it a "shear flow." Therefore, the curl appears to be zero. The vector field (B) rotates in the counterclockwise direction, hence we expect the curl to be positive. The vector field (C) rotates in a clockwise direction about the origin—we expect the curl to be negative. Finally, in the vector field (D), the fluid flows straight toward the origin without spiraling. We expect the curl to be zero.

28. Estimate the circulation of a vector field $\mathbf{F}$ around a circle of radius $R = 0.1$, assuming that $\text{curl}_z(\mathbf{F})$ takes the value 4 at the center of the circle.

SOLUTION We estimate the circulation by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx \text{curl}(\mathbf{F})(P)\text{Area}(\mathcal{D}) \qquad (1)$$



We are given that $\text{curl}(\mathbf{F})(P) = 4$. The area of the disk of radius $R = 0.1$ is $\pi \cdot 0.1^2 = \pi/100$. Substituting in (1) gives the estimation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx 4 \cdot \frac{\pi}{100} = \frac{\pi}{25}$$

29. Estimate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \left\langle x + 0.1y^2, y - 0.1x^2 \right\rangle$ and $\mathcal{C}$ encloses a small region of area 0.25 containing the point $P = (1, 1)$.

SOLUTION Use the following estimation:

$$\mathbf{F} \cdot d\mathbf{s} = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy \approx \text{curl}_z(\mathbf{F})(P) \cdot \text{Area}(\mathcal{D})$$

First computing $\text{curl}\mathbf{F}$ we have:

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x & \partial y & \partial z \\ x + 0.1y^2 & y - 0.1x^2 & 0 \end{vmatrix} = \langle 0, 0, -0.2x - 0.2y \rangle$$

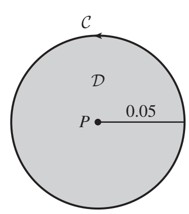
Thus $\text{curl}_z(\mathbf{F}) = -0.2x - 0.2y$ and $\text{curl}_z(\mathbf{F})(1, 1) = -0.2 - 0.2 = -0.4$. Also, we are given the area of the region is 0.25. Hence, we see:

$$\oint \mathbf{F} \cdot d\mathbf{s} \approx (-0.4)(0.25) = -0.10$$

30. Let $\mathbf{F}$ be the velocity field. Estimate the circulation of $\mathbf{F}$ around a circle of radius $R = 0.05$ with center P, assuming that $\text{curl}_z(\mathbf{F})(P) = -3$. In which direction would a small paddle placed at P rotate? How fast would it rotate (in radians per second) if $\mathbf{F}$ is expressed in meters per second?

SOLUTION We use the following estimation:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx \text{curl}(\mathbf{F})(P)\text{Area}(\mathcal{D}) \qquad (1)$$



We are given that $\text{curl}(\mathbf{F})(P) = -3$. Also, the area of the disk of radius $R = 0.05$ is $\pi \cdot 0.05^2 = 0.0025\pi$. Therefore, we obtain the following estimation:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx -3 \cdot 0.0025\pi \approx -0.024.$$

Since the curl is negative, the paddle would rotate in the clockwise direction. Using the formula $|\text{curl}(\mathbf{F})| = 2\omega$, we see that the angular speed is $\omega = 1.5$ radians per second.

31. Let $\mathcal{C}_R$ be the circle of radius R centered at the origin. Use Green's Theorem to find the value of R that maximizes $\oint_{\mathcal{C}_R} y^3\,dx + x\,dy$.

SOLUTION Using Green's Theorem we can write:

$$\oint_{\mathcal{C}_R} y^3\,dx + x\,dy = \iint_{\mathcal{D}} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$$
$$= \iint_{\mathcal{D}} 1 - 3y^2\,dA$$

Then we have the following, using polar coordinates:

$$\begin{aligned}
\oint_{\mathcal{C}_R} y^3\,dx + x\,dy &= \iint_{\mathcal{D}} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA \\
&= \iint_{\mathcal{D}} 1 - 3y^2\,dA \\
&= \int_0^{2\pi}\int_0^R (1 - 3r^2\sin^2\theta)(r)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^R r - 3r^3\sin^2\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{3}{4}r^4\sin^2\theta\Big|_0^R\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}R^2 - \frac{3}{8}R^4(1 - \cos 2\theta)\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{3R^4}{8}(\cos 2\theta) + \frac{1}{2}R^2 - \frac{3}{8}R^4\,d\theta \\
&= \frac{3R^4}{16}\sin 2\theta + \left(\frac{1}{2}R^2 - \frac{3}{8}R^4\right)\theta\Big|_0^{2\pi} \\
&= 0 + 2\pi\left(\frac{1}{2}R^2 - \frac{3}{8}R^4\right) = \pi\left(R^2 - \frac{3}{4}R^4\right)
\end{aligned}$$

Now to maximize this quantity, we need to let $f(R) = \pi(R^2 - 3/4R^4)$ and take the first derivative.

$$f'(R) = \pi(2R - 3R^3) = 0 \quad \Rightarrow R = 0, \pm\sqrt{2/3}$$

This quantity is maximized when $R = \pm\sqrt{\frac{2}{3}}$ (that is, $R = 0$ is a minimum).

32. Area of a Polygon Green's Theorem leads to a convenient formula for the area of a polygon.

(a) Let $\mathcal{C}$ be the line segment joining (x_1, y_1) to (x_2, y_2). Show that

$$\frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy = \frac{1}{2}(x_1y_2 - x_2y_1)$$

(b) Prove that the area of the polygon with vertices $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is equal [where we set $(x_{n+1}, y_{n+1}) = (x_1, y_1)$] to

$$\frac{1}{2}\sum_{i=1}^{n}(x_iy_{i+1} - x_{i+1}y_i)$$

SOLUTION

(a) We parametrize the segment from (x_1, y_1) to (x_2, y_2) by

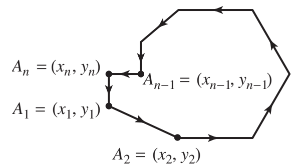
$$x = tx_2 + (1-t)x_1, \quad y = ty_2 + (1-t)y_1, \quad 0 \le t \le 1$$

Then, $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$. Therefore,

$$\begin{aligned}
-y\,dx + x\,dy &= (-ty_2 - (1-t)y_1)(x_2 - x_1)\,dt + (tx_2 + (1-t)x_1)(y_2 - y_1)\,dt \\
&= (-ty_2x_2 + ty_2x_1 - y_1x_2(1-t) + (1-t)y_1x_1 + tx_2y_2 - tx_2y_1 + (1-t)x_1y_2 - (1-t)x_1y_1)\,dt \\
&= (x_1y_2 - x_2y_1)\,dt
\end{aligned}$$

We obtain the following integral:

$$\frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy = \frac{1}{2}\int_0^1 (x_1y_2 - x_2y_1)\,dt = \frac{1}{2}(x_1y_2 - x_2y_1)t\Big|_{t=0}^{1} = \frac{1}{2}(x_1y_2 - x_2y_1)$$



(b) Let $A_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$, and let $\mathcal{C}$ be the closed curve determined by the polygon. By the formula for the area enclosed by a simple closed curve, the area of the polygon is

$$A = \frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy$$

We use additivity of line integrals and the result in part (a) to write the integral as follows:

$$\begin{aligned} A &= \frac{1}{2}\left(\sum_{i=1}^{n-1}\int_{\overline{A_iA_{i+1}}} -y\,dx + x\,dy + \int_{\overline{A_nA_1}} -y\,dx + x\,dy\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n-1}(x_iy_{i+1} - x_{i+1}y_i) + (x_ny_1 - x_1y_n)\right) \\ &= \frac{1}{2}\sum_{i=1}^{n-1}(x_iy_{i+1} - x_{i+1}y_i) + \frac{1}{2}(x_ny_1 - x_1y_n) \end{aligned}$$

If we define $(x_{n+1}, y_{n+1}) = (x_1, y_1)$, we obtain the sum

$$A = \frac{1}{2}\sum_{i=1}^{n}(x_iy_{i+1} - x_{i+1}y_i).$$

33. Use the result of Exercise 32 to compute the areas of the polygons in Figure 27. Check your result for the area of the triangle in (A) using geometry.

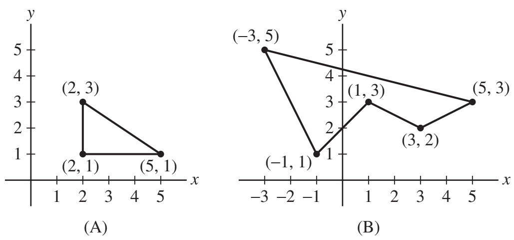
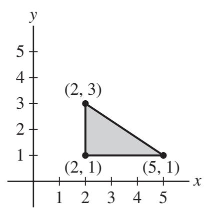


FIGURE 27

SOLUTION

(a) The vertices of the triangle are

$$(x_1, y_1) = (x_4, y_4) = (2, 1), \quad (x_2, y_2) = (5, 1), \quad (x_3, y_3) = (2, 3)$$



Using the formula obtained in Exercise 28, the area of the triangle is the following sum:

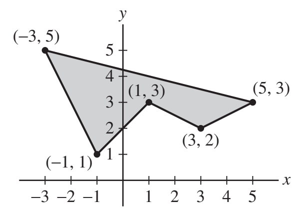
$$\begin{aligned} A &= \frac{1}{2}\big((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\big) \\ &= \frac{1}{2}\big((2\cdot 1 - 5\cdot 1) + (5\cdot 3 - 2\cdot 1) + (2\cdot 1 - 2\cdot 3)\big) = \frac{1}{2}(-3 + 13 - 4) = 3 \end{aligned}$$

We verify our result using the formula for the area of a triangle:

$$A = \frac{1}{2}bh = \frac{1}{2} \cdot (5-2) \cdot (3-1) = 3$$

(b) The vertices of the polygon are

$$(x_1, y_1) = (x_6, y_6) = (-1, 1)$$
$$(x_2, y_2) = (1, 3)$$
$$(x_3, y_3) = (3, 2)$$
$$(x_4, y_4) = (5, 3)$$
$$(x_5, y_5) = (-3, 5)$$



Using the formula in part (a), the area of the polygon is the following sum:

$$\begin{aligned} A &= \frac{1}{2}\big((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5)\big) \\ &= \frac{1}{2}\Big((-1\cdot 3 - 1\cdot 1) + (1\cdot 2 - 3\cdot 3) + (3\cdot 3 - 5\cdot 2) + \big(5\cdot 5 - (-3)\cdot 3\big) + \big(-3\cdot 1 - (-1)\cdot 5\big)\Big) \\ &= \frac{1}{2}(-4 - 7 - 1 + 34 + 2) = 12 \end{aligned}$$

Exercises 34–39: In Section 16.2, we defined the flux of $\mathbf{F}$ *across a curve* $\mathcal{C}$ *(Figure 28) as the integral of the normal component of* $\mathbf{F}$ *along* $\mathcal{C}$*, and we showed that if* $\mathbf{c}(t) = (x(t), y(t))$ *is a parametrization of* $\mathcal{C}$ *for* $a \le t \le b$*, then the flux is equal to*

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{n}(t)\, dt$$

where $\mathbf{n}(t) = \langle y'(t), -x'(t)\rangle$.

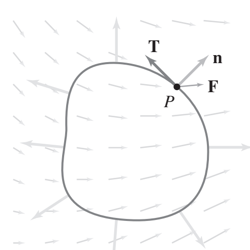


FIGURE 28 The flux of $\mathbf{F}$ is the integral of the normal component $\mathbf{F}\cdot\mathbf{n}$ around the curve.

34. Show that the flux of $\mathbf{F} = \langle P, Q\rangle$ across $\mathcal{C}$ is equal to $\oint_{\mathcal{C}} P\,dy - Q\,dx$.

SOLUTION First, let $\mathbf{F}^* = \langle Q, -P\rangle$. It is easy to show that $\mathbf{F}^*$ is a rotation of $\mathbf{F}$ by $\pi/2$ counterclockwise. The two are easily shown to be orthogonal since:

$$\mathbf{F}\cdot\mathbf{F}^* = \langle P, Q\rangle \cdot \langle Q, -P\rangle = 0$$

We must show that if $\mathcal{C}$ is a simple closed curve, then

$$\int_{\mathcal{C}} (\mathbf{F}\cdot\mathbf{n})d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F}^* \cdot d\mathbf{s}$$

By the definition of the vector line integral, the line integral of the vector field $\mathbf{F}^*$ over $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{F}^* \cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F}^* \cdot \mathbf{T}) d\mathbf{s} \tag{1}$$

Since $\mathbf{F}^*$ is a rotation of $\mathbf{F}$ by $\frac{\pi}{2}$ counterclockwise, the angle between $\mathbf{F}^*$ and the tangent $\mathbf{T}$ is equal to the angle between $\mathbf{F}$ and the normal $\mathbf{n}$. Also, $\|\mathbf{F}^*\| = \|\mathbf{F}\|$. Hence,

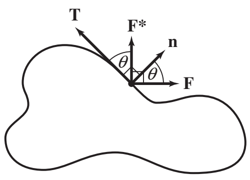
$$\mathbf{F}^* \cdot \mathbf{T} = \|\mathbf{F}^*\| \, \|\mathbf{T}\| \cos\theta = \|\mathbf{F}\| \cos\theta$$
$$\mathbf{F} \cdot \mathbf{n} = \|\mathbf{F}\| \|\mathbf{n}\| \cos\theta = \|\mathbf{F}\| \cos\theta$$

Since the dot products are equal, we conclude that

$$\int_{\mathcal{C}} (\mathbf{F}^* \cdot \mathbf{T}) d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) d\mathbf{s} \tag{2}$$

Combining (1) and (2) we obtain

$$\int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F}^* \cdot d\mathbf{s} = \int_{\mathcal{C}} \langle Q, -P \rangle \cdot \langle dx, dy \rangle = \int_{\mathcal{C}} Q\, dx - P\, dy$$



35. Define $\text{div}(\mathbf{F}) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Use Green's Theorem to prove that for any simple closed curve $\mathcal{C}$,

$$\text{Flux across } \mathcal{C} = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\, dA \tag{12}$$

where $\mathcal{D}$ is the region enclosed by $\mathcal{C}$. This is a two-dimensional version of the **Divergence Theorem** discussed in Section 17.3.

SOLUTION Since $\mathbf{F} = \langle P, Q \rangle$ and $\mathbf{F}^* = \langle -Q, P \rangle$, we have

$$\text{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$
$$\text{curl}(\mathbf{F}^*) = \frac{\partial P}{\partial x} - \frac{\partial}{\partial y}(-Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Therefore,

$$\text{div}(\mathbf{F}) = \text{curl}(\mathbf{F}^*) \tag{1}$$

Using Exercise 33, the flux of $\mathbf{F}$ across $\mathcal{C}$ is

$$\text{flux of } \mathbf{F} \text{ across } \mathcal{C} = \int_{\mathcal{C}} \mathbf{F}^* \cdot d\mathbf{s} \tag{2}$$

Green's Theorem and (1) imply that

$$\int_{\mathcal{C}} \mathbf{F}^* \cdot d\mathbf{s} = \iint_{\mathcal{D}} \text{curl}(\mathbf{F}^*)\, dA = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\, dA \tag{3}$$

Combining (2) and (3) we have

$$\text{flux of } \mathbf{F} \text{ across } \mathcal{C} = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\, dA.$$

36. Use Eq. (12) to compute the flux of $\mathbf{F} = \left\langle 2x + y^3, 3y - x^4 \right\rangle$ across the unit circle.

SOLUTION Using the result:

$$\text{flux} = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\, dA$$

we can compute the divergence:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2 + 3$$

Therefore,

$$\text{flux} = \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F})\, dA = \iint_{\mathcal{D}} (2+3)\, dA = 5(\text{Area of the Region}) = 5\pi$$

37. Use Eq. (12) to compute the flux of $\mathbf{F} = \langle \cos y, \sin y \rangle$ across the square $0 \le x \le 2$, $0 \le y \le \frac{\pi}{2}$.

SOLUTION Using the result:

$$\text{flux} = \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F})\, dA$$

we can compute the divergence:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 + \cos y$$

Therefore,

$$\begin{aligned}
\text{flux} &= \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F})\, dA \\
&= \iint_{\mathcal{D}} (0 + \cos y)\, dA \\
&= \int_0^2 \int_0^{\pi/2} \cos y\, dy\, dx \\
&= (2-0)\left(\sin y \Big|_0^{\pi/2} \right) = 2
\end{aligned}$$

38. If $\mathbf{v}$ is the velocity field of a fluid, the flux of $\mathbf{v}$ across $\mathcal{C}$ is equal to the flow rate (amount of fluid flowing across $\mathcal{C}$ in m^2/s). Find the flow rate across the circle of radius 2 centered at the origin if $\operatorname{div}(\mathbf{v}) = x^2$.

SOLUTION Using the result:

$$\text{flux} = \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F})\, dA$$

we have, using polar coordinates:

$$\begin{aligned}
\text{flux} &= \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F})\, dA \\
&= \iint_{\mathcal{D}} x^2\, dA \\
&= \int_0^{2\pi} \int_0^2 (r\cos\theta)^2 (r)\, dr\, d\theta \\
&= \int_0^2 \int_0^{2\pi} r^3 \cos^2\theta\, dr\, d\theta \\
&= \int_0^2 r^3\, dr \cdot \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)\, d\theta \\
&= \frac{r^4}{4}\Big|_0^2 \cdot \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} \\
&= (4) \cdot \frac{1}{2}(2\pi) = 4\pi
\end{aligned}$$

39. A buffalo (Figure 29) stampede is described by a velocity vector field $\mathbf{F} = \left\langle xy - y^3, x^2 + y \right\rangle$ km/h in the region $\mathcal{D}$ defined by $2 \le x \le 3$, $2 \le y \le 3$ in units of kilometers (Figure 30). Assuming a density is $\rho = 500$ buffalo per square kilometer, use Eq. (12) to determine the net number of buffalo leaving or entering $\mathcal{D}$ per minute (equal to ρ times the flux of $\mathbf{F}$ across the boundary of $\mathcal{D}$).

FIGURE 29 Buffalo stampede.

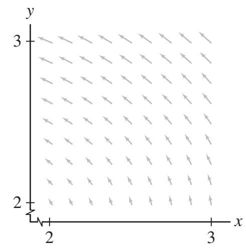


FIGURE 30 The vector field $\mathbf{F} = \left\langle xy - y^3, x^2 + y \right\rangle$.

SOLUTION The flux of $\mathbf{F}$ across the boundary ∂D has units of area per second. We multiply the buffalo density to obtain the number of buffalo per second crossing the boundary. Using Green's Theorem:

$$\begin{aligned}
\text{flux of buffalo} &= \int_{\partial D} 500\mathbf{F}\, d\mathbf{s} \\
&= 500 \int_{\partial D} \left\langle xy - y^3, x^2 + y \right\rangle d\mathbf{s} \\
&= 500 \int_2^3 \int_2^3 \operatorname{div}(\mathbf{F})\, dy\, dx \\
&= 500 \int_2^3 \int_2^3 (y+1)\, dy\, dx \\
&= 500 \int_2^3 dx \cdot \int_2^3 (y+1)\, dy \\
&= 500(3-2)\left(\frac{1}{2}y^2 + y\right)\Big|_2^3 \\
&= 500(1)\left(\frac{9}{2} + 3 - 2 - 2\right) \\
&= 500(3.5) \\
&= 1750 \text{ buffalos per second}
\end{aligned}$$

Further Insights and Challenges

In Exercises 40–43, the **Laplace operator** Δ *is defined by*

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \qquad \boxed{13}$$

For any vector field $\mathbf{F} = \langle F_1, F_2\rangle$*, define the conjugate vector field* $\mathbf{F}^* = \langle -F_2, F_1\rangle$.

40. Show that if $\mathbf{F} = \nabla\varphi$, then $\operatorname{curl}_z(\mathbf{F}^*) = \Delta\varphi$.

SOLUTION For a vector field $\mathbf{F} = \langle P, Q\rangle$, the conjugate vector field is $\mathbf{F}^* = \langle -Q, P\rangle$. By the given information,

$$\mathbf{F} = \nabla\varphi = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \right\rangle \quad \Rightarrow \quad \mathbf{F}^* = \left\langle -\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x} \right\rangle$$

We compute the curl of $\mathbf{F}^*$:

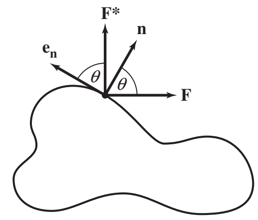
$$\operatorname{curl}(\mathbf{F}^*) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{-\partial\varphi}{\partial y}\right) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \Delta\varphi$$

41. Let $\mathbf{n}$ be the outward-pointing unit normal vector to a simple closed curve $\mathcal{C}$. The **normal derivative** of a function φ, denoted $\frac{\partial\varphi}{\partial\mathbf{n}}$, is the directional derivative $D_{\mathbf{n}}(\varphi) = \nabla\varphi \cdot \mathbf{n}$. Prove that

$$\oint_{\mathcal{C}} \frac{\partial\varphi}{\partial\mathbf{n}}\, ds = \iint_{\mathcal{D}} \Delta\varphi\, dA$$

where $\mathcal{D}$ is the domain enclosed by a simple closed curve $\mathcal{C}$. *Hint:* Let $\mathbf{F} = \nabla\varphi$. Show that $\frac{\partial\varphi}{\partial\mathbf{n}} = \mathbf{F}^* \cdot \mathbf{T}$ where $\mathbf{T}$ is the unit tangent vector, and apply Green's Theorem.

SOLUTION In Exercise 34 we showed that for any vector field $\mathbf{F}$, $\mathbf{F}^*$ is a rotation of $\mathbf{F}$ by $\frac{\pi}{2}$ counterclockwise. The unit tangent $\mathbf{e}_n$ is a rotation of $\mathbf{n}$ by $\frac{\pi}{2}$ counterclockwise.



These properties imply that the angle θ between $\mathbf{F}$ and $\mathbf{n}$ is equal to the angle between $\mathbf{F}^*$ and $\mathbf{e}_n$, and $\|\mathbf{F}\| = \left\|\mathbf{F}^*\right\|$. Therefore,

$$\begin{aligned} \mathbf{F}\cdot\mathbf{n} &= \|\mathbf{F}\|\|\mathbf{n}\|\cos\theta = \|\mathbf{F}\|\cos\theta \\ \mathbf{F}^*\cdot\mathbf{e}_n &= \left\|\mathbf{F}^*\right\|\|\mathbf{e}_n\|\cos\theta = \|\mathbf{F}\|\cos\theta \end{aligned} \quad \Rightarrow \quad \mathbf{F}\cdot\mathbf{n} = \mathbf{F}^*\cdot\mathbf{e}_n$$

Now, if $\mathbf{F} = \nabla\varphi$, then

$$\frac{\partial\varphi}{\partial\mathbf{n}} = \nabla\varphi\cdot\mathbf{n} = \mathbf{F}\cdot\mathbf{n} = \mathbf{F}^*\cdot\mathbf{e}_n$$

By the definition of the vector line integral $\int_{\mathcal{C}} \mathbf{F}^*\cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F}^*\cdot\mathbf{e}_n)\, ds$. Therefore,

$$\int_{\mathcal{C}} \frac{\partial\varphi}{\partial\mathbf{n}}\, ds = \int_{\mathcal{C}} (\mathbf{F}^*\cdot\mathbf{e}_n)\, ds = \int_{\mathcal{C}} \mathbf{F}^*\cdot d\mathbf{s}$$

Using Green's Theorem and the equality $\operatorname{curl}(\mathbf{F}^*) = \Delta\varphi$ obtained in Exercise 40, we get

$$\int_{\mathcal{C}} \frac{\partial\varphi}{\partial\mathbf{n}}\, ds = \int_{\mathcal{C}} \mathbf{F}^*\cdot d\mathbf{s} = \iint_{\mathcal{D}} \operatorname{curl}(\mathbf{F}^*)\, dA = \iint_{\mathcal{D}} \Delta\varphi\, dA.$$

42. Let $P = (a, b)$ and let $\mathcal{C}_r$ be the circle of radius r centered at P. The average value of a continuous function φ on $\mathcal{C}_r$ is defined as the integral

$$I_\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + r\cos\theta, b + r\sin\theta)\, d\theta$$

(a) Show that

$$\frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta) = \frac{\partial\varphi}{\partial r}(a + r\cos\theta, b + r\sin\theta)$$

(b) Use differentiation under the integral sign to prove that

$$\frac{d}{dr} I_\varphi(r) = \frac{1}{2\pi r}\int_{\mathcal{C}_r} \frac{\partial\varphi}{\partial\mathbf{n}}\, ds$$

(c) Use Exercise 41 to conclude that

$$\frac{d}{dr} I_\varphi(r) = \frac{1}{2\pi r}\iint_{\mathcal{D}(r)} \Delta\varphi\, dA$$

where $\mathcal{D}(r)$ is the interior of $\mathcal{C}_r$.

SOLUTION In this solution, $\varphi_r(a + r\cos\theta, b + r\sin\theta)$ denotes the partial derivative φ_r computed at $(a + r\cos\theta, b + r\sin\theta)$, whereas $\frac{\partial}{\partial r}\varphi(a + r\cos\theta, b + r\sin\theta)$ is the derivative of the composite function.

(a) Since $\frac{\partial\varphi}{\partial n} = \nabla\varphi\cdot\mathbf{n}$, we first express the gradient vector in terms of polar coordinates. We use the Chain Rule and the derivatives:

$$\theta_x = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}, \quad r_x = \cos\theta, \quad r_y = \sin\theta$$

We get

$$\begin{aligned} \varphi_x &= \varphi_r r_x + \varphi_\theta\theta_x = \varphi_r\cos\theta + \varphi_\theta\left(-\frac{\sin\theta}{r}\right) \\ \varphi_y &= \varphi_r r_y + \varphi_\theta\theta_y = \varphi_r\sin\theta + \varphi_\theta\left(\frac{\cos\theta}{r}\right) \end{aligned} \tag{1}$$

Hence,

$$\nabla\varphi = \left\langle \varphi_r \cos\theta - \varphi_\theta \frac{\sin\theta}{r}, \varphi_r \sin\theta + \varphi_\theta \frac{\cos\theta}{r} \right\rangle$$

We use the following parametrization for $\mathcal{C}_r$:

$$\mathcal{C}_r : \mathbf{c}(\theta) = \langle a + r\cos\theta,\ b + r\sin\theta\rangle, \quad 0 \le \theta \le 2\pi$$

The unit normal vector is

$$\mathbf{n} = \langle \cos\theta, \sin\theta\rangle.$$

We compute the dot product:

$$\begin{aligned}\frac{\partial\varphi}{\partial\mathbf{n}} &= \nabla\varphi \cdot \mathbf{n} = \left\langle \varphi_r \cos\theta - \varphi_\theta \frac{\sin\theta}{r}, \varphi_r \sin\theta + \varphi_\theta \frac{\cos\theta}{r} \right\rangle \cdot \langle \cos\theta, \sin\theta\rangle \\ &= \varphi_r \cos^2\theta - \varphi_\theta \frac{\sin\theta\cos\theta}{r} + \varphi_r \sin^2\theta + \varphi_\theta \frac{\cos\theta\sin\theta}{r} = \varphi_r\left(\cos^2\theta + \sin^2\theta\right) = \varphi_r\end{aligned}$$

That is,

$$\frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta) = \varphi_r(a + r\cos\theta, b + r\sin\theta) \tag{2}$$

(b) We compute the following derivative using the Chain Rule and (1):

$$\begin{aligned}\frac{\partial}{\partial r}\varphi(a + r\cos\theta, b + r\sin\theta) &= \varphi_x \frac{\partial}{\partial r}(a + r\cos\theta) + \varphi_y \frac{\partial}{\partial r}(b + r\sin\theta) \\ &= \left(\varphi_r\cos\theta - \varphi_\theta\frac{\sin\theta}{r}\right)\cos\theta + \left(\varphi_r\sin\theta + \varphi_\theta\frac{\cos\theta}{r}\right)\sin\theta \\ &= \varphi_r\cos^2\theta - \varphi_\theta\frac{\sin\theta\cos\theta}{r} + \varphi_r\sin^2\theta + \varphi_\theta\frac{\cos\theta\sin\theta}{r} \\ &= \varphi_r(a + r\cos\theta, b + r\sin\theta)\end{aligned} \tag{3}$$

We now differentiate $I_\varphi(r)$ under the integral sign, and use (3) and (2) to obtain

$$\begin{aligned}\frac{d}{dr}I_\varphi(r) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}\varphi(a + r\cos\theta, b + r\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \varphi_r(a + r\cos\theta, b + r\sin\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta)\,d\theta\end{aligned} \tag{4}$$

On the other hand, since $\mathbf{c}'(\theta) = \langle -r\sin\theta, r\cos\theta\rangle$, we have

$$\begin{aligned}\int_{\mathcal{C}_r} \frac{\partial\varphi}{\partial\mathbf{n}}\,ds &= \int_0^{2\pi} \frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta)\left\|\mathbf{c}'(\theta)\right\|\,d\theta = \int_0^{2\pi} \frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta) r\,d\theta \\ &= r\int_0^{2\pi} \frac{\partial\varphi}{\partial\mathbf{n}}(a + r\cos\theta, b + r\sin\theta)\,d\theta\end{aligned} \tag{5}$$

Combining (4) and (5) we get

$$\frac{d}{dr}I_\varphi(r) = \frac{1}{2\pi r}\int_{\mathcal{C}_r} \frac{\partial\varphi}{\partial\mathbf{n}}\,ds.$$

(c) We combine the result of part (b) and Exercise 36 to conclude

$$\frac{d}{dr}I_\varphi(r) = \frac{1}{2\pi r}\int_{\mathcal{C}_r} \frac{\partial\varphi}{\partial\mathbf{n}}\,ds = \frac{1}{2\pi r}\iint_{\mathcal{D}(r)} \Delta\varphi\,dA.$$

43. Prove that $m(r) \le I_\varphi(r) \le M(r)$, where $m(r)$ and $M(r)$ are the minimum and maximum values of φ on $\mathcal{C}_r$. Then use the continuity of φ to prove that $\lim_{r\to 0} I_\varphi(r) = \varphi(P)$.

SOLUTION $I_\varphi(r)$ is defined by

$$I_\varphi(r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + r\cos\theta, b + r\sin\theta)\,d\theta$$

The points on $\mathcal{C}_r$ have the form $(a + r\cos\theta, b + r\sin\theta)$. Therefore, since $m(r)$ and $M(r)$ are the minimum and maximum values of φ on $\mathcal{C}_r$, we have for all $0 \le \theta \le 2\pi$,

$$m(r) \le \varphi(a + r\cos\theta, b + r\sin\theta) \le M(r)$$

Using properties of integrals (Eq. (6) in Section 5.2), we conclude that

$$2\pi m(r) \le \int_0^{2\pi} \varphi(a + r\cos\theta + b + r\sin\theta) \le 2\pi M(r)$$

Dividing by 2π we obtain

$$m(r) \le I_\varphi(r) \le M(r) \tag{1}$$

Now, since φ is continuous and the functions $\sin\theta$ and $\cos\theta$ are bounded for all $0 \le \theta \le 2\pi$, the following holds:

$$\lim_{r\to 0} \varphi(a + r\cos\theta, b + r\sin\theta) = \varphi\left(\lim_{r\to 0}(a + r\cos\theta, b + r\sin\theta)\right) = \varphi(a, b)$$

which means that for $\epsilon > 0$ there exists $\delta > 0$ so that

$$|\varphi(a + r\cos\theta, b + r\sin\theta) - \varphi(a, b)| < \epsilon$$

for all $0 \le \theta \le 2\pi$, whenever $0 < r < \delta$. Hence also

$$\lim_{r\to 0} m(r) = \lim_{r\to 0} M(r) = \varphi(a, b) \tag{2}$$

Combining (1), (2), and the Squeeze Theorem, we obtain the following conclusion:

$$\lim_{r\to 0} I_\varphi(r) = \varphi(a, b).$$

In Exercises 44 and 45, let $\mathcal{D}$ be the region bounded by a simple closed curve $\mathcal{C}$. A function $\varphi(x, y)$ on $\mathcal{D}$ (whose second-order partial derivatives exist and are continuous) is called **harmonic** *if $\Delta\varphi = 0$, where $\Delta\varphi$ is the Laplace operator defined in Eq. (13).*

44. Use the results of Exercises 42 and 43 to prove the **mean-value property** of harmonic functions: If φ is harmonic, then $I_\varphi(r) = \varphi(P)$ for all r.

SOLUTION In Exercise 42 we showed that

$$\frac{d}{dr} I_\varphi(r) = \frac{1}{2\pi r} \iint_{\mathcal{D}} \Delta\varphi \, dA$$

If φ is harmonic, $\Delta\varphi = 0$. Therefore the right-hand side of the equality is zero, and we get

$$\frac{d}{dr} I_\varphi(r) = 0$$

We conclude that $I_\varphi(r)$ is constant, that is, $I_\varphi(r)$ has the same value for all r. The constant value is determined by the limit $\lim_{r\to 0} I_\varphi(r) = \varphi(P)$ obtained in Exercise 38. That is, $I_\varphi(r) = \varphi(P)$ for all r.

45. Show that $f(x, y) = x^2 - y^2$ is harmonic. Verify the mean-value property for $f(x, y)$ directly [expand $f(a + r\cos\theta, b + r\sin\theta)$ as a function of θ and compute $I_\varphi(r)$]. Show that $x^2 + y^2$ is not harmonic and does not satisfy the mean-value property.

SOLUTION We show that the function $f(x, y) = x^2 - y^2$ is harmonic by showing that $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. We have

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

Hence,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 - 2 = 0$$

We now verify the mean-value property for f. That is, we show that for all r,

$$I_f(r) = \frac{1}{2\pi} \int_0^{2\pi} f(a + r\cos\theta, b + r\sin\theta)\, d\theta = f(a, b)$$

We compute the integrand:

$$\begin{aligned} f(a + r\cos\theta, b + r\sin\theta) &= x^2 - y^2 = (a + r\cos\theta)^2 - (b + r\sin\theta)^2 \\ &= a^2 + 2ar\cos\theta + r^2\cos^2\theta - \left(b^2 + 2br\sin\theta + r^2\sin^2\theta\right) \\ &= a^2 - b^2 + 2r(a\cos\theta - b\sin\theta) + r^2\cos 2\theta \end{aligned}$$

We compute the integral:

$$\begin{aligned} 2\pi I_f(r) &= \int_0^{2\pi} \left(a^2 - b^2 + 2r(a\cos\theta - b\sin\theta) + r^2\cos 2\theta\right) d\theta \\ &= (a^2 - b^2)\theta + 2ar\sin\theta + 2br\cos\theta + \frac{r^2}{2}\sin 2\theta\Big|_{\theta=0}^{2\pi} = 2\pi(a^2 - b^2) \end{aligned}$$

Hence,

$$I_f(r) = a^2 - b^2$$

However, we have $f(a, b) = a^2 - b^2$. Hence, for all r, $I_f(r) = f(a, b)$, which proves the mean-value property for f. For $g(x, y) = x^2 + y^2$ we have

$$g_{xx} = 2, \quad g_{yy} = 2, \quad \text{and} \quad \Delta g = 2 + 2 = 4 \neq 0.$$

We check the mean value property:

$$\begin{aligned} I_g(r) &= \frac{1}{2\pi}\int_0^{2\pi} g(a + r\cos\theta, b + r\sin\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} (a + r\cos\theta)^2 + (b + r\sin\theta)^2\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} a^2 + b^2 + 2r(a\cos\theta + b\sin\theta) + r^2\, d\theta = a^2 + b^2 + r^2 \neq a^2 + b^2 = \varphi(a, b) \end{aligned}$$

The mean value property does not hold for g.

17.2 Stokes' Theorem (LT Section 18.2)

Preliminary Questions

1. Indicate with an arrow the boundary orientation of the boundary curves of the surfaces in Figure 14, oriented by the outward-pointing normal vectors.

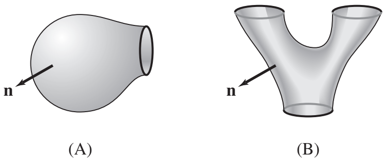
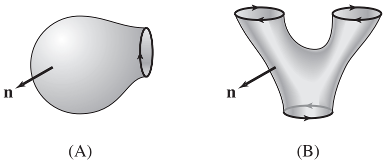


FIGURE 14

SOLUTION The indicated orientation is defined so that if the normal vector is moving along the boundary curve, the surface lies to the left. Since the surfaces are oriented by the outward-pointing normal vectors, the induced orientation is as shown in the figure:



2. Let $\mathbf{F} = \text{curl}(\mathbf{A})$. Which of the following are related by Stokes' Theorem?

(a) The circulation of $\mathbf{A}$ and flux of $\mathbf{F}$.

(b) The circulation of $\mathbf{F}$ and flux of $\mathbf{A}$.

SOLUTION Stokes' Theorem states that the circulation of $\mathbf{A}$ is equal to the flux of $\mathbf{F}$. The correct answer is (a).

3. What is the definition of a vector potential?

SOLUTION A vector field $\mathbf{A}$ such that $\mathbf{F} = \text{curl}(\mathbf{A})$ is a vector potential for $\mathbf{F}$.

4. Which of the following statements is correct?

(a) The flux of curl(**A**) through every oriented surface is zero.

(b) The flux of curl(**A**) through every closed, oriented surface is zero.

SOLUTION Statement (b) is the correct statement. The flux of curl(**F**) through an oriented surface is not necessarily zero, unless the surface is closed.

5. Which condition on **F** guarantees that the flux through $\mathcal{S}_1$ is equal to the flux through $\mathcal{S}_2$ for any two oriented surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same oriented boundary?

SOLUTION If **F** has a vector potential **A**, then by a corollary of Stokes' Theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s}$$

Therefore, if two oriented surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same oriented boundary curve, $\mathcal{C}$, then

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} \quad \text{and} \quad \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s}$$

Hence,

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{s}$$

Exercises

In Exercises 1–4, calculate curl(**F**).

1. $\mathbf{F} = \langle z - y^2, x + z^3, y + x^2 \rangle$

SOLUTION We have

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y^2 & x + z^3 & y + x^2 \end{vmatrix} = (1 - 3z^2)\mathbf{i} - (2x - 1)\mathbf{j} + (1 + 2y)\mathbf{k} = \langle 1 - 3z^2, 1 - 2x, 1 + 2y \rangle$$

2. $\mathbf{F} = \left\langle \frac{y}{x}, \frac{y}{z}, \frac{z}{x} \right\rangle$

SOLUTION The curl is the following vector:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x} & \frac{y}{z} & \frac{z}{x} \end{vmatrix} = \left(0 - \frac{-y}{z^2}\right)\mathbf{i} - \left(-\frac{z}{x^2} - 0\right)\mathbf{j} + \left(0 - \frac{1}{x}\right)\mathbf{k} = \left\langle \frac{y}{z^2}, \frac{z}{x^2}, -\frac{1}{x} \right\rangle$$

3. $\mathbf{F} = \langle e^y, \sin x, \cos x \rangle$

SOLUTION We have

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & \sin x & \cos x \end{vmatrix} = 0\mathbf{i} - (-\sin x)\mathbf{j} + (\cos x - e^y)\mathbf{k} = \langle 0, \sin x, \cos x - e^y \rangle$$

4. $\mathbf{F} = \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, 0 \right\rangle$

SOLUTION

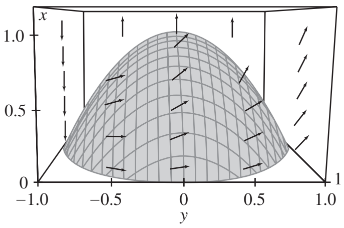
$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \left(\frac{-2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2}\right)\mathbf{k} = \mathbf{0}$$

In Exercises 5–8, verify Stokes' Theorem for the given vector field and surface, oriented with an upward-pointing normal.

5. $\mathbf{F} = \langle 2xy, x, y+z \rangle$, the surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \le 1$

SOLUTION We must show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$$



Step 1. Compute the line integral around the boundary curve. The boundary curve $\mathcal{C}$ is the unit circle oriented in the counterclockwise direction. We parametrize $\mathcal{C}$ by

$$\gamma(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$$

Then,

$$\begin{aligned}
\mathbf{F}(\gamma(t)) &= \langle 2\cos t \sin t, \cos t, \sin t \rangle \\
\gamma'(t) &= \langle -\sin t, \cos t, 0 \rangle \\
\mathbf{F}(\gamma(t)) \cdot \gamma'(t) &= \langle 2\cos t \sin t, \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = -2\cos t \sin^2 t + \cos^2 t
\end{aligned}$$

We obtain the following integral:

$$\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{s} = \int_0^{2\pi} \left(-2\cos t \sin^2 t + \cos^2 t\right) dt = -\frac{2\sin^3 t}{3} + \frac{t}{2} + \frac{\sin 2t}{4}\Bigg|_0^{2\pi} = \pi \tag{1}$$

Step 2. Compute the flux of the curl through the surface. We parametrize the surface by

$$\Phi(\theta, t) = \left(t\cos\theta, t\sin\theta, 1 - t^2\right), \quad 0 \le t \le 1, \quad 0 \le \theta \le 2\pi$$

We compute the normal vector:

$$\begin{aligned}
\mathbf{T}_\theta &= \frac{\partial \Phi}{\partial \theta} = \langle -t\sin\theta, t\cos\theta, 0 \rangle \\
\mathbf{T}_t &= \frac{\partial \Phi}{\partial t} = \langle \cos\theta, \sin\theta, -2t \rangle \\
\mathbf{T}_\theta \times \mathbf{T}_t &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & -2t \end{vmatrix} = (-2t^2\cos\theta)\mathbf{i} - (2t^2\sin\theta)\mathbf{j} - t(\sin^2\theta + \cos^2\theta)\mathbf{k} \\
&= (-2t^2\cos\theta)\mathbf{i} - (2t^2\sin\theta)\mathbf{j} - t\mathbf{k}
\end{aligned}$$

Since the normal is always supposed to be pointing upward, the z-coordinate of the normal vector must be positive. Therefore, the normal vector is

$$\mathbf{n} = \left\langle 2t^2\cos\theta, 2t^2\sin\theta, t \right\rangle$$

We compute the curl:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} = \mathbf{i} + (1-2x)\mathbf{k} = \langle 1, 0, 1-2x \rangle$$

We compute the curl in terms of the parameters:

$$\text{curl}(\mathbf{F}) = \langle 1, 0, 1 - 2t\cos\theta \rangle$$

Hence,

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle 1, 0, 1 - 2t\cos\theta \rangle \cdot \left\langle 2t^2\cos\theta, 2t^2\sin\theta, t \right\rangle = 2t^2\cos\theta + t - 2t^2\cos\theta = t$$

The surface integral is thus

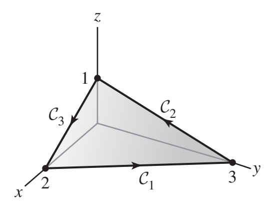
$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 t\,dt\,d\theta = 2\pi \int_0^1 t\,dt = 2\pi \cdot \left.\frac{t^2}{2}\right|_0^1 = \pi \tag{2}$$

The values of the integrals in (1) and (2) are equal, as stated in Stokes' Theorem.

6. $\mathbf{F} = \langle yz, 0, x \rangle$, the portion of the plane $\frac{x}{2} + \frac{y}{3} + z = 1$ where $x, y, z \geq 0$

SOLUTION

Step 1. Compute the integral around the boundary curve. The boundary curve $\mathcal{C}$ consists of the segments $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ shown in the figure:



We parametrize the segments by

$$\mathcal{C}_1 : \gamma_1(t) = (2 - 2t, 3t, 0), \quad t \text{ from 0 to 1}$$
$$\mathcal{C}_2 : \gamma_2(t) = (0, 3 - 3t, t), \quad t \text{ from 0 to 1}$$
$$\mathcal{C}_3 : \gamma_3(t) = (2t, 0, 1 - t), \quad t \text{ from 0 to 1}$$

We compute the following values:

$$\mathbf{F}(\gamma_1(t)) = \langle yz, 0, x \rangle = \langle 0, 0, 2 - 2t \rangle, \quad \gamma_1'(t) = \langle -2, 3, 0 \rangle$$
$$\mathbf{F}(\gamma_2(t)) = \langle yz, 0, x \rangle = \left\langle 3t - 3t^2, 0, 0 \right\rangle, \quad \gamma_2'(t) = \langle 0, -3, 1 \rangle$$
$$\mathbf{F}(\gamma_3(t)) = \langle yz, 0, x \rangle = \langle 0, 0, 2t \rangle, \quad \gamma_3'(t) = \langle 2, 0, -1 \rangle$$

Hence,

$$\mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) = \langle 0, 0, 2 - 2t \rangle \cdot \langle -2, 3, 0 \rangle = 0$$
$$\mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) = \left\langle 3t - 3t^2, 0, 0 \right\rangle \cdot \langle 0, -3, 1 \rangle = 0$$
$$\mathbf{F}(\gamma_3(t)) \cdot \gamma_3'(t) = \langle 0, 0, 2t \rangle \cdot \langle 2, 0, -1 \rangle = -2t$$

We obtain the following integral:

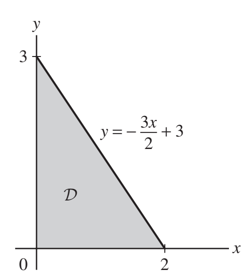
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = 0 + 0 + \int_0^1 -2t\,dt = -t^2\Big|_0^1 = -1 \tag{1}$$

Step 2. Compute the curl.

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 0 & x \end{vmatrix} = -(1 - y)\mathbf{j} + (0 - z)\mathbf{k} = \langle 0, y - 1, -z \rangle$$

Step 3. Compute the flux of the curl through the surface. We parametrize the portion of the plane by

$$\Phi(x, y) = \left(x, y, 1 - \frac{x}{2} - \frac{y}{3}\right) \quad \text{for } (x, y) \in \mathcal{D}$$



We find the normal vector:

$$\mathbf{T}_x = \frac{\partial \Phi}{\partial x} = \left\langle 1, 0, -\frac{1}{2} \right\rangle$$

$$\mathbf{T}_y = \frac{\partial \Phi}{\partial y} = \left\langle 0, 1, -\frac{1}{3} \right\rangle$$

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \end{vmatrix} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \mathbf{k} = \left\langle \frac{1}{2}, \frac{1}{3}, 1 \right\rangle$$

The upward pointing normal is

$$\mathbf{n} = \left\langle \frac{1}{2}, \frac{1}{3}, 1 \right\rangle.$$

We compute the curl($\mathbf{F}$) in terms of the parameters x and y:

$$\text{curl}(\mathbf{F}) = \left\langle 0, y-1, -\left(1 - \frac{x}{2} - \frac{y}{3}\right) \right\rangle = \left\langle 0, y-1, \frac{x}{2} + \frac{y}{3} - 1 \right\rangle$$

We obtain the following integral:

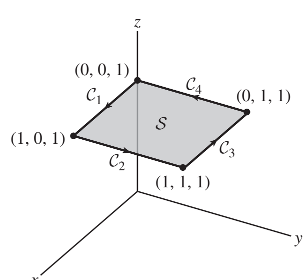
$$\begin{aligned} \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \iint_{\mathcal{D}} \left\langle 0, y-1, \frac{x}{2} + \frac{y}{3} - 1 \right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{3}, 1 \right\rangle dA = \iint_{\mathcal{D}} \left(\frac{x}{2} + \frac{2y}{3} - \frac{4}{3} \right) dA \\ &= \int_0^2 \int_0^{-3x/2+3} \left(\frac{x}{2} + \frac{2y}{3} - \frac{4}{3} \right) dy\, dx = \int_0^2 \frac{xy}{2} + \frac{y^2}{3} - \frac{4y}{3} \bigg|_{y=0}^{-3x/2+3} dx \\ &= \int_0^2 \left(\frac{x}{2} - 1 \right) dx = \frac{x^2}{4} - x \bigg|_0^2 = -1 \end{aligned} \tag{2}$$

The integrals in (1) and (2) are equal as stated in Stokes' Theorem.

7. $\mathbf{F} = \langle e^{y-z}, 0, 0 \rangle$, the square with vertices (1, 0, 1), (1, 1, 1), (0, 1, 1), and (0, 0, 1)

SOLUTION

Step 1. Compute the integral around the boundary curve. The boundary consists of four segments $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ shown in the figure:



We parametrize the segments by

$$\begin{aligned} \mathcal{C}_1 &: \gamma_1(t) = (t, 0, 1), \quad 0 \le t \le 1 \\ \mathcal{C}_2 &: \gamma_2(t) = (1, t, 1), \quad 0 \le t \le 1 \\ \mathcal{C}_3 &: \gamma_3(t) = (1-t, 1, 1), \quad 0 \le t \le 1 \\ \mathcal{C}_4 &: \gamma_4(t) = (0, 1-t, 1), \quad 0 \le t \le 1 \end{aligned}$$

We compute the following values:

$$\begin{aligned} \mathbf{F}(\gamma_1(t)) &= \langle e^{y-z}, 0, 0 \rangle = \left\langle e^{-1}, 0, 0 \right\rangle \\ \mathbf{F}(\gamma_2(t)) &= \langle e^{y-z}, 0, 0 \rangle = \left\langle e^{t-1}, 0, 0 \right\rangle \\ \mathbf{F}(\gamma_3(t)) &= \langle e^{y-z}, 0, 0 \rangle = \langle 1, 0, 0 \rangle \\ \mathbf{F}(\gamma_4(t)) &= \langle e^{y-z}, 0, 0 \rangle = \left\langle e^{-t-1}, 0, 0 \right\rangle \end{aligned}$$

Hence,

$$\mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) = \left\langle e^{-1}, 0, 0 \right\rangle \cdot \langle 1, 0, 0 \rangle = e^{-1}$$
$$\mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) = \left\langle e^{t-1}, 0, 0 \right\rangle \cdot \langle 0, 1, 0 \rangle = 0$$
$$\mathbf{F}(\gamma_3(t)) \cdot \gamma_3'(t) = \langle 1, 0, 0 \rangle \cdot \langle -1, 0, 0 \rangle = -1$$
$$\mathbf{F}(\gamma_4(t)) \cdot \gamma_4'(t) = \left\langle e^{-t-1}, 0, 0 \right\rangle \cdot \langle 0, -1, 0 \rangle = 0$$

We obtain the following integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{4} \int_{\mathcal{C}_i} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 e^{-1}\, dt + 0 + \int_0^1 (-1)\, dt + 0 = e^{-1} - 1$$

Step 2. Compute the curl.

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y-z} & 0 & 0 \end{vmatrix} = -e^{y-z}\ \mathbf{j}\ - e^{y-z}\ \mathbf{k}\ = \left\langle 0, -e^{y-z}, -e^{y-z} \right\rangle$$

Step 3. Compute the flux of the curl through the surface. We parametrize the surface by

$$\Phi(x, y) = (x, y, 1), \quad 0 \le x,\ y \le 1$$

The upward pointing normal is $\mathbf{n} = \langle 0, 0, 1 \rangle$. We express curl($\mathbf{F}$) in terms of the parameters x and y:

$$\operatorname{curl}(\mathbf{F})\,(\Phi(x, y)) = \left\langle 0, -e^{y-1}, -e^{y-1} \right\rangle$$

Hence,

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \left\langle 0, -e^{y-1}, -e^{y-1} \right\rangle \cdot \langle 0, 0, 1 \rangle = -e^{y-1}$$

The surface integral is thus

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot\, d\mathbf{S} = \iint_{\mathcal{D}} -e^{y-1}\, dA = \int_0^1 \int_0^1 -e^{y-1}\, dy\, dx = \int_0^1 -e^{y-1}\, dy = -e^{y-1}\Big|_0^1$$
$$= -1 + e^{-1} = e^{-1} - 1 \tag{1}$$

We see that the integrals in (1) and (2) are equal.

8. $\mathbf{F} = \left\langle y, x, x^2 + y^2 \right\rangle$, the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$

SOLUTION

Step 1. Compute the integral around the boundary curve. The boundary curve is the unit circle oriented in the counterclockwise direction. We use the parametrization $\gamma(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$. Then,

$$\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = \langle \sin t, \cos t, 1 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = -\sin^2 t + \cos^2 = \cos 2t$$

The line integral is

$$\int_{\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{s} = \int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt = \int_0^{2\pi} \cos 2t\, dt = \frac{1}{2} \sin 2t\Big|_0^{2\pi} = 0$$

Step 2. Compute the curl.

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & x^2 + y^2 \end{vmatrix} = \langle 2y, -2x, 0 \rangle$$

Step 3. Compute the flux of the curl through the surface. We parametrize the surface by

$$G(\phi, \theta) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$$

We compute the normal to the surface:

$$G_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle$$
$$G_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle$$
$$G_\theta \times G_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix}$$
$$= \left\langle -\cos\theta\ \sin^2\phi, -\sin\theta\ \sin^2\phi, -\frac{\sin^2\theta\sin 2\phi}{2} - \frac{\cos^2\theta\sin 2\phi}{2}\right\rangle$$

Since $\mathbf{n}$ points upward, we take $\mathbf{n} = -G_\theta \times G_\phi$. Thus,

$$\mathbf{n} = \sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

We express curl($\mathbf{F}$) in terms of $G(\phi, \theta)$:

$$\mathrm{curl}(\mathbf{F})(G(\phi,\theta)) = \langle 2\sin\theta\sin\phi, -2\cos\theta\sin\phi, 0\rangle$$

Then we obtain:

$$\mathrm{curl}(\mathbf{F})\cdot\mathbf{n} = \langle 2\sin\theta\sin\phi, -2\cos\theta\sin\phi, 0\rangle\cdot\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$
$$= \sin\phi(2\sin\theta\cos\theta\sin^2\phi - 2\sin\theta\cos\theta\sin^2\phi + 0) = 0$$

Therefore, the line integral around the unit circle is

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathcal{D}} 0\,dA = 0.$$

In Exercises 9 and 10, calculate curl($\mathbf{F}$) *and then use Stokes' Theorem to compute the flux of* curl($\mathbf{F}$) *through the given surface as a line integral.*

9. $\mathbf{F} = \left\langle e^{z^2} - y, e^{z^3} + x, \cos(xz)\right\rangle$, the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$ with outward-pointing normal

SOLUTION

Step 1. Compute the curl.

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x & \partial y & \partial z \\ e^{z^2} - y & e^{z^3} + x & \cos(xz) \end{vmatrix} = \left\langle -3z^2e^{z^3}, 2ze^{z^3} + z\sin(xz), 2\right\rangle$$

Step 2. Compute the flux of the curl through the surface. We will use Stokes' Theorem and compute the line integral around the boundary curve. The boundary curve is the unit circle oriented in the counterclockwise direction. We use the parametrization $\gamma(t) = \langle\cos t, \sin t, 0\rangle$, $0 \le t \le 2\pi$. Then

$$\mathbf{F}(\gamma(t))\cdot\gamma'(t) = \left\langle e^0 - \sin t, e^0 + \cos t, \cos(0)\right\rangle\cdot\langle -\sin t, \cos t, 0\rangle$$
$$= \langle 1 - \sin t, 1 + \cos t, 1\rangle\cdot\langle -\sin t, \cos t, 0\rangle$$
$$= -\sin t(1 - \sin t) + \cos t(1 + \cos t) + 0$$
$$= -\sin t + \sin^2 t + \cos t + \cos^2 t$$
$$= 1 - \sin t + \cos t$$

The line integral is:

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi}(1 - \sin t + \cos t)\,dt = t + \cos t + \sin t\Big|_0^{2\pi} = 2\pi$$

10. $\mathbf{F} = \left\langle x + y, z^2 - 4, x\sqrt{y^2 + 1} \right\rangle$, surface of the wedge-shaped box in Figure 15 (bottom included, top excluded) with outward pointing normal.

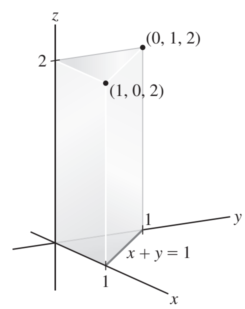


FIGURE 15

SOLUTION We are asked to calculate the flux of curl($\mathbf{F}$) through the walls of the wedge-shaped box (bottom included, top excluded) with outward pointing normal. The oriented boundary of this surface is the triangle at height $z = 2$, oriented clockwise (when viewed from above). By Stokes' Theorem, the flux of curl($\mathbf{F}$) is equal to the line integral of $\mathbf{F}$ around the oriented boundary. The restriction of $\mathbf{F}$ to the boundary, where $z = 2$ is

$$\mathbf{F} = \left\langle x + y, 0, x\sqrt{y^2 + 1} \right\rangle$$

We parametrize the three sides of this triangle for $0 \le t \le 1$:

$$\langle 1 - t, 0, 0 \rangle, \quad \langle 0, t, 0 \rangle, \quad \langle t, 1 - t, 0 \rangle$$

Thus $d\mathbf{s}$ on the three sides is

$$\langle -1, 0, 0 \rangle \; dt, \quad \langle 0, 1, 0 \rangle \; dt, \quad \langle 1, -1, 0 \rangle \; dt$$

and the dot products are (the z-component of $\mathbf{F}$ is not relevant because $d\mathbf{s}$ has zero z-component):

$$\begin{aligned}
\mathbf{F} \cdot d\mathbf{s} &= \langle 1 - t, 0, \star \rangle \cdot \langle -1, 0, 0 \rangle \; dt = (t - 1) \; dt \\
\mathbf{F} \cdot d\mathbf{s} &= \langle t, 0, \star \rangle \cdot \langle 0, 1, 0 \rangle \; dt = 0 \\
\mathbf{F} \cdot d\mathbf{s} &= \langle 1, 0, \star \rangle \cdot \langle 1, -1, 0 \rangle \; dt = \; dt
\end{aligned}$$

Therefore, the line integral around the oriented boundary is equal to

$$\int_0^1 (t - 1) \, dt + 0 + \int_0^1 \; dt = -\frac{1}{2} + 0 + 1 = \frac{1}{2}$$

Thus, the flux of curl($\mathbf{F}$) through the surface is $\frac{1}{2}$.

11. Let $\mathcal{S}$ be the surface of the cylinder (not including the top and bottom) of radius 2 for $1 \le z \le 6$, oriented with outward-pointing normal (Figure 16).

(a) Indicate with an arrow the orientation of $\partial\mathcal{S}$ (the top and bottom circles).

(b) Verify Stokes' Theorem for $\mathcal{S}$ and $\mathbf{F} = \left\langle yz^2, 0, 0 \right\rangle$.

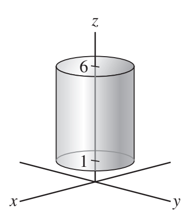
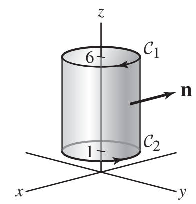


FIGURE 16

SOLUTION

(a) The induced orientation is defined so that as the normal vector travels along the boundary curve, the surface lies to its left. Therefore, the boundary circles on top and bottom have opposite orientations, which are shown in the figure.



(b) We verify Stokes' Theorem for $\mathcal{S}$ and $\mathbf{F} = \left\langle yz^2, 0, 0 \right\rangle$.

Step 1. Compute the integral around the boundary circles. We use the following parametrizations:

$$\mathcal{C}_1 : \gamma_1(t) = (2\cos t, 2\sin t, 6), \quad t \text{ from } 2\pi \text{ to } 0$$
$$\mathcal{C}_2 : \gamma_2(t) = (2\cos t, 2\sin t, 1), \quad t \text{ from } 0 \text{ to } 2\pi$$

We compute the following values:

$$\begin{aligned}
\mathbf{F}(\gamma_1(t)) &= \left\langle yz^2, 0, 0 \right\rangle = \langle 72\sin t, 0, 0 \rangle, \\
\gamma_1'(t) &= \langle -2\sin t, 2\cos t, 0 \rangle \\
\mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) &= \langle 72\sin t, 0, 0 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle = -144\sin^2 t \\
\mathbf{F}(\gamma_2(t)) &= \left\langle yz^2, 0, 0 \right\rangle = \langle 2\sin t, 0, 0 \rangle, \\
\gamma_2'(t) &= \langle -2\sin t, 2\cos t, 0 \rangle \\
\mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) &= \langle 2\sin t, 0, 0 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle = -4\sin^2 t
\end{aligned}$$

The line integral is thus

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{2\pi}^{0} (-144\sin^2 t)\, dt + \int_0^{2\pi} (-4\sin^2 t)\, dt \\
&= \int_0^{2\pi} 140\sin^2 t\, dt = 140 \int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt = 70 \cdot 2\pi - \left.\frac{70\sin 2t}{2}\right|_0^{2\pi} = 140\pi
\end{aligned}$$

Step 2. Compute the curl

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & 0 & 0 \end{vmatrix} = (2yz)\mathbf{j} - z^2\mathbf{k} = \left\langle 0, 2yz, -z^2 \right\rangle$$

Step 3. Compute the flux of the curl through the surface. We parametrize $\mathcal{S}$ by

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 1 \le z \le 6$$

In Example 2 in the text, it is shown that the outward pointing normal is

$$\mathbf{n} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$$

We compute the dot product:

$$\operatorname{curl}(\mathbf{F})(\Phi(\theta, z)) \cdot \mathbf{n} = \left\langle 0, 4z\sin\theta, -z^2 \right\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0 \rangle = 8z\sin^2\theta$$

We obtain the following integral (and use the integral we computed before):

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_1^6 \int_0^{2\pi} 8z\sin^2\theta\, d\theta\, dz = \left(\int_1^6 8z\, dz \right) \left(\int_0^{2\pi} \sin^2\theta\, d\theta \right) = \left. 4z^2 \right|_1^6 \cdot \pi = 140\pi$$

The line integral and the flux have the same value. This verifies Stokes' Theorem.

12. Let $\mathcal{S}$ be the portion of the plane $z = x$ contained in the half-cylinder of radius R depicted in Figure 17. Use Stokes' Theorem to calculate the circulation of $\mathbf{F} = \langle z, x, y + 2z \rangle$ around the boundary of $\mathcal{S}$ (a half-ellipse) in the counterclockwise direction when viewed from above. *Hint:* Show that curl($\mathbf{F}$) is orthogonal to the normal vector to the plane.

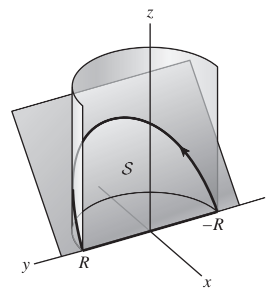


FIGURE 17

SOLUTION By Stokes' Theorem,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

We compute the curl:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y+2z \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle$$

The outward pointing vector of the plane $z = x$, or $x - z = 0$, is $\mathbf{n} = \langle -1, 0, 1 \rangle$. Therefore,

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle 1, 1, 1 \rangle \cdot \langle -1, 0, 1 \rangle = -1 + 0 + 1 = 0$$

We conclude that

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{s} = \iint_{\mathcal{D}} 0 \, dA = 0$$

This equation, along with Stokes' Theorem, allows us to conclude that the circulation of **F** around $\mathcal{C}$ is zero.

13. Let I be the flux of $\mathbf{F} = \left\langle e^y, 2xe^{x^2}, z^2 \right\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \left\langle e^y, 2xe^{x^2}, 0 \right\rangle$. Find a vector field **A** such that $\text{curl}(\mathbf{A}) = \mathbf{G}$.
(b) Use Stokes' Theorem to show that the flux of **G** through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of **A** around $\partial\mathcal{S}$.
(c) Calculate I. *Hint:* Use (b) to show that I is equal to the flux of $\left\langle 0, 0, z^2 \right\rangle$ through $\mathcal{S}$.

SOLUTION

(a) We search for a vector field **A** so that $\mathbf{G} = \text{curl}(\mathbf{A})$. That is,

$$\left\langle \frac{\partial \mathbf{A}_3}{\partial y} - \frac{\partial \mathbf{A}_2}{\partial z}, \frac{\partial \mathbf{A}_1}{\partial z} - \frac{\partial \mathbf{A}_3}{\partial x}, \frac{\partial \mathbf{A}_2}{\partial x} - \frac{\partial \mathbf{A}_1}{\partial y} \right\rangle = \left\langle e^y, 2xe^{x^2}, 0 \right\rangle$$

We note that the third coordinate of this curl vector must be zero; this can be satisfied if $\mathbf{A}_1 = 0$ and $\mathbf{A}_2 = 0$. With this in mind, we let $\mathbf{A} = \left\langle 0, 0, e^y - e^{x^2} \right\rangle$. The vector field $\mathbf{A} = \left\langle 0, 0, e^y - e^{x^2} \right\rangle$ satisfies this equality. Indeed,

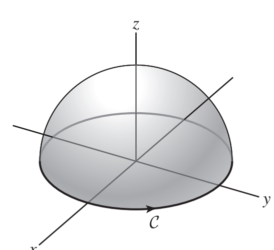
$$\frac{\partial \mathbf{A}_3}{\partial y} - \frac{\partial \mathbf{A}_2}{\partial z} = e^y, \quad \frac{\partial \mathbf{A}_1}{\partial z} - \frac{\partial \mathbf{A}_3}{\partial x} = 2xe^{x^2}, \quad \frac{\partial \mathbf{A}_2}{\partial x} - \frac{\partial \mathbf{A}_1}{\partial y} = 0$$

(b) We found that $\mathbf{G} = \text{curl}(\mathbf{A})$, where $\mathbf{A} = \left\langle 0, 0, e^y - e^{x^2} \right\rangle$. We compute the flux of **G** through $\mathcal{S}$. By Stokes' Theorem,

$$\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \text{curl}(\mathbf{A}) \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s}$$

The boundary $\mathcal{C}$ is the circle $x^2 + y^2 = 1$, parametrized by

$$\gamma(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$$



We compute the following values:

$$\mathbf{A}(\gamma(t)) = \left\langle 0, 0, e^y - e^{x^2} \right\rangle = \left\langle 0, 0, e^{\sin t} - e^{\cos^2 t} \right\rangle$$
$$\gamma'(t) = \langle -\sin t, \cos t, 0 \rangle$$
$$\mathbf{A}(\gamma(t)) \cdot \gamma'(t) = \left\langle 0, 0, e^{\sin t} - e^{\cos^2 t} \right\rangle \cdot \langle -\sin t, \cos t, 0 \rangle = 0$$

Therefore,

$$\int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} = \int_0^{2\pi} 0\, dt = 0$$

(c) We rewrite the vector field $\mathbf{F} = \left\langle e^y, 2xe^{x^2}, z^2 \right\rangle$ as

$$\mathbf{F} = \left\langle e^y, 2xe^{x^2}, z^2 \right\rangle = \left\langle e^y, 2xe^{x^2}, 0 \right\rangle + \left\langle 0, 0, z^2 \right\rangle = \operatorname{curl}(\mathbf{A}) + \left\langle 0, 0, z^2 \right\rangle$$

Therefore,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{A}) \cdot d\mathbf{S} + \iint_{\mathcal{S}} \left\langle 0, 0, z^2 \right\rangle \cdot d\mathbf{S} \tag{1}$$

In part (b) we showed that the first integral on the right-hand side is zero. Therefore,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \left\langle 0, 0, z^2 \right\rangle \cdot d\mathbf{S} \tag{2}$$

The upper hemisphere is parametrized by

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}.$$

with the outward pointing normal

$$\mathbf{n} = \sin\phi \, \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

See Example 4, Section 17.4. We have

$$\left\langle 0, 0, \cos^2\phi \right\rangle \cdot \mathbf{n} = \sin\phi \cos^3\phi$$

Therefore,

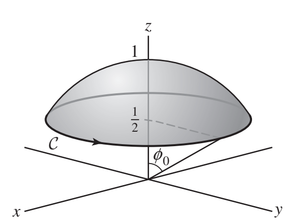
$$\iint_{\mathcal{S}} \left\langle 0, 0, z^2 \right\rangle \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \, \cos^3\phi \, d\phi \, d\theta = 2\pi \int_0^{\pi/2} \sin\phi \, \cos^3\phi \, d\phi$$
$$= 2\pi \left. \frac{-\cos^4\phi}{4} \right|_0^{\pi/2} = -\frac{\pi}{2}(0 - 1) = \frac{\pi}{2}$$

Combining with (2) we obtain the solution

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2}.$$

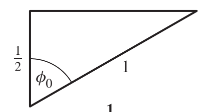
14. Let $\mathbf{F} = \langle 0, -z, 1 \rangle$. Let $\mathcal{S}$ be the spherical cap $x^2 + y^2 + z^2 \le 1$, where $z \ge \frac{1}{2}$. Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ directly as a surface integral. Then verify that $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, where $\mathbf{A} = (0, x, xz)$ and evaluate the surface integral again using Stokes' Theorem.

SOLUTION



We first compute the surface integral directly. The spherical cap is parametrized by

$$\Phi(\theta,\phi)=(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi),\quad 0\le\theta\le 2\pi,\quad 0\le\phi\le\frac{\pi}{3}$$



$$\cos\phi_0=\frac{1}{2}\Rightarrow\phi_0=\frac{\pi}{3}$$

The outward pointing normal is

$$\mathbf{n}=\sin\phi(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)$$

See Example 4, Section 17.4. Hence,

$$\mathbf{F}\left(\Phi(\theta,\phi)\right)\cdot\mathbf{n}=\langle 0,-\cos\phi,1\rangle\cdot\mathbf{n}=-\sin\theta\sin^2\phi\cos\phi+\sin\phi\cos\phi$$

We obtain the following integral:

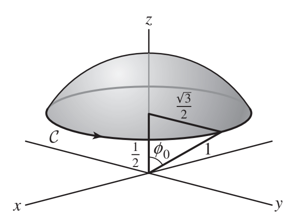
$$\begin{aligned}\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S}&=\int_0^{2\pi}\int_0^{\pi/3}\left(-\sin\theta\sin^2\phi\cos\phi+\sin\phi\cos\phi\right)\,d\phi\,d\theta\\&=\left(\int_0^{2\pi}-\sin\theta\,d\theta\right)\left(\int_0^{\pi/3}\sin^2\phi\cos\phi\,d\phi\right)+2\pi\int_0^{\pi/3}\sin\phi\cos\phi\,d\phi\\&=0+\pi\int_0^{\pi/3}\sin 2\phi\,d\phi=\pi\left(-\frac{\cos 2\phi}{2}\right)\Bigg|_0^{\pi/3}=\frac{3\pi}{4}\end{aligned}\tag{1}$$

We now evaluate the flux using Stokes' Theorem. We first notice that $\mathbf{F}=\mathrm{curl}(\mathbf{A})$, where $\mathbf{A}=\langle 0,x,xz\rangle$. We verify it:

$$\begin{aligned}\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}&=0-0=0\\\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}&=0-z=-z\\\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}&=1-0=1\end{aligned}$$

Indeed, we see that $\mathrm{curl}(\mathbf{A})=\mathbf{F}$. Applying Stokes' Theorem, we have

$$\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S}=\iint_{\mathcal{S}}\mathrm{curl}(\mathbf{A})\cdot d\mathbf{S}=\int_{\mathcal{C}}\mathbf{A}\cdot d\mathbf{s}\tag{2}$$



To compute the line integral, we notice that the boundary curve is the circle $x^2+y^2=\frac{3}{4}$ in the plane $z=\frac{1}{2}$. We parametrize $\mathcal{C}$ by

$$\gamma(t)=\left(\frac{\sqrt{3}}{2}\cos t,\frac{\sqrt{3}}{2}\sin t,\frac{1}{2}\right),\quad 0\le t\le 2\pi$$

We compute the following values:

$$\begin{aligned}\mathbf{A}\left(\gamma(t)\right)&=\langle 0,x,xz\rangle=\left\langle 0,\frac{\sqrt{3}}{2}\cos t,\frac{\sqrt{3}}{4}\cos t\right\rangle\\\gamma'(t)&=\left\langle -\frac{\sqrt{3}}{2}\sin t,\frac{\sqrt{3}}{2}\cos t,0\right\rangle\\\mathbf{A}\left(\gamma(t)\right)\cdot\gamma'(t)&=\left\langle 0,\frac{\sqrt{3}}{2}\cos t,\frac{\sqrt{3}}{4}\cos t\right\rangle\cdot\left\langle -\frac{\sqrt{3}}{2}\sin t,\frac{\sqrt{3}}{2}\cos t,0\right\rangle=\frac{3}{4}\cos^2 t\end{aligned}$$

We obtain the following line integral:

$$\int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} = \int_0^{2\pi} \frac{3}{4}\cos^2 t\, dt = \int_0^{2\pi} \left(\frac{3}{8} + \frac{3}{8}\cos 2t\right) dt = \frac{3t}{8} + \frac{3}{16}\sin 2t\Big|_0^{2\pi} = \frac{3\pi}{4} \tag{3}$$

Combining with (2), we obtain

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \frac{3\pi}{4}$$

This values agrees with the solution obtained in (1), as expected.

15. Let **A** be the vector potential and **B** the magnetic field of the infinite solenoid of radius R in Example 6. Use Stokes' Theorem to compute:

(a) The flux of **B** through a circle in the xy-plane of radius $r < R$

(b) The circulation of **A** around the boundary $\mathcal{C}$ of a surface lying outside the solenoid

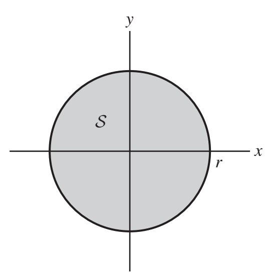
SOLUTION

(a) In Example 6 it is shown that $\mathbf{B} = \text{curl}(\mathbf{A})$, where

$$\mathbf{A} = \begin{cases} \frac{1}{2}R^2 B \left\langle -\frac{y}{r^2}, \frac{x}{r^2}, 0 \right\rangle & \text{if} \quad r > R \\ \frac{1}{2} B \langle -y, x, 0 \rangle & \text{if} \quad r < R \end{cases} \tag{1}$$

Therefore, using Stokes' Theorem, we have ($\mathcal{S}$ is the disk of radius r in the xy-plane)

$$\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \text{curl}(\mathbf{A}) \cdot d\mathbf{S} = \int_{\partial \mathcal{S}} \mathbf{A} \cdot d\mathbf{s} \tag{2}$$



We parametrize the circle $\mathcal{C} = \partial\mathcal{S}$ by $\mathbf{c}(t) = \langle r\cos t, r\sin t, 0\rangle$, $0 \le t \le 2\pi$. Then

$$\mathbf{c}'(t) = \langle -r\sin t, r\cos t, 0\rangle$$

By (1) for $r < R$,

$$\mathbf{A}\left(\mathbf{c}(t)\right) = \frac{1}{2}B\langle -r\sin t, r\cos t, 0\rangle$$

Hence,

$$\mathbf{A}\left(\mathbf{c}(t)\right) \cdot \mathbf{c}'(t) = \frac{1}{2}B\langle -r\sin t, r\cos t, 0\rangle \cdot \langle -r\sin t, r\cos t, 0\rangle = \frac{1}{2}B\left(r^2\sin^2 t + r^2\cos^2 t\right) = \frac{1}{2}r^2 B$$

Now, by (2) we get

$$\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{A} \cdot d\mathbf{S} = \int_0^{2\pi} \frac{1}{2}r^2\mathbf{B}\, dt = \frac{1}{2}r^2\mathbf{B}\int_0^{2\pi} dt = r^2\mathbf{B}\pi$$

(b) Outside the solenoid **B** is the zero field, hence $\mathbf{B} = \mathbf{0}$ on every domain lying outside the solenoid. Therefore, Stokes' Theorem implies that

$$\int_{\partial\mathcal{S}} \mathbf{A} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \text{curl}(\mathbf{A}) \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{0} \cdot d\mathbf{S} = 0.$$

16. The magnetic field $\mathbf{B}$ due to a small current loop (which we place at the origin) is called a **magnetic dipole** (Figure 18). Let $\rho = (x^2 + y^2 + z^2)^{1/2}$. For ρ large, $\mathbf{B} = \text{curl}(\mathbf{A})$, where

$$\mathbf{A} = \left\langle -\frac{y}{\rho^3}, \frac{x}{\rho^3}, 0 \right\rangle$$

(a) Let $\mathcal{C}$ be a horizontal circle of radius R with center $(0, 0, c)$, where c is large. Show that $\mathbf{A}$ is tangent to $\mathcal{C}$.

(b) Use Stokes' Theorem to calculate the flux of $\mathbf{B}$ through $\mathcal{C}$.

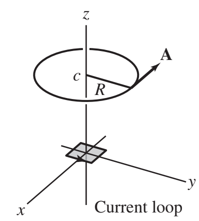


FIGURE 18

SOLUTION

(a) We parametrize $\mathcal{C}$ by

$$\mathbf{c}(t) = (R\cos t, R\sin t, c), \quad 0 \le t \le 2\pi$$

Then, the tangent to $\mathbf{c}(t)$ is in the direction of

$$\mathbf{c}'(t) = \langle -R\sin t, R\cos t, 0\rangle$$

We write $\mathbf{A} = \left\langle -\frac{y}{\rho^3}, \frac{x}{\rho^3}, 0 \right\rangle$ in terms of the parameter t:

$$\mathbf{A}(\mathbf{c}(t)) = \left\langle -\frac{R\sin t}{\rho^3}, \frac{R\cos t}{\rho^3}, 0 \right\rangle$$

$\mathbf{A}(\mathbf{c}(t)) = \frac{1}{\rho^3}\mathbf{c}'(t)$. Therefore, $\mathbf{A}$ is parallel to $\mathbf{c}'(t)$. We conclude that $\mathbf{A}$ is tangent to $\mathcal{C}$.

(b) Let $\mathcal{S}$ be the disc enclosed in $\mathcal{C}$. By Stokes' Theorem the flux of $\mathbf{B}$ through $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \text{curl}(\mathbf{A}) \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s}$$

We compute the line integral. Since $\mathbf{A}$ and $\mathbf{c}'(t)$ are parallel, we have

$$\mathbf{A}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \|\mathbf{A}(\mathbf{c}(t))\| \left\|\mathbf{c}'(t)\right\| = \sqrt{\frac{R^2\sin^2 t}{\rho^6} + \frac{R^2\cos^2 t}{\rho^6}}\sqrt{R^2\sin^2 t + R^2\cos^2 t}$$

$$= \frac{R}{\rho^3} \cdot R = \frac{R^2}{\rho^3} = \frac{R^2}{\left(R^2\cos^2 t + R^2\sin^2 t + c^2\right)^{3/2}}$$

$$= \frac{R^2}{\left(R^2\left(\cos^2 t + \sin^2 t + \left(\frac{c}{R}\right)^2\right)\right)^{3/2}} = \frac{1}{R\left(1 + \left(\frac{c}{R}\right)^2\right)^{3/2}}$$

Hence,

$$\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} = \int_0^{2\pi} \frac{dt}{R\left(1 + \left(\frac{c}{R}\right)^2\right)^{3/2}} = \frac{2\pi}{R\left(1 + \left(\frac{c}{R}\right)^2\right)^{3/2}}$$

17. A uniform magnetic field $\mathbf{B}$ has constant strength b in the z-direction [that is, $\mathbf{B} = \langle 0, 0, b\rangle$].

(a) Verify that $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ is a vector potential for $\mathbf{B}$, where $\mathbf{r} = \langle x, y, 0\rangle$.

(b) Calculate the flux of $\mathbf{B}$ through the rectangle with vertices A, B, C, and D in Figure 19.

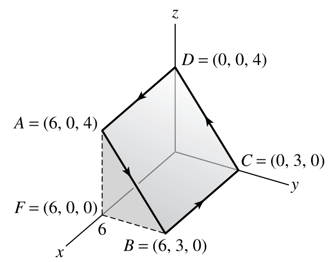


FIGURE 19

SOLUTION

(a) We compute the vector $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Since $\mathbf{B} = b\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, we have

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{1}{2}b\mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = \frac{1}{2}b(x\mathbf{k} \times \mathbf{i} + y\mathbf{k} \times \mathbf{j}) = \frac{1}{2}b(x\mathbf{j} - y\mathbf{i}) = \left\langle -\frac{by}{2}, \frac{bx}{2}, 0 \right\rangle$$

We now show that $\text{curl}(\mathbf{A}) = \mathbf{B}$. We compute the curl of $\mathbf{A}$:

$$\text{curl}\,(\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{by}{2} & \frac{bx}{2} & 0 \end{vmatrix} = \left\langle 0, 0, \frac{b}{2} + \frac{b}{2} \right\rangle = \langle 0, 0, b \rangle = \mathbf{B}$$

Therefore, $\mathbf{A}$ is a vector potential for $\mathbf{B}$.

(b) Let $\mathcal{S}$ be the rectangle $\square ABCD$ and let $\mathcal{C}$ be the boundary of $\mathcal{S}$. Since $\mathbf{B} = \text{Curl}(\mathbf{A})$, we see that $\mathbf{B}$ has a vector potential. It follows, as explained in this section, that the flux of $\mathbf{B}$ through rectangle $\mathcal{S}$ is equal to the flux of $\mathbf{B}$ through any surface with the same boundary $\mathcal{C}$. Let $\mathcal{S}'$ be the wedge-shaped box with four sides and open top. Since the boundary of $\mathcal{S}'$ is also $\mathcal{C}$, we have

$$\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \iint_{\mathcal{S}'} \mathbf{B} \cdot d\mathbf{S}$$

The vector field $\mathbf{B}$ points in the $\mathbf{k}$ direction, so it has zero flux through the three vertical sides of $\mathcal{S}'$. On the other hand, the unit normal vector to the bottom face of $\mathcal{S}'$ is $\mathbf{k}$, so the normal component of $\mathbf{B}$ along the bottom face is equal to b. We obtain

$$\iint_{\mathcal{S}'} \mathbf{B} \cdot d\mathbf{S} = \iint_{\text{Bottom Face of } \mathcal{S}'} b\,dA$$

$$= b(\text{Area of Bottom Face of } \mathcal{S}') = 18b$$

18. Let $\mathbf{F} = \langle -x^2y, x, 0 \rangle$. Referring to Figure 19, let $\mathcal{C}$ be the closed path $ABCD$. Use Stokes' Theorem to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ in two ways. First, regard $\mathcal{C}$ as the boundary of the rectangle with vertices A, B, C, and D. Then treat $\mathcal{C}$ as the boundary of the wedge-shaped box with open top.

SOLUTION Let $\mathcal{S}_1$ be the rectangle whose boundary is $\mathcal{C}$, and let $\mathcal{S}_2$ denote the wedge-shaped box. Then by Stokes' Theorem,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}_1} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \tag{1}$$

and:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \tag{2}$$

We find the curl of $\mathbf{F} = \left\langle -x^2y, x, 0 \right\rangle$:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2y & x & 0 \end{vmatrix} = \left\langle 0, 0, 1 + x^2 \right\rangle$$

We first compute the line integral via the surface integral in (1). To find a parametrization for $\mathcal{S}_1$, we compute the equation of the plane through $A = (6, 0, 4)$, $C = (0, 3, 0)$, $D = (0, 0, 4)$. A normal to the plane is

$$\begin{aligned}\overrightarrow{AD} \times \overrightarrow{AC} &= \langle -6, 0, 0\rangle \times \langle -6, 3, -4\rangle = -6\mathbf{i} \times (-6\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -18\mathbf{k} - 24\mathbf{j}\\ &= \langle 0, -24, -18\rangle = -6\,\langle 0, 4, 3\rangle\end{aligned}$$

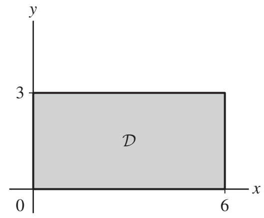
We use the point-normal equation of the plane:

$$\begin{aligned}0(x-0) + 4(y-3) + 3(z-0) &= 0\\ 4y - 12 + 3z = 0 \quad &\Rightarrow \quad z = 4 - \frac{4}{3}y\end{aligned}$$

We parametrize $\mathcal{S}_1$ by

$$\Phi(x, y) = \left(x, y, 4 - \frac{4y}{3}\right)$$

With the parameter domain, $\mathcal{D} = [0, 6] \times [0, 3]$ in the xy-plane.



Then

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = \langle 1, 0, 0\rangle \times \left\langle 0, 1, -\frac{4}{3}\right\rangle = \mathbf{i} \times (\mathbf{j} - \frac{4}{3}\mathbf{k}) = \mathbf{k} + \frac{4}{3}\mathbf{j} = \left\langle 0, \frac{4}{3}, 1\right\rangle$$

The upward pointing normal is

$$\mathbf{n} = \left\langle 0, \frac{4}{3}, 1\right\rangle$$

Also,

$$\operatorname{curl}(\mathbf{F})\,(\Phi(x, y)) = \left\langle 0, 0, 1 + x^2\right\rangle$$

Hence,

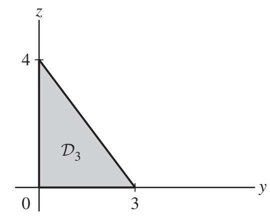
$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \left\langle 0, 0, 1 + x^2\right\rangle \cdot \left\langle 0, \frac{4}{3}, 1\right\rangle = 1 + x^2$$

The integral in (1) is thus

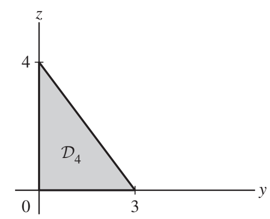
$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \iint_{\mathcal{S}_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} (1 + x^2)\, dA = \int_0^3 \int_0^6 (1 + x^2)\, dx\, dy = 3\int_0^6 (1 + x^2)\, dx\\ &= 3\left(x + \frac{x^3}{3}\bigg|_0^6\right) = 234 \qquad (3)\end{aligned}$$

We now compute the line integral via the surface integral (2). The surface $\mathcal{S}_2$ consists of two rectangles R_1 and R_2 and two triangles T_1 and T_2, parametrized by

$$\begin{aligned}R_1 :\ \Phi_1(x, z) &= (x, 0, z), \quad 0 \le x \le 6, \quad 0 \le z \le 4\\ n_1 &= \langle 0, 1, 0\rangle\\ R_2 :\ \Phi_2(x, y) &= (x, y, 0), \quad 0 \le x \le 6, \quad 0 \le y \le 3\\ n_2 &= \langle 0, 0, 1\rangle\\ T_1 :\ \Phi_3(y, z) &= (6, y, z), \quad (y, z) \in \mathcal{D}_3\\ n_3 &= \langle -1, 0, 0\rangle\end{aligned}$$



$$T_2: \ \Phi_4(y,z) = (0, y, z), \quad (y,z) \in \mathcal{D}_4$$
$$n_4 = \langle 1, 0, 0 \rangle$$



We compute each one of the surface integrals.

$$\begin{aligned}
\text{curl}(\mathbf{F})\,(\Phi_1(x,z)) \cdot \mathbf{n}_1 &= \left\langle 0, 0, 1 + x^2 \right\rangle \cdot \langle 0, 1, 0 \rangle = 0 \\
\text{curl}(\mathbf{F})\,(\Phi_2(x,y)) \cdot \mathbf{n}_2 &= \left\langle 0, 0, 1 + x^2 \right\rangle \cdot \langle 0, 0, 1 \rangle = 1 + x^2 \\
\text{curl}(\mathbf{F})\,(\Phi_3(x,y)) \cdot \mathbf{n}_3 &= \left\langle 0, 0, 1 + 6^2 \right\rangle \cdot \langle -1, 0, 0 \rangle = 0 \\
\text{curl}(\mathbf{F})\,(\Phi_4(y,z)) \cdot \mathbf{n}_4 &= \left\langle 0, 0, 1 + 0^2 \right\rangle \cdot \langle 1, 0, 0 \rangle = 0
\end{aligned}$$

Therefore the only nonzero integral is through R_2. We obtain

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \iint_{\mathcal{S}_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \\
&= \iint_{R_1} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} + \iint_{R_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} + \iint_{T_1} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} + \iint_{T_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \\
&= \iint_{R_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_0^3 \int_0^6 (1 + x^2)\, dx\, dy = 3 \int_0^6 (1 + x^2)\, dx \\
&= 3 \left(x + \frac{x^3}{3} \right) \Big|_0^6 = 234
\end{aligned} \tag{4}$$

The values in (3) and (4) match as expected.

19. Let $\mathbf{F} = \langle y^2, 2z + x, 2y^2 \rangle$. Use Stokes' Theorem to find a plane with equation $ax + by + cz = 0$ (where a, b, c are not all zero) such that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0$ for every closed $\mathcal{C}$ lying in the plane. *Hint:* Choose a, b, c so that curl($\mathbf{F}$) lies in the plane.

SOLUTION Since we are interested in $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, we can also consider $\iint \text{curl}\mathbf{F} \cdot d\mathbf{S}$, by Stokes' Theorem. The curl is $\langle 4y - 2, 0, 1 - 2y \rangle$ and the normal to the plane is $\mathbf{n} = \langle a, b, c \rangle$. They are orthogonal if

$$\langle 4y - 2, 0, 1 - 2y \rangle \cdot \langle a, b, c \rangle = a(4y - 2) + c(1 - 2y) = 0$$

which means:

$$4ay - 2a + c - 2cy = 0 \quad \Rightarrow \quad (4a - 2c) = 0, (c - 2a) = 0$$

This yields $c = 2a$ and b is arbitrary.

20. Let $\mathbf{F} = \left\langle -z^2, 2zx, 4y - x^2 \right\rangle$ and let $\mathcal{C}$ be a simple closed curve in the plane $x + y + z = 4$ that encloses a region of area 16 (Figure 20). Calculate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}$ is oriented in the counterclockwise direction (when viewed from above the plane).

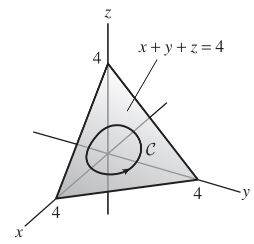


FIGURE 20

SOLUTION We denote by $\mathcal{S}$ the region enclosed by $\mathcal{C}$. Then by Stokes' Theorem,

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{s} \tag{1}$$

We compute the curl of $\mathbf{F} = \left\langle -z^2, 2zx, 4y - x^2 \right\rangle$:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z^2 & 2zx & 4y - x^2 \end{vmatrix} = \langle 4 - 2x, 2x - 2z, 2z \rangle$$

The plane $x + y + z = 4$ has the parametrization

$$\Phi(x, y) = \langle x, y, 4 - x - y \rangle$$

Hence,

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = (\mathbf{i} - \mathbf{k})(\mathbf{j} - \mathbf{k}) = \mathbf{k} + \mathbf{j} + \mathbf{i} = \langle 1, 1, 1 \rangle$$

The normal determined by the induced orientation is

$$\mathbf{n} = \langle 1, 1, 1 \rangle$$

Let $\mathcal{D}$ be the parameter domain in the parametrization $\Phi(x, y) = (x, y, 4 - x - y)$ of $\mathcal{S}$; that is, $\mathcal{D}$ will be the base triangle in the xy plane that lies underneath the pyramid in the picture. To compute the surface integral in (1) we compute the values

$$\text{curl}(\mathbf{F})\,(\Phi(x, y)) = \langle 4 - 2x, 2x - 2(4 - x - y), 2(4 - x - y) \rangle = \langle 4 - 2x, -8 + 4x + 2y, 8 - 2x - 2y \rangle$$

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle 4 - 2x, -8 + 4x + 2y, 8 - 2x - 2y \rangle \cdot \langle 1, 1, 1 \rangle = 4 - 2x - 8 + 4x + 2y + 8 - 2x - 2y = 4$$

Therefore, using (1) and (2) we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} 4\, dA = 4\text{Area}(\mathcal{D}) = 4 \cdot \frac{16}{2} = 32$$

21. Let $\mathbf{F} = \left\langle y^2, x^2, z^2 \right\rangle$. Show that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two closed curves lying on a cylinder whose central axis is the z-axis (Figure 21).

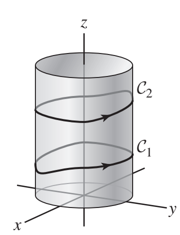


FIGURE 21

SOLUTION We denote by $\mathcal{S}$ the part of the cylinder for which $\mathcal{C}_1$ and $\mathcal{C}_2$ are boundary curves. Using Stokes' Theorem (notice that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same orientations), we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \tag{1}$$

We compute the curl:

$$\operatorname{curl}(\mathbf{F}) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle = \langle 0, 0, 2x - 2y \rangle$$

We parametrize $\mathcal{S}$ by

$$\Phi(\theta, z) = \langle R\cos\theta,\ R\sin\theta, z\rangle$$

where (θ, z) varies in a certain parameter domain $\mathcal{D}$. The outward-pointing normal is

$$\mathbf{n} = \langle R\cos\theta,\ R\sin\theta, 0\rangle$$

We compute curl($\mathbf{F}$) in terms of the parameters:

$$\operatorname{curl}(\mathbf{F}) = \langle 0, 0, 2x - 2y\rangle = \langle 0, 0, 2R\cos\theta - 2R\sin\theta\rangle$$

We compute the dot product:

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = 2R\,\langle 0, 0, \cos\theta - \sin\theta\rangle \cdot R\,\langle \cos\theta, \sin\theta, 0\rangle = 2R^2(0 + 0 + 0) = 0$$

Combining with (1) gives

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} 0\, d\theta\, dr = 0$$

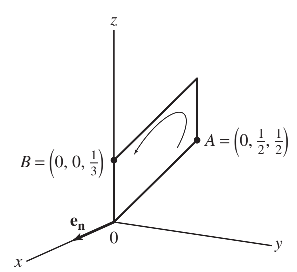
or

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}.$$

22. The curl of a vector field $\mathbf{F}$ at the origin is $\mathbf{v}_0 = \langle 3, 1, 4\rangle$. Estimate the circulation around the small parallelogram spanned by the vectors $\mathbf{A} = \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle$ and $\mathbf{B} = \left\langle 0, 0, \frac{1}{3}\right\rangle$.

SOLUTION We use the following approximation, relying on Stokes' Theorem:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx (\mathbf{v}_0 \cdot \mathbf{e}_n) A(P) \tag{1}$$



The unit normal vector in the positive x-direction is

$$\mathbf{e}_n = \langle 1, 0, 0\rangle \tag{2}$$

We compute the area of the parallelogram spanned by the vectors $\overrightarrow{OA} = \left\langle 0, \frac{1}{2}, \frac{1}{2}\right\rangle$ and $\overrightarrow{OB} = \left\langle 0, 0, \frac{1}{3}\right\rangle$:

$$\overrightarrow{OA} \times \overrightarrow{OB} = \frac{1}{2}(\mathbf{j} + \mathbf{k}) \times \frac{1}{3}\mathbf{k} = \frac{1}{6}(\mathbf{j} \times \mathbf{k} + \mathbf{k} \times \mathbf{k}) = \frac{1}{6}\mathbf{i}$$

$$A(P) = \left\| \frac{1}{6}\mathbf{i} \right\| = \frac{1}{6} \tag{3}$$

We now substitute $\mathbf{v}_0 = \langle 3, 1, 4\rangle$, (2), and (3) in (1) to obtain the approximation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx \langle 3, 1, 4\rangle \cdot \langle 1, 0, 0\rangle \cdot \frac{1}{6} = 3 \cdot \frac{1}{6} = \frac{1}{2}$$

23. You know two things about a vector field $\mathbf{F}$:

(i) $\mathbf{F}$ has a vector potential $\mathbf{A}$ (but $\mathbf{A}$ is unknown).

(ii) The circulation of $\mathbf{A}$ around the unit circle (oriented counterclockwise) is 25.

Determine the flux of $\mathbf{F}$ through the surface $\mathcal{S}$ in Figure 22, oriented with upward pointing normal.

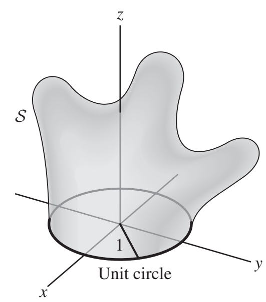
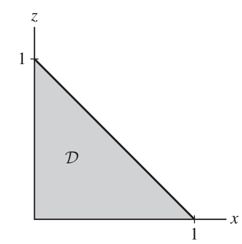


FIGURE 22 Surface $\mathcal{S}$ whose boundary is the unit circle.

SOLUTION Since $\mathbf{F}$ has a vector potential—that is, $\mathbf{F}$ is the curl of a vector field—the flux of $\mathbf{F}$ through a surface depends only on the boundary curve $\mathcal{C}$. Now, the surface $\mathcal{S}$ and the unit disc $\mathcal{S}_1$ in the xy-plane share the same boundary $\mathcal{C}$. Therefore,

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} \tag{1}$$



We compute the flux of $\mathbf{F}$ through $\mathcal{S}_1$, using the parametrization

$$\mathcal{S}_1:\ \Phi(r,\theta) = (r\cos\theta, r\sin\theta, 0), \quad 0\le r\le 1, \quad 0\le\theta\le 2\pi$$
$$\mathbf{n} = \langle 0,0,1\rangle$$

By the given information, we have

$$\mathbf{F}(\Phi(r,\theta)) = \mathbf{F}(r\cos\theta, r\sin\theta, 0) = \langle 0,0,1\rangle$$

Hence,

$$\mathbf{F}(\Phi(r,\theta))\cdot\mathbf{n} = \langle 0,0,1\rangle\cdot\langle 0,0,1\rangle = 1$$

We obtain the following integral:

$$\iint_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \mathbf{F}(\Phi(r,\theta))\cdot\mathbf{n}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 1\,dr\,d\theta = 2\pi$$

Combining with (1) we obtain

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = 2\pi$$

24. Suppose that $\mathbf{F}$ has a vector potential and that $\mathbf{F}(x,y,0) = \mathbf{k}$. Find the flux of $\mathbf{F}$ through the surface $\mathcal{S}$ in Figure 22, oriented with upward pointing normal.

SOLUTION The flux is equal to the flux through the lower hemisphere with outward pointing normal, and this is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi\,d\theta\,d\phi$$

25. Prove that $\mathrm{curl}(f\mathbf{a}) = \nabla f \times \mathbf{a}$, where f is a differentiable function and $\mathbf{a}$ is a constant vector.

SOLUTION Let us first write $\mathbf{a}$ as a constant vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $f = f(x, y, z)$. Then consider the following:

$$\mathrm{curl}(f\mathbf{a}) = \mathrm{curl}(f(x,y,z)\langle a_1,a_2,a_3\rangle) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 f(x,y,z) & a_2 f(x,y,z) & a_3 f(x,y,z) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(a_3 f) - \frac{\partial}{\partial z}(a_2 f), -\frac{\partial}{\partial z}(a_1 f) + \frac{\partial}{\partial x}(a_3 f), \frac{\partial}{\partial x}(a_2 f) - \frac{\partial}{\partial y}(a_1 f) \right\rangle$$

$$= \langle a_3 f_y - a_2 f_z, a_3 f_x - a_1 f_z, a_2 f_x - a_1 f_y \rangle$$

And now consider the following:

$$\nabla f \times \mathbf{a} = \langle f_x, f_y, f_z\rangle \times \langle a_1, a_2, a_3\rangle$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \langle a_3 f_y - a_2 f_z, a_3 f_x - a_1 f_z, a_2 f_x - a_1 f_y \rangle$$

Since the two expressions above are equal, we conclude

$$\mathrm{curl}(f\mathbf{a}) = \nabla f \times \mathbf{a}$$

26. Show that $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$ if $\mathbf{F}$ is **radial**, meaning that $\mathbf{F} = f(\rho)\langle x, y, z\rangle$ for some function $f(\rho)$, where $\rho = \sqrt{x^2+y^2+z^2}$. *Hint:* It is enough to show that one component of $\mathrm{curl}(\mathbf{F})$ is zero, because it will then follow for the other two components by symmetry.

SOLUTION Let $\mathbf{v} = \langle x, y, z\rangle$. We must show that $\mathrm{curl}(\mathbf{F}) = \mathrm{curl}(f\mathbf{v}) = \mathbf{0}$. We compute the curl of $\mathbf{v}$:

$$\mathrm{curl}(f\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x \cdot f & y \cdot f & z \cdot f \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(z\cdot f) - \frac{\partial}{\partial z}(y\cdot f), -\frac{\partial}{\partial z}(x\cdot f) + \frac{\partial}{\partial x}(z\cdot f), \frac{\partial}{\partial x}(y\cdot f) - \frac{\partial}{\partial y}(x\cdot f) \right\rangle$$

$$= \left\langle z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}, z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}, y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} \right\rangle$$

We now must prove that the cross product is zero. To compute the first order partials for f, we use the derivatives

$$\frac{\partial \rho}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\rho}$$

$$\frac{\partial \rho}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{\rho}$$

$$\frac{\partial \rho}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{\rho}$$

By the Chain Rule we get

$$\frac{\partial f}{\partial x} = \frac{df}{d\rho}\frac{\partial \rho}{\partial x} = \frac{df}{d\rho}\cdot\frac{x}{\rho}$$

$$\frac{\partial f}{\partial y} = \frac{df}{d\rho}\frac{\partial \rho}{\partial y} = \frac{df}{d\rho}\cdot\frac{y}{\rho}$$

$$\frac{\partial f}{\partial z} = \frac{df}{d\rho}\frac{\partial \rho}{\partial z} = \frac{df}{d\rho}\cdot\frac{z}{\rho}$$

Therefore,

$$\mathrm{curl}(f\mathbf{v}) = \left\langle z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}, z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}, y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} \right\rangle$$

$$= \left\langle \frac{df}{d\rho}\cdot\frac{yz}{\rho} - \frac{df}{d\rho}\cdot\frac{yz}{\rho}, \frac{df}{d\rho}\cdot\frac{xz}{\rho} - \frac{df}{d\rho}\cdot\frac{xz}{\rho}, \frac{df}{d\rho}\cdot\frac{xy}{\rho} - \frac{df}{d\rho}\cdot\frac{xy}{\rho} \right\rangle$$

$$= \mathbf{0}$$

Hence, we have shown, $\mathrm{curl}(\mathbf{F}) = \mathrm{curl}(f\mathbf{v}) = \mathbf{0}$.

27. Prove the following Product Rule:

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}(\mathbf{F}) + \nabla f \times \mathbf{F}$$

SOLUTION We evaluate the curl of $f\mathbf{F}$. Since $f\mathbf{F} = \langle fF_1, fF_2, fF_3\rangle$, using the Product Rule for scalar functions we have

$$\begin{aligned}
\operatorname{curl}(f\mathbf{F}) &= \left\langle \frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2), \frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3), \frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1)\right\rangle \\
&= \left\langle \frac{\partial f}{\partial y}F_3 + f\frac{\partial F_3}{\partial y} - \frac{\partial f}{\partial z}F_2 - f\frac{\partial F_2}{\partial z}, \frac{\partial f}{\partial z}F_1 + f\frac{\partial F_1}{\partial z} - \frac{\partial f}{\partial x}F_3 - f\frac{\partial F_3}{\partial x},\right. \\
&\qquad \left. \frac{\partial f}{\partial x}F_2 + f\frac{\partial F_2}{\partial x} - \frac{\partial f}{\partial y}F_1 - f\frac{\partial F_1}{\partial y}\right\rangle \\
&= f\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\rangle \\
&\quad + \left\langle \frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right\rangle \qquad (1)
\end{aligned}$$

The vector in the first term is $\operatorname{curl}(\mathbf{F})$. We show that the second term is the cross product $\nabla f \times \mathbf{F}$. We compute the cross product:

$$\begin{aligned}
\nabla f \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}F_3 - \frac{\partial f}{\partial z}F_1\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)\mathbf{k} \\
&= \left\langle \frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right\rangle
\end{aligned}$$

Therefore, (1) gives

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}(\mathbf{F}) + \nabla f \times \mathbf{F}$$

28. Assume that f and g have continuous partial derivatives of order 2. Prove that

$$\oint_{\partial\mathcal{S}} f\nabla(g)\cdot d\mathbf{s} = \iint_{\mathcal{S}} \nabla(f)\times\nabla(g)\cdot d\mathbf{s}$$

SOLUTION By Stokes' Theorem, we have

$$\int_{\partial\mathcal{S}} f\nabla(g)\cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(f\nabla g)\cdot d\mathbf{S}$$

We now use Eq.(8) to evaluate the curl of $f\nabla g$. That is,

$$\begin{aligned}
\int_{\partial\mathcal{S}} f\nabla(g)\cdot d\mathbf{s} &= \iint_{\mathcal{S}} (f\operatorname{curl}(\nabla g) + \nabla f\times\nabla g)\cdot d\mathbf{S} \\
&= \iint_{\mathcal{S}} f\operatorname{curl}(\nabla g)\cdot d\mathbf{S} + \iint_{\mathcal{S}} \nabla(g)\times\nabla(g)\cdot d\mathbf{S} \qquad (1)
\end{aligned}$$

Now, since the gradient field ∇f is conservative, this field satisfies the cross-partials condition. In other words,

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Combining with (1) we obtain

$$\int_{\partial\mathcal{S}} f(\nabla g)\cdot d\mathbf{s} = \iint_{\mathcal{S}} \mathbf{0}\cdot d\mathbf{S} + \iint_{\mathcal{S}} \nabla(f)\times\nabla(g)\cdot d\mathbf{S} = \iint_{\mathcal{S}} \nabla(f)\times\nabla(g)\cdot d\mathbf{S}$$

29. Verify that $\mathbf{B} = \operatorname{curl}(\mathbf{A})$ for $r > R$ in the setting of Example 6.

SOLUTION As observed in the example,

$$\operatorname{curl}(\langle f, g, 0\rangle) = \langle -g_z, f_z, g_x - f_y\rangle$$

and recall $r = x^2 + y^2$. For $r > R$, this yields

$$\operatorname{curl}(\mathbf{A}) = \frac{1}{2}R^2 B\left\langle 0, 0, \frac{\partial}{\partial x}(xr^{-2}) - \frac{\partial}{\partial y}(-yr^{-2})\right\rangle$$

The z-component on the right is also zero:

$$\frac{\partial}{\partial x}(xr^{-2}) + \frac{\partial}{\partial y}(yr^{-2}) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$
$$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$
$$= 0$$

Thus, $\text{curl}(\mathbf{A}) = \mathbf{0}$ when $r > R$ as required.

30. Explain carefully why Green's Theorem is a special case of Stokes' Theorem.

SOLUTION Let $\mathcal{C}$ be a simple closed curve enclosing a region $\mathcal{D}$ oriented counterclockwise in the xy-plane. We must show, using Stokes' Theorem, that for $\mathbf{F} = \langle F_1, F_2 \rangle$,

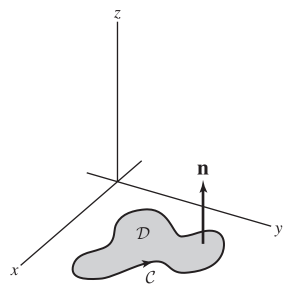
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

We consider $\mathcal{D}$ as a surface in three-space with parametrization

$$\Phi(x, y) = (x, y, 0), \quad (x, y) \in \mathcal{D}$$

The normal vector is

$$\mathbf{n} = \langle 0, 0, 1 \rangle$$



We compute the curl of $\mathbf{F} = \langle F_1, F_2, 0 \rangle$. Since $F_1 = F_1(x, y)$, $F_2 = F_2(x, y)$, and $F_3 = 0$, the curl of $\mathbf{F}$ is the vector

$$\text{curl}(\mathbf{F}) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle = \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Hence,

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \cdot \langle 0, 0, 1 \rangle = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

By Stokes' Theorem we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, dA = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

We thus showed that Green's Theorem is a special case of Stokes' Theorem for two dimensions.

Further Insights and Challenges

31. In this exercise, we use the notation of the proof of Theorem 1 and prove

$$\oint_{\mathcal{C}} F_3(x, y, z)\mathbf{k} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}(F_3(x, y, z)\mathbf{k}) \cdot d\mathbf{S} \qquad \boxed{11}$$

In particular, $\mathcal{S}$ is the graph of $z = f(x, y)$ over a domain $\mathcal{D}$, and $\mathcal{C}$ is the boundary of $\mathcal{S}$ with parametrization $(x(t), y(t), f(x(t), y(t)))$.

(a) Use the Chain Rule to show that

$$F_3(x, y, z)\mathbf{k} \cdot d\mathbf{s} = F_3(x(t), y(t), f(x(t), y(t)))\left(f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)\right) dt$$

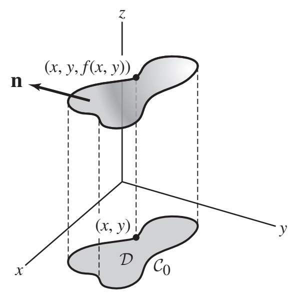
and verify that

$$\oint_{\mathcal{C}} F_3(x, y, z)\mathbf{k} \cdot d\mathbf{s} = \oint_{\mathcal{C}_0} \left\langle F_3(x, y, z) f_x(x, y), F_3(x, y, z) f_y(x, y) \right\rangle \cdot d\mathbf{s}$$

where $\mathcal{C}_0$ has parametrization $(x(t), y(t))$.

(b) Apply Green's Theorem to the line integral over $\mathcal{C}_0$ and show that the result is equal to the right-hand side of Eq. (11).

SOLUTION Let $(x(t), y(t))$, $a \le t \le b$ be a parametrization of the boundary curve $\mathcal{C}_0$ of the domain $\mathcal{D}$.



The boundary curve $\mathcal{C}$ of $\mathcal{S}$ projects on $\mathcal{C}_0$ and has the parametrization

$$\gamma(t) = (x(t), y(t), f(x(t), y(t))), \quad a \le t \le b$$

Let

$$\mathbf{F} = \langle 0, 0, F_3(x, y, z)\rangle$$

We must show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \tag{1}$$

We first compute the surface integral, using the parametrization

$$\mathcal{S} : \Phi(x, y) = (x, y, f(x, y))$$

The normal vector is

$$\mathbf{n} = \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = \langle 1, 0, f_x(x, y)\rangle \times \left\langle 0, 1, f_y(x, y)\right\rangle = (\mathbf{i} + f_x(x, y)\mathbf{k}) \times \left(\mathbf{j} + f_y(x, y)\mathbf{k}\right)$$
$$= -f_y(x, y)\mathbf{j} - f_x(x, y)\mathbf{i} + \mathbf{k} = \left\langle -f_x(x, y), -f_y(x, y), 1\right\rangle$$

We compute the curl of $\mathbf{F}$:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & F_3(x, y, z) \end{vmatrix} = \left\langle \frac{\partial F_3(x, y, z)}{\partial y}, -\frac{\partial F_3(x, y, z)}{\partial x}, 0\right\rangle$$

Hence,

$$\operatorname{curl}(\mathbf{F})\,(\Phi(x, y)) \cdot \mathbf{n} = \left\langle \frac{\partial F_3}{\partial y}\,(x, y, f(x, y)) - \frac{\partial F_3}{\partial x}\,(x, y, f(x, y))\,, 0\right\rangle \cdot \left\langle -f_x(x, y), -f_y(x, y), 1\right\rangle$$
$$= -\frac{\partial F_3\,(x, y, f(x, y))}{\partial y} f_x(x, y) + \frac{\partial F_3\,(x, y, f(x, y))}{\partial x} f_y(x, y)$$

The surface integral is thus

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} \left(-\frac{\partial F_3\,(x, y, f(x, y))}{\partial y} f_x(x, y) + \frac{\partial F_3\,(x, y, f(x, y))}{\partial x} f_y(x, y)\right) dx\,dy \tag{2}$$

We now evaluate the line integral in (1). We have

$$\mathbf{F}\big(\gamma(t)\big) \cdot \gamma'(t) = \left\langle 0, 0, \mathbf{F}_3\Big(x(t), y(t), f\big(x(t), y(t)\big)\Big)\right\rangle \cdot \left\langle x'(t), y'(t), \frac{d}{dt} f\big(x(t), y(t)\big)\right\rangle$$
$$= \mathbf{F}_3\Big(x(t), y(t), f\big(x(t), y(t)\big)\Big) \frac{d}{dt} f\big(x(t), y(t)\big) \tag{3}$$

Using the Chain Rule gives

$$\frac{d}{dt} f\big(x(t), y(t)\big) = f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t)$$

Substituting in (3), we conclude that the line integral is

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \Big(F_3\Big(x(t), y(t), f\big(x(t), y(t)\big)\Big) \cdot \Big(f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t)\Big)\Big)\, dt \tag{4}$$

We consider the following vector field:

$$\mathbf{G}(x, y) = \big\langle F_3\big(x, y, f(x, y)\big) f_x(x, y), F_3\big(x, y, f(x, y)\big) f_y(x, y)\big\rangle$$

Then the integral in (4) is the line integral of the planar vector field $\mathbf{G}$ over $\mathcal{C}_0$. That is,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_0} \mathbf{G} \cdot d\mathbf{s}$$

Therefore, we may apply Green's Theorem and write

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_0} \mathbf{G} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x}\Big(F_3\big(x, y, f(x, y)\big) f_y(x, y)\Big) - \frac{\partial}{\partial y}\Big(F_3\big(x, y, f(x, y)\big) f_x(x, y)\Big)\right) dx\, dy \tag{5}$$

We use the Product Rule to evaluate the integrand:

$$\frac{\partial F_3}{\partial x}(x, y, f(x, y))\, f_y(x, y) + F_3\big(x, y, f(x, y)\big) f_{yx}(x, y) - \frac{\partial F_3}{\partial y}\big(x, y, f(x, y)\big) f_x(x, y) - F_3\big(x, y, f(x, y)\big) f_{xy}(x, y)$$

$$= \frac{\partial F_3}{\partial x}\big(x, y, f(x, y)\big) f_y(x, y) - \frac{\partial F_3}{\partial y}\big(x, y, f(x, y)\big) f_x(x, y)$$

Substituting in (5) gives

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_3\,(x, y, f(x, y))}{\partial x} f_y(x, y) - \frac{\partial F_3\,(x, y, f(x, y))}{\partial y} f_x(x, y)\right) dx\, dy \tag{6}$$

Equations(2) and(6) give the same result, hence

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{s}$$

for

$$\mathbf{F} = \langle 0, 0, F_3(x, y, z)\rangle$$

32. Let $\mathbf{F}$ be a continuously differentiable vector field in $\mathbf{R}^3$, Q a point, and $\mathcal{S}$ a plane containing Q with unit normal vector $\mathbf{e}$. Let $\mathcal{C}_r$ be a circle of radius r centered at Q in $\mathcal{S}$, and let $\mathcal{S}_r$ be the disk enclosed by $\mathcal{C}_r$. Assume $\mathcal{S}_r$ is oriented with unit normal vector $\mathbf{e}$.

(a) Let $m(r)$ and $M(r)$ be the minimum and maximum values of $\operatorname{curl}(\mathbf{F}(P)) \cdot \mathbf{e}$ for $P \in \mathcal{S}_r$. Prove that

$$m(r) \le \frac{1}{\pi r^2} \iint_{\mathcal{S}_r} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \le M(r)$$

(b) Prove that

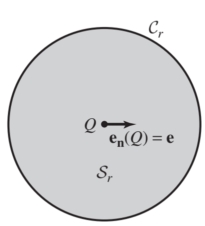
$$\operatorname{curl}(\mathbf{F}(Q)) \cdot \mathbf{e} = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s}$$

This proves that $\operatorname{curl}(\mathbf{F}(Q)) \cdot \mathbf{e}$ is the circulation per unit area in the plane $\mathcal{S}$.

SOLUTION

(a) We may assume that the circle lies on the xy-plane, and parametrize $\mathcal{S}_r$ by

$$\mathcal{S}_r : \Phi(x, y, z) = (x, y, 0), \quad (x, y) \in \mathcal{S}_r$$



Then, $\mathbf{n}$ is a unit vector and we have

$$\iint_{\mathcal{S}_r} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n(P)\, dA \tag{1}$$

We use the given information $m(r) \le \text{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n(P) \le M(r)$ for $P \in \mathcal{S}_r$ and properties of the double integral to write

$$A(\mathcal{S}_r)m(r) \le \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n(P)\, dA \le A(\mathcal{S}_r) \cdot M(r)$$

The area of the disk is $A(\mathcal{S}_r) = \pi r^2$. Therefore,

$$\pi r^2 m(r) \le \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n(P)\, dA \le \pi r^2 M(r)$$

or

$$m(r) \le \frac{1}{\pi r^2} \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n(P)\, dA \le M(r)$$

Combining with (1) we get

$$m(r) \le \frac{1}{\pi r^2} \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} \le M(r)$$

(b) By Stokes' Theorem,

$$\int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}_r} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

By part(a) we have

$$m(r) \le \frac{1}{\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s} \le M(r) \tag{2}$$

We take the limit over the circles of radius r centered at Q, as $r \to 0$. As $r \to 0$, the regions $\mathcal{S}_r$ are approaching the center Q. The continuity of the curl implies that

$$\lim_{r\to 0} m(r) = \lim_{r\to 0} M(r) = \text{curl}(\mathbf{F})(Q) \cdot \mathbf{e}_n(Q) = \text{curl}(\mathbf{F})(Q) \cdot \mathbf{e}$$

Therefore,

$$\lim_{r\to 0} m(r) \le \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s} \le \lim_{r\to 0} M(r)$$

$$\text{curl}(\mathbf{F})(Q) \cdot \mathbf{e} \le \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s} \le \text{curl}(\mathbf{F})(Q) \cdot \mathbf{e}$$

Hence,

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{s} = \text{curl}(\mathbf{F})(Q) \cdot \mathbf{e}$$

17.3 Divergence Theorem (LT Section 18.3)

Preliminary Questions

1. What is the flux of $\mathbf{F} = \langle 1, 0, 0 \rangle$ through a closed surface?

SOLUTION The divergence of $\mathbf{F} = \langle 1, 0, 0 \rangle$ is $\text{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$, therefore the Divergence Theorem implies that the flux of $\mathbf{F}$ through a closed surface $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 0\, dV = 0$$

2. Justify the following statement: The flux of $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ through every closed surface is positive.

SOLUTION The divergence of $\mathbf{F} = \left\langle x^3, y^3, z^3 \right\rangle$ is

$$\text{div}(\mathbf{F}) = 3x^2 + 3y^2 + 3z^2$$

Therefore, by the Divergence Theorem, the flux of **F** through a closed surface $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} (3x^2 + 3y^2 + 3z^2)\, dV$$

Since the integrand is positive for all $(x, y, z) \neq (0, 0, 0)$, the triple integral, hence also the flux, is positive.

3. Which of the following expressions are meaningful (where **F** is a vector field and f is a function)? Of those that are meaningful, which are automatically zero?

(a) $\operatorname{div}(\nabla f)$ **(b)** $\operatorname{curl}(\nabla f)$ **(c)** $\nabla\operatorname{curl}(f)$

(d) $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$ **(e)** $\operatorname{curl}(\operatorname{div}(\mathbf{F}))$ **(f)** $\nabla(\operatorname{div}(\mathbf{F}))$

SOLUTION

(a) The divergence is defined on vector fields. The gradient is a vector field, hence $\operatorname{div}(\nabla\varphi)$ is defined. It is not automatically zero since for $\varphi = x^2 + y^2 + z^2$ we have

$$\operatorname{div}(\nabla\varphi) = \operatorname{div}\langle 2x, 2y, 2z\rangle = 2 + 2 + 2 = 6 \neq 0$$

(b) The curl acts on vector valued functions, and $\nabla\varphi$ is such a function. Therefore, $\operatorname{curl}(\nabla\varphi)$ is defined. Since the gradient field $\nabla\varphi$ is conservative, the cross partials of $\nabla\varphi$ are equal, or equivalently, $\operatorname{curl}(\nabla\varphi)$ is the zero vector.

(c) The curl is defined on vector fields rather than on scalar functions. Therefore, $\operatorname{curl}(\varphi)$ is undefined. Obviously, $\nabla\operatorname{curl}(\varphi)$ is also undefined.

(d) The curl is defined on the vector field **F** and the divergence is defined on the vector field $\operatorname{curl}(\mathbf{F})$. Therefore the expression $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$ is meaningful. We show that this vector is automatically zero:

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}(\mathbf{F})) &= \operatorname{div}\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\rangle \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y} \\
&= \left(\frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial y\partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_2}{\partial x\partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_1}{\partial z\partial y}\right) \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

(e) The curl acts on vector valued functions, whereas $\operatorname{div}(\mathbf{F})$ is a scalar function. Therefore the expression $\operatorname{curl}(\operatorname{div}(\mathbf{F}))$ has no meaning.

(f) $\operatorname{div}(\mathbf{F})$ is a scalar function, hence $\nabla(\operatorname{div}\mathbf{F})$ is meaningful. It is not necessarily the zero vector as shown in the following example:

$$\begin{aligned}
\mathbf{F} &= \left\langle x^2, y^2, z^2\right\rangle \\
\operatorname{div}(\mathbf{F}) &= 2x + 2y + 2z \\
\nabla(\operatorname{div}\mathbf{F}) &= \langle 2, 2, 2\rangle \neq \langle 0, 0, 0\rangle
\end{aligned}$$

4. Which of the following statements is correct (where **F** is a continuously differentiable vector field defined everywhere)?

(a) The flux of $\operatorname{curl}(\mathbf{F})$ through all surfaces is zero.

(b) If $\mathbf{F} = \nabla\varphi$, then the flux of **F** through all surfaces is zero.

(c) The flux of $\operatorname{curl}(\mathbf{F})$ through all closed surfaces is zero.

SOLUTION

(a) This statement holds only for conservative fields. If **F** is not conservative, there exist closed curves such that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \neq 0$, hence by Stokes' Theorem $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \neq 0$.

(b) This statement is false. Consider the unit sphere $\mathcal{S}$ in the three-dimensional space and the function $\varphi(x, y, z) = x^2 + y^2 + z^2$. Then $\mathbf{F} = \nabla\varphi = \langle 2x, 2y, 2z\rangle$ and $\operatorname{div}(\mathbf{F}) = 2 + 2 + 2 = 6$. Using the Divergence Theorem, we have ($\mathcal{W}$ is the unit ball in R^3)

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 6\, dV = 6\iiint_{\mathcal{W}} dV = 6\,\operatorname{Vol}(\mathcal{W})$$

(c) This statement is correct, as stated in the corollary of Stokes' Theorem in section 18.2.

5. How does the Divergence Theorem imply that the flux of $\mathbf{F} = \langle x^2, y - e^z, y - 2zx \rangle$ through a closed surface is equal to the enclosed volume?

SOLUTION By the Divergence Theorem, the flux is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} (2x + 1 - 2x)\, dV = \iiint_{\mathcal{W}} 1\, dV = \text{Volume}(\mathcal{W})$$

Therefore the statement is true.

Exercises

In Exercises 1–4, compute the divergence of the vector field.

1. $\mathbf{F} = \langle xy, yz, y^2 - x^3 \rangle$

SOLUTION The divergence of $\mathbf{F}$ is

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(y^2 - x^3) = y + z + 0 = y + z$$

2. $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

SOLUTION

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

3. $\mathbf{F} = \langle x - 2zx^2, z - xy, z^2x^2 \rangle$

SOLUTION

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x - 2zx^2) + \frac{\partial}{\partial y}(z - xy) + \frac{\partial}{\partial z}(z^2x^2) = (1 - 4zx) + (-x) + (2zx^2) = 1 - 4zx - x + 2zx^2$$

4. $\sin(x + z)\mathbf{i} - ye^{xz}\mathbf{k}$

SOLUTION

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}\sin(x + z) + \frac{\partial}{\partial z}(-ye^{xz}) = \cos(x + z) - yxe^{xz}$$

5. Find a constant c for which the velocity field

$$\mathbf{v} = (cx - y)\mathbf{i} + (y - z)\mathbf{j} + (3x + 4cz)\mathbf{k}$$

of a fluid is incompressible [meaning that $\operatorname{div}(\mathbf{v}) = 0$].

SOLUTION We compute the divergence of $\mathbf{v}$:

$$\operatorname{div}(\mathbf{v}) = \frac{\partial}{\partial x}(cx - y) + \frac{\partial}{\partial y}(y - z) + \frac{\partial}{\partial z}(3x + 4cz) = c + 1 + 4c = 5c + 1$$

Therefore, $\operatorname{div}(\mathbf{v}) = 0$ if $5c + 1 = 0$ or $c = -\frac{1}{5}$.

6. Verify the identity $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$ where $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$. Assume that the components F_j have continuous second-order derivatives.

SOLUTION Let $\mathbf{F} = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$. We compute the curl of $\mathbf{F}$:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

The divergence of $\operatorname{curl}(\mathbf{F})$ is thus

$$\begin{aligned} \operatorname{div}(\operatorname{curl}(\mathbf{F})) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \end{aligned}$$

$$= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}\right)$$

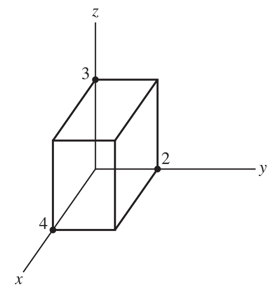
Since the second-order partials are continuous, the mixed partials are equal. Therefore,

$$\text{div}\,(\text{curl}(\mathbf{F})) = 0$$

In Exercises 7–10, verify the Divergence Theorem for the vector field and region.

7. $\mathbf{F} = \langle z, x, y \rangle$, the box $[0, 4] \times [0, 2] \times [0, 3]$

SOLUTION Let $\mathcal{S}$ be the surface of the box and $\mathcal{R}$ the region enclosed by $\mathcal{S}$.



We first compute the surface integral in the Divergence Theorem:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{R}} \text{div}(\mathbf{F})\, dV \tag{1}$$

We denote by $\mathcal{S}_i$, $i = 1, \dots, 6$, the faces of the box, starting at the face on the xz-plane and moving counterclockwise, then moving to the bottom and the top. We use parametrizations

$$\begin{aligned}
\mathcal{S}_1:\ & \Phi_1(x, z) = (x, 0, z), \quad 0 \le x \le 4, \quad 0 \le z \le 3 \\
& \mathbf{n} = \langle 0, -1, 0 \rangle \\
\mathcal{S}_2:\ & \Phi_2(y, z) = (0, y, z), \quad 0 \le y \le 2, \quad 0 \le z \le 3 \\
& \mathbf{n} = \langle -1, 0, 0 \rangle \\
\mathcal{S}_3:\ & \Phi_3(x, z) = (x, 2, z), \quad 0 \le x \le 4, \quad 0 \le z \le 3 \\
& \mathbf{n} = \langle 0, 1, 0 \rangle \\
\mathcal{S}_4:\ & \Phi_4(y, z) = (4, y, z), \quad 0 \le y \le 2, \quad 0 \le z \le 3 \\
& \mathbf{n} = \langle 1, 0, 0 \rangle \\
\mathcal{S}_5:\ & \Phi_5(x, y) = (x, y, 0), \quad 0 \le x \le 4, \quad 0 \le y \le 2 \\
& \mathbf{n} = \langle 0, 0, -1 \rangle \\
\mathcal{S}_6:\ & \Phi_6(x, y) = (x, y, 3), \quad 0 \le x \le 4, \quad 0 \le y \le 2 \\
& \mathbf{n} = \langle 0, 0, 1 \rangle
\end{aligned}$$

Then,

$$\begin{aligned}
\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^4 \mathbf{F}\,(\Phi_1(x, z)) \cdot \langle 0, -1, 0 \rangle\, dx\, dz = \int_0^3 \int_0^4 \langle z, x, 0 \rangle \cdot \langle 0, -1, 0 \rangle\, dx\, dz \\
&= \int_0^3 \int_0^4 -x\, dx\, dz = 3 \frac{-x^2}{2}\bigg|_0^4 = -24 \\
\iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^2 \mathbf{F}\,(\Phi_2(y, z)) \cdot \langle -1, 0, 0 \rangle\, dy\, dz = \int_0^3 \int_0^2 \langle z, 0, y \rangle \cdot \langle -1, 0, 0 \rangle\, dy\, dz \\
&= \int_0^3 \int_0^2 -z\, dy\, dz = 2 \cdot \frac{-z^2}{2}\bigg|_0^3 = -9 \\
\iint_{\mathcal{S}_3} \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^4 \mathbf{F}\,(\Phi_3(x, z)) \cdot \langle 0, 1, 0 \rangle\, dx\, dz = \int_0^3 \int_0^4 \langle z, x, 2 \rangle \cdot \langle 0, 1, 0 \rangle\, dx\, dz
\end{aligned}$$

$$= \int_0^3 \int_0^4 x\,dx\,dz = 3 \cdot \frac{x^2}{2}\Big|_0^4 = 24$$

$$\iint_{\mathcal{S}_4} \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_0^2 \mathbf{F}\left(\Phi_4(y,z)\right) \cdot \langle 1,0,0\rangle\,dy\,dz = \int_0^3 \int_0^2 \langle z,4,y\rangle \cdot \langle 1,0,0\rangle\,dy\,dz$$

$$= \int_0^3 \int_0^2 z\,dy\,dz = 2 \cdot \frac{z^2}{2}\Big|_0^3 = 9$$

$$\iint_{\mathcal{S}_5} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^4 \mathbf{F}\left(\Phi_5(x,y)\right) \cdot \langle 0,0,-1\rangle\,dx\,dy = \int_0^2 \int_0^4 \langle 0,x,y\rangle \cdot \langle 0,0,-1\rangle\,dx\,dy$$

$$= \int_0^2 \int_0^4 -y\,dx\,dy = 4 \cdot \frac{-y^2}{2}\Big|_0^2 = -8$$

$$\iint_{\mathcal{S}_6} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^4 \mathbf{F}\left(\Phi_6(x,y)\right) \cdot \mathbf{n}\,dx\,dy = \int_0^2 \int_0^4 \langle 3,x,y\rangle \cdot \langle 0,0,1\rangle\,dx\,dy$$

$$= \int_0^2 \int_0^4 y\,dx\,dy = 4 \cdot \frac{y^2}{2}\Big|_0^2 = 8$$

We add the integrals to obtain the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{\mathcal{S}_i} \mathbf{F} \cdot d\mathbf{S} = -24 - 9 + 24 + 9 - 8 + 8 = 0 \qquad (2)$$

We now evaluate the triple integral in (1). We compute the divergence of $\mathbf{F} = \langle z, x, y\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = 0$$

Hence,

$$\iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{R}} 0\,dV = 0 \qquad (3)$$

The equality of the integrals in (2) and (3) verifies the Divergence Theorem.

8. $\mathbf{F} = \langle y, x, z\rangle$, the region $x^2 + y^2 + z^2 \le 4$

SOLUTION Let $\mathcal{S}$ be the surface of the sphere and $\mathcal{R}$ the ball enclosed by $\mathcal{S}$. We compute both sides of the Divergence Theorem:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\,dV \qquad (1)$$

Step 1. Integral over sphere. We use the parametrization

$$\mathcal{S}: \Phi(\theta,\phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

$$\mathbf{n} = 4\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

Then,

$$\mathbf{F}\left(\Phi(\theta,\phi)\right) \cdot \mathbf{n} = 2\,\langle\sin\theta\sin\phi, \cos\theta\sin\phi, \cos\phi\rangle \cdot 4\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

$$= 8\left(\sin\theta\cos\theta\sin^3\phi + \cos\theta\sin\theta\sin^3\phi + \cos^2\phi\sin\phi\right)$$

$$= 8\sin 2\theta\sin^3\phi + 8\cos^2\phi\sin\phi$$

The surface integral is thus

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} \mathbf{F}\left(\Phi(\theta,\phi)\right) \cdot \mathbf{n}\,d\theta\,d\phi = \int_0^{\pi} \int_0^{2\pi} \left(8\sin 2\theta\,\sin^3\phi + 8\,\cos^2\phi\sin\phi\right)\,d\theta\,d\phi$$

$$= \left(8\int_0^{2\pi} \sin 2\theta\,d\theta\right)\left(\int_0^{\pi} \sin^3\phi d\phi\right) + 16\pi\int_0^{\pi} \cos^2\phi\sin\phi\,d\phi$$

$$= 0 + 16\pi\left(-\frac{\cos^3\phi}{3}\Big|_0^{\pi}\right) = -\frac{16\pi}{3}(-1-1) = \frac{32\pi}{3}$$

We compute the triple integral in (1):

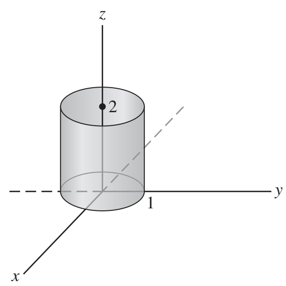
$$\operatorname{div}\mathbf{F} = \operatorname{div}\langle y, x, z\rangle = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 1$$

$$\iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{R}} 1\, dV = \text{Volume}(\mathcal{R}) = \frac{4\pi \cdot 2^3}{3} = \frac{32\pi}{3} \tag{2}$$

The equality of the integrals in (2) and (3) verifies the Divergence Theorem.

9. $\mathbf{F} = \langle 2x, 3z, 3y\rangle$, the region $x^2 + y^2 \le 1, 0 \le z \le 2$

SOLUTION



Let $\mathcal{S}$ be the surface of the cylinder and $\mathcal{R}$ the region enclosed by $\mathcal{S}$. We compute the two sides of the Divergence Theorem:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\, dV \tag{1}$$

We first calculate the surface integral.

Step 1. Integral over the side of the cylinder. The side of the cylinder is parametrized by

$$\Phi(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2$$
$$\mathbf{n} = \langle \cos\theta, \sin\theta, 0\rangle$$

Then,

$$\mathbf{F}(\Phi(\theta, z)) \cdot \mathbf{n} = \langle 2\cos\theta, 3z, 3\sin\theta\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle = 2\cos^2\theta + 3z\sin\theta$$

We obtain the integral

$$\iint_{\text{side}} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^{2\pi} \left(2\cos^2\theta + 3z\sin\theta\right) d\theta\, dz = 4\int_0^{2\pi} \cos^2\theta\, d\theta + \left(\int_0^2 3z\, dz\right)\left(\int_0^{2\pi} \sin\theta\, d\theta\right)$$
$$= 4 \cdot \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\bigg|_0^{2\pi}\right) + 0 = 4\pi$$

Step 2. Integral over the top of the cylinder. The top of the cylinder is parametrized by

$$\Phi(x, y) = (x, y, 2)$$

with parameter domain $\mathcal{D} = \left\{(x, y) : x^2 + y^2 \le 1\right\}$. The upward pointing normal is

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \mathbf{i} \times \mathbf{j} = \mathbf{k} = \langle 0, 0, 1\rangle$$

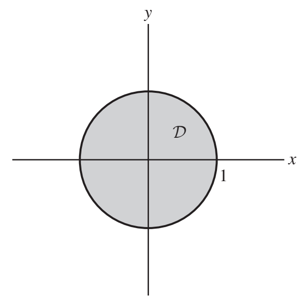
Also,

$$\mathbf{F}(\Phi(x, y)) \cdot \mathbf{n} = \langle 2x, 6, 3y\rangle \cdot \langle 0, 0, 1\rangle = 3y$$

Hence,

$$\iint_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} 3y\, dA = 0$$

The last integral is zero due to symmetry.



Step 3. Integral over the bottom of the cylinder. We parametrize the bottom by

$$\Phi(x, y) = (x, y, 0), \quad (x, y) \in \mathcal{D}$$

The downward pointing normal is $\mathbf{n} = \langle 0, 0, -1\rangle$. Then

$$\mathbf{F}(\Phi(x, y)) \cdot \mathbf{n} = \langle 2x, 0, 3y\rangle \cdot \langle 0, 0, -1\rangle = -3y$$

We obtain the following integral, which is zero due to symmetry:

$$\iint_{\text{bottom}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} -3y\, dA = 0$$

Adding the integrals we get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\text{side}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\text{top}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\text{bottom}} \mathbf{F} \cdot d\mathbf{S} = 4\pi + 0 + 0 = 4\pi \tag{2}$$

Step 4. Compare with integral of divergence.

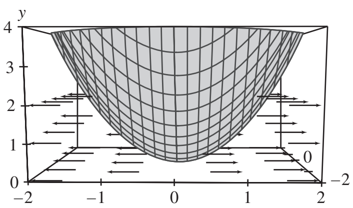
$$\operatorname{div}(\mathbf{F}) = \operatorname{div}\langle 2x, 3z, 3y\rangle = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3z) + \frac{\partial}{\partial z}(3y) = 2$$

$$\iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{R}} 2\, dV = 2\iiint_{\mathcal{R}} dV = 2\,\mathrm{Vol}(\mathcal{R}) = 2 \cdot \pi \cdot 2 = 4\pi \tag{3}$$

The equality of (2) and (3) verifies the Divergence Theorem.

10. $\mathbf{F} = \langle x, 0, 0\rangle$, the region $x^2 + y^2 \le z \le 4$

SOLUTION



Let $\mathcal{S}$ be the surface enclosing the given region $\mathcal{R}$. We must verify the equality

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F})\, dV \tag{1}$$

We first compute the surface integral on the left-hand side.

Step 1. Integral over the side of the surface. The side of the surface is parametrized by

$$\Phi(\theta, t) = \left(t\cos\theta, t\sin\theta, t^2\right), \quad 0 \le t \le 2, \quad 0 \le \theta \le 2\pi$$

The outward pointing normal is

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & 2t \end{vmatrix} = \left\langle 2t^2\cos\theta, 2t^2\sin\theta, -t\right\rangle$$

Also,

$$\mathbf{F}(\Phi(\theta, t)) \cdot \mathbf{n} = \langle t\cos\theta, 0, 0\rangle \cdot \left\langle 2t^2\cos\theta, 2t^2\sin\theta, -t\right\rangle = 2t^3\cos^2\theta$$

The surface integral over the side is

$$\iint_{\text{side}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 2t^3 \cos^2\theta \, dt \, d\theta = \left(\int_0^{2\pi} \cos^2\theta \, d\theta\right)\left(\int_0^2 2t^3 \, dt\right)$$

$$= \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_{\theta=0}^{2\pi}\right)\left(\frac{t^4}{2}\Big|_{t=0}^{2}\right) = 8\pi \tag{2}$$

Step 2. Integral over the top of the surface. The top of the surface is parametrized by $\Phi(x, y) = (x, y, 4)$ with parameter domain $\mathcal{D} = \left\{(x, y) : x^2 + y^2 \le 2\right\}$. The upward pointing normal vector is

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \mathbf{i} \times \mathbf{j} = \mathbf{k} = \langle 0, 0, 1\rangle$$

Also,

$$\mathbf{F}\left(\Phi(x, y)\right) \cdot \mathbf{n} = \langle x, 0, 0\rangle \cdot \langle 0, 0, 1\rangle = 0$$

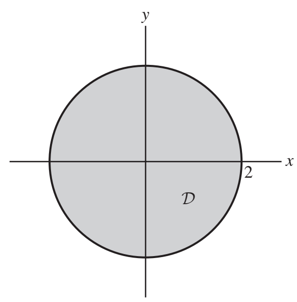
Hence,

$$\iint_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} 0 \, dA = 0 \tag{3}$$

Adding the surface integrals (2) and (3) we get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\text{side}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = 8\pi + 0 = 8\pi$$

Step 3. Compare with integral of divergence.



We compute the divergence:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 1$$

We obtain the following triple integral:

$$\iiint_{\mathcal{R}} \operatorname{div}(\mathbf{F}) \cdot dV = \iiint_{\mathcal{R}} 1 \, dV = \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{4} dz\right) dx\, dy = \iint_{\mathcal{D}} \left(4 - (x^2 + y^2)\right) dx\, dy$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = 2\pi \int_0^2 \left(4r - r^3\right) dr = 2\pi \left(2r^2 - \frac{r^4}{4}\Big|_0^2\right)$$

$$= 2\pi \cdot 4 = 8\pi \tag{4}$$

The equality of (2) and (4) verifies the Divergence Theorem.

In Exercises 11–18, use the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

11. $\mathbf{F} = \left\langle 0, 0, z^3/3\right\rangle$, $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION We compute the divergence of $\mathbf{F} = \left\langle 0, 0, z^3/3\right\rangle$:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z^3/3) = z^2$$

Hence, by the Divergence Theorem ($\mathcal{W}$ is the unit ball),

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}} z^2\,dV$$

Computing this integral we see:

$$\begin{aligned}
\iiint_{\mathcal{W}} z^2\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cos^2\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi} d\theta\cdot\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\cdot\int_0^1\rho^4\,d\rho \\
&= (2\pi)\cdot\left(-\frac{\cos^3\phi}{3}\bigg|_0^{\pi}\right)\cdot\left(\frac{\rho^5}{5}\bigg|_0^1\right) \\
&= 2\pi\left(-\frac{1}{3}(-1-1)\right)\left(\frac{1}{5}\right) \\
&= 2\pi\left(\frac{2}{3}\right)\left(\frac{1}{5}\right) = \frac{4\pi}{15}
\end{aligned}$$

12. $\mathbf{F} = \langle y, z, x\rangle$, $\mathcal{S}$ is the sphere $x^2+y^2+z^2=1$.

SOLUTION We compute the divergence of $\mathbf{F} = \langle y, z, x\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(x) = 0+0+0 = 0$$

Hence, by the Divergence Theorem ($\mathcal{W}$ is the unit ball),

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}} 0\,dV = 0$$

13. $\mathbf{F} = \left\langle x^3, 0, z^3\right\rangle$, $\mathcal{S}$ is the octant of the sphere $x^2+y^2+z^2=4$, in the first octant $x\ge 0$, $y\ge 0$, $z\ge 0$.

SOLUTION We compute the divergence of $\mathbf{F} = \left\langle x^3, 0, z^3\right\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z^3) = 3x^2+3z^2 = 3(x^2+z^2)$$

Using the Divergence Theorem we obtain ($\mathcal{W}$ is the region inside the sphere)

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}} 3(x^2+z^2)\,dV$$

We convert the integral to spherical coordinates. We have

$$\begin{aligned}
x^2+z^2 &= \rho^2\cos^2\theta\sin^2\phi + \rho^2\cos^2\phi = \rho^2\cos^2\theta\sin^2\phi + \rho^2(1-\sin^2\phi) \\
&= -\rho^2\sin^2\phi(1-\cos^2\theta) + \rho^2 = -\rho^2\sin^2\phi\sin^2\theta + \rho^2 = \rho^2(1-\sin^2\phi\sin^2\theta)
\end{aligned}$$

We obtain the following integral:

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2(1-\sin^2\phi\sin^2\theta)\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4(\sin\phi - \sin^3\phi\sin^2\theta)d\rho\,d\phi\,d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4\sin\phi\,d\rho\,d\phi\,d\theta - 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4\sin^3\phi\sin^2\theta\,d\rho\,d\phi\,d\theta \\
&= 6\pi\left(\int_0^{\pi/2}\sin\phi\,d\phi\right)\left(\int_0^2\rho^4\,d\rho\right) - 3\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\,d\phi\right)\left(\int_0^2\rho^4\,d\rho\right) \\
&= 6\pi\left(-\cos\phi\bigg|_{\phi=0}^{\pi/2}\right)\left(\frac{\rho^5}{5}\bigg|_{\rho=0}^{2}\right)\left(-3\frac{\theta}{2} - \frac{\sin 2\theta}{4}\bigg|_{\theta=0}^{2\pi}\right)\cdot\left(-\frac{\sin^2\phi\cos\phi}{3} - \frac{2}{3}\cos\phi\bigg|_{\phi=0}^{\pi/2}\right)\left(\frac{\rho^5}{5}\bigg|_{\rho=0}^{2}\right) \\
&= 6\pi\cdot\frac{32}{5} - 3\pi\cdot\frac{2}{3}\cdot\frac{32}{5} = \frac{128\pi}{5}
\end{aligned}$$

14. $\mathbf{F} = \langle e^{x+y}, e^{x+z}, e^{x+y} \rangle$, $\mathcal{S}$ is the boundary of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

SOLUTION Let $\mathcal{W}$ denote the box $[0,1] \times [0,1] \times [0,1]$. By the Divergence Theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV$$

We compute the divergence of $\mathbf{F} = \langle e^{x+y}, e^{x+z}, e^{x+y} \rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(e^{x+y}) + \frac{\partial}{\partial y}(e^{x+z}) + \frac{\partial}{\partial z}(e^{x+y}) = e^{x+y} + 0 + 0 = e^{x+y}$$

By the Divergence Theorem we have

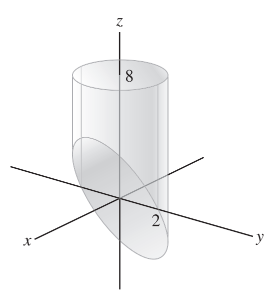
$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} e^{x+y}\, dV$$

We compute the triple integral:

$$\begin{aligned} \iiint_{\mathcal{W}} e^{x+y}\, dV &= \int_0^1 \int_0^1 \int_0^1 e^x \cdot e^y \, dx\,dy\,dz \\ &= \int_0^1 dz \cdot \int_0^1 e^x\, dx \cdot \int_0^1 e^y\, dy \\ &= (1) \cdot e^x \Big|_0^1 \cdot e^y \Big|_0^1 = (e-1)^2 \end{aligned}$$

15. $\mathbf{F} = \langle x, y^2, z+y \rangle$, $\mathcal{S}$ is the boundary of the region contained in the cylinder $x^2 + y^2 = 4$ between the planes $z = x$ and $z = 8$.

SOLUTION Let $\mathcal{W}$ be the region enclosed by $\mathcal{S}$.



We compute the divergence of $\mathbf{F} = \langle x, y^2, z+y \rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z+y) = 1 + 2y + 1 = 2 + 2y.$$

By the Divergence Theorem we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} (2+2y)\, dV$$

We compute the triple integral. Denoting by $\mathcal{D}$ the disk $x^2 + y^2 \le 4$ in the xy-plane, we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \int_x^8 (2+2y)\, dz\, dx\, dy = \iint_{\mathcal{D}} (2+2y) z \Big|_{z=x}^{8} dx\, dy = \iint_{\mathcal{D}} (2+2y)(8-x)\, dx\, dy$$

We convert the integral to polar coordinates:

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^2 (2 + 2r\sin\theta)(8 - r\cos\theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left(16r + 2r^2(8\sin\theta - \cos\theta) - r^3 \sin 2\theta\right) dr\, d\theta \\ &= \int_0^{2\pi} 8r^2 + \frac{2}{3} r^3 (8\sin\theta - \cos\theta) - \frac{r^4}{4} \sin 2\theta \Big|_{r=0}^{2} d\theta \end{aligned}$$

$$= \int_0^{2\pi} \left(32 + \frac{16}{3}(8\sin\theta - \cos\theta) - 4\sin 2\theta \right) d\theta$$

$$= 64\pi + \frac{128}{3}\int_0^{2\pi} \sin\theta\, d\theta - \frac{16}{3}\int_0^{2\pi} \cos\theta\, d\theta - \int_0^{2\pi} 4\sin 2\theta\, d\theta = 64\pi$$

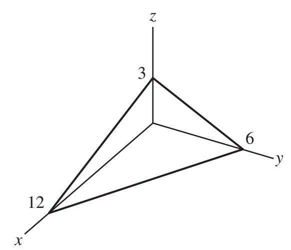
16. $\mathbf{F} = \left\langle x^2 - z^2, e^{z^2} - \cos x, y^3 \right\rangle$, $\mathcal{S}$ is the boundary of the region bounded by $x + 2y + 4z = 12$ and the coordinate planes in the first octant.

SOLUTION We compute the divergence of $\mathbf{F} = \left\langle x^2 - z^2, e^{z^2} - \cos x, y^3 \right\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^2 - z^2) + \frac{\partial}{\partial y}\left(e^{z^2} - \cos x\right) + \frac{\partial}{\partial z}(y^3) = 2x.$$

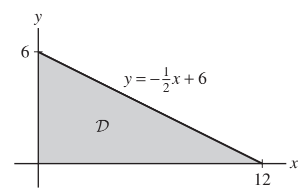
By the Divergence Theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 2x\, dV.$$



To compute the triple integral, we describe the region $\mathcal{W}$ by the inequalities

$$0 \le x \le 12, \quad 0 \le y \le -\frac{x}{2} + 6, \quad 0 \le z \le 3 - \frac{y}{2} - \frac{x}{4}.$$



Thus,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{12} \int_0^{-x/2+6} \int_0^{3-\frac{y}{2}-\frac{x}{4}} 2x\, dz\, dy\, dx = \int_0^{12} \int_0^{-x/2+6} 2xz \Big|_{z=0}^{3-\frac{y}{2}-\frac{x}{4}} dy\, dx$$

$$= \int_0^{12} \int_0^{-x/2+6} 2x\left(3 - \frac{y}{2} - \frac{x}{4}\right) dy\, dx = \int_0^{12} 2x \left(3y - \frac{y^2}{4} - \frac{xy}{4}\right) \Big|_{y=0}^{-x/2+6} dx$$

$$= \int_0^{12} 2x \left(\left(3 - \frac{x}{4}\right)\left(6 - \frac{x}{2}\right) - \frac{\left(6 - \frac{x}{2}\right)^2}{4} \right) dx$$

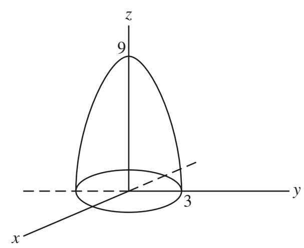
We let $u = 6 - \frac{x}{2}$ and $du = -\frac{1}{2}\, dx$:

$$\int_0^{12} 2x \cdot \frac{\left(6 - \frac{x}{2}\right)^2}{4}\, dx = \int_0^6 2(6-u)u^2\, du = 4u^3 - \frac{1}{2}u^4 \Big|_0^6 = 216.$$

17. $\mathbf{F} = \langle x + y, z, z - x \rangle$, $\mathcal{S}$ is the boundary of the region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

SOLUTION We compute the divergence of $\mathbf{F} = \langle x + y, z, z - x \rangle$,

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(z - x) = 1 + 0 + 1 = 2.$$

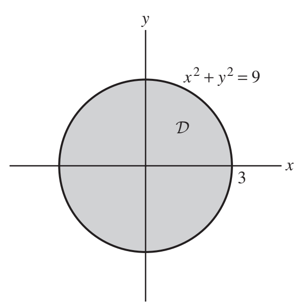


Using the Divergence Theorem we have

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}} 2\,dV$$

We compute the triple integral:

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} 2\,dV = \iint_{\mathcal{D}} \int_0^{9-x^2-y^2} 2\,dz\,dx\,dy = \iint_{\mathcal{D}} 2z\Big|_0^{9-x^2-y^2}\,dx\,dy$$
$$= \iint_{\mathcal{W}} 2(9-x^2-y^2)\,dx\,dy$$



We convert the integral to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^3 2\left(9-r^2\right) r\,dr\,d\theta = 4\pi\int_0^3 (9r-r^3)\,dr = 4\pi\left(\frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^3\right) = 81\pi$$

18. $\mathbf{F} = \left\langle e^{z^2}, 2y+\sin(x^2z), 4z+\sqrt{x^2+9y^2}\right\rangle$, $\mathcal{S}$ is the region $x^2+y^2 \le z \le 8-x^2-y^2$.

SOLUTION First, let us solve for the boundary (or intersection) of the two surfaces:

$$8-x^2-y^2 = x^2+y^2$$
$$8 = 2x^2+2y^2$$
$$x^2+y^2 = 4$$

The intersection of the two surfaces is a circle of radius 2 centered at the origin.

We compute the divergence of $\mathbf{F} = \left\langle e^{z^2}, 2y+\sin(x^2z), 4z+\sqrt{x^2+9y^2}\right\rangle$:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^{z^2}) + \frac{\partial}{\partial y}(2y+\sin(x^2z)) + \frac{\partial}{\partial z}(4z+\sqrt{x^2+9y^2}) = 0+2+4 = 6$$

Using the Divergence Theorem we have

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}} 6\,dV$$

We compute the triple integral:

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} 6\,dV = \iint_{\mathcal{D}} \int_{x^2+y^2}^{8-x^2-y^2} 6\,dz\,dx\,dy$$

We convert this triple integral to cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le r \le 2, 0 \le \theta \le 2\pi$$

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= 6 \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\, dz\, dr\, d\theta \\
&= 6 \int_0^{2\pi} d\theta \cdot \int_0^2 r[(8-r^2) - r^2]\, dr \\
&= 6(2\pi) \int_0^2 8r - 2r^3\, dr = 12\pi \left(4r^2 - \frac{1}{2} r^4 \Big|_0^2 \right) \\
&= 12\pi(16-8) = 96\pi
\end{aligned}$$

19. Calculate the flux of the vector field $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + \mathbf{k}$ through the surface $\mathcal{S}$ in Figure 18. *Hint:* Apply the Divergence Theorem to the closed surface consisting of $\mathcal{S}$ and the unit disk.

SOLUTION From the diagram in the book, $\mathcal{S}$ is the surface in question bounded by the unit circle. Let T be the union of $\mathcal{S}$ and the unit disk D. Then T is a closed surface, and we may apply the Divergence Theorem:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_D \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F}) \cdot d\mathbf{S}$$

where $\mathcal{W}$ is the region enclosed by T. Now we observe that $\mathbf{F} = \left\langle 2xy, -y^2, 1 \right\rangle$ and we compute the divergence of $\mathbf{F}$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(1) = 2y - 2y + 0 = 0$$

Therefore, the triple integral is zero and we obtain:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \mathbf{F} \cdot d\mathbf{S} \tag{1}$$

where D is oriented with a downward pointing normal. Let $\Phi(r, \theta) = (r\cos\theta, r\sin\theta, 0)$ be the parametrization of D with polar coordinates. Then

$$\mathbf{F}(\Phi(r,\theta)) = \left\langle 2r^2 \cos\theta \sin\theta, -r^2 \sin^2\theta, 1 \right\rangle$$

Furthermore,

$$\Phi_r(r,\theta) = \langle \cos\theta, \sin\theta, 0 \rangle, \quad \Phi_\theta(r,\theta) = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

and $\Phi_r \times \Phi_\theta = \langle 0, 0, r \rangle$ is an upward pointing normal. Finally,

$$\mathbf{F} \cdot d\mathbf{S} = \mathbf{F}(\Phi(r,\theta)) \cdot (\Phi_r \times \Phi_\theta)\, dr\, d\theta = r\, dr\, d\theta$$

The integral on the right in (1) uses a downward pointing normal, so we may drop the minus sign and use the upward-pointing normal to obtain:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \pi$$

20. Let $\mathcal{S}_1$ be the closed surface consisting of $\mathcal{S}$ in Figure 18 together with the unit disk. Find the volume enclosed by $\mathcal{S}_1$, assuming that

$$\iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\mathbf{S} = 72$$

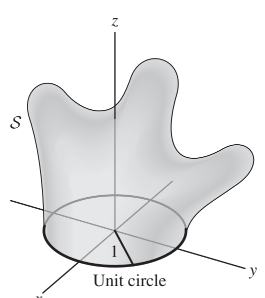


FIGURE 18 Surface $\mathcal{S}$ whose boundary is the unit circle.

SOLUTION First, let $\mathbf{F} = \langle x, 2y, 3z \rangle$. By the Divergence Theorem we can conclude

$$\iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}\mathbf{F}\, dV$$

where

$$\text{div}\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1 + 2 + 3 = 6$$

So then

$$72 = \iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}\mathbf{F}\, dV = \iiint_{\mathcal{W}} 6\, dV = 6 \cdot \text{Volume}(\mathcal{W})$$

Therefore, Volume($\mathcal{W}$) = 12.

21. Let $\mathcal{S}$ be the half-cylinder $x^2 + y^2 = 1, x \geq 0, 0 \leq z \leq 1$. Assume that $\mathbf{F}$ is a horizontal vector field (the z-component is zero) such that $\mathbf{F}(0, y, z) = zy^2\mathbf{i}$. Let $\mathcal{W}$ be the solid region enclosed by $\mathcal{S}$, and assume that

$$\iiint_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = 4$$

Find the flux of $\mathbf{F}$ through the curved side of $\mathcal{S}$.

SOLUTION The flux through the top and bottom of the surface are zero. The flux through the flat side (with outward normal $-\mathbf{i}$) is

$$-\int_{z=0}^{1} \int_{y=-1}^{1} zy^2\, dy dz = -\frac{1}{2}\left(\frac{2}{3}\right) = -\frac{1}{3}$$

The flux through the curved side is $4 + \frac{1}{3} = \frac{13}{3}$.

22. Volume as a Surface Integral Let $\mathbf{F} = \langle x, y, z \rangle$. Prove that if $\mathcal{W}$ is a region $\mathbf{R}^3$ with a smooth boundary $\mathcal{S}$, then

$$\text{Volume}(\mathcal{W}) = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \qquad \boxed{10}$$

SOLUTION Using the volume as a triple integral we have

$$\text{Volume}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\, dV \qquad (1)$$

We compute the surface integral of $\mathbf{F}$ over $\mathcal{S}$, using the Divergence Theorem. Since $\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$, we get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 3\, dV = 3 \iiint_{\mathcal{W}} 1\, dV \qquad (2)$$

We combine (1) and (2) to obtain

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = 3 \cdot \text{volume}(\mathcal{W})$$

or

$$\text{Volume}(\mathcal{W}) = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

23. Use Eq. (10) to calculate the volume of the unit ball as a surface integral over the unit sphere.

SOLUTION Let $\mathcal{S}$ be the unit sphere and $\mathcal{W}$ is the unit ball. By Eq. (10) we have

$$\text{Volume}(\mathcal{W}) = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},\ \mathbf{F} = \langle x, y, z \rangle$$

To compute the surface integral, we parametrize $\mathcal{S}$ by

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$$\mathbf{n} = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

Then

$$\mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n} = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\cdot\left\langle\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos\phi\sin\phi\right\rangle$$
$$= \cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \cos^2\phi\sin\phi = \sin^3\phi(\cos^2\theta+\sin^2\theta) + \cos^2\phi\sin\phi$$
$$= \sin^3\phi + \cos^2\phi\sin\phi = \sin^3\phi + (1-\sin^2\phi)\sin\phi = \sin\phi$$

We obtain the following integral:

$$\text{Volume}(\mathcal{W}) = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{1}{3}\cdot 2\pi\int_0^{\pi}\sin\phi\,d\phi = \frac{2\pi}{3}\left(-\cos\phi\Big|_0^{\pi}\right) = \frac{2\pi}{3}(1+1) = \frac{4\pi}{3}$$

24. Verify that Eq. (10) applied to the box $[0,a]\times[0,b]\times[0,c]$ yields the volume $V = abc$.

SOLUTION Recall the result

$$\begin{aligned}\text{Volume}(\mathcal{W}) &= \frac{1}{3}\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S}\\ &= \frac{1}{3}\iiint_{\mathcal{W}}\operatorname{div}(\mathbf{F})\,dV\\ &= \frac{1}{3}\iiint_{\mathcal{W}}(1+1+1)\,dV = \iiint_{\mathcal{W}}dV\\ &= \int_0^c\int_0^b\int_0^a dV = abc\end{aligned}$$

25. Let $\mathcal{W}$ be the region in Figure 19 bounded by the cylinder $x^2+y^2=4$, the plane $z=x+1$, and the xy-plane. Use the Divergence Theorem to compute the flux of $\mathbf{F}=\left\langle z, x, y+z^2\right\rangle$ through the boundary of $\mathcal{W}$.

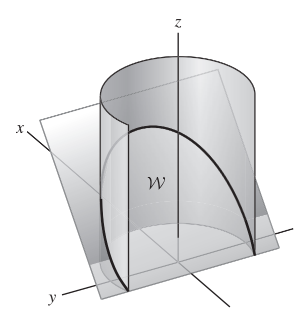


FIGURE 19

SOLUTION We compute the divergence of $\mathbf{F}=\left\langle z, x, y+z^2\right\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y+z^2) = 2z$$

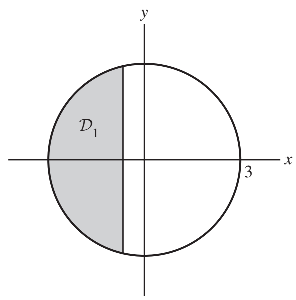
By the Divergence Theorem we have

$$\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}}\operatorname{div}(\mathbf{F})\,dV = \iiint_{\mathcal{W}}2\,dV$$

To compute the triple integral, we identify the projection $\mathcal{D}$ of the region on the xy-plane. $\mathcal{D}$ is the region in the xy plane enclosed by the circle $x^2+y^2=4$ and the line $0=x+1$ or $x=-1$. We obtain the following integral:

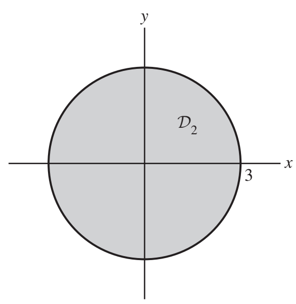
$$\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}}2z\,dV = \iint_{\mathcal{D}}\int_0^{x+1}2z\,dz\,dx\,dy = \iint_{\mathcal{D}}z^2\Big|_{z=0}^{x+1}dx\,dy = \iint_{\mathcal{D}}(x+1)^2\,dx\,dy$$

We compute the double integral as the difference of two integrals: the integral over the disk $\mathcal{D}_2$ of radius 2, and the integral over the part $\mathcal{D}_1$ of the disk, shown in the figure.



We obtain

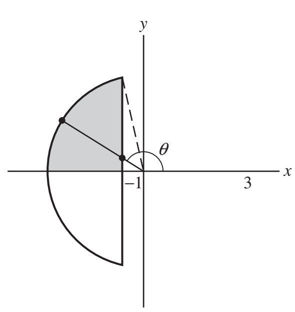
$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}_2} (x+1)^2\, dx\, dy - \iint_{\mathcal{D}_1} (x+1)^2\, dx\, dy$$



We compute the first integral, converting to polar coordinates:

$$\begin{aligned}
\iint_{\mathcal{D}_2} (x+1)^2\, dx\, dy &= \int_0^{2\pi} \int_0^2 (r\cos\theta + 1)^2 r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r^3 \cos^2\theta + 2r^2\cos\theta + r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{r^4}{4}\cos^2\theta + \frac{2}{3}r^3\cos\theta + \frac{1}{2}r^2 \Big|_0^2\, d\theta \\
&= \int_0^{2\pi} 4\cos^2\theta + \frac{16}{3}\cos\theta + 2\, d\theta \\
&= \int_0^{2\pi} 2\cos 2\theta + \frac{16}{3}\cos\theta + 4\, d\theta \\
&= \sin 2\theta + \frac{16}{3}\sin\theta + 4\theta \Big|_0^{2\pi} = 8\pi
\end{aligned}$$

We compute the second integral over the upper part of $\mathcal{D}_1$. Due to symmetry, this integral is equal to half of the integral over $\mathcal{D}_1$.



We describe the region in polar coordinates:

$$\frac{2\pi}{3} \le \theta \le \pi, \quad \frac{-1}{\cos\theta} \le r \le 2$$

Then

$$\begin{aligned}
\iint_{\mathcal{D}_1} (x+1)^2\,dx\,dy &= 2\int_{2\pi/3}^{\pi}\int_{-1/\cos\theta}^{2} (r\cos\theta+1)^2 r\,dr\,d\theta \\
&= \int_{2\pi/3}^{\pi}\int_{-1/\cos\theta}^{2} (r^3\cos^2\theta + 2r^2\cos\theta + r)\,dr\,d\theta \\
&= \int_{2\pi/3}^{\pi} \frac{r^4}{4}\cos^2\theta + \frac{2}{3}r^3\cos\theta + \frac{1}{2}r^2\Big|_{r=\frac{-1}{\cos\theta}}^{2}\,d\theta \\
&= 2\int_{2\pi/3}^{\pi} (4\cos^2\theta + \frac{16}{3}\cos\theta + 2) - \left(\frac{\cos^2\theta}{4\cos^4\theta} - \frac{2\cos\theta}{3\cos^3\theta} + \frac{1}{2\cos^2\theta}\right)d\theta \\
&= 2\int_{2\pi/3}^{\pi} 2\cos 2\theta + 4 + \frac{16}{3}\cos\theta - \frac{1}{4}\sec^2\theta + \frac{2}{3}\sec^2\theta - \frac{1}{2}\sec^2\theta\,d\theta \\
&= 2\int_{2\pi/3}^{\pi} 2\cos 2\theta + 4 + \frac{16}{3}\cos\theta - \frac{1}{12}\sec^2\theta\,d\theta \\
&= \sin 2\theta + 4\theta + \frac{16}{3}\sin\theta - \frac{1}{12}\tan\theta\Big|_{2\pi/3}^{\pi} \\
&= 2(4\pi) - 2\left(\sin\frac{4\pi}{3} + \frac{8\pi}{3} + \frac{16}{3}\sin\frac{2\pi}{3} - \frac{1}{12}\tan\frac{2\pi}{3}\right) \\
&= 8\pi - 2\left(-\frac{\sqrt{3}}{2} + \frac{8\pi}{3} + \frac{16\sqrt{3}}{6} + \frac{\sqrt{3}}{12}\right) \\
&= 8\pi + \sqrt{3} - \frac{16\pi}{3} - \frac{16\sqrt{3}}{3} - \frac{\sqrt{3}}{6} \\
&= \frac{8\pi}{3} + \sqrt{3}\left(1 - \frac{16}{3} - \frac{1}{6}\right) \\
&= \frac{8\pi}{3} - \frac{9}{2}\sqrt{3}
\end{aligned}$$

so we have

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = 8\pi - \iint_{\mathcal{D}_1} (x+1)^2\,dx\,dy \approx 8\pi - \left(\frac{8\pi}{3} - \frac{9}{2}\sqrt{3}\right) = \frac{16\pi}{3} + \frac{9}{2}\sqrt{3} \approx 24.550.$$

26. Let $I = \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F} = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2} \right\rangle$$

$(r = \sqrt{x^2+y^2+z^2})$ and $\mathcal{S}$ is the boundary of a region $\mathcal{W}$.

(a) Check that $\mathbf{F}$ is divergence-free.

(b) Show that $I = 0$ if $\mathcal{S}$ is a sphere centered at the origin. Explain, however, why the Divergence Theorem cannot be used to prove this.

SOLUTION

(a) To find $\mathrm{div}(\mathbf{F})$, we first compute the partial derivatives of $r = \sqrt{x^2+y^2+z^2}$:

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

We compute the partial derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{2yz}{r^2}\right) &= 2yz\frac{\partial}{\partial x}(r^{-2}) = 2yz\cdot(-2)r^{-3}\frac{\partial r}{\partial x} = -4yz\cdot r^{-3}\frac{x}{r} = -4xyzr^{-4} \\
\frac{\partial}{\partial y}\left(-\frac{xz}{r^2}\right) &= -xz\frac{\partial}{\partial y}(r^{-2}) = -xz\cdot(-2)r^{-3}\frac{\partial r}{\partial y} = 2zx\cdot r^{-3}\frac{y}{r} = 2xyzr^{-4} \\
\frac{\partial}{\partial z}\left(-\frac{xy}{r^2}\right) &= -xy\frac{\partial}{\partial z}(r^{-2}) = -xy\cdot(-2)r^{-3}\frac{\partial r}{\partial z} = 2xyr^{-3}\frac{z}{r} = 2xyzr^{-4}
\end{aligned}$$

The divergence of **F** is the sum of these partials. That is,

$$\operatorname{div}(\mathbf{F}) = -4xyzr^{-4} + 2xyzr^{-4} + 2xyzr^{-4} = 0$$

We conclude that **F** is divergence-free.

(b) We compute the flux of **F** over $\mathcal{S}$, using the following parametrization:

$$\mathcal{S} : \Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$
$$\mathbf{n} = R^2 \sin\phi\, \mathbf{e}_r \quad \text{where } \mathbf{e}_r = r^{-1}\langle x, y, z\rangle$$

We compute the dot product:

$$\mathbf{F} \cdot \mathbf{n} = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2} \right\rangle \cdot \langle x, y, z\rangle\, r^{-1} \cdot R^2 \sin\phi = (2xyz - xyz - xyz)r^{-3} \cdot R^2 \sin\phi = 0$$

Therefore, $\mathbf{F}(\Phi(\theta, \phi)) \cdot \mathbf{n} = 0$, so we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \mathbf{F}(\Phi(\theta, \phi)) \cdot \mathbf{n}\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} 0\, d\phi\, d\theta = 0$$

The Divergence Theorem cannot be used since **F** is not defined at the origin, which is inside the ball with the boundary $\mathcal{S}$.

27. The velocity field of a fluid **v** (in meters per second) has divergence $\operatorname{div}(\mathbf{v})(P) = 3$ at the point $P = (2, 2, 2)$. Estimate the flow rate out of the sphere of radius 0.5 centered at P.

SOLUTION

$$\text{flow rate through the box} \approx \operatorname{div}(\mathbf{v})(P) \cdot \left(\frac{4}{3}\pi(0.5)^3\right) = \frac{\pi}{2} \approx 1.57\ \text{m}^3/\text{s}$$

28. A hose feeds into a small screen box of volume 10 cm^3 that is suspended in a swimming pool. Water flows across the surface of the box at a rate of 12 cm^3/s. Estimate $\operatorname{div}(\mathbf{v})(P)$, where **v** is the velocity field of the water in the pool and P is the center of the box. What are the units of $\operatorname{div}(\mathbf{v})(P)$?

SOLUTION

$$\text{flow rate through the box} = 12 \approx \operatorname{div}(\mathbf{v})(P) \cdot (10)$$

Thus $\operatorname{div}(\mathbf{v})(P) \approx 1.2\ \text{sec}^{-1}$.

29. The electric field due to a unit electric dipole oriented in the **k**-direction is $\mathbf{E} = \nabla(z/r^3)$, where $r = (x^2 + y^2 + z^2)^{1/2}$ (Figure 20). Let $\mathbf{e}_r = r^{-1}\langle x, y, z\rangle$.

(a) Show that $\mathbf{E} = r^{-3}\mathbf{k} - 3zr^{-4}\mathbf{e}_r$.

(b) Calculate the flux of **E** through a sphere centered at the origin.

(c) Calculate $\operatorname{div}(\mathbf{E})$.

(d) Can we use the Divergence Theorem to compute the flux of **E** through a sphere centered at the origin?

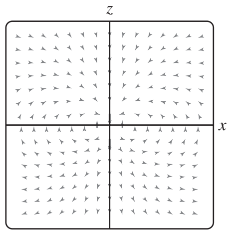


FIGURE 20 The dipole vector field restricted to the xz-plane.

SOLUTION

(a) We first compute the partial derivatives of r:

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x = \frac{x}{r}$$
$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y = \frac{y}{r}$$
$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2z = \frac{z}{r} \tag{1}$$

We compute the partial derivatives of $\frac{z}{r^3}$, using the Chain Rule and the partial derivatives in (1):

$$\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = z\frac{\partial}{\partial x}(r^{-3}) = z\cdot(-3)r^{-4}\frac{\partial r}{\partial x} = -3z\cdot r^{-4}\frac{x}{r} = -\frac{3zx}{r^5} = -3zr^{-5}x$$

$$\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) = z\frac{\partial}{\partial y}(r^{-3}) = z\cdot(-3)r^{-4}\frac{\partial r}{\partial y} = -3z\cdot r^{-4}\frac{y}{r} = -3zr^{-5}y$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{\partial}{\partial z}(z\cdot r^{-3}) = 1\cdot r^{-3} + z\cdot(-3)r^{-4}\frac{\partial r}{\partial z} = r^{-3} - 3z\cdot r^{-4}\cdot\frac{z}{r} = r^{-3} - 3z^2r^{-5}$$

Therefore,

$$\begin{aligned}\mathbf{E} = \nabla\left(\frac{z}{r^3}\right) &= -3zr^{-5}x\mathbf{i} - 3zr^{-5}y\mathbf{j} + (r^{-3} - 3z^2r^{-5})\mathbf{k}\\ &= r^{-3}\mathbf{k} - 3zr^{-4}\cdot r^{-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = r^{-3}\mathbf{k} - 3zr^{-4}\mathbf{e}_r\end{aligned}$$

(b) To compute the flux $\iint_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{S}$ we use the parametrization $\Phi(\theta,\phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$, $0\le\theta\le 2\pi$, $0\le\phi\le\pi$:

$$\mathbf{n} = R^2\sin\phi\mathbf{e}_r$$

We compute $\mathbf{E}(\Phi(\theta,\phi))\cdot\mathbf{n}$. Since $r = R$ on $\mathcal{S}$, we get

$$\begin{aligned}\mathbf{E}(\Phi(\theta,\phi))\cdot\mathbf{n} &= \left(R^{-3}\mathbf{k} - 3zR^{-4}\mathbf{e}_r\right)\cdot R^2\sin\phi\mathbf{e}_r = R^{-1}\sin\phi\mathbf{k}\cdot\mathbf{e}_r - 3zR^{-2}\sin\phi\\ &= R^{-1}\sin\phi\mathbf{k}\cdot R^{-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - 3zR^{-2}\sin\phi\\ &= R^{-2}z\sin\phi - 3zR^{-2}\sin\phi = -2zR^{-2}\sin\phi\\ &= -2R\cos\phi\cdot R^{-2}\sin\phi = -R^{-1}\sin 2\phi\end{aligned}$$

Hence,

$$\iint_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} -R^{-1}\sin 2\phi\,d\phi\,d\theta = -\frac{2\pi}{R}\int_0^{\pi}\sin 2\phi\,d\phi = \frac{\pi}{R}\cos 2\phi\Big|_{\phi=0}^{\pi} = 0$$

(c) We use part (a) to write the vector $\mathbf{E}$ componentwise:

$$\mathbf{E} = r^{-3}\mathbf{k} - 3zr^{-4}\mathbf{e}_r = r^{-3}\mathbf{k} - 3zr^{-4}r^{-1}\langle x, y, z\rangle = \left\langle -3zr^{-5}x, -3zr^{-5}y, -3z^2r^{-5} + r^{-3}\right\rangle$$

To find div($\mathbf{E}$) we compute the following derivatives, using (1) and the laws of differentiation. This gives

$$\begin{aligned}\frac{\partial}{\partial x}(-3zr^{-5}x) &= -3z\frac{\partial}{\partial x}(r^{-5}x) = -3z\left(-5r^{-6}\frac{\partial r}{\partial x}x + r^{-5}\cdot 1\right)\\ &= -3z\left(-5r^{-6}x\frac{x}{r} + r^{-5}\right) = 3zr^{-7}(5x^2 - r^2)\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}(-3zr^{-5}y) = 3zr^{-7}(5y^2 - r^2)$$

and

$$\begin{aligned}\frac{\partial}{\partial z}(-3z^2r^{-5} + r^{-3}) &= -6zr^{-5} - 3z^2(-5)r^{-6}\frac{\partial r}{\partial z} - 3r^{-4}\frac{\partial r}{\partial z}\\ &= -6zr^{-5} + 15z^2r^{-6}\frac{z}{r} - 3r^{-4}\frac{z}{r} = 3zr^{-7}(5z^2 - 3r^2)\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{div}(\mathbf{E}) &= 3zr^{-7}(5x^2 - r^2 + 5y^2 - r^2 + 5z^2 - 3r^2) = 15zr^{-7}(x^2 + y^2 + z^2 - r^2)\\ &= 15zr^{-7}(r^2 - r^2) = 0\end{aligned}$$

(d) Since $\mathbf{E}$ is not defined at the origin, which is inside the ball $\mathcal{W}$, we cannot use the Divergence Theorem to compute the flux of $\mathbf{E}$ through the sphere.

30. Let $\mathbf{E}$ be the electric field due to a long, uniformly charged rod of radius R with charge density δ per unit length (Figure 21). By symmetry, we may assume that $\mathbf{E}$ is everywhere perpendicular to the rod and its magnitude $E(d)$ depends only on the distance d to the rod (strictly speaking, this would hold only if the rod were infinite, but it is nearly true if the rod is long enough). Show that $E(d) = \delta/2\pi\epsilon_0 d$ for $d > R$. *Hint:* Apply Gauss's Law to a cylinder of radius R and of unit length with its axis along the rod.

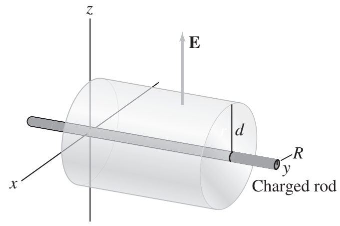


FIGURE 21

SOLUTION Gauss's Theorem asserts that if $\mathcal{S}$ is a closed surface, then the total charge Q enclosed by $\mathcal{S}$ is given by

$$Q = \iint_{\mathcal{S}} \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

where ϵ_0 is the dielectric coefficient in vacuum. Here, we take $\mathcal{S}$ is a closed cylinder of radius d and unit length that encloses a charge $Q = \delta \cdot 1 = \delta$. By symmetry, $\mathbf{E}$ is directed radially outward from the rod and its magnitude depends only on the distance d:

$$\mathbf{E} = E(d)\,\langle \cos\theta, 0, \sin\theta\rangle$$

By Gauss's Theorem, for $d > R$,

$$\delta = \iint_{\mathcal{S}} \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \iint_{\text{integral cylinder}} \epsilon_0 \mathbf{E} \cdot d\mathbf{S} + \iint_{\text{bases}} \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

The second integral is zero because $\mathbf{E}$ is perpendicular to the normal vectors of the two bases. To compute the surface integral over the side of the cylinder, we parametrize the side of $\mathcal{S}$ by $\Phi(y,\theta) = (d\cos\theta, y, d\sin\theta)$ for $0 \le y \le 1$ and $0 \le \theta \le 2\pi$. It can be checked that the normal vector in this parametrization is

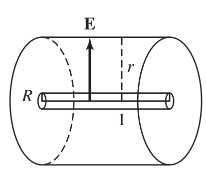
$$\mathbf{n} = \Phi_y \times \Phi_\theta = \langle d\cos\theta, 0, d\sin\theta\rangle$$

Therefore,

$$\mathbf{E} \cdot \mathbf{n} = dE(d)$$

and we

$$\delta = \iint_{\mathcal{S}} \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \epsilon_0\, dE(d)\, dy\, d\theta = 2\pi\epsilon_0\, dE(d) \tag{1}$$



We obtain

$$E(d) = \frac{\delta}{2\pi\epsilon_0 d}.$$

31. Let $\mathcal{W}$ be the region between the sphere of radius 4 and the cube of side 1, both centered at the origin. What is the flux through the boundary $\mathcal{S} = \partial\mathcal{W}$ of a vector field $\mathbf{F}$ whose divergence has the constant value $\mathrm{div}(\mathbf{F}) = -4$?

SOLUTION Recall,

$$\text{flux} = \iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F})dV$$

Using this fact we see:

$$\text{flux} = \iiint_{\mathcal{W}} (-4)dV = -4 \cdot V(\mathcal{W}) = (-4)\left(\frac{256\pi}{3} - 1\right)$$

32. Let $\mathcal{W}$ be the region between the sphere of radius 3 and the sphere of radius 2, both centered at the origin. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = x\mathbf{i}$ through the boundary $\mathcal{S} = \partial\mathcal{W}$.

SOLUTION We first calculate the divergence of $\mathbf{F} = \langle x, 0, 0\rangle$:

$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 1$$

Then to calculate flux, we use the Divergence Theorem:

$$\text{flux} = \iint_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = \iint_{\mathcal{W}} 1\, dV = \text{Volume}(\mathcal{W}) = \frac{4}{3}\pi(3)^3 - \frac{4}{3}\pi(2)^3 = \frac{76\pi}{3}$$

33. Find and prove a Product Rule expressing $\text{div}(f\mathbf{F})$ in terms of $\text{div}(\mathbf{F})$ and ∇f.

SOLUTION Let $\mathbf{F} = \langle P, Q, R\rangle$. We compute $\text{div}(f\mathbf{F})$:

$$\text{div}(f\mathbf{F}) = \text{div}\,\langle fP, fQ, fR\rangle = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

Applying the product rule for scalar functions we obtain

$$\begin{aligned}\text{div}(f\mathbf{F}) &= \left(f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial x}P\right) + \left(f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial y}Q\right) + \left(f\frac{\partial R}{\partial z} + \frac{\partial f}{\partial z}R\right)\\ &= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = f\,\text{div}(\mathbf{F}) + \mathbf{F}\cdot\nabla f\end{aligned}$$

We thus proved the following identity:

$$\text{div}(f\mathbf{F}) = f\,\text{div}(\mathbf{F}) + \mathbf{F}\cdot\nabla f$$

34. Prove the identity

$$\text{div}(\mathbf{F}\times\mathbf{G}) = \text{curl}(\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot\text{curl}(\mathbf{G})$$

Then prove that the cross product of two irrotational vector fields is incompressible [$\mathbf{F}$ is called **irrotational** if $\text{curl}(\mathbf{F}) = 0$ and **incompressible** if $\text{div}(\mathbf{F}) = 0$].

SOLUTION We compute the left-hand side of the identity. For $F = \langle P, Q, R\rangle$ and $G = \langle S, T, U\rangle$ we have

$$\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ P & Q & R\\ S & T & U\end{vmatrix} = (QU - RT)\mathbf{i} - (PU - RS)\mathbf{j} + (PT - QS)\mathbf{k}$$

$$\begin{aligned}\text{div}(\mathbf{F}\times\mathbf{G}) &= \frac{\partial}{\partial x}(QU - RT) - \frac{\partial}{\partial y}(PU - RS) + \frac{\partial}{\partial z}(PT - QS)\\ &= (Q_xU + QU_x - R_xT - RT_x) - \left(P_yU + PU_y - R_yS - RS_y\right) + (P_zT + PT_z - Q_zS - QS_z)\\ &= S(R_y - Q_z) + T(P_z - R_x) + U(Q_x - P_y) - P(U_y - T_z) - Q(S_z - U_x) - R(T_x - S_y)\end{aligned}$$

We compute the right hand side of the given identity. We have

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y\right\rangle\\ \text{curl}(\mathbf{G}) &= \left\langle U_y - T_z, S_z - U_x, T_x - S_y\right\rangle\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{G}\cdot\text{curl}(\mathbf{F}) - \mathbf{F}\cdot\text{curl}(\mathbf{G}) &= \langle S, T, U\rangle\cdot\left\langle R_y - Q_z, P_z - R_x, Q_x - P_y\right\rangle\\ &\quad - \langle P, Q, R\rangle\cdot\left\langle U_y - T_z, S_z - U_x, T_x - S_y\right\rangle\\ &= S(R_y - Q_z) + T(P_z - R_x) + U(Q_x - P_y)\\ &\quad - P(U_y - T_z) - Q(S_z - U_x) - R(T_x - S_y)\end{aligned}$$

This, with (1), proves the identity

$$\text{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\text{curl}(\mathbf{F}) - \mathbf{F}\cdot\text{curl}(\mathbf{G}).$$

Thus, if both $\mathbf{F}$ and $\mathbf{G}$ are irrotational (that is, with curl zero), then their cross product is source-free (that is, with divergence zero), as $\text{div}\,\mathbf{F}\times\mathbf{G} = \mathbf{G}\cdot\mathbf{0} - \mathbf{F}\cdot\mathbf{0} = 0$.

35. Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

SOLUTION We compute the cross product:

$$\nabla f \times \nabla g = \langle f_x, f_y, f_z \rangle \times \langle g_x, g_y, g_z \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix}$$
$$= \langle f_y g_z - f_z g_y, f_z g_x - f_x g_z, f_x g_y - f_y g_x \rangle$$

We now compute the divergence of this vector. Using the Product Rule for scalar functions and the equality of the mixed partials, we obtain

$$\begin{aligned}\operatorname{div}(\nabla f \times \nabla g) &= \frac{\partial}{\partial x}(f_y g_z - f_z g_y) + \frac{\partial}{\partial y}(f_z g_x - f_x g_z) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x) \\ &= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} + f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy} + f_{xz} g_y + f_x g_{yz} \\ &\quad - f_{yz} g_x - f_y g_{xz} \\ &= (f_{yx} - f_{xy}) g_z + (g_{zx} - g_{xz}) f_y + (f_{xz} - f_{zx}) g_y + (g_{xy} - g_{yx}) f_z \\ &\quad + (f_{zy} - f_{yz}) g_x + (g_{yz} - g_{zy}) f_x = 0\end{aligned}$$

In Exercises 36–38, Δ denotes the Laplace operator defined by

$$\boxed{\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}}$$

36. Prove the identity

$$\operatorname{curl}(\operatorname{curl}(\mathbf{F})) = \nabla(\operatorname{div}(\mathbf{F})) - \Delta \mathbf{F}$$

where $\Delta \mathbf{F}$ denotes $\langle \Delta F_1, \Delta F_2, \Delta F_3 \rangle$.

SOLUTION We compute the left-hand side of the identity. We have

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Hence,

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}(\mathbf{F})) &= \left\langle \frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right), \right. \\ &\quad \left. \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \right\rangle \\ &= \left\langle \frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x}, \frac{\partial^2 F_3}{\partial z \partial y} - \frac{\partial^2 F_2}{\partial z^2} - \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial y}, \right. \\ &\quad \left. \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z} \right\rangle \end{aligned} \tag{1}$$

We now compute the right-hand side of the given identity:

$$\begin{aligned}\nabla(\operatorname{div}(\mathbf{F})) &= \nabla\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\ &= \left\langle \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right), \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right), \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \right\rangle \\ &= \left\langle \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}, \frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}, \frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^2 F_3}{\partial z^2} \right\rangle\end{aligned}$$

Therefore,

$$\nabla(\operatorname{div}(\mathbf{F})) - \Delta \mathbf{F} = \nabla(\operatorname{div}(\mathbf{F})) - \left\langle \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right\rangle$$

$$= \left\langle \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}, \right.$$

$$\left. \frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} \right\rangle \qquad (2)$$

Since the mixed partials are equal, the expressions obtained in (1) and (2) are the same. This proves the desired identity.

37. A function φ satisfying $\Delta\varphi = 0$ is called **harmonic**.

(a) Show that $\Delta\varphi = \operatorname{div}(\nabla\varphi)$ for any function φ.

(b) Show that φ is harmonic if and only if $\operatorname{div}(\nabla\varphi) = 0$.

(c) Show that if **F** is the gradient of a harmonic function, then $\operatorname{curl}(F) = 0$ and $\operatorname{div}(F) = 0$.

(d) Show that $\mathbf{F} = \left\langle xz, -yz, \frac{1}{2}(x^2 - y^2)\right\rangle$ is the gradient of a harmonic function. What is the flux of **F** through a closed surface?

SOLUTION

(a) We compute the divergence of $\nabla\varphi$:

$$\operatorname{div}(\nabla\varphi) = \operatorname{div}\left(\left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\rangle\right) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \Delta\varphi$$

(b) In part (a) we showed that $\Delta\varphi = \operatorname{div}(\nabla\varphi)$. Therefore $\Delta\varphi = 0$ if and only if $\operatorname{div}(\nabla\varphi) = 0$. That is, φ is harmonic if and only if $\nabla\varphi$ is divergence free.

(c) We are given that $\mathbf{F} = \nabla\varphi$, where $\Delta\varphi = 0$. In part (b) we showed that

$$\operatorname{div}(\mathbf{F}) = \operatorname{div}(\nabla\varphi) = 0$$

We now show that $\operatorname{curl}(\mathbf{F}) = 0$. We have

$$\operatorname{curl}(\mathbf{F}) = \operatorname{curl}(\nabla\varphi) = \operatorname{curl}\left\langle \varphi_x, \varphi_y, \varphi_z\right\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \left\langle \varphi_{zy} - \varphi_{yz}, \varphi_{xz} - \varphi_{zx}, \varphi_{yx} - \varphi_{xy}\right\rangle = \langle 0, 0, 0\rangle = \mathbf{0}$$

The last equality is due to the equality of the mixed partials.

(d) We first show that $\mathbf{F} = \left\langle xz, -yz, \frac{x^2-y^2}{2}\right\rangle$ is the gradient of a harmonic function. We let $\varphi = \frac{x^2 z}{2} - \frac{y^2 z}{2}$ such that $\mathbf{F} = \nabla\varphi$. Indeed,

$$\nabla\varphi = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\rangle = \left\langle xz, -yz, \frac{x^2 - y^2}{2}\right\rangle = \mathbf{F}$$

We show that φ is harmonic, that is, $\Delta\varphi = 0$. We compute the partial derivatives:

$$\frac{\partial\varphi}{\partial x} = xz \quad \Rightarrow \quad \frac{\partial^2\varphi}{\partial x^2} = z$$

$$\frac{\partial\varphi}{\partial y} = -yz \quad \Rightarrow \quad \frac{\partial^2\varphi}{\partial y^2} = -z$$

$$\frac{\partial\varphi}{\partial z} = \frac{x^2 - y^2}{2} \quad \Rightarrow \quad \frac{\partial^2\varphi}{\partial z^2} = 0$$

Therefore,

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = z - z + 0 = 0$$

Since **F** is the gradient of a harmonic function, we know by part (c) that $\operatorname{div}(\mathbf{F}) = 0$. Therefore, by the Divergence Theorem, the flux of **F** through a closed surface is zero:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 0\, dV = 0$$

38. Let $\mathbf{F} = r^n \mathbf{e}_r$, where n is any number, $r = (x^2 + y^2 + z^2)^{1/2}$, and $\mathbf{e}_r = r^{-1} \langle x, y, z \rangle$ is the unit radial vector.

(a) Calculate $\operatorname{div}(\mathbf{F})$.

(b) Calculate the flux of $\mathbf{F}$ through the surface of a sphere of radius R centered at the origin. For which values of n is this flux independent of R?

(c) Prove that $\nabla(r^n) = n\, r^{n-1}\mathbf{e}_r$.

(d) Use (c) to show that $\mathbf{F}$ is conservative for $n \neq -1$. Then show that $\mathbf{F} = r^{-1}\mathbf{e}_r$ is also conservative by computing the gradient of $\ln r$.

(e) What is the value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}$ is a closed curve that does not pass through the origin?

(f) Find the values of n for which the function $\varphi = r^n$ is harmonic.

SOLUTION

(a) $\mathbf{F}$ is the vector field:

$$\mathbf{F}(x, y, z) = r^n r^{-1} \langle x, y, z \rangle = (x^2 + y^2 + z^2)^{(n-1)/2} \langle x, y, z \rangle \tag{1}$$

Hence,

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{(n-1)/2}x\right) = \left(\frac{n-1}{2}\right)(x^2+y^2+z^2)^{(n-3)/2}\cdot 2x\cdot x + (x^2+y^2+z^2)^{(n-1)/2}\\ &= (x^2+y^2+z^2)^{(n-3)/2}\left((n-1)x^2 + x^2 + y^2 + z^2\right) = (x^2+y^2+z^2)^{(n-3)/2}(nx^2+y^2+z^2)\end{aligned}$$

Similarly,

$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{(n-1)/2}y\right) = (x^2+y^2+z^2)^{(n-3)/2}(x^2+ny^2+z^2)$$

and

$$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{(n-1)/2}z\right) = (x^2+y^2+z^2)^{(n-3)/2}(x^2+y^2+nz^2)$$

The divergence of $\mathbf{F}$ is the sum

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= (x^2+y^2+z^2)^{(n-3)/2} = (nx^2+y^2+z^2+x^2+ny^2+z^2+x^2+y^2+nz^2)\\ &= (x^2+y^2+z^2)^{(n-3)/2}(n+2)(x^2+y^2+z^2) = (n+2)(x^2+y^2+z^2)^{(n-1)/2} = (n+2)r^{n-1}\end{aligned}$$

(b) Let $\mathcal{W}$ denote the ball inside the sphere of radius R. We may apply the Divergence Theorem when the components of $\mathbf{F}$ in (1) are defined and have continuous derivatives in $\mathcal{W}$, that is, when $\frac{n-1}{2} \geq 0$ and $\frac{n-3}{2} \geq 0$ or when $n \geq 3$. In this case, we have

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(F)\, dV = \iiint_{\mathcal{W}} (n+2)r^{n-1}\, dV$$

We compute the triple integral by converting it to spherical coordinates. Since $r = \rho$, we obtain

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(n+2)r^{n-1}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi(n+2)\int_0^{\pi}\int_0^{R}\rho^{n+1}\sin\phi\, d\rho\, d\phi\\ &= 2\pi(n+2)\left(\int_0^R \rho^{n+1}d\rho\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right) = 2\pi(n+2)\left(-\cos\phi\Big|_{\phi=0}^{\pi}\right)\int_0^R\rho^{n+1}d\rho\\ &= 4\pi(n+2)\int_0^R\rho^{n+1}\, d\rho = 4\pi(n+2)\frac{\rho^{n+2}}{n+2}\Big|_{\rho=0}^{R} = 4\pi(n+2)\frac{R^{n+2}}{n+2} = 4\pi R^{n+2}\end{aligned}$$

That is,

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = 4\pi R^{n+2}, \quad n \geq 3.$$

We now consider the case $n < 3$. We evaluate the surface integral directly, using the parametrization, $\Phi(\theta, \phi) = (R\cos\theta\sin\phi,\, R\sin\theta\sin\phi,\, R\cos\phi)$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

$$\mathbf{n} = R^2\sin\phi\mathbf{e}_r$$

Then,

$$\begin{aligned}\mathbf{F}\left(\Phi(\theta,\phi)\right)\cdot\mathbf{n} &= r^n\mathbf{e}_r\cdot R^2\sin\phi\mathbf{e}_r = \rho^n R^2\sin\phi\\ &= R^2\sin\phi\left(R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta\sin^2\phi + R^2\cos^2\phi\right)^{n/2} = R^{n+2}\sin\phi\end{aligned}$$

Hence,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} R^{n+2} \sin\phi \, d\phi \, d\theta = 2\pi R^{n+2} \left(-\cos\phi \Big|_0^{\pi} \right) = 4\pi R^{n+2}$$

We conclude that, for all values of n,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = 4\pi R^{n+2}$$

The flux is independent of R when $n + 2 = 0$ or $n = -2$.

(c) We compute the gradient of r^n. We first compute the partial derivatives,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r} \qquad (2)$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Therefore,

$$\frac{\partial}{\partial x}(r^n) = nr^{n-1} \frac{\partial r}{\partial x} = nr^{n-1} \cdot \frac{x}{r} = nr^{n-2} x$$
$$\frac{\partial}{\partial y}(r^n) = nr^{n-1} \frac{\partial r}{\partial y} = nr^{n-1} \cdot \frac{y}{r} = nr^{n-2} y$$
$$\frac{\partial}{\partial z}(r^n) = nr^{n-1} \frac{\partial r}{\partial z} = nr^{n-1} \cdot \frac{z}{r} = nr^{n-2} z$$

The gradient of r^n is thus

$$\nabla(r^n) = \left\langle nr^{n-2}x, nr^{n-2}y, nr^{n-2}z \right\rangle = nr^{n-2} \langle x, y, z \rangle = nr^{n-1} \cdot r^{-1} \langle x, y, z \rangle = nr^{n-1} \mathbf{e}_r$$

(d) Replacing n by $n + 1$ in the equality of part (c), we have

$$\nabla(r^{n+1}) = (n+1) r^n \mathbf{e}_r = (n+1)\mathbf{F}$$

Therefore, if $n \neq -1$, then

$$\mathbf{F} = \nabla \left(\frac{r^{n+1}}{n+1} \right)$$

We now show that $\mathbf{F} = r^{-1}\mathbf{e}_r$ is also conservative. We compute the gradient of $\ln r$. Using the Chain Rule and the partial derivatives (1), we have

$$\frac{\partial}{\partial x}(\ln r) = \frac{1}{r} \frac{\partial r}{\partial x} = \frac{1}{r} \frac{x}{r} = \frac{x}{r^2}$$
$$\frac{\partial}{\partial y}(\ln r) = \frac{1}{r} \frac{\partial r}{\partial y} = \frac{1}{r} \frac{y}{r} = \frac{y}{r^2}$$
$$\frac{\partial}{\partial z}(\ln r) = \frac{1}{r} \frac{\partial r}{\partial z} = \frac{1}{r} \frac{z}{r} = \frac{z}{r^2}$$

Therefore, the gradient of $\ln r$ is

$$\nabla(\ln r) = \left\langle \frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right\rangle = r^{-1} \cdot r^{-1} \langle x, y, z \rangle = r^{-1}\mathbf{e}_r = \mathbf{F}.$$

We conclude that $\mathbf{F}$ is conservative for all values of n.

(e) Since $\mathbf{F}$ is conservative, the line integral of $\mathbf{F}$ is zero over every closed curve in the domain of $\mathbf{F}$, that is, over every closed curve not passing through the origin.

(f) Using Exercise 37 part (b), $\varphi = r^n$ is harmonic if and only if

$$\operatorname{div}\left(\nabla(r^n)\right) = 0$$

That is, by part (c),

$$\operatorname{div}(nr^{n-1}\mathbf{e}_r) = n \operatorname{div}\left(r^{n-1}\mathbf{e}_r\right) = 0$$

or

$$\operatorname{div}(r^{n-1}\mathbf{e}_r) = 0 \quad \text{or} \quad n = 0$$

Using part (a) for n replaced by $n-1$, we have

$$\operatorname{div}(r^{n-1}\mathbf{e}_r) = (n+1)r^{n-2} = 0 \quad \Rightarrow \quad n = -1 \quad \text{or} \quad n = 0.$$

We conclude that $\varphi = r^n$ is harmonic for $n = -1$ or $n = 0$.

Further Insights and Challenges

39. Let $\mathcal{S}$ be the boundary surface of a region $\mathcal{W}$ in $\mathbf{R}^3$ and let $D_{\mathbf{e_n}}\varphi$ denote the directional derivative of φ, where $\mathbf{e_n}$ is the outward unit normal vector. Let Δ be the Laplace operator defined earlier.

(a) Use the Divergence Theorem to prove that

$$\iint_{\mathcal{S}} D_{\mathbf{e_n}}\varphi \, dS = \iiint_{\mathcal{W}} \Delta\varphi \, dV$$

(b) Show that if φ is a harmonic function (defined in Exercise 37), then

$$\iint_{\mathcal{S}} D_{\mathbf{e_n}}\varphi \, dS = 0$$

SOLUTION

(a) By the theorem on evaluating directional derivatives, $D_{\mathbf{e}_n}\varphi = \nabla\varphi \cdot \mathbf{e}_n$, hence,

$$\iint_{\mathcal{S}} D_{\mathbf{e}_n}\varphi \, dS = \iint_{\mathcal{S}} \nabla\varphi \cdot \mathbf{e}_n \, dS \tag{1}$$

By the definition of the vector surface integral, we have

$$\iint_{\mathcal{S}} \nabla\varphi \cdot d\mathbf{S} = \iint_{\mathcal{S}} (\nabla\varphi \cdot \mathbf{e}_n) \, dS$$

Combining with (1) gives

$$\iint_{\mathcal{S}} D_{\mathbf{e}_n}\varphi \, dS = \iint_{\mathcal{S}} \nabla\varphi \cdot d\mathbf{S}$$

We now apply the Divergence Theorem and the identity $\operatorname{div}(\nabla\varphi) = \Delta\varphi$ shown in part (a) of Exercise 27, to write

$$\iint_{\mathcal{S}} D_{\mathbf{e}_n}\varphi \, dS = \iint_{\mathcal{S}} \nabla\varphi \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\nabla\varphi) \, dV = \iiint_{\mathcal{W}} \Delta\varphi \, dV$$

(b) If φ is harmonic, then $\Delta\varphi = 0$; therefore, by the equality of part (a) we have

$$\iint_{\mathcal{S}} D_{\mathbf{e}_n}\varphi \, dS = \iiint_{\mathcal{W}} \Delta\varphi \cdot \, dV = \iiint_{\mathcal{W}} 0 \, dV = 0.$$

40. Assume that φ is harmonic. Show that $\operatorname{div}(\varphi\nabla\varphi) = \|\nabla\varphi\|^2$ and conclude that

$$\iint_{\mathcal{S}} \varphi D_{\mathbf{e_n}}\varphi \, dS = \iiint_{\mathcal{W}} \|\nabla\varphi\|^2 \, dV$$

SOLUTION In Exercise 33 we proved the following Product Rule:

$$\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div}(\mathbf{F})$$

We use this rule for $f = \varphi$ and $\mathbf{F} = \nabla\varphi$ to obtain

$$\operatorname{div}(\varphi\nabla\varphi) = \nabla\varphi \cdot \nabla\varphi + \varphi \operatorname{div}(\nabla\varphi) = \|\nabla\varphi\|^2 + \varphi \operatorname{div}(\nabla\varphi) \tag{1}$$

By Exercise 37 part (a),

$$\operatorname{div}(\nabla\varphi) = \Delta\varphi \tag{2}$$

Also, since φ is harmonic,

$$\Delta\varphi = 0 \tag{3}$$

Combining (1), (2), and (3), we obtain

$$\operatorname{div}(\varphi\nabla\varphi) = \|\nabla\varphi\|^2 + \varphi\cdot 0 = \|\nabla\varphi\|^2 \tag{4}$$

Now, by the Theorem on evaluating directional derivatives,

$$D_{\mathbf{e}_n}\varphi = \nabla\varphi\cdot\mathbf{e}_n$$

Hence,

$$\iint_{\mathcal{S}} \varphi D_{\mathbf{e}_n}\varphi\, dS = \iint_{\mathcal{S}} (\varphi\nabla\varphi\cdot\mathbf{e}_n)\, dS \tag{5}$$

By the definition of the vector surface integral we have

$$\iint_{\mathcal{S}} \varphi\nabla\varphi\cdot d\mathbf{S} = \iint_{\mathcal{S}} (\varphi\nabla\varphi\cdot\mathbf{e}_n)\, dS \tag{6}$$

Combining (5) and (6) and using the Divergence Theorem and equality (4), we get

$$\iint_{\mathcal{S}} \varphi D_{\mathbf{e}_n}\varphi\, dS = \iint_{\mathcal{S}} \varphi\nabla\varphi\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\varphi\nabla\varphi)\, dV = \iiint_{\mathcal{W}} \|\nabla\varphi\|^2\, dV$$

41. Let $\mathbf{F} = \langle P, Q, R\rangle$ be a vector field defined on $\mathbf{R}^3$ such that $\operatorname{div}(\mathbf{F}) = 0$. Use the following steps to show that $\mathbf{F}$ has a vector potential.

(a) Let $\mathbf{A} = \langle f, 0, g\rangle$. Show that

$$\operatorname{curl}(\mathbf{A}) = \left\langle \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} - \frac{\partial g}{\partial x}, -\frac{\partial f}{\partial y}\right\rangle$$

(b) Fix any value y_0 and show that if we define

$$f(x,y,z) = -\int_{y_0}^{y} R(x,t,z)\, dt + \alpha(x,z)$$
$$g(x,y,z) = \int_{y_0}^{y} P(x,t,z)\, dt + \beta(x,z)$$

where α and β are any functions of x and z, then $\partial g/\partial y = P$ and $-\partial f/\partial y = R$.

(c) It remains for us to show that α and β can be chosen so $Q = \partial f/\partial z - \partial g/\partial x$. Verify that the following choice works (for any choice of z_0):

$$\alpha(x,z) = \int_{z_0}^{z} Q(x, y_0, t)\, dt, \qquad \beta(x,z) = 0$$

Hint: You will need to use the relation $\operatorname{div}(\mathbf{F}) = 0$.

SOLUTION

(a) If $\mathbf{A} = \langle f, 0, g\rangle$, then the curl of $\mathbf{A}$ is the following vector field:

$$\operatorname{curl}(\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & 0 & g \end{vmatrix} = \left(\frac{\partial g}{\partial y} - 0\right)\mathbf{i} - \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(0 - \frac{\partial f}{\partial y}\right)\mathbf{k} = \left\langle \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} - \frac{\partial g}{\partial x}, -\frac{\partial f}{\partial y}\right\rangle$$

(b) Using the Fundamental Theorem of Calculus, we have

$$\frac{\partial g}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\int_{y_0}^{y} P(x,t,z)\, dt + \frac{\partial}{\partial y}\beta(x,z) = P(x,y,z) + 0 = P(x,y,z)$$
$$-\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\int_{y_0}^{y} R(x,t,z)\, dt + \frac{\partial}{\partial y}\alpha(x,z) = R(x,y,z) + 0 = R(x,y,z)$$

(c) We verify that the functions

$$\alpha(x,z) = \int_{z_0}^{z} Q(x, y_0, t)\, dt, \quad \beta(x,z) = 0$$

satisfy the equality

$$Q = \frac{\partial f}{\partial z} - \frac{\partial g}{\partial x}$$

We differentiate to obtain

$$\frac{\partial f}{\partial z} - \frac{\partial g}{\partial x} = -\int_{y_0}^{y} R_z(x,t,z)\,dt + \alpha_z(x,z) - \int_{y_0}^{y} P_x(x,t,z)\,dz - \beta_x(x,z)$$
$$= -\int_{y_0}^{y} \left(P_x(x,t,z) + R_z(x,t,z)\right)\,dt + \alpha_z(x,z) \qquad (1)$$

By the Fundamental Theorem of Calculus,

$$\alpha_z(x,z) = \frac{\partial}{\partial z}\int_{z_0}^{z} Q(x,y_0,t)\,dt = Q(x,y_0,z) \qquad (2)$$

Also, since $\operatorname{div}(\mathbf{F}) = 0$, we have

$$\operatorname{div}(\mathbf{F}) = P_x + Q_y + R_z = 0 \quad \Rightarrow \quad P_x + R_z = -Q_y \qquad (3)$$

Substituting (2) and (3) in (1) gives

$$\frac{\partial f}{\partial z} - \frac{\partial g}{\partial x} = \int_{y_0}^{y} Q_y(x,t,z)\,dt + Q(x,y_0,z) = Q(x,y,z) - Q(x,y_0,z) + Q(x,y_0,z) = Q(x,y,z)$$

Parts (a)–(c) prove that $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, or $\mathbf{A}$ is a vector potential for $\mathbf{F}$.

42. Show that

$$\mathbf{F} = \left\langle 2y - 1, 3z^2, 2xy \right\rangle$$

has a vector potential and find one.

SOLUTION Since $\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(2y-1) + \frac{\partial}{\partial y}(3z^2) + \frac{\partial}{\partial z}(2xy) = 0$, we know by Exercise 41 that $\mathbf{F}$ has a vector potential $\mathbf{A}$, which is

$$\mathbf{A} = \langle f, 0, g \rangle \qquad (1)$$
$$f(x,y,z) = -\int_{y_0}^{y} R(x,t,z)\,dt + \int_{z_0}^{z} Q(x,y_0,t)\,dt$$
$$g(x,y,z) = \int_{y_0}^{y} P(x,t,z)\,dt$$

Hence, $P(x,y,z) = 2y - 1$, $Q(x,y,z) = 3z^2$, and $R(x,y,z) = 2xy$. We choose $z_0 = y_0 = 0$ and find f and g:

$$f(x,y,z) = -\int_0^y 2xt\,dt + \int_0^z 3t^2\,dt = -xt^2\Big|_{t=0}^{y} + t^3\Big|_{t=0}^{z} = -xy^2 + z^3$$
$$g(x,y,z) = \int_0^y (2t-1)\,dt = t^2 - t\Big|_{t=0}^{y} = y^2 - y$$

Substituting in (1) we obtain the vector potential

$$\mathbf{A} = \left\langle z^3 - xy^2, 0, y^2 - y \right\rangle$$

43. Show that

$$\mathbf{F} = \left\langle 2ye^z - xy, y, yz - z \right\rangle$$

has a vector potential and find one.

SOLUTION As shown in Exercise 41, if $\mathbf{F}$ is divergence free, then $\mathbf{F}$ has a vector potential. We show that $\operatorname{div}(\mathbf{F}) = 0$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(2ye^z - xy) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(yz - z) = -y + 1 + y - 1 = 0$$

We find a vector potential $\mathbf{A}$, using the result in Exercise 41:

$$\mathbf{A} = \langle f, 0, g \rangle \qquad (1)$$

Using $z_0 = 0$, we have

$$f(x,y,z) = -\int_{y_0}^{y} R(x,t,z)\,dt + \int_0^z Q(x,y_0,t)\,dt$$

$$g(x, y, z) = \int_{y_0}^{y} P(x, t, z)\, dt$$

Hence, $P(x, y, z) = 2ye^z - xy$, $Q(x, y, z) = y$, and $R(x, y, z) = yz - z$. We choose $y_0 = 0$ and compute the functions f and g:

$$f(x, y, z) = -\int_0^y (tz - z)\, dt + \int_0^z 0\, dt = -\left(\frac{t^2 z}{2} - zt\right)\Bigg|_{t=0}^{y} = zy - \frac{y^2 z}{2} = z\left(y - \frac{y^2}{2}\right)$$

$$g(x, y, z) = \int_0^y (2te^z - xt)\, dt = t^2 e^z - \frac{xt^2}{2}\Bigg|_{t=0}^{y} = y^2 e^z - \frac{xy^2}{2} = y^2\left(e^z - \frac{x}{2}\right)$$

Substituting in (1) we obtain

$$\mathbf{A} = \left\langle z\left(y - \frac{y^2}{2}\right), 0, y^2\left(e^z - \frac{x}{2}\right)\right\rangle$$

44. In the text, we observed that although the inverse-square radial vector field $\mathbf{F} = \frac{\mathbf{e}_r}{r^2}$ satisfies $\mathrm{div}(\mathbf{F}) = 0$, $\mathbf{F}$ cannot have a vector potential on its domain $\{(x, y, z) \neq (0, 0, 0)\}$ because the flux of $\mathbf{F}$ through a sphere containing the origin is nonzero.

(a) Show that the method of Exercise 41 produces a vector potential $\mathbf{A}$ such that $\mathbf{F} = \mathrm{curl}(\mathbf{A})$ on the restricted domain $\mathcal{D}$ consisting of $\mathbf{R}^3$ with the y-axis removed.

(b) Show that $\mathbf{F}$ also has a vector potential on the domains obtained by removing either the x-axis or the z-axis from $\mathbf{R}^3$.

(c) Does the existence of a vector potential on these restricted domains contradict the fact that the flux of $\mathbf{F}$ through a sphere containing the origin is nonzero?

SOLUTION

(a) We have $\mathbf{F}(x, y, z) = \frac{\mathbf{e}_r}{r^2} = \frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$, hence

$$P(x, y, z) = \frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$Q(x, y, z) = \frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$R(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

In Exercise 41, we defined the functions (taking $y_0 = z_0 = 0$)

$$f(x, y, z) = -\int_0^y \frac{z}{(x^2 + t^2 + z^2)^{3/2}} dt + \int_0^z Q(x, 0, t)\, dt = -\int_0^y \frac{z}{(x^2 + t^2 + z^2)^{3/2}} dt$$

$$g(x, y, z) = \int_0^y \frac{x}{(x^2 + t^2 + z^2)^{3/2}} dt$$

We obtain

$$f(x, y, z) = -\frac{yz}{(x^2 + z^2)(x^2 + y^2 + z^2)^{1/2}}$$

$$g(x, y, z) = \frac{xy}{(x^2 + z^2)(x^2 + y^2 + z^2)^{1/2}}$$

We can check directly that $\mathbf{A} = \langle f, 0, g\rangle$ is a vector potential, without using the FTC.

These functions are defined for $(x, z) \neq (0, 0)$, since the points with $x = 0$ and $z = 0$ are on the y-axis. (Notice that for any fixed $(x, z) \neq (0, 0)$ the interval of integration do not intersect the y-axis, therefore they are contained in the domain $\mathcal{D}$.) For $(x, z) \neq (0, 0)$ we have by the Fundamental Theorem of Calculus

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\int_0^y \frac{x}{(x^2 + t^2 + z^2)^{3/2}} dt = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = P(x, y, z)$$

$$\frac{\partial f}{\partial z} - \frac{\partial g}{\partial x} = -\int_0^y \frac{(x^2 + t^2 + z^2)^{3/2} - z \cdot \frac{3}{2}(x^2 + t^2 + z^2)^{1/2} \cdot 2z}{(x^2 + t^2 + z^2)^3} dt$$

$$-\int_0^y \frac{(x^2 + t^2 + z^2)^{3/2} - x \cdot \frac{3}{2}(x^2 + t^2 + z^2)^{1/2} \cdot 2x}{(x^2 + t^2 + z^2)^3} dt$$

$$= -\int_0^y \frac{(x^2+t^2+z^2)^{1/2}(x^2+t^2+z^2-3z^2)}{(x^2+t^2+z^2)^3}\,dt$$

$$-\int_0^y \frac{(x^2+t^2+z^2)^{1/2}(x^2+t^2+z^2-3x^2)}{(x^2+t^2+z^2)^3}\,dt$$

$$= -\int_0^y \frac{x^2+t^2-2z^2+t^2+z^2-2x^2}{(x^2+t^2+z^2)^{5/2}}\,dt$$

$$= \int_0^y \frac{x^2-2t^2+z^2}{(x^2+t^2+z^2)^{5/2}}\,dt = \frac{y}{(x^2+y^2+z^2)^{3/2}} = Q(x,y,z)$$

The last integral can be verified by showing that

$$\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

and

$$\frac{\partial f}{\partial y} = -\frac{\partial}{\partial y}\int_0^y \frac{z}{(x^2+t^2+z^2)^{3/2}}\,dt = -\frac{z}{(x^2+y^2+z^2)^{3/2}} = -R(x,y,z)$$

We conclude that the vector $\mathbf{A} = \langle f, 0, g\rangle$ is a vector potential of $\mathbf{F}$ in $\mathcal{D}$, since

$$\operatorname{curl}(\mathbf{A}) = \left\langle \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} - \frac{\partial g}{\partial x}, -\frac{\partial f}{\partial y}\right\rangle = \langle P, Q, R\rangle = \mathbf{F}.$$

(b) Suppose we remove the x-axis. In this case, we let

$$\mathbf{A} = \langle 0, f, g\rangle$$

$$g(x,y,z) = -\int_{x_0}^x Q(t,y,z)\,dt + \int_{y_0}^y P(x_0,t,z)\,dt$$

$$f(x,y,z) = \int_{x_0}^x R(t,y,z)\,dt$$

Using similar procedure to that in Exercise 41, one can show that

$$\mathbf{F} = \operatorname{curl}(\mathbf{A}).$$

In removing the z-axis the proof is similar, with corresponding modifications of the functions in Exercise 41.
(c) The ball inside any sphere containing the origin must intersect the x, y, and z axes; therefore, $\mathbf{F}$ does not have a vector potential in the ball, and the flux of $\mathbf{F}$ through the sphere may differ from zero, as in our example.

CHAPTER REVIEW EXERCISES

1. Let $\mathbf{F}(x,y) = \langle x+y^2, x^2-y\rangle$ and let $\mathcal{C}$ be the unit circle, oriented counterclockwise. Evaluate $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ directly as a line integral and using Green's Theorem.

SOLUTION We parametrize the unit circle by $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Then, $\mathbf{c}'(t) = \langle -\sin t, \cos t\rangle$ and $\mathbf{F}(\mathbf{c}(t)) = (\cos t + \sin^2 t, \cos^2 t - \sin t)$. We compute the dot product:

$$\mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t) = \left\langle \cos t + \sin^2 t, \cos^2 t - \sin t\right\rangle \cdot \langle -\sin t, \cos t\rangle$$
$$= (-\sin t)(\cos t + \sin^2 t) + \cos t(\cos^2 t - \sin t)$$
$$= \cos^3 t - \sin^3 t - 2\sin t\cos t$$

The line integral is thus

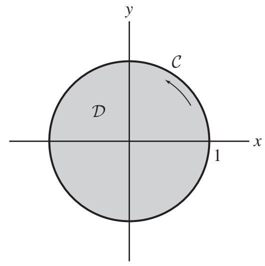
$$\int_{\mathcal{C}} \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt = \int_0^{2\pi}\left(\cos^3 t - \sin^3 t - 2\sin t\cos t\right)dt$$
$$= \int_0^{2\pi}\cos^3 t\,dt - \int_0^{2\pi}\sin^3 t\,dt - \int_0^{2\pi}\sin 2t\,dt$$
$$= \frac{\cos^2 t\sin t}{3} + \frac{2\sin t}{3}\Big|_0^{2\pi} + \left(\frac{\sin^2 t\cos t}{3} + \frac{2\cos t}{3}\right)\Big|_0^{2\pi} + \frac{\cos 2t}{2}\Big|_0^{2\pi} = 0$$

We now compute the integral using Green's Theorem. We compute the curl of $\mathbf{F}$. Since $P = x + y^2$ and $Q = x^2 - y$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$$

Thus,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} (2x - 2y)\, dx\, dy$$



We compute the double integral by converting to polar coordinates. We get

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^1 (2r\cos\theta - 2r\sin\theta) r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$
$$= \left(\int_0^1 2r^2\, dr\right)\left(\int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta\right) = \left(\frac{2}{3}r^3\Big|_0^1\right)\left(\sin\theta + \cos\theta\Big|_0^{2\pi}\right) = \frac{2}{3}(1-1) = 0$$

2. Let $\partial\mathcal{R}$ be the boundary of the rectangle in Figure 1 and let $\partial\mathcal{R}_1$ and $\partial\mathcal{R}_2$ be the boundaries of the two triangles, all oriented counterclockwise.

(a) Determine $\oint_{\partial\mathcal{R}_1} \mathbf{F} \cdot d\mathbf{s}$ if $\oint_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{s} = 4$ and $\oint_{\partial\mathcal{R}_2} \mathbf{F} \cdot d\mathbf{s} = -2$.

(b) What is the value of $\oint_{\partial\mathcal{R}} \mathbf{F}\, d\mathbf{s}$ if $\partial\mathcal{R}$ is oriented clockwise?

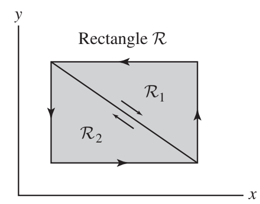
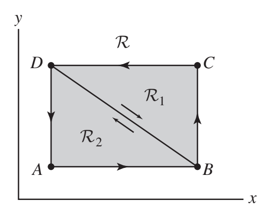


FIGURE 1

SOLUTION

(a) Since all boundaries are oriented counterclockwise, the segment $\overline{DB}$ is oriented in opposite directions as part of the boundaries ∂R_1 and ∂R_2.



Therefore, the contributions of this segment to the sum of the line integrals over ∂R_1 and ∂R_2 cancel each other and the following equality holds:

$$\int_{\partial R} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial R_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\partial R_2} \mathbf{F} \cdot d\mathbf{s}$$

Substituting the given information, we get

$$4 = \int_{\partial R_1} \mathbf{F} \cdot d\mathbf{s} - 2 \quad \text{or} \quad \int_{\partial R_1} \mathbf{F} \cdot d\mathbf{s} = 6.$$

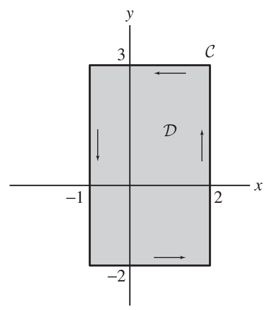
(b) Reversing the orientation of the curve gives the opposite integral. Therefore if ∂R is oriented clockwise, the line integral is the opposite of 4; that is,

$$\int_{\partial R} \mathbf{F} \cdot d\mathbf{s} = -4.$$

In Exercises 3–6, use Green's Theorem to evaluate the line integral around the given closed curve.

3. $\oint_{\mathcal{C}} xy^3\,dx + x^3y\,dy$, where $\mathcal{C}$ is the rectangle $-1 \le x \le 2$, $-2 \le y \le 3$, oriented counterclockwise.

SOLUTION



Since $P = xy^3$, $Q = x^3y$ the curl of $\mathbf{F}$ is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2y - 3xy^2$$

By Green's Theorem we obtain

$$\int_{\mathcal{C}} xy^3\,dx + x^3y\,dy = \iint_{\mathcal{D}} (3x^2y - 3xy^2)\,dx\,dy = \int_{-2}^{3}\int_{-1}^{2} (3x^2y - 3xy^2)\,dx\,dy$$

$$= \int_{-2}^{3} x^3y - \frac{3x^2y^2}{2}\bigg|_{x=-1}^{2} dy = \int_{-2}^{3} \left((8y - 6y^2) - \left(-y - \frac{3y^2}{2}\right)\right) dy$$

$$= \int_{-2}^{3} \left(-\frac{9y^2}{2} + 9y\right) dy = -\frac{3y^3}{2} + \frac{9y^2}{2}\bigg|_{-2}^{3} = \left(-\frac{81}{2} + \frac{81}{2}\right) - (12 + 18) = -30$$

4. $\oint_{\mathcal{C}} (3x + 5y - \cos y)\,dx + x \sin y\,dy$, where $\mathcal{C}$ is any closed curve enclosing a region with area 4, oriented counterclockwise.

SOLUTION The components of $\mathbf{F}$ are $P = 3x + 5y - \cos y$ and $Q = x \sin y$. Therefore the curl of $\mathbf{F}$ is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin y - (5 + \sin y) = -5$$

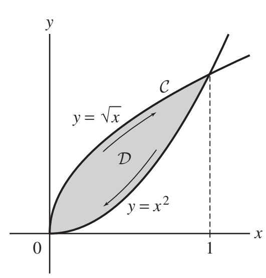
Using Green's Theorem we obtain

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (-5)\,dA$$

$$= -5 \iint_{\mathcal{D}} 1 \cdot dA = 5\text{Area}(A) = -5 \cdot 4 = -20$$

5. $\oint_{\mathcal{C}} y^2\,dx - x^2\,dy$, where $\mathcal{C}$ consists of the arcs $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, oriented clockwise.

SOLUTION We compute the curl of $\mathbf{F}$.



We have $P = y^2$ and $Q = -x^2$, hence

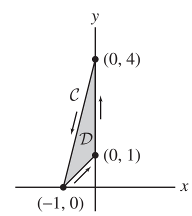
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x - 2y$$

We now compute the line integral using Green's Theorem. Since the curve is oriented clockwise, we consider the negative of the double integrals. We get

$$\begin{aligned}\int_{\mathcal{C}} y^2\,dx - x^2\,dy &= -\iint_{\mathcal{D}} (-2x-2y)\,dA = -\int_0^1 \int_{x^2}^{\sqrt{x}} (-2x-2y)\,dy\,dx \\ &= \int_0^1 2xy + y^2\Big|_{y=x^2}^{\sqrt{x}} dx = \int_0^1 \left(\left(2x\sqrt{x}+x\right) - (2x\cdot x^2 + x^4)\right) dx \\ &= \int_0^1 (-x^4 - 2x^3 + 2x^{3/2} + x)\,dx = -\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^{5/2}}{5} + \frac{x^2}{2}\Big|_0^1 \\ &= -\frac{1}{5} - \frac{1}{2} + \frac{4}{5} + \frac{1}{2} = \frac{3}{5}\end{aligned}$$

6. $\oint_{\mathcal{C}} ye^x\,dx + xe^y\,dy$, where $\mathcal{C}$ is the triangle with vertices $(-1, 0)$, $(0, 4)$, and $(0, 1)$, oriented counterclockwise.

SOLUTION

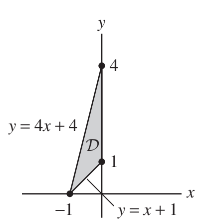


The components of the vector field are $P = ye^x$ and $Q = xe^y$, hence the flux is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^y - e^x.$$

Green's Theorem implies that

$$\begin{aligned}\int_{\mathcal{C}_0} ye^x\,dx + xe^y\,dy &= \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (e^y - e^x)\,dA \\ &= \int_{-1}^0 \int_{x+1}^{4x+4} (e^y - e^x)\,dy\,dx = \int_{-1}^0 e^y - ye^x\Big|_{y=x+1}^{4x+4} dx \\ &= \int_{-1}^0 \left(\left(e^{4x+4} - (4x+4)e^x\right) - \left(e^{x+1} - (x+1)e^x\right)\right) dx \\ &= \int_{-1}^0 \left(e^{4x+4} - e^{x+1} - (3x+3)e^x\right) dx \\ &= \int_{-1}^0 (e^{4x+4} - e^{x+1} - 3e^x)\,dx - \int_{-1}^0 3xe^x\,dx\end{aligned}$$



The second integral is computed by parts. We obtain

$$\int_{\mathcal{C}} ye^x\,dx + xe^y\,dy = \frac{e^{4x+4}}{4} - e^{x+1} - 3e^x\Big|_{-1}^0 - 3e^x(x-1)\Big|_{-1}^0$$

$$= \left(\frac{e^4}{4} - e - 3\right) - \left(\frac{1}{4} - 1 - 3e^{-1}\right) - (-3 + 6e^{-1})$$

$$= \frac{e^4}{4} - e + \frac{3}{4} - 3e^{-1} \approx 10.58$$

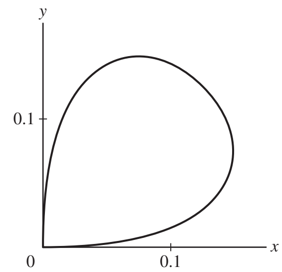
7. Let $\mathbf{c}(t) = \left(t^2(1-t), t(t-1)^2\right)$.

(a) $\boxed{\text{GU}}$ Plot the path $\mathbf{c}(t)$ for $0 \le t \le 1$.

(b) Calculate the area A of the region enclosed by $\mathbf{c}(t)$ for $0 \le t \le 1$ using the formula $A = \frac{1}{2}\oint_{\mathcal{C}} (x\,dy - y\,dx)$.

SOLUTION

(a) The path $\mathbf{c}(t)$ for $0 \le t \le 1$ is shown in the figure:



Note that the path is traced out clockwise as t goes from 0 to 1.

(b) We use the formula for the area enclosed by a closed curve,

$$A = \frac{1}{2}\int_{\mathcal{C}} (x\,dy - y\,dx)$$

We compute the line integral. Since $x = t^2(1-t)$ and $y = t(t-1)^2$, we have

$$dx = \left(2t(1-t) - t^2\right)\,dt = \left(2t - 3t^2\right)\,dt$$

$$dy = (t-1)^2 + t \cdot 2(t-1) = (t-1)(3t-1)\,dt$$

Therefore,

$$x\,dy - y\,dx = t^2(1-t)\cdot(t-1)(3t-1)\,dt - t(t-1)^2\cdot(2t-3t^2)\,dt = t^2(t-1)^2\,dt$$

We obtain the following integral (note that the path must be counterclockwise):

$$A = \frac{1}{2}\int_1^0 -t^2(t-1)^2\,dt = \frac{1}{2}\int_0^1 (t^4 - 2t^3 + t^2)\,dt = \frac{1}{2}\left(\frac{t^5}{5} - \frac{t^4}{2} + \frac{t^3}{3}\Big|_0^1\right) = \frac{1}{60}$$

8. In (a)–(d), state whether the equation is an identity (valid for all $\mathbf{F}$ or V). If it is not, provide an example in which the equation does not hold.

(a) $\mathrm{curl}(\nabla V) = 0$

(b) $\mathrm{div}(\nabla V) = 0$

(c) $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$

(d) $\nabla(\mathrm{div}(\mathbf{F})) = 0$

SOLUTION

(a) This equality is valid for all V since

$$\nabla V = \left\langle V_x, V_y, V_z\right\rangle$$

$$\mathrm{curl}(\nabla V) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = (V_{zy} - V_{yz})\mathbf{i} - (V_{zx} - V_{xz})\mathbf{j} - (V_{yx} - V_{xy})\mathbf{k}$$

By the equality of the mixed partials, we conclude that $\mathrm{curl}(\nabla\varphi)$ is the zero vector.

(b) This equation is not an identity. Take $V(x, y, z) = x^2 + y + z$. Then $\nabla V = \langle 2x, 1, 1\rangle$ and

$$\mathrm{div}(\nabla V) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(1) = 2 + 0 + 0 = 2 \ne 0.$$

(c) The equality div (curl($\mathbf{F}$)) = 0 is an identity. To prove it, we let $\mathbf{F} = \langle P, Q, R\rangle$. Then

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

$$\begin{aligned}\text{div}\,(\text{curl}(\mathbf{F})) &= \frac{\partial}{\partial x}(R_y - Q_z) - \frac{\partial}{\partial y}(R_x - P_z) + \frac{\partial}{\partial z}(Q_x - P_y) \\ &= R_{yx} - Q_{zx} - (R_{xy} - P_{zy}) + Q_{xz} - P_{yz} \\ &= (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) = 0\end{aligned}$$

The last equality is due the equality of the mixed partials.

(d) The equality $\nabla\,(\text{div}(\mathbf{F})) = \mathbf{0}$ is not an identity. Take $\mathbf{F} = \left\langle x^2, y, z\right\rangle$. Then

$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 2x + 2$$

$$\nabla\,(\text{div}(\mathbf{F})) = \left\langle \frac{\partial}{\partial x}(2x+2), \frac{\partial}{\partial y}(2x+2), \frac{\partial}{\partial z}(2x+2)\right\rangle = \langle 2, 0, 0\rangle \neq \mathbf{0}$$

In Exercises 9–12, calculate the curl and divergence of the vector field.

9. $\mathbf{F} = y\mathbf{i} - z\mathbf{k}$

SOLUTION We compute the curl of the vector field,

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & -z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(-z) - \frac{\partial}{\partial z}(0)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial z}(y)\right)\mathbf{j} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(y)}{\partial y}\right)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} - 1\mathbf{k} = -\mathbf{k}\end{aligned}$$

The divergence of $\mathbf{F}$ is

$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-z) = 0 + 0 - 1 = -1.$$

10. $\mathbf{F} = \left\langle e^{x+y}, e^{y+z}, xyz\right\rangle$

SOLUTION The curl of $\mathbf{F} = \left\langle e^{x+y}, e^{y+z}, xyz\right\rangle$ is the following vector:

$$\begin{aligned}\text{curl}(F) = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y} & e^{y+z} & xyz \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}e^{y+z}\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}e^{x+y}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}e^{y+z} - \frac{\partial}{\partial y}e^{x+y}\right)\mathbf{k} \\ &= (xz - e^{y+z})\mathbf{i} - (yz)\mathbf{j} - e^{x+y}\mathbf{k} = \left\langle xz - e^{y+z}, -yz, -e^{x+y}\right\rangle\end{aligned}$$

The divergence of $\mathbf{F}$ is

$$\text{div}\mathbf{F} = \frac{\partial}{\partial x}(e^{x+y}) + \frac{\partial}{\partial y}(e^{y+z}) + \frac{\partial}{\partial z}(xyz) = e^{x+y} + e^{y+z} + xy.$$

11. $\mathbf{F} = \nabla(e^{-x^2-y^2-z^2})$

SOLUTION In Exercise 8 we proved the identity $\text{curl}(\nabla\varphi) = \mathbf{0}$. Here, $\varphi = e^{-x^2-y^2-z^2}$, and we have $\text{curl}\left(\nabla\left(e^{-x^2-y^2-z^2}\right)\right) = \mathbf{0}$. To compute div $\mathbf{F}$, we first write $\mathbf{F}$ explicitly:

$$\mathbf{F} = \nabla\left(e^{-x^2-y^2-z^2}\right) = \left\langle -2xe^{-x^2-y^2-z^2}, -2ye^{-x^2-y^2-z^2}, -2ze^{-x^2-y^2-z^2}\right\rangle = \langle P, Q, R\rangle$$

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= \left(-2e^{-x^2-y^2-z^2} + 4x^2e^{-x^2-y^2-z^2}\right) + \left(-2e^{-x^2-y^2-z^2} + 4y^2e^{-x^2-y^2-z^2}\right) \\ &\quad + \left(-2e^{-x^2-y^2-z^2} + 4z^2e^{-x^2-y^2-z^2}\right) \\ &= 2e^{-x^2-y^2-z^2}\left(2(x^2+y^2+z^2) - 3\right)\end{aligned}$$

12. $\mathbf{e}_r = r^{-1}\langle x, y, z\rangle \ \left(r = \sqrt{x^2+y^2+z^2}\right)$

SOLUTION It can be easily verified that $\mathbf{e}_r = \nabla\varphi$ for $\varphi(x, y, z) = \sqrt{x^2+y^2+z^2} = r$. Therefore, by the identity $\operatorname{curl}(\nabla\varphi) = \mathbf{0}$ (provided in Exercise 8), we have

$$\operatorname{curl}(\mathbf{e}_r) = \operatorname{curl}(\nabla\varphi) = \mathbf{0}$$

We compute the divergence of $\mathbf{e}_r$. Since $r_x = \frac{x}{r}$, $r_y = \frac{y}{r}$, $r_z = \frac{z}{r}$, we have

$$\begin{aligned}\operatorname{div}(\mathbf{e}_r) &= \frac{\partial}{\partial x}(xr^{-1}) + \frac{\partial}{\partial y}(yr^{-1}) + \frac{\partial}{\partial z}(zr^{-1}) = (r^{-1} - xr^{-2}r_x) + (r^{-1} - yr^{-2}r_y) + (r^{-1} - zr^{-2}r_z) \\ &= 3r^{-1} - r^{-2}(xr_x + yr_y + zr_z) = 3r^{-1} - r^{-2}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) = 3r^{-1} - r^{-2}\cdot\frac{r^2}{r} = 2r^{-1}\end{aligned}$$

13. Recall that if F_1, F_2, and F_3 are differentiable functions of one variable, then

$$\operatorname{curl}\left(\langle F_1(x), F_2(y), F_3(z)\rangle\right) = \mathbf{0}$$

Use this to calculate the curl of

$$\mathbf{F} = \left\langle x^2+y^2, \ln y + z^2, z^3\sin(z^2)e^{z^3}\right\rangle$$

SOLUTION We use the linearity of the curl and the property mentioned in the exercise to compute the curl of $\mathbf{F}$:

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \operatorname{curl}\left(\left\langle x^2+y^2, \ln y + z^2, z^3\sin\left(z^2\right)e^{z^3}\right\rangle\right) = \operatorname{curl}\left(\left\langle x^2, \ln y, z^3\sin(z^2)e^{z^3}\right\rangle\right) + \operatorname{curl}\left(\left\langle y^2, z^2, 0\right\rangle\right) \\ &= 0 + \operatorname{curl}\left\langle y^2, z^2, 0\right\rangle = \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}z^2, \frac{\partial}{\partial z}y^2 - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial y}y^2\right\rangle = \langle -2z, 0, -2y\rangle\end{aligned}$$

14. Give an example of a nonzero vector field $\mathbf{F}$ such that $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ and $\operatorname{div}(\mathbf{F}) = 0$.

SOLUTION Let $\mathbf{F} = \langle x, -y, 0\rangle$. Then

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x)\right)\mathbf{k} = \mathbf{0}$$

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(0) = 1 - 1 + 0 = 0$$

15. Verify the identities of Exercises 6 and 34 in Section 17.3 for the vector fields $\mathbf{F} = \left\langle xz, ye^x, yz\right\rangle$ and $\mathbf{G} = \left\langle z^2, xy^3, x^2y\right\rangle$.

SOLUTION We first show $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$. Let $\mathbf{F} = \langle P, Q, R\rangle = \left\langle xz, ye^x, yz\right\rangle$. We compute the curl of $\mathbf{F}$:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

Substituting in the appropriate values for P, Q, R and taking derivatives, we get

$$\operatorname{curl}(\mathbf{F}) = \left\langle z - 0, x - 0, ye^x - 0\right\rangle$$

Thus,

$$\operatorname{div}\left(\operatorname{curl}(\mathbf{F})\right) = (z)_x + (x)_y + (ye^x)_z = 0 + 0 + 0 = 0.$$

Likewise, for $\mathbf{G} = \langle P, Q, R\rangle = \left\langle z^2, xy^3x^2y\right\rangle$, we compute the curl of $\mathbf{G}$:

$$\text{curl}(\mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

Substituting in the appropriate values for P, Q, R and taking derivatives, we get

$$\text{curl}(\mathbf{G}) = \left\langle x^2 - 0, 2z - 2xy, y^3 - 0\right\rangle$$

Thus,

$$\text{div}\,(\text{curl}(\mathbf{G})) = (x^2)_x + (2z - 2xy)_y + (y^3)_z = 2x - 2x = 0.$$

We now work on the second identity. For $\mathbf{F} = \langle xz, ye^x, yz\rangle$ and $\mathbf{G} = \langle z^2, xy^3, x^2y\rangle$, it is easy to calculate

$$\mathbf{F} \times \mathbf{G} = \langle x^2y^2e^x - xy^4z, yz^3 - x^3yz, x^2y^3z - yz^2e^x\rangle$$

Thus,

$$\text{div}(\mathbf{F} \times \mathbf{G}) = (2xy^2e^x + x^2y^2e^x - y^4z) + (z^3 - x^3z) + (x^2y^3 - 2yze^x)$$

On the other hand, from our work above,

$$\text{curl}(\mathbf{F}) = \langle z, x, ye^x\rangle$$
$$\text{curl}(\mathbf{G}) = \left\langle x^2, 2z - 2xy, y^3\right\rangle$$

So, we calculate

$$\begin{aligned} \mathbf{G} \cdot \text{curl}(\mathbf{F} - \mathbf{F}) \cdot \text{curl}(\mathbf{G}) &= z^2 \cdot z + xy^3 \cdot x + x^2y \cdot ye^x - xz \cdot x^2 - ye^x \cdot (2z - 2xy) - yz \cdot y^3 \\ &= z^3 + x^2y^3 + x^2y^2e^x + 2xy^2e^x - x^3z - 2yze^x - y^4z \\ &= (2xy^2e^x + x^2y^2e^x - y^4z) + (z^3 - x^3z) + (x^2y^3 - 2yze^x) = \text{div}(\mathbf{F} \times \mathbf{G}) \end{aligned}$$

16. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are surfaces with the same oriented boundary curve $\mathcal{C}$. Which of the following conditions guarantees that the flux of $\mathbf{F}$ through $\mathcal{S}_1$ is equal to the flux of $\mathbf{F}$ through $\mathcal{S}_2$?

(i) $\mathbf{F} = \nabla V$ for some function V

(ii) $\mathbf{F} = \text{curl}(\mathbf{G})$ for some vector field $\mathbf{G}$

SOLUTION If $\mathbf{F} = \text{curl}(\mathbf{G})$, then by the Theorem on Surface Independence for Curl Vector Fields, the flux of $\mathbf{F}$ through a surface $\mathcal{S}$ depends only the oriented boundary $\partial\mathcal{S}$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same oriented boundary curve, we conclude that the flux of $\mathbf{F}$ through $\mathcal{S}_1$ is equal to the flux of $\mathbf{F}$ through $\mathcal{S}_2$. The condition in (i) that $\mathbf{F}$ is conservative does not guarantee that the flux of $\mathbf{F}$ through $\mathcal{S}_1$ is equal to the flux through $\mathcal{S}_2$.

17. Prove that if $\mathbf{F}$ is a gradient vector field, then the flux of $\text{curl}(\mathbf{F})$ through a smooth surface $\mathcal{S}$ (whether closed or not) is equal to zero.

SOLUTION If $\mathbf{F}$ is a gradient vector field, then $\mathbf{F}$ is conservative; therefore the line integral of $\mathbf{F}$ over any closed curve is zero. Combining with Stokes' Theorem yields

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = 0$$

18. Verify Stokes' Theorem for $\mathbf{F} = \langle y, z - x, 0\rangle$ and the surface $z = 4 - x^2 - y^2$, $z \geq 0$, oriented by outward-pointing normals.

SOLUTION We begin by computing the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$. The boundary curve is the circle $x^2 + y^2 = 4$ (in the xy plane) oriented in the counterclockwise direction. We use the parametrization

$$\mathcal{C} : r(t) = (2\cos t, 2\sin t, 0), \quad 0 \leq t \leq 2\pi$$

Then,

$$\begin{aligned} \mathbf{F}\,(r(t)) &= \langle 2\sin t, -2\cos t, 0\rangle \\ r'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \\ \mathbf{F}\,(r(t)) \cdot r'(t) &= \langle 2\sin t, -2\cos t, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle = -4\sin^2 t - 4\cos^2 t = -4 \end{aligned}$$

The line integral is thus

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -4\,dt = -8\pi \tag{1}$$

We now compute the integral $\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$. We find the curl of $\mathbf{F}$:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z-x & 0 \end{vmatrix} = \left(0 - \frac{\partial}{\partial z}(z-x)\right)\mathbf{i} - \left(0 - \frac{\partial}{\partial z}y\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(z-x) - \frac{\partial}{\partial y}y\right)\mathbf{k}$$
$$= -\mathbf{i} - 2\mathbf{k} = \langle -1, 0, -2\rangle$$

We parametrized the surface by

$$\Phi(u,v) = \left(u\cos v, u\sin v, 4-u^2\right), \quad 0 \le v < 2\pi, \quad 0 \le u \le 2.$$

Then,

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \langle \cos v, \sin v, -2u\rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \langle -u\sin v, u\cos v, 0\rangle$$
$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u^2\cos v)\mathbf{i} + (2u^2\sin v)\mathbf{j} + (u\cos^2 v + u\sin^2 v)\mathbf{k}$$
$$= \left\langle 2u^2\cos v, 2u^2\sin v, u\right\rangle$$

The surface is oriented outwards, hence the z-component of the normal vector is nonnegative. Therefore, the normal vector is (recall that $u \ge 0$)

$$\mathbf{n} = \left\langle 2u^2\cos v, 2u^2\sin v, u\right\rangle$$

We compute the dot product:

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle -1, 0, -2\rangle \cdot \left\langle 2u^2\cos v, 2u^2\sin v, u\right\rangle = -2u^2\cos v - 2u$$

We obtain the following integral:

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot \mathbf{n}\,ds = \int_0^{2\pi}\int_0^2 (-2u^2\cos v - 2u)\,du\,dv = \int_0^{2\pi} \left.\frac{-2u^3\cos v}{3} - u^2\right|_{u=0}^2 dv$$
$$= \int_0^{2\pi} \left(\frac{-16\cos v}{3} - 4\right) dv = \left.\frac{-16\sin v}{3} - 4v\right|_0^{2\pi} = -8\pi \tag{2}$$

By (1) and (2), both the line integral and the flux of the curl are equal to -8π. Thus, this example verifies Stokes' Theorem.

19. Let $\mathbf{F} = \left\langle z^2, x+z, y^2\right\rangle$ and let $\mathcal{S}$ be the upper half of the ellipsoid

$$\frac{x^2}{4} + y^2 + z^2 = 1$$

oriented by outward-pointing normals. Use Stokes' Theorem to compute $\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

SOLUTION We compute the curl of $\mathbf{F} = \left\langle z^2, x+z, y^2\right\rangle$:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x+z & y^2 \end{vmatrix} = (2y-1)\mathbf{i} - (0-2z)\mathbf{j} + (1-0)\mathbf{k} = \langle 2y-1, 2z, 1\rangle$$

Let $\mathcal{C}$ denote the boundary of $\mathcal{S}$, that is, the ellipse $\frac{x^2}{4} + y^2 = 1$ in the xy-plane, oriented counterclockwise. Then by Stoke's Theorem we have

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$

We parametrize $\mathcal{C}$ by

$$\mathcal{C}: r(t) = (2\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$$

Then

$$\mathbf{F}(r(t)) \cdot r'(t) = \left\langle 0, 2\cos t, \sin^2 t \right\rangle \cdot \langle -2\sin t, \cos t, 0\rangle = 2\cos^2 t$$

Combining with (1) gives

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{s} = \int_0^{2\pi} 2\cos^2 t\, dt = t + \frac{\sin 2t}{2}\bigg|_0^{2\pi} = 2\pi$$

20. Use Stokes' Theorem to evaluate $\oint_{\mathcal{C}} \langle y, z, x\rangle \cdot d\mathbf{s}$, where $\mathcal{C}$ is the curve in Figure 2.

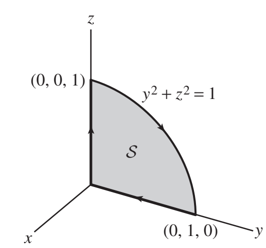


FIGURE 2

SOLUTION We compute the curl of $\mathbf{F} = \langle y, z, x\rangle$:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1\rangle$$

By Stokes' Theorem, we have

$$\int_{\mathcal{C}} \langle y, z, x\rangle \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{S}} (\operatorname{curl}(\mathbf{F}) \cdot \mathbf{e}_n)\, d\mathbf{S}$$

Since the boundary $\mathcal{C}$ of the quarter circle $\mathcal{S}$ is oriented clockwise, the induced orientation on $\mathcal{S}$ is normal pointing in the negative x direction. Thus,

$$\mathbf{e}_n = \langle -1, 0, 0\rangle.$$

Hence,

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{e}_n = \langle -1, -1, -1\rangle \cdot \langle -1, 0, 0\rangle = 1.$$

Combining with (1) we get

$$\int_{\mathcal{C}} \langle y, z, x\rangle \cdot d\mathbf{s} = \iint_{\mathcal{S}} 1\, ds = \text{Area}(\mathcal{S}) = \frac{\pi}{4}$$

21. Let $\mathcal{S}$ be the side of the cylinder $x^2 + y^2 = 4$, $0 \le z \le 2$ (not including the top and bottom of the cylinder). Use Stokes' Theorem to compute the flux of $\mathbf{F} = \langle 0, y, -z\rangle$ through $\mathcal{S}$ (with outward pointing normal) by finding a vector potential $\mathbf{A}$ such that $\operatorname{curl}(\mathbf{A}) = \mathbf{F}$.

SOLUTION We can write $\mathbf{F} = \operatorname{curl}(\mathbf{A})$ where $\mathbf{A} = \langle yz, 0, 0\rangle$. The flux of $\mathbf{F}$ through $\mathcal{S}$ is equal to the line integral of A around the oriented boundary which consists of two circles of radius 2 with center on the z-axis (one at height $z = 0$ and one at height $z = 2$).

However, the line integrals of A about both circles are zero. This is clear for the circle at $z = 0$ because then $A = 0$, but it is also true at $z = 2$ because the vector field $A = \langle 2y, 0, 0\rangle$ integrates to zero around the circle.

22. Verify the Divergence Theorem for $\mathbf{F} = \langle 0, 0, z \rangle$ and the region $x^2 + y^2 + z^2 = 1$.

SOLUTION Let $\mathcal{S}$ be the boundary of the unit sphere $\mathcal{W}$. We calculate both sides of the equation:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV \tag{1}$$

We start with the surface integral. We parametrize $\mathcal{S}$ by

$$\Phi(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle, \quad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi$$

Then (see Example 1 in Section 17.5)

$$\mathbf{n} = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

Hence,

$$\mathbf{F}(\Phi(\theta, \phi)) \cdot \mathbf{n} = \langle 0, 0, \cos\phi \rangle \cdot \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle = \cos^2\phi \sin\phi$$

We obtain the following integral:

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} \mathbf{F}(\Phi(\theta, \phi)) \cdot \mathbf{n}\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi\, d\theta \\ &= 2\pi \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi = 2\pi \left(-\frac{\cos^3\phi}{3} \Big|_0^{\pi} \right) = 2\pi \left(\frac{1+1}{3} \right) = \frac{4\pi}{3} \end{aligned} \tag{2}$$

We now compute the triple integral in (1). We find the divergence of $\mathbf{F}$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 1$$

Hence,

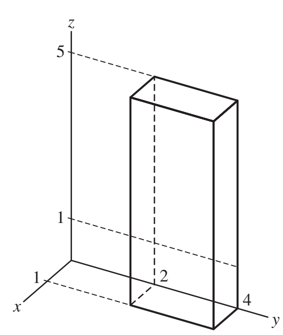
$$\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F}) \cdot dV = \iiint_{\mathcal{W}} 1\, dV = \text{Volume}(\mathcal{W}) = \frac{4\pi}{3} \tag{3}$$

The integrals in (2) and (3) are equal, as follows from the Divergence Theorem.

In Exercises 23–26, use the Divergence Theorem to calculate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ *for the given vector field and surface.*

23. $\mathbf{F} = \langle xy, yz, x^2z + z^2 \rangle$, $\mathcal{S}$ is the boundary of the box $[0, 1] \times [2, 4] \times [1, 5]$.

SOLUTION



We compute the divergence of $\mathbf{F} = \left\langle xy, yz, x^2z + z^2 \right\rangle$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} yz + \frac{\partial}{\partial z}(x^2z + z^2) = y + z + x^2 + 2z = x^2 + y + 3z$$

The Divergence Theorem gives

$$\begin{aligned} \iint_{\mathcal{S}} \left\langle xy, yz, x^2z + z^2 \right\rangle \cdot d\mathbf{S} &= \int_1^5 \int_2^4 \int_0^1 (x^2 + y + 3z)\, dx\, dy\, dz = \int_1^5 \int_2^4 \frac{x^3}{3} + (y + 3z)x \Big|_{x=0}^{1} dy\, dz \\ &= \int_1^5 \int_2^4 \left(\frac{1}{3} + y + 3z \right) dy\, dz = \int_1^5 \frac{1}{3}y + \frac{1}{2}y^2 + 3zy \Big|_{y=2}^{4} dz \end{aligned}$$

$$= \int_1^5 \left(\left(\frac{4}{3} + \frac{16}{2} + 12z \right) - \left(\frac{2}{3} + 2 + 6z \right) \right) dz = \int_1^5 \left(\frac{20}{3} + 6z \right) dz$$

$$= \frac{20z}{3} + \frac{3z^2}{2} \Big|_1^5 = \left(75 + \frac{100}{3} \right) - \left(3 + \frac{20}{3} \right) = \frac{296}{3}$$

24. $\mathbf{F} = \langle xy, yz, x^2z + z^2 \rangle$, $\mathcal{S}$ is the boundary of the unit sphere.

SOLUTION We use spherical coordinates:

$$x = \rho \cos\theta \sin\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho\cos\phi$$

with

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi, \quad 0 \le \rho \le 1$$

We obtain

$$\iint_{\mathcal{S}} \left\langle xy, yz, x^2z + z^2 \right\rangle \cdot d\mathbf{S} = \iiint_V \operatorname{div}(\mathbf{F})\, dV = \iiint_V (x^2 + y + 3z)\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho^2 \cos^2\theta \sin^2\phi + \rho \sin\theta \sin\phi + 3\rho\cos\phi) \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi} \cos^2\theta\, d\theta \right) \left(\int_0^{\pi} \sin^3\phi\, d\phi \right) \left(\int_0^1 \rho^4\, d\rho \right)$$

$$+ \left(\int_0^{2\pi} \sin\theta\, d\theta \right) \left(\int_0^{\pi} \sin^2\phi\, d\phi \right) \left(\int_0^1 \rho^3\, d\rho \right)$$

$$+ 6\pi \left(\int_0^{\pi} \cos\phi \sin\phi\, d\phi \right) \left(\int_0^1 \rho^3 d\rho \right)$$

$$= \pi \cdot \frac{4}{3} \cdot \frac{1}{5} + 0 + 0 = \frac{4\pi}{15}$$

25. $\mathbf{F} = \langle xyz + xy, \frac{1}{2}y^2(1 - z) + e^x, e^{x^2+y^2} \rangle$, $\mathcal{S}$ is the boundary of the solid bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = y - 4$.

SOLUTION We compute the divergence of $\mathbf{F}$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(xyz + xy) + \frac{\partial}{\partial y}\left(\frac{y^2}{2}(1 - z) + e^x \right) + \frac{\partial}{\partial z}(e^{x^2+y^2}) = yz + y + y(1 - z) = 2y$$

Let $\mathcal{S}$ denote the surface of the solid $\mathcal{W}$. The Divergence Theorem gives

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 2y\, dV = \iint_{\mathcal{D}} \int_{y-4}^{0} 2y\, dz\, dx\, dy$$

$$= \iint_{\mathcal{D}} 2yz \Big|_{z=y-4}^{0} dx\, dy = \iint_{\mathcal{D}} 2y\,(0 - (y - 4))\, dx\, dy = \iint_{\mathcal{D}} (8y - 2y^2)\, dx\, dy$$

We convert the integral to polar coordinates:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^4 (8r\cos\theta - 2r^2\cos^2\theta) r\, dr\, d\theta$$

$$= 8 \left(\int_0^4 r^2\, dr \right) \left(\int_0^{2\pi} \cos\theta\, d\theta \right) - \left(\int_0^4 r^3\, dr \right) \left(\int_0^{2\pi} 2\cos^2\theta\, d\theta \right)$$

$$= 0 - \left(\frac{r^4}{4} \Big|_0^4 \right) \left(\theta + \frac{\sin 2\theta}{2} \Big|_0^{2\pi} \right) = -\frac{4^4}{4} \cdot 2\pi = -128\pi$$

26. $\mathbf{F} = \left\langle \sin(yz), \sqrt{x^2+z^4}, x\cos(x-y) \right\rangle$, $\mathcal{S}$ is any smooth closed surface that is the boundary of a region in $\mathbf{R}^3$.

SOLUTION We compute the divergence of $\mathbf{F}$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(\sin(yz)) + \frac{\partial}{\partial y}\left(\sqrt{x^2+z^4}\right) + \frac{\partial}{\partial z}(x\cos x(x-y)) = 0$$

Let $\mathcal{W}$ denote the solid inside $\mathcal{S}$. The Divergence Theorem gives

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iiint_{\mathcal{W}} 0\, dV = 0$$

27. Find the volume of a region $\mathcal{W}$ if

$$\iint_{\partial \mathcal{W}} \left\langle x + xy + z, x + 3y - \frac{1}{2}y^2, 4z \right\rangle \cdot d\mathbf{S} = 16$$

SOLUTION Let $\mathbf{F} = \left\langle x + xy + z, x + 3y - \frac{1}{2}y^2, 4z \right\rangle$. We compute the divergence of $\mathbf{F}$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x + xy + z) + \frac{\partial}{\partial y}\left(x + 3y - \frac{1}{2}y^2\right) + \frac{\partial}{\partial z}(4z) = 1 + y + 3 - y + 4 = 8$$

Using the Divergence Theorem and the given information, we obtain

$$16 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = \iint_{\mathcal{W}} 8\, dV = 8 \iint_{\mathcal{W}} 1\, dV = 8 \text{ Volume } (\mathcal{W})$$

That is,

$$16 = 8 \text{ Volume } (\mathcal{W})$$

or

$$\text{Volume } (\mathcal{W}) = 2$$

28. Show that the circulation of $\mathbf{F} = \left\langle x^2, y^2, z(x^2+y^2) \right\rangle$ around any curve $\mathcal{C}$ on the surface of the cone $z^2 = x^2 + y^2$ is equal to zero (Figure 3).

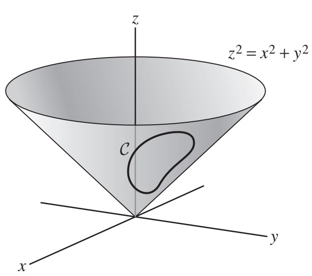


FIGURE 3

SOLUTION Let $\mathcal{S}$ be the part of the cone that is inside $\mathcal{C}$. Then by Stoke's Theorem, the circulation of $\mathbf{F}$ around $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \tag{1}$$

We parametrize the cone by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r)$$

Then,

$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \langle \cos\theta, \sin\theta, 1 \rangle$$

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \langle r\cos\theta, r\sin\theta, -r \rangle$$

We compute the curl of **F** and express it in terms of the parameters:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z(x^2+y^2) \end{vmatrix} = \langle 2yz, -2xz, 0\rangle$$

$$\text{curl}(\mathbf{F})\,(\Phi(r,\theta)) = \langle 2r\sin\theta\cdot r, -2r\cos\theta\cdot r, 0\rangle = \left\langle 2r^2\sin\theta, -2r^2\cos\theta, 0\right\rangle$$

The dot product is thus

$$\begin{aligned}\text{curl}(\mathbf{F})\big(\Phi(r,\theta)\big)\cdot\mathbf{n} &= \left\langle 2r^2\sin\theta, -2r^2\cos\theta, 0\right\rangle\cdot\langle r\cos\theta, r\sin\theta, -r\rangle \\ &= 2r^3\sin\theta\cos\theta - 2r^3\cos\theta\sin\theta + 0 = 0\end{aligned}$$

We see that curl(**F**) is tangent to the cone at all points on the cone, hence the surface integral in (1) is zero. We conclude that the circulation of **F** around any curve $\mathcal{C}$ on the cone is zero.

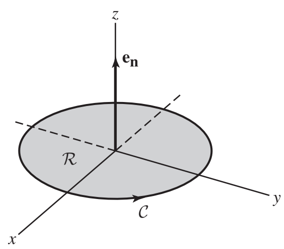
In Exercises 29–32, let **F** *be a vector field whose curl and divergence at the origin are*

$$\text{curl}(\mathbf{F})(0,0,0) = \langle 2,-1,4\rangle, \qquad \text{div}(\mathbf{F})(0,0,0) = -2$$

29. Estimate $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$, where $\mathcal{C}$ is the circle of radius 0.03 in the xy-plane centered at the origin.

SOLUTION We use the estimation

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} \approx (\text{curl}(\mathbf{F})(\mathbf{0})\cdot\mathbf{e}_n)\,\text{Area}(\mathcal{R})$$



The unit normal vector to the disk $\mathcal{R}$ is $\mathbf{e}_n = \mathbf{k} = \langle 0,0,1\rangle$. The area of the disk is

$$\text{Area}\,(\mathcal{R}) = \pi\cdot 0.03^2 = 0.0009\pi.$$

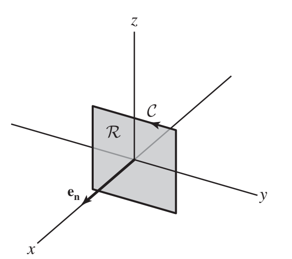
Using the given curl at the origin, we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} \approx \langle 2,-1,4\rangle\cdot\langle 0,0,1\rangle\cdot 0.0009\pi = 4\cdot 0.0009\pi \approx 0.0113$$

30. Estimate $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$, where $\mathcal{C}$ is the boundary of the square of side 0.03 in the yz-plane centered at the origin. Does the estimate depend on how the square is oriented within the yz-plane? Might the actual circulation depend on how it is oriented?

SOLUTION We use the estimation

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} \approx (\text{curl}(\mathbf{F})(\mathbf{0})\cdot\mathbf{e}_n)\,\text{Area}\,(\mathcal{R}) \tag{1}$$



If we orient $\mathcal{C}$ counterclockwise, then the unit normal vector is

$$\mathbf{e}_n = \mathbf{i} = \langle 1, 0, 0 \rangle .$$

The area of the square is Area$(\mathcal{R}) = 0.03^2 = 0.0009$, and by the given information the curl at the origin is $\langle 2, -1, 4 \rangle$. Therefore (1) gives the estimation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx \langle 2, -1, 4 \rangle \cdot \langle 1, 0, 0 \rangle \cdot 0.0009 = 2 \cdot 0.0009 = 0.0018$$

The estimate depend on curl$(\mathbf{F})$, $\mathbf{e}_n$ and the area of the square. Hence, if we flip the square over (such that $\mathbf{e}_n$ points along the negative x-axis), then we will get a different answer.

31. Suppose that $\mathbf{v}$ is the velocity field of a fluid and imagine placing a small paddle wheel at the origin. Find the equation of the plane in which the paddle wheel should be placed to make it rotate as quickly as possible.

SOLUTION The paddle wheel has the maximum spin when the circulation of the velocity field $\mathbf{v}$ around the wheel is maximum. The maximum circulation occurs when $\mathbf{e}_n$, and the curl of $\mathbf{v}$ at the origin (i.e., the vector $\langle 2, -1, 4 \rangle$) point in the same direction. Therefore, the plane in which the paddle wheel should be placed is the plane through the origin with the normal $\langle 2, -1, 4 \rangle$. This plane has the equation, $2x - y + 4z = 0$.

32. Estimate the flux of $\mathbf{F}$ through the box of side 0.5 in Figure 4. Does the result depend on how the box is oriented relative to the coordinate axes?

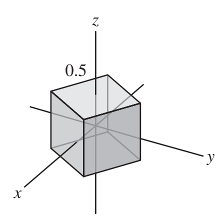


FIGURE 4

SOLUTION We use the following estimation:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \approx \operatorname{div} \mathbf{F}(0) \text{ Volume } (\mathcal{W})$$

The volume of the box $\mathcal{W}$ is 0.5^3, and we are given that $\operatorname{div}(\mathbf{F})(0) = -2$. This gives the estimation

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \approx -2 \cdot 0.5^3 = -0.25.$$

The negative sign shows that there is a net inflow across the box. Our estimation of the flux does not depend on the orientation of the box; rather, it depends on the magnitude of the divergence of $\mathbf{F}$.

33. The velocity vector field of a fluid (in meters per second) is

$$\mathbf{F} = \left\langle x^2 + y^2, 0, z^2 \right\rangle$$

Let $\mathcal{W}$ be the region between the hemisphere

$$\mathcal{S} = \left\{ (x, y, z) : x^2 + y^2 + z^2 = 1, \quad x, y, z \geq 0 \right\}$$

and the disk $\mathcal{D} = \left\{ (x, y, 0) : x^2 + y^2 \leq 1 \right\}$ in the xy-plane. Recall that the flow rate of a fluid across a surface is equal to the flux of $\mathbf{F}$ through the surface.

(a) Show that the flow rate across $\mathcal{D}$ is zero.

(b) Use the Divergence Theorem to show that the flow rate across $\mathcal{S}$, oriented with outward-pointing normal, is equal to $\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F}) \, dV$. Then compute this triple integral.

SOLUTION

(a) To show that no fluid flows across $\mathcal{D}$, we show that the normal component of $\mathbf{F}$ at each point on $\mathcal{D}$ is zero. At each point $P = (x, y, 0)$ on the xy-plane,

$$\mathbf{F}(P) = \left\langle x^2 + y^2, 0, 0^2 \right\rangle = \left\langle x^2 + y^2, 0, 0 \right\rangle.$$

Moreover, the unit normal vector to the xy-plane is $\mathbf{e}_n = (0, 0, 1)$. Therefore,

$$\mathbf{F}(P) \cdot \mathbf{e}_n = \left\langle x^2 + y^2, 0, 0 \right\rangle \cdot \langle 0, 0, 1 \rangle = 0.$$

Since $\mathcal{D}$ is contained in the xy-plane, we conclude that the normal component of $\mathbf{F}$ at each point on $\mathcal{D}$ is zero. Therefore, no fluid flows across $\mathcal{D}$.

(b) By the Divergence Theorem and the linearity of the flux we have

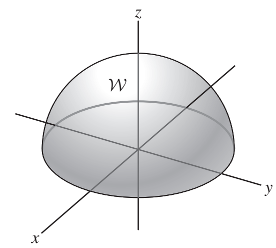
$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV$$

Since the flux through the disk $\mathcal{D}$ is zero, we have

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV \tag{1}$$

To compute the triple integral, we first compute $\operatorname{div}(\mathbf{F})$:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z^2) = 2x + 2z = 2(x + z).$$



Using spherical coordinates we get

$$\begin{aligned}
\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV &= 2\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 (\rho \sin\phi\cos\theta + \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi \\
&= 2\int_0^1 \rho^3 d\rho \left(\left(\int_0^{\pi/2} \sin^2\phi\, d\phi\right)\left(\int_0^{2\pi} \cos\theta\, d\theta\right) + 2\pi \int_0^{\pi/2} \cos\phi \sin\phi\, d\rho\right) \\
&= \frac{1}{2}\left(0 + \pi \int_0^{\pi/2} \sin 2\phi\, d\phi\right) = \frac{\pi}{2}\left(-\frac{\cos 2\phi}{2}\right)\Bigg|_0^{\pi/2} = -\frac{\pi}{4}(-1-1) = \frac{\pi}{2}
\end{aligned}$$

Combining with (1) we obtain the flux:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2}$$

34. The velocity field of a fluid (in meters per second) is

$$\mathbf{F} = (3y - 4)\mathbf{i} + e^{-y(z+1)}\mathbf{j} + (x^2 + y^2)\mathbf{k}$$

(a) Estimate the flow rate (in cubic meters per second) through a small surface $\mathcal{S}$ around the origin if $\mathcal{S}$ encloses a region of volume 0.01 m^3.

(b) Estimate the circulation of $\mathbf{F}$ about a circle in the xy-plane of radius $r = 0.1$ m centered at the origin (oriented counterclockwise when viewed from above).

(c) Estimate the circulation of $\mathbf{F}$ about a circle in the yz-plane of radius $r = 0.1$ m centered at the origin (oriented counterclockwise when viewed from the positive x-axis).

SOLUTION

(a) We use the approximation

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \approx \operatorname{div}(\mathbf{F})(\mathbf{0})\operatorname{Vol}(\mathcal{W}) \tag{1}$$

Here, $\operatorname{Vol}(\mathcal{W}) = 0.01\ \text{m}^3$. We compute the divergence at the origin:

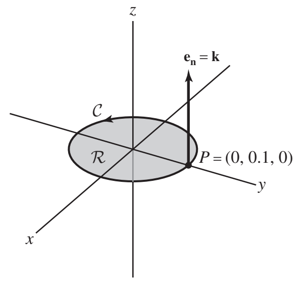
$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(3y - 4) + \frac{\partial}{\partial y}e^{-y(z+1)} + \frac{\partial}{\partial z}(x^2 + y^2) = -(z+1)e^{-y(z+1)} \quad \Rightarrow \quad \operatorname{div}(\mathbf{F})(\mathbf{0}) = -1$$

Substituting in (1) gives the estimation

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \approx -1 \cdot 0.01 = -0.01\ \text{m}^3/\text{s}$$

(b) We use the estimation for the circulation using the point $P = (0, 0.1, 0)$:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx (\operatorname{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n) \text{ Area } (\mathcal{R}) \tag{2}$$



The unit normal vector to $\mathcal{R}$ is $\mathbf{e}_n = \langle 0, 0, 1\rangle = \mathbf{k}$ and the area of the disc is $\pi \cdot 0.1^2 = 0.01\pi$. We compute the curl at P:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (3y-4) & e^{-y(z+1)} & x^2+y^2 \end{vmatrix} = \left(2y - e^{-y(z+1)} \cdot (-y)\right)\mathbf{i} - (2x-0)\mathbf{j} + (0-3)\mathbf{k}$$

$$= \left(2y + ye^{-y(z+1)}\right)\mathbf{i} - 2x\mathbf{j} - 3\mathbf{k}$$

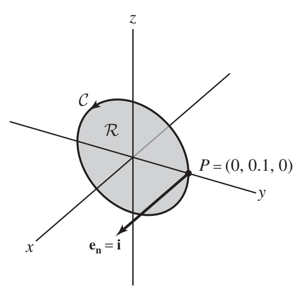
Hence,

$$\operatorname{curl}(\mathbf{F})(P) = (0.2 + 0.1e^{-0.1})\mathbf{i} - 3\mathbf{k}$$
$$D\operatorname{curl}(\mathbf{F})(P) \cdot \mathbf{e}_n = \left((0.2 + 0.1e^{-0.1})\mathbf{i} - 3\mathbf{k}\right) \cdot \mathbf{k} = -3$$

Combining with (2) gives the estimation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx -3 \cdot 0.01\pi = -0.03\pi$$

(c) We use estimation (2), only that now the unit normal vector is $\mathbf{e}_n = \mathbf{i}$.



We get

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \approx \left(\left((0.2 + 0.1e^{-0.1})\mathbf{i} - 3\mathbf{k}\right) \cdot \mathbf{i}\right) 0.01\pi = (0.2 + 0.1e^{-0.1}) \cdot 0.01\pi = 0.009$$

35. Let $V(x, y) = x + \dfrac{x}{x^2 + y^2}$. The vector field $\mathbf{F} = \nabla V$ (Figure 5) provides a model in the plane of the velocity field of an incompressible, irrotational fluid flowing past a cylindrical obstacle (in this case, the obstacle is the unit circle $x^2 + y^2 = 1$).

(a) Verify that $\mathbf{F}$ is irrotational [by definition, $\mathbf{F}$ is irrotational if $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$].

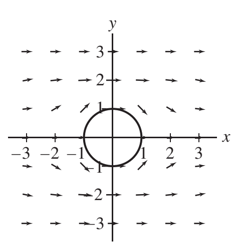


FIGURE 5 The vector field ∇V for $V(x, y) = x + \dfrac{x}{x^2 + y^2}$.

(b) Verify that $\mathbf{F}$ is tangent to the unit circle at each point along the unit circle except $(1, 0)$ and $(-1, 0)$ (where $\mathbf{F} = \mathbf{0}$).

(c) What is the circulation of $\mathbf{F}$ around the unit circle?

(d) Calculate the line integral of $\mathbf{F}$ along the upper and lower halves of the unit circle separately.

SOLUTION

(a) In Exercise 8, we proved the identity $\operatorname{curl}(\nabla\varphi) = \mathbf{0}$. Since $\mathbf{F}$ is a gradient vector field, it is irrotational; that is, $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ for $(x, y) \neq (0, 0)$, where $\mathbf{F}$ is defined.

(b) We compute $\mathbf{F}$ explicitly:

$$\mathbf{F} = \nabla\varphi = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \right\rangle = \left\langle 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2} \right\rangle$$

Now, using $x = \cos t$ and $y = \sin t$ as a parametrization of the circle, we see that

$$\mathbf{F} = \left\langle 1 + \sin^2 t - \cos^2 t, -2\cos t \sin t \right\rangle = \left\langle 2\sin^2 t, -2\cos t \sin t \right\rangle,$$

and so

$$\mathbf{F} = 2\sin t \langle \sin t, -\cos t \rangle = 2\sin t \langle y, -x \rangle,$$

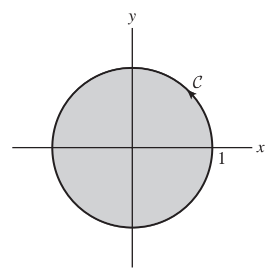
which is clearly perpendicular to the radial vector $\langle x, y \rangle$ for the circle.

(c) We use our expression of $\mathbf{F}$ from Part (b):

$$\mathbf{F} = \nabla\varphi = \left\langle 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2} \right\rangle$$

Now, $\mathbf{F}$ is not defined at the origin and therefore we cannot use Green's Theorem to compute the line integral along the unit circle. We thus compute the integral directly, using the parametrization

$$\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi.$$



Then,

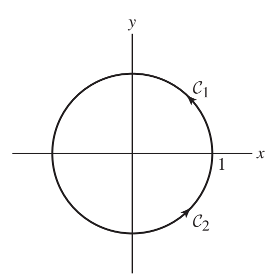
$$\begin{aligned}\mathbf{F}(\mathbf{c}(t)) \cdot c'(t) &= \left\langle 1 + \frac{\sin^2 t - \cos^2 t}{(\cos^2 t + \sin^2 t)^2}, -\frac{2\cos t \sin t}{(\cos^2 t + \sin^2 t)^2} \right\rangle \cdot \langle -\sin t, \cos t \rangle \\ &= \left\langle 1 + \sin^2 t - \cos^2 t, -2\cos t, \sin t \right\rangle \cdot \langle -\sin t, \cos t \rangle = \left\langle 2\sin^2 t, -2\cos t \sin t \right\rangle \cdot \langle -\sin t, \cos t \rangle \\ &= -2\sin^3 t - 2\cos^2 t \sin t = -2\sin t(\sin^2 t + \cos^2 t) = -2\sin t\end{aligned}$$

Hence,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -2\sin t \, dt = 0$$

(d) We denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the upper and lower halves of the unit circle. Using part (c) we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 0 \quad \Rightarrow \quad \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} \tag{1}$$



To compute the circulation along $\mathcal{C}_1$, we compute the integral as in part (c), only that the limits of integration are now $t = 0$ and $t = \pi$. Using the computations in part (c) we obtain

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi} -2\sin^2 t \, dt = -4$$

Therefore, by (1),

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 4.$$

36. Figure 6 shows the vector field $\mathbf{F} = \nabla V$, where

$$V(x, y) = \ln\left(x^2 + (y-1)^2\right) + \ln\left(x^2 + (y+1)^2\right)$$

which is the velocity field for the flow of a fluid with sources of equal strength at $(0, \pm 1)$ (note that V is undefined at these two points). Show that $\mathbf{F}$ is both irrotational and incompressible—that is, $\text{curl}_z(\mathbf{F}) = 0$ and $\text{div}(\mathbf{F}) = 0$ [in computing $\text{div}(\mathbf{F})$, treat $\mathbf{F}$ as a vector field in $\mathbf{R}^3$ with a zero z-component]. Is it necessary to compute $\text{curl}_z(\mathbf{F})$ to conclude that it is zero?

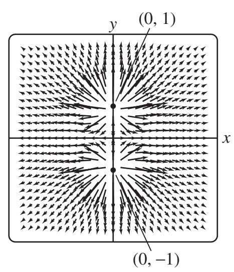


FIGURE 6 The vector field ∇V for $V(x, y) = \ln(x^2 + (y-1)^2) + \ln(x^2 + (y+1)^2)$.

SOLUTION Since $\mathbf{F}$ is a gradient field it is irrotational. This property was proved in Exercise 8, where we showed that $\text{curl}(\nabla\varphi) = \mathbf{0}$ for all φ. To show that $\mathbf{F}$ is incompressible, we first find $\mathbf{F}$ explicitly.

$$\begin{aligned}\mathbf{F}(x, y, z) &= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle = \left\langle \frac{2x}{x^2 + (y-1)^2} + \frac{2x}{x^2 + (y+1)^2}, \frac{2(y-1)}{x^2 + (y-1)^2} + \frac{2(y+1)}{x^2 + (y+1)^2}, 0 \right\rangle \\ &= \langle F_1, F_2, F_3 \rangle\end{aligned}$$

Hence,

$$\begin{aligned}\text{div}\,\mathbf{F}(x, y, z) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{2\left(x^2 + (y-1)^2\right) - 2x \cdot 2x}{\left(x^2 + (y-1)^2\right)} + \frac{2\left(x^2 + (y+1)^2\right) - 2x \cdot 2x}{\left(x^2 + (y+1)^2\right)} \\ &\quad + \frac{2\left(x^2 + (y-1)^2\right) - 2(y-1) \cdot 2(y-1)}{\left(x^2 + (y-1)^2\right)} + \frac{2\left(x^2 + (y+1)^2\right) - 2 \cdot 2(y+1)^2}{\left(x^2 + (y+1)^2\right)} \\ &= 0 + 0 = 0\end{aligned}$$

Note that, again by Exercise 8, the divergence of $\nabla\varphi$ is zero, and hence so also is the divergence of $\mathbf{F}$.

37. In Section 17.1, we showed that if $\mathcal{C}$ is a simple closed curve, oriented counterclockwise, then the line integral is

$$\text{Area enclosed by } \mathcal{C} = \frac{1}{2} \oint_{\mathcal{C}} x\,dy - y\,dx \qquad \boxed{1}$$

Suppose that $\mathcal{C}$ is a path from P to Q that is not closed but has the property that every line through the origin intersects $\mathcal{C}$ in at most one point, as in Figure 7. Let $\mathcal{R}$ be the region enclosed by $\mathcal{C}$ and the two radial segments joining P and Q to the origin. Show that the line integral in Eq. (1) is equal to the area of $\mathcal{R}$. *Hint:* Show that the line integral of $\mathbf{F} = \langle -y, x \rangle$ along the two radial segments is zero and apply Green's Theorem.

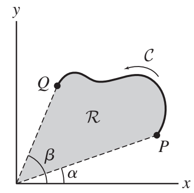
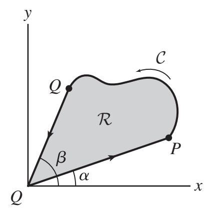


FIGURE 7

SOLUTION



Let $\mathbf{F} = \langle -y, x \rangle$. Then $P = -y$ and $Q = x$, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$. By Green's Theorem, we have

$$\int_{\mathcal{C}} -y\,dx + x\,dy + \int_{\overline{QO}} -y\,dx + x\,dy + \int_{\overline{OP}} -y\,dx + x\,dy = \iint_{\mathcal{R}} 2\,dA = 2\iint_{\mathcal{R}} dA$$

Denoting by A the area of the region $\mathcal{R}$, we obtain

$$A = \frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy + \frac{1}{2}\int_{\overline{QO}} -y\,dx + x\,dy + \frac{1}{2}\int_{\overline{OP}} -y\,dx + x\,dy \tag{1}$$

We parametrize the two segments by

$$\begin{aligned} \overline{QO} &: \mathbf{c}(t) = (t, t\tan\beta) \\ \overline{OP} &: \mathbf{d}(t) = (t, t\tan\alpha) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{c}'(t) &= \langle 1, \tan\beta \rangle \\ \mathbf{d}'(t) &= \langle 1, \tan\alpha \rangle \end{aligned}$$

Then,

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \langle -t\tan\beta, t \rangle \cdot \langle 1, \tan\beta \rangle = -t\tan\beta + t\tan\beta = 0$$
$$\mathbf{F}(\mathbf{d}(t)) \cdot \mathbf{d}'(t) = \langle -t\tan\alpha, t \rangle \cdot \langle 1, \tan\alpha \rangle = -t\tan\alpha + t\tan\alpha = 0$$

Therefore,

$$\int_{\overline{QO}} \mathbf{F} \cdot d\mathbf{s} = \int_{\overline{OP}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

Combining with (1) gives

$$A = \frac{1}{2}\int_{\mathcal{C}} -y\,dx + x\,dy.$$

38. Suppose that the curve $\mathcal{C}$ in Figure 7 has the polar equation $r = f(\theta)$.

(a) Show that $\mathbf{c}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$ is a counterclockwise parametrization of $\mathcal{C}$.

(b) In Section 11.4, we showed that the area of the region $\mathcal{R}$ is given by the formula

$$\text{Area of } \mathcal{R} = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta$$

Use the result of Exercise 37 to give a new proof of this formula. *Hint:* Evaluate the line integral in Eq. (1) using $\mathbf{c}(\theta)$.

SOLUTION

(a) The curve $r = f(\theta)$ in polar coordinates can be parametrized using θ as a parameter. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

When θ varies from α to β, the path $\mathcal{C}$ is traversed counterclockwise.

(b) In Exercise 37 we showed that

$$\text{area of } \mathcal{R} = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx \tag{1}$$

We evaluate the line integral using the parametrization in (a):

$$\mathbf{c}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta), \quad \alpha \le \theta \le \beta.$$

We have

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta \quad \Rightarrow \quad dy = \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)d\theta$$
$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad \Rightarrow \quad dx = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)d\theta$$

Hence

$$\begin{aligned}\int_{\mathcal{C}} x\,dy - y\,dx &= \int_\alpha^\beta \left(f(\theta)\cos\theta\left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right) - f(\theta)\sin\theta\left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)\right)d\theta\\ &= \int_\alpha^\beta \left(f(\theta)f'(\theta)\cos\theta\sin\theta + f^2(\theta)\cos^2\theta - f(\theta)f'(\theta)\sin\theta\cos\theta + f^2(\theta)\sin^2\theta\right)d\theta\\ &= \int_\alpha^\beta f^2(\theta)\left(\cos^2\theta + \sin^2\theta\right)d\theta = \int_\alpha^\beta f^2(\theta)\,d\theta\end{aligned}$$

Substituting in (1) we obtain

$$\text{area of } \mathcal{R} = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta$$

39. Prove the following generalization of Eq. (1). Let $\mathcal{C}$ be a simple closed curve in the plane (Figure 8)

$$\mathcal{S}: \quad ax + by + cz + d = 0$$

Then the area of the region R enclosed by $\mathcal{C}$ is equal to

$$\frac{1}{2\|\mathbf{n}\|}\oint_{\mathcal{C}} (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$$

where $\mathbf{n} = \langle a, b, c\rangle$ is the normal to $\mathcal{S}$, and $\mathcal{C}$ is oriented as the boundary of $\mathcal{R}$ (relative to the normal vector $\mathbf{n}$). *Hint:* Apply Stokes' Theorem to $\mathbf{F} = \langle bz - cy, cx - az, ay - bx\rangle$.

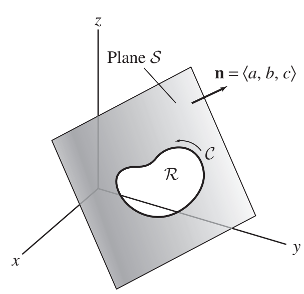


FIGURE 8

SOLUTION By Stokes' Theorem,

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathcal{S}} (\text{curl}(\mathbf{F})\cdot \mathbf{e}_n)\;dS = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} \tag{1}$$

We compute the curl of $\mathbf{F}$:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ bz - cy & cx - az & ay - bx\end{vmatrix} = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k} = 2\langle a, b, c\rangle$$

The unit normal to the plane $ax + by + cz + d = 0$ is

$$\mathbf{e}_n = \frac{\langle a, b, c\rangle}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore,

$$\begin{aligned}\operatorname{curl}(\mathbf{F}) \cdot \mathbf{e}_n &= 2\langle a, b, c\rangle \cdot \frac{1}{\sqrt{a^2+b^2+c^2}}\langle a, b, c\rangle \\ &= \frac{2}{\sqrt{a^2+b^2+c^2}}(a^2+b^2+c^2) = 2\sqrt{a^2+b^2+c^2}\end{aligned}$$

Hence,

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{e}_n \, dS = \iint_{\mathcal{S}} 2\sqrt{a^2+b^2+c^2}\, dS = 2\sqrt{a^2+b^2+c^2} \iint_{\mathcal{S}} 1 \, dS \tag{2}$$

The sign of $\iint_S 1\, d\mathbf{S}$ is determined by the orientation of $\mathcal{S}$. Since the area is a positive value, we have

$$\left| \iint_{\mathcal{S}} 1 \, ds \right| = \text{Area}\,(\mathcal{S})$$

Therefore, (2) gives

$$\left| \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \right| = 2\sqrt{a^2+b^2+c^2}\, \text{Area}(S)$$

Combining with (1) we obtain

$$2\sqrt{a^2+b^2+c^2}\, \text{Area}(\mathcal{S}) = \left| \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \right|$$

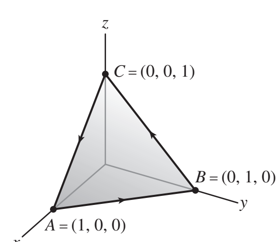
or

$$\text{Area}(\mathcal{S}) = \frac{1}{2\sqrt{a^2+b^2+c^2}} = \frac{1}{2\|\mathbf{n}\|} \cdot \left| \int_{\mathcal{C}} (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz \right|$$

40. Use the result of Exercise 39 to calculate the area of the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1) as a line integral. Verify your result using geometry.

SOLUTION In Exercise 39 we showed that if $\mathcal{C}$ is a simple closed curve in the plane $ax + by + cz + d = 0$, then the area of the region $\mathcal{R}$ enclosed by $\mathcal{C}$ is equal to

$$\frac{1}{2\|\mathbf{n}\|} \int_{\mathcal{C}} (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz, \ \mathbf{n} = \langle a, b, c\rangle \tag{1}$$



We use this formula where $\mathcal{C}$ is the triangle ABC parametrized counterclockwise. We compute the upward-pointing normal to the plane of the triangle:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i}) = \mathbf{i} + \mathbf{k} + \mathbf{j} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We substitute $\|\mathbf{n}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ and $a = b = c = 1$ in (1) to obtain:

$$\text{area of } \mathcal{R} = \frac{1}{2\sqrt{3}} \int_{\mathcal{C}} (z - y)\, dx + (x - z)\, dy + (y - x)\, dz \tag{2}$$

We parametrized the oriented segments by

$$\begin{aligned}
\overline{AB} &: \mathbf{c}_1(t) = (1-t, t, 0), \quad 0 \le t \le 1 \quad \Rightarrow \quad dx = -dt, \quad dy = dt, \quad dz = 0 \\
\overline{BC} &: \mathbf{c}_2(t) = (0, 1-t, t), \quad 0 \le t \le 1 \quad \Rightarrow \quad dx = 0, \quad dy = -dt, \quad dz = dt \\
\overline{CA} &: \mathbf{c}_3(t) = (t, 0, 1-t), \quad 0 \le t \le 1 \quad \Rightarrow \quad dx = dt, \quad dy = 0, \quad dz = -dt
\end{aligned}$$

We compute the line integral along each segment separately:

$$\int_{\overline{AB}} (z-y)\,dx + (x-z)\,dy + (y-x)\,dz = \int_0^1 (0-t)(-dt) + (1-t-0)\,dt = \int_0^1 1\,dt = 1$$
$$\int_{\overline{BC}} (z-y)\,dx + (x-z)\,dy + (y-x)\,dz = \int_0^1 (0-t)(-dt) + (1-t-0)\,dt = \int_0^1 1\,dt = 1$$
$$\int_{\overline{CA}} (z-y)\,dx + (x-z)\,dy + (y-x)\,dz = \int_0^1 (1-t-0)\,dt + (0-t)(-dt) = \int_0^1 1\,dt = 1$$

The integral along $\mathcal{C}$ is the sum of these three integrals. That is,

$$\int_{\mathcal{C}} (z-c)\,dx + (x-z)\,dy + (y-x)\,dz = 1+1+1 = 3$$

We combine with (2) to obtain the following area of the triangle:

$$\text{area of } \mathcal{R} = \frac{1}{2\sqrt{3}} \cdot 3 = \frac{\sqrt{3}}{2}$$

We verify this solution geometrically. The triangle spanned by the vectors $\overrightarrow{AB} = \mathbf{j} - \mathbf{i}$ and $\overrightarrow{AC} = \mathbf{k} - \mathbf{i}$ has area

$$\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\|\mathbf{i}+\mathbf{j}+\mathbf{k}\| = \frac{1}{2}\sqrt{3}.$$

The two answers match.

41. Show that $G(\theta,\phi) = (a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi)$ is a parametrization of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Then calculate the volume of the ellipsoid as the surface integral of $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$ (this surface integral is equal to the volume by the Divergence Theorem).

SOLUTION For the given parametrization,

$$x = a\cos\theta\sin\phi, \quad y = b\sin\theta\sin\phi, \quad z = c\cos\phi \tag{1}$$

We show that it satisfies the equation of the ellipsoid

$$\begin{aligned}\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 &= \left(\frac{a\cos\theta\sin\phi}{a}\right)^2 + \left(\frac{b\sin\theta\sin\phi}{b}\right)^2 + \left(\frac{c\cos\phi}{c}\right)^2 \\ &= \cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi \\ &= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi \\ &= \sin^2\phi + \cos^2\phi = 1\end{aligned}$$

Conversely, for each (x, y, z) on the ellipsoid, there exists θ and ϕ so that (1) holds. Therefore $\Phi(\theta,\phi)$ parametrizes the whole ellipsoid. Let $\mathcal{W}$ be the interior of the ellipsoid $\mathcal{S}$. Then by Eq. (10):

$$\text{Volume}(\mathcal{W}) = \frac{1}{3}\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}, \quad \mathbf{F} = \langle x, y, z\rangle$$

We compute the surface integral, using the given parametrization. We first compute the normal vector:

$$\begin{aligned}\frac{\partial\Phi}{\partial\theta} &= \langle -a\sin\theta\sin\phi, b\cos\theta\sin\phi, 0\rangle \\ \frac{\partial\Phi}{\partial\phi} &= \langle a\cos\theta\cos\phi, b\sin\theta\cos\phi, -c\sin\phi\rangle \\ \frac{\partial\Phi}{\partial\theta}\times\frac{\partial\Phi}{\partial\phi} &= -ab\sin^2\theta\sin\phi\cos\phi\mathbf{k} - ac\sin\theta\sin^2\phi\mathbf{j} - ab\cos^2\theta\sin\phi\cos\phi\mathbf{k} - bc\cos\theta\sin^2\phi\mathbf{i} \\ &= \left\langle -bc\cos\theta\sin^2\phi, -ac\sin\theta\sin^2\phi, -ab\sin\phi\cos\phi\right\rangle\end{aligned}$$

Hence, the outward pointing normal is

$$\mathbf{n} = \left\langle bc\cos\theta\sin^2\phi, ac\sin\theta\sin^2\phi, ab\sin\phi\cos\phi\right\rangle$$

$$\begin{aligned}
\mathbf{F}(\Phi(\theta,\phi))\cdot\mathbf{n} &= \langle a\cos\theta\sin\phi,\, b\sin\theta\sin\phi,\, c\cos\phi\rangle\cdot\left\langle bc\cos\theta\sin^2\phi,\, ac\sin\theta\sin^2\phi,\, ab\sin\phi\cos\phi\right\rangle\\
&= abc\cos^2\theta\sin^3\phi + abc\sin^2\theta\sin^3\phi + abc\sin\phi\cos^2\phi\\
&= abc\sin^3\phi(\cos^2\theta+\sin^2\theta) + abc\sin\phi\cos^2\phi\\
&= abc\sin^3\phi + abc\sin\phi\cos^2\phi = abc\sin^3\phi + abc\sin\phi(1-\sin^2\phi)\\
&= abc\sin\phi
\end{aligned}$$

We obtain the following integral:

$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \frac{1}{3}\iint_{\mathcal{S}}\mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} abc\sin\phi\,d\phi\,d\theta\\
&= \frac{2\pi abc}{3}\int_0^{\pi}\sin\phi\,d\varphi = \frac{2\pi abc}{3}\left(-\cos\phi\Big|_0^{\pi}\right) = \frac{4\pi abc}{3}
\end{aligned}$$