20W-MATH32B-1 Midterm 2

JEFFREY MA

TOTAL POINTS

40 / 40

QUESTION 1

1 Question 1 10 / 10

✓ **- 0 pts** Correct

QUESTION 2

2 Question 2 10 / 10

✓ **- 0 pts** d: Correct

✓ **- 0 pts** c: Correct

✓ **- 0 pts** b: Correct

✓ **- 0 pts** a: Correct

QUESTION 3

Question 3 12 pts

3.1 Question 3(a) 5 / 5

✓ **- 0 pts** Correct

3.2 Question 3(b) 5 / 5

✓ **- 0 pts** Correct

3.3 Question 3(c) 2 / 2

✓ **- 0 pts** Correct

QUESTION 4

4 Question 4(a) 2 / 2

✓ **+ 1 pts** T_u correct

✓ **+ 1 pts** T_v correct

+ 0 pts No points

QUESTION 5

5 Question 4(b) 3 / 3

✓ **+ 1 pts** Correct normal vector

✓ **+ 1 pts** Correct length of normal vector given computed normal vector

✓ **+ 1 pts** Correct integral given length

+ 0 pts No points

QUESTION 6

6 Question 4(c) 3 / 3

✓ **+ 1 pts** Correct normal vector

✓ **+ 1 pts** Uses point (2, 1, 1) correctly for plane eqn

✓ **+ 1 pts** Correct equation of plane

+ 0 pts No points

Midterm 2

Name: Jeffrey Ma

UID: 305309403

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Benjamin Johnsrude
	1C	1D	TA: Alexander Kastner
	1E	1F	TA: Max Zhou

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. You may not use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You must **show all your work** to receive credit. Simplify your answers as much as possible.

You may use the front and back of the page for your answers. Do not write answers for one question on the page of another question.

Please do not write below this line on this page.

Question	Points	Score
1	10	
2	10	
3	12	
4	8	
Total:	40	

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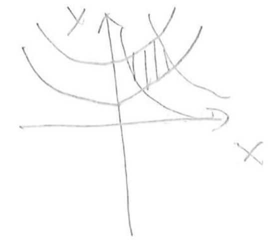
1. (10 points) Calculate the integral $\displaystyle\iint_{\mathcal{D}} \frac{x^2+y^2}{x^2-y^2}\,dx\,dy$ where $\mathcal{D}$ is the region in the first quadrant enclosed by the curves $x^2-y^2=16$, $x^2-y^2=9$, $xy=4$, and $xy=1$.

$D: \{(x,y) \mid 9 \le x^2-y^2 \le 16 \quad 1 \le xy \le 4\}$



$G^{-1} = F(x,y) = (x^2-y^2,\ xy) = (u(x,y), v(x,y))$

$\mathrm{Jac}(F) = \frac{\partial(u,v)}{\partial(x,y)}$

$\mathrm{Jac}(F) = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2 = 2(x^2+y^2)$

$\mathrm{Jac}(G) = \frac{1}{\mathrm{Jac}(F)} = \frac{1}{2(x^2+y^2)}$

$D_0: \{(u,v) \mid 9 \le u \le 16 \quad 1 \le v \le 4\}$

$\iint_D \frac{x^2+y^2}{x^2-y^2}\,dx\,dy = \iint_{D_0} \frac{x^2+y^2}{u}\,|\mathrm{Jac}(G)|\,du\,dv$

$= \int_1^4\int_9^{16} \frac{\cancel{x^2+y^2}}{u}\,\frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv = \frac{1}{2}\int_1^4 \ln u\Big|_9^{16}\,dv$

$= \frac{1}{2}\int_1^4 \ln\frac{16}{9}\,dv = \frac{1}{2}(4-1)\ln\frac{16}{9}$

$= \ln\left(\frac{16}{9}\right)^{3/2} = \boxed{\ln\left(\frac{64}{27}\right)}$

$\frac{(4^2)^{3/2}}{(3^2)^{3/2}} = \frac{4^3}{3^3}$

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2. Let $\mathbf{F}(x, y, z) = \langle \cos z, 2y, -x \sin z \rangle$, and let C be the curve parametrized by $\mathbf{r}(t) = \langle t^2 + 1, te^{t^2 - t}, \pi t^5 \rangle$ for $0 \le t \le 1$.

   (a) (4 points) Calculate $\operatorname{div}(\mathbf{F})$ and $\operatorname{curl}(\mathbf{F})$.

   (b) (2 points) Show that $\mathbf{F}$ is conservative without calculating a potential function.

   (c) (2 points) Find a function f such that $\mathbf{F} = \nabla f$.

   (d) (2 points) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

a) $\operatorname{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \cos z \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + (-x \sin z) \frac{\partial}{\partial z}$

$= 0 + 2 - x\cos z = 2 - x\cos z$

$\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos z & 2y & -x\sin z \end{vmatrix} = \hat{i}\left(-x\sin z \frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}\right) - \hat{j}\left(\frac{\partial}{\partial x}(-x\sin z) - \frac{\partial}{\partial z}\cos z\right) + \hat{k}\left(\frac{\partial}{\partial x} 2y - \frac{\partial}{\partial y}\cos z\right)$

$= \langle 0, -\sin z - (-\sin z), 0 \rangle = \vec{0}$

b) $\vec{F}(x,y,z)$ is defined / continuous on $\mathbb{R}^3$ (w/ cont. partial derivatives)

$\mathbb{R}^3$ is a simply connected domain [sin/cos/polynomials always cont. w/ cont. partial derivatives]

$\therefore$ b/c the $\operatorname{curl}(\vec{F})$ calculated earlier is $\vec{0}$, F is a conservative function

Given that the domain is simply connected, the converse is true: $\operatorname{curl}(\vec{F}) = \vec{0}$ indicates that $\vec{F}$ is conservative

c) $f(x,y,z) = \int F_1\, dx = x\cos z + g(y,z)$

$= \int F_2\, dy = y^2 + j(x,z)$

$= \int F_3\, dz = x\cos z + h(x,y)$

One potential func. of $\vec{F}$ is $f(x,y,z) = x\cos z + y^2$

$\nabla f(x,y,z) = \langle \cos z, 2y, -x\sin z \rangle$ ✓

d) $\int_C F \cdot dr = f(Q) - f(P)$ where Q – endpoint, P – start point

$= f(\vec{r}(1)) - f(\vec{r}(0))$

$\vec{r}(1) = \langle 2, 1, \pi \rangle$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$= 2\cos\pi + (1)^2 - (\underbrace{1\cos(0)}_{1} + 0^2) = \boxed{-2}$

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3. (a) (5 points) Let $\mathcal{C}_0$ be the curve given by $y = \dfrac{x^2}{2}$ for $0 \le x \le 1$ in the plane $z = 5$. Calculate $\displaystyle\int_{\mathcal{C}_0} xz\, ds$.

(b) (5 points) Suppose $\mathcal{C}$ is the curve with positive parametrization $\mathbf{r}(t) = \langle \cos t, 1, \sin t\rangle$ for $0 \le t \le 2\pi$. Calculate the work done by the force field $\mathbf{F} = \langle -z, xy, x\rangle$ in moving a particle along $\mathcal{C}$.

(c) (2 points) Suppose $\mathbf{G}$ is a conservative vector field and $\mathcal{C}$ is the curve from part (b). Do we have enough information to calculate $\displaystyle\int_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{r}$? If yes, justify and evaluate the integral. If not, explain what information is lacking.

a) C_0: $y=\frac{x^2}{2}$ $0\le x\le 1$ in plane $z=5$

parametrized curve $\vec{r}(t) = \langle t, \frac{t^2}{2}, 5\rangle$ $0 \le t \le 1$ $\quad \vec{r}'(t) = \langle 1, t, 0\rangle$

$\|\vec{r}'(t)\| = \sqrt{t^2+1}$

$f(x,y,z) = xz \qquad ds = \sqrt{t^2+1}\,dt$

$\displaystyle\int_{C_0} xz\,ds = \int_0^1 f(\vec{r}(t))\|\vec{r}'(t)\|\,dt = \int_0^1 (5t)\sqrt{t^2+1}\,dt$; $u = t^2+1$, $\frac{du}{2} = t\,dt$

$\displaystyle = \frac{5}{2}\int_1^2 \sqrt{u}\,du = \frac{5}{\cancel{2}}\left[\frac{\cancel{2}}{3}u^{3/2}\right]_1^2 = \frac{5}{3}(2^{3/2}-1)$

b) Work $= \displaystyle\int_C \vec{F}\cdot d\vec{r}$ $\quad \vec{r}(t) = \langle \cos t, 1, \sin t\rangle$ $\quad \vec{r}'(t) = \langle -\sin t, 0, \cos t\rangle$

$0 \le t \le 2\pi$

$\displaystyle = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^{2\pi} \langle -\sin t, \cos t, \cos t\rangle\cdot\langle -\sin t, 0, \cos t\rangle\,dt$

$\displaystyle = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$

c) $\vec{r}(0) = \langle 0,0,1\rangle$ $\quad \vec{r}(2\pi) = \langle 0,0,1\rangle$

We do have enough information — yes

The curve parametrized in part b is a closed curve — it starts and ends on the same point.

$\therefore$, given that G is conservative, the circulation is 0

$\displaystyle\int_C \vec{G}\cdot d\vec{r} = \oint_C \vec{G}\cdot d\vec{r} = 0$

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4. Suppose a surface $\mathcal{S}$ is parametrized by $G(u,v) = (u+v^2, u^2, uv)$ for $-3 \leq u \leq 5$, $0 \leq v \leq 4$.

   (a) (2 points) Calculate $\mathbf{T_u}$ and $\mathbf{T_v}$.

   (b) (3 points) Set up the iterated integral in the u- and v-variables to calculate the surface area of $\mathcal{S}$. You do *not* need to evaluate the integral.

   (c) (3 points) Find the tangent plane to $\mathcal{S}$ at $(2,1,1)$.

a) $\vec{T}_u = \langle g_1 \frac{\partial u}{\partial x}, g_2 \frac{\partial u}{\partial y}, g_3 \frac{\partial u}{\partial z} \rangle = \langle 1, 2u, v \rangle$

$\vec{T}_v = \langle 2v, 0, u \rangle$

b) $SA = \iint_D \|\vec{N}(u,v)\| \, du dv$

$\vec{N}(u,v) = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & v \\ 2v & 0 & u \end{vmatrix} = \hat{i}(2u^2) - \hat{j}(u - 2v^2) + \hat{k}(-4uv)$

$= \langle 2u^2, 2v^2 - u, -4uv \rangle$

$D: \{(u,v) \mid -3 \leq u \leq 5, 0 \leq v \leq 4\}$

$\therefore SA = \iint_D \|\vec{N}(u,v)\| \, du dv = \int_0^4 \int_{-3}^5 \sqrt{4u^4 + (2v^2-u)^2 + 16u^2v^2} \, du dv$

c) Tangent plane: $\vec{N} \cdot \vec{OP} = 0$

$ax + by + cz = ax_0 + by_0 + cz_0$

$\vec{N}(u,v) = \langle 2u^2, 2v^2 - u, -4uv \rangle \quad \vec{N}(1,1) = \langle 2, 1, -4 \rangle$

$(2,1,1) = G(1,1) \quad u=1 \; v=1$

$2x + y - 4z = 4 + 1 + (-4) = 1$

$2x + y - 4z = 1$