Midterm 1

Name: ______________________________

UID: ______________________________

Section:

Tuesday:	Thursday:	
1A	1B	TA: Benjamin Johnsrude
1C	1D	TA: Alexander Kastner
1E	1F	TA: Max Zhou

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. You may not use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You must **show all your work** to receive credit. Simplify your answers as much as possible.

You may use the front and back of the page for your answers. Do not write answers for one question on the page of another question.

Please do not write below this line on this page.

Question	Points	Score
1	8	
2	8	
3	12	
4	12	
Total:	40	

1. (8 points) This question is multiple choice. Indicate your answers clearly in the following table. Only this table will be graded for this question.

Part	A	B	C	D
(a)	✓			
(b)		✓		
(c)		✓		
(d)		✓		
(e)	✓			
(f)				✓
(g)	✓			
(h)		✓		

(a) If $\mathcal{R} = [-2, 0] \times [-1, 3]$, then $\iint_{\mathcal{R}} 3\, dA$ is equal to

A. 24

B. 8

C. 18

D. 12

(b) If $\mathcal{R} = [0, 1] \times [-1, 1]$, then $\iint_{\mathcal{R}} y \sin(xy^2)\, dA$ is equal to

A. -1

B. 0

C. π

D. π^2

(c) Let $\mathcal{R} = [0, 2] \times [0, 2]$. The Riemann sum $S_{2,2}$ for estimating $\iint_{\mathcal{R}} (x + y)\, dA$ where we divide each interval in half and use upper left corners as sample points is equal to

A. 12

B. 8

C. 6

D. 4

Question 1 continues on the next page. . .

(d) If $\mathcal{D}$ is the region bounded by the curves $y = 2x^2$ and $y = 1 - x^2$, then $\mathcal{D}$ has the description

A. $-\sqrt{3} \le x \le \sqrt{3}$, and $2x^2 \le y \le 1 - x^2$
B. $-1/\sqrt{3} \le x \le 1/\sqrt{3}$, and $2x^2 \le y \le 1 - x^2$
C. $-\sqrt{3} \le x \le \sqrt{3}$, and $1 - x^2 \le y \le 2x^2$
D. $-1/\sqrt{3} \le x \le 1/\sqrt{3}$, and $1 - x^2 \le y \le 2x^2$

(e) If $\mathcal{D}$ is the disc $x^2 + y^2 \le 9$, then after changing to polar coordinates, the integral $\iint_{\mathcal{D}} xy\, dA$ becomes

A. $\int_0^{2\pi} \int_0^3 r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta$
B. $\int_0^{2\pi} \int_0^3 r^2 \sin(\theta)\cos(\theta)\, dr\, d\theta$
C. $\int_0^{\pi} \int_0^3 r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta$
D. $\int_0^{\pi} \int_0^3 r^2 \sin(\theta)\cos(\theta)\, dr\, d\theta$

(f) The triangle with vertices $(0,0)$, $(1,1)$, $(1,\sqrt{3})$ is described in polar coordinates by

A. $\pi/6 \le \theta \le \pi/4$, and $0 \le r \le 1/\sin(\theta)$
B. $\pi/4 \le \theta \le \pi/3$, and $0 \le r \le 1/\sin(\theta)$
C. $\pi/6 \le \theta \le \pi/4$, and $0 \le r \le 1/\cos(\theta)$
D. $\pi/4 \le \theta \le \pi/3$, and $0 \le r \le 1/\cos(\theta)$

(g) The domain $\{(x,y) \mid y \ge x, (x-1)^2 + y^2 \le 1\}$ in $\mathbb{R}^2$ is described in polar coordinates by

A. $\pi/4 \le \theta \le \pi/2$, and $0 \le r \le 2\cos(\theta)$
B. $0 \le \theta \le \pi/4$, and $0 \le r \le 2\cos(\theta)$
C. $\pi/4 \le \theta \le \pi/2$, and $0 \le r \le 2\sin(\theta)$
D. $0 \le \theta \le \pi/4$, and $0 \le r \le 2\sin(\theta)$

(h) Every integral of the form $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$ can also be represented in the form $\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$.

A. True
B. False

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2. (8 points) Find a real number C such that

$$\int_0^2 \int_{y/2}^1 C\cos(Cx^2)\,dx\,dy = 1.$$

Solution:

$$\begin{aligned}\int_0^2 \int_{y/2}^1 C\cos(Cx^2)\,dx\,dy &= \int_0^1 \int_0^{2x} C\cos(Cx^2)\,dy\,dx \\ &= \int_0^1 2xC\cos(Cx^2)\,dx \\ &= \big(\sin(Cx^2)\big)_0^1 \\ &= \sin(C).\end{aligned}$$

Hence for instance $C = \frac{\pi}{2}$ will do.

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3. Let $\mathcal{W}$ be the region in $\mathbb{R}^3$ bounded by the planes $y = 0$, $y = 1$, $z = 0$, $z = 1$ and between the surfaces given by $x = 5 - y^2$ and $x = 2y^2$.
   (a) (4 points) Find a domain $\mathcal{D}$ in the yz-plane and functions $f_1(y, z)$ and $f_2(y, z)$ such that $\mathcal{W} = \{(x, y, z) \mid (y, z) \in \mathcal{D},\ f_1(y, z) \leq x \leq f_2(y, z)\}$.
   (b) (4 points) Calculate the volume of $\mathcal{W}$.
   (c) (4 points) Is the region $\mathcal{W}$ z-simple? If not, explain why not. If yes, describe a domain $\mathcal{E}$ in the xy-plane and functions $g_1(x, y)$, $g_2(x, y)$ such that $\mathcal{W} = \{(x, y, z) \mid (x, y) \in \mathcal{E},\ g_1(x, y) \leq z \leq g_2(x, y)\}$.

Solution:

(a)
- $\mathcal{D} = [0, 1] \times [0, 1] = \{(y, z) \mid 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$,
- $f_1(y, z) = 2y^2$,
- $f_2(y, z) = 5 - y^2$.

(b)

$$\begin{aligned}
\mathrm{Vol}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1\, dV \\
&= \int_0^1 \int_0^1 \int_{2y^2}^{5-y^2} 1\, dx\, dy\, dz \\
&= \int_0^1 \int_0^1 (5 - 3y^2)\, dy\, dz \\
&= \int_0^1 4\, dz \\
&= 4.
\end{aligned}$$

(c) Yes. One can take

- $\mathcal{E} = \{(x, y) \mid 0 \leq y \leq 1,\ 2y^2 \leq x \leq 5 - y^2\}$,
- $g_1(x, y) = 0$, and
- $g_2(x, y) = 1$.

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4. A spiky mountain is described by the region bounded by the xy-plane and the surface $z = 5 - \sqrt{x^2 + y^2}$.
   (a) (3 points) Find the projection of the mountain onto the xy-plane.
   (b) (5 points) Use a suitable integral to find the average height of the surface of the mountain.
   (c) (2 points) Without calculation, can we know if there is a point on the surface of the mountain where the average height is attained? Justify your answer.
   (d) (2 points) Suppose another mountain is described by a surface above a disk of radius 3 in the xy-plane, which has average height 10. What is the volume of this other mountain? Justify your answer.

Solution:

(a) $\mathcal{D} = \{(x, y) \mid x^2 + y^2 \leq 25\}$.

(b) The average height $\bar{h}$ is given by the integral of the function giving the surface, divided by the area of the domain. It turns out polar coordinates help us out:

$$\begin{aligned}\bar{h} &= \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} (5 - \sqrt{x^2 + y^2})\, dA \\ &= \frac{1}{25\pi} \int_0^{2\pi} \int_0^5 (5 - r) r\, dr\, d\theta \\ &= \frac{1}{25\pi} \int_0^{2\pi} \frac{125}{6}\, d\theta \\ &= \frac{5}{3}\end{aligned}$$

(c) Yes, because $f(x, y) = 5 - \sqrt{x^2 + y^2}$ is continuous on $\mathcal{D}$ and $\mathcal{D}$ is closed, bounded and connected. Thus, this follows from the mean value theorem.

(d) Say this other mountain lies under the surface $f(x, y)$ above domain $\mathcal{E}$, then

$$\text{Vol}(\textit{Other Mountain}) = \iint_{\mathcal{E}} f(x, y)\, dA$$

which by definition equals

$$\bar{f} \cdot \text{Area}(\mathcal{E}) = 90\pi.$$