Final exam

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Instructions:

- This is a take-home exam, open book and open notes. You have 24 hours to complete it. Your solutions must be uploaded to gradescope by Tuesday, March 17, at 5pm PDT. The document you upload must match the answer template which has been provided to you. See also the announcement on CCLE for the instructions.
- If you have any questions while working on this exam, you should direct them by email to me: `pspaas@math.ucla.edu`. In particular, the TAs will just forward any emails they get to me, hence there is no need to email them.
- While working on this exam, you may *not* use any additional materials other than the textbook and your own notes. Academic integrity violations will be taken very seriously. As a reminder: your answer sheet must contain the verbatim signed academic integrity statement that has been provided to you. Also, we reserve the right to contact you after the exam to ask for additional explanation of solutions for problems if we deem it necessary.
- As usual, you must *show all your work* to receive credit. Correct answers without justification will not be awarded any points.

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
5	15	
6	10	
7	15	
8	10	
Total:	100	

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Some useful formulas.

Spherical coordinates are given by:

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho\sin\phi\cos\theta, \\ y(\rho,\theta,\phi) &= \rho\sin\phi\sin\theta, \\ z(\rho,\theta,\phi) &= \rho\cos\phi. \end{aligned}$$

The Jacobian for the change of variables to spherical coordinates is $\rho^2\sin\phi$.

Some trigonometric usefulness:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2},$$
$$2\sin\theta\cos\theta = \sin(2\theta).$$

1. (10 points) This question is multiple choice. Indicate your answers clearly in the table on your answer sheet. No explanations required.

   (a) The Jacobian of the change of variables $G(u, v, w) = (u^2, v - u, ve^w)$ is

   A. $2uve^w$

   B. $2u + 1 + ve^w$

   C. $\langle e^w, 0, -1\rangle$

   D. $\sqrt{e^{2w} + 1}$

   (b) Let $f(x, y)$ be a function on $\mathbb{R}^2$ and let $\mathbf{F} = \nabla f$. Suppose $\int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r} = 1$ where $\mathcal{C}'$ is the line segment from $(-2, 0)$ to $(2, 0)$. If $\mathcal{C}$ is the bottom half of the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$ oriented clockwise, then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is

   A. -1

   B. 0

   C. 1

   D. Not enough information.

   (c) Let $\mathcal{R} = [0, 2] \times [-1, 0]$, and let $S_{5,5}$ be a Riemann sum for estimating $\iint_{\mathcal{R}} 5\, dA$ by partitioning each interval in 5 parts and choosing sample points. Then $S_{5,5}$

   A. depends only on the sample points.

   B. depends only on the way we partition each interval.

   C. depends both on the sample points and the way we partition each interval.

   D. is exactly equal to $\iint_{\mathcal{R}} 5\, dA$.

   (d) Suppose we have the domain $\mathcal{D} = \{(r, \theta) \mid -\pi/4 \leq \theta \leq \pi/4,\ 1/\cos\theta \leq r \leq 2\cos\theta\}$ in polar coordinates. In the xy-plane, $\mathcal{D}$ is

   A. a disk of radius 1.

   B. half a disk of radius 1.

   C. a triangle.

   D. a "pizza slice".

   (e) Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ such that $\text{div}(\mathbf{F}) = 0$. Then the outward flux of $\mathbf{F}$ through the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

   A. 0

   B. πabc

   C. $-\pi abc$

   D. $\sqrt{a^2 + b^2 + c^2}$

Question 1 continues on the next page...

(f) Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ such that $\mathbf{F}(x, y, z) \cdot \langle x, y, 0\rangle > 0$ for all z and all $(x, y) \neq (0, 0)$. Let $\mathcal{S}$ be the cone $z^2 = x^2 + y^2$ for $0 \leq z \leq 5$ with downward normal (i.e. having negative z-coordinate). Then the flux of $\mathbf{F}$ through $\mathcal{S}$

A. is always strictly positive.
B. is always strictly negative.
C. could be positive or zero.
D. could be either positive, negative or zero.

(g) In $\mathbb{R}^2$, let $\mathcal{C}$ be the unit circle oriented clockwise and $\mathcal{D}$ be the unit disk, both centered at the origin. Then $\oint_{\mathcal{C}} \langle xy, x + y\rangle \cdot d\mathbf{r}$ is equal to

A. $\iint_{\mathcal{D}} (1 - x)\, dA$
B. $\iint_{\mathcal{D}} (x - 1)\, dA$
C. $\iint_{\mathcal{D}} (1 - y)\, dA$
D. $\iint_{\mathcal{D}} (y - 1)\, dA$

(h) Suppose a surface $\mathcal{S}$ with surface area 5 has a parametrization $G(u, v)$ with $\mathbf{T}_u = \langle 1, -2, 0\rangle$ and $\mathbf{T}_v = \langle 0, 2, -1\rangle$. Then $\iint_{\mathcal{S}} 2\, dS$ is

A. $5\langle 4, 2, 4\rangle$
B. 10
C. 30
D. 50

(i) Every domain in $\mathbb{R}^2$ which is both vertically simple and radially simple, is also horizontally simple.

A. True
B. False

(j) Suppose $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ except the origin, and whose partial derivatives are continuous where $\mathbf{F}$ is defined. Let $\mathcal{S}_1$ be the upper hemisphere $x^2+y^2+z^2 = 9$ for $z \geq 0$, and let $\mathcal{S}_2$ be the part of the surface $z = 9-x^2-y^2$ where $z \geq 0$. Orient both surfaces with upward normal (i.e. with positive z-coordinate). Then we must have $\iint_{\mathcal{S}_1} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{S}_2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

A. True
B. False

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2. A polar bear is happily living in the first quadrant of the xy-plane, his habitat being a piece of ice enclosed by the curve described by $(x^2+y^2)^2-2xy=0$, see figure 1 below.

   (a) (2 points) Check that this curve in polar coordinates is given by $r^2 = 2\cos\theta\sin\theta$ and find the bounds for θ.

   (b) (5 points) Find the area of the polar bear's habitat.

   (c) (5 points) Chasing a butterfly, our enthusiastic polar bear runs along the part of the circle $2x^2+2y^2 = 1$ *contained in his habitat* in counterclockwise direction. Meanwhile, the wind is blowing according to the vector field $\mathbf{F} = \langle y, -x\rangle$. Compute the work the polar bear has to perform against the wind.

   (d) (3 points) Suppose the polar bear's habitat is surrounded by water. Given that the mass density of water is $\delta_{water} = 10$, and the mass density of ice is $\delta_{ice} = 9.2$, find the total mass of water and ice present in the square $-1 \leq x, y \leq 1$. (Recall: the total mass of a domain is given by the integral over the domain of the mass density.)

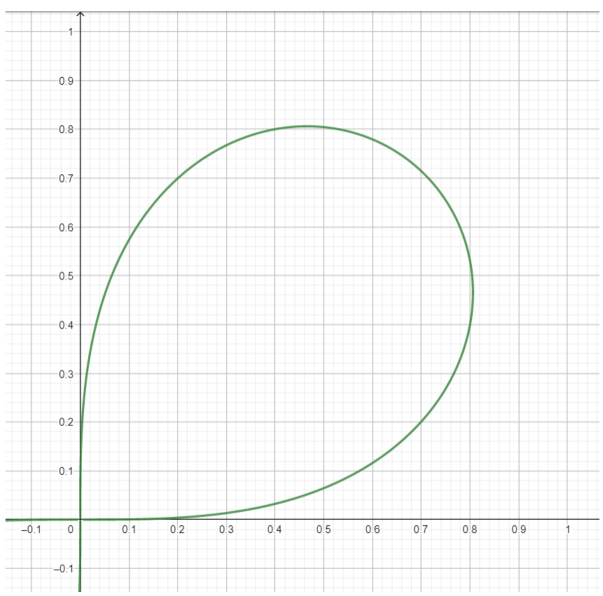

Figure 1

3. Let $\mathcal{T}$ be the triangle in the uv-plane given by $\{(u, v) \mid 0 \le u \le 2,\ u \le v \le 2\}$ and consider the change of variables $G(u, v) = (x(u, v), y(u, v))$, where $x(u, v) = v + 2u^3$, $y(u, v) = u$. Let $\mathcal{D} = G(\mathcal{T})$ be the image of $\mathcal{T}$.

   (a) (3 points) Find real numbers a, b and functions $f(y)$, $g(y)$ such that $\mathcal{D} = \{(x, y) \mid a \le y \le b,\ f(y) \le x \le g(y)\}$.

   (b) (7 points) Calculate $\displaystyle\iint_{\mathcal{D}} e^{(x-2y^3)^2}\, dx\, dy$ using the change of variables $G(u, v)$.

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4. Let $\mathcal{S}$ be the surface parametrized by $G(u,v) = (u\cos v, u\sin v, u^2)$ with domain $\mathcal{D} = \{(u,v) \mid 0 \le u \le 2,\ 0 \le v \le \pi\}$.

   (a) (6 points) Find the surface area of $\mathcal{S}$ by calculating a suitable surface integral.

   (b) (3 points) Find the tangent plane to $\mathcal{S}$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$.

   (c) (6 points) Compute $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = \langle -xy - y, y^2, yz - y^2e^y \rangle$, and $\mathcal{S}$ is given upward orientation (i.e. the normal vector has positive z-coordinate).

   Hint: What does $\mathcal{S}$ look like? Use a theorem to rewrite the integral.

5. Consider the vector field $\mathbf{F} = \langle \frac{-z}{x^2+z^2}, 0, \frac{x}{x^2+z^2} \rangle$ defined on all of $\mathbb{R}^3$ except for the y-axis.

   (a) (4 points) Compute $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the circle of radius 1 centered at the origin contained in the xz-plane, oriented clockwise when seen from the positive y-axis. (Note: this means the circle is oriented counterclockwise in the xz-plane like we would usually draw it.)

   (b) (2 points) Calculate curl($\mathbf{F}$).

   (c) (4 points) Explain why $\mathbf{F}$ is not conservative. Can you find a (non-empty) domain on which it is conservative? Explain.

   (d) (5 points) Calculate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the path in the xz-plane consisting of the line segment from $(-4, 0, 0)$ to $(0, 0, -4)$, the line segment from $(0, 0, -4)$ to $(4, 0, 0)$, and the semicircle in the positive xz-plane from $(4, 0, 0)$ to $(-4, 0, 0)$ in that order.

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6. Suppose $\mathcal{S}$ is the surface given by the part of the graph of $z = e^{-x^2-y^2}$ which is contained inside the cylinder $x^2 + y^2 \leq 1$ with upward orientation (i.e. the normal vector has positive z-coordinate). Let $\mathbf{F} = \langle y, -x, x+z \rangle$.

   (a) (2 points) Either find a vector potential for $\mathbf{F}$ (i.e. a vector field $\mathbf{G}$ such that $\operatorname{curl}(\mathbf{G}) = \mathbf{F}$) or explain why one doesn't exist.

   (b) (3 points) Find a parametrization of $\mathcal{S}$.

   (c) (5 points) Find the flux of $\mathbf{F}$ through $\mathcal{S}$, i.e. compute $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

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7. Let $\mathcal{W}$ be the solid region in $\mathbb{R}^3$ that lies within the sphere $x^2+y^2+z^2=4$ and above the cone $z=\sqrt{3x^2+3y^2}$. Let $\mathcal{S}$ be the boundary surface of $\mathcal{W}$.
   (a) (6 points) Describe $\mathcal{W}$ in spherical coordinates and find the volume of $\mathcal{W}$.
   (b) (3 points) Calculate the outward flux of $\mathbf{F}=\langle x, y+x^2, 4z-5y\rangle$ through $\mathcal{S}$.
   (c) (6 points) Suppose $\mathcal{S}'$ is the part of $\mathcal{S}$ belonging to the sphere $x^2+y^2+z^2=4$ with upward orientation (i.e. the normal has positive z-coordinate). Compute $\iint_{\mathcal{S}'} \operatorname{curl}(\mathbf{G}) \cdot d\mathbf{S}$, where $\mathbf{G}=\langle 2y+x^3, e^{yz}, x^2z\rangle$.

8. Let $\mathcal{W}$ be the solid region in $\mathbb{R}^3$ bounded by the xy-plane and the graph of $z = 4 - x^2 - y^2$. Let $\mathcal{S}$ be the boundary surface of $\mathcal{W}$ with outward orientation and let $\mathbf{F} = \langle xz\sin(yz) + x^3, \cos(yz), 3y^2z + 1\rangle$.

   (a) (5 points) Find $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

   (b) (5 points) Let $\mathcal{S}_1$ be the surface $z = 4 - x^2 - y^2$ for $z \geq 0$ with upward orientation (i.e. the normal has positive z-coordinate). Find $\displaystyle\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S}$.

   Hint: Use part (a)!