Midterm 1
Multivariable Calculus
(Math 32B-001)

Show your work to receive partial credits. Use of calculator is NOT allowed for this exam.

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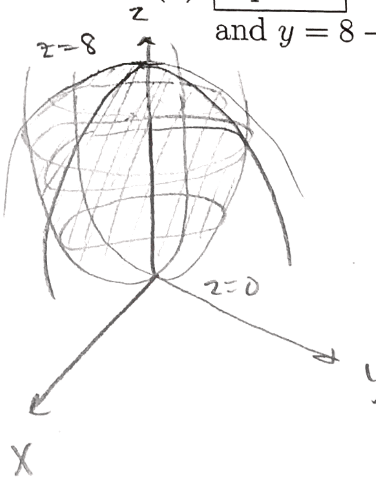
TA's Name: Casey TA Meeting Day: ~~Friday~~ Thurs 11am

Question:	1	2	3	4	Total
Points:	10	5	5	5	25
Score:	10	~~4~~ 2	5	5	~~24~~ 22

1. (a) $\boxed{\text{5 points}}$ Sketch the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Then set up a **double integral** which gives the volume of this region. (Do not evaluate the double integral).

(b) $\boxed{\text{5 points}}$ Compute the volume of the region bounded by $z = 16 - y$, $z = y$, $y = x^2$, and $y = 8 - x^2$

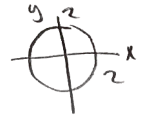


$x^2+y^2 = 8-x^2-y^2$

$2x^2+2y^2=8$

$x^2+y^2=4$

5

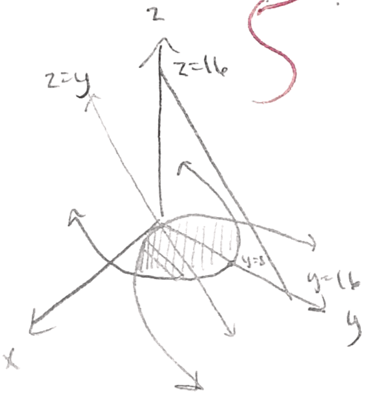
D= 

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 8-x^2-y^2-x^2-y^2 \, dy\, dx$

$\boxed{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 8-2x^2-2y^2 \, dy\, dx}$

b. $\int_{x=-2}^{2}\int_{y=x^2}^{8-x^2} 16-y-y \, dy\, dx$

5



inner: $\int_{x^2}^{8-x^2} 16-2y \, dy$

$16y - \frac{2}{2}y^2 \Big|_{x^2}^{8-x^2}$

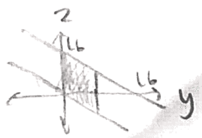
$16(8-x^2) - (8-x^2)^2 - 16x^2 + (x^4)$

$16\cdot 8 - 16x^2 - 64 + \cancel{16x^2} - \cancel{x^4} - 16x^2 + \cancel{x^4}$

$-16x^2 - 64 + 128$

outer: $\int_{-2}^{2} -16x^2 + 64 \, dx$

side view



$-\frac{16}{3}x^3 + 64x \Big|_{-2}^{2}$

$-\frac{16}{3}(8) + 64(2) + \frac{16}{3}(-8) - 64(-2)$

$-\frac{128}{3} + 128 - \frac{128}{3} + 128$

$\boxed{256 - \frac{256}{3}}$

$x^2 = 8-x^2$

$2x^2 = 8$

$x^2 = 4$

$\begin{array}{r} 16 \\ \times 8 \\ \hline 128 \\ -64 \\ \hline 64 \end{array}$

$\begin{array}{r} 128 \\ +128 \\ \hline 256 \end{array}$

$3\overline{)256}$ 8, -24, 16

2. $\boxed{\text{5 points}}$ Prove the inequality $\displaystyle\iint_{\mathcal{D}} \frac{2}{1+x^2+y^2}\, dA \le 8\pi$, where $\mathcal{D}$ is the disk $x^2+y^2 \le 4$.

$\iint \frac{2}{1+x^2+y^2} \le 8\pi$

$\iint \frac{1}{1+x^2+y^2} \le 4\pi$

$x^2+y^2 \le 4$

$1+x^2+y^2 \le 5$

$\iint \frac{1}{1+x^2+y^2} \ge \iint \frac{1}{5}$

$2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{5}\, dy\, dx$

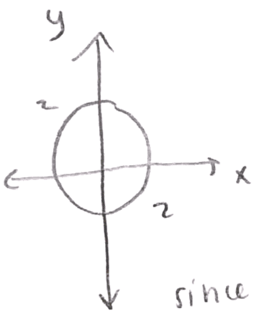
$\frac{1}{5} y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$

$\frac{1}{5}(4-x^2)^{1/2} + \frac{1}{5}(4-x^2)^{1/2}$

$2\int_{-2}^{2} \frac{2}{5}(4-x^2)^{1/2}\, dx$

$\frac{4}{5}\int_{-2}^{2}\sqrt{4-x^2}\, dx$

try w/ polar



since $1+x^2+y^2$ is distance from point above origin

min distance = 1

max distance = 5

polar: $\displaystyle\int_0^{2\pi}\int_0^2 \frac{2}{1+r^2}\, r\,dr\,d\theta$ ← solve this, the answer ×5 will be less than 4π, thus proved

$u = 1+r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$1+1=2$

I divided both sides by 2 to get

$\iint \frac{1}{1+x^2+y^2} dA \le 4\pi$

$\frac{1}{2}\int_0^2 \frac{1}{u}$

$\frac{1}{2}\ln u \Big|_0^2$

$\frac{1}{2}\ln(1+r^2)\Big|_0^2$

$\frac{1}{2}(\ln 5 - \cancel{\ln 1})$

$\int_0^{2\pi} \frac{1}{2}\ln 5\, d\theta$

$\frac{1}{2}\ln 5\, \theta \Big|_0^{2\pi}$

$\pi \ln 5$

but max distance = 5 ∴ max volume = $5\pi \ln 5$

Despite solving by polar coordinates to get my answer, I still completed the proof by showing $\ln 5 \le 4$

$5 \le e^4$

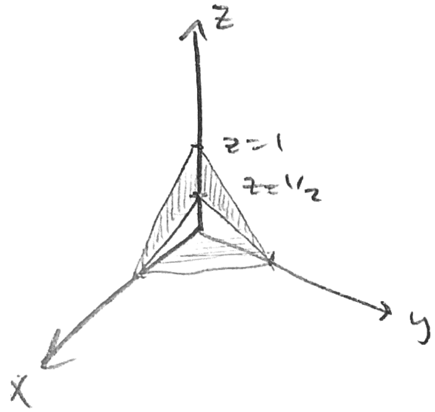
even though I made a mistake in multiplying one side by 5

$5\pi \ln 5 \le 4\pi$

$e^{5\pi} e^{\ln 5} \le e^{4\pi}$

$5e^{5\pi} \le e^{4\pi}$

3. $\boxed{\text{5 points}}$ Using **triple integration** to compute the volume of the solid region W which lies in the first octant: $x \geq 0, y \geq 0, z \geq 0$ and bounded by the planes $x + y + z = 1$ and $x + y + 2z = 1$.



$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=\frac{1}{2}-\frac{x}{2}-\frac{y}{2}}^{1-x-y} 1 \, dz \, dy \, dx$$

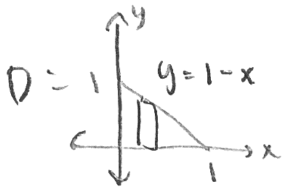
$$\int_{x=0}^{1} \int_{y=0}^{1-x} 1-x-y-\frac{1}{2}+\frac{x}{2}+\frac{y}{2} \, dy \, dx$$

inner: dy $\int_{y=0}^{1-x} \frac{1}{2}-\frac{x}{2}-\frac{y}{2} \, dy$

$$\frac{1}{2}\left(y - xy - \frac{1}{2}y^2\right)\Big|_0^{1-x}$$

$$\frac{1}{2}\left(1-x-x(1-x)-\frac{1}{2}(1-x)^2\right)$$

$$\frac{1}{2}\left(1-x-\cancel{x}+x^2-\frac{1}{2}+\cancel{x}-\frac{1}{2}x^2\right)$$



outer: dx $\frac{1}{2}\int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2} \, dx$

$$\frac{1}{2}\left[\frac{1}{2}\cdot\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x \Big|_0^1\right]$$

$$\frac{1}{2}\left[\frac{1}{2}\cdot\frac{1}{3} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right]$$

$$\frac{1}{2}\left[\frac{1}{6}\right] = \boxed{\frac{1}{12}}$$

$x+y+2z=1$

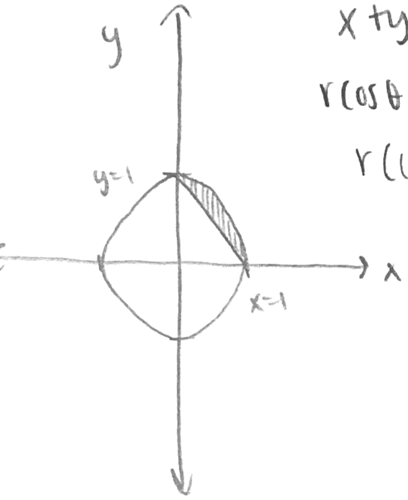
$2z = 1-x-y$

$z = \frac{1}{2}-\frac{x}{2}-\frac{y}{2}$

4. [5 points] Compute the following integral using **polar co-ordinates.**

5

$$\int\int_D f(x,y),\ dA, \quad \text{where} \quad f(x,y) = (x^2+y^2)^{-\frac{3}{2}} \text{ and } D: x^2+y^2 \le 1, x+y \ge 1.$$



$x + y \geq 1$

$r\cos\theta + r\sin\theta \geq 1$

$r(\cos\theta + \sin\theta) \geq 1$

$r = \frac{1}{\cos\theta + \sin\theta}$

$$\int_{\theta=0}^{\pi/2} \int_{r=\frac{1}{\cos\theta+\sin\theta}}^{1} (r^2)^{-3/2}\, r\, dr\, d\theta$$

inner $\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^{-2}\, dr\, d\theta$

$s \to c \to -s \to -c$

$-r^{-1} \Big|_{\frac{1}{\cos\theta+\sin\theta}}^{1}$

outer: $\int_0^{\pi/2} -1 + \cos\theta + \sin\theta \, d\theta$

$-\theta + \sin\theta - \cos\theta \Big|_0^{\pi/2}$

$-\frac{\pi}{2} + 1 - \cancel{0} + \cancel{0} - \cancel{0} + 1$

$\boxed{2 - \frac{\pi}{2}}$