Course 32B | UCLA Department of Mathematics

Spring 2021 | Instructor: Oleg Gleizer

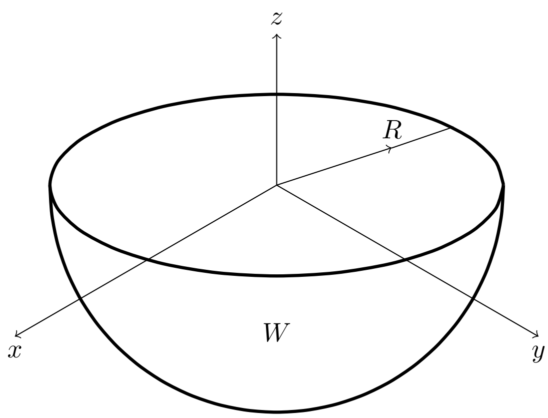
Student: | Student ID:

Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Total
$\overline{}$ 10	14	6	10	18	10	12	10	10	100

Midterm 2

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

In Problems 1–3, you will be asked to find various characteristics of a homogeneous lower hemi-sphere W of radius R and mass M centered at the origin, its equatorial plane coinciding with the (xy)-plane as shown on the picture below.



Problem 1 Find the centroid of W. **10 pts**

Volume of $W = \frac{1}{\cancel{2}_{1}} \cdot \frac{\cancel{4}^{2}}{3} \pi R^3 = \frac{2\pi R^3}{3}$

$\mu_W = \frac{\text{Mass}}{\text{Volume}}$ homogenous means μ is a constant

$\mu = \frac{3M}{2\pi R^3}$ $\qquad \mu \cdot \frac{1}{M} = \frac{3\cancel{M}}{2\pi R^3} \cdot \frac{1}{\cancel{M}} = \frac{3}{2\pi R^3}$

$x_c = y_c = 0$ by symmetry

$\frac{\pi}{2} \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

Centroid: $\left(\emptyset, \emptyset, \frac{-3R}{8}\right)$

$0 \leq \rho \leq R$

$z = \rho\cos\varphi$

$\rho^3\cos\varphi\sin\varphi$

$$\frac{3}{2\pi R^3}\int_0^{2\pi}\int_0^{R}\int_{\pi/2}^{\pi} \rho^3 \underbrace{\cos\varphi\sin\varphi}_{\frac{\sin 2\varphi}{2}} \, d\varphi\, d\rho\, d\theta$$

$$\frac{3}{4\pi R^3}\int_0^{2\pi}\int_0^{R}\int_{\pi/2}^{\pi} \rho^3 \sin 2\varphi \, d\varphi\, d\rho\, d\theta$$

$$-\frac{\cos 2\varphi}{2}\Big]_{\pi/2}^{\pi}$$

$\cos\pi = -1$

1

$$-\frac{1}{2} - \left(\frac{-(-1)}{2}\right)$$

$$-\frac{1}{2} - \frac{1}{2} = -1$$

$$\frac{-3}{4\pi R^3}\int_0^{2\pi}\int_0^{R} \rho^3 \, d\rho\, d\theta \qquad \frac{\rho^4}{4}\Big]_0^R \qquad \frac{R^4}{4}$$

$$-\frac{3R}{16\pi}\int_0^{2\pi} d\theta \qquad \theta\Big]_0^{2\pi} \qquad \cancel{2\pi}^{1} \cdot \frac{-3R}{\cancel{16\pi}_{8}}$$

Problem 2 Find the moments of inertia of W with respect to the x, y, and z axes. Hint: it will help a lot if you think about the symmetries of the sphere! **14 pts**

$y = \rho\sin\varphi\sin\theta \quad z = \rho\cos\varphi \quad dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

- I_x **6 pts**

$$I_x = \iiint_W (y^2+z^2)\,\mu(x,y,z)\,dV$$

$(\rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\varphi)\rho^2\sin\varphi$

$\mu = \frac{3M}{2\pi R^3}$

$$I_x = \frac{3M}{2\pi R^3}\int_0^{2\pi}\int_0^{R}\int_{\pi/2}^{\pi} \rho^4\sin^2\theta\sin^3\varphi + \rho^4\sin\varphi\cos^2\varphi \; d\varphi\, d\rho\, d\theta$$

$$\rho^4\sin^2\theta\int_{\pi/2}^{\pi}\sin\varphi(1-\cos^2\varphi)\,d\varphi + \rho^4\int_{\pi/2}^{\pi}\sin\varphi\cos^2\varphi\,d\varphi$$

$u = \cos\varphi$

$\frac{du}{d\varphi} = -\sin\varphi$

$$\rho^4\sin^2\theta\int_{-1}^{0} 1-u^2\,du$$

$$u - \frac{u^3}{3}\Big]_{-1}^{0}$$

$$-\left(-1-\frac{(-1)}{3}\right)$$

$\frac{2}{3} \qquad -\frac{3}{3}+\frac{1}{3}$

$u = \cos\varphi$

$$\rho^4\int_{-1}^{0} u^2\,du$$

$$\frac{u^3}{3}\Big]_{-1}^{0}$$

$$+\frac{2}{3}\rho^4\sin^2\theta + \frac{\rho^4}{3}$$

$$I_x = \frac{M}{2\pi R^3}\int_0^{2\pi}\int_0^{R} 2\rho^4\sin^2\theta + \rho^4\,d\rho\,d\theta$$

$$\frac{2\sin^2\theta\,\rho^5}{5} + \frac{\rho^5}{5}\Big]_0^R \qquad \frac{2\sin^2\theta R^5}{5} + \frac{R^5}{5}$$

The problem continues to the next page.

$$I_x = \frac{MR^2}{10\pi}\int_0^{2\pi} 2\sin^2\theta + 1 \; d\theta$$

$$I_x = \frac{MR^2}{10\pi} \int_0^{2\pi} 1 - \cos 2\theta + 1 \; d\theta$$

$$2\theta - \frac{\sin 2\theta}{2} \Big]_0^{2\pi}$$

$$4\pi - 0$$

$$I_x = \cancel{4\pi}^{\,2\;1} \cdot \frac{MR^2}{\cancel{16}_{5}\,\cancel{\pi}_{1}}$$

$$I_x = \frac{2MR^2}{5}$$

- I_y $I_y = I_x$ by symmetry **2 pts**

$$I_y = \frac{2MR^2}{5}$$

- I_z **6 pts**

$$\iiint_W (x^2+y^2)\mu(x,y,z)\, dV$$

$$\frac{3M}{2\pi R^3}\int_0^{2\pi}\int_0^R\int_{\pi/2}^{\pi} (\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)\rho^2\sin\varphi\, d\varphi\, d\rho\, d\theta$$

$$\rho^2\sin^2\varphi(\underbrace{\cancel{\cos^2\theta+\sin^2\theta}}_{1})$$

$$\frac{3M}{2\pi R^3}\int_0^{2\pi} d\theta\int_0^R \rho^4 d\rho\int_{\pi/2}^{\pi} \sin^3\varphi\, d\varphi$$

$$I_z = \frac{2MR^2}{5}$$

$$\frac{3M\cdot \cancel{2\pi}\cdot R^{\cancel{5}\,2}}{\cancel{2\pi}\cdot \cancel{R^3}\cdot 5}\int_{\pi/2}^{\pi} \sin\varphi(\sin^2\varphi)\, d\varphi$$

$$\sin\varphi(1-\cos^2\varphi)$$

$$u = \cos\varphi$$

$$\frac{du}{d\varphi} = -\sin\varphi$$

$$\frac{3MR^2}{5}\int_{-1}^{0} 1-u^2\, du$$

$$u - \frac{u^3}{3}\Big]_{-1}^{0} \qquad -\left(-1+\frac{1}{3}\right)$$

$$-\left(\frac{-2}{3}\right)$$

$$\frac{2}{\cancel{3}}\cdot\frac{\cancel{3}MR^2}{5}$$

Problem 3 Find the radii of gyration of W with respect to the x, y, and z axes. **6 pts**

- R_x **2 pts**

$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{2\cancel{M}R^2}{5\cancel{M}}} \quad \sqrt{\frac{2}{5}}\,R$$

- R_y **2 pts**

$$\sqrt{\frac{2}{5}}\,R$$

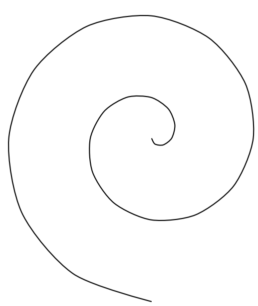
- R_z **2 pts**

$$\sqrt{\frac{2}{5}}\,R$$

Problem 4 Find the length of a *circle involute* **10 pts**

$$x = R(\cos t + t\sin t), \qquad y = R(\sin t - t\cos t), \qquad 0 \le t \le 2.$$

$$r(t) = \langle R\cos t + Rt\sin t,\ R\sin t - Rt\cos t \rangle$$



$$r'(t) = \langle \cancel{-R\sin t} \cancel{+R\sin t} + Rt\cos t,\ \cancel{R\cos t} \cancel{-R\cos t} + Rt\sin t \rangle$$

$$\| r'(t) \| = \sqrt{(Rt\cos t)^2 + (Rt\sin t)^2}$$

$$\sqrt{R^2 t^2 \underbrace{\cancel{(\sin^2 t + \cos^2 t)}}_{1}}$$

$$\| r'(t) \| = Rt$$

$$2R$$

$$\int_0^2 Rt\,dt = \frac{Rt^2}{2}\Big]_0^2 \quad 2R - 0$$

Problem 5 **18 pts**

- Use vector notations to formulate Coulomb's Law **2 pts**
for the electric point-charges Q and q.

$$\|\overrightarrow{Force}_{\text{2 charges}}\| = k\frac{Qq}{r^2} \qquad r^2 = x^2+y^2+z^2 \qquad \frac{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\sqrt{x^2+y^2+z^2}}$$

Force as a vector: multiply magnitude by direction vector

$$\overrightarrow{Force} = \frac{kQq}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Find the potential U an electric charge Q **6 pts**
placed at the origin creates at a point $P = (x, y, z)$.
Check that the found U is the potential by computing ∇U.

$$\vec{F} = \frac{kQ}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{F} = kQ\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

By inspection: $$U = \frac{-kQ}{(x^2+y^2+z^2)^{1/2}}$$

To confirm:

$$\nabla U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$$

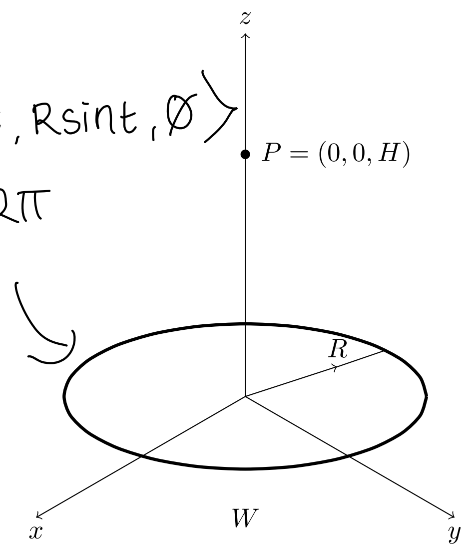
The problem continues to the next page.

$$\left\langle \frac{-1\cdot -\frac{1}{\cancel{2}}\cdot \cancel{2}x\cdot kQ}{(x^2+y^2+z^2)^{3/2}}, \frac{+1\cdot +\frac{1}{\cancel{2}}\cdot \cancel{2}y\cdot kQ}{(x^2+y^2+z^2)^{3/2}}, \frac{+1\cdot +\frac{1}{\cancel{2}}\cdot \cancel{2}z\cdot kQ}{(x^2+y^2+z^2)^{3/2}} \right\rangle = \vec{F}$$

- An electric charge Q is evenly distributed along a thin circular wire of radius R located in the xy-plane and centered at the origin as on the picture below. Find the potential of the wire at the point $P = (0, 0, H)$. **10 pts**

$r(t) = \langle R\cos t, R\sin t, 0 \rangle$

$0 \le t \le 2\pi$



charge density along wire =

$\mu = \dfrac{Q}{2\pi R} = \dfrac{\text{charge}}{\text{length}}$

$d\ell = \|r'(t)\|\, dt = R\, dt$

$U = \dfrac{-kQ}{r} \Rightarrow dU = \dfrac{\frac{-kQ}{2\pi R} \cdot R\,dt}{r}$

$r'(t) = \langle -R\sin t, R\cos t, 0 \rangle$

$\|r'(t)\| = \sqrt{\underbrace{R^2\sin^2 t + R^2\cos^2 t}_{1}}$

$= R$

$r = \sqrt{(0 - R\cos t)^2 + (0 - R\sin t)^2 + H^2}$

$R^2\underbrace{(\cos^2 t + \sin^2 t)}_{1} + H^2$

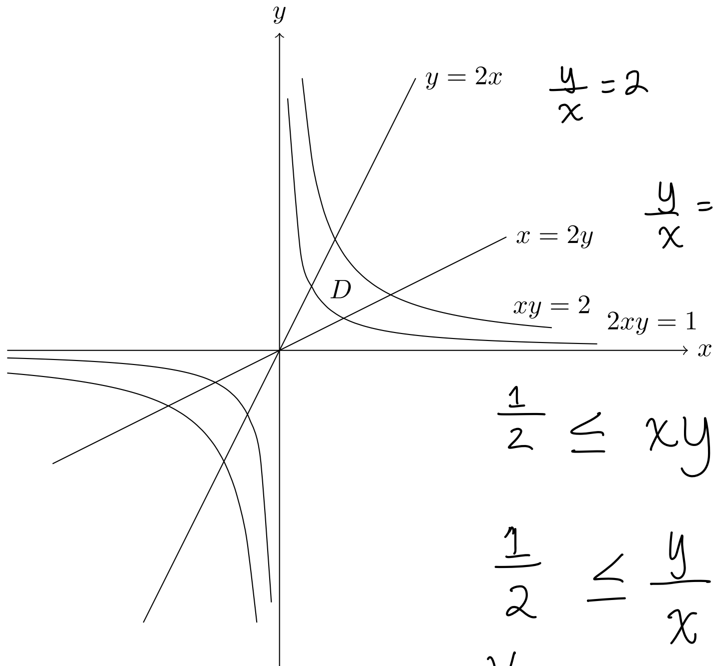
$r = \sqrt{R^2 + H^2}$

$$\int_0^{2\pi} \frac{\frac{-kQ}{2\pi \cancel{R}} \cdot \cancel{R}\, dt}{\sqrt{R^2 + H^2}} \quad \frac{-kQ}{2\pi\sqrt{R^2+H^2}} \cdot t \Big]_0^{2\pi}$$

$$\frac{-kQ}{\cancel{2\pi}\sqrt{R^2+H^2}} \cdot \cancel{2\pi} = \frac{-kQ}{\sqrt{R^2+H^2}}$$

Problem 6 **10 pts**

Find the area of the region D in the first quadrant bounded by the curves on the picture below. Leave the answer in the symbolic form, but simplify it as much as possible.



$\frac{y}{x} = 2$

$\frac{y}{x} = \frac{1}{2}$

$\frac{1}{2} \leq xy \leq 2$

$\frac{1}{2} \leq \frac{y}{x} \leq 2$

$$G(x,y) \xmapsto{u,v} \Big(\overset{u}{xy}, \overset{v}{\frac{y}{x}}\Big)$$

$$dudv = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} dxdy$$

$$\frac{y}{x} + \frac{y}{x}$$

$$dxdy = \frac{x}{2y} dudv$$

$$dxdy = \frac{1}{2v} dudv$$

$$dudv = \frac{2y}{x} dxdy$$

$$\frac{1}{2}\int_{1/2}^{2}\int_{1/2}^{2}\frac{1}{v}\,du\,dv$$

$$\ln v\Big]_{1/2}^{2}$$

$$\ln 2-\ln\tfrac{1}{2}=\ \ln(2\cdot 2)=\ln 4$$

$$\frac{\ln 4}{2}\int_{1/2}^{2}dv$$

$$v\Big]_{1/2}^{2}$$

$$\frac{4}{2}-\frac{1}{2}=\frac{3}{2}$$

$$\frac{3}{2}\cdot\frac{\ln 4}{2}=\frac{3}{4}\ln 4$$

Problem 7 **12 pts**

- Write down the formulas that express the rectangular coordinates x, y, and z as functions of the spherical coordinates ρ, ϕ, and θ. **2 pts**

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

- Find the Jacobian of the coordinate change. **10 pts**

$$G(\rho, \varphi, \theta) \xrightarrow{x, y, z} (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

$$dx\,dy\,dz = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} d\rho\, d\varphi\, d\theta$$

$$\cos\varphi \begin{vmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} + \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$\cos\varphi\left(\rho^2\sin\varphi\cos\varphi\cos^2\theta + \rho^2\sin\varphi\cos\varphi\sin^2\theta\right) \quad 1$$

$$\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \quad 1$$

$$\cos\varphi\left(\rho^2\sin\varphi\cos\varphi\right)$$

$$\rho\sin\varphi\left(\rho\sin^2\varphi\right)$$

$$\rho^2\sin^3\varphi$$

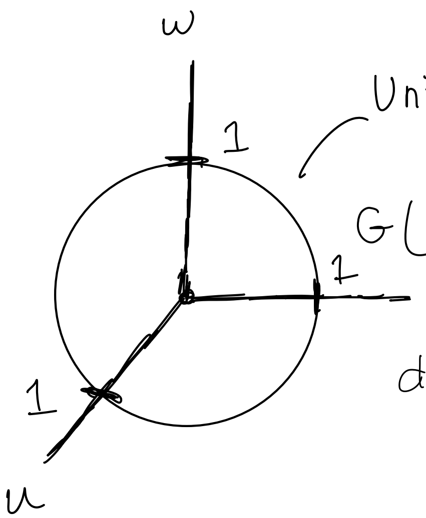
$$\rho^2\sin\varphi\cos^2\varphi + \rho^2\sin^3\varphi$$

$$\rho^2\sin\varphi\,(\underbrace{\cancel{\cos^2\varphi + \sin^2\varphi}}_{1}) = \rho^2\sin\varphi$$

Problem 8 **10 pts**

Find the volume of the following ellipsoid. Hint: a coordinate change helps!

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Unit sphere : $dV = du\,dv\,dw = \frac{4}{3}\pi$

$$G(u,v,w) \xmapsto{x,y,z} (\overset{x}{au}, \overset{y}{bv}, \overset{z}{cw})$$

$$dx\,dy\,dz = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} du\,dv\,dw$$

$$= a \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} du\,dv\,dw$$

$$\underbrace{dx\,dy\,dz}_{\text{volume of ellipsoid is abc * volume of unit sphere}} = abc \quad du\,dv\,dw = abc \cdot \frac{4}{3}\pi = \frac{4\pi abc}{3}$$

$$\int_0^{2\pi}\int_0^{1}\int_0^{\pi} abc\, du\,dv\,dw$$

$$abc \underbrace{\int_0^{2\pi}\int_0^{1}\int_0^{\pi} du\,dv\,dw}_{\text{volume of unit sphere}}$$

Problem 9 **10 pts**

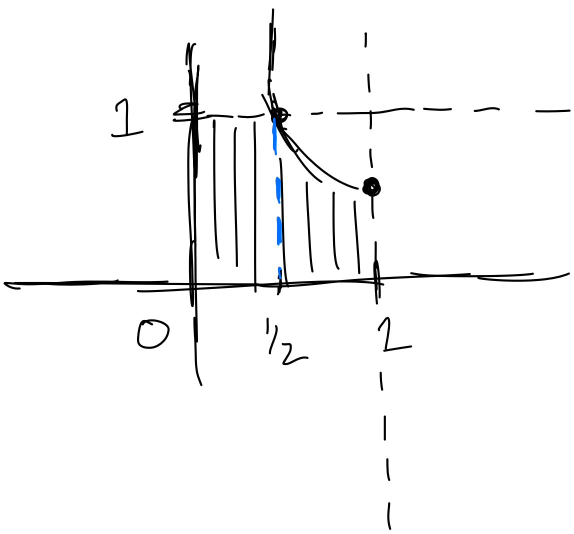
The real numbers X and Y are randomly and independently chosen between zero and one. The joint probability density is

$$p(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability P that the product XY is at most 1/2.

$xy \leq \frac{1}{2}$

$y \leq \frac{1}{2x}$



$1 = \frac{1}{2x}$

$2x = 1$

$x = \frac{1}{2}$

$y = \frac{1}{2(1)}$

$y = \frac{1}{2}$

$\int_0^{1/2} \int_0^1 1 \, dy \, dx = \frac{1}{2}$

$\frac{1}{2} + \int_{1/2}^{1} \int_0^{\frac{1}{2x}} 1 \, dy \, dx$

$\int_{1/2}^{1} \frac{1}{2x} \, dx$

$\frac{1}{2} \ln x \Big|_{1/2}^{1}$

$\frac{1}{2} \cdot 0 - \frac{1}{2} \ln \frac{1}{2}$

$\frac{1}{2} - \frac{1}{2} \ln \frac{1}{2}$